TITLE

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Our momentum map is

$$J_L(q,p) = p_{ai}\delta_{q^a} - p_{ai}{}^{k} q^{a\ell}{}_{k} \partial_\ell \delta_{q^a}$$

To declutter formulas let us define the $(1,1)$ tensors on the ath particle

$$\mu_{ai}{}^{\ell} := p_{ai}{}^{k} q^{a\ell}{}_{k}$$

so that J_L can be written as

$$J_L(q,p) = p_{ai}\delta_{q^a} - \mu_{ai}{}^{\ell} \partial_\ell \delta_{q^a}.$$

The velocity field is

$$u^i(x) = \left(K * J(q,p)\right)^i(x) = \int K^{ij}(x-y)\left(p_{aj}\delta_{q^a}(y) - \mu_{aj}{}^{\ell}\partial_\ell \delta_{q_a}(y)\right) dy$$

by the definition of $\partial_\ell \delta_{q^a}$ we find

$$= p_{aj} K^{ij}(x - q^a) + \mu_{aj}{}^{\ell} \frac{\partial}{\partial y^\ell}|_{y=q^a} K^{ij}(x-y)$$

and by the chain rule we find

$$= p_{aj} K^{ij}(x - q^a) - \mu_{aj}{}^{\ell} \partial_\ell K^{ij}(x - q^a)$$

The Hamiltonian is

$$\begin{aligned} H(q,p) = h(J_L(q,p)) &= \frac{1}{2}\langle J_L(q,p) \mid K * J_L(q,p)\rangle \\ &= \frac{1}{2}\langle J_L(q,p) \mid u\rangle \\ &= \frac{1}{2}\langle p_{ai}\delta_{q^a} - \mu_{ai}{}^{\ell}\partial_\ell\delta_{q^a} \mid p_{bj}K^{ij}(x-q^b) - \mu_{bj}{}^{m}\partial_m K^{ij}(x-q^b)\rangle \\ &= \frac{1}{2}p_{ai}p_{bj}K^{ij}(q^a-q^b) + \frac{1}{2}\mu_{ai}{}^{\ell}p_{bj}\partial_\ell K^{ij}(q^a-q^b) - \frac{1}{2}p_{ai}\mu_{bj}{}^{m}\partial_m K^{ij}(q^a-q^b) \\ &\quad - \frac{1}{2}\mu_{ai}{}^{\ell}\mu_{bj}{}^{m}\partial_\ell\partial_m K^{ij}(q^a-q^b) \end{aligned}$$

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The index shuffling you did before still works, except this time the terms which couple μ and p do not cancel, but add together. In particular, let us do some index shuffling of the third term

$$T_3 = \frac{-1}{2} p_{ai} {\mu_{bj}}^m \partial_m K^{ij}(q^a - q^b)$$

The symmetry $K^{ij} = K^{ji}$ allows us to swap the indices i and j as follows

$$T_3 = \frac{-1}{2} p_{aj} {\mu_{bi}}^m \partial_m K^{ij}(q^a - q^b).$$

We can then swap the dummy indices a and b to obtain

$$T_3 = \frac{-1}{2} p_{bj} {\mu_{ai}}^m \partial_m K^{ij}(q^b - q^a).$$

Finally, we can use the fact that $\partial_m K^{ij}(x) = -\partial_m K^{ij}(-x)$ to obtain

$$T_3 = \frac{1}{2} p_{bj} {\mu_{ai}}^m \partial_m K^{ij}(q^a - q^b).$$

Just for a completeness, replace the dummy index m with ℓ to obtain

$$T_3 = \frac{1}{2} p_{bj} {\mu_{ai}}^\ell \partial_\ell K^{ij}(q^a - q^b).$$

We see this is exactly the second term in $H(q,p)$. So the Hamiltonian is

$$H(q,p) = \frac{1}{2} p_{ai} p_{bj} K^{ij}(q^a - q^b) + {\mu_{ai}}^\ell p_{bj} \partial_\ell K^{ij}(q^a - q^b) - \frac{1}{2} {\mu_{ai}}^\ell {\mu_{bj}}^m \partial_\ell \partial_m K^{ij}(q^a - q^b)$$