AMATH 352: Problem Set 6

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Norms and Inner products:

1. Let $\boldsymbol{W}$ be an invertible matrix. Show that the map

$$\|\boldsymbol{x}\|_{\boldsymbol{W}} = \|\boldsymbol{W}\boldsymbol{x}\|_2$$

is a norm on $\mathbb{R}^m$. Is it still a norm if $\boldsymbol{W}$ is singular? Why or why not?

Solution:

If $\boldsymbol{W}$ is invertible, $\|\cdot\|_{\boldsymbol{W}}$ satisfies the 4 properties of a norm:

(a) $\|\boldsymbol{x}\|_{\boldsymbol{W}} = \|\boldsymbol{W}\boldsymbol{x}\|_2 \geq 0 \ \forall\ \boldsymbol{x} \neq \boldsymbol{0}$ because $\|\cdot\|_2$ is a norm on $\mathbb{R}^m$, and null$(\boldsymbol{W}) = \{\boldsymbol{0}\}$.

(b) Since $\boldsymbol{W}$ is invertible, $\boldsymbol{W}\boldsymbol{x} = \boldsymbol{0} \implies \boldsymbol{x} = \boldsymbol{0}$, so $\|\boldsymbol{x}\|_{\boldsymbol{W}} = \|\boldsymbol{W}\boldsymbol{x}\|_2 = \boldsymbol{0} \implies \boldsymbol{x} = \boldsymbol{0}$.

(c)

$$\begin{aligned}
\|\alpha\boldsymbol{x}\|_{\boldsymbol{W}} &= \|\boldsymbol{W}(\alpha\boldsymbol{x})\|_2 \\
&= \|\alpha\boldsymbol{W}\boldsymbol{x}\|_2 \\
&= |\alpha|\|\boldsymbol{W}\boldsymbol{x}\|_2 \\
&= |\alpha|\|\boldsymbol{x}\|_{\boldsymbol{W}}
\end{aligned}$$

(d) By distributing the multiplication by $\boldsymbol{W}$, we see that all of the following statements are equivalent:

$$\begin{aligned}
\|\boldsymbol{x}+\boldsymbol{y}\|_{\boldsymbol{W}} &\leq \|\boldsymbol{x}\|_{\boldsymbol{W}} + \|\boldsymbol{y}\|_{\boldsymbol{W}} \\
\|\boldsymbol{W}(\boldsymbol{x}+\boldsymbol{y})\|_2 &\leq \|\boldsymbol{W}\boldsymbol{x}\|_2 + \|\boldsymbol{W}\boldsymbol{y}\|_2 \\
\|\boldsymbol{W}\boldsymbol{x}+\boldsymbol{W}\boldsymbol{y}\|_2 &\leq \|\boldsymbol{W}\boldsymbol{x}\|_2 + \|\boldsymbol{W}\boldsymbol{y}\|_2
\end{aligned}$$

We know the last version is satisfied because $\|\cdot\|_2$ satisfies the triangle inequality.

If $\boldsymbol{W}$ is singular, then $\|\cdot\|_{\boldsymbol{W}}$ is *not* a norm; by selecting a non-zero $\boldsymbol{x} \in$ null$(\boldsymbol{W})$, we can violate the first, second, or third properties of a norm.

2. Consider a real square matrix $\boldsymbol{M} \in \mathbb{R}^{n\times n}$. Suppose $\boldsymbol{M}$ is symmetric and full rank. Furthermore, suppose $\boldsymbol{M}$ is positive definite, i.e. it satisfies

$$\forall \boldsymbol{x} \in \mathbb{R}^n,\ \boldsymbol{x} \neq \boldsymbol{0} \implies \boldsymbol{x}^T\boldsymbol{M}\boldsymbol{x} > 0.$$

Show that the map $\langle\cdot,\cdot\rangle_{\boldsymbol{M}} : \mathbb{R}^n \to \mathbb{R}$ given by

$$\langle\boldsymbol{x},\boldsymbol{y}\rangle_{\boldsymbol{M}} = \boldsymbol{x}^T\boldsymbol{M}\boldsymbol{y}$$

is an inner product on $\mathbb{R}^n$.

Solution:

Checking the five properties of an inner product:

(a) $\langle \boldsymbol{x}, \boldsymbol{x} \rangle_{\boldsymbol{M}} \geq 0 \; \forall \; \boldsymbol{x}$ is true by the definition of a positive definite matrix.

(b) $\langle \boldsymbol{x}, \boldsymbol{x} \rangle_{\boldsymbol{M}} = 0 \implies \boldsymbol{x} = \boldsymbol{0}$ is also true because $\boldsymbol{M}$ is a square matrix of full rank, $\text{null}(\boldsymbol{M}) = \{\boldsymbol{0}\}$.

(c)

$$\begin{aligned} \langle \alpha \boldsymbol{x}, \boldsymbol{y} \rangle_{\boldsymbol{M}} &= (\alpha \boldsymbol{x})^T \boldsymbol{M} \boldsymbol{y} \\ &= \alpha (\boldsymbol{x}^T \boldsymbol{M} \boldsymbol{y}) \\ &= \alpha \langle \boldsymbol{x}, \boldsymbol{y} \rangle_{\boldsymbol{M}} \end{aligned}$$

(d)

$$\begin{aligned} \langle \boldsymbol{x}, \boldsymbol{y} \rangle_{\boldsymbol{M}} &= \boldsymbol{x}^T \boldsymbol{M} \boldsymbol{y} \\ &= \boldsymbol{y}^T \boldsymbol{M}^T \boldsymbol{x} \\ &= \boldsymbol{y}^T \boldsymbol{M} \boldsymbol{x} \\ &= \langle \boldsymbol{y}, \boldsymbol{x} \rangle_{\boldsymbol{M}} \end{aligned}$$

Since $\langle \cdot, \cdot \rangle_{\boldsymbol{M}} \in \mathbb{R}$, and a scalar is equal to it's transpose, we rewrite the expression in step 2 using the rule of matrix products and the transpose operator. Since $\boldsymbol{M}$ is symmetric, $\boldsymbol{M}^T = \boldsymbol{M}$ in the third step.

(e)

$$\begin{aligned} \langle \boldsymbol{x} + \boldsymbol{z}, \boldsymbol{y} \rangle_{\boldsymbol{M}} &= (\boldsymbol{x} + \boldsymbol{z})^T \boldsymbol{M} \boldsymbol{y} \\ &= \boldsymbol{x}^T \boldsymbol{M} \boldsymbol{y} + \boldsymbol{z}^T \boldsymbol{M} \boldsymbol{y} \\ &= \langle \boldsymbol{x}, \boldsymbol{y} \rangle_{\boldsymbol{M}} + \langle \boldsymbol{z}, \boldsymbol{y} \rangle_{\boldsymbol{M}} \end{aligned}$$

$\langle \cdot, \cdot \rangle_{\boldsymbol{M}} : \mathbb{R}^n \to \mathbb{R}$ therefore satisfies all five properties required of an inner product.

3. Show that the function $\langle \cdot, \cdot \rangle : C^0([-1, 1], \mathbb{R}) \to \mathbb{R}$ given by

$$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) dx$$

is an inner product on $C^0([-1, 1], \mathbb{R})$. Note that $C^0([-1, 1], \mathbb{R})$ denotes the space of continuous functions that take input from [-1,1] and produce output in $\mathbb{R}$.

Solution:

Checking the five properties of an inner product:

(a) $\forall \; f \neq 0, \langle f, f \rangle \geq 0$, since the product $f(x)f(x) = f^2(x)$ is always non-negative. ($f(x) \in \mathbb{R}$.)

(b) From the above we know that $\langle f, f \rangle = 0 \implies f(x) = 0$, (which is the zero element for this operator) since if f were anywhere $\neq 0$ on the interval $[-1, 1]$, the integral would be positive.

(c)

$$\begin{aligned} \langle \alpha f, g \rangle &= \int_{-1}^{1} \alpha f(x) g(x) dx \\ &= \alpha \int_{-1}^{1} f(x) g(x) dx \\ &= \alpha \langle f, g \rangle \end{aligned}$$

(d)

$$\begin{aligned}\langle f, g\rangle &= \int_{-1}^{1} f(x)g(x)dx \\ &= \int_{-1}^{1} g(x)f(x)dx \\ &= \langle g, f\rangle\end{aligned}$$

(e)

$$\begin{aligned}\langle f+h, g\rangle &= \int_{-1}^{1} (f(x)+h(x))g(x)dx \\ &= \int_{-1}^{1} (f(x)g(x)+h(x)g(x))dx \\ &= \int_{-1}^{1} f(x)g(x)dx + \int_{-1}^{1} h(x)g(x)dx \\ &= \langle f, g\rangle + \langle h, g\rangle\end{aligned}$$

$\langle f, g\rangle$ therefore satisfies all five properties required of an inner product.

Conditioning:

4. In this problem we show that orthogonal matrices are "perfectly conditioned" in the sense that their condition numbers with respect to the 2-norm are always 1. Suppose $\boldsymbol{O} \in \mathbb{R}^{n\times n}$ is an orthogonal matrix.

   (a) Show that $\|\boldsymbol{O}\|_2 = 1$. (Hint: recall that orthogonal matrices preserve the 2-norms of vectors).

   (b) Show that $\boldsymbol{O}^T$ is an orthogonal matrix.

   (c) Show that $\kappa_2(\boldsymbol{O}) = 1$.

Solution:

(a) $\|\boldsymbol{O}\|_2 = \max\frac{\|\boldsymbol{O}\boldsymbol{x}\|_2}{\|\boldsymbol{x}\|_2}$, but since $\boldsymbol{O}$ is orthogonal, we know that $\|\boldsymbol{O}\boldsymbol{x}\|_2 = \|\boldsymbol{x}\|_2$ for all $\boldsymbol{x}$. So this reduces to $\|\boldsymbol{O}\|_2 = \max\frac{\|\boldsymbol{O}\boldsymbol{x}\|_2}{\|\boldsymbol{x}\|_2} = \max\frac{\|\boldsymbol{x}\|_2}{\|\boldsymbol{x}\|_2} = 1$.

(b) By definition, an orthogonal matrix is a square matrix with $\boldsymbol{O}^T = \boldsymbol{O}^{-1}$. By inverting both sides, we can see that $(\boldsymbol{O}^T)^{-1} = \boldsymbol{O}$. Since the inverse of $\boldsymbol{O}^T$ is equal to its tranpose, $\boldsymbol{O}^T$ is orthogonal.

(c) $\kappa_2(\boldsymbol{O}) = \|\boldsymbol{O}\|_2\|\boldsymbol{O}^{-1}\|_2$, but by definition, $\boldsymbol{O}^{-1} = \boldsymbol{O}^T$, so this is equivalent to $\|\boldsymbol{O}\|_2\|\boldsymbol{O}^T\|_2$. From the previous problems, we know that $\boldsymbol{O}^T$ is orthogonal and that the the 2-norm of any orthogonal matrix is 1. So this reduces to

$$\kappa_2(\boldsymbol{O}) = \|\boldsymbol{O}\|_2\|\boldsymbol{O}^T\|_2 = 1$$

Operation count:

5. Find the number of necessary floating point operations required to compute the following operations (using the big-oh notation introduced in class). Explain your reasoning in each case.

   (a) Compute the sum $\boldsymbol{A}+\boldsymbol{B}$ for $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{m\times n}$.

   (b) Compute the outer product $\boldsymbol{u}\boldsymbol{v}^T$ for $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$.

(c) Compute the product $\boldsymbol{Ax}$ for $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ where $\boldsymbol{A}$ is upper triangular.

Solution:

(a) Each element of the sum requires a single addition, and there are $m \times n$ elements in $\boldsymbol{A}+\boldsymbol{B} \in \mathbb{R}^{m \times n}$, so this is $\mathrm{O}(mn)$.

(b) $\boldsymbol{uv}^T \in \mathbb{R}^{n \times n}$, and each element of the product requires a single multiplication, so the total operation count is $\mathrm{O}(n^2)$.

(c) This is equivalent to n dot-products of n-dimensional vectors. Each dot-product requires n multiplies and $n-1$ additions, or $2n-1$ total flops. So the total operation count is $2n^2 - n$, or $\mathrm{O}(n^2)$.

Note that because $\boldsymbol{A}$ is upper-triangular, only half of the operations will be with non-zero components, but the resulting reduction in flops of $\frac{1}{2}$ is irrelevant for big-O.

6. **Comparing growth rates:** This exercise is meant to give you an idea of how quickly the number of flops required to solve a problem can increase when one increases the problem size, depending on the complexity of the algorithm used. Construct a table comparing $n, n\log_2(n), n^2, n^3, 2^n$, and $n!$ for $n = 2, 4, 8, 16, 64, 512$. (You may need to use something like Wolfram Alpha to compute some of these quantities).

Solution:

	2	4	8	16	64	512
n	2	4	8	16	64	512
$n\log_2(n)$	2	8	24	64	384	4608
n^2	4	16	64	256	4096	2.621×10^5
n^3	8	64	512	4096	2.621×10^5	1.342×10^8
2^n	4	16	256	65536	1.844×10^{19}	1.340×10^{154}
$n!$	2	24	40320	2.092×10^{13}	1.268×10^{89}	3.477×10^{1166}