Graph Sparsification

- Sparse graphs:
  Graphs with less # of edges
  $\Rightarrow$ less space to store the graphs
  $\Rightarrow$ less processing time

- IDEA: Given an undirected weighted graph $G = (V, E, w)$, where $|V| = n$, $|E| = n$, the goal is to output the graph H, a subgraph of G with fewer edges, where H may be reweighted, while preserving "interesting quantities"

Interesting quantities:
e.g. extremal (min, max) cuts, eigenvalues, random walk properties.
(typically captured by graph Laplacian)

Graph Sparsification w.r.t. cuts

- For simplicity, let's assume we are given an undirected unweighted graph G. The goal is to approximate G by a sparse graph while preserving the cut size for all possible cuts with small errors.
- More precisely, a cut $(S, \bar{S})$ is a partition of V into two subsets S, $\bar{S} = V \setminus S$. Let $E_G(S, \bar{S})$ denote the set of edges crossing the cut $(S, \bar{S})$ in G, i.e., $E_G(S, \bar{S}) = \{u, v \in E(G) \mid u \in S \text{ and } v \notin S\}$
  The capacity (or size) of the cut $(S, \bar{S})$ is denoted by $|E_G(S, \bar{S})|$. If G is weighted by a weight function $w: E \to \mathbb{R}$, then
  $|E_G(S, \bar{S})| = \sum_{e \in E_G(S, \bar{S})} w(e)$.

Goal: Construct a graph $H = (V, E')$, where $E' \subset E$ and $|E'| << |E|$, and H is potentially reweight by a function $w': E' \to \mathbb{R}$ s.t. $\forall U \subseteq V$

$$|E_H(U, \bar{U})| = (1 \pm \epsilon) |E_G(U, \bar{U})|$$

for a small fixed $\epsilon > 0$. Such a graph H is call a "cut sparsifier" of G

Cut Sparsifier of Complete Graphs

- Given a complete graph $G = K_n$, how can we construct a cut sparsifier H of K_n? (not that $|E(K_n)| = O(n^2)$.)

· <u>Sub sampling</u>: Consider the following proces:

- Sample (keep) every edge independenty with some probability p. Then,

$$\mathbb{E}[\,|E(H)|\,] = p\,|E(K_n)|\ , \text{ and}$$

$$\forall U \subseteq V(K_n),\ \mathbb{E}[\,|E_H(U,\bar{U})|\,] = p\,|E_{K_n}(U,\bar{U})|$$

- Let's assign weight $\frac{1}{p}$ to each edge of H so that

$$\forall U \subseteq V(K_n),\ \mathbb{E}[\,|E_H(U,\bar{U})|\,] = \cancel{p}\,\frac{1}{\cancel{p}}\,|E_{K_n}(U,\bar{U})|.$$

- So in expectation, cuts are preserved !! Let's analyze how likely the cut capacity is close to its expectation.

<u>Theorem</u> [Chernoff-Heffding Concentration Bound]

Let $X = \sum_{i \in [n]} X_i$, where each $X_i \in [0,1]$ is an indicator random variable, and $(X_i : i \in [n])$ are independently distributed. Then,

- $\forall t > 0$, $\Pr\{|X - \mathbb{E}[X]| > t\} \le e^{-2t^2/n}$
- $\forall 0 < \varepsilon < 1$, $\Pr\{X < (1-\varepsilon)\mathbb{E}[X]\} \le e^{-\varepsilon^2 \mathbb{E}[X]/2}$
- $\forall 0 < \varepsilon < 1$, $\Pr\{X > (1+\varepsilon)\mathbb{E}[X]\} \le e^{-\varepsilon^2 \mathbb{E}[X]/3}$
- $\forall t > 2e\,\mathbb{E}[X]$, $\Pr\{X > t\} \le 2^{-t}$.

Analysis of Subsampling

- To simplify the analysis, we consider H in the unweighted version.

For a subset $U \subseteq V$, let $q = |U|$, $C_H = |E_H(U, \bar{U})|$, and $C_G = |E_G(U,\bar{U})| \geq \frac{qn}{2}$, thus $\mathbb{E}[C_H] = PC_G \geq \frac{pqn}{2}$. Using the concentration bound from above,

$$\Pr\{C_H > (1+\varepsilon)\mathbb{E}[C_H]\} \leq e^{-\varepsilon^2 \mathbb{E}[C_H]/3} = e^{-\frac{\varepsilon^2 p C_G}{3}} \leq e^{-\varepsilon^2 \frac{pqn}{6}}$$

- Suppose we want the RHS to be at most $1/n^{dq}$ for some fixed $d > 1$ (say $d=5$)

Then, we shoud set $p \geq \frac{6d\log n}{\varepsilon^2 n}$

$$\Rightarrow e^{-\frac{\varepsilon^2 pqn}{6}} \leq e^{-\cancel{\varepsilon^2}\left(\frac{\cancel{6}d\log n}{\cancel{\varepsilon^2}\cancel{n}}\right)\frac{q\cancel{n}}{\cancel{6}}} = e^{-dq\log n} = n^{-dq} = \frac{1}{n^{dq}}$$

- Note that a similar bound applies to deviation in the other direction, we get

$$\Pr\{C_H \notin (1 \pm \varepsilon)\mathbb{E}[C_H]\} \leq \frac{2}{n^{dq}}$$

- Also, note that the failure probability above is for a single cut. The probability for any cut to fail is obtained by the following analysis.

$$\Pr\{\text{any cut fails}\} \leq \Pr\{\text{any cut fails independently}\}$$

$$= \sum_{1<q<N} (\#\text{ of cut of size } q) \cdot \Pr\{\text{cut of size } q \text{ fails}\}$$

$$\leq \sum_{1<q<N} \underbrace{\binom{n}{q}}_{\Theta(n^q)} \cdot \frac{2}{n^{dq}}$$

$$\leq \sum_{1<q<N} \cancel{n^q} \cdot \frac{2}{n^d \cdot \cancel{n^q}} \leq \frac{2n}{n^d} = \frac{2}{n^{d-1}}.$$

- The **subsampling** will fail with probability at most $2/n^{d-1}$, if $p \geq \frac{6d\log n}{\varepsilon^2 n}$

- If we set $d = 5$ then, the sub sampling will success with <u>high probability</u> (i.e., the success probability is at least $1 - 2/n^4$), and the graph sparsifier H will have $p|E(K_n)| = \frac{(6 \times 5)\log n}{\varepsilon^2 n} \times \Theta(n^2) = \Theta\left(\frac{n\log n}{\varepsilon^2}\right)$

$$\Rightarrow \mathbb{E}[|E(H)|] = \tilde{\Theta}\left(\frac{n}{\varepsilon^2}\right)$$

- Clearly, the **sub** sampling of H takes time $O(m)$.

<u>Theorem 1:</u> The subsampling can construct a cut sparsifier H of a complete graph K_n in time $O(m)$, where $|E(H)| = \tilde{\Theta}\left(\frac{n}{\varepsilon^2}\right)$, with high probability of success.

Cut Sparsifier of Arbitrary Graphs

- Let's check if the same subsampling also works for arbitrary graph $G = (V, E)$
- Same as before,

$$\mathbb{E}\left[|E(H)|\right] = p\,|E(G)|,$$

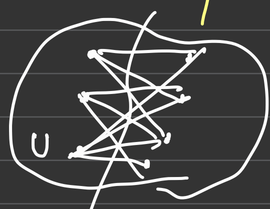
$$\forall U \subseteq V,\ \mathbb{E}\left[|E_H(U,\bar{U})|\right] = \frac{1}{p}\,|E_G(U,\bar{U})|$$

$\Rightarrow$ All cuts are preserved up to scaling by a factor $1/p$ in expectation !!

- Next, we use the concentration bound to analyze the followings:

$$\Rightarrow \Pr\left\{ C_H > (1+\varepsilon)\,\mathbb{E}[C_H] \right\} \leq e^{-\varepsilon^2 \frac{\mathbb{E}[C_H]}{2}} = e^{-\frac{\varepsilon^2 p c_G}{2}}$$

  - Note that this failure probability is exponentially decreaseing in C_G.
  - In the case of G being a complete graph, we can simply obtain the bound $\boxed{C_G \geq \frac{nq}{2}}$ ***



$G = K_n$

$C_G = |U|\cdot|\bar{U}|$ and we know that either $|U| = q \geq \frac{n}{2}$ or $|\bar{U}| = n-q \geq \frac{n}{2}$.

However, for the case of arbitrary graphs, we cannot obtain this kind of bound because no assumption can be made about the structure of G.

- Instead, we will use the fact about the number of cuts of small size.

Lemma 2 [Karger]: Let G be a graph with n vertices. Let C_G^* denote the size of minimum cuts of G. Then, for every $\alpha \geq 1$, the number of cuts of size at most $\alpha\, C_G^*$ is at most $n^{2\alpha}$.

Observation: A cut of size αC_G^* fails with probability at most $e^{-\frac{\varepsilon^2 p \alpha C_G^*}{3}}$

- Suppose among all cuts of size $\alpha \cdot C_G^*$ we want each cut to deviate from its mean more than a factor of $1+\varepsilon$, independently, with propability at most $\frac{1}{n^{\alpha d}}$. Then,

$$n^{2\alpha} \cdot e^{-\frac{\varepsilon^2 p \alpha C_G^*}{3}} \leq \frac{1}{n^{\alpha d}}$$

$$\Leftrightarrow \quad n^{2\alpha + \alpha d} \leq e^{\varepsilon^2 p \alpha C_G^*}$$

$$\Leftrightarrow \quad \cancel{\alpha}(d+2) \ln n \leq \varepsilon^2 p \cancel{\alpha} C_G^*$$

$$\Leftrightarrow \quad p \geq \frac{(d+2) \ln n}{\varepsilon^2 C_G^*}$$

Note that for each cut to deviate more than a factor of $(1 \pm \varepsilon)$, independenty, we have $\frac{2}{n^{\alpha d}}$ for the bouded probability.

- Now, with $p \geq \frac{(d+2)\ln n}{\varepsilon^2 c_G^*}$

$$\Pr\{\text{any cut fails}\} \leq \Pr\{\text{all cut fails independently}\}$$

$$\leq \int_1^{\infty} 2\left(n^{2\alpha} \cdot e^{-\frac{\varepsilon^2 p \cdot \alpha c_G^*}{3}}\right) d\alpha \leq \int_1^{\infty} \frac{2}{n^{d\alpha}} d\alpha = O\left(\frac{1}{n^d}\right)$$

Therefore, the subsampling succeeds w. h. p.

- In addition, given the value of c_G^* beforehand, the subsampling works in $O(m)$ time, and the resulting sparsifier H has

$$\mathbb{E}[|E(H)|] \geq \frac{(d+2)}{\varepsilon^2 c_G^*} \log n \cdot m$$

$$= \tilde{\Theta}\left(\frac{m}{\varepsilon^2 c_G^*}\right)$$

So, the number of edges decreases roughly by a factor of $\Theta\left(\frac{\log n}{c_G^*}\right)$ ← We shall discuss the limitation due to this result later.

Theorem 3:

The subsampling can construct a spasifier H of an arbitrary graph G, where $|E(H)| = \tilde{\Theta}\left(\frac{m}{\varepsilon^2 c_G^*}\right)$, with high probability of success.

Appendix : $\int_{1}^{\infty} \frac{1}{n^{\alpha}} \, d\alpha = \int_{1}^{\infty} n^{-\alpha} \, d\alpha$

$$= \frac{n^{-\alpha}}{\ln n} + c \Big|_{1}^{\infty}$$

$$= O\left(\frac{n^{-1}}{\ln n}\right) = O\left(\frac{1}{n \log n}\right)$$

$$= O\left(\frac{1}{n}\right)$$