AMATH 352: Problem Set 2

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Norms:

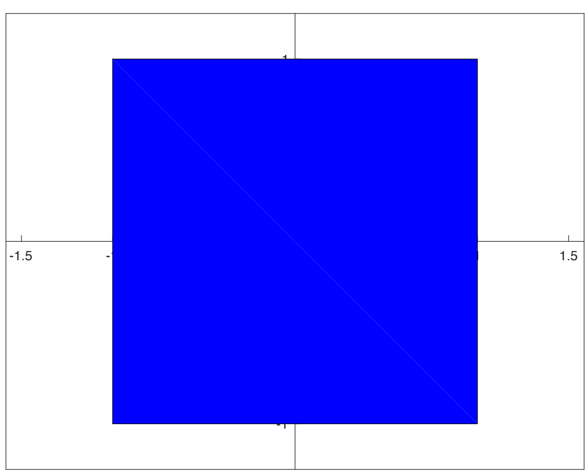
1. Find and sketch the closed unit ball in $\mathbb{R}^2$ for the infinity norm. Justify your drawing (your answer for this problem should be more than just a drawing).

Solution:

The closed unit ball in $\mathbb{R}^2$ under $\|\cdot\|_\infty$ in $\mathbb{R}^2$ is the set of points that satisfy $\|\boldsymbol{x}\|_\infty \leq 1$. Since $\|\boldsymbol{x}\|_\infty := max(|x_i|)$, a vector is in this set iff

$$-1 \leq x_1 \geq 1$$
$$-1 \leq x_2 \geq 1$$

This is just a square with sides of length 2 centered at the origin.



Real Linear Spaces:

2. Verify that $\mathbb{C}^2$ (the set of column vectors with two entries which are both in $\mathbb{C}$) is a real linear space, i.e. show that it satisfies the 10 defining properties of a real linear space.

Solution:

(a) $\forall a, b \in \mathbb{C}^2$

$$a + b = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix}$$

Because the complex numbers are closed under addition, each $a_i + b_i$ term is also a complex number. The resulting vector has 2 elements, and is in $\mathbb{C}^2$, so $\mathbb{C}^2$ is closed under addition.

(b) Addition in $\mathbb{C}^2$ is commutative because the complex numbers are commutative under addition:

$$a + b == \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix} = \begin{bmatrix} b_1 + a_1 \\ b_2 + a_2 \end{bmatrix} = b + a$$

(c) Addition in $\mathbb{C}^2$ is associative because the complex numbers are associative:

$$a + (b + c) = \begin{bmatrix} a_1 + (b_1 + c_1) \\ a_2 + (b_2 + c_2) \end{bmatrix} = \begin{bmatrix} (a_1 + b_1) + c_1 \\ (a_2 + b_2) + c_2 \end{bmatrix} = (a + b) + c$$

(d) The zero vector in $\mathbb{C}^2$ is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$:

$$a + \mathbf{0} = \begin{bmatrix} a_1 + 0 \\ a_2 + 0 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0$$

(e) Because every element $x \in \mathbb{C}$ has an additive inverse $-x \in \mathbb{C}$, every element of $\mathbb{C}^2$ also has an additive inverse:

$$a + -a = \begin{bmatrix} a_1 + -a_1 \\ a_2 + -a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \mathbf{0}$$

(f) Because $\mathbb{C}$ is closed under scalar multiplication, $\forall x \in R, z \in C, xz \in \mathbb{C}$, $\mathbb{C}^2$ is similarly closed under scalar multiplication

$$x \in \mathbb{R} \tag{1}$$

$$a \in \mathbb{C}^2 \tag{2}$$

$$xa = \begin{bmatrix} xa_1 \\ xa_2 \end{bmatrix} \tag{3}$$

The xa_i terms are complex numbers, so this 2-element vector is in $\mathbb{C}^2$.

(g) $\forall a, b \in \mathbb{R}, x \in \mathbb{C}^2$

$$(a+b)x = \begin{bmatrix}(a+b)x_1 \\ (a+b)x_1\end{bmatrix} = \begin{bmatrix}ax_1 + bx_1 \\ ax_2 + bx_2\end{bmatrix} = ax + bx$$

(h) $\forall a \in \mathbb{R}, u, v \in \mathbb{C}^2$

$$a(u+v) = a\begin{bmatrix}(u_1 + v_1) \\ (u_2 + v_2)\end{bmatrix} = \begin{bmatrix}a(u_1 + v_1) \\ a(u_2 + v_2)\end{bmatrix} = \begin{bmatrix}au_1 + av_1 \\ au_2 + av_2\end{bmatrix} = au + av$$

(i) $\forall x \in \mathbb{C}^2$

$$1x = \begin{bmatrix}1x_1 \\ 1x_2\end{bmatrix} = \begin{bmatrix}x_1 \\ x_2\end{bmatrix} = x$$

$\mathbb{C}^2$ therefore satsifies all ten requirements of a real linear space.

3. Which of the following sets W are subspaces of V (and hence are linear spaces themselves)? Justify your answers with arguments showing they are closed under addition and scalar multiplication or counterexamples showing they are not.

   (a) $V = \mathbb{R}^4$ and $W = \left\{ \boldsymbol{x} = \begin{bmatrix}x_1 \\ x_2 \\ x_3 \\ x_4\end{bmatrix} \in \mathbb{R}^4 : x_1 + 2x_2 = 0 \text{ and } x_3 - x_4 = 0 \right\}$

   (b) $V = C^0(\mathbb{R}, \mathbb{R})$ and $W = \{f \in V : f(3) = 2\}$

   (c) $V = \mathcal{F}(\mathbb{R}, \mathbb{R})$ and W is the set of all periodic functions of period 1, i.e. the set of all functions f such that $f(x+1) = f(x)$ for all $x \in \mathbb{R}$.

Solution:

(a) This set is closed under addition and scalar multiplication, and is therefore a linear subspace of $\mathbb{R}^4$. To show that the set is closed under addition, we want to show that $\forall x, y \in W, z = (x+y) \in W$.

$$z = (x+y) = \begin{bmatrix}x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_2 \\ x_4 + y_4\end{bmatrix}$$

We can see that

$$\begin{aligned} z_1 + 2z_2 &= (x_1 + y_1) + 2(x_2 + y_2) \\ &= (x_1 + 2x_2) + (y_1 + 2y_2) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} z_3 - z_4 &= (x_3 + y_3) - (x_4 + y_4) \\ &= (x_3 - x_4) + (y_3 - y_4) \\ &= 0 \end{aligned}$$

To show that it is closed under scalar multiplication, need to show that $\forall \alpha \in \mathbb{R}, x \in W, \alpha x \in W$

$$\alpha x = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \\ \alpha x_4 \end{bmatrix}$$

We want to show that αx satisfies the set criteria:

$$\begin{aligned} \alpha x_1 + 2\alpha x_2 &= \alpha(x_1 + 2x_2) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \alpha x_3 - \alpha x_4 &= \alpha(x_3 - x_4) \\ &= 0 \end{aligned}$$

The set is therefore closed under scalar multiplication.

(b) This set is not closed under addition. If we take $f(x) = 2$, $(f+f)(2) = 4$, and is therefore not in W.

(c) W is closed under addition and scalar multiplication and is therefore a subspace of V. For $f, g \in W$, we need to show that $h = (f+g) \in W$.

$$\begin{aligned} h(x) &= h(x+1) \\ f(x) + g(x) &= f(x+1) + g(x+1) \end{aligned}$$

Since both f and g are in W we know that

$$\begin{aligned} f(x) &= f(x+1) \\ g(x) &= g(x+1) \end{aligned}$$

The sum of these equations

$$f(x) + g(x) = f(x+1) + g(x+1)$$

shows that $h \in W$. We also need to show that $\alpha f \in W$ for $\alpha \in \mathbb{R}$.

$$\alpha f = \alpha f(x+1)$$

For $\alpha = 0$, this is obviously true, and for $alpha \neq 0$, we can simply divide through, which leaves $f = f(x+1)$, which is true since $f \in W$.

4. Explain why the following set is not a real linear space:

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : x_1 + x_2 x_3 = 1 \right\}.$$

Solution:
This set is not closed under addition. If we take two elements of the set

$$a = \begin{bmatrix} 7 \\ -2 \\ 3 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

we can see that their sum

$$c = a + b = \begin{bmatrix} 7 \\ -2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \\ 3 \end{bmatrix}$$

is not in W. $(8 + -2 * 3 = 2 \neq 1)$

Span:

5. Answer the following (justify your answers):

(a) Is $\begin{bmatrix} 1 \\ 7 \end{bmatrix}$ in span $\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right)$?

(b) Is $\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$ in span $\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right)$?

(c) Is $f(x) = 1$ (the constant function that is 1 for any input x) in span$(2x^2 - 2, x + 3)$?

Solution:

(a) $\begin{bmatrix} 1 \\ 7 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, so it is in the span. More generally, since the basis vectors are linearly independent (observe that $1/3 \neq 2/1$), they span all of $\mathbb{R}^2$.

(b) $\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, it is in the span of this basis.

(c) $f(x) = 1$ is not in $\text{span}(2x^2 - 2, x + 3)$. f must have zero coefficients for x^2 and x, and we can see that any linear combination of the basis functions

$$\alpha(2x^2 - 2) + \beta(x + 3) = 2\alpha x^2 + \beta x - (2\alpha + 3\beta)$$

will have non-zero coefficients for x^2 and x unless $\alpha = 0$ and $\beta = 0$. This would force the $(2\alpha + 3\beta)$ term to 0 as well, yielding $f(x) = 0$ rather than $f(x) = 1$.