ECO 2302 - Final Exam

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1 Question 1

(a) Normalize wage to be 1. Firm i's problem

$$\max_{L_i, X_{ij}} \left(P_i X_i - L_i - \sum_{j=1}^{N} P_j X_{ij} \right)$$
$$\text{s.t } X_i = T_i \left(\frac{L_i}{\lambda} \right)^{\lambda} \left(\frac{M_i}{1-\lambda} \right)^{1-\lambda}$$
$$M_i = \prod_{j=1}^{N} \left(\frac{X_{ij}}{\beta_{ij}} \right)^{\beta_{ij}}$$

FOCs are

$$L_i = \lambda P_i X_i \tag{1}$$
$$P_i X_{ij} = (1-\lambda)\beta_{ij} P_i X_i \qquad \forall j = 1, ..., N \tag{2}$$

Substituting equation (1) and (2) into production function yields

$$\begin{aligned} X_i &= T_i(P_iX_i)^{\lambda}\left[\prod_{j=1}^{N}\left(\frac{P_iX_i}{P_j}\right)^{\beta_{ij}}\right]^{1-\lambda} \\ &= T_iP_iX_i\prod_{j=1}^{N}P_j^{-(1-\lambda)\beta_{ij}} \end{aligned}$$

Further simplification yields

$$P_i = \frac{1}{T_i}\prod_{j=1}^{N}P_j^{(1-\lambda)\beta_{ij}}$$

In log terms, where $p_i = log(P_i)$ and $t_i = log(T_i)$:

$$p_i = -t_i + \sum_{j=1}^{N}(1-\lambda)\beta_{ij}p_j$$

In matrix form, we can solve for $\mathbf{p}$ as function of $\mathbf{t}$:

$$\begin{aligned} &\mathbf{p} = -\mathbf{t} + (1-\lambda)\boldsymbol{\beta}\mathbf{p} \\ \Rightarrow\quad &\mathbf{p} = -\left[I-(1-\lambda)\boldsymbol{\beta}\right]^{-1}\mathbf{t} \end{aligned}$$

(b) Household's problem

$$\begin{aligned} &\underset{C_i}{max}\quad \prod_{i=1}^{N}\left(\frac{C_i}{\alpha_i}\right)^{\alpha_i} \\ &\text{s.t.}\quad \sum_{i=1}^{N}P_iC_i = 1 \end{aligned}$$

FOCs are

$$\begin{aligned} &P_iC_i = \alpha_i \qquad \forall i = 1, ..., N \\ \Rightarrow\quad &C_i = \frac{\alpha_i}{P_i} \qquad \forall i = 1, ..., N \end{aligned}$$

Substitute into household' s utility:

$$U = \prod_{i=1}^{N} \left(\frac{C_i}{\alpha_i} \right)^{\alpha_i} = \prod_{i=1}^{N} P_i^{-\alpha_i} = \left(\prod_{i=1}^{N} P_i \right)^{-\frac{1}{N}}$$

In log:

$$log(U) = -\frac{1}{N} \sum_{i=1}^{N} p_i = -\frac{1}{N} \mathbf{1}'\mathbf{p} = \mathbf{v}'\mathbf{t}$$

$$\text{where} \quad \mathbf{v} \equiv \frac{1}{N} \left[I - (1-\lambda)\boldsymbol{\beta} \right]^{-1} \mathbf{1}$$

Let v_i denote the i element of influence vector v. As a result, we have

$$var(log(U)) = var(t_i) \sum_{i=1}^{N} v_i^2 = \sigma^2 ||\mathbf{v}||^2$$

To solve for $var(log(U))$ explicitly as a function of $\sigma, \lambda, \beta, N$, we will first derive the expression of $\mathbf{v}$. Denote $A = I - (1-\lambda)\boldsymbol{\beta}$. As a result,

$$A = I - (1-\lambda) \begin{pmatrix} (1-\beta) & \frac{\beta}{N-1} & \cdots & \frac{\beta}{N-1} \\ \frac{\beta}{N-1} & (1-\beta) & \cdots & \frac{\beta}{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\beta}{N-1} & \frac{\beta}{N-1} & \cdots & (1-\beta) \end{pmatrix}$$

$$= \begin{pmatrix} 1-(1-\lambda)(1-\beta) & -\frac{(1-\lambda)\beta}{N-1} & \cdots & -\frac{(1-\lambda)\beta}{N-1} \\ -\frac{(1-\lambda)\beta}{N-1} & 1-(1-\lambda)(1-\beta) & \cdots & -\frac{(1-\lambda)\beta}{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{(1-\lambda)\beta}{N-1} & -\frac{(1-\lambda)\beta}{N-1} & \cdots & 1-(1-\lambda)(1-\beta) \end{pmatrix}$$

Denote $a = 1 - (1-\lambda)(1-\beta)$ and $b = -\frac{(1-\lambda)\beta}{N-1}$.

$$A = \begin{pmatrix} a & b & \cdots & b \\ b & a & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \cdots & a \end{pmatrix}$$

We have

$$A^{-1} = \frac{1}{(a-b)[a+(N-1)b]} \begin{pmatrix} a+(N-2)b & -b & \cdots & -b \\ -b & a+(N-2)b & \cdots & -b \\ \vdots & \vdots & \ddots & \vdots \\ -b & -b & \cdots & a+(N-2)b \end{pmatrix}$$

As a result,

$$\begin{aligned} \mathbf{v} = \frac{1}{N}A^{-1}\mathbf{1} &= \frac{1}{N}\frac{1}{(a-b)[a+(N-1)b]} \begin{pmatrix} a+(N-2)b-(N-1)b \\ a+(N-2)b-(N-1)b \\ \vdots \\ a+(N-2)b-(N-1)b \end{pmatrix} \\ &= \frac{1}{N}\frac{1}{(a-b)[a+(N-1)b]} \begin{pmatrix} a-b \\ \vdots \\ a-b \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{N}\frac{1}{[a+(N-1)b]} \\ \vdots \\ \frac{1}{N}\frac{1}{[a+(N-1)b]} \end{pmatrix} \end{aligned}$$

We have $a+(N-1)b = 1-(1-\lambda)(1-\beta)-(1-\lambda)\beta = 1-(1-\lambda) = \lambda$. Therefore,

$$\begin{aligned} & \mathbf{v} = \frac{1}{N\lambda}\mathbf{1} \\ \Rightarrow \quad & ||\mathbf{v}||^2 = N\frac{1}{(N\lambda)^2} = \frac{1}{N\lambda^2} \\ \Rightarrow \quad & var(log(U)) = \frac{\sigma^2}{N\lambda^2} \end{aligned}$$

(c) Variance of log household utility decreases in N and λ and does not depend on β.

- Aggregate shock can be thought of as average of idiosyncratic shock. Given that idiosyncratic shocks are identical and independent, volatility of aggregate shock will decay as N grows.
- The higher value of λ, the less important intermediate input and network linkage is. The less important network linkage is, the weaker idiosyncratic shock propagates to

aggregate shock. In the limit case when $\lambda = 1$, there's no linkage across sector and aggregate shock is simply average of idiosyncratic shocks and there's no amplification of idiosyncratic shocks via network.

- Aggregate volatility does not depend on β because this network is symmetric. All sectors are symmetric in terms of consumption share, intermediate input share and TFP distribution. As a result, the intensity of intermediate inputs in form of its own sector (β) or other sectors ($\frac{1-\beta}{N-1}$) does not affect the aggregate volatility.

2 Question 2

(a) Firm i's problem

$$\max_{L_i, X_{ij}} \left(P_i X_i - L_i - \sum_{j=1}^{N} P_j X_{ij} \right)$$

$$\text{s.t } X_i = T_i \left(\frac{L_i}{\lambda}\right)^{\lambda} \left(\frac{M_i}{1-\lambda}\right)^{1-\lambda}$$

$$M_i = \prod_{j=1}^{N+1} \left(\frac{X_{ij}}{\beta_{ij}}\right)^{\beta_{ij}}$$

FOCs are

$$L_i = \lambda P_i X_i \tag{3}$$

$$P_i X_{ij} = (1-\lambda)\beta_{ij} P_i X_i \qquad \forall j = 1, ..., N+1 \tag{4}$$

Substituting equation (3) and (4) into production function yields

$$X_i = T_i (P_i X_i)^{\lambda} \left[\prod_{j=1}^{N+1} \left(\frac{P_i X_i}{P_j}\right)^{\beta_{ij}} \right]^{1-\lambda}$$

$$= T_i P_i X_i \prod_{j=1}^{N+1} P_j^{-(1-\lambda)\beta_{ij}}$$

Further simplification yields

$$P_i = \frac{1}{T_i} \prod_{j=1}^{N+1} P_j^{(1-\lambda)\beta_{ij}}$$

In log terms, where $p_i = log(P_i)$, $p^* = log(P^*)$ and $t_i = log(T_i)$:

$$\begin{aligned} p_i &= -t_i + \sum_{j=1}^{N+1} (1-\lambda)\beta_{ij} p_j \\ &= -t_i + \sum_{j=1}^{N} (1-\lambda)\beta_{ij} p_j + (1-\lambda)\beta_{i,N+1} p^* \end{aligned}$$

Denote $\mathbf{p} \equiv \begin{bmatrix} p_1 & p_2 & \cdots & p_N \end{bmatrix}'$

Denote $\mathbf{t} \equiv \begin{bmatrix} t_1 & t_2 & \cdots & t_N \end{bmatrix}'$

Denote $\boldsymbol{\beta^*} \equiv \begin{bmatrix} \beta_{1,N+1} & \beta_{2,N+1} & \cdots & \beta_{N,N+1} \end{bmatrix}'$

Denote $\boldsymbol{\beta}$ as $N \times N$ matrix in which the $ij-$element is β_{ij}

We can then solve for $\mathbf{p}$

$$\begin{aligned} & \mathbf{p} = (1-\lambda)\boldsymbol{\beta^*} p^* - \mathbf{t} + (1-\lambda)\boldsymbol{\beta}\mathbf{p} \\ \Rightarrow \quad & \mathbf{p} = [I - (1-\lambda)\boldsymbol{\beta}]^{-1} \left[(1-\lambda)\boldsymbol{\beta^*} p^* - \mathbf{t}\right] \end{aligned}$$

Denote $\Omega \equiv [I - (1-\lambda)\boldsymbol{\beta}]^{-1}$. As a result,

$$\begin{aligned} p_i &= \sum_{j=1}^{N} \Omega_{ij} \left[(1-\lambda)\beta_{j,N+1} p^* - t_j\right] \\ &= \sum_{j=1}^{N} \Omega_{ij} (1-\lambda)\beta_{j,N+1} p^* - \sum_{j=1}^{N} \Omega_{ij} t_j \end{aligned} \tag{5}$$

Household's problem

$$\begin{aligned} \underset{C_i}{max} \quad & \prod_{i=1}^{N+1} \left(\frac{C_i}{\alpha_i}\right)^{\alpha_i} \\ \text{s.t.} \quad & \sum_{i=1}^{N+1} P_i C_i = L \end{aligned}$$

FOCs are

$$\begin{aligned} P_iC_i &= \alpha_i L \qquad \forall i = 1, ..., N+1 \\ \Rightarrow \quad C_i &= \frac{\alpha_i L}{P_i} \qquad \forall i = 1, ..., N+1 \end{aligned}$$

Substitute into household' s utility:

$$U = \prod_{i=1}^{N} \left(\frac{C_i}{\alpha_i}\right)^{\alpha_i} = \prod_{i=1}^{N} \left(\frac{L}{P_i}\right)^{\alpha_i} = L \prod_{i=1}^{N} P_i^{-\alpha_i}$$

As a result,

$$log\left(\frac{U}{L}\right) = -\sum_{i=1}^{N+1} \alpha_i p_i = -\sum_{i=1}^{N} \alpha_i p_i - \alpha_{N+1} p^* \tag{6}$$

Combining goods market clearing condition for good $i \in \{1, ..., N\}$ with FOCs from firms' and household's problem yields

$$\begin{aligned} P_iX_i &= P_iC_i + \sum_{j}^{N} P_i X_{ji} \\ &= \alpha_i L + \sum_{j}^{N} (1-\lambda)\beta_{ji} P_j X_j \end{aligned}$$

Equivalently, by denoting Domar weight $D_i \equiv \frac{P_iX_i}{L}$, we have

$$\begin{aligned} \frac{P_iX_i}{L} &= \alpha_i + \sum_{j}^{N} (1-\lambda)\beta_{ji} \frac{P_jX_j}{L} \\ D_i &= \alpha_i + \sum_{j}^{N} (1-\lambda)\beta_{ji} D_j \end{aligned}$$

In matrix form,

$$\begin{aligned} \mathbf{D} &= \boldsymbol{\alpha} + (1-\lambda)\boldsymbol{\beta}'\mathbf{D} \\ \mathbf{D} &= \left[I - (1-\lambda)\boldsymbol{\beta}'\right]^{-1} \boldsymbol{\alpha} \end{aligned}$$

We have $I - (1-\lambda)\boldsymbol{\beta}' = [I - (1-\lambda)\boldsymbol{\beta}]'$. As a result,

$$\left[I - (1-\lambda)\boldsymbol{\beta}'\right]^{-1} = \left[(I - (1-\lambda)\boldsymbol{\beta})'\right]^{-1} = \left[(I - (1-\lambda)\boldsymbol{\beta})^{-1}\right]' = \Omega'$$

Therefore, we have

$$D_i = \sum_{j=1}^{N} \Omega_{ji}\alpha_j \tag{7}$$

Next, we combine this condition with equation (5). Multiply both sides of equation (5) with α_i

$$\alpha_i p_i = \sum_{j=1}^{N} \alpha_i \Omega_{ij}(1-\lambda)\beta_{j,N+1}p^* - \sum_{j=1}^{N} \alpha_i \Omega_{ij} t_j$$

Summing over all goods $i \in \{1, ..., N\}$

$$\begin{aligned}
\sum_{i=1}^{N} \alpha_i p_i &= \sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i \Omega_{ij}(1-\lambda)\beta_{j,N+1}p^* - \sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i \Omega_{ij} t_j \\
&= \sum_{j=1}^{N}(1-\lambda)\beta_{j,N+1}p^* \sum_{i=1}^{N} \alpha_i \Omega_{ij} - \sum_{j=1}^{N} t_j \sum_{i=1}^{N} \alpha_i \Omega_{ij} \\
&\text{By equation (7):} \quad \sum_{i=1}^{N} \alpha_i \Omega_{ij} = D_j \\
&= \sum_{j=1}^{N}(1-\lambda)\beta_{j,N+1}p^* D_j - \sum_{j=1}^{N} t_j D_j
\end{aligned}$$

Adding $\alpha_{N+1}p^*$ to both sides yields

$$\sum_{i=1}^{N+1} \alpha_i p_i = \alpha_{N+1}p^* + \sum_{j=1}^{N}(1-\lambda)\beta_{j,N+1}p^* D_j - \sum_{j=1}^{N} t_j D_j$$

The left-handed side is equal to $-log(\frac{U}{L})$ by equation (6):

$$\begin{aligned}
log\left(\frac{U}{L}\right) &= \sum_{j=1}^{N} t_j D_j - \alpha_{N+1}p^* - \sum_{j=1}^{N}(1-\lambda)\beta_{j,N+1}D_j p^* \\
&= \sum_{j=1}^{N} D_j t_j - \frac{P^* C_{N+1}}{L}p^* - \sum_{j=1}^{N}(1-\lambda)\beta_{j,N+1}\frac{X_j}{L}p^* \\
&= \sum_{j=1}^{N} D_j t_j - \frac{P^* C_{N+1}}{L}p^* - \sum_{j=1}^{N} \frac{P^* X_{i,N+1}}{L}p^*
\end{aligned}$$

Therefore, we can finally have

$$log(U) = log(L) + \sum_{j=1}^{N} D_j t_j - \left(\frac{P^* C_{N+1}}{L} + \sum_{j=1}^{N} \frac{P^* X_{i,N+1}}{L} \right) p^*$$

As a result, first-order effect of a change in import price on household welfare is

$$\frac{\partial log(U)}{\partial log(P^*)} = -\left(\frac{P^* C_{N+1}}{L} + \sum_{j=1}^{N} \frac{P^* X_{i,N+1}}{L} \right) = -\frac{5}{50} - \frac{10}{50} = -\frac{3}{10}$$

This is just the negative of Domar weight for foreign sector.

(b) If every sector exports a fixed fraction $\kappa \in (0, 1)$, firms' and household's problem does not change. As a result, FOCs for firms and households and equations (5) and (6) do not change:

$$p_i = \sum_{j=1}^{N} \Omega_{ij}(1 - \lambda)\beta_{j,N+1} p^* - \sum_{j=1}^{N} \Omega_{ij} t_j$$

and

$$log\left(\frac{U}{L}\right) = -\sum_{i=1}^{N+1} \alpha_i p_i = -\sum_{i=1}^{N} \alpha_i p_i - \alpha_{N+1} p^*$$

Given no changes in parameters, the set of prices does not change and U consequently remains the same. Consumption C_i does not change as well as no changes in wage and prices.

Only goods market clearing condition changes.

$$(1 - \kappa) P_i X_i = P_i C_i + \sum_{j=1}^{N} P_i X_{ji}$$

Given that the economy consumes the same and exports, revenue and Domar weight for each sectors will increase. This gives rise to the use of foreign goods as intermediate and results in higher Domar weight for foreign sector.

Therefore, the higher κ is, the more domestic firms export. Given no change in price and consumption, domestic economy will produce more and subsequently imports more foreign goods as intermediate inputs. This results in higher Domar weight for foreign sector and larger elasticity of domestic welfare on foreign prices in absolute term (lower first-order effect of import price on domestic welfare as it is the negative of foreign Domar weight).

3 Question 3

(a) Empty network is network E in which $e_{ij} = 0\ \forall i \neq j$. In empty network, we have $\phi_i = T_i = T$ and $\pi_i = A\phi_i = AT\ \forall i \in \{1, ...N\}$.

Empty network is stable iff any firm i cannot achieve higher profit by changing e_{ij} holding constant the set of active suppliers for all other firms.

Equivalently, for all $k \in \{1, ..., N-1\}$ (k is the number of suppliers)

$$\begin{aligned}
& AT \geq A(T + \alpha k\phi_j) - kf \\
\Leftrightarrow \quad & AT \geq A(T + \alpha kT) - kf \\
\Leftrightarrow \quad & AT \geq AT + A\alpha kT - kf \\
\Leftrightarrow \quad & kf \geq A\alpha kT \\
\Leftrightarrow \quad & \alpha \leq \frac{f}{AT}
\end{aligned}$$

Combining with condition $\alpha \leq \frac{1}{N-1}$, we have empty network is stable if $\alpha \leq min\{\frac{1}{N-1}, \frac{f}{AT}\}$.

The maximum value of α is nonincreasing in A, T and N and nondecreasing in f

- A and T: The higher the aggregate productivity shifter and firm productivity, the more productivity/profit firm gain from sourcing. As a result, we need a lower input suitability (α) so that firm will not source to have a stable empty network.
- F: The higher the fixed cost of sourcing, the lower incentive firms have to source. As a result, input suitability (α) can have a lower value in which empty network is still stable.
- N: Condition of α in the question.

(b) Complete network is a network E in which $e_{ij} = 1\ \forall i \neq j$. In complete network, we have

$$\phi_i = T_i + \alpha \sum_{j \neq i} \phi_j = T + \alpha \sum_{j \neq i} \phi_j$$

As a result, we have $\phi_i = \phi_j \ \forall i \neq j$. And we can solve for ϕ_i as

$$\phi_i = \frac{T}{1-\alpha(N-1)}$$

Firm i profit can be derived as

$$\begin{aligned}\pi_i &= A\phi_i - (N-1)f \\ &= \frac{AT}{1-\alpha(N-1)} - (N-1)f\end{aligned}$$

Complete network is stable iff any firm i cannot achieve higher profit by changing e_{ij} holding constant the set of active suppliers for all other firms.

Consider any firm i, we'll compare its profit in complete network with the profit if it only sources from k suppliers: $k \in \{0, ..., N-2\}$.

Case 1: k $= 0$. Firm i does not have any supplier. As a result, $\phi_i = T$ and profit will be $\pi_i = AT$. Firm i will not switch to this strategy iff

$$\begin{aligned}& \frac{AT}{1-\alpha(N-1)} - (N-1)f \geq AT \\ \Leftrightarrow \quad & \frac{AT}{1-\alpha(N-1)} \geq AT + (N-1)f \\ \Leftrightarrow \quad & 1-\alpha(N-1) \leq \frac{AT}{AT+(N-1)f} \\ \Leftrightarrow \quad & \alpha(N-1) \geq \frac{(N-1)f}{AT+(N-1)f} \\ \Leftrightarrow \quad & \alpha \geq \frac{f}{AT+(N-1)f}\end{aligned}$$

Case 2: $k \in \{1, ..., N-2\}$. There are three types of firms in this network

- There are k number of firms j from which firm i chooses to source. All firms j are similar due to symmetry.

$$\phi_j = T + \alpha \sum_{l \neq j} \phi_l = T + \alpha(k-1)\phi_j + \alpha(N-1-k)\phi_m + \alpha\phi_i \tag{8}$$

- There are $N-1-k$ number of firms m which firm i don't choose to source. All firms m are similar due to symmetry.

$$\phi_m = T + \alpha \sum_{l \neq m} \phi_l = T + \alpha k \phi_j + \alpha(N-2-k)\phi_m + \alpha \phi_i \tag{9}$$

- Firm i which sources from k number of firms j

$$\phi_i = T + \alpha \sum_j \phi_j = T + \alpha k \phi_j \tag{10}$$

Subtracting equation (8) from equation (9) yields

$$\begin{aligned} & \phi_j - \phi_m = \alpha(\phi_m - \phi_j) \\ \Rightarrow \quad & \phi_j = \phi_m \quad \text{(as } \alpha > 0 \text{ following Case 1)} \end{aligned}$$

As a result, we have

$$\phi_j = \phi_m = T + \alpha(N-2)\phi_j + \alpha\phi_i \tag{11}$$

$$\begin{aligned} \Rightarrow \quad & [1-\alpha(N-2)]\phi_j = T + \alpha\phi_i \\ \Rightarrow \quad & \phi_j = \frac{T+\alpha\phi_i}{1-\alpha(N-2)} \end{aligned}$$

Substitute into equation (10), we have

$$\begin{aligned} & \phi_i = T + \alpha k \phi_j = T + \alpha k \frac{T+\alpha\phi_i}{1-\alpha(N-2)} \\ \Rightarrow \quad & \left(1 - \frac{\alpha^2 k}{1-\alpha(N-2)}\right)\phi_i = \left(1 + \frac{\alpha k}{1-\alpha(N-2)}\right)T \\ \Rightarrow \quad & \frac{1-\alpha(N-2)-\alpha^2 k}{1-\alpha(N-2)}\phi_i = \frac{1-\alpha(N-2-k)}{1-\alpha(N-2)}T \\ \Rightarrow \quad & \phi_i = \frac{1-\alpha(N-2-k)}{1-\alpha(N-2)-\alpha^2 k}T \end{aligned}$$

This formula also holds true for $k=0$ (no suppliers) and $k=N-1$ (complete network).

Firms will not deviate from $k = N-1$ to other $k \in \{1, ..., N-2\}$ iff

$$
\begin{aligned}
& \frac{AT}{1-\alpha(N-1)} - (N-1)f \geq \frac{1-\alpha(N-2-k)}{1-\alpha(N-2)-\alpha^2 k} AT - kf \\
\Leftrightarrow \quad & \left[\frac{1}{1-\alpha(N-1)} - \frac{1-\alpha(N-2-k)}{1-\alpha(N-2)-\alpha^2 k}\right] AT \geq (N-1-k)f \\
\Leftrightarrow \quad & \frac{1}{1-\alpha(N-1)} - \frac{1-\alpha(N-2-k)}{1-\alpha(N-2)-\alpha^2 k} \geq \frac{(N-1-k)f}{AT}
\end{aligned}
\tag{12}
$$

For each value of k, we can solve for a range of values for α such that inequality (12) holds. The final condition for α will be the set of values such that inequality (12) holds for all $k \in \{0, .., N-2\}$.

(c) The range of values α in which both empty and complete networks are stable is the intersection of

- Stable empty network: $\alpha \leq min\{\frac{1}{N-1}, \frac{f}{AT}\}$.
- Stable complete network:
  - No deviation to no sourcing strategy: $\alpha \geq \frac{f}{AT+(N-1)f}$
  - No deviations to another sourcing strategies with $k < N-1$ suppliers:

$$\frac{1}{1-\alpha(N-1)} - \frac{1-\alpha(N-2-k)}{1-\alpha(N-2)-\alpha^2 k} \geq \frac{(N-1-k)f}{AT}$$

In the range of values for α in which both empty and complete networks are stable, we have

$$
\begin{aligned}
&\text{Empty Network:} && \pi_i = AT \\
&\text{Complete Network:} && \pi_i = \frac{AT}{1-\alpha(N-1)} - (N-1)f
\end{aligned}
$$

From **Case 1** in part (b), we know that one necessary condition for complete network to be stable is

$$\frac{AT}{1-\alpha(N-1)} - (N-1)f \geq AT$$

As a result, in a range of values for α such that both empty and complete networks are stable, complete network yields higher profit for each firm as a necessary condition for firm not to deviate to no sourcing strategy in complete network.

4 Question 4

(a) Let M denote probability matrix where M_{ij} is the probability of a route from i to j in 1 step.

$$M = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

The probability for 2-step route from i to j is $\sum_{k=1} M_{ik}M_{kj}$ which is the $ij-$element of M^2. As a result, generally, the probability of k-step route from i to j is the the $ij-$element of M^k. Therefore, the expected trade cost from i to j for $j \neq i$ is

$$T_{ij} = \sum_{k=0}^{\infty} M_{ij}^k k$$

Denote T as the matrix of expected trade cost. We have

$$\begin{aligned} T &= \sum_{k=0}^{\infty} M^k k \\ &= \left(\sum_{k=0}^{\infty} M^k\right)\left(\sum_{k=1}^{\infty} M^k\right) \\ &\text{(the total number of } M^k \text{ in the product is k)} \\ &= \left(\sum_{k=0}^{\infty} M^k\right)\left(\sum_{k=0}^{\infty} M^k - I\right) \\ &= (I-M)^{-1}\left[(I-M)^{-1} - I\right] \end{aligned}$$

With the matrix T, we can then calculate average expected trade cost of any location i

to all other locations as an average of row i of T (except element T_{ii})

$$\tilde{T}_i = \frac{1}{N-1} \sum_{j \neq i} T_{ij}$$

Please see the Python code for detailed calculations. The result shows that location 7 has the highest average expected trade cost ($\tilde{T}_i$) and location 4 has the lowest average expected trade cost ($\tilde{T}_i$).

(b) To find the link with highest importance, we can calculate the average expected trade costs between all pairs of locations in the network considering all scenarios with removal of any pair of existing links. The algorithm is as follows

- Loop through through all pairs of locations, check if the link exists.
- If the link between i and j exists, change probability matrix M
  - Set $M_{ij} = M_{ji} = 0$: Remove the link
  - Update all positive probability in row i and row j of matrix M
  - For all $k \neq j$ and $M_{ik} > 0$, set M_{ik} be $\frac{1}{\frac{1}{M_{ik}} - 1}$ which is the new probability from i to k when link between i and j is removed
  - Perform similar update for row j of matrix M
- Repeat exercises in part (a) to calculate expected trade cost matrix T, average expected trade costs $\tilde{T}$ and average trade costs between all pairs Θ
- Find the scenario that yields highest Θ. The link which is removed in that scenario has the highest importance.

Please see the Python code for detailed calculations. The result shows that the link between 1 and 3 has the highest importance.