

ECO 2302 - Problem Set 2

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1 Part 1

1. Solve profit maximization problem for downstream firm

$$\begin{aligned} & \max_{p(Z)} \{p(Z)X(Z) - \eta(Z)X(Z)\} \quad \text{s.t.} \quad X(Z) = Ap(Z)^{-\rho} \\ \Leftrightarrow & \max_{p(Z)} \{Ap(Z)^{1-\rho} - \eta(Z)Ap(Z)^{-\rho}\} \end{aligned}$$

FOC is

$$\begin{aligned} & (1 - \rho)Ap(Z)^{-\rho} + \eta(Z)A(-\rho)p(Z)^{-\rho-1} = 0 \\ \Rightarrow & p(Z)(\rho - 1) = \eta(Z)\rho \\ \Rightarrow & p(Z) = \frac{\rho}{\rho - 1}\eta(Z) \end{aligned}$$

Revenues $R(Z)$ can be derived as

$$R(Z) = p(Z)X(Z) = Ap(Z)^{1-\rho} = A \left[\frac{\rho}{\rho - 1}\eta(Z) \right]^{1-\rho}$$

Total Costs $C(Z)$ can be derived as

$$C(Z) = \eta(Z)X(Z) = \eta(Z)Ap(Z)^{-\rho} = \eta(Z)A \left(\frac{\rho}{\rho - 1}\eta(Z) \right)^{-\rho} = A \left(\frac{\rho}{\rho - 1} \right)^{-\rho} \eta(Z)^{1-\rho}$$

2. Variable profit by a z-seller from selling to a Z-buyer can be expressed as

$$\begin{aligned}
\pi(z, Z) &= \frac{1}{\sigma} C(Z) \left[\frac{p(z, Z)}{q(Z)} \right]^{1-\sigma} \\
&= \frac{1}{\sigma} A \left(\frac{\rho}{\rho-1} \right)^{-\rho} \eta(Z)^{1-\rho} \left[\frac{\sigma}{\sigma-1} \frac{w}{z} \frac{1}{q(Z)} \right]^{1-\sigma} \\
&= \frac{1}{\sigma} A \left(\frac{\rho}{\rho-1} \right)^{-\rho} \left[\frac{q(Z)}{Z} \right]^{1-\rho} \left[\frac{\sigma}{\sigma-1} \frac{w}{z} \frac{1}{q(Z)} \right]^{1-\sigma} \\
&= \frac{1}{\sigma} \left(\frac{\rho}{\rho-1} \right)^{-\rho} \left(\frac{\sigma}{\sigma-1} \right)^{1-\sigma} A w^{1-\sigma} q(Z)^{\sigma-\rho} Z^{\rho-1} z^{\sigma-1} \\
&= \gamma A w^{1-\sigma} q(Z)^{\sigma-\rho} Z^{\rho-1} z^{\sigma-1}
\end{aligned}$$

where $\gamma = \frac{1}{\sigma} \left(\frac{\rho}{\rho-1} \right)^{-\rho} \left(\frac{\sigma}{\sigma-1} \right)^{1-\sigma}$.

3. Consider the case $\rho = \sigma$. From the equilibrium sorting condition, we have

$$\begin{aligned}
\pi(\underline{z}(Z), Z) &= wf \\
\Leftrightarrow \gamma A w^{1-\sigma} Z^{\rho-1} \underline{z}(Z)^{\sigma-1} &= wf \\
\Leftrightarrow \underline{z}(Z) &= \left(\frac{w^\sigma f}{\gamma A} \right)^{\frac{1}{\sigma-1}} \frac{1}{Z}
\end{aligned}$$

In this case, the sorting function $\underline{z}(Z)$ is increasing in w, f and decreasing in A, Z .

2 Part 2

I have written the code in Python (using *root* function in *scipy.optimize* library) to find $\underline{z}(Z)$ given Z, A, w and f by solving nonlinear equation

$$r(\underline{z}) \equiv \gamma A w^{1-\rho} Z^{1-\rho} \left[\int_{\underline{z}(Z)}^{\infty} z^{\sigma-1} dF(z) \right]^{\frac{\sigma-\rho}{1-\sigma}} \underline{z}(Z)^{\sigma-1} - wf = 0$$

As $z \sim \log \mathcal{N}(0, 1)$, we have $z^{\sigma-1} \sim \log \mathcal{N}(0, \sigma-1)$. The integral term $\int_{\underline{z}(Z)}^{\infty} z^{\sigma-1} dF(z)$ is then the partial expectation of $\log \mathcal{N}(0, \sigma-1)$ which is

$$\int_{\underline{z}(Z)}^{\infty} z^{\sigma-1} dF(z) = e^{\frac{(\sigma-1)^2}{2}} \Phi \left(\frac{(\sigma-1)^2 - \ln \underline{z}(Z)}{\sigma-1} \right)$$

4. I first fix $A = w = f = 1$ and solve for $\underline{z}(Z)$ given different values of $Z \in [0.1, 1]$. The plot of $\underline{z}(Z)$ against Z is shown in Figure (1). The equilibrium sorting function $\underline{z}(Z)$ is decreasing in Z suggesting that more productive downstream firms match with more upstream suppliers (more productive downstream firms also match with less productive upstream suppliers).

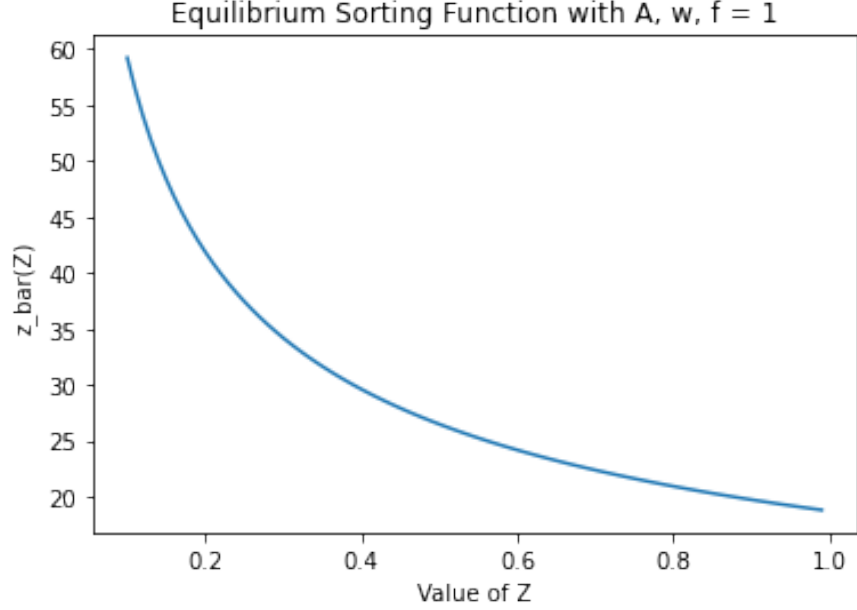


Figure 1: Equilibrium Sorting Function with $A, w, f = 1$

5. In this part, I solve for the equilibrium sorting function $\underline{z}(Z)$ given different values of A, w, f . Figure (2), (3), and (4) consequently present $\underline{z}(Z)$ when A, w , and f takes different values. Qualitatively, there's no difference in direction with the analytic solution in part (3): $\underline{z}(Z)$ is decreasing in A and increasing in w, f .

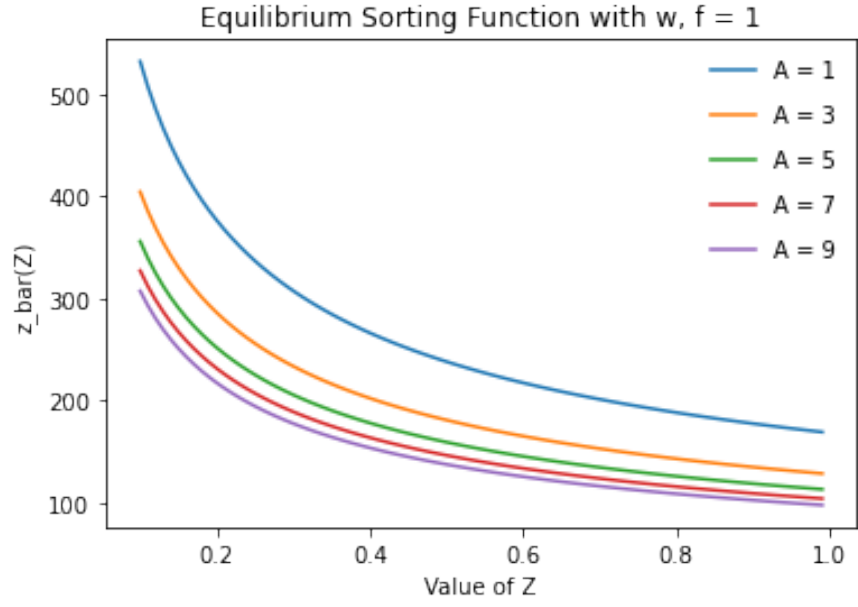


Figure 2: Equilibrium Sorting Function with $w, f = 1$

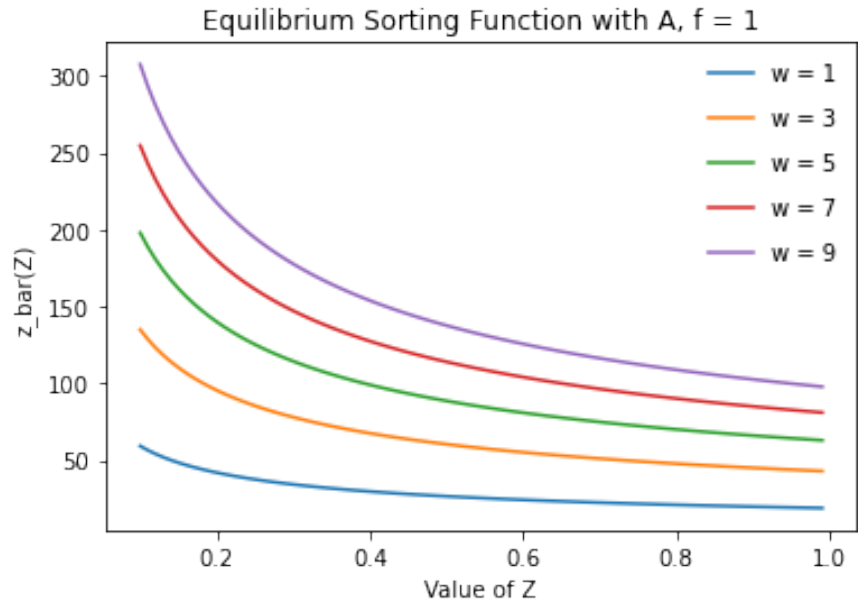


Figure 3: Equilibrium Sorting Function with $A, f = 1$

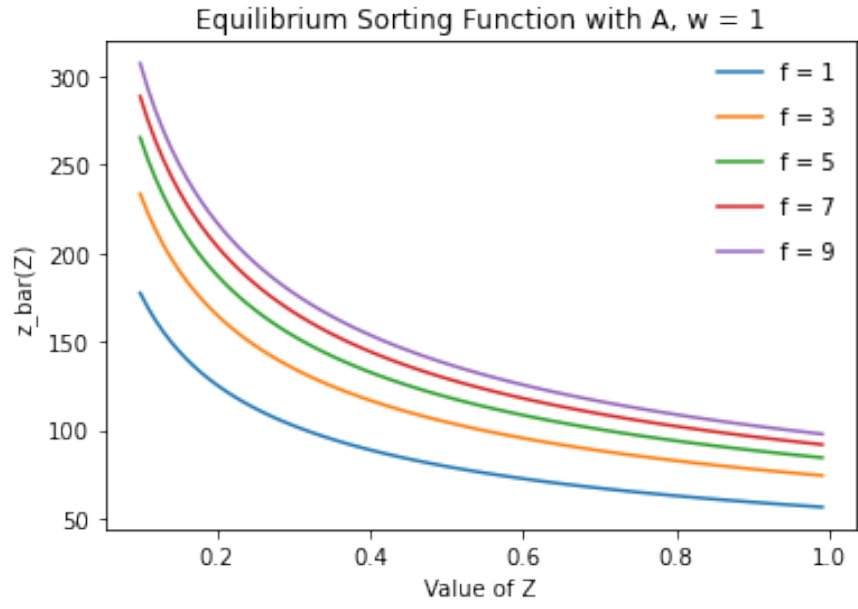


Figure 4: Equilibrium Sorting Function with $A, w = 1$