AMATH 352: Problem Set 3

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Due: Friday January 27, 2017

Instructions:

Complete the following problems. Turn in a write up of these problems digitally (via Canvas) by 5:00pm of the due date.

Linear dependence and independence:

1. Determine whether the given vectors or functions are linearly independent. Justify your answers.

   (a) $\begin{bmatrix} -1 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

   (c) $f_1(x) = x^2 + 3$, $f_2(x) = 1 - x$, and $f_3(x) = (x+1)^2$

   (d) $f_1(x) = 1$, $f_2(x) = \sin(\pi x)$,and $f_3(x) = \cos(\pi x)$

   ### Solution:

   (a) These vectors are linearly independent, since we can see that a linear combination equal to 0

   $$\alpha \begin{bmatrix} -1 \\ -3 \\ 3 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix} = \mathbf{0}$$

means that $-\alpha + -\beta = 0 \implies \alpha = \beta$ from the first component of the sum. But if that is true, the second component

$$\begin{aligned} -3\alpha + 5\beta &= 0 \\ -3\alpha + 5\alpha &= 0 \\ 2\alpha &= 0 \end{aligned}$$

implies that α must be zero. Since we know that $\alpha = \beta$, only the trivial combination satisfies the original equation and the vectors are linearly independent.

(b)
$$\left(3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Is a non-trivial combination that yields $\mathbf{0}$, so these vectors are not linearly independent.

(c) These elements of $\mathcal{P}^2$ are not linearly independent. If we evaluate

$$\alpha f_1 + \beta f_2 + \delta f_3$$

with $\alpha = 1, \beta = -2, \delta = -1$, we find

$$\begin{aligned} \alpha f_1 + \beta f_2 + \delta f_3 &= (x^2 + 3) - 2(1 - x) - (x + 1)^2 \\ &= x^2 + 3 - 2 + 2x - (x^2 + 2x + 1) \\ &= x^2 - x^2 + 2x - 2x + 3 - 2 - 1 \\ &= 0 \end{aligned}$$

Since this is a non-trivial combination, the functions are not linearly independent.

(d) Looking for $\alpha, \beta, \delta \in \mathbb{R}$ such that

$$\alpha + \beta sin(\pi x) + \delta cos(\pi x) = 0$$

This must hold $\forall x \in \mathbb{R}$. If we take $x = 0$, we find that

$$\alpha + \beta 0 + \delta 1 = 0 \implies \delta = -\alpha$$

If we take $x = 1$, we find

$$\alpha + \beta 1 + \delta 0 = 0 \implies \beta = -\alpha$$

Evaluating at, say $x = 1/4$, we find

$$\begin{aligned} \alpha - \frac{2}{\sqrt{2}}\beta - \frac{2}{\sqrt{2}}\delta &= 0 \\ (1 - \sqrt{2})\alpha &= 0 \end{aligned}$$

Only $\alpha = 0$, and therefore $\beta = 0, \delta = 0$ satisfies this equation. Since only the trivial combination yields $\mathbf{0}$, the functions are linearly independent.

2. Show that the following vectors are linearly independent:

$$\begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} b \\ c \\ 0 \end{bmatrix}, \begin{bmatrix} d \\ e \\ f \end{bmatrix},$$

where $a, b, c, d, e, f \in \mathbb{R}$, provided that $a, b, f \neq 0$. Does this set form a basis for $\mathbb{R}^3$? Why or why not?

Solution:

Since $dim(\mathbb{R}^3) = 3$, and we have 3 basis vectors, it suffices to show that they are linearly independent. By working backwards, we can show that no $\alpha, \beta, \delta \in \mathbb{R}$ satisfies the equation

$$\alpha \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} b \\ c \\ 0 \end{bmatrix} + \delta \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

other than the trivial solution. If we examine only the x_3 component of the sum, we can see that $\delta f = 0 \implies \delta = 0$. But in that case, $\beta c = 0$ as well, since the second component becomes

$$\beta c + \delta e = \beta c = 0 \implies \beta = 0$$

We can similarly argue that α must 0, so the set is linearly indepedent, and forms a basis for $\mathbb{R}^3$.

Basis and dimension:

3. Find the dimension of and a basis for the following real linear spaces. Justify your answers.

(a) $S = \{\boldsymbol{x} \in \mathbb{R}^3 : x_1 + 3x_2 - 5x_3 = 0\}$

(b) $S = \{\boldsymbol{x} \in \mathbb{R}^3 : x_1 = x_3\}$

(c) $S = \{p \in \mathcal{P}_3 : p(1) = 0\}$

(d) $S = \{p \in \mathcal{P}_3 : p(-1) = 0, p'(1) = 0\}$

Solution:

(a) By starting with $x_1 = 1$, we can immediately find two basis vectors by holding x_2 or x_3 to 0.

$$\begin{bmatrix} 1 \\ -1/3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1/5 \end{bmatrix}$$

We started with $dim(\mathbb{R}^3) = 3$ and added one constraint, so we would expect $dim(S) = 2$, but let's attempt to add another vector to the set.

$$\alpha \begin{bmatrix} 1 \\ -1/3 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 0 \\ 1/5 \end{bmatrix} + \delta \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

so we have

$$\begin{aligned} \alpha + \beta + \delta a &= 0 \\ -\alpha + 3\delta b &= 0 \\ \beta + 5\delta c &= 0 \end{aligned}$$

The last two equations imply

$$\begin{aligned} \alpha &= 3\delta b \\ \beta &= -5\delta c \end{aligned}$$

By substitution, we find that the first equation becomes

$$3\delta b - 5\delta c + \delta a = \delta(a + 3b - 5c) = 0$$

Because we know that $(a + 3b - 5c) = 0$, since $[abc]^t \in S$, we can find a non-trivial combination with $(\alpha, \beta, \delta \neq 0)$ with this arbitrary element of S. So the two vectors already form a basis for S.

(b) If we chose $x_1 = 1$ and $x_1 = 0$ as our starting points, we can immediately generate two basis vectors

$$\begin{bmatrix} 1 \\ \alpha \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ \beta \\ 0 \end{bmatrix}$$

we know that $\beta \neq 0$, if it were, the second basis vector would be $\mathbf{0}$ which is disallowed. We can arbitrarily choose $\alpha = 0$ and add $[101]^t$ to the set. To show that we then cannot add another basis vector for $\alpha \neq 0$, observe that if we did, we could form the sum

$$a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ \alpha \\ 1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0$$

By selecting $a = -b$ and $c = -\alpha$, we can create a non-trivial combination summing to $\mathbf{0}$. Therefore

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

is a basis for S.

(c) p has the form $a + bx + cx^2 + dx^3$. If we evaluate this at 1, we find that our constraint is $a + b + c + d = 0$. By setting $a = 1$, and holding all but one of the other components to zero, we can find 3 basis functions

$$\begin{aligned} f_1 &= 1 - x \\ f_2 &= 1 - x^2 \\ f_3 &= 1 - x^3 \end{aligned}$$

To prove this is a sufficient basis, let's see if we are unable do find a non-trivial combination if we add another arbitrary member of S to this set. So for $f = \alpha + \beta x + \gamma x^2 + \delta x^3 \in S$, we need to show that

$$af_1 + bf_2 + cf_3 + df = 0$$

Has no solutions other than the trivial one. If we consider each coefficient of the resulting polynomial indepedently, we find that the following relations must hold for all x.

$$\begin{aligned} a + b + c + d\alpha &= 0 \\ -a + d\beta &= 0 \\ -b + d\gamma &= 0 \\ -c + d\delta &= 0 \end{aligned}$$

We can satisfy the last 3 equations for $d \neq 0$ with

$$\begin{aligned} a &= d\beta \\ b &= d\gamma \\ c &= d\delta \end{aligned}$$

By substitution in the first equation, it reduces to $d(\alpha + \beta + \gamma + \delta) = 0$ Because $f \in S$, this will hold for any value, including $d \neq 0$. This means that we cannot add any further members of S to the set without making it linearly dependent. f_{1-3} form a basis for S.

(d) p has the form $a+bx+cx^2+dx^3$, which implies that $p' = b+2cx+3dx^2$. By evaluating our constraints, we find that

$$\begin{aligned} a - b + c - d &= 0 \\ b + 2c + 3d &= 0 \end{aligned}$$

Beginning with $a = 1, c = -1$, we can see that by substitution, $b+d = 0$ and $b + 3d = 2$. By subtracting, we see that $2d = 2 \implies d = 1$, which means that $b = -1$, giving us

$$f_1 = 1 - x - x^2 + x^3$$

as our first basis function.

Choosing $a = 1, b = 1$, we conclude that $c = d$ and $b = -5c \implies c = -1/5$ which yields (scaling by 5 for simplicity)

$$f_2 = 5 + 5x - x^2 - x^3$$

as our second basis function.

Let's confirm that these are a complete basis by adding an arbitrary element and see if we can form a non-trivial combination yielding $\mathbf{0}$. If we examine the coefficients of the sum with $f = \alpha+\beta x+\gamma x^2+\delta x^3 \in S$

$$af_1 + bf_2 + cf = 0$$

We find the following equations must hold

$$\begin{aligned} a + 5b + c\alpha &= 0 \\ -a + 5b + c\beta &= 0 \\ -a - b + c\gamma &= 0 \\ a - b + c\delta &= 0 \end{aligned}$$

and by adding

$$\begin{aligned} 10b + c(\alpha+\beta) = 0 &\implies b = \frac{-c(\alpha+\beta)}{10} \\ -2a + c(\gamma-\delta) = 0 &\implies a = \frac{c(\gamma-\delta)}{2} \end{aligned}$$

Turning the crank and substituting in the original set of equations, we find

$$\begin{aligned} \frac{c(\gamma-\delta)}{2} + 5\frac{-c(\alpha+\beta)}{10} + c\alpha &= 0 \\ -\frac{c(\gamma-\delta)}{2} + 5\frac{-c(\alpha+\beta)}{10} + c\beta &= 0 \\ -\frac{c(\gamma-\delta)}{2} - \frac{-c(\alpha+\beta)}{10} + c\gamma &= 0 \\ \frac{c(\gamma-\delta)}{2} - \frac{-c(\alpha+\beta)}{10} + c\delta &= 0 \end{aligned}$$

Pulling out c and simplifying:

$$\begin{aligned} c\left((\gamma-\delta) - (\alpha+\beta) + 2\alpha\right) &= c\left(\alpha - \beta + \gamma - \delta\right) = 0 \\ c\left(-(\gamma-\delta) - (\alpha+\beta) + 2\beta\right) &= -c\left(\alpha - \beta + \gamma - \delta\right) = 0 \\ c\left(-5(\gamma-\delta) + (\alpha+\beta) + 10\gamma\right) &= c\left(\alpha + \beta + 5\gamma + 5\delta\right) = 0 \\ c\left(5(\gamma-\delta) + (\alpha+\beta) + 10\delta\right) &= c(\alpha + \beta + 5\gamma + 5\delta) = 0 \end{aligned}$$

Note that the first two lines are immediately of the form $c \times 0$, since we know all of the parts in parentheses are 0, since $f \in S$. So those two lines are satisfied for any c.

For the last two parentheticals, we can subtract $(\alpha - \beta + \gamma - \delta)$ since we know $f \in S$. So for instance, the last line becomes

$$(\alpha + \beta + 5\gamma + 5\delta) - (\alpha - \beta + \gamma - \delta) = (2\beta + 4\gamma + 6\delta)$$

The last parentheticals are simply $2(\beta + 2\gamma + 3\delta)$, which we know equals 0. So we can find a non-trivial combination summing to 0, since these are satisfied for $c \neq 0$, and f_1, f_2 form a basis for S.