AMATH 352: Problem Set 6

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Norms and Inner products:

1. Let $\boldsymbol{W}$ be an invertible matrix. Show that the map

$$\|\boldsymbol{x}\|_{\boldsymbol{W}} = \|\boldsymbol{W}\boldsymbol{x}\|_2$$

is a norm on $\mathbb{R}^m$. Is it still a norm if $\boldsymbol{W}$ is singular? Why or why not?

Solution:

Since $\boldsymbol{W}$ is invertible, its range is all of $\mathbb{R}^m$, which means that we can simply observe that $\|\cdot\|_2$ is a norm on $\mathbb{R}^m$, and therefore $\|\cdot\|_{\boldsymbol{W}}$ satisfies the 5 properties of a norm. However, if $\boldsymbol{W}$ were singular, the property that $\|\boldsymbol{x}\|_{\boldsymbol{W}} = 0 \implies \boldsymbol{x} = \boldsymbol{0}$ would be violated, since we could choose some $\boldsymbol{y} \neq 0 \in \text{null}(\boldsymbol{A})$ to obtain $\|\boldsymbol{y}\|_{\boldsymbol{W}} = \|\boldsymbol{W}\boldsymbol{y}\|_2 = \|\boldsymbol{0}\|_2 = 0$.

2. Consider a real square matrix $\boldsymbol{M} \in \mathbb{R}^{n\times n}$. Suppose $\boldsymbol{M}$ is symmetric and full rank. Furthermore, suppose $\boldsymbol{M}$ is positive definite, i.e. it satisfies

$$\forall \boldsymbol{x} \in \mathbb{R}^n,\ \boldsymbol{x} \neq \boldsymbol{0} \implies \boldsymbol{x}^T\boldsymbol{M}\boldsymbol{x} > 0.$$

Show that the map $\langle\cdot,\cdot\rangle_{\boldsymbol{M}} : \mathbb{R}^n \to \mathbb{R}$ given by

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle_{\boldsymbol{M}} = \boldsymbol{x}^T\boldsymbol{M}\boldsymbol{y}$$

is an inner product on $\mathbb{R}^n$.

Solution:

Your solution.

3. Show that the function $\langle\cdot,\cdot\rangle : C^0([-1,1],\mathbb{R}) \to \mathbb{R}$ given by

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx$$

is an inner product on $C^0([-1,1],\mathbb{R})$. Note that $C^0([-1,1],\mathbb{R})$ denotes the space of continuous functions that take input from [-1,1] and produce output in $\mathbb{R}$.

Solution:

Your solution.

Conditioning:

4. In this problem we show that orthogonal matrices are "perfectly conditioned" in the sense that their condition numbers with respect to the 2-norm are always 1. Suppose $\boldsymbol{O} \in \mathbb{R}^{n\times n}$ is an orthogonal matrix.

   (a) Show that $\|\boldsymbol{O}\|_2 = 1$. (Hint: recall that orthogonal matrices preserve the 2-norms of vectors).
   (b) Show that $\boldsymbol{O}^T$ is an orthogonal matrix.
   (c) Show that $\kappa_2(\boldsymbol{O}) = 1$.

Solution:

   (a) Your solution.
   (b) Your solution.
   (c) Your solution.

Operation count:

5. Find the number of necessary floating point operations required to compute the following operations (using the big-oh notation introduced in class). Explain your reasoning in each case.

   (a) Compute the sum $\boldsymbol{A} + \boldsymbol{B}$ for $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{m\times n}$.
   (b) Compute the outer product $\boldsymbol{u}\boldsymbol{v}^T$ for $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$.
   (c) Compute the product $\boldsymbol{A}\boldsymbol{x}$ for $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ where $\boldsymbol{A}$ is upper triangular.

Solution:

   (a) Each element of the sum requires a single addition, and there are $m\times n$ elements in $\boldsymbol{A}+\boldsymbol{B} \in \mathbb{R}^{m\times n}$, so this is $\mathrm{O}(m \times n)$.
   (b) $\boldsymbol{u}\boldsymbol{v}^T \in \mathbb{R}^{n\times n}$, and each element of the product requires a single multiplication, so the total operation count is $\mathrm{O}(n^2)$.
   (c) This is equivalent to n dot-products of n-dimensional vectors. Each dot-product requires n multiplies and $n-1$ additions, or $2n-1$ total flops. So the time complexity for n dot-products is $\mathrm{O}(n^2)$.
   Note that because $\boldsymbol{A}$ is upper-triangular, only half of the multiplies will be with non-zero components, but the resulting reduction in flops of $\frac{1}{2}$ is irrelevant for big-oh.

6. **Comparing growth rates:** This exercise is meant to give you an idea of how quickly the number of flops required to solve a problem can increase when one increases the problem size, depending on the complexity of the algorithm used. Construct a table comparing $n, n\log_2(n), n^2, n^3, 2^n$, and $n!$ for $n = 2, 4, 8, 16, 64, 512$. (You may need to use something like Wolfram Alpha to compute some of these quantities).

Solution:

	2	4	8	16	64	512
n	2	4	8	16	64	512
$nlog_2(n)$	2	8	24	64	384	4608
n^2	4	16	64	256	4096	2.621×10^5
n^3	8	64	512	4096	2.621×10^5	1.342×10^8
2^n	4	16	256	65536	1.844×10^{19}	1.340×10^{154}
$n!$	2	24	40320	2.092×10^{13}	1.268×10^{89}	3.477×10^{1166}