TEMA 7: Contraste de hipótesis

7.1. Planteamiento del problema y conceptos básicos.

7.2. Test de Neyman-Pearson.

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7.5. Contrastes sobre los parámetros de dos poblaciones normales.

7.6. Dualidad entre estimación por intervalos y contraste de hipótesis.

7.1. PLANTEAMIENTO DEL PROBLEMA Y CONCEPTOS BÁSICOS

$(X_1, \dots, X_n) \in \chi^n$ muestra aleatoria simple de $X \to \{P_\theta;\ \theta \in \Theta\}$, $\Theta = \Theta_0 \cup \Theta_1$

$H_0 : \theta \in \Theta_0$ *Hipótesis nula*
$H_1 : \theta \in \Theta_1$ *Hipótesis alternativa*

Test de hipótesis: Es un estadístico, $\varphi(X_1, \dots, X_n)$, con valores en [0,1], que especifica la probabilidad de rechazar H_0 a partir de $X_1, \dots, X_n$.

- *Test no aleatorizado*: $\varphi : \chi^n \longrightarrow \{0, 1\}$.

$$\varphi(X_1, \dots, X_n) = \begin{cases} 1 & (X_1, \dots, X_n) \in C \qquad C \subseteq \chi^n \text{ región crítica o de rechazo} \\ 0 & (X_1, \dots, X_n) \notin C \qquad C^c = \chi^n - C \text{ región de aceptación.} \end{cases}$$

- *Test aleatorizado*: Toma algún valor distinto de $0, 1$.

Tipos de errores asociados a un test de hipótesis:

- *Error de tipo 1*: Rechazar H_0 siendo cierta.
- *Error de tipo 2*: Aceptar H_0 siendo falsa.

Función de potencia de $\varphi(X_1, \dots, X_n)$:

$$\begin{array}{rcl} \beta_\varphi : \Theta & \longrightarrow & [0, 1] \\ \theta & \longmapsto & \beta_\varphi(\theta) = E_\theta\left[\varphi(X_1, \dots, X_n)\right] \end{array}$$ (probabilidad media de rechazar H_0 bajo P_θ.)

Tamaño de $\varphi(X_1, \dots, X_n)$: $\sup_{\theta \in \Theta_0} \beta_\varphi(\theta)$ (máxima probabilidad media de cometer un error de tipo 1).

Nivel de significación de un test: $\varphi(X_1, \dots X_n)$ tiene nivel de significación α ($\in [0, 1]$) si su tamaño es menor o igual que α (cota superior de las probabilidades medias de cometer error de tipo 1):

$$\forall \theta \in \Theta_0,\ \beta_\varphi(\theta) = E_\theta\left[\varphi(X_1, \dots, X_n)\right] \leq \alpha.$$

Test uniformemente más potente: Un test $\varphi(X_1, \dots X_n)$ con nivel de significación α es uniformemente más potente a dicho nivel si para cualquier otro test, $\varphi'(X_1, \dots X_n)$, con nivel de significación α, se tiene:

$$\beta_{\varphi'}(\theta) \leq \beta_\varphi(\theta), \quad \forall \theta \in \Theta_1.$$

Resolución de un problema de contraste: *fijado un nivel de significación, encontrar el test uniformemente más potente a dicho nivel.*

7.2. LEMA DE NEYMAN-PEARSON (H_0, H_1 simples)

Sea $X \to \{P_\theta;\ \theta \in \{\theta_0, \theta_1\}\}$ y $(X_1, \ldots, X_n)$ una muestra aleatoria simple con funciones de densidad (o funciones masa de probabilidad) $f_0^n(x_1, \ldots, x_n)$ $(\theta = \theta_0)$ y $f_1^n(x_1, \ldots, x_n)$ $(\theta = \theta_1)$. Consideremos el problema de contraste

$$\begin{gathered} H_0 : \theta = \theta_0 \\ H_1 : \theta = \theta_1. \end{gathered}$$

a) *Sea $\varphi(X_1, \ldots, X_n)$ un test de la forma*

$$\varphi(X_1, \ldots, X_n) = \begin{cases} 1, & si \ \ f_1^n(X_1, \ldots, X_n) > k f_0^n(X_1, \ldots, X_n) \\ \gamma(X_1, \ldots, X_n), & si \ \ f_1^n(X_1, \ldots, X_n) = k f_0^n(X_1, \ldots, X_n) \\ 0, & si \ \ f_1^n(X_1, \ldots, X_n) < k f_0^n(X_1, \ldots, X_n), \end{cases}$$

con $k \in \mathbb{R}^+ \cup \{0\}$ y $\gamma(X_1, \ldots, X_n) \in [0, 1]$. Si $\varphi(X_1, \ldots, X_n)$ tiene tamaño α, es de máxima potencia a nivel de significación α. Un test de esta forma se denomina test de Neyman-Pearson.

b) *Para todo $\alpha \in (0, 1]$ existe un test de Neyman-Pearson de tamaño α, con $\gamma(X_1, \ldots, X_n) = \gamma$ constante.*

c) *Si $\varphi'(X_1, \ldots, X_n)$ es un test de tamaño α y es de máxima potencia a nivel de significación α, $\varphi'(X_1, \ldots, X_n)$ es un test de Neyman-Pearson.*

d) *El test de máxima potencia entre todos los de nivel de significación 0 (tamaño 0) es:*

$$\varphi_0(X_1, \ldots, X_n) = \begin{cases} 1, & si \ \ f_0^n(X_1, \ldots, X_n) = 0 \\ 0, & si \ \ f_0^n(X_1, \ldots, X_n) > 0 \end{cases}$$

7.3. TEST DE LA RAZÓN DE VEROSIMILITUDES (H_0, H_1 arbitrarias)

Sea $(X_1, \ldots, X_n) \in \chi^n$ una muestra aleatoria simple de $X \to \{P_\theta;\ \theta \in \Theta = \Theta_0 \cup \Theta_1\}$. El test de razón de verosimilitudes para el problema de contraste

$$\begin{gathered} H_0 : \theta \in \Theta_0 \\ H_1 : \theta \in \Theta_1 \end{gathered}$$

se define como:

$$\varphi(X_1, \ldots, X_n) = \begin{cases} 1 & si \ \ \lambda(X_1, \ldots, X_n) < c \\ 0 & si \ \ \lambda(X_1, \ldots, X_n) \geq c \end{cases} \quad con\ \lambda(x_1, \ldots, x_n) = \frac{\sup\limits_{\theta \in \Theta_0} L_{x_1, \ldots, x_n}(\theta)}{\sup\limits_{\theta \in \Theta} L_{x_1, \ldots, x_n}(\theta)}, \ \ \forall (x_1, \ldots, x_n) \in \chi^n,$$

siendo $L_{x_1, \ldots, x_n}$ la función de verosimilitud asociada a $(x_1, \ldots, x_n)$, y $c \in (0, 1]$ una constante que se determina imponiendo el tamaño o nivel de significación requerido.

7.4. CONTRASTES SOBRE LOS PARÁMETROS DE UNA NORMAL

Contrastes sobre la media con varianza conocida

$(X_1, \dots, X_n)$ muestra aleatoria simple de $X \longrightarrow \{\mathcal{N}(\mu, \sigma_0^2);\ \mu \in \mathbb{R}\}$

Función de verosimilitud:

$$(x_1, \dots, x_n) \in \mathbb{R}^n \longrightarrow L_{x_1,\dots,x_n}(\mu) = \frac{1}{(\sigma_0^2)^{n/2}(2\pi)^{n/2}} e^{-\sum\limits_{i=1}^{n}(x_i-\mu)^2/2\sigma_0^2}, \quad \mu \in \mathbb{R}$$

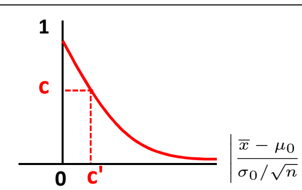
- $\sup\limits_{\mu \in \mathbb{R}} L_{x_1,\dots,x_n}(\mu) = L_{x_1,\dots,x_n}(\overline{x})$

- $$\sup_{\mu \le \mu_0} L_{x_1,\dots,x_n}(\mu) = \begin{cases} L_{x_1,\dots,x_n}(\overline{x}), & \overline{x} \le \mu_0 \left(\Leftrightarrow \dfrac{\overline{x}-\mu_0}{\sigma_0/\sqrt{n}} \le 0 \right) \\[2ex] L_{x_1,\dots,x_n}(\mu_0), & \overline{x} \ge \mu_0 \left(\Leftrightarrow \dfrac{\overline{x}-\mu_0}{\sigma_0/\sqrt{n}} \ge 0 \right) \end{cases}$$

- $$\sup_{\mu \ge \mu_0} L_{x_1,\dots,x_n}(\mu) = \begin{cases} L_{x_1,\dots,x_n}(\mu_0), & \overline{x} \le \mu_0 \left(\Leftrightarrow \dfrac{\overline{x}-\mu_0}{\sigma_0/\sqrt{n}} \le 0 \right) \\[2ex] L_{x_1,\dots,x_n}(\overline{x}), & \overline{x} \ge \mu_0 \left(\Leftrightarrow \dfrac{\overline{x}-\mu_0}{\sigma_0/\sqrt{n}} \ge 0 \right) \end{cases}$$

$$\boxed{\begin{array}{l} H_0 : \mu = \mu_0 \\ H_1 : \mu \neq \mu_0 \end{array}} \text{ TRV de tamaño } \alpha \in [0,1] \ \rightarrow\ \varphi(X_1, \dots, X_n) = \begin{cases} 1 & \text{si } \left| \dfrac{\overline{X}-\mu_0}{\sigma_0/\sqrt{n}} \right| > z_{\alpha/2} \\[2ex] 0 & \text{si } \left| \dfrac{\overline{X}-\mu_0}{\sigma_0/\sqrt{n}} \right| \le z_{\alpha/2} \end{cases}$$

$$\lambda(x_1, \dots, x_n) = \frac{\sup\limits_{\mu=\mu_0} L_{x_1,\dots,x_n}(\mu)}{\sup\limits_{\mu \in \mathbb{R}} L_{x_1,\dots,x_n}(\mu)} = \frac{L_{x_1,\dots,x_n}(\mu_0)}{L_{x_1,\dots,x_n}(\overline{x})} = \exp\left\{ \frac{-n(\overline{x}-\mu_0)^2}{2\sigma_0^2} \right\}$$

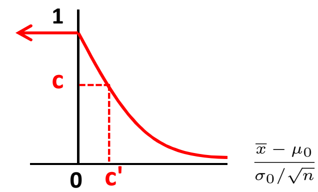


$$\varphi(x_1, \dots, x_n) = \begin{cases} 1, & \lambda(x_1, \dots, x_n) < c \ \ (\in (0,1]) \\ 0, & \lambda(x_1, \dots, x_n) \ge c \end{cases} \Leftrightarrow \varphi(x_1, \dots, x_n) = \begin{cases} 1, & \left| \dfrac{\overline{x}-\mu_0}{\sigma_0/\sqrt{n}} \right| > c' \ \ (\ge 0) \\[2ex] 0, & \left| \dfrac{\overline{x}-\mu_0}{\sigma_0/\sqrt{n}} \right| \le c' \end{cases}$$

$$\alpha = P_{\mu_0}\left(\left| \frac{\overline{X}-\mu_0}{\sigma_0/\sqrt{n}} \right| > c' \right) = \left[Z \rightarrow \mathcal{N}(0,1) \right] = P\big(|Z| > c'\big) \Rightarrow c' = z_{\alpha/2} \ge 0,\ \forall \alpha \in [0,1].$$

$$\boxed{\begin{array}{l} H_0 : \mu \leq \mu_0 \\ H_1 : \mu > \mu_0 \end{array}} \quad \text{TRV de tamaño } \alpha \leq 1/2 \ \rightarrow \ \varphi(X_1, \dots, X_n) = \begin{cases} 1 & \text{si } \dfrac{\overline{X} - \mu_0}{\sigma_0/\sqrt{n}} > z_\alpha \\ 0 & \text{si } \dfrac{\overline{X} - \mu_0}{\sigma_0/\sqrt{n}} \leq z_\alpha \end{cases}$$

$$\lambda(x_1, \dots, x_n) = \frac{\sup\limits_{\mu \leq \mu_0} L_{x_1,\dots,x_n}(\mu)}{L_{x_1,\dots,x_n}(\overline{x})} = \begin{cases} 1, & \dfrac{\overline{x} - \mu_0}{\sigma_0/\sqrt{n}} \leq 0 \\ \dfrac{L_{x_1,\dots,x_n}(\mu_0)}{L_{x_1,\dots,x_n}(\overline{x})}, & \dfrac{\overline{x} - \mu_0}{\sigma_0/\sqrt{n}} \geq 0 \end{cases}$$



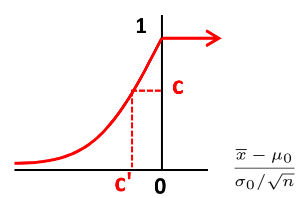
$$\varphi(x_1, \dots, x_n) = \begin{cases} 1, & \lambda(x_1, \dots, x_n) < c \ \ (\in (0,1]) \\ 0, & \lambda(x_1, \dots, x_n) \geq c \end{cases} \Leftrightarrow \ \varphi(x_1, \dots, x_n) = \begin{cases} 1, & \dfrac{\overline{x} - \mu_0}{\sigma_0/\sqrt{n}} > c' \quad (c' \geq 0) \\ 0, & \dfrac{\overline{x} - \mu_0}{\sigma_0/\sqrt{n}} \leq c' \end{cases}$$

$$\alpha = \sup_{\mu \leq \mu_0} P_\mu\left(\frac{\overline{X} - \mu_0}{\sigma_0/\sqrt{n}} > c'\right) = \sup_{\mu \leq \mu_0} P_\mu\left(\overline{X} > \mu_0 + c'\frac{\sigma_0}{\sqrt{n}}\right) = \sup_{\mu \leq \mu_0} P_\mu\left(\frac{\overline{X} - \mu}{\sigma_0/\sqrt{n}} > \frac{\mu_0 - \mu}{\sigma_0/\sqrt{n}} + c'\right) =$$

$$\left[Z \rightarrow \mathcal{N}(0,1)\right] \quad = \sup_{\mu \leq \mu_0} P\left(Z > \frac{\mu_0 - \mu}{\sigma_0/\sqrt{n}} + c'\right) = P\left(Z > c'\right) = \alpha \ \Rightarrow \ c' = z_\alpha \ (\geq 0 \Leftrightarrow \alpha \leq 1/2)$$

$$\boxed{\begin{array}{l} H_0 : \mu \geq \mu_0 \\ H_1 : \mu < \mu_0 \end{array}} \quad \text{TRV de tamaño } \alpha \leq 1/2 \ \rightarrow \ \varphi(X_1, \dots, X_n) = \begin{cases} 1 & \text{si } \dfrac{\overline{X} - \mu_0}{\sigma_0/\sqrt{n}} < z_{1-\alpha} \\ 0 & \text{si } \dfrac{\overline{X} - \mu_0}{\sigma_0/\sqrt{n}} \geq z_{1-\alpha} \end{cases}$$

$$\lambda(x_1, \dots, x_n) = \frac{\sup\limits_{\mu \geq \mu_0} L_{x_1,\dots,x_n}(\mu)}{L_{x_1,\dots,x_n}(\overline{x})} = \begin{cases} \dfrac{L_{x_1,\dots,x_n}(\mu_0)}{L_{x_1,\dots,x_n}(\overline{x})}, & \dfrac{\overline{x} - \mu_0}{\sigma_0/\sqrt{n}} \leq 0 \\ 1, & \dfrac{\overline{x} - \mu_0}{\sigma_0/\sqrt{n}} \geq 0 \end{cases}$$



$$\varphi(x_1, \dots, x_n) = \begin{cases} 1, & \lambda(x_1, \dots, x_n) < c \ \ (\in (0,1]) \\ 0, & \lambda(x_1, \dots, x_n) \geq c \end{cases} \Leftrightarrow \ \varphi(x_1, \dots, x_n) = \begin{cases} 1, & \dfrac{\overline{x} - \mu_0}{\sigma_0/\sqrt{n}} < c' \quad (c' \leq 0) \\ 0, & \dfrac{\overline{x} - \mu_0}{\sigma_0/\sqrt{n}} \geq c' \end{cases}$$

$$\alpha = \sup_{\mu \geq \mu_0} P_\mu\left(\frac{\overline{X} - \mu_0}{\sigma_0/\sqrt{n}} < c'\right) = \sup_{\mu \geq \mu_0} P_\mu\left(\overline{X} < \mu_0 + c'\frac{\sigma_0}{\sqrt{n}}\right) = \sup_{\mu \geq \mu_0} P_\mu\left(\frac{\overline{X} - \mu}{\sigma_0/\sqrt{n}} < \frac{\mu_0 - \mu}{\sigma_0/\sqrt{n}} + c'\right) =$$

$$\left[Z \rightarrow \mathcal{N}(0,1)\right] \quad = \sup_{\mu \geq \mu_0} P\left(Z < \frac{\mu_0 - \mu}{\sigma_0/\sqrt{n}} + c'\right) = P\left(Z < c'\right) = \alpha \ \Rightarrow \ c' = z_{1-\alpha} \ (\leq 0 \Leftrightarrow \alpha \leq 1/2)$$

Contrastes sobre la media con varianza desconocida

$(X_1,\ldots,X_n)$ muestra aleatoria simple de $X \longrightarrow \{\mathcal{N}(\mu,\sigma^2);\ \mu\in\mathbb{R},\ \sigma^2\in\mathbb{R}^+\}$

Función de verosimilitud:

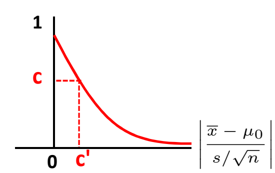
$$(x_1,\ldots,x_n)\in\mathbb{R}^n \ \longrightarrow\ L_{x_1,\ldots,x_n}(\mu,\sigma^2)=\frac{1}{(\sigma^2)^{n/2}(2\pi)^{n/2}}e^{-\sum\limits_{i=1}^{n}(x_i-\mu)^2/2\sigma^2},\quad \mu\in\mathbb{R},\ \sigma^2\in\mathbb{R}^+$$

- $$\sup_{\mu\in\mathbb{R},\sigma^2\in\mathbb{R}^+} L_{x_1,\ldots,x_n}(\mu,\sigma^2)=L_{x_1,\ldots,x_n}(\overline{x},\widehat{\sigma}^2),\quad \widehat{\sigma}^2=\frac{\sum\limits_{i=1}^{n}(x_i-\overline{x})^2}{n}$$
- $$\sup_{\mu=\mu_0,\sigma^2\in\mathbb{R}^+} L_{x_1,\ldots,x_n}(\mu,\sigma^2)=L_{x_1,\ldots,x_n}(\mu_0,\widehat{\sigma}_0^2),\quad \widehat{\sigma}_0^2=\frac{\sum\limits_{i=1}^{n}(x_i-\mu_0)^2}{n}$$
- $$\sup_{\mu\leq\mu_0,\sigma^2\in\mathbb{R}^+} L_{x_1,\ldots,x_n}(\mu,\sigma^2)=\begin{cases} L_{x_1,\ldots,x_n}(\overline{x},\widehat{\sigma}^2), & \overline{x}\leq\mu_0 \ \left(\Leftrightarrow \dfrac{\overline{x}-\mu_0}{s/\sqrt{n}}\leq 0\right) \\[2ex] L_{x_1,\ldots,x_n}(\mu_0,\widehat{\sigma}_0^2), & \overline{x}\geq\mu_0 \ \left(\Leftrightarrow \dfrac{\overline{x}-\mu_0}{s/\sqrt{n}}\geq 0\right)\end{cases}$$
- $$\sup_{\mu\geq\mu_0,\sigma^2\in\mathbb{R}^+} L_{x_1,\ldots,x_n}(\mu,\sigma^2)=\begin{cases} L_{x_1,\ldots,x_n}(\mu_0,\widehat{\sigma}_0^2), & \overline{x}\leq\mu_0 \ \left(\Leftrightarrow \dfrac{\overline{x}-\mu_0}{s/\sqrt{n}}\leq 0\right) \\[2ex] L_{x_1,\ldots,x_n}(\overline{x},\widehat{\sigma}^2), & \overline{x}\geq\mu_0 \ \left(\Leftrightarrow \dfrac{\overline{x}-\mu_0}{s/\sqrt{n}}\geq 0\right)\end{cases}$$

$$\boxed{\begin{array}{l} H_0:\mu=\mu_0 \\ H_1:\mu\neq\mu_0\end{array}} \quad \text{TRV de tamaño } \alpha\in[0,1] \ \rightarrow\ \varphi(X_1,\ldots,X_n)=\begin{cases} 1 & \text{si} \ \left|\dfrac{\overline{X}-\mu_0}{S/\sqrt{n}}\right| > t_{n-1;\ \alpha/2} \\[2ex] 0 & \text{si} \ \left|\dfrac{\overline{X}-\mu_0}{S/\sqrt{n}}\right| \leq t_{n-1;\ \alpha/2}\end{cases}$$

$$\lambda(x_1,\ldots,x_n)=\frac{\sup\limits_{\mu=\mu_0,\sigma^2\in\mathbb{R}^+} L_{x_1,\ldots,x_n}(\mu,\sigma^2)}{\sup\limits_{\mu\in\mathbb{R},\sigma^2\in\mathbb{R}^+} L_{x_1,\ldots,x_n}(\mu,\sigma^2)}=\frac{L_{x_1,\ldots,x_n}(\mu_0,\widehat{\sigma}_0^2)}{L_{x_1,\ldots,x_n}(\overline{x},\widehat{\sigma}^2)}=$$

$$=\left[\frac{\sum\limits_{i=1}^{n}(x_i-\overline{x})^2}{\sum\limits_{i=1}^{n}(x_i-\mu_0)^2}\right]^{n/2}=\left[\frac{\sum\limits_{i=1}^{n}(x_i-\overline{x})^2}{\sum\limits_{i=1}^{n}(x_i-\overline{x})^2+n(\overline{x}-\mu_0)^2}\right]^{n/2}=\left[\frac{1}{1+\frac{n(\overline{x}-\mu_0)^2}{(n-1)s^2}}\right]^{n/2}$$

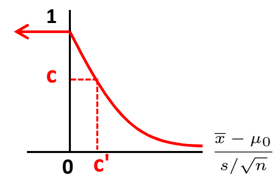


$$\varphi(x_1,\ldots,x_n)=\begin{cases} 1, & \lambda(x_1,\ldots,x_n)<c \ \ (\in(0,1]) \\ 0, & \lambda(x_1,\ldots,x_n)\geq c\end{cases} \ \Leftrightarrow\ \varphi(x_1,\ldots,x_n)=\begin{cases} 1, & \left|\dfrac{\overline{x}-\mu_0}{s/\sqrt{n}}\right|>c' \quad (c'\geq 0) \\[2ex] 0, & \left|\dfrac{\overline{x}-\mu_0}{s/\sqrt{n}}\right|\leq c'.\end{cases}$$

$$\alpha=\sup_{\sigma^2\in\mathbb{R}^+} P_{\mu_0,\sigma^2}\left(\left|\frac{\overline{X}-\mu_0}{S/\sqrt{n}}\right|>c'\right)=\Big[T\rightarrow t(n-1)\Big]=P\big(|T|>c'\big)\Rightarrow c'=t_{n-1;\ \alpha/2}\geq 0,\ \forall\alpha\in[0,1].$$

$\boxed{\begin{array}{l} H_0 : \mu \leq \mu_0 \\ H_1 : \mu > \mu_0 \end{array}}$ TRV de tamaño $\alpha \leq 1/2 \;\rightarrow\; \varphi(X_1, \ldots, X_n) = \begin{cases} 1 & \text{si } \dfrac{\overline{X} - \mu_0}{S/\sqrt{n}} > t_{n-1;\ \alpha} \\[2ex] 0 & \text{si } \dfrac{\overline{X} - \mu_0}{S/\sqrt{n}} \leq t_{n-1;\ \alpha} \end{cases}$

$$\lambda(x_1, \ldots, x_n) = \frac{\sup\limits_{\mu \leq \mu_0, \sigma^2 \in \mathbb{R}^+} L_{x_1,\ldots,x_n}(\mu, \sigma^2)}{L_{x_1,\ldots,x_n}(\overline{x}, \widehat{\sigma}^2)} = \begin{cases} 1, & \dfrac{\overline{x} - \mu_0}{s/\sqrt{n}} \leq 0 \\[2ex] \dfrac{L_{x_1,\ldots,x_n}(\mu_0, \widehat{\sigma}_0^2)}{L_{x_1,\ldots,x_n}(\overline{x}, \widehat{\sigma}^2)}, & \dfrac{\overline{x} - \mu_0}{s/\sqrt{n}} \geq 0 \end{cases}$$



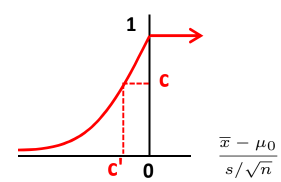
$$\varphi(x_1, \ldots, x_n) = \begin{cases} 1, & \lambda(x_1, \ldots, x_n) < c \;\; (\in (0,1]) \\ 0, & \lambda(x_1, \ldots, x_n) \geq c \end{cases} \Leftrightarrow \; \varphi(x_1, \ldots, x_n) = \begin{cases} 1, & \dfrac{\overline{x} - \mu_0}{s/\sqrt{n}} > c' \quad (c' \geq 0) \\[2ex] 0, & \dfrac{\overline{x} - \mu_0}{s/\sqrt{n}} \leq c'. \end{cases}$$

$$\alpha = \sup_{\mu \leq \mu_0, \sigma^2 \in \mathbb{R}^+} P_{\mu,\sigma^2}\left(\frac{\overline{X} - \mu_0}{S/\sqrt{n}} > c'\right) = \sup_{\mu \leq \mu_0, \sigma^2 \in \mathbb{R}^+} P_{\mu,\sigma^2}\left(\overline{X} > \mu_0 + c'\frac{S}{\sqrt{n}}\right) = \sup_{\mu \leq \mu_0, \sigma^2 \in \mathbb{R}^+} P_{\mu,\sigma^2}\left(\frac{\overline{X} - \mu}{S/\sqrt{n}} > \frac{\mu_0 - \mu}{S/\sqrt{n}} + c'\right) =$$

$$\Big[T \rightarrow t(n-1)\Big] \quad = \sup_{\mu \leq \mu_0} P\left(T > \frac{\mu_0 - \mu}{S/\sqrt{n}} + c'\right) = P\left(T > c'\right) = \alpha \;\Rightarrow\; c' = t_{n-1;\ \alpha} \; (\geq 0 \Leftrightarrow \alpha \leq 1/2)$$

$\boxed{\begin{array}{l} H_0 : \mu \geq \mu_0 \\ H_1 : \mu < \mu_0 \end{array}}$ TRV de tamaño $\alpha \leq 1/2 \;\rightarrow\; \varphi(X_1, \ldots, X_n) = \begin{cases} 1 & \text{si } \dfrac{\overline{X} - \mu_0}{S/\sqrt{n}} < t_{n-1;\ 1-\alpha} \\[2ex] 0 & \text{si } \dfrac{\overline{X} - \mu_0}{S/\sqrt{n}} \geq t_{n-1;\ 1-\alpha} \end{cases}$

$$\lambda(x_1, \ldots, x_n) = \frac{\sup\limits_{\mu \geq \mu_0, \sigma^2 \in \mathbb{R}^+} L_{x_1,\ldots,x_n}(\mu, \sigma^2)}{L_{x_1,\ldots,x_n}(\overline{x}, \widehat{\sigma}^2)} = \begin{cases} \dfrac{L_{x_1,\ldots,x_n}(\mu_0, \widehat{\sigma}_0^2)}{L_{x_1,\ldots,x_n}(\overline{x}, \widehat{\sigma}^2)}, & \dfrac{\overline{x} - \mu_0}{s/\sqrt{n}} \leq 0 \\[2ex] 1, & \dfrac{\overline{x} - \mu_0}{s/\sqrt{n}} \geq 0 \end{cases}$$



$$\varphi(x_1, \ldots, x_n) = \begin{cases} 1, & \lambda(x_1, \ldots, x_n) < c \;\; (\in (0,1]) \\ 0, & \lambda(x_1, \ldots, x_n) \geq c \end{cases} \Leftrightarrow \; \varphi(x_1, \ldots, x_n) = \begin{cases} 1, & \dfrac{\overline{x} - \mu_0}{s/\sqrt{n}} < c' \quad (c' \leq 0) \\[2ex] 0, & \dfrac{\overline{x} - \mu_0}{s/\sqrt{n}} \geq c'. \end{cases}$$

$$\alpha = \sup_{\mu \geq \mu_0, \sigma^2 \in \mathbb{R}^+} P_{\mu,\sigma^2}\left(\frac{\overline{X} - \mu_0}{S/\sqrt{n}} < c'\right) = \sup_{\mu \geq \mu_0, \sigma^2 \in \mathbb{R}^+} P_{\mu,\sigma^2}\left(\overline{X} < \mu_0 + c'\frac{S}{\sqrt{n}}\right) = \sup_{\mu \geq \mu_0, \sigma^2 \in \mathbb{R}^+} P_{\mu,\sigma^2}\left(\frac{\overline{X} - \mu}{S/\sqrt{n}} < \frac{\mu_0 - \mu}{S/\sqrt{n}} + c'\right) =$$

$$\Big[T \rightarrow t(n-1)\Big] \quad = \sup_{\mu \geq \mu_0} P\left(T < \frac{\mu_0 - \mu}{S/\sqrt{n}} + c'\right) = P\left(T < c'\right) = \alpha \;\Rightarrow\; c' = t_{n-1;\ 1-\alpha} \; (\geq 0 \Leftrightarrow \alpha \leq 1/2)$$

Contrastes sobre la varianza con media conocida

$(X_1,\ldots,X_n)$ muestra aleatoria simple de $X \longrightarrow \{\mathcal{N}(\mu_0,\sigma^2);\ \sigma^2\in\mathbb{R}^+\}$

Función de verosimilitud:

$$(x_1,\ldots,x_n)\in\mathbb{R}^n \longrightarrow L_{x_1,\ldots,x_n}(\sigma^2)=\frac{1}{(\sigma^2)^{n/2}(2\pi)^{n/2}}e^{-\sum\limits_{i=1}^{n}(x_i-\mu_0)^2/2\sigma^2},\quad \sigma^2\in\mathbb{R}^+$$

- $$\sup_{\sigma^2\in\mathbb{R}^+} L_{x_1,\ldots,x_n}(\sigma^2)=L_{x_1,\ldots,x_n}(\widehat{\sigma}_0^2),\qquad \widehat{\sigma}_0^2=\frac{\sum\limits_{i=1}^{n}(x_i-\mu_0)^2}{n}$$

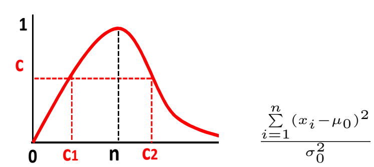
- $$\sup_{\sigma^2\le\sigma_0^2} L_{x_1,\ldots,x_n}(\sigma^2)=\begin{cases} L_{x_1,\ldots,x_n}(\widehat{\sigma}_0^2), & \widehat{\sigma}_0^2\le\sigma_0^2\ \left(\Leftrightarrow \dfrac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{\sigma_0^2}\le n\right)\\[2ex] L_{x_1,\ldots,x_n}(\sigma_0^2), & \widehat{\sigma}_0^2\ge\sigma_0^2\ \left(\Leftrightarrow \dfrac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{\sigma_0^2}\ge n\right)\end{cases}$$

- $$\sup_{\sigma^2\ge\sigma_0^2} L_{x_1,\ldots,x_n}(\sigma^2)=\begin{cases} L_{x_1,\ldots,x_n}(\sigma_0^2), & \widehat{\sigma}_0^2\le\sigma_0^2\ \left(\Leftrightarrow \dfrac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{\sigma_0^2}\le n\right)\\[2ex] L_{x_1,\ldots,x_n}(\widehat{\sigma}_0^2), & \widehat{\sigma}_0^2\ge\sigma_0^2\ \left(\Leftrightarrow \dfrac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{\sigma_0^2}\ge n\right)\end{cases}$$

$$\boxed{\begin{array}{l} H_0:\sigma^2=\sigma_0^2\\ H_1:\sigma^2\neq\sigma_0^2\end{array}}$$

TRV($\approx$) de tamaño $\alpha\in[0,1]$ $\to$ $$\varphi(X_1,\ldots,X_n)=\begin{cases}1 & \text{si}\quad \dfrac{\sum\limits_{i=1}^{n}(X_i-\mu_0)^2}{\sigma_0^2}<\chi^2_{n;\,1-\alpha/2}\ \text{ó}\ >\chi^2_{n;\,\alpha/2}\\[3ex] 0 & \text{si}\quad \chi^2_{n;\,1-\alpha/2}\le\dfrac{\sum\limits_{i=1}^{n}(X_i-\mu_0)^2}{\sigma_0^2}\le\chi^2_{n;\,\alpha/2}.\end{cases}$$

$$\lambda(x_1,\ldots,x_n)=\frac{\sup\limits_{\sigma^2=\sigma_0^2}L_{x_1,\ldots,x_n}(\sigma^2)}{\sup\limits_{\sigma^2\in\mathbb{R}^+}L_{x_1,\ldots,x_n}(\sigma^2)}=\frac{L_{x_1,\ldots,x_n}(\sigma_0^2)}{L_{x_1,\ldots,x_n}(\widehat{\sigma}_0^2)}=\left(\frac{\widehat{\sigma}_0^2}{\sigma_0^2}\right)^{n/2}\exp\left\{\frac{-n\widehat{\sigma}_0^2}{2\sigma_0^2}+\frac{n}{2}\right\}$$



$$\varphi(x_1,\ldots,x_n)=\begin{cases}1, & \lambda(x_1,\ldots,x_n)<c\ \ (\in(0,1])\\ 0, & \lambda(x_1,\ldots,x_n)\ge c\end{cases}\ \Leftrightarrow\ \varphi(x_1,\ldots,x_n)=\begin{cases}1, & \dfrac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{\sigma_0^2}<c_1\ \text{ó}\ \dfrac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{\sigma_0^2}>c_2\\[3ex] 0, & c_1\le\dfrac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{\sigma_0^2}\le c_2\end{cases}$$

donde $c_1\le n$ y $c_2\ge n$ son tales que $(c_1/n)^{n/2}e^{-c_1/2+n/2}=(c_2/n)^{n/2}e^{-c_2/2+n/2}$ y

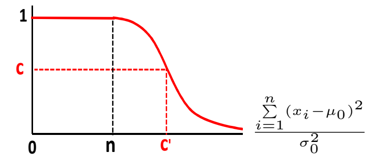
$$\alpha=P_{\sigma_0^2}\left(\frac{\sum_{i=1}^{n}(X_i-\mu_0)^2}{\sigma_0^2}<c_1\right)+P_{\sigma_0^2}\left(\frac{\sum_{i=1}^{n}(X_i-\mu_0)^2}{\sigma_0^2}>c_2\right)=P\left(Y<c_1\right)+P\left(Y>c_2\right),\ \text{con } Y\to\chi^2(n).$$

En la práctica, se toma el test de colas iguales, $c_1=\chi^2_{n;\,1-\alpha/2}$, $c_2=\chi^2_{n;\,\alpha/2}$ $\left(\forall\alpha\in[0,1],\ \chi^2_{n;\,1-\alpha/2}\le\chi^2_{n;\,\alpha/2}\right)$.

$$\boxed{\begin{array}{l} H_0 : \sigma^2 \leq \sigma_0^2 \\ H_1 : \sigma^2 > \sigma_0^2 \end{array}}$$

$$\text{TRV de tamaño } \alpha \leq P\left(Y > n\right)\ \left(Y \to \chi^2(n)\right) \longrightarrow \varphi(X_1,\ldots,X_n) = \begin{cases} 1 & \text{si } \dfrac{\sum\limits_{i=1}^{n}(X_i-\mu_0)^2}{\sigma_0^2} > \chi^2_{n;\ \alpha} \\[2ex] 0 & \text{si } \dfrac{\sum\limits_{i=1}^{n}(X_i-\mu_0)^2}{\sigma_0^2} \leq \chi^2_{n;\ \alpha} \end{cases}$$

$$\lambda(x_1,\ldots,x_n) = \frac{\sup\limits_{\sigma^2\leq\sigma_0^2} L_{x_1,\ldots,x_n}(\sigma^2)}{L_{x_1,\ldots,x_n}(\widehat{\sigma}_0^2)} = \begin{cases} 1, & \dfrac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{\sigma_0^2} \leq n \\[1ex] \dfrac{L_{x_1,\ldots,x_n}(\sigma_0^2)}{L_{x_1,\ldots,x_n}(\widehat{\sigma}_0^2)}, & \dfrac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{\sigma_0^2} \geq n \end{cases}$$



$$\varphi(x_1,\ldots,x_n) = \begin{cases} 1, & \lambda(x_1,\ldots,x_n) < c\ \ (\in (0,1]) \\ 0, & \lambda(x_1,\ldots,x_n) \geq c \end{cases} \Leftrightarrow \varphi(x_1,\ldots,x_n) = \begin{cases} 1, & \dfrac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{\sigma_0^2} > c' \quad (c' \geq n) \\[1ex] 0, & \dfrac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{\sigma_0^2} \leq c' \end{cases}$$

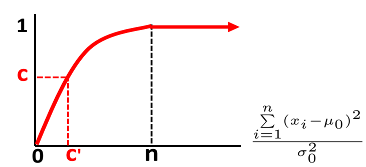
$$\alpha = \sup_{\sigma^2\leq\sigma_0^2} P_{\sigma^2}\left(\frac{\sum_{i=1}^{n}(X_i-\mu_0)^2}{\sigma_0^2} > c'\right) = \sup_{\sigma^2\leq\sigma_0^2} P_{\sigma^2}\left(\frac{\sum_{i=1}^{n}(X_i-\mu_0)^2}{\sigma^2} > c'\frac{\sigma_0^2}{\sigma^2}\right) = \left[Y \to \chi^2(n)\right] = \sup_{\sigma^2\leq\sigma_0^2} P\left(Y > c'\frac{\sigma_0^2}{\sigma^2}\right) =$$

$$= P\left(Y > c'\right) \ \Rightarrow\ c' = \chi^2_{n;\ \alpha}\ \left(\geq n \Leftrightarrow P\big(Y > n\big) \geq \alpha\right).$$

$$\boxed{\begin{array}{l} H_0 : \sigma^2 \geq \sigma_0^2 \\ H_1 : \sigma^2 < \sigma_0^2 \end{array}}$$

$$\text{TRV de tamaño } \alpha \leq P\left(Y \leq n\right)\ \left(Y \to \chi^2(n)\right) \longrightarrow \varphi(X_1,\ldots,X_n) = \begin{cases} 1 & \text{si } \dfrac{\sum\limits_{i=1}^{n}(X_i-\mu_0)^2}{\sigma_0^2} < \chi^2_{n;\ 1-\alpha} \\[2ex] 0 & \text{si } \dfrac{\sum\limits_{i=1}^{n}(X_i-\mu_0)^2}{\sigma_0^2} \geq \chi^2_{n;\ 1-\alpha} \end{cases}$$

$$\lambda(x_1,\ldots,x_n) = \frac{\sup\limits_{\sigma^2\geq\sigma_0^2} L_{x_1,\ldots,x_n}(\sigma^2)}{L_{x_1,\ldots,x_n}(\widehat{\sigma}_0^2)} = \begin{cases} \dfrac{L_{x_1,\ldots,x_n}(\sigma_0^2)}{L_{x_1,\ldots,x_n}(\widehat{\sigma}_0^2)}, & \dfrac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{\sigma_0^2} \leq n \\[1ex] 1, & \dfrac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{\sigma_0^2} \geq n \end{cases}$$



$$\varphi(x_1,\ldots,x_n) = \begin{cases} 1, & \lambda(x_1,\ldots,x_n) < c\ \ (\in (0,1]) \\ 0, & \lambda(x_1,\ldots,x_n) \geq c \end{cases} \Leftrightarrow \varphi(x_1,\ldots,x_n) = \begin{cases} 1, & \dfrac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{\sigma_0^2} < c' \quad (c' \leq n) \\[1ex] 0, & \dfrac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{\sigma_0^2} \geq c'. \end{cases}$$

$$\alpha = \sup_{\sigma^2\geq\sigma_0^2} P_{\sigma^2}\left(\frac{\sum_{i=1}^{n}(X_i-\mu_0)^2}{\sigma_0^2} < c'\right) = \sup_{\sigma^2\geq\sigma_0^2} P_{\sigma^2}\left(\frac{\sum_{i=1}^{n}(X_i-\mu_0)^2}{\sigma^2} < c'\frac{\sigma_0^2}{\sigma^2}\right) = \left[Y \to \chi^2(n)\right] = \sup_{\sigma^2\geq\sigma_0^2} P\left(Y < c'\frac{\sigma_0^2}{\sigma^2}\right) =$$

$$= P\left(Y < c'\right) \ \Rightarrow\ c' = \chi^2_{n;\ 1-\alpha}\ \left(\leq n \Leftrightarrow P\big(Y < n\big) \geq \alpha\right).$$

Contrastes sobre la varianza con media desconocida

$(X_1,\dots,X_n)$ muestra aleatoria simple de $X \longrightarrow \{\mathcal{N}(\mu,\sigma^2);\ \mu\in\mathbb{R},\ \sigma^2\in\mathbb{R}^+\}$

Función de verosimilitud:

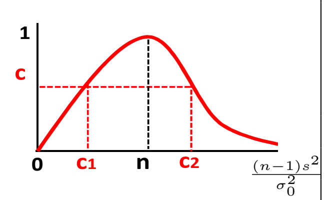
$$(x_1,\dots,x_n)\in\mathbb{R}^n \longrightarrow L_{x_1,\dots,x_n}(\mu,\sigma^2)=\frac{1}{(\sigma^2)^{n/2}(2\pi)^{n/2}}e^{-\sum_{i=1}^{n}(x_i-\mu)^2/2\sigma^2},\quad \mu\in\mathbb{R},\ \sigma^2\in\mathbb{R}^+$$

- $\sup\limits_{\mu\in\mathbb{R},\sigma^2\in\mathbb{R}^+} L_{x_1,\dots,x_n}(\mu,\sigma^2)=L_{x_1,\dots,x_n}(\overline{x},\widehat{\sigma}^2),\quad \widehat{\sigma}^2=\dfrac{\sum_{i=1}^{n}(x_i-\overline{x})^2}{n}=\dfrac{(n-1)s^2}{n}.$
- $\sup\limits_{\mu\in\mathbb{R},\sigma^2=\sigma_0^2} L_{x_1,\dots,x_n}(\mu,\sigma^2)=L_{x_1,\dots,x_n}(\overline{x},\sigma_0^2)$
- $\sup\limits_{\mu\in\mathbb{R},\sigma^2\leq\sigma_0^2} L_{x_1,\dots,x_n}(\mu,\sigma^2)=\begin{cases} L_{x_1,\dots,x_n}(\overline{x},\widehat{\sigma}^2), & \widehat{\sigma}^2\leq\sigma_0^2\ \left(\Leftrightarrow \frac{(n-1)s^2}{\sigma_0^2}\leq n\right)\\ L_{x_1,\dots,x_n}(\overline{x},\sigma_0^2), & \widehat{\sigma}^2\geq\sigma_0^2\ \left(\Leftrightarrow \frac{(n-1)s^2}{\sigma_0^2}\geq n\right)\end{cases}$
- $\sup\limits_{\mu\in\mathbb{R},\sigma^2\geq\sigma_0^2} L_{x_1,\dots,x_n}(\mu,\sigma^2)=\begin{cases} L_{x_1,\dots,x_n}(\overline{x},\sigma_0^2), & \widehat{\sigma}^2\leq\sigma_0^2\ \left(\Leftrightarrow \frac{(n-1)s^2}{\sigma_0^2}\leq n\right)\\ L_{x_1,\dots,x_n}(\overline{x},\widehat{\sigma}^2), & \widehat{\sigma}^2\geq\sigma_0^2\ \left(\Leftrightarrow \frac{(n-1)s^2}{\sigma_0^2}\geq n\right)\end{cases}$

$$\boxed{\begin{array}{l} H_0:\sigma^2=\sigma_0^2\\ H_1:\sigma^2\neq\sigma_0^2\end{array}}$$

TRV($\approx$) de tamaño $\alpha\in[0,1]$ $\to$ $\varphi(X_1,\dots,X_n)=\begin{cases}1 & \text{si}\ \ \dfrac{(n-1)S^2}{\sigma_0^2}<\chi^2_{n-1;\ 1-\alpha/2}\ \text{ó}\ >\chi^2_{n-1;\ \alpha/2}\\ 0 & \text{si}\ \ \chi^2_{n-1;\ 1-\alpha/2}\leq\dfrac{(n-1)S^2}{\sigma_0^2}\leq\chi^2_{n-1;\ \alpha/2}.\end{cases}$

$$\lambda(x_1,\dots,x_n)=\frac{\sup\limits_{\mu\in\mathbb{R},\sigma^2=\sigma_0^2} L_{x_1,\dots,x_n}(\mu,\sigma^2)}{\sup\limits_{\mu\in\mathbb{R},\sigma^2\in\mathbb{R}^+} L_{x_1,\dots,x_n}(\mu,\sigma^2)}=\frac{L_{x_1,\dots,x_n}(\overline{x},\sigma_0^2)}{L_{x_1,\dots,x_n}(\overline{x},\widehat{\sigma}^2)}=\left(\frac{\widehat{\sigma}^2}{\sigma_0^2}\right)^{n/2}\exp\left\{\frac{-n\widehat{\sigma}^2}{2\sigma_0^2}+\frac{n}{2}\right\}$$



$$\varphi(x_1,\dots,x_n)=\begin{cases}1, & \lambda(x_1,\dots,x_n)<c\ \ (\in(0,1])\\ 0, & \lambda(x_1,\dots,x_n)\geq c\end{cases}\Leftrightarrow \varphi(x_1,\dots,x_n)=\begin{cases}1 & \text{si}\ \frac{(n-1)s^2}{\sigma_0^2}<c_1\ \text{ó}\ \frac{(n-1)s^2}{\sigma_0^2}>c_2\\ 0 & \text{si}\ c_1\leq\frac{(n-1)s^2}{\sigma_0^2}\leq c_2\end{cases}$$

donde $c_1\leq n,\ c_2\geq n,\ (c_1/n)^{n/2}e^{-c_1/2+n/2}=(c_2/n)^{n/2}e^{-c_2/2+n/2}$ son tales que

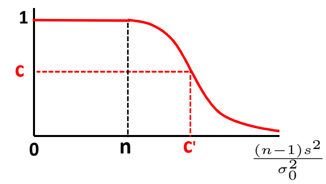
$$\alpha=\sup_{\mu\in\mathbb{R}}P_{\mu,\sigma_0^2}\left(\frac{(n-1)S^2}{\sigma_0^2}<c_1\right)+\sup_{\mu\in\mathbb{R}}P_{\mu,\sigma_0^2}\left(\frac{(n-1)S^2}{\sigma_0^2}>c_2\right)=\left[Y\to\chi^2(n-1)\right]=P\left(Y<c_1\right)+P\left(Y>c_2\right)$$

En la práctica, se toma el test de colas iguales, $c_1=\chi^2_{n-1;1-\alpha/2},\ c_2=\chi^2_{n-1;\alpha/2}$ $\left(\forall\alpha\in[0,1],\ \chi^2_{n-1;\ 1-\alpha/2}\leq\chi^2_{n-1;\ \alpha/2}\right)$.

$$\boxed{\begin{array}{l} H_0 : \sigma^2 \leq \sigma_0^2 \\ H_1 : \sigma^2 > \sigma_0^2 \end{array}}$$

TRV de tamaño $\alpha \leq P\left(Y > n\right)$ $(Y \to \chi^2(n-1)) \longrightarrow \varphi(X_1, \ldots, X_n) = \begin{cases} 1 & \text{si } \dfrac{(n-1)S^2}{\sigma_0^2} > \chi^2_{n-1;\ \alpha} \\ 0 & \text{si } \dfrac{(n-1)S^2}{\sigma_0^2} \leq \chi^2_{n-1;\ \alpha} \end{cases}$

$$\lambda(x_1, \ldots, x_n) = \frac{\sup\limits_{\mu \in \mathbb{R}, \sigma^2 \leq \sigma_0^2} L_{x_1,\ldots,x_n}(\mu, \sigma^2)}{L_{x_1,\ldots,x_n}(\overline{x}, \widehat{\sigma}^2)} = \begin{cases} 1, & \frac{(n-1)s^2}{\sigma_0^2} \leq n \\ \frac{L_{x_1,\ldots,x_n}(\overline{x}, \sigma_0^2)}{L_{x_1,\ldots,x_n}(\overline{x}, \widehat{\sigma}^2)}, & \frac{(n-1)s^2}{\sigma_0^2} \geq n \end{cases}$$



$$\varphi(x_1, \ldots, x_n) = \begin{cases} 1, & \lambda(x_1, \ldots, x_n) < c \quad (\in (0, 1]) \\ 0, & \lambda(x_1, \ldots, x_n) \geq c \end{cases} \Leftrightarrow \varphi(x_1, \ldots, x_n) = \begin{cases} 1 & \text{si } \frac{(n-1)s^2}{\sigma_0^2} > c' \quad (c' \geq n) \\ 0 & \text{si } \frac{(n-1)s^2}{\sigma_0^2} \leq c'. \end{cases}$$

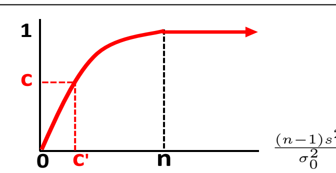
$$\alpha = \sup_{\mu \in \mathbb{R}, \sigma^2 \leq \sigma_0^2} P_{\mu,\sigma^2}\left(\frac{(n-1)S^2}{\sigma_0^2} > c'\right) = \sup_{\mu \in \mathbb{R}, \sigma^2 \leq \sigma_0^2} P_{\mu,\sigma^2}\left(\frac{(n-1)S^2}{\sigma^2} > c'\frac{\sigma_0^2}{\sigma^2}\right) = \left[Y \to \chi^2(n-1)\right] =$$

$$= \sup_{\sigma^2 \leq \sigma_0^2} P\left(Y > c'\frac{\sigma_0^2}{\sigma^2}\right) = P\left(Y > c'\right) \Rightarrow c' = \chi^2_{n-1;\ \alpha} \ \left(\geq n \Leftrightarrow P\big(Y > n\big) \geq \alpha\right).$$

$$\boxed{\begin{array}{l} H_0 : \sigma^2 \geq \sigma_0^2 \\ H_1 : \sigma^2 < \sigma_0^2 \end{array}}$$

TRV de tamaño $\alpha \leq P\left(Y \leq n\right)$ $(Y \to \chi^2(n-1)) \longrightarrow \varphi(X_1, \ldots, X_n) = \begin{cases} 1 & \text{si } \dfrac{(n-1)S^2}{\sigma_0^2} < \chi^2_{n-1;\ 1-\alpha} \\ 0 & \text{si } \dfrac{(n-1)S^2}{\sigma_0^2} \geq \chi^2_{n-1;\ 1-\alpha} \end{cases}$

$$\lambda(x_1, \ldots, x_n) = \frac{\sup\limits_{\mu \in \mathbb{R}, \sigma^2 \geq \sigma_0^2} L_{x_1,\ldots,x_n}(\mu, \sigma^2)}{L_{x_1,\ldots,x_n}(\overline{x}, \widehat{\sigma}^2)} = \begin{cases} \dfrac{L_{x_1,\ldots,x_n}(\overline{x}, \sigma_0^2)}{L_{x_1,\ldots,x_n}(\overline{x}, \widehat{\sigma}^2)}, & \frac{(n-1)s^2}{\sigma_0^2} \leq n \\ 1, & \frac{(n-1)s^2}{\sigma_0^2} \geq n \end{cases}$$



$$\varphi(x_1, \ldots, x_n) = \begin{cases} 1, & \lambda(x_1, \ldots, x_n) < c \quad (\in (0, 1]) \\ 0, & \lambda(x_1, \ldots, x_n) \geq c \end{cases} \Leftrightarrow \varphi(x_1, \ldots, x_n) = \begin{cases} 1 & \text{si } \frac{(n-1)s^2}{\sigma_0^2} < c' \quad (c' \leq n) \\ 0 & \text{si } \frac{(n-1)s^2}{\sigma_0^2} \geq c'. \end{cases}$$

$$\alpha = \sup_{\mu \in \mathbb{R}, \sigma^2 \geq \sigma_0^2} P_{\mu,\sigma^2}\left(\frac{(n-1)S^2}{\sigma_0^2} < c'\right) = \sup_{\mu \in \mathbb{R}, \sigma^2 \geq \sigma_0^2} P_{\mu,\sigma^2}\left(\frac{(n-1)S^2}{\sigma^2} < c'\frac{\sigma_0^2}{\sigma^2}\right) = \left[Y \to \chi^2(n-1)\right] =$$

$$= \sup_{\sigma^2 \geq \sigma_0^2} P\left(Y < c'\frac{\sigma_0^2}{\sigma^2}\right) = P\left(Y < c'\right) \Rightarrow c' = \chi^2_{n-1;\ 1-\alpha} \ \left(\geq n \Leftrightarrow P\big(Y < n\big) \geq \alpha\right).$$

TESTS E INTERVALOS EN POBLACIONES NORMALES

Contraste	Región de rechazo $\sigma^2 = \sigma_0^2$ conocida	Región de rechazo σ^2 desconocida
$H_0 : \mu = \mu_0$ $H_1 : \mu \neq \mu_0$	$\left\lvert \dfrac{\overline{X} - \mu_0}{\sigma_0/\sqrt{n}} \right\rvert > z_{\alpha/2}$ $\mu_0 \notin \left(\overline{X} - z_{\alpha/2}\dfrac{\sigma_0}{\sqrt{n}},\ \overline{X} + z_{\alpha/2}\dfrac{\sigma_0}{\sqrt{n}} \right)$	$\left\lvert \dfrac{\overline{X} - \mu_0}{S/\sqrt{n}} \right\rvert > t_{n-1;\ \alpha/2}$ $\mu_0 \notin \left(\overline{X} - t_{n-1;\ \alpha/2}\dfrac{S}{\sqrt{n}},\ \overline{X} + t_{n-1;\ \alpha/2}\dfrac{S}{\sqrt{n}} \right)$
$H_0 : \mu \leq \mu_0$ $H_1 : \mu > \mu_0$	$\dfrac{\overline{X} - \mu_0}{\sigma_0/\sqrt{n}} > z_{\alpha}$ $\mu_0 \notin \left(\overline{X} - z_{\alpha}\dfrac{\sigma_0}{\sqrt{n}},\ +\infty \right)$	$\dfrac{\overline{X} - \mu_0}{S/\sqrt{n}} > t_{n-1;\ \alpha}$ $\mu_0 \notin \left(\overline{X} - t_{n-1;\ \alpha}\dfrac{S}{\sqrt{n}},\ +\infty \right)$
$H_0 : \mu \geq \mu_0$ $H_1 : \mu < \mu_0$	$\dfrac{\overline{X} - \mu_0}{\sigma_0/\sqrt{n}} < z_{1-\alpha}$ $\mu_0 \notin \left(-\infty,\ \overline{X} + z_{\alpha}\dfrac{\sigma_0}{\sqrt{n}} \right)$	$\dfrac{\overline{X} - \mu_0}{S/\sqrt{n}} < t_{n-1;\ 1-\alpha}$ $\mu_0 \notin \left(-\infty,\ \overline{X} + t_{n-1;\ \alpha}\dfrac{S}{\sqrt{n}} \right)$

Contraste	Región de rechazo $\mu = \mu_0$ conocida	Región de rechazo μ desconocida
$H_0 : \sigma^2 = \sigma_0^2$ $H_1 : \sigma^2 \neq \sigma_0^2$	$\dfrac{\sum_{i=1}^{n} (X_i - \mu_0)^2}{\sigma_0^2} < \chi^2_{n;\ 1-\alpha/2}$ ó $> \chi^2_{n;\ \alpha/2}$ $\sigma_0^2 \notin \left(\dfrac{\sum_{i=1}^{n} (X_i - \mu_0)^2}{\chi^2_{n;\ \alpha/2}},\ \dfrac{\sum_{i=1}^{n} (X_i - \mu_0)^2}{\chi^2_{n;\ 1-\alpha/2}} \right)$	$\dfrac{(n-1)S^2}{\sigma_0^2} < \chi^2_{n-1;\ 1-\alpha/2}$ ó $> \chi^2_{n-1;\ \alpha/2}$ $\sigma_0^2 \notin \left(\dfrac{(n-1)S^2}{\chi^2_{n-1;\ \alpha/2}},\ \dfrac{(n-1)S^2}{\chi^2_{n-1;\ 1-\alpha/2}} \right)$
$H_0 : \sigma^2 \leq \sigma_0^2$ $H_1 : \sigma^2 > \sigma_0^2$	$\dfrac{\sum_{i=1}^{n} (X_i - \mu_0)^2}{\sigma_0^2} > \chi^2_{n;\ \alpha}$ $\sigma_0^2 \notin \left(\dfrac{\sum_{i=1}^{n} (X_i - \mu_0)^2}{\chi^2_{n;\ \alpha}},\ +\infty \right)$	$\dfrac{(n-1)S^2}{\sigma_0^2} > \chi^2_{n-1;\ \alpha}$ $\sigma_0^2 \notin \left(\dfrac{(n-1)S^2}{\chi^2_{n-1;\ \alpha}},\ +\infty \right)$
$H_0 : \sigma^2 \geq \sigma_0^2$ $H_1 : \sigma^2 < \sigma_0^2$	$\dfrac{\sum_{i=1}^{n} (X_i - \mu_0)^2}{\sigma_0^2} < \chi^2_{n;\ 1-\alpha}$ $\sigma_0^2 \notin \left(0,\ \dfrac{\sum_{i=1}^{n} (X_i - \mu_0)^2}{\chi^2_{n;\ 1-\alpha}} \right)$	$\dfrac{(n-1)S^2}{\sigma_0^2} < \chi^2_{n-1;\ 1-\alpha}$ $\sigma_0^2 \notin \left(0,\ \dfrac{(n-1)S^2}{\chi^2_{n-1;\ 1-\alpha}} \right)$

7.5. TESTS DE HIPÓTESIS E INTERVALOS DE CONFIANZA PARA LOS PARÁMETROS DE DOS POBLACIONES NORMALES

Contraste	Región de rechazo, σ_1^2, σ_2^2 conocidas	Región de rechazo, $\sigma_1^2 = \sigma_2^2$ desconocida
$H_0 : \mu_1 = \mu_2$ $H_1 : \mu_1 \neq \mu_2$	$\dfrac{\lvert\overline{X}-\overline{Y}\rvert}{\sqrt{\dfrac{\sigma_1^2}{n_1}+\dfrac{\sigma_2^2}{n_2}}} > z_{\alpha/2}$ $0 \notin \left(\overline{X}-\overline{Y} \mp z_{\alpha/2}\sqrt{\dfrac{\sigma_1^2}{n_1}+\dfrac{\sigma_2^2}{n_2}}\right)$	$\dfrac{\lvert\overline{X}-\overline{Y}\rvert}{\sqrt{\dfrac{(n_1-1)S_1^2+(n_2-1)S_2^2}{n_1+n_2-2}}\sqrt{\dfrac{1}{n_1}+\dfrac{1}{n_2}}} > t_{n_1+n_2-2;\ \alpha/2}$ $0 \notin \left(\overline{X}-\overline{Y} \mp t_{n_1+n_2-2;\ \alpha/2}\sqrt{\dfrac{(n_1-1)S_1^2+(n_2-1)S_2^2}{n_1+n_2-2}}\sqrt{\dfrac{1}{n_1}+\dfrac{1}{n_2}}\right)$
$H_0 : \mu_1 \leq \mu_2$ $H_1 : \mu_1 > \mu_2$	$\dfrac{\overline{X}-\overline{Y}}{\sqrt{\dfrac{\sigma_1^2}{n_1}+\dfrac{\sigma_2^2}{n_2}}} > z_{\alpha}$ $0 \notin \left(\overline{X}-\overline{Y} - z_{\alpha}\sqrt{\dfrac{\sigma_1^2}{n_1}+\dfrac{\sigma_2^2}{n_2}},\ \ +\infty\right)$	$\dfrac{\overline{X}-\overline{Y}}{\sqrt{\dfrac{(n_1-1)S_1^2+(n_2-1)S_2^2}{n_1+n_2-2}}\sqrt{\dfrac{1}{n_1}+\dfrac{1}{n_2}}} > t_{n_1+n_2-2;\ \alpha}$ $0 \notin \left(\overline{X}-\overline{Y} - t_{n_1+n_2-2;\ \alpha}\sqrt{\dfrac{(n_1-1)S_1^2+(n_2-1)S_2^2}{n_1+n_2-2}}\sqrt{\dfrac{1}{n_1}+\dfrac{1}{n_2}},\ \ +\infty\right)$

Contraste	Región de rechazo, μ_1, μ_2 conocidas	Región de rechazo, μ_1, μ_2 desconocidas
$H_0 : \sigma_1^2 = \sigma_2^2$ $H_1 : \sigma_1^2 \neq \sigma_2^2$	$\dfrac{\sum_{i=1}^{n_1}(X_i-\mu_1)^2/n_1}{\sum_{i=1}^{n_2}(Y_i-\mu_2)^2/n_2} < F_{n_1,n_2;\ 1-\alpha/2}$ ó $> F_{n_1,n_2;\ \alpha/2}$ $1 \notin \left(F_{n_2,n_1;\ 1-\alpha/2}\dfrac{\sum_{i=1}^{n_1}(X_i-\mu_1)^2/n_1}{\sum_{i=1}^{n_2}(Y_i-\mu_2)^2/n_2},\ \ F_{n_2,n_1;\ \alpha/2}\dfrac{\sum_{i=1}^{n_1}(X_i-\mu_1)^2/n_1}{\sum_{i=1}^{n_2}(Y_i-\mu_2)^2/n_2}\right)$	$\dfrac{S_1^2}{S_2^2} < F_{n_1-1,n_2-1;\ 1-\alpha/2}$ ó $> F_{n_1-1,n_2-1;\ \alpha/2}$ $1 \notin \left(F_{n_2-1,n_1-1;\ 1-\alpha/2}\dfrac{S_1^2}{S_2^2},\ \ F_{n_2-1,n_1-1;\ \alpha/2}\dfrac{S_1^2}{S_2^2}\right)$
$H_0 : \sigma_1^2 \leq \sigma_2^2$ $H_1 : \sigma_1^2 > \sigma_2^2$	$\dfrac{\sum_{i=1}^{n_1}(X_i-\mu_1)^2/n_1}{\sum_{i=1}^{n_2}(Y_i-\mu_2)^2/n_2} > F_{n_1,n_2;\ \alpha}$ $1 \notin \left(F_{n_2,n_1;\ 1-\alpha}\dfrac{\sum_{i=1}^{n_1}(X_i-\mu_1)^2/n_1}{\sum_{i=1}^{n_2}(Y_i-\mu_2)^2/n_2},\ \ +\infty\right)$	$\dfrac{S_1^2}{S_2^2} > F_{n_1-1,n_2-1;\ \alpha}$ $1 \notin \left(F_{n_2-1,n_1-1;\ 1-\alpha}\dfrac{S_1^2}{S_2^2},\ \ +\infty\right)$

7.6. DUALIDAD ENTRE TESTS DE HIPÓTESIS Y REGIONES DE CONFIANZA

Sea $X \to \{P_\theta;\ \theta \in \Theta\}$ *y* $(X_1, \dots, X_n)$ *una muestra aleatoria simple de* X*. Para cada* $\theta_0 \in \Theta$ *consideramos un conjunto* $A(\theta_0) \subseteq \chi^n$ *y, para cada realización muestral,* $(x_1, \dots, x_n) \in \chi^n$*, definimos:*

$$\varphi_{\theta_0}(x_1, \dots, x_n) = \begin{cases} 1 & si \ \ (x_1, \dots, x_n) \notin A(\theta_0) \\ \\ 0 & si \ \ (x_1, \dots, x_n) \in A(\theta_0) \end{cases}$$

$$S(x_1, \dots, x_n) = \{\theta \in \Theta \ / \ (x_1, \dots, x_n) \in A(\theta)\} \subseteq \Theta.$$

Cada uno de lo tests $\varphi_{\theta_0}(X_1, \dots, X_n)$ *aplicado al problema de contrastar* $H_0 : \theta = \theta_0$ *frente a* $H_1 : \theta \neq \theta_0$ *tiene nivel de significación* α *si, y sólo si,* $S(X_1, \dots, X_n)$ *es una región de confianza para* θ *al nivel de confianza* $1 - \alpha$*.*

DEMOSTRACIÓN LEMA DE NEYMAN-PEARSON

Sea $X \to \{P_\theta;\ \theta \in \{\theta_0, \theta_1\}\}$ y $(X_1, \ldots, X_n)$ una muestra aleatoria simple con funciones de densidad (o funciones masa de probabilidad) $f_0^n(x_1, \ldots, x_n)$ $(\theta = \theta_0)$ y $f_1^n(x_1, \ldots, x_n)$ $(\theta = \theta_1)$. Consideremos el problema de contraste

$$H_0 : \theta = \theta_0$$
$$H_1 : \theta = \theta_1.$$

a) *Sea $\varphi(X_1, \ldots, X_n)$ un test de la forma*

$$\varphi(X_1, \ldots, X_n) = \begin{cases} 1, & si\ \ f_1^n(X_1, \ldots, X_n) > k f_0^n(X_1, \ldots, X_n) \\ \gamma(X_1, \ldots, X_n), & si\ \ f_1^n(X_1, \ldots, X_n) = k f_0^n(X_1, \ldots, X_n) \\ 0, & si\ \ f_1^n(X_1, \ldots, X_n) < k f_0^n(X_1, \ldots, X_n), \end{cases}$$

con $k \in \mathbb{R}^+ \cup \{0\}$ y $\gamma(X_1, \ldots, X_n) \in [0, 1]$. Si $\varphi(X_1, \ldots, X_n)$ tiene tamaño α, es de máxima potencia a nivel de significación α.

Demostración: Supondremos que X es de tipo continuo y, para simplificar, notaremos $\boldsymbol{X} = (X_1, \ldots, X_n)$ a la muestra aleatoria simple y $\mathbf{x} = (x_1, \ldots, x_n) \in \chi^n$ a las realizaciones muestrales. Así,

$$\beta_\varphi(\theta_i) = E_{\theta_i}[\varphi(\boldsymbol{X})] = \int_{\chi^n} \varphi(\mathbf{x}) f_i^n(\mathbf{x}) d\mathbf{x}, \quad i = 0, 1.$$

Una demostración totalmente similar puede hacerse con variables discretas, sustituyendo las densidades por las funciones masa de probabilidad y las integrales por sumas.

Consideremos la siguiente integral:

$$I = \int_{\chi^n} h(\mathbf{x}) d\mathbf{x}, \quad h(\mathbf{x}) = [\varphi(\mathbf{x}) - \varphi'(\mathbf{x})][f_1^n(\mathbf{x}) - k f_0^n(\mathbf{x})],\ \mathbf{x} \in \chi^n.$$

Teniendo en cuenta la forma de φ y que $\varphi' \in [0, 1]$ tenemos:

- $f_1^n(\mathbf{x}) > k f_0^n(\mathbf{x}) \Rightarrow h(\mathbf{x}) = [1 - \varphi'(\mathbf{x})][f_1^n(\mathbf{x}) - k f_0^n(\mathbf{x})] \geq 0.$
- $f_1^n(\mathbf{x}) < k f_0^n(\mathbf{x}) \Rightarrow h(\mathbf{x}) = [-\varphi'(\mathbf{x})][f_1^n(\mathbf{x}) - k f_0^n(\mathbf{x})] \geq 0.$
- $f_1^n(\mathbf{x}) = k f_0^n(\mathbf{x}) \Rightarrow h(\mathbf{x}) = 0.$

Esto es, $h \geq 0$ y, consecuentemente, $I \geq 0$. Entonces, desarrollando h tenemos:

$$\begin{aligned} I &= \int_{\chi^n} \varphi(\mathbf{x}) f_1^n(\mathbf{x}) d\mathbf{x} - \int_{\chi^n} \varphi'(\mathbf{x}) f_1^n(\mathbf{x}) d\mathbf{x} - k \left(\int_{\chi^n} \varphi(\mathbf{x}) f_0^n(\mathbf{x}) d\mathbf{x} - \int_{\chi^n} \varphi'(\mathbf{x}) f_0^n(\mathbf{x}) d\mathbf{x} \right) \\ &= \beta_\varphi(\theta_1) - \beta_{\varphi'}(\theta_1) - k(\alpha - E_{\theta_0}[\varphi'(\boldsymbol{X})]) \geq 0 \Rightarrow \beta_\varphi(\theta_1) - \beta_{\varphi'}(\theta_1) \geq k(\alpha - E_{\theta_0}[\varphi'(\boldsymbol{X})]). \end{aligned}$$

Por tanto, como $k \geq 0$ y $E_{\theta_0}[\varphi'(\boldsymbol{X})] \leq \alpha$, el segundo miembro es no negativo y, por tanto, el primero también. Esto es, $\beta_\varphi(\theta_1) \geq \beta_{\varphi'}(\theta_1)$.

b) Para todo $\alpha \in (0,1]$ existe un test de Neyman-Pearson de tamaño α, con $\gamma(\boldsymbol{X}) = \gamma$ constante.

Demostración: Dado $\alpha \in (0,1]$, hacemos $\gamma(\boldsymbol{X}) = \gamma$ en el test de Neyman-Pearson, $\varphi(\boldsymbol{X})$, y probamos que existen $k \geq 0$ y $\gamma \in [0,1]$ tales que el test tiene tamaño α. Esto es:

$$\alpha = E_{\theta_0}[\varphi(\boldsymbol{X})] = P_{\theta_0}(f_1^n(\boldsymbol{X}) > k f_0^n(\boldsymbol{X})) + \gamma P_{\theta_0}(f_1^n(\boldsymbol{X}) = k f_0^n(\boldsymbol{X}))$$ [1]

$$= P_{\theta_0}\left(\frac{f_1^n(\boldsymbol{X})}{f_0^n(\boldsymbol{X})} > k\right) + \gamma P_{\theta_0}\left(\frac{f_1^n(\boldsymbol{X})}{f_0^n(\boldsymbol{X})} = k\right).$$

Equivalentemente, notando H_{θ_0} la función de distribución de $f_1^n(\boldsymbol{X})/f_0^n(\boldsymbol{X}) \geq 0$ bajo P_{θ_0}, dado $\alpha \in (0,1]$, debemos encontrar $k \geq 0$ y $\gamma \in [0,1]$ tales que:

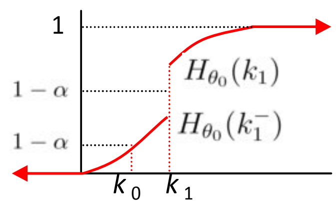
$$1 - \alpha = H_{\theta_0}(k) - \gamma(H_{\theta_0}(k) - H_{\theta_0}(k^-)).$$

Existen dos posibilidades:

a) $\exists k_0 \geq 0 \;/\; H_{\theta_0}(k_0) = 1 - \alpha \;\to\; k = k_0,\; \gamma = 0.$

b) $\exists k_1 \geq 0 \;/\; H_{\theta_0}(k_1^-) \leq 1 - \alpha < H_{\theta_0}(k_1)$

$$\to\; k = k_1,\; \gamma = \frac{H_{\theta_0}(k_1) - (1-\alpha)}{H_{\theta_0}(k_1) - H_{\theta_0}(k_1^-)} \in (0,1).$$



c) Si $\varphi'(\boldsymbol{X})$ es un test de tamaño α y es de máxima potencia a nivel de significación α, $\varphi'(\boldsymbol{X})$ es un test de Neyman-Pearson.

Demostración: Por *b*), dado $\alpha = E_{\theta_0}[\varphi'(\boldsymbol{X})]$, podemos encontrar un test de Neyman-Pearson de tamaño α, $\varphi(\boldsymbol{X})$. Puesto que $\varphi(\boldsymbol{X})$ y $\varphi'(\boldsymbol{X})$ son de máxima potencia, ésta debe ser la misma, y ambos tests tienen el mismo tamaño y la misma potencia. Por tanto:

$$I = \int_{\chi^n} [\varphi(\mathbf{x}) - \varphi'(\mathbf{x})]\,[f_1^n(\mathbf{x}) - k f_0^n(\mathbf{x})]\,d\mathbf{x} = \beta_\varphi(\theta_1) - \beta_{\varphi'}(\theta_1) - k\Big(\beta_\varphi(\theta_0) - \beta_{\varphi'}(\theta_0)\Big) = 0.$$

Ya que, según se probó en el apartado *a*), el integrando es una función no negativa, $I = 0$ significa que el integrando es nulo (salvo, quizás, en conjuntos con medida de Lebesgue nula, que tienen probabilidades nulas). Esto es:

- $f_1^n(\mathbf{x}) > k f_0^n(\mathbf{x}) \Rightarrow \varphi'(\mathbf{x}) = \varphi(\mathbf{x}) = 1.$
- $f_1^n(\mathbf{x}) < k f_0^n(\mathbf{x}) \Rightarrow \varphi'(\mathbf{x}) = \varphi(\mathbf{x}) = 0.$

Por tanto:

$$\varphi'(\boldsymbol{X}) = \begin{cases} 1, & f_1^n(\boldsymbol{X}) > k f_0^n(\boldsymbol{X}) \\ \gamma'(\boldsymbol{X}), & f_1^n(\boldsymbol{X}) = k f_0^n(\boldsymbol{X}) \\ 0, & f_1^n(\boldsymbol{X}) < k f_0^n(\boldsymbol{X}). \end{cases}$$

[1] $P_{\theta_0}(f_0^n(\boldsymbol{X}) = 0) = \int_{\{\mathbf{x} \in \chi^n \;/\; f_0^n(\mathbf{x}) = 0\}} f_0^n(\mathbf{x}) d\mathbf{x} = 0.$

d) *El test de máxima potencia entre todos los de nivel de significación 0 (tamaño 0) es:*

$$\varphi_0(\boldsymbol{X}) = \begin{cases} 1, & f_0^n(\boldsymbol{X}) = 0 \\ 0, & f_0^n(\boldsymbol{X}) > 0. \end{cases}$$

Demostración: Puesto que $P_{\theta_0}(f_0^n(\boldsymbol{X}) = 0)$, es inmediato que el test $\varphi_0(\boldsymbol{X})$ tiene tamaño 0.

Si $\varphi_0'(\boldsymbol{X})$ es cualquier otro test de tamaño cero, $E_{\theta_0}[\varphi_0'(\boldsymbol{X})] = 0$, y al ser una función no negativa, debe anularse en $\{\mathbf{x} \in \chi^n \ / \ f_0^n(\mathbf{x}) > 0\}$. Esto es:

$$\varphi_0'(\boldsymbol{X}) = \begin{cases} \gamma(\boldsymbol{X}), & f_0^n(\boldsymbol{X}) = 0, \\ 0, & f_0^n(\boldsymbol{X}) > 0, \end{cases} \qquad \gamma(\boldsymbol{X}) \in [0, 1].$$

Por tanto, $\varphi_0'(\boldsymbol{X}) \leq \varphi_0(\boldsymbol{X})$ y, consecuentemente:

$$\beta_{\varphi_0'}(\theta_1) = E_{\theta_1}\left[\varphi_0'(\boldsymbol{X})\right] \leq E_{\theta_1}\left[\varphi_0(\boldsymbol{X})\right] = \beta_{\varphi_0}(\theta_1).$$

□

Expresión del test para su resolución en diferentes situaciones prácticas:

$$\chi_0 = \{x / f_0(x) > 0\}; \quad \chi_1 = \{x / f_1(x) > 0\}.$$

- $\chi_0 \supseteq \chi_1 \ \Rightarrow \ \chi^n = \chi_0^n = \{(x_1, \ldots, x_n) \ / \ f_0^n(x_1, \ldots x_n) \neq 0\}.$

  En esta situación, se puede dividir siempre por $f_0^n(x_1, \ldots x_n)$ y la función test queda:

$$\varphi(x_1, \ldots, x_n) = \begin{cases} 1 & \text{si } \ \lambda(x_1, \ldots, x_n) > k \\ \gamma & \text{si } \ \lambda(x_1, \ldots, x_n) = k \\ 0 & \text{si } \ \lambda(x_1, \ldots, x_n) < k, \end{cases} \quad \text{con } \ \lambda(x_1, \ldots, x_n) = \frac{f_1^n(x_1, \ldots, x_n)}{f_0^n(x_1, \ldots, x_n)}.$$

- $\chi_0 \subset \chi_1 \ \Rightarrow \ \chi^n = \chi_1^n = \{(x_1, \ldots, x_n) \ / \ f_1^n(x_1, \ldots x_n) \neq 0\}.$

  Existen realizaciones muestrales para las que $f_0^n(x_1, \ldots x_n) = 0$ y no se puede dividir. Sin embargo, en estos casos es obvio que, $f_1^n(x_1, \ldots x_n) > k f_0^n(x_1, \ldots x_n)$, $\forall k \geq 0$, lo que significa que tales realizaciones conducen al rechazo de H_0 en cualquier test de Neyman-Pearson, y éste se expresa como:

$$\varphi(x_1, \ldots, x_n) = \begin{cases} 1 & \text{si } \ f_0^n(x_1, \ldots, x_n) = 0 \\ 1 & \text{si } \ f_0^n(x_1, \ldots, x_n) \neq 0 \ \text{ y } \ \lambda(x_1, \ldots, x_n) > k \\ \gamma & \text{si } \ f_0^n(x_1, \ldots, x_n) \neq 0 \ \text{ y } \ \lambda(x_1, \ldots, x_n) = k \\ 0 & \text{si } \ f_0^n(x_1, \ldots, x_n) \neq 0 \ \text{ y } \ \lambda(x_1, \ldots, x_n) < k. \end{cases} \quad \lambda(x_1, \ldots, x_n) = \frac{f_1^n(x_1, \ldots, x_n)}{f_0^n(x_1, \ldots, x_n)}.$$