ECO 2302 - Problem Set 4

Duc Nguyen

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1 Benchmark Equilibrium

1. Cost minimization problem in sector i

$$\begin{aligned}
&\underset{L_i, X_{ij}}{max} \left(P_i X_i - L_i - \sum_{j=1}^{2} P_j X_{ij} \right) \\
\text{s.t } X_i &= \left(\frac{L_i}{\lambda} \right)^{\lambda} \left(\frac{M_i}{1-\lambda} \right)^{1-\lambda} \\
M_i &= \left(\frac{X_{ij}}{\beta_{ij}} \right)^{\beta_{ij}} \left(\frac{X_{ii}}{\beta_{ii}} \right)^{\beta_{ii}}
\end{aligned}$$

FOCs are

$$L_i = \lambda P_i X_i \tag{1}$$

$$P_j X_{ij} = (1-\lambda)\beta_{ij} P_i X_i \tag{2}$$

Substituting equation (1) and (2) into production function yields

$$\begin{aligned}
X_i &= (P_i X_i)^{\lambda} \left[\left(\frac{P_i X_i}{P_j} \right)^{\beta_{ij}} \left(\frac{P_i X_i}{P_i} \right)^{\beta_{ii}} \right]^{1-\lambda} \\
&= P_i X_i P_j^{-(1-\lambda)\beta_{ij}} P_i^{-(1-\lambda)\beta_{ii}}
\end{aligned}$$

Further simplification yields

$$\begin{aligned} P_i &= P_j^{(1-\lambda)\beta_{ij}} P_i^{(1-\lambda)\beta_{ii}} \\ &= (P_i P_j)^{(1-\lambda)\beta} \end{aligned}$$

As a result, we have

$$P_1 = P_2 = 1 \tag{3}$$

2. Substitute $P_1 = P_2 = 1$ into equation (1) and (2), we will have

$$L_i = \lambda X_i$$

$$X_{ii} = X_{ij} = (1-\lambda)\beta X_i$$

From the household's problem, we have $P_1 C_1 = P_2 C_2 = \frac{1}{2}$ (total income $= wL = 1$) which implies $C_1 = C_2 = \frac{1}{2}$

Goods market clearing conditions imply that

$$X_i = X_{ii} + X_{ji} + C_i = (1-\lambda)\beta X_i + (1-\lambda)\beta X_j + \frac{1}{2}$$

As a result, $X_1 = X_2$ and subsequently $L_1 = L_2 = \frac{1}{2}$.

We can finally solve for X_i, X_{ii}, X_{ij}

$$\begin{aligned} X_i &= \frac{L_i}{\lambda} = \frac{3}{2} \\ X_{ij} = X_{ii} &= (1-\lambda)\beta X_i = \frac{1}{2} \end{aligned}$$

3. Substituting $C_1 = C_2 = \frac{1}{2}$ and $\alpha = 1\frac{1}{2}$ into utility function, we have

$$U = \left(\frac{C_1}{\alpha}\right)^{\alpha} \left(\frac{C_2}{1-\alpha}\right)^{1-\alpha} = 1$$

2 Policy in Benchmark Equilibrium

4. Cost minimization problem in sector i

$$
\begin{aligned}
&\underset{L_i, X_{ij}}{max} \left(P_i X_i - L_i - P_i X_{ii} - (1-s) P_j X_{ij}\right) \\
\text{s.t } X_i &= \left(\frac{L_i}{\lambda}\right)^{\lambda} \left(\frac{M_i}{1-\lambda}\right)^{1-\lambda} \\
M_i &= \left(\frac{X_{ij}}{\beta_{ij}}\right)^{\beta_{ij}} \left(\frac{X_{ii}}{\beta_{ii}}\right)^{\beta_{ii}}
\end{aligned}
$$

FOCs are

$$L_i = \lambda P_i X_i \tag{4}$$

$$P_i X_{ii} = (1-\lambda)\beta_{ii} P_i X_i \tag{5}$$

$$(1-s) P_j X_{ij} = (1-\lambda)\beta_{ij} P_i X_i \tag{6}$$

Substituting equation (4), (5) and (6) into production function yields

$$
\begin{aligned}
X_i &= (P_i X_i)^{\lambda} \left[\left(\frac{P_i X_i}{(1-s)P_j}\right)^{\beta_{ij}} \left(\frac{P_i X_i}{P_i}\right)^{\beta_{ii}}\right]^{1-\lambda} \\
&= P_i X_i P_j^{-(1-\lambda)\beta_{ij}} P_i^{-(1-\lambda)\beta_{ii}} (1-s)^{-(1-\lambda)\beta_{ij}}
\end{aligned}
$$

Further simplification yields

$$
\begin{aligned}
P_i &= P_j^{(1-\lambda)\beta_{ij}} P_i^{(1-\lambda)\beta_{ii}} (1-s)^{(1-\lambda)\beta_{ij}} \\
&= [P_i P_j (1-s)]^{(1-\lambda)\beta}
\end{aligned}
$$

As a result, we have $P_1 = P_2$ and then solve for P_i as

$$P_i = (1-s)^{\frac{\beta(1-\lambda)}{\lambda}} = 1 - s \tag{7}$$

5. Similar to part 1), due to the symmetry of the two sectors, we will have $C_1 = C_2$, $X_1 = X_2$ and $L_1 = L_2$ in equilibrium.
   From the labor marketing clearing condition, we can solve for labor $L_1 = L_2 = \frac{1}{2}$.
   From equation (4), we can solve for X_i as

$$X_i = \frac{L_i}{\lambda P_i} = \frac{3}{2(1-s)} \tag{8}$$

From equation (5), we can solve for X_{ii} as

$$X_{ii} = (1-\lambda)\beta X_i = \frac{1}{3}\frac{3}{2(1-s)} = \frac{1}{2(1-s)} \tag{9}$$

From equation (6), we can solve for X_{ii} as

$$X_{ij} = \frac{(1-\lambda)\beta}{1-s}X_i = \frac{1}{3(1-s)}\frac{3}{2(1-s)} = \frac{1}{2(1-s)^2} \tag{10}$$

6. From the expression of prices and X_{ij}, we can solve for T as

$$T = sP_2X_{12} + sP_1X_{21} = 2s(1-s)\frac{1}{2(1-s)^2} = \frac{s}{1-s}$$

   The tax value T is an increasing function of s.

7. Either from good marketing clearing ($X_i = X_{ii} + X_{ji} + C_i$) or from household's problem ($P_iC_i = \frac{1-T}{2}$), we can solve for C_i as

$$C_i = \frac{1-T}{2P_i} = \frac{1-\frac{s}{1-s}}{2(1-s)} = \frac{1-2s}{2(1-s)^2} \tag{11}$$

   Given subsidy s, household's utility is then derived as

$$U = \left(\frac{C_1}{\alpha}\right)^{\alpha}\left(\frac{C_2}{1-\alpha}\right)^{1-\alpha} = \frac{1-2s}{(1-s)^2}$$

8. From the expression of household's utility given s, we have

$$U = \frac{1-2s}{(1-s)^2} = \frac{1-2s}{1-2s+s^2} \leq 1$$

   Equality occurs when $s = 0$ (no subsidy). The intuition behind this result is that the subsidy creates incentive for firms in both sectors to use more intermediate inputs from other sectors. Even though total production in each sector (X_i) rises, most of the products are used as intermediate inputs (X_{ij}) and results in lower amount of final consumption (C_i).

3 Policy in an Equilibrium with Trade Costs

9. Cost minimization problem in sector i

$$\max_{L_i, X_{ij}} \left(P_i X_i - L_i - P_i X_{ii} - (1+\tau-s)P_j X_{ij}\right)$$
$$\text{s.t } X_i = \left(\frac{L_i}{\lambda}\right)^{\lambda}\left(\frac{M_i}{1-\lambda}\right)^{1-\lambda}$$
$$M_i = \left(\frac{X_{ij}}{\beta_{ij}}\right)^{\beta_{ij}}\left(\frac{X_{ii}}{\beta_{ii}}\right)^{\beta_{ii}}$$

FOCs are

$$L_i = \lambda P_i X_i \tag{12}$$

$$P_i X_{ii} = (1-\lambda)\beta_{ii} P_i X_i \tag{13}$$

$$(1+\tau-s)P_j X_{ij} = (1-\lambda)\beta_{ij} P_i X_i \tag{14}$$

Substituting equation (12), (13) and (14) into production function yields

$$\begin{aligned} X_i &= (P_i X_i)^{\lambda}\left[\left(\frac{P_i X_i}{(1+\tau-s)P_j}\right)^{\beta_{ij}}\left(\frac{P_i X_i}{P_i}\right)^{\beta_{ii}}\right]^{1-\lambda} \\ &= P_i X_i P_j^{-(1-\lambda)\beta_{ij}} P_i^{-(1-\lambda)\beta_{ii}}(1+\tau-s)^{-(1-\lambda)\beta_{ij}} \end{aligned}$$

Further simplification yields

$$\begin{aligned} P_i &= P_j^{(1-\lambda)\beta_{ij}} P_i^{(1-\lambda)\beta_{ii}}(1+\tau-s)^{(1-\lambda)\beta_{ij}} \\ &= [P_i P_j (1+\tau-s)]^{(1-\lambda)\beta} \end{aligned}$$

As a result, we have $P_1 = P_2$ and then solve for P_i as

$$P_i = (1+\tau-s)^{\frac{\beta(1-\lambda)}{\lambda}} = 1+\tau-s \tag{15}$$

Similar to part 1), due to the symmetry of the two sectors, we will have $C_1 = C_2$, $X_1 = X_2$ and $L_1 = L_2$ in equilibrium.
From the labor marketing clearing condition, we can solve for labor $L_1 = L_2 = \frac{1}{2}$.
From equation (4), we can solve for X_i as

$$X_i = \frac{L_i}{\lambda P_i} = \frac{3}{2(1+\tau-s)} \tag{16}$$

From equation (5), we can solve for X_{ii} as

$$X_{ii} = (1-\lambda)\beta X_i = \frac{1}{3}\frac{3}{2(1+\tau-s)} = \frac{1}{2(1+\tau-s)} \tag{17}$$

From equation (6), we can solve for X_{ii} as

$$X_{ij} = \frac{(1-\lambda)\beta}{1-s}X_i = \frac{1}{3(1+\tau-s)}\frac{3}{2(1+\tau-s)} = \frac{1}{2(1+\tau-s)^2} \tag{18}$$

From the expression of prices and X_{ij}, we can solve for T as

$$T = sP_2X_{12} + sP_1X_{21} = 2s(1+\tau-s)\frac{1}{2(1+\tau-s)^2} = \frac{s}{1+\tau-s}$$

The tax value T is an increasing function of s.

Either from good marketing clearing ($X_i = X_{ii} + (1+\tau)X_{ji} + C_i$) or from household's problem ($P_iC_i = \frac{1-T}{2}$), we can solve for C_i as

$$C_i = \frac{1-T}{2P_i} = \frac{1-\frac{s}{1+\tau-s}}{2(1+\tau-s)} = \frac{1+\tau-2s}{2(1+\tau-s)^2} \tag{19}$$

Given subsidy s, household's utility is then derived as

$$U = \left(\frac{C_1}{\alpha}\right)^{\alpha}\left(\frac{C_2}{1-\alpha}\right)^{1-\alpha} = \frac{1+\tau-2s}{(1+\tau-s)^2}$$

10. From the expression of household's utility given s, we have

$$U(s) = \frac{1+\tau-2s}{(1+\tau-s)^2} = \frac{1}{(1+\tau-s)^2} - \frac{s}{(1+\tau-s)^2}$$

We have

$$\begin{aligned} U'(s) &= \frac{-2(1+\tau-s)^2 + 2(1+\tau-s)(1+\tau-2s)}{(1+\tau-s)^2} \\ &= \frac{-2s(1+\tau-s)}{(1+\tau-s)^2} \end{aligned}$$

As $s \in [0,1]$, the condition $U'(s) = 0$ is equivalent to $s = 0$. As $U''(s) < 0$, the function is concave and maximized when $s = 0$ (no subsidy).

This case is similar to previous part with no trade costs. The trade costs are frictions similar to technological constraint of the economy. The subsidy cannot mitigate the impact of trade costs but creates more distortions and results in lower consumption/welfare of households.

4 Policy in a Distorted Equilibrium

11. Because the technology and cost structure are similar, firm's problems from which we can derive prices (P_i), intermediate inputs (X_{ii}, X_{ij}) and total production quantities (X_i) are similar to previous parts.

    The only differences are the goods market clearing condition and household's budget constraint which result in different consumption (C_i). Household's income can be derived as

$$
\begin{aligned}
I &= 1 - T + \tau P_2 X_{12} + \tau P_1 X_{21} \\
&= 1 - \frac{s}{1+\tau-s} + 2\tau(1+\tau-s)\frac{1}{2(1+\tau-s)^2} \\
&= 1 - \frac{s}{1+\tau-s} + \frac{\tau}{1+\tau-s} \\
&= \frac{1+2(\tau-s)}{1+\tau-s}
\end{aligned}
$$

    As a result, consumption C_i is

$$
C_i = \frac{I}{2P_i} = \frac{\frac{1+2(\tau-s)}{1+\tau-s}}{2(1+\tau-s)} = \frac{1+2(\tau-s)}{2(1+\tau-s)^2}
$$

    Household's utility can then be derived as

$$
\begin{aligned}
U &= \left(\frac{C_1}{\alpha}\right)^{\alpha}\left(\frac{C_2}{1-\alpha}\right)^{1-\alpha} \\
&= \frac{1+2(\tau-s)}{(1+\tau-s)^2} \\
&= \frac{1+2(\tau-s)}{1+2(\tau-s)+(\tau-s)^2} \\
&\leq 1
\end{aligned}
$$

    Equality occurs when $\tau - s = 0$ or $s = \tau$. In this part, the economy is not subject to trade costs (which cannot be mitigated by subsidy like part 3). Instead, the economy is subject to distortions which lead to lower welfare compared to efficient economy in part 2. As a result, a subsidy with equal size to the distortion ($s = \tau$) is similar to removal of distortions and bring the economy back to efficient one in part 2.