# AMATH 352: Problem Set 6

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## Due: Friday February 24, 2017

## Instructions:

Complete the following problems. Submit two Matlab files to Scorelator; one which generates the dat files containing your solutions to the problems in the Matlab section and `rel_err.m`, your implementation of the relative error function. You have five chances to submit your assignment. Your score for the Matlab portion of the assignment will be the highest score from your five submissions. The remaining problems (1-6) make up the written portion of this assignment. Turn in a write up of these problems digitally (via Canvas) by 5:00pm of the due date.

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## Norms and Inner products:

1. Let $\boldsymbol{W}$ be an invertible matrix. Show that the map

$$\|\boldsymbol{x}\|_{\boldsymbol{W}} = \|\boldsymbol{W}\boldsymbol{x}\|_2$$

   is a norm on $\mathbb{R}^m$. Is it still a norm if $\boldsymbol{W}$ is singular? Why or why not?

2. Consider a real square matrix $\boldsymbol{M} \in \mathbb{R}^{n\times n}$. Suppose $\boldsymbol{M}$ is symmetric and full rank. Furthermore, suppose $\boldsymbol{M}$ is positive definite, i.e. it satisfies

$$\forall \boldsymbol{x} \in \mathbb{R}^n,\ \boldsymbol{x} \neq \boldsymbol{0} \implies \boldsymbol{x}^T\boldsymbol{M}\boldsymbol{x} > 0.$$

   Show that the map $\langle\cdot,\cdot\rangle_{\boldsymbol{M}} : \mathbb{R}^n \to \mathbb{R}$ given by

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle_{\boldsymbol{M}} = \boldsymbol{x}^T\boldsymbol{M}\boldsymbol{y}$$

   is an inner product on $\mathbb{R}^n$.

3. Show that the function $\langle\cdot,\cdot\rangle : C^0([-1,1],\mathbb{R}) \to \mathbb{R}$ given by

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx$$

   is an inner product on $C^0([-1,1],\mathbb{R})$. Note that $C^0([-1,1],\mathbb{R})$ denotes the space of continuous functions that take input from [-1,1] and produce output in $\mathbb{R}$.

# Conditioning:

4. In this problem we show that orthogonal matrices are "perfectly conditioned" in the sense that their condition numbers with respect to the 2-norm are always 1. Suppose $\boldsymbol{O} \in \mathbb{R}^{n \times n}$ is an orthogonal matrix.

   (a) Show that $\|\boldsymbol{O}\|_2 = 1$. (Hint: recall that orthogonal matrices preserve the 2-norms of vectors).

   (b) Show that $\boldsymbol{O}^T$ is an orthogonal matrix.

   (c) Show that $\kappa_2(\boldsymbol{O}) = 1$.

# Operation count:

5. Find the number of necessary floating point operations required to compute the following operations (using the big-oh notation introduced in class). Explain your reasoning in each case.

   (a) Compute the sum $\boldsymbol{A} + \boldsymbol{B}$ for $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{m \times n}$.

   (b) Compute the outer product $\boldsymbol{u}\boldsymbol{v}^T$ for $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$.

   (c) Compute the product $\boldsymbol{A}\boldsymbol{x}$ for $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ where $\boldsymbol{A}$ is upper triangular.

6. **Comparing growth rates:** This exercise is meant to give you an idea of how quickly the number of flops required to solve a problem can increase when one increases the problem size, depending on the complexity of the algorithm used. Construct a table comparing $n, n\log_2(n), n^2, n^3, 2^n$, and $n!$ for $n = 2, 4, 8, 16, 64, 512$. (You may need to use something like Wolfram Alpha to compute some of these quantities).

# Matlab:

Create and submit to Scorelator a single Matlab script (.m file) which performs the following tasks along with the function file `rel_err.m` which contains the implementation of your relative error function. Your code should generate the dat files `A1.dat`, `A2.dat`,..., and `A7.dat`.

7. There is a relationship between the (2-norm) condition number of a square matrix $\boldsymbol{A}$ and the condition number of the matrix $\boldsymbol{A}^T\boldsymbol{A}$. Perform any numerical experiments you like to try to determine this relationship (the `cond` command should prove useful here–you can read about it at `http://www.mathworks.com/help/matlab/ref/cond.html`). Suppose the 2-norm condition number of $\boldsymbol{A}$ were 34. What would the 2-norm condition number of $\boldsymbol{A}^T\boldsymbol{A}$ be? Store your answer (the condition number of $\boldsymbol{A}^T\boldsymbol{A}$) in `A1.dat`.

8. Create a function called `rel_err` which takes as its input two vectors `b_exact` and `b_appx` and returns (gives as output) the relative error between the two vectors in the 2-norm. Below is some skeleton code to get you started:

```
function output = rel_err(b_exact, b_appx)

% Your code goes here

end
```

   Use this function to compute the relative error between the following two vectors

$$\boldsymbol{b} = \begin{bmatrix} 1 \\ 4 \\ 7 \\ 3 \\ 0 \\ -8 \end{bmatrix}, \quad \tilde{\boldsymbol{b}} = \begin{bmatrix} 1.001 \\ 4 \\ 7 \\ \pi \\ -0.00003 \\ -8.06 \end{bmatrix},$$

where $\boldsymbol{b}$ is to be treated as the exact solution and $\tilde{\boldsymbol{b}}$ as the approximation. Store the result in the file `A2.dat`. When you submit your homework on Scorelator you will need to upload **both** your main homework script and the m file containing your function, `rel_err.m`. Before clicking submit you must **highlight your main homework script** (the one that creates the dat files). If you highlight `rel_err.m` Scorelator will give you a 0.

9. This problem is meant to illustrate different situations which can arise when working with a poorly conditioned matrix. Define the following matrix and vector in Matlab:

$$\boldsymbol{A} = \begin{bmatrix} 10 & 3 & -13 \\ 4 & -5 & 1 \\ 16 & -16 & 10^{-6} \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Compute the 2-norm condition number of $\boldsymbol{A}$, $\kappa_2(\boldsymbol{A})$, and store it in the file `A3.dat`. Would you say that $\boldsymbol{A}$ is well conditioned or ill conditioned (you do not need to submit anything for this question–it is rhetorical)? Compute the vector $\boldsymbol{b}$ obtained by multiplying $\boldsymbol{A}$ and $\boldsymbol{x}$; $\boldsymbol{Ax} = \boldsymbol{b}$. Store $\boldsymbol{b}$ in the file `A4.dat`. Next define in Matlab the perturbed vectors $\boldsymbol{x_1}$, $\boldsymbol{x_2}$, and $\boldsymbol{x_3}$ given below:

$$\boldsymbol{x_1} = \begin{bmatrix} 1+0.1 \\ 2-0.1 \\ 3+0.1 \end{bmatrix}, \quad \boldsymbol{x_2} = \begin{bmatrix} 1+0.1 \\ 2+0.1 \\ 3-0.1 \end{bmatrix}, \quad \boldsymbol{x_3} = \begin{bmatrix} 1+10 \\ 2+10 \\ 3+10 \end{bmatrix}.$$

Compute the perturbed right-hand-side vectors $\boldsymbol{b_1}$, $\boldsymbol{b_2}$, and $\boldsymbol{b_3}$, defined by $\boldsymbol{Ax_i} = \boldsymbol{b_i}$, for $i = 1, 2, 3$. Treating each of these right-hand-side vectors as approximations to $\boldsymbol{b}$, calculate the relative error (with respect to the 2-norm) for each and store the results in `A5.dat`, `A6.dat`, and `A7.dat`, respectively. That is (perhaps using the function you defined earlier), compute

$$\frac{\|\boldsymbol{b} - \boldsymbol{b_i}\|_2}{\|\boldsymbol{b}\|_2} \quad \text{for } i = 1, 2, 3.$$

You should find that in some cases a small perturbation in $\boldsymbol{x}$ leads to a large change in the output, while in others a large perturbation in the input leads to a tiny change in the output.