AMATH 352: Problem set 1 solutions

Dave Moore, dmmfix@uw.edu

January 10, 2017

Column Vectors:

1. Show that for any $\alpha \in \mathbb{R}$ and any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^m$, $\alpha(\boldsymbol{x} + \boldsymbol{y}) = \alpha\boldsymbol{x} + \alpha\boldsymbol{y}$.

   Solution:
   By definitions (1.2) and (1.3):

$$\begin{aligned}
\alpha(\boldsymbol{x}+\boldsymbol{y}) &= \begin{bmatrix} \alpha(x_1+y_1) \\ \alpha(x_2+y_2) \\ \vdots \\ \alpha(x_m+y_m) \end{bmatrix} \\
&= \begin{bmatrix} \alpha x_1 + \alpha y_1 \\ \alpha x_2 + \alpha y_2 \\ \vdots \\ \alpha x_m + \alpha y_m \end{bmatrix} \\
&= \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_m \end{bmatrix} + \begin{bmatrix} \alpha y_1 \\ \alpha y_2 \\ \vdots \\ \alpha y_m \end{bmatrix} \\
&= \alpha\boldsymbol{x} + \alpha\boldsymbol{y}
\end{aligned}$$

2. Show that for every vector $\boldsymbol{x} \in \mathbb{R}^m$, $1\boldsymbol{x} = \boldsymbol{x}$.

   Solution:
   By definition (1.3):

$$1\boldsymbol{x} = \begin{bmatrix} 1x_1 \\ 1x_2 \\ \vdots \\ 1x_m \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \boldsymbol{x}$$

Norms and inner products:

3. Determine whether each of the following functions is a norm. Justify your answer, i.e. if you claim it is a norm, show that it satisfies the five criteria discussed in class, and if not, give a concrete example that shows it doesn't satisfy one of the criteria.

   (a) $\boldsymbol{x} \in \mathbb{R}^3$, $\|\boldsymbol{x}\| := \|\boldsymbol{x}\|_2 - \|\boldsymbol{x}\|_1$

   (b) $\boldsymbol{x} \in \mathbb{R}^3$, $\|\boldsymbol{x}\| := \|\boldsymbol{x}\|_2 + \|\boldsymbol{x}\|_1$

   (c) $\boldsymbol{x} \in \mathbb{R}^3$, $\|\boldsymbol{x}\| :=$ the number of nonzero entries in $\boldsymbol{x}$.

   (d) $\boldsymbol{x} \in \mathbb{R}^3$, $\|\boldsymbol{x}\| := 4|x_1| + |x_1 - x_2 + x_3| + |x_2 + x_3|$

Solution:

(a) $\|\boldsymbol{x}\|_2 - \|\boldsymbol{x}\|_1$ violates (1.4b). If $\boldsymbol{x} = \begin{bmatrix}1 & 0 & 0\end{bmatrix}^t$, then

$$\left\|\boldsymbol{x}\right\|_2 - \left\|\boldsymbol{x}\right\|_1 = 1 - 1 = 0$$

(b) $\|\boldsymbol{x}\|_{2+1} = \|\boldsymbol{x}\|_2 + \|\boldsymbol{x}\|_1$ is a valid norm. Since both $\|\cdot\|_2$ and $\|\cdot\|_1$ are valid norms, we can observe that

   i. $\forall \boldsymbol{x}, \|\boldsymbol{x}\|_{2+1} \geq 0$ because both the sub-components are ≥ 0.

   ii. Since both components are ≥ 0, $\|\boldsymbol{x}\|_{2+1} = 0 \implies \|\boldsymbol{x}\|_2 = 0$ and $\|\boldsymbol{x}\|_1 = 0$. These are satisified only when $\boldsymbol{x} = \mathbf{0}$.

   iii. $\|\mathbf{0}\|_{2+1} = \|\mathbf{0}\|_2 + \|\mathbf{0}\|_1 = 0$

   iv. We know that the sub-components already satisfy (1.4d) so we have

$$\begin{aligned}\|\alpha\boldsymbol{x}\|_{2+1} &= \|\alpha\boldsymbol{x}\|_2 + \|\alpha\boldsymbol{x}\|_1 \\ &= |\alpha|\|\boldsymbol{x}\|_2 + |\alpha|\|\boldsymbol{x}\|_1 \\ &= |\alpha|(\|\boldsymbol{x}\|_2 + \|\boldsymbol{x}\|_1) \\ &= |\alpha|\|\boldsymbol{x}\|_{2+1}\end{aligned}$$

   v. It satisfies the triangle inequality, since

$$\|\boldsymbol{x} + \boldsymbol{y}\|_{2+1} \leq \|\boldsymbol{x}\|_{2+1} + \|\boldsymbol{y}\|_{2+1}$$

   can be rewritten as

$$\|\boldsymbol{x} + \boldsymbol{y}\|_2 + \|\boldsymbol{x} + \boldsymbol{y}\|_1 \leq (\|\boldsymbol{x}\|_2 + \|\boldsymbol{y}\|_2) + (\|\boldsymbol{x}\|_1 + \|\boldsymbol{y}\|_1)$$

   Which is the sum of two inequalities we already know are satisfied

$$\begin{aligned}\|\boldsymbol{x} + \boldsymbol{y}\|_2 &\leq \|\boldsymbol{x}\|_2 + \|\boldsymbol{y}\|_2 \\ \|\boldsymbol{x} + \boldsymbol{y}\|_1 &\leq \|\boldsymbol{x}\|_1 + \|\boldsymbol{y}\|_1\end{aligned}$$

   by the properties of the $\|\cdot\|_2$ and $\|\cdot\|_1$ norms.

(c) "Number of nonzeros" violates (1.4d), since the number of non-zero entries is unchanged when multiplied by a non-zero scalar and therefore

$$\left\|\alpha \boldsymbol{x}\right\|_{nonzero} \neq |\alpha| \left\|\boldsymbol{x}\right\|_{nonzero}$$

(d) $4|x_1| + |x_1 - x_2 + x_3| + |x_2 + x_3|$ is a valid norm. (Written as $\|.\|_{sum}$ below.)

i. $\forall \boldsymbol{x}$, $\|\boldsymbol{x}\|_{sum} \geq 0$ because the absolute value is always non-negative.

ii. Because the absolute value is 0 iff its input is 0, in order to verify that $\|\boldsymbol{x}\|_{sum} = 0 \implies \boldsymbol{x} = 0$ we can treat the components as independent and look for solutions to $A\boldsymbol{x} = \mathbf{0}$ where

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

That is, we're asking if A has a null space. Since $det(A) = -8$, it has no null space, and the only solution for $\|\boldsymbol{x}\|_{sum} = 0$ is $\boldsymbol{x} = \mathbf{0}$.

iii. If $\boldsymbol{x} = 0$,

$$\begin{aligned} \|\boldsymbol{x}\|_{sum} &= (4|0| + |0 - 0 + 0| + |0 + 0|) \\ &= 0 \end{aligned}$$

iv. Recalling that for scalars, $|\alpha s| = |\alpha||s|$

$$\begin{aligned} \|\alpha\boldsymbol{x}\|_{sum} &= 4|\alpha x_1| + |\alpha x_1 - \alpha x_2 + \alpha x_3| + |\alpha x_2 + \alpha x_3| \\ &= 4|\alpha||x_1| + |\alpha(x_1 - x_2 + x_3)| + |\alpha(x_2 + x_3)| \\ &= 4|\alpha||x_1| + |\alpha||x_1 - x_2 + x_3| + |\alpha||x_2 + x_3| \\ &= |\alpha|(4|x_1| + |x_1 - x_2 + x_3| + |x_2 + x_3|) \\ &= |\alpha|\|\boldsymbol{x}\|_{sum} \end{aligned}$$

v. Let's first consider the sub-terms in isolation. If we can show that they individually satisfy the triangle inequality, then their sum will as well. So we want to show that

$$4|x_1 + y_1| \leq 4|x_1| + 4|y_1| \tag{1}$$

$$|x_1 + y_1 - x_2 - y_2 + x_3 + y_3| \leq |x_1 - x_2 + x_3| + |y_1 - y_2 + y_3| \tag{2}$$

$$|x_2 + y_2 + x_3 + y_3| \leq |x_2 + x_3| + |y_2 + y_3| \tag{3}$$

Recalling that for scalars, $|a + b| \leq |a| + |b|$ is satisfied by the absolute value, we can see that (1) is straighforwardly true. By substitution

$$\begin{aligned} a &= x_1 - x_2 + x_3 \\ b &= y_1 - y_2 + y_3 \end{aligned}$$

we can rewrite (2) as $|a + b| \leq |a| + |b|$. By similar substitution, we can similarly verify (3) is satisfied. Therefore their sum

$$\|\boldsymbol{x} + \boldsymbol{y}\|_{sum} \leq \|\boldsymbol{x}\|_{sum} + \|\boldsymbol{y}\|_{sum}$$

is a valid inequality.

4. Find and sketch the closed unit ball in $\mathbb{R}^2$ for the infinity norm. Justify your drawing (your answer for this problem should be more than just a drawing).

Solution:

The set of points in $\mathbb{R}^2$ that satisfy $\|\boldsymbol{x}\|_\infty = 1$ are those for which either $x_1 = \pm 1$ and $|x_2| \leq |x_1|$, or vice versa. This is just a square with sides of length 2 centered at the origin.