ECO 2302: Networks in Trade and Macroeconomics

Problem Set 3

Due date: 5:00 PM, 25 March 2021

The goal of this assignment is to give you some practice working with models of endogenous networks. In particular, you will see that there are two important challenges that we usually face in developing models of endogenous networks: (1) multiple equilibria, and (2) the curse of dimensionality. To illustrate these challenges, we will consider a very simple endogenous network model.

A simple model of endogenous networks

Suppose that the economy is populated by a *discrete* set of firms, $\Omega = \{1, \cdots, N\}$. Let T_i denote a measure of endogenous productivity for firm i, and suppose that the production technologies and market structure in the economy are such that firm productivities are related by:

$$T_i = A_i + \alpha \sum_{j=1}^{N} e_{ij} T_j \tag{1}$$

Here, A_i denotes a measure of exogenous technology for firm i. The production network is characterized by $\{e_{ij}\}_{i,j=1}^{N}$, where $e_{ij} = 1$ if firm j supplies firm i and $e_{ij} = 0$ otherwise. The parameter α captures how suitable one firm's output is as an input for another firm's production process. Equation (1) is the standard supply-side production network system: firm i has high productivity T_i if it has a high level of exogenous technology A_i, or if it sources inputs from suppliers with high productivities T_j themselves. To model the extensive margin of firm-to-firm relationships, we will also assume that each active relationship requires payment of a fixed cost f. For simplicity, we will assume that this cost is fully paid for by the buying firm.

On the demand side, suppose that consumer preferences in the model are such that profits for a firm i with productivity T_i and supplier network $\{e_{ij}\}_{j=1}^N$ are given by:

$$\pi_i = BT_i - f \sum_{j=1}^{N} e_{ij} \tag{2}$$

Note that profits are comprised of variable profits BT_i (where B is a profit shifter that is usually determined in general equilibrium) net of the total relationship costs $f \sum_{j=1}^{N} e_{ij}$ that the firm has to pay to maintain its supplier network.

Given the network $\{e_{ij}\}_{i,j=1}^N$, this simple model is fully specified by equations (1) and (2): equation (1) allows us to compute firm productivities T_i, and equation (2) then allows us to compute firm profits.

Now, suppose that we allow each firm i to choose its optimal supplier network $\{e_{ij}\}_{j=1}^N$. We first assume that self-links in the network are not possible, so that:

$$e_{ii} = 0, \ \ \forall i \in \Omega \tag{3}$$

We then look for a Nash equilibrium of the model. Speceifically, suppose that firm i takes as given the supplier networks of all other firms (i.e. $\{e_{kj}\}_{j=1}^N$ for all $k \neq i$), and then chooses its own supplier network to solve the following problem:

$$\max_{\{e_{ij}\}_{j=1}^N} \ \pi_i \tag{4}$$

subject to equations (1)-(3). We can then define a stable network of the model as follows.

> A **stable network** is a network $\left\{e_{ij}^*\right\}_{i,j=1}^N$ such that for each firm $i \in \Omega$, taking as given $\left\{e_{kj}^*\right\}_{j=1}^N$ for all $k \neq i$, the supplier network $\left\{e_{ij}^*\right\}_{j=1}^N$ solves equation (4) subject to equations (1)-(3).

Although the model is very simple, you should be able to see immediately that finding a stable network is not a trivial problem. For starters, the choice variables $\{e_{ij}\}_{j=1}^N$ in equation (4) are binary. Hence, the objective function is not differentiable in these arguments, and we cannot use standard first-order conditions to characterize the solution. We will now try to make progress in finding Nash equilibria of the model, and we will see what kinds of problems we run into.

Brute force solution

First, we will use Matlab to solve for a Nash equilibrium of the model by checking the stability of all potential networks $\{e_{ij}\}_{i,j=1}^N$. This is what we might refer to as a *brute force solution*. In what follows, first assume that there are only $N = 2$ firms. For simplicity, suppose that the baseline parameters of the model are:

$$\begin{aligned}
\alpha &= 0.1 \\
f &= 0.1 \\
A_i &= i \\
B &= 1
\end{aligned}$$

(Note that $A_i = i$ implies that the index of the firm is also its level of technology.)

1. How many unique supplier networks can each firm possibly choose (i.e. for a given firm i, how many unique vectors $\{e_{ij}\}_{j=1}^N$ are there with $e_{ii} = 0$)? How many unique networks are there for the economy as a whole (i.e. how many unique matrices $\{e_{ij}\}_{i,j=1}^N$ are there with $e_{ii} = 0 \ \forall i \in \Omega$)?

2. Next, using Matlab, cycle through each of the possible networks of the economy $\{e_{ij}\}_{i,j=1}^N$ (excluding networks with self-links). For each network, do the following:

   (a) Solve equations (1) and (2) to compute profits $\{\pi_i\}_{i=1}^N$ for each firm.

   (b) For each firm i, check whether it is profitable to add or drop a supplier. Specifically, for each potential supplier j, consider an alternative network $\{\hat{e}_{ij}\}_{i,j=1}^N$ that is identical to $\{e_{ij}\}_{i,j=1}^N$ excecpt that $\hat{e}_{ij} = |e_{ij} - 1|$ (i.e. adding the relationship $i \leftarrow j$ if it did not exist, or dropping it if it did exist). Re-solve equations (1) and (2) under the alternative network $\{\hat{e}_{ij}\}_{i,j=1}^N$ to determine what profits $\hat{\pi}_i$ for firm i would be under the alternative network.

   (c) Check whether $\hat{\pi}_i > \pi_i$. If $\hat{\pi}_i > \pi_i$, then the network $\{e_{ij}\}_{i,j=1}^N$ cannot be stable (because it is optimal for at least one firm to change its supplier set).

   (d) On the other hand, if $\hat{\pi}_i \leq \pi_i$ for all firms $i \in \Omega$ and all potential suppliers $j \in \Omega$, the network $\{e_{ij}\}_{i,j=1}^N$ is stable.

   Using the above algorithm, you should find that there is a single stable network $\{e^*_{ij}\}_{i,j=1}^N$. What is this network?

3. Repeat the exercise in part 2 for different values of the relationship cost, $f = 0.15$ and $f = 0.25$. In each case, you should again find that there is a single stable network. How does the stable network change as the relationship cost f increases?

Multiple Equilibria

For small networks (as in the cases above), solving for a Nash equilibrium using brute force techniques is still feasible (if inelegant). However, even if the size of the problem is small enough to implement a brute force solution, one important issue in many models of endogenous networks is that there may multiple Nash equilibria. We will now examine this in the context of our simple model.

4. Repeat the exercise in part 2 under the baseline parameters, but now assume that $A_i = 1$ for all $i \in \Omega$ (so that the firms are symmetric). How many stable networks are there? What are the stable networks of the model?

5. Why is there more than one stable network of the model?

The existence of multiple equilibria may seem like a technical issue, but it is in fact a fundamental theoretical problem. Suppose that we write down a theory of endogenous network formation and want to use the theory to make some predictions about the real world. If there are multiple equilbria in the model, and if the theoretical predictions under the various equilibria are drastically different, then how would we know which prediction is the right one?

The Curse of Dimensionality

A second important challenge that we often face in working with endogenous network models is that brute force solutions only work for very small networks. This is sometimes referred to as the "curse of dimentionality".

6. Repeat the exercise in part 2 under the baseline parameters with $N = 2$, but this time, time how long the Matlab code takes to run. You can do this by placing the command **tic** before your first line of code, and the command **toc** after your last line of code. How long does the brute force solution take with $N = 2$?

7. Now repeat the exercise in part 6 under the baseline parameters, but with $N = 3$, $N = 4$, and $N = 5$. How long does the brute force solution take in each case? How does the computation time scale with the size of the network? Why do you think this occurs?

The curse of dimensionality is a major reason why many existing models of endogenous networks have to make simplifying assumptions that may seem somewhat unreasonable but that are essential for computational tractability, e.g. no firm-to-firm markups, infinitesimally small firms, exogenous restrictions on which relationships in the network can form, and so on.