AMATH 352: Problem Set 3

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Due: Friday January 27, 2017

Instructions:

Complete the following problems. Turn in a write up of these problems digitally (via Canvas) by 5:00pm of the due date.

Linear dependence and independence:

1. Determine whether the given vectors or functions are linearly independent. Justify your answers.

   (a) $\begin{bmatrix} -1 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

   (c) $f_1(x) = x^2 + 3$, $f_2(x) = 1 - x$, and $f_3(x) = (x+1)^2$

   (d) $f_1(x) = 1$, $f_2(x) = \sin(\pi x)$,and $f_3(x) = \cos(\pi x)$

Solution:

(a) These vectors are linearly independent, since we can see that a linear combination equal to 0

$$\alpha \begin{bmatrix} -1 \\ -3 \\ 3 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix} = 0$$

means that $-\alpha + -\beta = 0 \implies \alpha = \beta$ from the first component of the sum. But if that is true, the second component

$$\begin{aligned} -3\alpha + 5\beta &= 0 \\ -3\alpha + 5\alpha &= 0 \\ 2\alpha &= 0 \end{aligned}$$

implies that α must be zero. Since we know that $\alpha = \beta$, only the trivial combination satisfies the original equation and the vectors are linearly independent.

(b)

$$\left(3\begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}1\\2\\1\end{bmatrix} - \begin{bmatrix}2\\1\\2\end{bmatrix}\right) = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Is a non-trivial combination that yields $\mathbf{0}$, so these vectors are not linearly independent.

(c) These elements of $\mathcal{P}^2$ are not linearly dependent. If we evaluate

$$\alpha f_1 + \beta f_2 + \delta f_3$$

with $\alpha = 1, \beta = -2, \delta = -1$, we find

$$\begin{aligned} \alpha f_1 + \beta f_2 + \delta f_3 &= (x^2 + 3) - 2(1 - x) - (x + 1)^2 \\ &= x^2 + 3 - 2 + 2x - (x^2 + 2x + 1) \\ &= x^2 - x^2 + 2x - 2x + 3 - 2 - 1 \\ &= 0 \end{aligned}$$

Since this is a non-trivial combination, the functions are not linearly independent.

(d) Looking for $\alpha, \beta, \delta \in \mathbb{R}$ such that

$$\alpha + \beta sin(\pi x) + \delta cos(\pi x) = 0$$

This must hold $\forall x \in \mathbb{R}$. If we take $x = 0$, we find that

$$\alpha + \beta 0 + \delta 1 = 0 \implies \delta = -\alpha$$

If we take $x = 1$, we find

$$\alpha + \beta 1 + \delta 0 = 0 \implies \beta = -\alpha$$

Evaluating at, say $x = 1/4$, we find

$$\begin{aligned} \alpha + \frac{2}{\sqrt{2}}\beta + \frac{2}{\sqrt{2}}\delta &= 0 \\ (1 - \sqrt{2})\alpha &= 0 \end{aligned}$$

Only $\alpha = 0$, and therefore $\beta = 0, \delta = 0$ satisfy this equation. Since only the trivial combination yields $\mathbf{0}$, the functions are linearly independent.

2. Show that the following vectors are linearly independent:

$$\begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} b \\ c \\ 0 \end{bmatrix}, \begin{bmatrix} d \\ e \\ f \end{bmatrix},$$

where $a, b, c, d, e, f \in \mathbb{R}$, provided that $a, b, f \neq 0$. Does this set form a basis for $\mathbb{R}^3$? Why or why not?

Solution:

Since $dim(\mathbb{R}^3) = 3$, and we have 3 basis vectors, it suffices to show that they are linearly independent. By working backwards, we can show that no $\alpha, \beta, \delta \in \mathbb{R}$ satisfies the equation

$$\alpha \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} b \\ c \\ 0 \end{bmatrix} + \delta \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

other than the trivial solution. If we examine only the x_3 component of the sum, we can see that $\delta f = 0 \implies \delta = 0$. But in that case, $\beta c = 0$ as well, since the second component becomes

$$\beta c + \delta e = \beta c = 0 \implies \beta = 0$$

We can similarly argue that α must 0. Only the trivial combination satisfies the relation, the set is linearly indepedent, and forms a basis for $\mathbb{R}^3$.

Basis and dimension:

3. Find the dimension of and a basis for the following real linear spaces. Justify your answers.

   (a) $S = \{\boldsymbol{x} \in \mathbb{R}^3 : x_1 + 3x_2 - 5x_3 = 0\}$
   (b) $S = \{\boldsymbol{x} \in \mathbb{R}^3 : x_1 = x_3\}$
   (c) $S = \{p \in \mathcal{P}_3 : p(1) = 0\}$
   (d) $S = \{p \in \mathcal{P}_3 : p(-1) = 0, p'(1) = 0\}$

Solution:

(a) By starting with $x_1 = 1$, we can immediately find two basis vectors by holding x_2 or x_3 to 0.

$$\begin{bmatrix} 1 \\ -1/3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1/5 \end{bmatrix}$$

We started with $dim(\mathbb{R}^3) = 3$ and added one constraint, so we would expect $dim(S) = 2$.

Checking: TODO!

(b) If we chose $x_1 = 1$ and $x_1 = 0$ as our starting points, we can immediately generate two basis vectors

$$\begin{bmatrix} 1 \\ \alpha \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ \beta \\ 0 \end{bmatrix}$$

we know that $\beta \neq 0$, if it were, the second basis vector would be $\mathbf{0}$ which is disallowed. We can arbitrarily add $[101]^t$ to the set. To show that we then cannot add another basis vector for $\alpha \neq 0$, observe that if we did, we could form the sum

$$a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ \alpha \\ 1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0$$

By selecting $a = -b$ and $c = -\alpha$, we can create a non-trivial combination summing to $\mathbf{0}$. Therefore

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Is a basis for S.

(c) p has the form $a + bx + cx^2 + dx^3$. If we evaluate this at 1, we find that our constraint is $a + b + c + d = 0$. By setting $a = 1$, and holding all but one of the other components to zero, we can find 3 basis functions

$$1 - x, 1 - x^2, 1 - x^3$$

Checking: TODO!

(d) p has the form $a+bx+cx^2+dx^3$, which implies that $p' = b+2cx+3dx^2$. By evaluating our constraints, we find that

$$a - b + c - d = 0$$
$$b + 2c + 3d = 0$$

Beginning with $a = 1, c = -1$, we can see that by substitution, $b+d = 0$ and $b + 3d = 2$. By subtracting, we see that $2d = 2 \implies d = 1$, which means that $b = -1$, giving us

$$1 - x - x^2 + x^3$$

as our first basis function.

Choosing $a = 1, b = 1$, we conclude that $c = d$ and $b = -5c \implies c = -1/5$ which yields

$$1 + x - \frac{1}{5}x^2 - \frac{1}{5}x^3$$

as our second basis function.

Checking: TODO!