# AMATH 352: Problem Set 5

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## Matrices:

1. Consider a real matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ with $m \geq n$ which is full rank, i.e. $\operatorname{rank}(\boldsymbol{A}) = n$.

   (a) Find the nullspace of $\boldsymbol{A}$.

   (b) Show that for any vector $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{x}^T \boldsymbol{A}^T \boldsymbol{A} \boldsymbol{x} = \|\boldsymbol{A}\boldsymbol{x}\|_2^2$.

   (c) Show that the matrix $\boldsymbol{A}^T \boldsymbol{A}$ is invertible. Hint: show the nullspace of $\boldsymbol{A}^T \boldsymbol{A}$ is $\{\boldsymbol{0}\}$ by assuming $\boldsymbol{z} \in \operatorname{null}(\boldsymbol{A}^T \boldsymbol{A})$ then using the previous parts of this question to show that $\boldsymbol{z} = \boldsymbol{0}$. (Second hint: try multiplying by $\boldsymbol{z}^T$. You will also need one of the defining properties of a norm).

### Solution:

(a) By rank-nullity, $\operatorname{rank}(A) + \dim(\operatorname{null}(\boldsymbol{A})) = n$, so the dimension of the nullspace is 0 which implies $\operatorname{null}(\boldsymbol{A}) = \{\boldsymbol{0}\}$.

(b) From the properties of the tranpose operator, we can express $\boldsymbol{x}^T \boldsymbol{A}^T \boldsymbol{A} \boldsymbol{x}$ as $(\boldsymbol{A}\boldsymbol{x})^T \boldsymbol{A} x$. In other words, it is the inner product of $\boldsymbol{A}\boldsymbol{x}$ with itself. But we already know that the inner product of vector with itself is equivalent to the square of the 2-norm, so

$$\boldsymbol{x}^T \boldsymbol{A}^T \boldsymbol{A} \boldsymbol{x} = (\boldsymbol{A}\boldsymbol{x})^T \boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x} \cdot \boldsymbol{A}\boldsymbol{x} = \|\boldsymbol{A}\boldsymbol{x}\|_2^2$$

(c) From the previous answer, we know that if we create the product $\boldsymbol{x}^T (\boldsymbol{A}^T \boldsymbol{A}) \boldsymbol{x}$ with some arbitrary $\boldsymbol{x} \in \mathbb{R}^n$, that it is equivalent to $\|\boldsymbol{A}\boldsymbol{x}\|_2^2$. Assume $\boldsymbol{x} \in \operatorname{null}(\boldsymbol{A}^T \boldsymbol{A})$. Then

$$\boldsymbol{x}^T (\boldsymbol{A}^T \boldsymbol{A}) \boldsymbol{x} = \boldsymbol{x}^T \boldsymbol{0} = 0$$

But this means that $\|\boldsymbol{Ax}\|_2^2 = 0$, which implies that $\boldsymbol{Ax} = \boldsymbol{0}$ by the properties of the 2-norm. Since we know that $\boldsymbol{A}$ is full-rank, $\boldsymbol{Ax} = \boldsymbol{0} \implies \boldsymbol{x} = \boldsymbol{0}$. Therefore $\text{null}(\boldsymbol{A}^T\boldsymbol{A}) = \{\boldsymbol{0}\}$, which shows that $\boldsymbol{A}^T\boldsymbol{A}$ is invertible, since it is a square matrix of full rank.

2. Suppose $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is invertible. Find the inverse of $\boldsymbol{A}^T\boldsymbol{A}$ and show that it is symmetric.

## Solution:

Using the associative property of matrix multiplication, we can see that the inverse of $\boldsymbol{A}^T\boldsymbol{A}$ is $\boldsymbol{A}^{-1}(\boldsymbol{A}^T)^{-1}$ since

$$\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{A}^{-1}(\boldsymbol{A}^T)^{-1} = \boldsymbol{A}^T\boldsymbol{I}(\boldsymbol{A}^T)^{-1} = \boldsymbol{I}$$

But the properties of the matrix inverse allows us to express $(\boldsymbol{A}^T)^{-1}$ as $(\boldsymbol{A}^{-1})^T$, so the inverse is equivalently $\boldsymbol{A}^{-1}(\boldsymbol{A}^{-1})^T$, in other words, it's the product of a matrix with its transpose. Any such matrix product must be symmetric, since for $\boldsymbol{C} = \boldsymbol{B}\boldsymbol{B}^T$,

$$\boldsymbol{C}_{ij} = \sum_{k=1}^{m} \boldsymbol{B}_{ik}\boldsymbol{B}^T_{kj}$$

but $\boldsymbol{B}^T_{kj} = \boldsymbol{B}_{jk}$ so

$$\boldsymbol{C}_{ij} = \sum_{k=1}^{m} \boldsymbol{B}_{ik}\boldsymbol{B}_{jk} = \sum_{k=1}^{m} \boldsymbol{B}_{jk}\boldsymbol{B}_{ik} = \boldsymbol{C}_{ji}$$

3. Suppose $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{n\times n}$. Show that $\boldsymbol{AB}$ is invertible if and only if both $\boldsymbol{A}$ and $\boldsymbol{B}$ are invertible. This means you must show that if $\boldsymbol{AB}$ is invertible, so are $\boldsymbol{A}$ and $\boldsymbol{B}$ and you must also show that if $\boldsymbol{A}$ and $\boldsymbol{B}$ are invertible, so is $\boldsymbol{AB}$. Hint: use the determinant.

## Solution:

The statement that $\boldsymbol{AB}$ is invertible is equivalent to stating that its determinant is non-zero. By the properties of the determinant, we know that

$$det(\boldsymbol{AB}) = det(\boldsymbol{A})\det(\boldsymbol{B})$$

so if $det(\boldsymbol{AB}) \neq 0$, then both $\boldsymbol{A}$ and $\boldsymbol{B}$ must have non-zero determinants, and are therefore invertible. If either of $\boldsymbol{A}$ or $\boldsymbol{B}$ are non-invertible, then it follows that $det(\boldsymbol{AB}) = 0$, and $\boldsymbol{AB}$ is non-invertible.

4. Suppose $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathbb{R}^{n\times n}$ are all invertible. Show that $\boldsymbol{ABC}$ is invertible by finding a matrix $\boldsymbol{D}$ such that $(\boldsymbol{ABC})\boldsymbol{D} = \boldsymbol{D}(\boldsymbol{ABC}) = \boldsymbol{I}$.

**Solution:**

$\boldsymbol{D} = \boldsymbol{C}^{-1}\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$ satisfies the requirement, since the associative property of matrix multiplication shows that

$$\boldsymbol{ABCC}^{-1}\boldsymbol{B}^{-1}\boldsymbol{A}^{-1} = \boldsymbol{ABIB}^{-1}\boldsymbol{A}^{-1} = \boldsymbol{AIA}^{-1} = \boldsymbol{I}$$

and

$$\boldsymbol{C}^{-1}\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}\boldsymbol{ABC} = \boldsymbol{C}^{-1}\boldsymbol{B}^{-1}\boldsymbol{IBC} = \boldsymbol{C}^{-1}\boldsymbol{IC} = \boldsymbol{I}$$

5. Based on your answer to the previous problem, what do you think the inverse of $\boldsymbol{A}_1\boldsymbol{A}_2\cdots\boldsymbol{A}_k$ would be, assuming $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_k \in \mathbb{R}^{n\times n}$ are all invertible? You do not have to provide a proof, but you should briefly explain your reasoning.

**Solution:**

If we define $\boldsymbol{A} = \prod_{k=1}^{n} \boldsymbol{A}_k$, then $\boldsymbol{A}^{-1} = \prod_{k=n}^{1} \boldsymbol{A}_k^{-1}$, that is, we multiply the inverses in reverse order. This allows associative cancellation under either left- or right-multiplication by $\boldsymbol{A}^{-1}$.

6. Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ be invertible and let $\boldsymbol{B} \in \mathbb{R}^{n\times r}$ for some positive integer $r$. Show that $\boldsymbol{AX} = \boldsymbol{B}$ has a unique solution. Note that $\boldsymbol{X} \in \mathbb{R}^{n\times r}$ is a matrix.

**Solution:**
Assume that there were two solutions, $\boldsymbol{AX}_1 = \boldsymbol{B}$ and $\boldsymbol{AX}_2 = \boldsymbol{B}$. Then we can see by equating them, and subtracting, that the columns of $\boldsymbol{X}_1 - \boldsymbol{X}_2$ must be in the null space of $\boldsymbol{A}$.

$$\begin{aligned} \boldsymbol{AX}_1 &= \boldsymbol{AX}_2 \\ \boldsymbol{AX}_1 - \boldsymbol{AX}_2 &= \boldsymbol{0} \\ \boldsymbol{A}(\boldsymbol{X}_1 - \boldsymbol{X}_2) &= \boldsymbol{0} \end{aligned}$$

Since $\boldsymbol{A}$ is invertible, it has no non-trival null space. So it must be the case that $\boldsymbol{X}_1 = \boldsymbol{X}_2$ and only a unique solution is possible. Therefore

$$\begin{aligned} \boldsymbol{A}^{-1}\boldsymbol{AX} &= \boldsymbol{A}^{-1}\boldsymbol{B} \\ \boldsymbol{X} &= \boldsymbol{A}^{-1}\boldsymbol{B} \end{aligned}$$

uniquely solves the system of equations.

7. We saw in class that multiplication by an orthogonal matrix preserves lengths (with respect to the 2-norm). In this problem you will show that

they also preserve the dot product and orthogonality, that is, the dot product between two vectors is the same as the dot product of any orthogonal matrix applied to the two vectors.

Let $\boldsymbol{U} \in \mathbb{R}^{n\times n}$ be an orthogonal matrix. Show that for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$

(a) $(\boldsymbol{Ux}) \cdot (\boldsymbol{Uy}) = \boldsymbol{x} \cdot \boldsymbol{y}$,

(b) $(\boldsymbol{Ux}) \cdot (\boldsymbol{Uy}) = 0$ if and only if $\boldsymbol{x} \cdot \boldsymbol{y} = 0$.
(This means you must show that $(\boldsymbol{Ux}) \cdot (\boldsymbol{Uy}) = 0$ implies $\boldsymbol{x} \cdot \boldsymbol{y} = 0$ and that $\boldsymbol{x} \cdot \boldsymbol{y} = 0$ implies $(\boldsymbol{Ux}) \cdot (\boldsymbol{Uy}) = 0$).

## Solution:

(a) Recall that the dot product is distributive, so that

$$\boldsymbol{x} \cdot (\boldsymbol{y} + \boldsymbol{z}) = \boldsymbol{x} \cdot \boldsymbol{y} + \boldsymbol{x} \cdot \boldsymbol{z}$$

If we denote the columns of $\boldsymbol{U}$ as $\boldsymbol{U}_i$, then we can rewrite the matrix-vector product $\boldsymbol{Ux}$ as $\sum_i \boldsymbol{U}_i x_i$. This allows us to rewrite

$$\begin{aligned}(\boldsymbol{Ux}) \cdot (\boldsymbol{Uy}) &= \left(\sum_i \boldsymbol{U}_i x_i\right) \cdot \left(\sum_j \boldsymbol{U}_j y_j\right) \\ &= \sum_i \sum_j (\boldsymbol{U}_i x_i) \cdot (\boldsymbol{U}_j y_j) \\ &= \sum_i \sum_j x_i y_j (\boldsymbol{U}_i \cdot \boldsymbol{U}_j)\end{aligned}$$

However, because $\boldsymbol{U}$ is orthogonal, we know that $\boldsymbol{U}_i \cdot \boldsymbol{U}_j$ is only non-zero when $i = j$, and further, that when $i = j$, $\boldsymbol{U}_i \cdot \boldsymbol{U}_j = 1$. So all of the summation terms except $i = j$ drop out, leaving us with

$$\begin{aligned}(\boldsymbol{Ux}) \cdot (\boldsymbol{Uy}) &= \sum_i \sum_j x_i y_j (\boldsymbol{U}_i \cdot \boldsymbol{U}_j) \\ &= \sum_i x_i y_i (\boldsymbol{U}_i \cdot \boldsymbol{U}_i) \\ &= \sum_i x_i y_i \\ &= \boldsymbol{x} \cdot \boldsymbol{y}\end{aligned}$$

(b) Since we demonstrated the full equality in the previous step, there's nothing remaining to show here. Since $(\boldsymbol{Ux}) \cdot (\boldsymbol{Uy}) = \boldsymbol{x} \cdot \boldsymbol{y}$, both

$$\begin{aligned}(\boldsymbol{Ux}) \cdot (\boldsymbol{Uy}) = 0 &\implies \boldsymbol{x} \cdot \boldsymbol{y} = 0 \\ \boldsymbol{x} \cdot \boldsymbol{y} = 0 &\implies (\boldsymbol{Ux}) \cdot (\boldsymbol{Uy}) = 0\end{aligned}$$

are true.

| Nutrient | Food 1 | Food 2 | Food 3 | Total nutrients required (mg) |
|---|---|---|---|---|
| Vitamin C | 10 | 20 | 20 | 100 |
| Calcium | 50 | 40 | 10 | 300 |
| Magnesium | 30 | 10 | 40 | 200 |

Figure 1: Milligrams (mg) of nutrients per unit of food

## Systems of equations

8. A dietician is planning a meal that supplies certain quantities of vitamin C, calcium, and magnesium. Three foods will be included in the diet, their quantities measured in appropriate units. The nutrients supplied by these foods and the dietary requirements are given in Figure 1. Write a linear system of equations which represents the problem of choosing the appropriate amounts of each food that should be consumed to get the desired nutrients. Write the linear system of equations as a matrix-vector equation.

   ### Solution:

   We will denote the vector of supplied foods as $\boldsymbol{x} \in \mathbb{R}^3$, where $x_i$ represents the amount of Food $i$. As a system of equations, we are looking for solutions to

   $$\begin{aligned} 10x_1 + 20x_2 + 20x_3 &= 100 \\ 50x_1 + 40x_2 + 10x_3 &= 300 \\ 30x_1 + 10x_2 + 10x_3 &= 200 \end{aligned}$$

   or as a matrix-vector equation, we are looking for solutions to $\boldsymbol{Ax} = \boldsymbol{b}$ for

   $$\boldsymbol{A} = \begin{bmatrix} 10 & 20 & 20 \\ 50 & 40 & 10 \\ 30 & 10 & 10 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 100 \\ 300 \\ 200 \end{bmatrix}$$

9. For each of the following matrices, $\boldsymbol{A}$, and vectors, $\boldsymbol{b}$, determine the number of solutions to $\boldsymbol{Ax} = \boldsymbol{b}$.

   (a) $\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

   (b) $\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

   (c) $\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$

   (d) $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$

(e) $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 5 \\ -3 \\ 5 \end{bmatrix}$

## Solution:

(a) By inspection, $\text{range}(\boldsymbol{A}) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ since it's columns are identical. Since $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is not a scalar multiple of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\boldsymbol{b} \notin \text{range}(\boldsymbol{A})$, and there are no solutions.

(b) $\boldsymbol{A} \in \mathbb{R}^{2x2}$ and is full rank, since its columns are non-zero and not scalar multiples. So $\text{range}(\boldsymbol{A}) = \mathbb{R}^2$, and $\dim(\text{null}(\boldsymbol{A})) = 0$, which means there is a single valid solution for this equation.

(c) Recalling that for $\boldsymbol{A} \in \mathbb{R}^{nxn}$, $det(\boldsymbol{A}) \neq 0$ implies that $\boldsymbol{A}$ has full rank and a trivial nullspace, here we see that $det(\boldsymbol{A}) = 1$, which means this equation has a single valid solution.

(d) By inspection, $\boldsymbol{A}$ does not have full rank, since

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \boldsymbol{0}$$

Since the first two columns *are* linearly independent (no scalar multiple of the first column will have a non-zero second component), we need to determine if there are solutions to

$$\alpha \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

However, by examining the first and third component of the sum, we see that $\alpha = 2$ *and* $\alpha = 1$ are required to satisfy the equation, so $\boldsymbol{b} \notin \text{range}(\boldsymbol{A})$, and there are zero solutions.

(e) Noting that $\boldsymbol{A}$ has the same properties as in the previous problem, we are looking for solutions to

$$\alpha \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 5 \end{bmatrix}$$

which we can simply observe is satisfied by $\alpha = 5,\ \beta = -3$. Since $\dim(\text{null}(\boldsymbol{A})) = 1$ by the rank-nullity theorem, and $\boldsymbol{b} \in \text{range}(\boldsymbol{A})$, there are infinitely many solutions to $\boldsymbol{Ax} = \boldsymbol{b}$ as given in this problem.