AMATH 352: Problem Set 4

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February 3, 2017

Due: Friday February 10, 2017

Instructions:

Complete the following problems. Submit a single Matlab script to Scorelator with your solutions to the problems in the Matlab section. You have five chances to submit your assignment. Your score for the Matlab portion of the assignment will be the highest score from your five submissions. The remaining problems (1-5) make up the written portion of this assignment. Turn in a write up of these problems digitally (via Canvas) by 5:00pm of the due date.

Linear functions

1. The following linear functions can be written as $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$ for some matrix $\boldsymbol{A}$ (not necessarily square). In each case, determine $\boldsymbol{A}$.

   (a) $\boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} 3x_1 - 2x_2 \\ 4x_2 - x_3 \end{bmatrix}, \quad \forall \boldsymbol{x} \in \mathbb{R}^3.$

   (b) $\boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} x_3 \\ x_4 \\ x_2 \\ x_1 \end{bmatrix}, \quad \forall \boldsymbol{x} \in \mathbb{R}^4.$

   (c) $\boldsymbol{f}(\boldsymbol{x}) = 2x_1 - 4x_3 + 5x_4, \quad \forall \boldsymbol{x} \in \mathbb{R}^4.$

   (d) $\boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} x_1\cos(\theta) + x_3\sin(\theta) \\ x_2 \\ -x_1\sin(\theta) + x_3\cos(\theta) \end{bmatrix}, \quad \forall \boldsymbol{x} \in \mathbb{R}^3$, for some fixed $\theta \in [0, 2\pi)$. This type of matrix is called a rotation matrix.

Range, Rank, and Nullspace of a matrix:

2. Compute the rank, dimension of the nullspace, and a basis of the nullspace for the following matrices:

   (a) $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

   (b) $\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

   (c) $\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \\ 0 & 1 \end{bmatrix}$

   (d) $\boldsymbol{A} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

3. Consider the matrix
$$\boldsymbol{A} = \boldsymbol{u}\boldsymbol{v}$$
   where $\boldsymbol{u} \in \mathbb{R}^{n\times 1}$, $\boldsymbol{v} \in \mathbb{R}^{1\times m}$, and neither $\boldsymbol{u}$ nor $\boldsymbol{v}$ is the zero vector. If we view $\boldsymbol{u}$ and $\boldsymbol{v}$ as column and row vectors, respectively, then $\boldsymbol{A}$ is called the *outer product* of $\boldsymbol{u}$ and $\boldsymbol{v}$. Find the rank of $\boldsymbol{A}$ and a basis for the range of $\boldsymbol{A}$.

4. Consider a real matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ with $m \geq n$. Suppose $\boldsymbol{A}$ is full rank, i.e. $\text{rank}(\boldsymbol{A}) = n$. Find the nullspace of $\boldsymbol{A}$.

Transpose and adjoint of a matrix:

5. What can you say about the diagonal entries of a skew-symmetric matrix? How about the diagonal entries of a complex hermitian matrix?

Matlab:

Create and submit to Scorelator a single Matlab script (.m file) which performs the following tasks. Your code should generate the dat files `A1.dat`, `A2.dat`,..., and `A9.dat`.

6. Watch the video tutorial on working with matrices in Matlab found at `https://youtu.be/DoW6aduQLt0`. You do not need to submit anything for this problem. You can find the code used in the tutorial in the file `matrix_demo.m` on the course website

7. Use the `rank` and `null` Matlab commands to check your answers in problem 2. You do not need to submit anything for this problem.

8. Define the following matrices in Matlab (the `zeros` and `ones` commands and for loops might be useful here):

   - $\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 & \dots & 9 & 10 \\ 1 & 2 & 3 & \dots & 9 & 10 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 2 & 3 & \dots & 9 & 10 \end{bmatrix}$,
   - $\boldsymbol{B} = \begin{bmatrix} 1 & 8 & -1 & 0 \\ 4 & 2 & 3 & 3 \\ 5 & -3 & 0 & \pi \\ 4 & 1 & -6 & 2 \end{bmatrix}$,
   - $\boldsymbol{C} = (99)^2 \begin{bmatrix} -2 & 1 & 0 & \dots & 0 \\ 1 & -2 & 1 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & 1 & -2 & 1 \\ 0 & \dots & 0 & 1 & -2 \end{bmatrix}$

   with $\boldsymbol{A} \in \mathbb{R}^{100\times10}, \boldsymbol{C} \in \mathbb{R}^{100\times100}$. Note the factor of $(99)^2$ in the definition of $\boldsymbol{C}$. $\boldsymbol{C}$ is called a *tridiagonal* matrix since only its diagonal entries are nonzero. In fact it fits a more specific description: it is also a *Toeplitz* matrix.

   Next define the following vector in Matlab (you may wish to familiarize yourself with the `linspace` command):

   - $\boldsymbol{x} \in \mathbb{R}^{1\times100}$ where $\boldsymbol{x}$ is the row vector of 100 equally spaced points between 1 and 2 (inclusive).

   Complete the following exercises:

   (a) Save $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ in the files `A1.dat`, `A2.dat`, and `A3.dat`, respectively.

   (b) Compute $6\boldsymbol{B}^4 + \frac{1}{2}\sin(\boldsymbol{B}) - 2\boldsymbol{B}$ and store the result in `A4.dat`.

   (c) Compute the values of the function $f(x) = x^3 + \frac{2}{x^2} - 4\cos(x)$ at 100 equally spaced points between 1 and 2 (inclusive) by manipulating the vector `x` you defined earlier. Store the results in a 1×100 row vector and save this vector in `A5.dat`. If you save the output in a row vector called `z`, you can visualize f with the `plot` command: `plot(x,z)`. Be sure to comment out or remove the plot command before submitting the assignment to Scorelator.

9. **Powers of matrices:**. Consider the following matrix:

$$\boldsymbol{M} = \frac{1}{65}\begin{bmatrix} 17 & 24 & 1 & 8 & 15 \\ 23 & 5 & 7 & 14 & 16 \\ 4 & 6 & 13 & 20 & 22 \\ 10 & 12 & 19 & 21 & 3 \\ 11 & 18 & 25 & 2 & 9 \end{bmatrix}.$$

Observe that all the rows and columns of $\boldsymbol{M}$ sum to 1.

(a) Define $\boldsymbol{M}$ in Matlab and save the result in `A6.dat`.

(b) In Matlab, compute $\boldsymbol{M}^2, \boldsymbol{M}^3, \boldsymbol{M}^4, \boldsymbol{M}^{10}, \boldsymbol{M}^{20}$, etc. Do you see a pattern emerging? What number do all the entries of $\boldsymbol{M}^n$ approach as $n \to \infty$? Store your answer in `A7.dat` (it should be a single number).

(c) What happens if we change the constant factor of $\frac{1}{65}$ slightly? Define a new matrix $\boldsymbol{M_1}$ the same way as $\boldsymbol{M}$, but with the constant factor $\frac{1}{65}$ replaced by $\frac{1}{10}$. Experiment with $\boldsymbol{M_1}$ raised to various powers, as you did with $\boldsymbol{M}$. What number do the entries of $(\boldsymbol{M_1})^n$ approach as $n \to \infty$? Choose one of the answers in the left column of the table in Figure 1. Store the number corresponding to your answer in `A8.dat` (choose a number in the left column and save the corresponding number in the right column).

(d) Now change the constant in front to $\frac{1}{100}$ to define a new matrix, $\boldsymbol{M_2}$, and perform the same experiments. What number do the entries of $(\boldsymbol{M_2})^n$ approach now as $n \to \infty$? Save your answer in `A9.dat` (it should be a single number).

Later in the course we will learn how to characterize the behavior of matrices when they are raised to higher and higher powers when we study *eigenvalues* and *eigenvectors* of matrices.

Your answer	Number to save in `A8.dat`
0	0
1	1
-1	2
0.2	3
∞	4
$-\infty$	5
$\frac{1}{10}$	6

Figure 1: Answer legend for problem 9(c)