ECO 2302: Networks in Trade and Macroeconomics

Problem Set 2

Due date: 5:00 PM, 26 February 2021

In this assignment, we will practice solving a simplified closed-economy version of the endogenous bipartite network formation model in Bernard, Moxnes, and Ulltveit-Moe (2018).

Basic environment. Consider a single country. There is a unit mass of upstream firms and a unit mass of downstream firms. Upstream firms produce intermediate inputs using labor as the only factor, while downstream firms produce final goods using a combination of upstream intermediate inputs. Upstream firms are heterogeneous in productivity z with cumulative distribution function F, while downstream firms are heterogeneous in productivity Z.

Relationship costs and sorting. To sell to a downstream firm, each upstream firm must pay a nominal relationship cost wf, where w is the price of labor and f is the real relationship cost. Hence, a downstream firm with productivity Z will source inputs only from upstream firms with productivity $z \geq \underline{z}(Z)$, where $\underline{z}(\cdot)$ is the sorting function of the economy.

Production technologies. The production technology for an upstream firm with productivity z is:

$$x(z) = zl(z) \tag{1}$$

where $x(z)$ is output and $l(z)$ is the quantity of labor hired. The production technology for a downstream firm with productivity Z is a CES aggregate of upstream inputs:

$$X(Z) = Z\left[\int_{\underline{z}(Z)}^{\infty} m(z,Z)^{\frac{\sigma-1}{\sigma}}\, dF(Z)\right]^{\frac{\sigma}{\sigma-1}} \tag{2}$$

where $X(Z)$ is output, $m(z,Z)$ is the quantity of inputs purchased from upstream firms of productivity z, and $\sigma > 1$ is the elasticity of substitution across inputs.

Market structure, pricing, and variable profits. The market structure for all firms is monopolistic competition. Given the CES downstream technology, this implies certain standard results.[1]

- CES marginal cost. The marginal cost $\eta(Z)$ for a downstream firm of productivity Z inherits the CES structure of the production function:

$$\eta(Z) = \frac{1}{Z} q(Z) \tag{3}$$

$$q(Z) = \left[\int_{\underline{z}(Z)}^{\infty} p(z,Z)^{1-\sigma} \, dF(Z) \right]^{\frac{1}{1-\sigma}} \tag{4}$$

  where $p(z,Z)$ is the price charged by a z-supplier to a Z-buyer and $q(Z)$ is the ideal price index (i.e. unit cost) of the intermediate input bundle.

- Constant markup pricing by suppliers. The price $p(z,Z)$ is a constant markup $\mu \equiv \frac{\sigma}{\sigma-1}$ over the supplier's marginal cost:

$$p(z,Z) = \mu(w/z) \tag{5}$$

  where w denotes the price of labor.

- Profit function with constant price elasticity. The variable profit earned by a z-seller from selling to a Z-buyer is:

$$\pi(z,Z) = \frac{1}{\sigma} C(Z) \left[\frac{p(z,Z)}{q(Z)} \right]^{1-\sigma} \tag{6}$$

  where $C(Z)$ is total expenditure by the Z-buyer on intermediate inputs.

Final demand. We will abstract from general equilibrium and assume that each downstream firm of productivity Z faces some exogenous final demand of the form:

$$X(Z) = Ap(Z)^{-\rho} \tag{7}$$

where A is an exogenous demand shifter, $p(Z)$ is the final goods price charged by the downstream firm, and $\rho > 1$ is the price elasticity of demand.

[1] It is fairly straightforward to derive these results - they follow from cost minimization by downstream firms and profit maximization by upstream firms. However, this isn't really the point of the assignment, so I am giving you these results to start with.

Solving the network formation problem

Given the assumptions above, our goal is to solve for the sorting function of the economy, $\underline{z}(\cdot)$. We will do this in several steps.

1. First, solve the profit maximization problem for a downstream firm of productivity Z given its marginal cost $\eta(Z)$:

$$\max_{p(Z)} \{p(Z) X(Z) - \eta(Z) X(Z)\} \tag{8}$$

   subject to the final demand function (7). What are revenues $R(Z) = p(Z) X(Z)$ and total costs $C(Z) = \eta(Z) X(Z)$ for the firm given marginal cost $\eta(Z)$?

2. Next, using the expression that you derive for $C(Z)$ from part 1, show that the variable profit earned by a z-seller from selling to a Z-buyer can be expressed as:

$$\pi(z, Z) = \gamma A w^{1-\sigma} q(Z)^{\sigma-\rho} Z^{\rho-1} z^{\sigma-1} \tag{9}$$

   where $\gamma \equiv \frac{1}{\sigma}\left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma}\left(\frac{\rho}{\rho-1}\right)^{-\rho}$ is a constant that depends only on $\{\sigma, \rho\}$.

Now, recall that the equilibrium sorting function is defined implicitly by:

$$\pi(\underline{z}(Z), Z) = wf$$

Consider first the knife-edge case in which the price elasticity of final demand is equal to the elasticity of substitution across intermediate inputs, so that $\rho = \sigma$.

3. What is the solution for the equilibrium sorting function $\underline{z}(Z)$? How does this depend on the wage w, the demand shifter A, the relationship cost f, and the productivity of the downstream firm Z?

Next consider the more general case in which $\rho \neq \sigma$. In this case the equilibrium sorting function is defined implicitly by:

$$\gamma A w^{1-\rho} Z^{\rho-1} \left[\int_{\underline{z}(Z)}^{\infty} z^{\sigma-1} dF(z)\right]^{\frac{\sigma-\rho}{1-\sigma}} \underline{z}(Z)^{\sigma-1} = wf \tag{10}$$

For a general distribution F of upstream firm productivities, there is no closed-form solution for $\underline{z}(Z)$. Nonetheless, given a distribution F, values for the model parameters, and a value for Z, equation (10) is a single equation in the unknown $\underline{z}(Z)$ and hence is straightforward to solve numerically.

To solve this equation, we will use Matlab's **fsolve** routine. First, assume that $\sigma = 5$, $\rho = 3$, and F is the CDF of a lognormal $(0, 1)$ distribution.[2] Next, write a Matlab function (let's call this **sorting.m**) that takes four inputs:

- Z: the productivity of the downstream firm
- A: the value of the final demand shifter
- w: the wage
- f: the fixed relationship cost

Given these inputs, we can define within the function **sorting.m** the residual of equation (10) as a function of the cutoff upstream productivity $\underline{z}$:

$$r\left(\underline{z}\right) \equiv \gamma A w^{1-\rho} Z^{\rho-1} \left[\int_{\underline{z}(Z)}^{\infty} z^{\sigma-1} dF\left(z\right)\right]^{\frac{\sigma-\rho}{1-\sigma}} \underline{z}\left(Z\right)^{\sigma-1} - wf \tag{11}$$

We can then use **fsolve** to numerically determine the value of $\underline{z}$ such that $r\left(\underline{z}\right) = 0$.[3] This is the minimum productivity for an upstream firm that a Z-buyer matches with.

One technical issue that you will encounter in defining the function r in equation (11) is the following: evaluating the integral $\int_{\underline{z}(Z)}^{\infty} z^{\sigma-1} dF\left(z\right)$. Note that since z is a lognormal$(0,1)$ random variable, then $z^{\sigma-1}$ is a lognormal$(0, \sigma-1)$ random variable. Hence, the integral $\int_{\underline{z}(Z)}^{\infty} z^{\sigma-1} dF\left(z\right)$ is just the partial expectation of a lognormal$(0, \sigma-1)$ random variable over $[\underline{z}, \infty)$. There is a standard formula for this.[4]

Now, we will use the function **sorting.m** to study how matching between upstream sellers and downstream buyers varies with the model's parameters.

4. First, fix $A = 1$, $w = 1$, and $f = 1$. Use the function **sorting.m** to compute $\underline{z}\left(Z\right)$ for a range of values for $Z \in [0.1, 1]$. Plot $\underline{z}\left(Z\right)$ against Z. Do more productive downstream firms match with more or fewer upstream suppliers?

5. Next, repeat the exercise in part 4. for different values of $\{A, w, f\}$. How does the equilibrium sorting function vary with these parameters? Why? Is there are qualitative difference in this behavior as compared with the analytic solution for the sorting function that you derived in part 3.?

[2] That is, $\log z$ is normally distributed with mean 0 and unit standard deviation.

[3] Read the Matlab help file on fsolve to see how to specify the syntax if you are not familiar with this. This is one of the most important functions in Matlab for solving economic models, so it is good to get some practice with it.

[4] See, for example, http://en.wikipedia.org/wiki/Log-normal_distribution#Partial_expectation.