AMATH 352: Problem Set 4

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Linear functions

1. The following linear functions can be written as $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$ for some matrix $\boldsymbol{A}$ (not necessarily square). In each case, determine $\boldsymbol{A}$.

   (a) $\boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} 3x_1 - 2x_2 \\ 4x_2 - x_3 \end{bmatrix}, \quad \forall \boldsymbol{x} \in \mathbb{R}^3.$

   (b) $\boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} x_3 \\ x_4 \\ x_2 \\ x_1 \end{bmatrix}, \quad \forall \boldsymbol{x} \in \mathbb{R}^4.$

   (c) $\boldsymbol{f}(\boldsymbol{x}) = 2x_1 - 4x_3 + 5x_4, \quad \forall \boldsymbol{x} \in \mathbb{R}^4.$

   (d) $\boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} x_1 \cos(\theta) + x_3 \sin(\theta) \\ x_2 \\ -x_1 \sin(\theta) + x_3 \cos(\theta) \end{bmatrix}, \quad \forall \boldsymbol{x} \in \mathbb{R}^3$, for some fixed $\theta \in [0, 2\pi)$. This type of matrix is called a rotation matrix.

   ### Solution:

   (a) $\boldsymbol{A} = \begin{bmatrix} 3 & -2 & 0 \\ 0 & 4 & -1 \end{bmatrix}$

   (b) $\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

   (c) $\boldsymbol{A} = \begin{bmatrix} 2 & 0 & -4 & 5 \end{bmatrix}$

(d) $\boldsymbol{A} = \begin{bmatrix} cos(\theta) & 0 & sin(\theta) \\ 0 & 1 & 0 \\ -sin(\theta) & 0 & cos(\theta) \end{bmatrix}$

Range, Rank, and Nullspace of a matrix:

2. Compute the rank, dimension of the nullspace, and a basis of the nullspace for the following matrices:

   (a) $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

   (b) $\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

   (c) $\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \\ 0 & 1 \end{bmatrix}$

   (d) $\boldsymbol{A} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

Solution:

(a) rank($\boldsymbol{A}$) = 3, since the columns are simply $\boldsymbol{e}_{1-3}$, which are linearly independent. There is therefore no non-trivial null-space for A by the rank-nullity theorem.

(b) Column 1 is simply the zero vector, which is always linearly dependent. Cols 2 and 3 are $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, which are independent, so rank($\boldsymbol{A}$) = 2.

Since there are no non-zero entries in the first column, we can see that any vector of the form $\begin{bmatrix} \alpha & 0 & 0 \end{bmatrix}^T$ will map to $\mathbf{0}$. Since dim(null($\boldsymbol{A}$)) = 1 by the rank-nullity theorem, $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ is a basis for the nullspace of $\boldsymbol{A}$.

(c) The columns are not scalar multiples of each other, so rank($\boldsymbol{A}$) = 2. By the rank-nullity theorem, the dimension of the nullspace must be zero, since $\boldsymbol{A} \in \mathbb{R}^{3x2}$.

(d) rank($\boldsymbol{A}$) = 2, since

$$-\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = \mathbf{0}$$

so the columns are not linearly independent. However, $\begin{bmatrix}1 & 2 & 3\end{bmatrix}^T$ and $\begin{bmatrix}4 & 5 & 6\end{bmatrix}^T$ are not scalar multiples, so *are* linearly independent. Rank-nullity shows that the dimension of the nullspace must be 1, since $\boldsymbol{A} \in \mathbb{R}^{3x3}$.

We note from the above equation that any vector of the form

$$\boldsymbol{x} = \alpha \begin{bmatrix}-1 \\ 2 \\ -1\end{bmatrix}$$

will yield $\boldsymbol{Ax} = \mathbf{0}$, so $\begin{bmatrix}-1 & 2 & -1\end{bmatrix}^T \in null(\boldsymbol{A})$. Since we know the dimension of the null space is 1, this forms a basis for the null space of $\boldsymbol{A}$.

3. Consider the matrix

$$\boldsymbol{A} = \boldsymbol{uv}$$

where $\boldsymbol{u} \in \mathbb{R}^{n\times 1}$, $\boldsymbol{v} \in \mathbb{R}^{1\times m}$, and neither $\boldsymbol{u}$ nor $\boldsymbol{v}$ is the zero vector. If we view $\boldsymbol{u}$ and $\boldsymbol{v}$ as column and row vectors, respectively, then $\boldsymbol{A}$ is called the *outer product* of $\boldsymbol{u}$ and $\boldsymbol{v}$. Find the rank of $\boldsymbol{A}$ and a basis for the range of $\boldsymbol{A}$.

Solution:

The columns of $\boldsymbol{A}$ are simply $\boldsymbol{u} * v_i$, They are linearly dependent since they are simply scalar multiples of $\boldsymbol{u}$. This implies that rank($\boldsymbol{A}$) = 1, since $\boldsymbol{u} \neq \mathbf{0}$, and that $\boldsymbol{u}$ is a basis for the range of $\boldsymbol{A}$.

4. Consider a real matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ with $m \geq n$. Suppose $\boldsymbol{A}$ is full rank, i.e. rank($\boldsymbol{A}$) = n. Find the nullspace of $\boldsymbol{A}$.

Solution:

By the rank-nullity theorem, $dim(null(\boldsymbol{A})) + rank(\boldsymbol{A}) = n$, since $\boldsymbol{A} \in \mathbb{R}^{mxn}$. This means that there is no non-trivial nullspace for $\boldsymbol{A}$, that is, $null(\boldsymbol{A}) = \mathbf{0}$.

Transpose and adjoint of a matrix:

5. What can you say about the diagonal entries of a skew-symmetric matrix? How about the diagonal entries of a complex hermitian matrix?

Solution:

For a skew-symmetric matrix, where $\boldsymbol{A} = -\boldsymbol{A}^t \implies x_{ij} = -x_{ji}$, is satisfied, so any diagonal entry (where $i = j$), must be 0, since only $x_{ii} = 0$ satisfies $x_{ii} = -x_{ii}$.

A complex hermitian matrix, where $\boldsymbol{A} = \boldsymbol{A}^*$, satisfies $x_{ij} = \overline{x_{ji}}$, so the diagonal entries satisfy $x_{ii} = \overline{x_{ii}}$. That is, each diagonal entry is equal to its complex conjugate. This describes any complex number with only a real component, $a + i0, \forall a \in \mathbb{R}$.