ECO 2302 - Problem Set 2

Duc Nguyen

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1 Part 1

1. Solve profit maximization problem for downstream firm

$$\begin{aligned} &\max_{p(Z)} \quad \{p(Z)X(Z) - \eta(Z)X(Z)\} \quad \text{s.t.} \quad X(Z) = Ap(Z)^{-\rho} \\ \Leftrightarrow \quad &\max_{p(Z)} \quad \{Ap(Z)^{1-\rho} - \eta(Z)Ap(Z)^{-\rho}\} \end{aligned}$$

FOC is

$$\begin{aligned} &(1-\rho)Ap(Z)^{-\rho} + \eta(Z)A(-\rho)p(Z)^{-\rho-1} = 0 \\ \Rightarrow \quad &p(Z)(\rho - 1) = \eta(Z)\rho \\ \Rightarrow \quad &p(Z) = \frac{\rho}{\rho - 1}\eta(Z) \end{aligned}$$

Revenues R(Z) can be derived as

$$R(Z) = p(Z)X(Z) = Ap(Z)^{1-\rho} = A\left[\frac{\rho}{\rho-1}\eta(Z)\right]^{1-\rho}$$

Total Costs C(Z) can be derived as

$$C(Z) = \eta(Z)X(Z) = \eta(Z)Ap(Z)^{-\rho} = \eta(Z)A\left(\frac{\rho}{\rho-1}\eta(Z)\right)^{-\rho} = A\left(\frac{\rho}{\rho-1}\right)^{-\rho}\eta(Z)^{1-\rho}$$

2. Variable profit by a z-seller from selling to a Z-buyer can be expressed as

$$
\begin{aligned}
\pi(z,Z) &= \frac{1}{\sigma}C(Z)\left[\frac{p(z,Z)}{q(Z}\right]^{1-\sigma} \\
&= \frac{1}{\sigma}A\left(\frac{\rho}{\rho-1}\right)^{-\rho}\eta(Z)^{1-\rho}\left[\frac{\sigma}{\sigma-1}\frac{w}{z}\frac{1}{q(Z)}\right]^{1-\sigma} \\
&= \frac{1}{\sigma}A\left(\frac{\rho}{\rho-1}\right)^{-\rho}\left[\frac{q(Z)}{Z}\right]^{1-\rho}\left[\frac{\sigma}{\sigma-1}\frac{w}{z}\frac{1}{q(Z)}\right]^{1-\sigma} \\
&= \frac{1}{\sigma}\left(\frac{\rho}{\rho-1}\right)^{-\rho}\left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma}Aw^{1-\sigma}q(Z)^{\sigma-\rho}Z^{\rho-1}z^{\sigma-1} \\
&= \gamma Aw^{1-\sigma}q(Z)^{\sigma-\rho}Z^{\rho-1}z^{\sigma-1}
\end{aligned}
$$

where $\gamma = \frac{1}{\sigma}\left(\frac{\rho}{\rho-1}\right)^{-\rho}\left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma}$.

3. Consider the case $\rho = \sigma$. From the equilibrium sorting condition, we have

$$
\begin{aligned}
& \pi(\underline{z}(Z), Z) = wf \\
\Leftrightarrow\quad & \gamma Aw^{1-\sigma}Z^{\rho-1}\underline{z}(Z)^{\sigma-1} = wf \\
\Leftrightarrow\quad & \underline{z}(Z) = \left(\frac{w^{\sigma}f}{\gamma A}\right)^{\frac{1}{\sigma-1}}\frac{1}{Z}
\end{aligned}
$$

In this case, the sorting function $\underline{z}(Z)$ is increasing in w, f and decreasing in A, Z.

2 Part 2

I have written the code in Python (using *root* function in *scipy.optimize* library) to find $\underline{z}(Z)$ given Z, A, w and f by solving nonlinear equation

$$
r(\underline{z}) \equiv \gamma Aw^{1-\rho}Z^{1-\rho}\left[\int_{\underline{z}(Z)}^{\infty} z^{\sigma-1}dF(z)\right]^{\frac{\sigma-\rho}{1-\sigma}}\underline{z}(Z)^{\sigma-1} - wf = 0
$$

As $z \sim log\mathcal{N}(0,1)$, we have $z^{\sigma-1} \sim log\mathcal{N}(0,\sigma-1)$. The integral term $\int_{\underline{z}(Z)}^{\infty} z^{\sigma-1}dF(z)$ is then the partial expectation of log $\mathcal{N}(0,\sigma-1)$ which is

$$
\int_{\underline{z}(Z)}^{\infty} z^{\sigma-1}dF(z) = e^{\frac{(\sigma-1)^2}{2}}\Phi\left(\frac{(\sigma-1)^2 - ln\ \underline{z}(Z)}{\sigma-1}\right)
$$

4. I first fix $A = w = f = 1$ and solve for $\underline{z}(Z)$ given different values of $Z \in [0.1, 1]$. The plot of $\underline{z}(Z)$ against Z is shown in Figure (1).

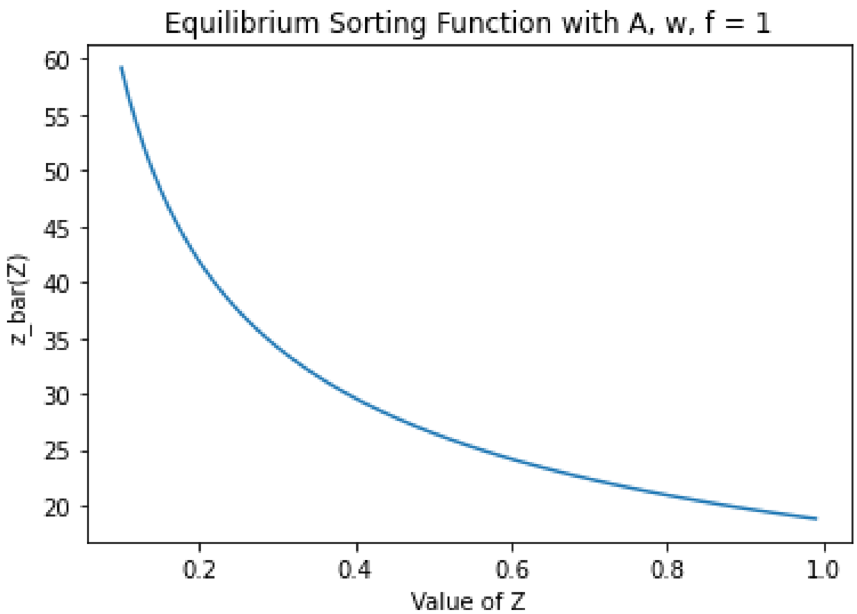

Figure 1: Equilibrium Sorting Function with $A, w, f = 1$

5. In this part, I solve for the equilibrium sorting function $\underline{z}(Z)$ given different values of A, w, f. Figure (2), (3), and (4) consequently present $\underline{z}(Z)$ when A, w, and f takes different values. Qualitatively, there's no difference in direction with the analytic solution in part (3): $\underline{z}(Z)$ is decreasing in A and increasing in w, f.

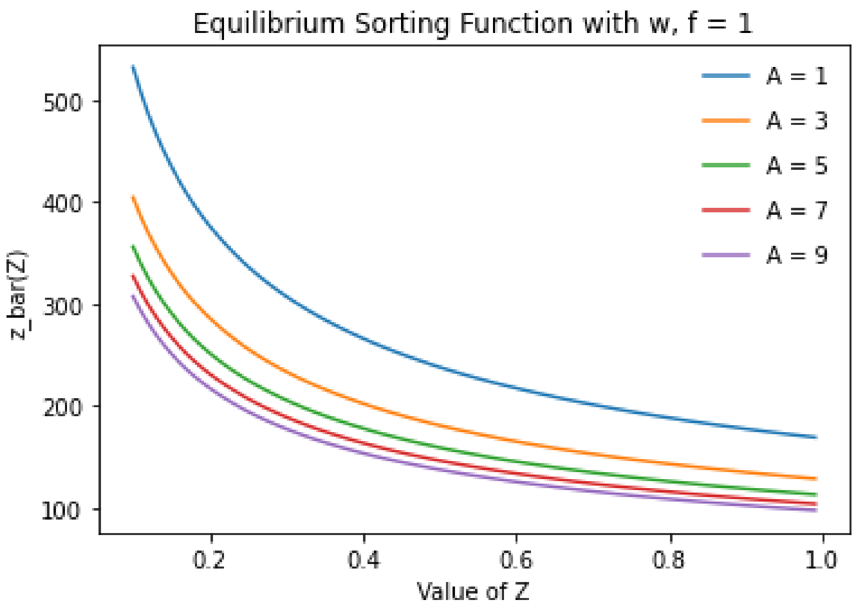

Figure 2: Equilibrium Sorting Function with $w, f = 1$

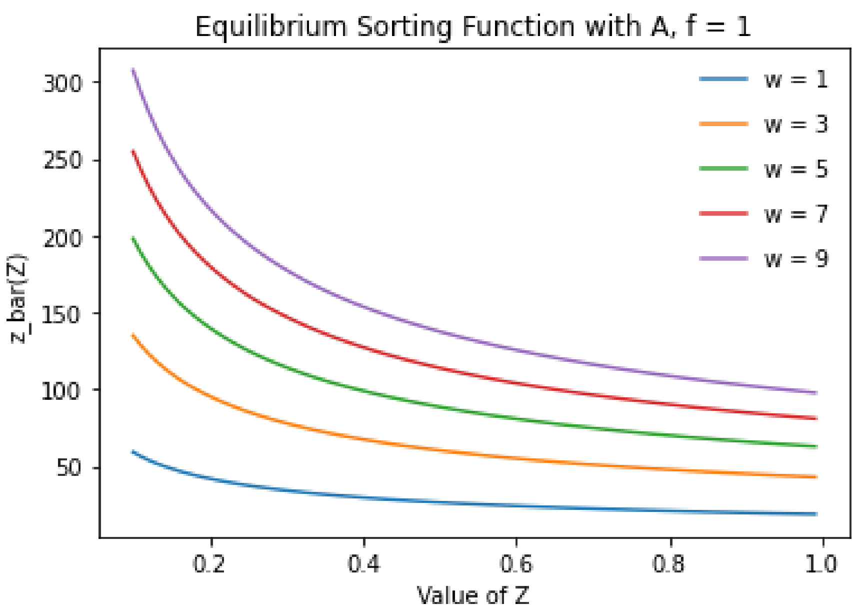

Figure 3: Equilibrium Sorting Function with $A, f = 1$

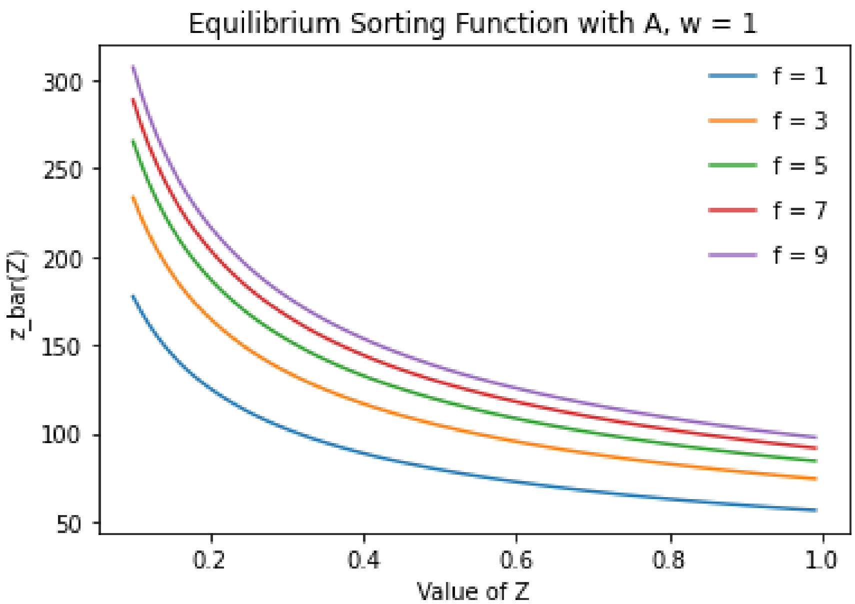

Figure 4: Equilibrium Sorting Function with $A, w = 1$