

# AMATH 352: Problem Set 2

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**Due: Friday January 20, 2017**

## Norms:

1. Find and sketch the closed unit ball in  $\mathbb{R}^2$  for the infinity norm. Justify your drawing (your answer for this problem should be more than just a drawing).

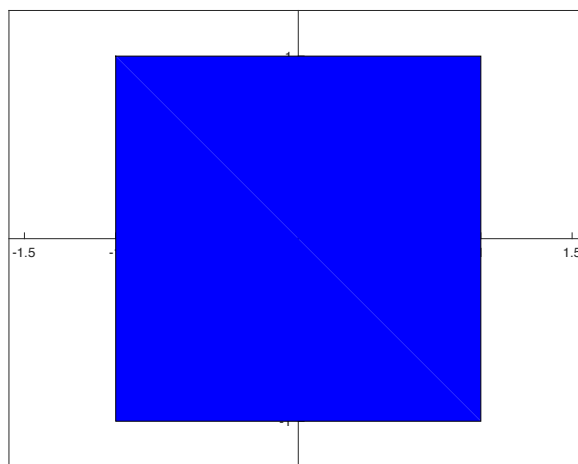
## Solution:

The closed unit ball in  $\mathbb{R}^2$  under  $\|\cdot\|_\infty$  in  $\mathbb{R}^2$  is the set of points that satisfy  $\|\mathbf{x}\|_\infty \leq 1$ . Since  $\|\mathbf{x}\|_\infty := \max(|x_i|)$ , a vector is in this set iff

$$-1 \leq x_1 \leq 1$$

$$-1 \leq x_2 \leq 1$$

This is just a square with sides of length 2 centered at the origin.



## Real Linear Spaces:

2. Verify that  $\mathbb{C}^2$  (the set of column vectors with two entries which are both in  $\mathbb{C}$ ) is a real linear space, i.e. show that it satisfies the 10 defining properties of a real linear space.

### Solution:

- (a)  $\forall a, b \in \mathbb{C}^2$

$$a + b = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix}$$

Because the complex numbers are closed under addition, each  $a_i + b_i$  term is also a complex number. The resulting vector has 2 elements, and is in  $\mathbb{C}^2$ , so  $\mathbb{C}^2$  is closed under addition.

- (b) Addition in  $\mathbb{C}^2$  is commutative because the complex numbers are commutative under addition:

$$a + b = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix} = \begin{bmatrix} b_1 + a_1 \\ b_2 + a_2 \end{bmatrix} = b + a$$

- (c) Addition in  $\mathbb{C}^2$  is associative because the complex numbers are associative:

$$a + (b + c) = \begin{bmatrix} a_1 + (b_1 + c_1) \\ a_2 + (b_2 + c_2) \end{bmatrix} = \begin{bmatrix} (a_1 + b_1) + c_1 \\ (a_2 + b_2) + c_2 \end{bmatrix} = (a + b) + c$$

- (d) The zero vector in  $\mathbb{C}^2$  is  $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ :

$$a + \mathbf{0} = \begin{bmatrix} a_1 + 0 \\ a_2 + 0 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = a$$

- (e) Because every element  $x \in \mathbb{C}$  has an additive inverse  $-x \in \mathbb{C}$ , every element of  $\mathbb{C}^2$  also has an additive inverse:

$$a + -a = \begin{bmatrix} a_1 + -a_1 \\ a_2 + -a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \mathbf{0}$$

- (f)  $\forall x \in \mathbb{R}, z \in \mathbb{C}, xz \in \mathbb{C}$ , so  $\mathbb{C}^2$  is similarly closed under scalar multiplication

$$x \in \mathbb{R}, a \in \mathbb{C}^2 \tag{1}$$

$$xa = \begin{bmatrix} xa_1 \\ xa_2 \end{bmatrix} \tag{2}$$

The  $xa_i$  terms are complex numbers, so this 2-element vector is in  $\mathbb{C}^2$ .

$$(g) \quad \forall a, b \in \mathbb{R}, x \in \mathbb{C}^2$$

$$(a + b)x = \begin{bmatrix} (a + b)x_1 \\ (a + b)x_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_1 \\ ax_2 + bx_2 \end{bmatrix} = ax + bx$$

$$(h) \quad \forall a \in \mathbb{R}, u, v \in \mathbb{C}^2$$

$$a(u + v) = a \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix} = \begin{bmatrix} a(u_1 + v_1) \\ a(u_2 + v_2) \end{bmatrix} = \begin{bmatrix} au_1 + av_1 \\ au_2 + av_2 \end{bmatrix} = au + av$$

$$(i) \quad \forall x \in \mathbb{C}^2$$

$$1x = \begin{bmatrix} 1x_1 \\ 1x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x$$

$\mathbb{C}^2$  therefore satisfies all ten requirements of a real linear space.

3. Which of the following sets  $W$  are subspaces of  $V$  (and hence are linear spaces themselves)? Justify your answers with arguments showing they are closed under addition and scalar multiplication or counterexamples showing they are not.

$$(a) \quad V = \mathbb{R}^4 \text{ and } W = \left\{ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 : x_1 + 2x_2 = 0 \text{ and } x_3 - x_4 = 0 \right\}$$

$$(b) \quad V = C^0(\mathbb{R}, \mathbb{R}) \text{ and } W = \{f \in V : f(3) = 2\}$$

$$(c) \quad V = \mathcal{F}(\mathbb{R}, \mathbb{R}) \text{ and } W \text{ is the set of all periodic functions of period 1, i.e. the set of all functions } f \text{ such that } f(x+1) = f(x) \text{ for all } x \in \mathbb{R}.$$

### Solution:

- (a) This set is closed under addition and scalar multiplication, and is therefore a subspace of  $\mathbb{R}^4$ . To show that the set is closed under addition, we want to show that  $\forall x, y \in W, z = (x + y) \in W$ .

$$z = (x + y) = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \\ x_4 + y_4 \end{bmatrix}$$

We can see that

$$\begin{aligned} z_1 + 2z_2 &= (x_1 + y_1) + 2(x_2 + y_2) \\ &= (x_1 + 2x_2) + (y_1 + 2y_2) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} z_3 - z_4 &= (x_3 + y_3) - (x_4 + y_4) \\ &= (x_3 - x_4) + (y_3 - y_4) \\ &= 0 \end{aligned}$$

To show that it is closed under scalar multiplication, need to show that  $\forall \alpha \in \mathbb{R}, x \in W$

$$\alpha x = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \\ \alpha x_4 \end{bmatrix}$$

We want to show that  $\alpha x$  satisfies the set criteria:

$$\begin{aligned} \alpha x_1 + 2\alpha x_2 &= \alpha(x_1 + 2x_2) \\ &= \alpha 0 \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \alpha x_3 - \alpha x_4 &= \alpha(x_3 - x_4) \\ &= \alpha 0 \\ &= 0 \end{aligned}$$

The set is therefore closed under scalar multiplication.

- (b) This set is not closed under addition. If we take  $f(x) = 2$ ,  $f$  is clearly in  $W$ , but

$$\begin{aligned} (f + f)(3) &= f(3) + f(3) \\ &= 2 + 2 \\ &= 4 \end{aligned}$$

is not in  $W$ .

- (c)  $W$  is closed under addition and scalar multiplication and is therefore a subspace of  $V$ . For  $f, g \in W$ , we need to show that  $h = (f+g) \in W$ .

$$\begin{aligned} h(x) &= h(x+1) \\ f(x) + g(x) &= f(x+1) + g(x+1) \end{aligned}$$

Since  $f, g \in W$  we know that

$$\begin{aligned} f(x) &= f(x+1) \\ g(x) &= g(x+1) \end{aligned}$$

The sum of these equations

$$f(x) + g(x) = f(x+1) + g(x+1)$$

shows that  $h \in W$ . We also need to show that  $\alpha f \in W$  for  $\alpha \in \mathbb{R}$ .

$$\alpha f = \alpha f(x+1)$$

For  $\alpha = 0$ , this is obviously true, and for  $\alpha \neq 0$ , we can simply divide through, which leaves  $f = f(x+1)$ , which is true since  $f \in W$ .

4. Explain why the following set is not a real linear space:

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : x_1 + x_2 x_3 = 1 \right\}.$$

**Solution:**

This set is not closed under addition. If we take two elements of the set

$$a = \begin{bmatrix} 7 \\ -2 \\ 3 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

we can see that their sum

$$c = a + b = \begin{bmatrix} 7 \\ -2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \\ 3 \end{bmatrix}$$

is not in  $W$ . ( $8 + -2 * 3 = 2 \neq 1$ )

**Span:**

5. Answer the following (justify your answers):

(a) Is  $\begin{bmatrix} 1 \\ 7 \end{bmatrix}$  in  $\text{span} \left( \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right)$ ?

(b) Is  $\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$  in  $\text{span} \left( \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right)$ ?

(c) Is  $f(x) = 1$  (the constant function that is 1 for any input  $x$ ) in  $\text{span}(2x^2 - 2, x + 3)$ ?

**Solution:**

(a)  $\begin{bmatrix} 1 \\ 7 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ , so it is in the span.

- (b)  $\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = -\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ , it is in the span of this basis.
- (c)  $f(x) = 1$  is not in  $\text{span}(2x^2 - 2, x + 3)$ .  $f$  must have zero coefficients for  $x^2$  and  $x$ , and we can see that any linear combination of the basis functions

$$\alpha(2x^2 - 2) + \beta(x + 3) = 2\alpha x^2 + \beta x - (2\alpha + 3\beta)$$

will have non-zero coefficients for  $x^2$  and  $x$  unless  $\alpha = 0$  and  $\beta = 0$ . This would force the  $(2\alpha + 3\beta)$  term to 0 as well, yielding  $f(x) = 0$  rather than  $f(x) = 1$ .