AMATH 352: Problem Set 7

Your name

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Due: Friday March 3, 2017

LU factorization:

1. In deriving the LU factorization we implicitly relied on the fact that the product of two lower triangular matrices is also lower triangular. Prove this result for the specific case of lower triangular matrices in $\mathbb{R}^{3\times 3}$. That is prove that the product of two lower triangular 3×3 matrices is also lower triangular.

2. Using the results from lecture, what is the inverse of each of the following matrices?

   (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & m_1 & 1 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & 0 & 0 \\ m_1 & 1 & 0 \\ m_2 & 0 & 1 \end{bmatrix}$

   (c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 9 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 6 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 8 & 0 & 0 & 1 \end{bmatrix}$

   Solution:

   (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & m_1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -m_1 & 1 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & 0 & 0 \\ m_1 & 1 & 0 \\ m_2 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -m_1 & 1 & 0 \\ -m_2 & 0 & 1 \end{bmatrix}$

   (c) $\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 9 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 6 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 8 & 0 & 0 & 1 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -6 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 \\ -8 & -9 & -5 & 1 \end{bmatrix}$

3. Compute, by hand, the LU decomposition of the following matrices. Show your work.

   (a) $\begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 2 \\ 0 & 7 & 8 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 1 & 2 & 3 \\ 4 & 0 & 5 & 11 \\ 12 & 10 & 15 & -13 \\ 8 & 6 & 12 & -1 \end{bmatrix}$

Solution:

(a)

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 2 \\ 0 & 7 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & -4 \\ 0 & 7 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 7 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 2 \\ 0 & 7 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & -4 \\ 0 & 0 & -20 \end{bmatrix}$$

$$\boldsymbol{L} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -7 & 1 \end{bmatrix}, \ \boldsymbol{U} = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & -4 \\ 0 & 0 & -20 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -6 & 0 & 1 & 0 \\ -4 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 & 3 \\ 4 & 0 & 5 & 11 \\ 12 & 10 & 15 & -13 \\ 8 & 6 & 12 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 & 3 \\ 0 & -2 & 1 & 5 \\ 0 & 4 & 3 & -31 \\ 0 & 2 & 4 & -13 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -6 & 0 & 1 & 0 \\ -4 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 & 3 \\ 4 & 0 & 5 & 11 \\ 12 & 10 & 15 & -13 \\ 8 & 6 & 12 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 & 3 \\ 0 & -2 & 1 & 5 \\ 0 & 0 & 5 & -21 \\ 0 & 0 & 5 & -8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -6 & 0 & 1 & 0 \\ -4 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 & 3 \\ 4 & 0 & 5 & 11 \\ 12 & 10 & 15 & -13 \\ 8 & 6 & 12 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 & 3 \\ 0 & -2 & 1 & 5 \\ 0 & 0 & 5 & -21 \\ 0 & 0 & 0 & 13 \end{bmatrix}$$

$$\boldsymbol{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 6 & -2 & 1 & 0 \\ 4 & -1 & 1 & 1 \end{bmatrix}, \ \boldsymbol{U} = \begin{bmatrix} 2 & 1 & 2 & 3 \\ 0 & -2 & 1 & 5 \\ 0 & 0 & 5 & -21 \\ 0 & 0 & 0 & 13 \end{bmatrix}$$

4. Given the following LU factorization $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$ and the vectors $\boldsymbol{x}, \boldsymbol{y}$:

$$\boldsymbol{L} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad \boldsymbol{U} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

compute

$$\boldsymbol{A}^{-1}\boldsymbol{x} + \boldsymbol{A}^{-2}\boldsymbol{y}$$

without forming $\boldsymbol{A}$, $\boldsymbol{A}^2$, $\boldsymbol{A}^{-1}$, or $\boldsymbol{A}^{-2}$ explicitly. Show your work.

Solution:

$\boldsymbol{A}^{-1}\boldsymbol{x} = (\boldsymbol{L}\boldsymbol{U})^{-1}\boldsymbol{x} = \boldsymbol{U}^{-1}\boldsymbol{L}^{-1}\boldsymbol{x}$. We may solve for $\boldsymbol{x}' = \boldsymbol{L}^{-1}\boldsymbol{x}$ by forward substituion using $\boldsymbol{L}$, and then solve for $\boldsymbol{U}^{-1}\boldsymbol{x}'$ by using back-substitution through $\boldsymbol{U}$.

$$\boldsymbol{x}' = \boldsymbol{L}^{-1}\boldsymbol{x}$$

$$\boldsymbol{x}'_1 = 1,\ \boldsymbol{x}'_2 = \frac{0-1}{1} = -1,\ \boldsymbol{x}'_3 = \frac{1--1}{1} = 2$$

$$\boldsymbol{x}' = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$\boldsymbol{A}^{-1}\boldsymbol{x} = \boldsymbol{z} = \boldsymbol{U}^{-1}\boldsymbol{x}'$$

$$\boldsymbol{z}_3 = 2,\ \boldsymbol{z}_2 = \frac{-1-2}{1} = -3,\ \boldsymbol{z}_1 = \frac{1-2}{1} = -1$$

$$\boldsymbol{A}^{-1}\boldsymbol{x} = \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}$$

To form $\boldsymbol{A}^{-2}\boldsymbol{y}$, let's instead evaluate the substitution steps symbolically, and then simply turn the crank four times. For $\boldsymbol{L}^{-1}$ we want to find $\boldsymbol{x}$, where

$$\boldsymbol{L}\boldsymbol{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{aligned} \boldsymbol{x}_1 &= a \\ \boldsymbol{x}_2 &= b - \boldsymbol{x}_1 = b - a \\ \boldsymbol{x}_3 &= c - \boldsymbol{x}_2 = c - (b - a) \end{aligned}$$

Since all of the pivot values are 1, these are straightforward to write down. Similarly for $\boldsymbol{U}^{-1}$ we want

$$\boldsymbol{U}\boldsymbol{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{aligned} \boldsymbol{x}_3 &= c \\ \boldsymbol{x}_2 &= b - \boldsymbol{x}_3 = b - c \\ \boldsymbol{x}_1 &= a - \boldsymbol{x}_3 = a - c \end{aligned}$$

We are now ready to compute the last term of the sum:

$$\begin{aligned} \boldsymbol{A}^{-2}\boldsymbol{y} &= \boldsymbol{U}^{-1}\boldsymbol{L}^{-1}\boldsymbol{U}^{-1}\boldsymbol{L}^{-1}\boldsymbol{y} \\ &= \boldsymbol{U}^{-1}\boldsymbol{L}^{-1}\boldsymbol{U}^{-1} \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \\ &= \boldsymbol{U}^{-1}\boldsymbol{L}^{-1} \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix} \\ &= \boldsymbol{U}^{-1} \begin{bmatrix} 0 \\ 3 \\ -4 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 7 \\ -4 \end{bmatrix} \end{aligned}$$

And finally:

$$\boldsymbol{A}^{-1}\boldsymbol{x} + \boldsymbol{A}^{-2}\boldsymbol{y} = \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 7 \\ -4 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}$$