19 abril

Series de Fourier

Base de Fourier en $[-\pi,\pi]$: (asociada a $\left.\begin{array}{l} y''+\lambda y=0 \\ y(-\pi)=y(\pi) \\ y'(-\pi)=y'(\pi) \end{array}\right\}$)

$$\{1, \cos(nx), \operatorname{sen}(nx) \ / \ n\geq 1\}$$

Sabemos que para cualquier $f\in L^2(-\pi,\pi)$,

$$f=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos(nx)+\sum_{n=1}^{\infty} b_n \operatorname{sen}(nx)$$

en sentido de convergencia $L^2(-\pi,\pi)$.

* Coeficientes:

$$(m\geq 1) \quad \int_{-\pi}^{\pi} f(x)\cos(mx)\,dx = \frac{a_0}{2}\cancelto{0}{\int_{-\pi}^{\pi}\cos(mx)\,dx} + \sum_{n=1}^{\infty} a_n \underbrace{\int_{-\pi}^{\pi}\cos(nx)\cos(mx)\,dx}_{0 \text{ salvo cuando } m=n}$$

$$+\sum_{n=1}^{\infty} b_n \cancelto{0}{\int_{-\pi}^{\pi}\operatorname{sen}(nx)\cos(mx)\,dx}$$

$$= a_m\int_{-\pi}^{\pi}(\cos(mx))^2\,dx = a_m\pi.$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \qquad n\geq 1.$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\operatorname{sen}(nx)\,dx \qquad n\geq 1.$$

$$\int_{-\pi}^{\pi} f(x)\,dx=\pi a_0 \Rightarrow a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx.$$

$$f = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \operatorname{sen}(nx)$$

Serie de Fourier de f.

$(f \in L^1(-\pi,\pi))$.

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \operatorname{sen}(nx)\, dx$$

Completitud base de Fourier

Si $f, g \in L^1(-\pi,\pi)$ tienen los mismos coeficientes de Fourier

$\Rightarrow f = g$ en casi todo punto de $[-\pi,\pi]$.

Forma alternativa

$f \in L^2(-\pi,\pi)$. Base $\{ e^{inx}, n \in \mathbb{Z} \}$.

$$e^{inx} = \cos(nx) + i \operatorname{sen}(nx).$$

Son ortogonales: si $n \neq m \in \mathbb{Z}$, $\displaystyle\int_{-\pi}^{+\pi} e^{inx} e^{-imx} \, dx = 0.$

$$f(x) = \sum_{n \in \mathbb{Z}} \hat{f}_n e^{inx} \qquad \text{(en sentido } L^2(-\pi,\pi)\text{)}.$$

coeficientes:

$$\int_{-\pi}^{\pi} f(x) e^{-inx} \, dx = \hat{f}_n \int_{-\pi}^{\pi} e^{inx} e^{-inx} = \hat{f}_n \cdot 2\pi$$

$$\hat{f}_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx$$

Cambio de base: para $n \geq 0$:

$$\hat{f}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\cdot\big(\cos(nx) - i\,\mathrm{sen}(nx)\big)\, dx =$$

$$= \frac{1}{2} a_n - \frac{i}{2} b_n$$

para $n \leq 0$.

$$\hat{f}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\,\big(\cos(-nx) + i\,\mathrm{sen}(-nx)\big)\, dx = \frac{1}{2} a_{-n} + \frac{i}{2} b_{-n}.$$

Identidad de Parseval

$$f = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \operatorname{sen}(nx). \qquad f \in L^2(-\pi,\pi).$$

$$\int_{-\pi}^{\pi} |f(x)|^2 dx = \int_{-\pi}^{\pi} \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \operatorname{sen}(nx)\right)$$

$$\left(\frac{\overline{a_0}}{2} + \sum_{m=1}^{\infty} \overline{a_m} \cos(mx) + \sum_{m=1}^{\infty} \overline{b_m} \operatorname{sen}(mx)\right) dx$$

$$= \sum_{n=1}^{\infty} |a_n|^2 \pi + \sum_{n=1}^{\infty} |b_n|^2 \pi + \frac{|a_0|^2}{4} \cdot 2\pi$$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} |a_n|^2 + \sum_{n=1}^{\infty} |b_n|^2 \qquad \text{Parseval.}$$

Teo. $f \in L^1(-\pi,\pi)$.

$$f \in L^2(-\pi,\pi) \iff \sum_{n=1}^{\infty} |a_n|^2 + \sum_{n=1}^{\infty} |b_n|^2 < +\infty.$$

$$f = \sum_{n\in\mathbb{Z}} \hat{f}_n e^{inx}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{n\in\mathbb{Z}} |\hat{f}_n|^2$$

Ejemplos: ⋆ $a_0 = 0$, $b_n = 0$, $a_n = \frac{1}{n}$ $(n \geq 1)$.

$\sum_{n=1}^{\infty} a_n^2 < +\infty$. $\Rightarrow$ la f correspondiente está en L^2.

⋆ $a_0 = 0$, $b_n = 0$ $a_n = \frac{1}{\sqrt{n}}$ $(n \geq 1)$.

no sabemos si hay una función f con estos coeficientes.

Series de Fourier en otros intervalos

$[a,b] \subseteq \mathbb{R}$, $f \in L^1(a,b)$.

Defino $\phi : [a,b] \to [-\pi,\pi]$, afín, $\phi(a) = -\pi$, $\phi(b) = \pi$.

$$\phi(x) = \pi \cdot \frac{2x - b - a}{b-a}$$

Base: $\{1, \cos(n\phi(x)), \operatorname{sen}(n\phi(x)) \,/\, n \geq 1 \text{ entero}\}$.

* Ejemplo:

$[0, L]$. $\quad \phi(x) = \pi \cdot \dfrac{2x - L}{L} = \pi\left(\dfrac{2x}{L} - 1\right)$

$$\left\{1, \cos\left(\pi n\left(\frac{2x}{L} - 1\right)\right), \operatorname{sen}\left(\pi n\left(\frac{2x}{L} - 1\right)\right) \,/\, n \geq 1\right\}$$

mejor: $\left\{1, \cos\left(\dfrac{2\pi n x}{L}\right), \operatorname{sen}\left(\dfrac{2\pi n x}{L}\right) \,/\, n \geq 1\right\}$

$$f = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{L}\right) + \sum_{n=1}^{\infty} b_n \operatorname{sen}\left(\frac{2\pi n x}{L}\right)$$

$$\int_0^L f \cdot \cos\left(\frac{2\pi n x}{L}\right) dx = a_n \frac{L}{2} \quad \Rightarrow \quad a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{2\pi n x}{L}\right) dx$$

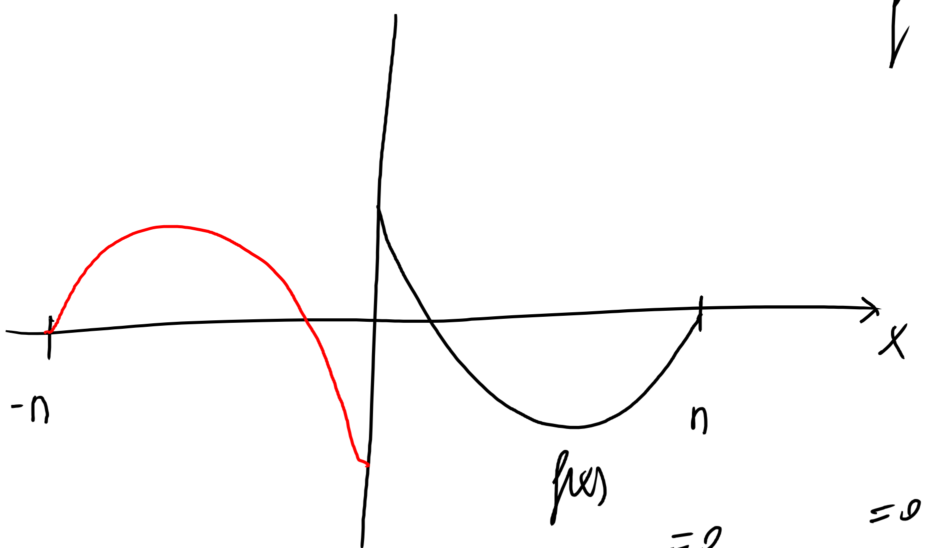
$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{2\pi n x}{L}\right)dx \qquad n \geq 0 \text{ entero}$$

$$b_n = \frac{2}{L}\int_0^L f(x)\,\text{sen}\left(\frac{2\pi n x}{L}\right)dx \qquad n \geq 1 \text{ entero.}$$

Serie de Fourier en senos

$f : [0,\pi] \to \mathbb{R}$, $\quad f \in L^2(0,\pi)$.

$$\tilde{f}(x) := \begin{cases} -f(-x) & \text{si } x \in (-\pi,0) \\ f(x) & \text{si } x \in (0,\pi). \end{cases}$$



Sé que $\quad \tilde{f} = \overset{=0}{\frac{a_0}{2}} + \sum_{n=1}^{\infty} \overset{=0}{a_n} \cos(nx) + \sum_{n=1}^{\infty} b_n \operatorname{sen}(nx) \quad$ en $L^2(-\pi,\pi)$.

$$\Rightarrow \quad f = \sum_{n=1}^{\infty} b_n \operatorname{sen}(nx) \quad \text{en} \quad L^2(0,\pi).$$

$$\pi\, b_n = \int_{-\pi}^{\pi} \tilde{f}(x) \operatorname{sen}(nx)\, dx = -\int_{-\pi}^{0} f(-x) \operatorname{sen}(nx)\, dx + \int_{0}^{\pi} f(x) \operatorname{sen}(nx)\, dx = \quad (y=-x)$$

$$= +\int_{0}^{\pi} f(y) \operatorname{sen}(+ny)\, dy + \int_{0}^{\pi} f(x) \operatorname{sen}(nx)\, dx = 2\int_{0}^{\pi} f(x) \operatorname{sen}(nx)\, dx$$

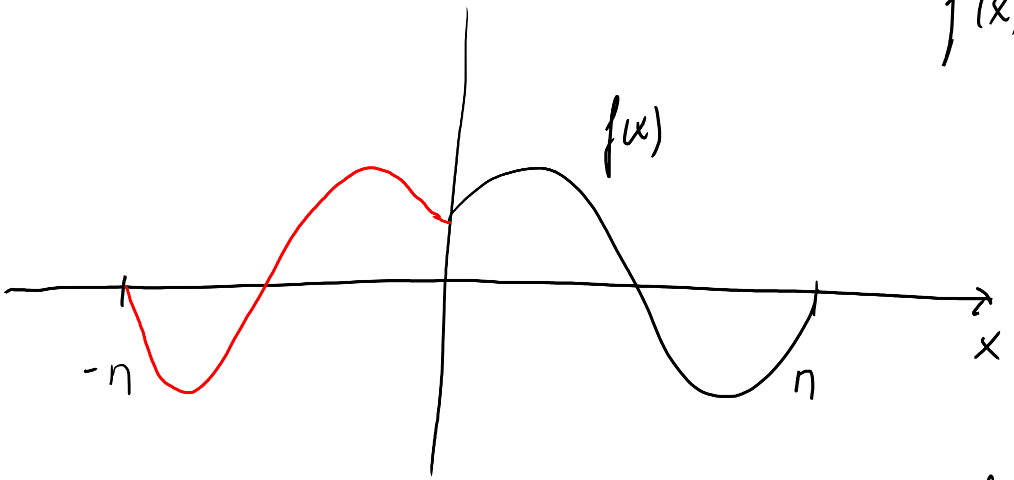
$$b_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \operatorname{sen}(nx)\, dx$$

Serie de Fourier en senos en $(0,\pi)$.

$$f = \sum_{n=1}^{\infty} b_n \operatorname{sen}(nx) \quad \text{en} \quad L^2(0,\pi).$$

Fourier en cosenos

$f : [0,\pi] \to \mathbb{R}$ $\qquad f \in L^2(0,\pi)$.



$$\tilde{f}(x) = \begin{cases} f(x) & \text{si } x \in [0,\pi] \\ f(-x) & \text{si } x \in [-\pi,0). \end{cases}$$

simétrica

$b_n = 0 \quad \forall\, n \geq 1$

$$\tilde{f} = \frac{a_0}{2} + \sum_{n\geq 1} a_n \cos(nx) + \sum_{n \geq 1} b_n \operatorname{sen}(nx) \qquad \text{en } L^2(-\pi,\pi).$$

$$\pi\, a_n = \int_{-\pi}^{\pi} \tilde{f}(x)\cos(nx)\,dx = \int_{-\pi}^{0} f(-x)\cos(nx)\,dx + \int_{0}^{\pi} f(x)\cos(nx)\,dx = [y=-x]$$

$$= 2\int_0^{\pi} f(x)\cos(nx)\,dx$$

$$f = \frac{a_0}{2} + \sum_{n\geq 1} a_n \cos(nx) \quad \text{en} \quad L^2(0,\pi).$$

$$a_n := \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)\,dx \qquad n\geq 0$$

Serie de Fourier en cosenos.

Serie de Fourier y derivadas

$f \in \mathcal{C}^1[-\pi,\pi]$, $f(\pi)=f(-\pi)$.

$$\begin{cases} f = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \operatorname{sen}(nx) \\ f' = -\sum_{n=1}^{\infty} n a_n \operatorname{sen}(nx) + \sum_{n=1}^{\infty} n b_n \cos(nx) \end{cases}$$

$$\begin{cases} f = \sum_{n\in\mathbb{Z}} \hat{f}_n e^{inx} \\ f' = \sum_{n\in\mathbb{Z}} in \hat{f}_n e^{inx} \end{cases}$$

$$\hat{f}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$$

$$\widehat{f'}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)\, e^{-inx}\, dx = \left[\begin{array}{l} f'(x)\,dx = dv, \quad v = f(x) \\ u = e^{-inx} \\ du = -in\, e^{-inx}\, dx \end{array}\right]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, in\, e^{-inx}\, dx + \cancelto{0}{\frac{1}{2\pi} f(x)\, e^{-inx}\Big|_{x=-\pi}^{x=\pi}}$$

$$= in\, \hat{f}_n .$$

Teo. $f \in \mathcal{C}^1[-\pi,\pi]$, $f(-\pi) = f(\pi)$

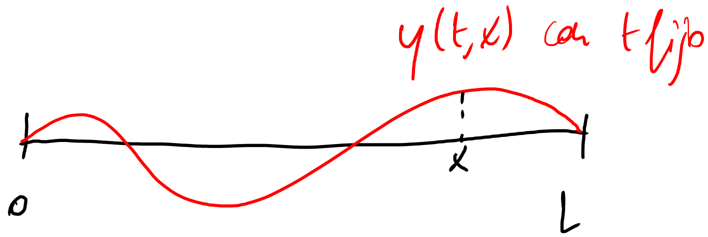
$\widehat{f'}_n = in\, \hat{f}_n$, y la serie de Fourier se puede derivar término a término.

Consecuencia: si $f \in \mathcal{C}^1[-\pi,\pi]$

$$\Rightarrow \sum_{n\in\mathbb{Z}} n^2 |\hat{f}_n|^2 < \infty .$$

Ecuación de ondas

$$\left[\partial_t^2 u = \partial_x^2 u \right] \qquad \partial_{tt} u = \partial_{xx} u \qquad u_{tt} = u_{xx}$$



$t \equiv$ tiempo

$x \equiv$ posición en $[0,L]$

$y(t,x) \equiv$ desplazamiento de la cuerda sobre x, en tiempo t.

Quiero escribir la energía cinética y la energía potencial (elástica) en tiempo t.

E. cinética $\equiv T = \displaystyle\int_0^L \frac{1}{2} m \left(\partial_t y(t,x)\right)^2 dx$

densidad de masa de la cuerda.

E. elástica $\equiv U = \displaystyle\int_0^L \frac{1}{2} K \left(\sqrt{1+(\partial_x y(t,x))^2}\right)^2 dx$

$$= \int_0^L \frac{1}{2} K \left(1+(\partial_x y(t,x))^2\right) dx$$

Salvo constante aditiva, puedo tomar

$$U = \int_0^L \frac{1}{2} K \left(\partial_x y(t,x)\right)^2 dx$$

27 abril

* Cambio de escala

y solución de $\partial_t^2 y = \partial_x^2 y$, $\lambda, \mu > 0$.

$z(t,x) = y(\lambda t, \mu x)$

$$\partial_t^2 z(t,x) = \lambda^2 \partial_t^2 y(\lambda t, \mu x)$$
$$\partial_x^2 z(t,x) = \mu^2 \partial_x^2 y(\lambda t, \mu x)$$

$$\frac{1}{\lambda^2} \partial_t^2 z = \frac{1}{\mu^2} \partial_x^2 z$$

$$\partial_t^2 z = \frac{\lambda^2}{\mu^2} \partial_x^2 z$$

Otra forma de escribirlo:

$y = y(t,x)$ solución de $\partial_t^2 y = \partial_x^2 y$

$t = \lambda \tau$, $x = \mu w$

$$\partial_\tau y = \partial_t y \frac{dt}{d\tau} = \lambda \partial_t y$$
$$\partial_\tau^2 y = \cdots = \lambda^2 \partial_t^2 y$$

$$\partial_w y = \partial_x y \frac{dx}{dw} = \partial_x y \cdot \mu$$
$$\partial_w^2 y = \mu^2 \cdot \partial_x^2 y$$

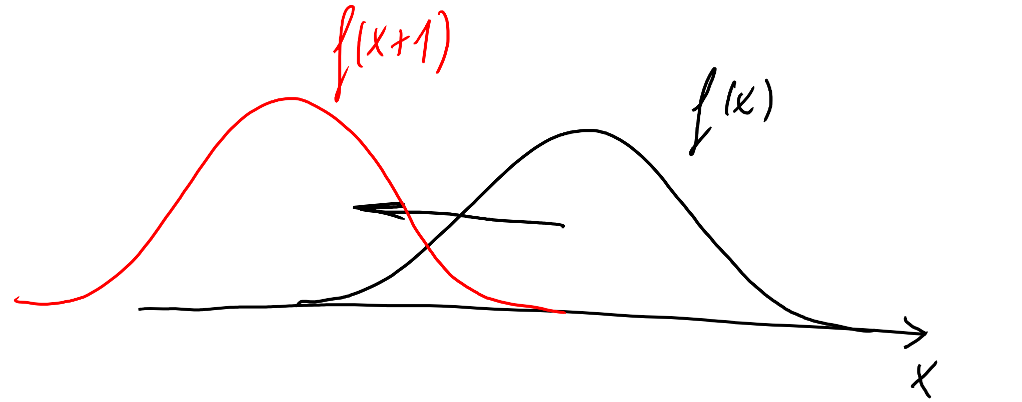
* Solucions particulars:

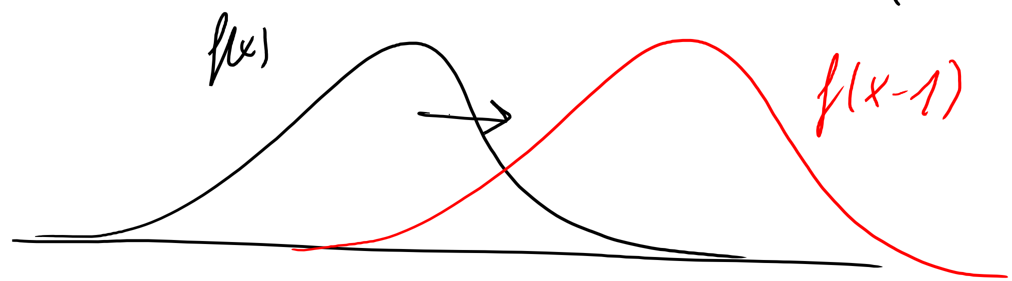
$$\partial_t^2 y = \partial_x^2 y$$

* Constants

* $y(t,x) = f(t+x)$

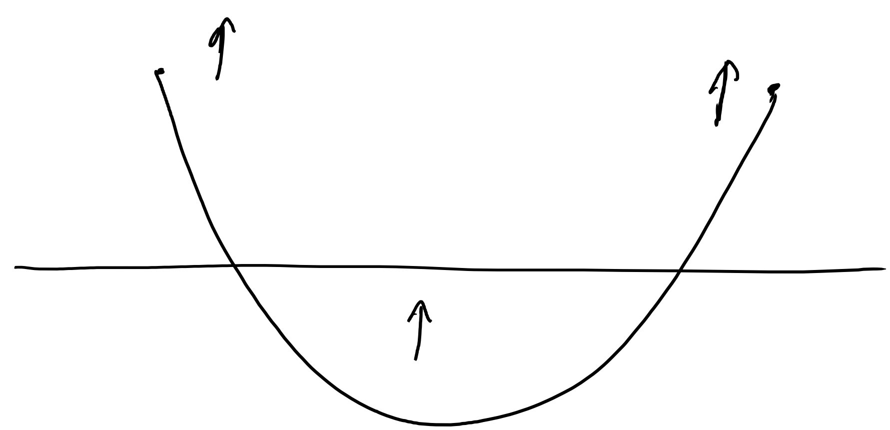
* $y(t,x) = g(x-t)$





* $y(t,x) = A + Bt + Cx$

* $y(t,x) = t^2 + x^2$



* $y(t,x) = f(t^2+x^2)$

$$\partial_t y = 2t\, f'(t^2+x^2)$$

$$\partial_t^2 y = 2\, f'(t^2+x^2) + 4t^2 f''(t^2+x^2)$$

Soluciones en $\mathbb{R}$

$$\left.\begin{array}{ll} \partial_t^2 y = \partial_x^2 y & \text{en } (t,x) \in \mathbb{R}\times\mathbb{R} \\ y(0,x) = f(x), & x \in \mathbb{R} \\ \partial_t y(0,x) = g(x), & x \in \mathbb{R} \end{array}\right\}$$

$$\partial_t^2 - \partial_x^2 = (\partial_t - \partial_x)(\partial_t + \partial_x)$$
$$= (\partial_t + \partial_x)(\partial_t - \partial_x).$$

Soluciones particulares:

$$\partial_t y + \partial_x y = 0, \qquad \partial_t y - \partial_x y = 0.$$

$$\partial_t y + \partial_x y = 0 \implies y(t,x) = F(x+t)$$

$$\partial_t y - \partial_x y = 0 \implies y(t,x) = G(x-t)$$

Buscamos una solución del tipo

$$y(t,x) = F(x+t) + G(x-t), \qquad \partial_t y = F'(x+t) - G'(x-t)$$

$$\left.\begin{array}{l} y(0,x) = F(x) + G(x) = f(x) \\ \partial_t y(0,x) = F'(x) - G'(x) = g(x) \end{array}\right\}$$

Elijo H una primitiva de f de forma que:

$$\left.\begin{array}{l} F(x)-G(x)=H(x) \\ F(x)+G(x)=f(x) \end{array}\right\} \qquad \begin{array}{l} F(x)=\frac{1}{2}(H(x)+f(x)) \\ G(x)=\frac{1}{2}(f(x)-H(x)) \end{array}$$

$$y(t,x)=F(t+x)+G(x-t)=$$

$$=\frac{1}{2}(H(t+x)+f(t+x))+\frac{1}{2}(f(x-t)-H(x-t))$$

$$=\frac{1}{2}(f(t+x)+f(x-t))+\frac{1}{2}(H(t+x)-H(x-t))$$

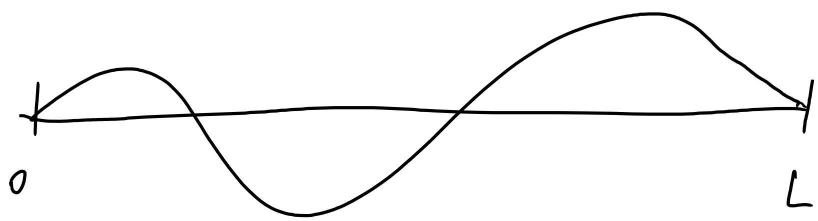
$$\circledast \quad y(t,x)=\frac{1}{2}(f(t+x)+f(x-t))+\frac{1}{2}\int_{x-t}^{x+t} f(s)\,ds$$

Fórmula de d'Alembert

Teo. $f \in \mathcal{C}^2(\mathbb{R})$, $g \in \mathcal{C}^1(\mathbb{R})$.

$\Rightarrow$ ⊛ es solución del PVI. para la ecuación de ondas en $\mathbb{R}$.

Ecuación de ondas en $[0,L]$



$$\partial_t^2 y = \partial_x^2 y \quad \text{en} \quad \mathbb{R}\times(0,L)$$

$$\left.\begin{aligned} y(0,x) &= f(x) \\ \partial_t y(0,x) &= g(x) \end{aligned}\right\} \quad x \in [0,L]$$

$$y(t,0) = y(t,L) = 0 \quad , \quad t \in \mathbb{R}.$$

Método de variables separadas

Busco soluciones particulares del tipo

$$y(t,x) = \varphi(t)\,\psi(x)$$

$$y(t,0)=0 \ \forall t \Rightarrow \varphi(t)\psi(0)=0 \ \forall t \xRightarrow{y \text{ es solución no trivial}} \psi(0)=0$$

$$y(t,L)=0 \ \forall t \Rightarrow \varphi(t)\psi(L)=0 \ \forall t \Rightarrow \psi(L)=0.$$

28 abril

$$y(t,x) = \varphi(t)\,\psi(x)$$

$$\left.\begin{aligned} \partial_t^2 y(t,x) &= \varphi''(t)\,\psi(x) \\ \partial_x^2 y(t,x) &= \varphi(t)\,\psi''(x) \end{aligned}\right\}$$

$$\varphi''(t)\,\psi(x) = \varphi(t)\,\psi''(x)$$

$$\underbrace{\frac{\varphi''(t)}{\varphi(t)}} = \underbrace{\frac{\psi''(x)}{\psi(x)}}$$

const. $= -\lambda$

$$\left.\begin{aligned} \psi''(x) &= -\lambda\,\psi(x) \\ \psi(0) &= \psi(L) = 0 \end{aligned}\right\} \text{ S-L.}$$

$$\varphi''(t) = -\lambda\,\varphi(t)$$

valores propios: $\lambda = \dfrac{n^2\pi^2}{L^2}$, $n \geq 1$ entero.

funciones propias:

$$\psi(x) = A \operatorname{sen}\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow \varphi(t) = B \cdot \cos\left(\frac{n\pi t}{L}\right) + C \operatorname{sen}\left(\frac{n\pi t}{L}\right)$$

$$y_n(t,x) := \cos\left(\frac{n\pi t}{L}\right) \operatorname{sen}\left(\frac{n\pi x}{L}\right)$$

$$z_n(t,x) := \operatorname{sen}\left(\frac{n\pi t}{L}\right) \operatorname{sen}\left(\frac{n\pi x}{L}\right)$$

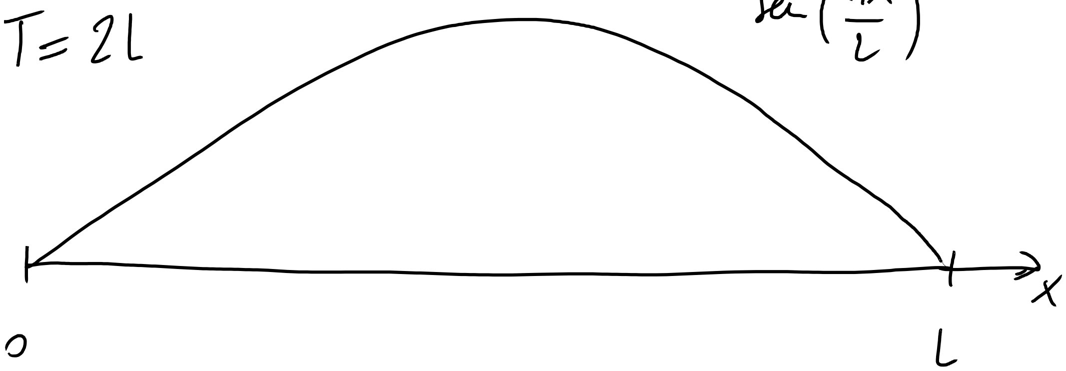
(por ejemplo).

modos de oscilación de la cuerda.

$\operatorname{sen}\left(\frac{\pi x}{L}\right)$ $(n=1)$

periodo: $\frac{\pi T}{L} = 2\pi$, $T = 2L$

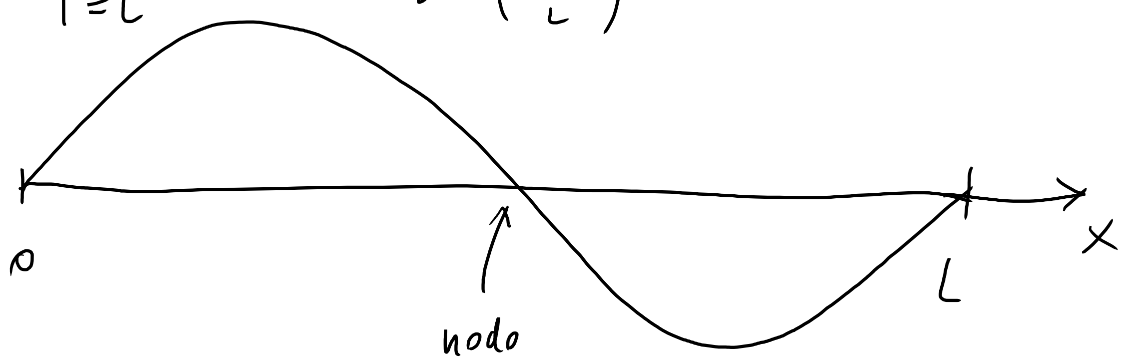
frecuencia: $\frac{1}{2L}$



$\operatorname{sen}\left(\frac{2\pi x}{L}\right)$

periodo: $\frac{2\pi T}{L} = 2\pi$, $T = L$

frecuencia: $\frac{1}{L}$



en general:

periodo: $\frac{2L}{n}$

frecuencia: $\frac{n}{2L}$

Problema de valores iniciales

$$
\left.\begin{aligned}
&\partial_t^2 y = \partial_x^2 y \qquad t \in \mathbb{R},\ x \in (0,L) \\
&y(t,0) = y(t,L) = 0, \qquad t \in \mathbb{R} \\
&y(0,x) = f(x), \qquad x \in [0,L] \\
&\partial_t y(0,x) = g(x), \qquad x \in [0,L]
\end{aligned}\right\}
$$

Intento: $$y(t,x) = \sum_{n=1}^{\infty} A_n\, y_n(t,x) + \sum_{n=1}^{\infty} B_n\, z_n(t,x)$$

$$y_n(t,x) := \cos\left(\frac{n\pi t}{L}\right) \operatorname{sen}\left(\frac{n\pi x}{L}\right)$$

$$z_n(t,x) := \operatorname{sen}\left(\frac{n\pi t}{L}\right) \operatorname{sen}\left(\frac{n\pi x}{L}\right)$$

$$\partial_t y_n(t,x) = -\frac{n\pi}{L} \operatorname{sen}\left(\frac{n\pi t}{L}\right) \operatorname{sen}\left(\frac{n\pi x}{L}\right)$$

$$\partial_t z_n(t,x) = \frac{n\pi}{L} \cos\left(\frac{n\pi t}{L}\right) \operatorname{sen}\left(\frac{n\pi x}{2}\right)$$

$$y(0,x) = \sum_{n=1}^{\infty} A_n \operatorname{sen}\left(\frac{n\pi x}{L}\right)$$

Elijo A_n = coef. de la serie de Fourier en senos de f.

$$\partial_t y(0,x) = \sum_{n=1}^{\infty} B_n \frac{n\pi}{L} \operatorname{sen}\left(\frac{n\pi x}{L}\right)$$

Si tenemos $\displaystyle\sum_{n=1}^{\infty} b_n \operatorname{sen}\left(\frac{n\pi x}{L}\right) = g(x)$

elijo $B_n := \dfrac{L}{n\pi} \cdot b_n$.

<u>Teo.</u> $f \in \mathcal{C}^3[0,L]$, $g \in \mathcal{C}^2[0,L]$. $f(0) = f(L) = g(0) = g(L) = 0$.

$f''(0) = f''(L) = 0$.

$a_n, b_n \equiv$ coef. de la serie de Fourier de f, g en senos.

$A_n := a_n$, $B_n := \dfrac{L}{n\pi} b_n$

Entonces la serie

$$\sum_{n=1}^{\infty} A_n y_n + \sum_{n=1}^{\infty} B_n z_n$$

converge a una función $\mathcal{C}^2$ en $\mathbb{R} \times [0,L]$.

que es solución del PVI. para la ecuación de ondas en $[0,L]$.

3 de mayo

Teo. $f \in \mathcal{C}^3[0,L]$, $g \in \mathcal{C}^2[0,L]$. $f(0) = f(L) = g(0) = g(L) = 0$.

$f''(0) = f''(L) = 0$.

$a_n, b_n \equiv$ coef. de la serie de Fourier de f, g en senos.

$$A_n := a_n, \quad B_n := \frac{L}{n\pi} b_n$$

Entonces la serie

$$\sum_{n=1}^{\infty} A_n y_n + \sum_{n=1}^{\infty} B_n z_n$$

converge a una función $\mathcal{C}^2$ en $\mathbb{R} \times [0,L]$.

que es solución del PVI. para la ecuación de ondas en $[0,L]$.

Criterio Weierstrass

$f_n : [a,b] \to \mathbb{R}$ $(n \geq 1)$ $|f_n(x)| \leq a_n$ $\forall\, n \geq 1,\ \forall\, x \in [a,b]$

con $\sum_{n=1}^{\infty} a_n < \infty$.

$\Rightarrow$ $\sum_{n\geq 1} f_n(x)$ converge absoluta- uniformemente.

$\left[\ \sum_{n=m}^{\infty} |f_n(x)| \to 0\right.$ cuando $m \to \infty$, uniformemente en $\left. x \in [a,b]\ \right]$.

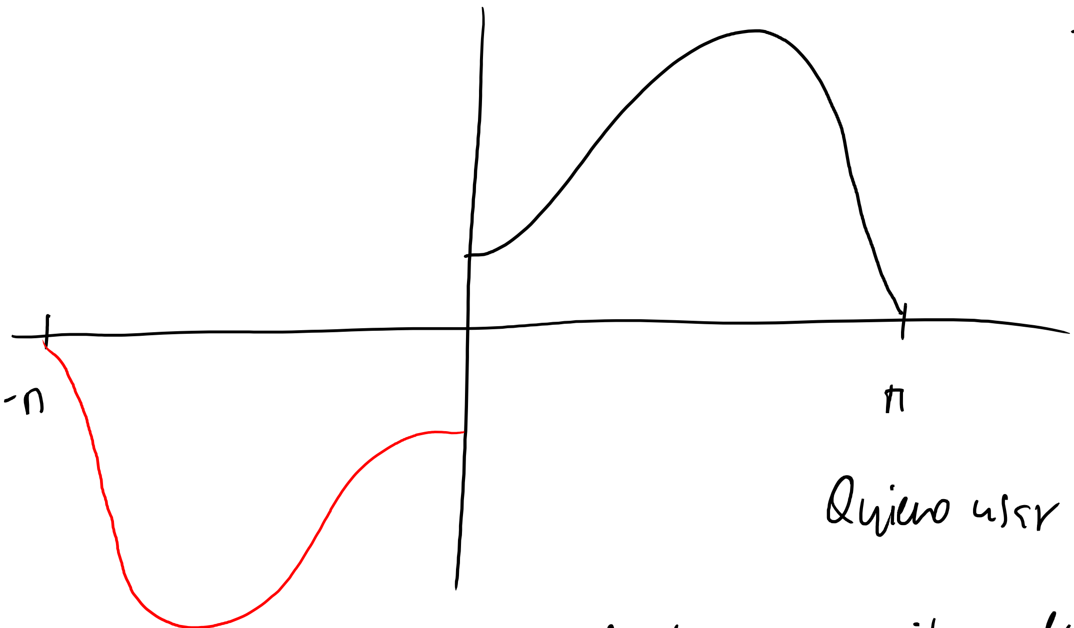
Decrecimiento / convergencia uniforme series de Fourier

Teo. $f : [-\pi,\pi] \to \mathbb{R}$, $f \in \mathcal{C}^k[-\pi,\pi]$ con $f^{(j)}(-\pi) = f^{(j)}(\pi)$, $j = 0,\dots,k$.

a_n, b_n coeficientes de Fourier de f, $k \geq 0$ entero.

$$\Rightarrow \quad \sum_{n=1}^{\infty} |a_n|^2 n^{2k} + \sum_{n=1}^{\infty} |b_n|^2 n^{2k} < \infty.$$

- para la serie de Fourier en senos:



$f : [0,\pi] \longrightarrow \mathbb{R}$

$$\tilde{f}(x) = \begin{cases} f(x), & x \in [0,\pi] \\ -f(-x), & x \in [-\pi, 0) \end{cases}$$

Quiero usar el teorema para $\tilde{f}$:

* $k=0$: necesito $f(0) = f(\pi) = 0$.

* $k=1$: necesito $f(0) = f(\pi) = 0$.

$$\tilde{f}'(x) = \begin{cases} f'(x) & x \in (0,\pi] \\ f'(-x) & x \in [-\pi, 0) \end{cases}$$

✓

* $k=2$, necesito $f(0) = f(\pi) = 0$,

$f''(0) = f''(\pi) = 0$

Teo. $f:[0,\pi]\to\mathbb{R}$, $f\in\mathcal{C}^k[0,\pi]$ con $f^{(j)}(0)=f^{(j)}(\pi)=0$ $\quad 0\le j\le k$, j par

b_n coeficientes de Fourier en senos de f, $k\ge 0$ entero.

$$\Rightarrow \quad \sum_{n=1}^{\infty}|b_n|^2 n^{2k}<\infty.$$

Convergencia uniforme:

Supongamos $\sum_{n=1}^{\infty}|b_n|^2 n^2<\infty$.

Serie de Fourier en senos de $f:[0,\pi]\to\mathbb{R}$:

$$\sum_{n=1}^{\infty} b_n \operatorname{sen}(nx) \quad , \quad \text{¿converge uniformemente?}$$

$$\sum_{n=1}^{\infty}|b_n| = \sum_{n=1}^{\infty}|b_n|n\cdot\frac{1}{n} \le \underbrace{\left(\sum_{n=1}^{\infty}|b_n|^2n^2\right)^{1/2}}_{\substack{<\infty \text{ si} \\ \text{se cumplen las} \\ \text{condiciones de teo. con } k=1.}} \underbrace{\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{1/2}}_{<\infty.}$$

Teo. $f:[0,\pi]\to\mathbb{R}$, $f\in \mathcal{C}^1[0,\pi]$ con $f(0)=f(\pi)=0$,
b_n coeficientes de Fourier en senos de f,

$$\Rightarrow \quad \sum_{n\geq 1}^{\infty} |b_n| < \infty \,,$$ y la serie de Fourier converge uniformemente

Derivación series de funciones

$$f_n:[a,b]\to\mathbb{R}, \quad n\geq 1.$$

$$\left(\sum_{n\geq 1} f_n(x)\right)' = \sum_{n\geq 1} f_n'(x) \quad ?$$

Teo. $f_n:[a,b]\to\mathbb{R}$, $n\geq 1$.

* $f_n\in\mathcal{C}^1[a,b]$, $\forall\, n\geq 1$.
* $\sum_{n\geq 1} f_n'(x)$ converge uniformemente.
* $\sum_{n\geq 1} f_n(x_0)$ converge para algún $x_0\in[a,b]$.

Entonces: $\sum_{n\geq 1} f_n(x)$ c. unif., y además

$$\left(\sum_{n\geq 1} f_n(x)\right)' = \sum_{n\geq 1} f_n'(x).$$

Dem. del teorema del principio: $\left(\sum |B_n| < \infty\right)$

* la serie converge, porque $\sum_{n=1}^{\infty} |A_n| < \infty$, $\sum \frac{1}{n} |b_n| < \infty$.

* las series de las derivadas convergen?

$$y(t,x) = \sum_{n=1}^{\infty} A_n y_n(t,x) + \sum_{n=1}^{\infty} B_n z_n(t,x)$$

$$y_n(t,x) = \cos\left(\frac{\pi n t}{L}\right) \operatorname{sen}\left(\frac{\pi n x}{L}\right)$$

$$z_n(t,x) = \operatorname{sen}\left(\frac{\pi n t}{L}\right) \operatorname{sen}\left(\frac{\pi n x}{L}\right)$$

$$\partial_t y_n(t,x) = -\frac{\pi n}{L} z_n(t,x)$$

$$\partial_t z_n(t,x) = \frac{\pi n}{L} y_n(t,x)$$

$$\partial_t y_n(t,x) = -\sum_{n=1}^{\infty} \frac{\pi n}{L} A_n z_n(t,x) + \sum_{n=1}^{\infty} \frac{\pi n}{L} B_n y_n(t,x) \quad ?$$

$$\sum_{n=1}^{\infty} n |A_n| < \infty \; ? \qquad \sum n |B_n| < \infty \; ?$$

$$\sum n |A_n| = \sum n |a_n| \leq \underbrace{\left(\sum n^4 |a_n|^2\right)^{1/2}}_{<\infty} \underbrace{\left(\sum \frac{1}{n^2}\right)^{1/2}}_{<\infty} \quad \checkmark$$

si se cumplen condiciones con $K = 2$.

$$\sum n |B_n| = \frac{L}{\pi} \sum |b_n| < \infty$$ por resultado con $K = 1$.

Con las otras derivadas $\longrightarrow$ análogo.

Ec. Laplace / Poisson

$(*)$ $\Delta u = 0$ en Ω $\qquad$ $\Omega \subseteq \mathbb{R}^d$ abierto.

Ec. Laplace. $\qquad$ $u : \Omega \to \mathbb{R}$, $u = u(x)$

$x = (x_1, \dots, x_d) \in \Omega$.

$$\Delta u(x) := \partial^2_{x_1} u(x) + \partial^2_{x_2} u(x) + \dots + \partial^2_{x_d} u(x).$$

Las funciones que cumplen $(*)$ se llaman armónicas. (en Ω).

Forma variacional:

$$F(u) := \int_\Omega |\nabla u(x)|^2 dx = \int_\Omega \sum_{n=1}^{d} |\partial_{x_n} u(x)|^2 dx$$

dada.

$D = \{ u \in \mathcal{C}^2(\bar\Omega) \}$ o bien: $D = \{ u \in \mathcal{C}^2(\bar\Omega) / u(x) = f(x) \ \forall x \in \partial\Omega \}$.

Euler-Lag: $\quad \partial_u F = \operatorname{div}(\nabla_z F)$. $\qquad F = F(x, \underbrace{u}_{u(x)}, \underbrace{z}_{\nabla u(x)})$

$$F(x,u,z) = |z|^2 = z_1^2 + \dots + z_d^2 .$$

$$\partial_u F = 0, \qquad \nabla_z F = (2z_1, \dots, 2z_d) = 2z$$

$$0 = \operatorname{div}(2 \nabla u(x)) = 2 \Delta u(x).$$

(L) $\left.\begin{array}{lll} \Delta u = 0 & \text{en} & \Omega \\ u = f & \text{en} & \partial\Omega \end{array}\right\}$ problema de contorno ec. de Laplace

$\left.\begin{array}{lll} \Delta u = f & \text{en} & \Omega \\ u = 0 & \text{en} & \partial\Omega \end{array}\right\}$ problema de contorno ec. de Poisson.

__Def.__ $\Omega \subseteq \mathbb{R}^d$ abierto, $f : \partial\Omega \to \mathbb{R}$.

$u : \overline{\Omega} \to \mathbb{R}$ es sol. de (L) si $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$

y se cumple (L).

★ __Soluciones particulares:__

★ $\Omega = \mathbb{R}$. $\quad u : \mathbb{R} \to \mathbb{R}$, $u''(x) = 0 \quad \forall x \in \mathbb{R}$.

$u(x) = Ax + B$.

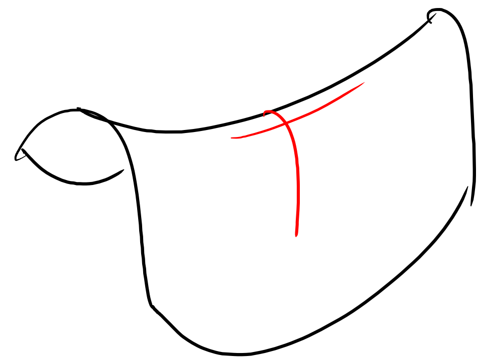
★ $\Omega = (a,b)$, $\left.\begin{array}{l} u(x) = Ax + B \\ u(a) = f(a), \ u(b) = f(b) \end{array}\right\}$ hay solución única.

* $\Omega = \mathbb{R}^2$: $u(x,y) = A + Bx + Cy$.

$u(x,y) = \operatorname{sen} x \cdot e^y$

$u(x,y) = x^2 - y^2$

$u(x,y) = xy$



$F: \mathbb{C} \to \mathbb{C}$ holomorfa

$[\, F: \mathbb{R}^2 \to \mathbb{C} \,]$. $\operatorname{Re} F: \mathbb{R}^2 \to \mathbb{R}$, $\operatorname{Im} F: \mathbb{R}^2 \to \mathbb{R}$.

F holomorfa $\Rightarrow$ $\operatorname{Re} F$, $\operatorname{Im} F$ armónicas.

Ejemplo: $F(z) = e^z = e^{\operatorname{Re} z}(\cos \operatorname{Im} z + i \operatorname{sen} \operatorname{Im} z)$.

$z = x + iy$, $F(z) = e^x(\cos y + i \operatorname{sen} y)$.

$\operatorname{Re} F = e^x \cos y$

$\operatorname{Im} F = e^x \operatorname{sen} y$

Propiedades

① Linealidad: u, v armónicas $\Rightarrow \lambda u + \mu v$ también.

② Invariancias: traslaciones: u armónica $\Rightarrow v(x) := u(x + x_0)$ también.
$(x_0 \in \mathbb{R}^d)$.

movimientos rígidos (isometrías de $\mathbb{R}^d$).

$$\left.\begin{array}{l} R : \mathbb{R}^d \to \mathbb{R}^d \text{ m rígido} \\ u \text{ armónica} \end{array}\right\} \quad v(x) = u(Rx) \quad \text{también}$$

③ Cambio de escala: u armónica $\Rightarrow v(x) = u(\lambda x)$ también. (en $\mathbb{R}^d$)

En $\mathbb{R}^2$: $v(x,y) = u(\lambda x, \mu y) \qquad \lambda, \mu \neq 0.$

$$\left.\begin{array}{l} \partial_x^2 v(x,y) = \lambda^2 \partial_x^2 u(\lambda x, \mu y) \\ \partial_y^2 v(x,y) = \mu^2 \partial_y^2 u(\lambda x, \mu y) \end{array}\right\} \quad \frac{1}{\lambda^2}\partial_x^2 v + \frac{1}{\mu^2}\partial_y^2 v = 0.$$

Solución ec. Laplace en el cuadrado

$\Omega = (0,L)\times(0,L)$, $f : \partial\Omega \to \mathbb{R}$ $\qquad u = u(x,y)$.

$$\left.\begin{array}{ll} \partial_x^2 u + \partial_y^2 u = 0 & \text{en } \Omega \\ u = f & \text{en } \partial\Omega \end{array}\right\}$$

Variables separadas: busco soluciones del tipo

$$u(x,y) = \psi(x)\varphi(y)$$

$$\partial_x^2 u = \psi''(x)\varphi(y) \quad , \quad \partial_y^2 u = \psi(x)\varphi''(y).$$

$$\psi''(x)\varphi(y) + \psi(x)\varphi''(y) = 0.$$

$$\underbrace{\frac{\psi''(x)}{\psi(x)}}_{\text{const} = +\lambda} = -\underbrace{\frac{\varphi''(y)}{\varphi(y)}}_{\text{const} = +\lambda.}$$

$$\psi''(x) = +\lambda\psi(x)$$

$$\varphi''(y) = -\lambda\varphi(y)$$

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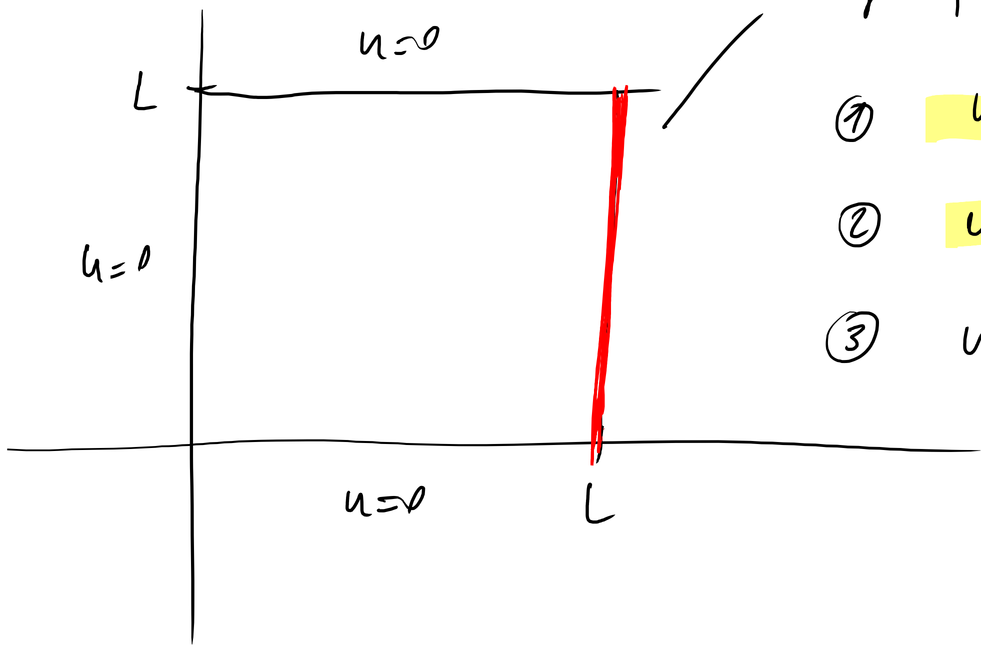

u=f aquí $\qquad f : [0,L] \longrightarrow \mathbb{R}.$

① $u(0,y) = 0 \qquad y \in [0,L]$

② $u(x,0) = u(x,L) = 0 \qquad x \in [0,L]$

③ $u(L,y) = f(y) \qquad y \in [0,L].$

Por ahora sólo usamos ①, ②:

$$u(x,y) = \psi(x)\varphi(y)$$

$0 = u(0,y) = \psi(0)\varphi(y)$. Si u es no trivial $\Rightarrow \psi(0) = 0$.

$0 = u(x,0) = \psi(x)\varphi(0) \Rightarrow \varphi(0) = 0.$

$0 = u(x,L) = \psi(x)\varphi(L) \Rightarrow \varphi(L) = 0.$

Así que:

$$\varphi''(x) = +\lambda \varphi(x), \qquad \varphi(0) = 0.$$

$$\left[\psi''(y) = -\lambda \psi(y), \qquad \psi(0) = \psi(L) = 0 \right] \text{ Sturm-Liouville.}$$

Valores propios: $\lambda = \dfrac{n^2\pi^2}{L^2}$, $\quad \psi(y) = A \operatorname{sen}\left(\dfrac{n\pi y}{L}\right)$.

$$\varphi(x) = B e^{\frac{n\pi x}{L}} + C e^{-\frac{n\pi x}{L}}$$

$$0 = \varphi(0) = B + C \Rightarrow B = -C$$

$$\varphi(x) = B\left(e^{\frac{n\pi x}{L}} - e^{-\frac{n\pi x}{L}}\right) = 2B \operatorname{senh}\left(\frac{n\pi x}{L}\right).$$

$$u_n(x,y) := \operatorname{sen}\left(\frac{n\pi y}{L}\right) \operatorname{senh}\left(\frac{n\pi x}{L}\right)$$

Intentemos el problema completo:

$$\Delta u = 0 \quad \text{en} \quad \Omega \qquad \checkmark$$

① $u(0,y) = 0 \qquad y \in [0,L]$ ✓

② $u(x,0) = u(x,L) = 0 \qquad x \in [0,L]$ ✓

③ $u(L,y) = f(y) \qquad y \in [0,L]$. ←

Probamos: $\displaystyle u(x,y) := \sum_{n=1}^{\infty} B_n\, u_n(x,y)$.

$$u_n(L,y) = \operatorname{senh}(n\pi) \cdot \operatorname{sen} \frac{n\pi y}{L}$$

$$u(L,y) = \sum_{n=1}^{\infty} \operatorname{senh}(n\pi) \cdot \operatorname{sen} \frac{n\pi y}{L} \cdot B_n$$

Si $b_n \equiv$ coef. de la serie de Fourier en senos de f

$$B_n := \frac{1}{\operatorname{senh}(n\pi)}\, b_n$$

Teo. $f \in \mathcal{C}^3[0,L]$, $\quad f(0) = f(L) = f''(0) = f''(L) = 0.$

Si $b_n \equiv$ coef. de la serie de Fourier en senos de f

$$B_n := \frac{1}{\sinh(n\pi)} b_n$$

$$u(x,y) := \sum_{n=1}^{\infty} B_n \, u_n(x,y).$$

Esta serie converge a una función $\mathcal{C}^2([0,L]\times[0,L])$ solución de la ec. de Laplace:

$$\left.\begin{array}{lll} & \Delta u = 0 \quad \text{en } \Omega & \checkmark \\ ① & u(0,y) = 0 \qquad y \in [0,L] & \checkmark \\ ② & u(x,0) = u(x,L) = 0 \qquad x \in [0,L] & \checkmark \\ ③ & u(L,y) = f(y) \qquad y \in [0,L]. & \leftarrow \end{array}\right\}$$

<u>Dem.</u> (Esquema): condiciones sobre $f \Rightarrow \sum_{n=1}^{\infty} |b_n|^2 n^6 < \infty$.

* Criterio de Weierstrass a la serie de u:

$$u(x,y) := \sum_{n=1}^{\infty} B_n\, u_n(x,y).$$

$$|u_n(x,y)| \leq \operatorname{senh}(n\pi) \qquad B_n := \frac{1}{\operatorname{senh}(n\pi)}\, b_n.$$

$$|B_n\, u_n(x,y)| \leq \frac{|b_n|}{\operatorname{senh}(n\pi)}\operatorname{senh}(n\pi) = |b_n|$$

$$\sum |b_n| \leq \left(\sum |b_n|^2 n^2\right)^{1/2} \left(\sum \frac{1}{n^2}\right)^{1/2} < \infty.$$

$$\Rightarrow \sum_{n\geq 1} B_n\, u_n(x,y) \text{ converge unif.}$$

* Serie correspondiente a $\partial_x u$:

$$\partial_x u \overset{?}{=} \sum_{n=1}^{\infty} B_n \operatorname{sen}\left(\frac{n\pi y}{L}\right)\cosh\left(\frac{n\pi x}{L}\right)\cdot\frac{n\pi}{L}$$

Weierstrass:

$$\left| B_n \operatorname{sen}\left(\frac{n\pi y}{L}\right)\cosh\left(\frac{n\pi x}{L}\right)\cdot\frac{n\pi}{L}\right| \leq |b_n| \overbrace{\frac{1}{\operatorname{senh}(n\pi)}\cosh(n\pi)}^{\leq \text{ const.}}\, \frac{n\pi}{L}.$$

$$\frac{1}{\operatorname{senh}(n\pi)}\cosh(n\pi) = \frac{e^{n\pi}+e^{-n\pi}}{e^{n\pi}-e^{-n\pi}} \leq \frac{2e^{n\pi}}{e^{n\pi}-e^{-n\pi}} = \frac{2}{1-e^{-2n\pi}}$$

$$\leq \frac{2}{1-e^{-2\pi}} = \text{const.}$$

$$\sum |b_n|\, n \leq \left(\sum |b_n|^2 n^4\right)^{1/2}\left(\sum \frac{1}{n^2}\right)^{1/2} < \infty$$

$\partial_x^2 u,\ \partial_y u,\ \partial_y^2 u \longrightarrow$ parecido. ✓

Resumen:

* Ec. Laplace en el cuadrado.
* En otros dominios $\to$ no sabemos.

Ec. Poisson:

$$\left.\begin{array}{ll} \Delta u = f & \text{en } \Omega \\ u = 0 & \text{en } \partial\Omega \end{array}\right\} \text{ Poisson}$$

Si podemos resolver Poisson $\Rightarrow$ podemos resolver Laplace

Dem. Quiero resolver

$$\left.\begin{array}{ll} \Delta u = 0 & \text{en } \Omega \\ u = f & \text{en } \partial\Omega \end{array}\right\} \text{(L)}$$

Elijo $v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ con $v = f$ en $\partial\Omega$.

(tiene que existir si (L) tiene solución).

$w := u - v$

$$\left.\begin{aligned} \Delta w &= \Delta u - \Delta v = -\Delta v =: f \\ w &= 0 \quad \text{in } \partial\Omega \end{aligned}\right\} \text{ Poisson. } \checkmark$$

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$$\left.\begin{array}{ll}\Delta u = f & \text{en } \Omega \\ u = 0 & \text{en } \partial\Omega\end{array}\right\} \text{Poisson} \qquad \left.\begin{array}{ll}\Delta u = 0 & \text{en } \Omega \\ u = f & \text{en } \partial\Omega\end{array}\right\} \text{Laplace}$$

Si sé resolver Laplace $\Rightarrow$ puedo resolver Poisson?

Quiero resolver $\Delta u = f$ en Ω, $u = 0$ en $\partial\Omega$.

* Necesito v con $\Delta v = f$ (que valga cualquier cosa en $\partial\Omega$).

$w := u - v$, $\left.\begin{array}{ll}\Delta w = 0 & \text{en } \Omega \\ w = -v & \text{en } \partial\Omega\end{array}\right\}$ Laplace.

Queremos resolver esto.

Solución fundamental de la ec. de Laplace

Intentamos buscar soluciones de $\Delta u = 0$ en $\mathbb{R}^d$ radialmente simétricas:

$$u(x) = \varphi(|x|) \quad , \quad \varphi : [0,\infty) \to \mathbb{R}.$$

$$\partial_{x_i} u(x) = \varphi'(|x|) \frac{\partial |x|}{\partial x_i} = \varphi'(|x|) \frac{x_i}{|x|} \quad \Bigg\| \quad \frac{\partial |x|}{\partial x_i} = \frac{\partial \sqrt{x_1^2 + \dots + x_d^2}}{\partial x_i} = \frac{x_i}{|x|}$$

$$\left[\partial_{x_i} u = \varphi' \cdot \frac{x_i}{|x|} \right]$$

$$\partial_{x_i}^2 u(x) = \varphi''(|x|) \frac{x_i^2}{|x|^2} + \varphi'(|x|) \frac{1}{|x|} - \varphi'(|x|) \frac{x_i}{|x|^2} \frac{x_i}{|x|} =$$

$$= \varphi'' \frac{x_i^2}{|x|^2} + \varphi' \cdot \frac{1}{|x|} - \varphi' \cdot \frac{x_i^2}{|x|^3}.$$

$$0 = \Delta u = \sum_{i=1}^{d} \partial_{x_i}^2 u = \varphi'' + \frac{d}{|x|} \varphi' - \frac{1}{|x|} \varphi'$$

$$= \varphi'' + \frac{(d-1)}{|x|} \varphi'.$$

$r \equiv |x|$. $\qquad 0 = \psi'' + \frac{(d-1)}{r}\psi' \qquad (r>0)$.

$$0 = \psi''(r) + \frac{d-1}{r}\psi'(r)$$

$$\varphi := \psi', \qquad 0 = \varphi' + \frac{d-1}{r}\varphi \qquad \left(e^{\log r}\right)^{-(d-1)}$$

$$\varphi' = -\frac{d-1}{r}\varphi \Rightarrow \varphi(r) = C\, e^{-(d-1)\log r}$$

$$= C\, r^{-(d-1)}$$

$$\psi' = \varphi \Rightarrow \psi(r) = A - \frac{C}{d-2} r^{-(d-2)} \qquad (d \neq 2).$$

$$\psi(r) = A + C \log r \qquad (d=2).$$

$$\left.\begin{aligned} u(x) &= A + B|x|^{-(d-2)} \qquad (d \neq 2) \\ u(x) &= A + C\log|x| \qquad (d=2) \end{aligned}\right\}$$

Def_ $\Psi(x) := \frac{1}{(d-2)\omega_d} |x|^{-(d-2)}$, $x \in \mathbb{R}^d \setminus \{0\}$, $d \neq 2$.

$\Psi(x) := \frac{1}{2\pi} \log |x|$, $x \in \mathbb{R}^d \setminus \{0\}$, $d = 2$.

$\omega_d \equiv$ superficie de la esfera unidad en $\mathbb{R}^d$.

$\omega_2 = 2\pi$, $\omega_3 = 4\pi$.

$\frac{1}{(d-2)\omega_d} \equiv K_d$.

$\Psi \equiv$ solución fundamental ec. de Laplace.

$\int \delta_0(x) f(x)\, dx = f(0).$

$\int \delta_0(x-y) f(y)\, dy = f(x)$

"Intuición": $\Delta \Psi = \delta_0$.

$x \longmapsto \Psi(x - x_0)$ $\Delta_x \Psi(x - x_0) = \delta_{x_0}$

$u(x) = \int_{\mathbb{R}^d} \Psi(x-y) f(y)\, dy$, $\Delta u = \Delta \int_{\mathbb{R}^d} \Psi(x-y) f(y)\, dy =$

$$= \int_{\mathbb{R}^d} \Delta \Psi(x-y) f(y)\, dy = \int_{\mathbb{R}^d} \delta_x(y) f(y)\, dy = f(x)$$

Teo. $f:\mathbb{R}^d \to \mathbb{R}$, $\quad f \in \mathcal{C}^2_c(\mathbb{R}^d)$. $\quad d \geq 2$.

$$u(x) := \int_{\mathbb{R}^d} \Phi(x-y) f(y)\, dy \quad , \; x \in \mathbb{R}^d$$

sol. fundamental.

Entonces $u \in \mathcal{C}^2(\mathbb{R}^d)$, $\; -\Delta u(x) = f(x) \quad \forall\, x \in \mathbb{R}^d$.

Dem. $\displaystyle\int_{B(0,1)} |\Phi(x)|\, dx < \infty$? $\qquad \Phi(x) = K_d \cdot |x|^{2-d}$

Observación: $\displaystyle\int_{B(0,1)} |x|^{-\alpha}\, dx < \infty \iff \alpha < d$.

$\Phi(x-y) f(y)$ integrable $\Rightarrow$ u está bien definida.

Quiero calcular Δu:

$$\cancel{\text{"} \; \Delta \int_{\mathbb{R}^d} \Psi(x-y)\, u(y)\, dy = \int_{\mathbb{R}^d} \overbrace{\Delta \Psi(x-y)}^{=0}\, u(y)\, dy = 0. \; \text{"}}$$

$$\parallel$$

$$\Delta_x \int_{\mathbb{R}^d} u(x-y)\, \Psi(y)\, dy = \int_{\mathbb{R}^d} \Delta_x u(x-y)\, \Psi(y)\, dy = \int_{\mathbb{R}^d} \Delta_y (u(x-y))\, \Psi(y)\, dy.$$

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Teo. $f : \mathbb{R}^d \to \mathbb{R}$, $\quad f \in \mathcal{C}_c^2(\mathbb{R}^d)$. $\quad d \geq 2$.

$$u(x) := \int_{\mathbb{R}^d} \Psi(x-y) f(y)\, dy, \quad x \in \mathbb{R}^d$$

Ψ: sol. fundamental.

Entonces $u \in \mathcal{C}^2(\mathbb{R}^d)$, $\quad -\Delta u(x) = f(x) \quad \forall\, x \in \mathbb{R}^d$.

Dem.

$$u(x) = \int_{\mathbb{R}^d} f(x-y) \Psi(y)\, dy, \quad x \in \mathbb{R}^d.$$

$$\Delta_x u(x) = \int_{\mathbb{R}^d} \Delta_x f(x-y) \Psi(y)\, dy = \int_{\mathbb{R}^d} \Delta_y f(x-y) \Psi(y)\, dy =$$

$B_\varepsilon \equiv B(0,\varepsilon)$.

$$= \underbrace{\int_{\mathbb{R}^d \setminus B_\varepsilon} \Delta_y f(x-y) \Psi(y)\, dy}_{I_1} + \underbrace{\int_{B_\varepsilon} \Delta_y f(x-y) \Psi(y)\, dy}_{\to 0 \text{ cuando } \varepsilon \to 0.}$$

$$I_1 = -\underbrace{\int_{\mathbb{R}^d\setminus B_\varepsilon} \nabla_y f(x-y)\, \nabla\Psi(y)\, dy}_{I_2.} + \underbrace{\int_{\partial B_\varepsilon} \Psi(y)\, \nabla_y f(x-y)\, N(y)\, dy}_{\to 0 \text{ cuando } \varepsilon \to 0.}$$

$\uparrow$ normal exterior a $\mathbb{R}^d\setminus B_\varepsilon$.

$$I_2 = \underbrace{\int_{\mathbb{R}^d\setminus B_\varepsilon} f(x-y)\, \underbrace{\Delta\Psi(y)}_{=0}\, dy}_{=0} - \underbrace{\int_{\partial B_\varepsilon} f(x-y)\, \nabla\Psi(y)\, N(y)\, dy}$$

* Cálculo en $d \geq 3$: $\quad \Psi(y) = K_d\, |y|^{2-d} \qquad K_d = \dfrac{1}{\omega_d (d-2)}.$

$$\nabla\Psi(y) = K_d\, (2-d)\, |y|^{-d}\, y$$

$$N(y) = -\frac{y}{|y|}$$

$$-\int_{\partial B_\varepsilon} f(x-y)\,\nabla\Psi(y)\,N(y)\,dy = +\int_{\partial B_\varepsilon} f(x-y)\,K_d\,(2-d)\,|y|^{-d}\,y\cdot\frac{y}{|y|}\,dS(y) =$$

$$= K_d\,(2-d)\int_{\partial B_\varepsilon} f(x-y)\,\underbrace{|y|}_{=\varepsilon}{}^{1-d}\,dS(y) = K_d\,(2-d)\,\varepsilon^{1-d}\int_{\partial B_\varepsilon} f(x-y)\,dS(y)$$

f continue $\Rightarrow$ $$\frac{1}{\omega_d\,\varepsilon^{d-1}}\int_{\partial B_\varepsilon} f(x-y)\,dS(y) \xrightarrow{\varepsilon\to 0} f(x)$$

$$K_d\,(2-d)\,\varepsilon^{1-d}\int_{\partial B_\varepsilon} f(x-y)\,dS(y) = -\frac{1}{\omega_d\,\varepsilon^{d-1}}\int_{\partial B_\varepsilon} f(x-y)\,dS(y) \to -f(x). \quad \checkmark$$

$$\Rightarrow -\Delta u(x) = f(x)\ ,\ x\in\mathbb{R}^d.$$

Comentario: Si sé resolver la ec. de Laplace en un dominio:

$$\left.\begin{array}{ll} \Delta u = 0 & \text{en } \Omega \\ u = f & \text{en } \partial\Omega \end{array}\right\}$$

y quiero resolver la de Poisson,

$$\left.\begin{array}{ll} \Delta u = f & \text{en } \Omega \\ u = 0 & \text{en } \partial\Omega \end{array}\right\}$$

Si f se extiende a una función $f \in \mathcal{C}^2_c(\mathbb{R}^d)$, entonces:

$$v(x) := -\int_{\mathbb{R}^d} \Phi(x-y) f(y)\, dy$$

$w := u - v.$

$$\left.\begin{array}{ll} \Delta w = 0 & \text{en } \Omega \\ w = -v & \text{en } \partial\Omega \end{array}\right\} \text{ Laplace}$$

Interpretación física

$$\underset{m_1 \equiv m}{\bullet} \longrightarrow \qquad \longleftarrow \underset{m_2}{\bullet}$$

$$|\text{fuerza}| = \frac{G \cdot m_1 m_2}{(\text{dist})^2} = \frac{G m_1 m_2}{r^2}$$

potencial creado por m_1: $\quad -\dfrac{G m_1}{r}$

* potencial: $V(x) = -\dfrac{G m}{|x - x_0|}$ si m está en x_0.

(masa puntual)

* $\rho = \rho(x) \equiv$ densidad de masa en $x \in \mathbb{R}^3$.

$$V(x) = -\int_{\mathbb{R}^3} \frac{G \rho(y)\, dy}{|x-y|} = -4\pi G \int_{\mathbb{R}^3} \Phi(x-y) \rho(y)\, dy$$

$$\Phi(x) = \frac{1}{4\pi} \frac{1}{|x|} \qquad \Bigg\| \qquad \Delta V(x) = 4\pi G \rho(x), \quad x \in \mathbb{R}^3.$$

Ejemplo

Campo gravitatorio creado por esfera uniforme

$$\rho(x) = \begin{cases} D & \text{si } |x| \leq R \\ 0 & \text{si } x > R. \end{cases}$$

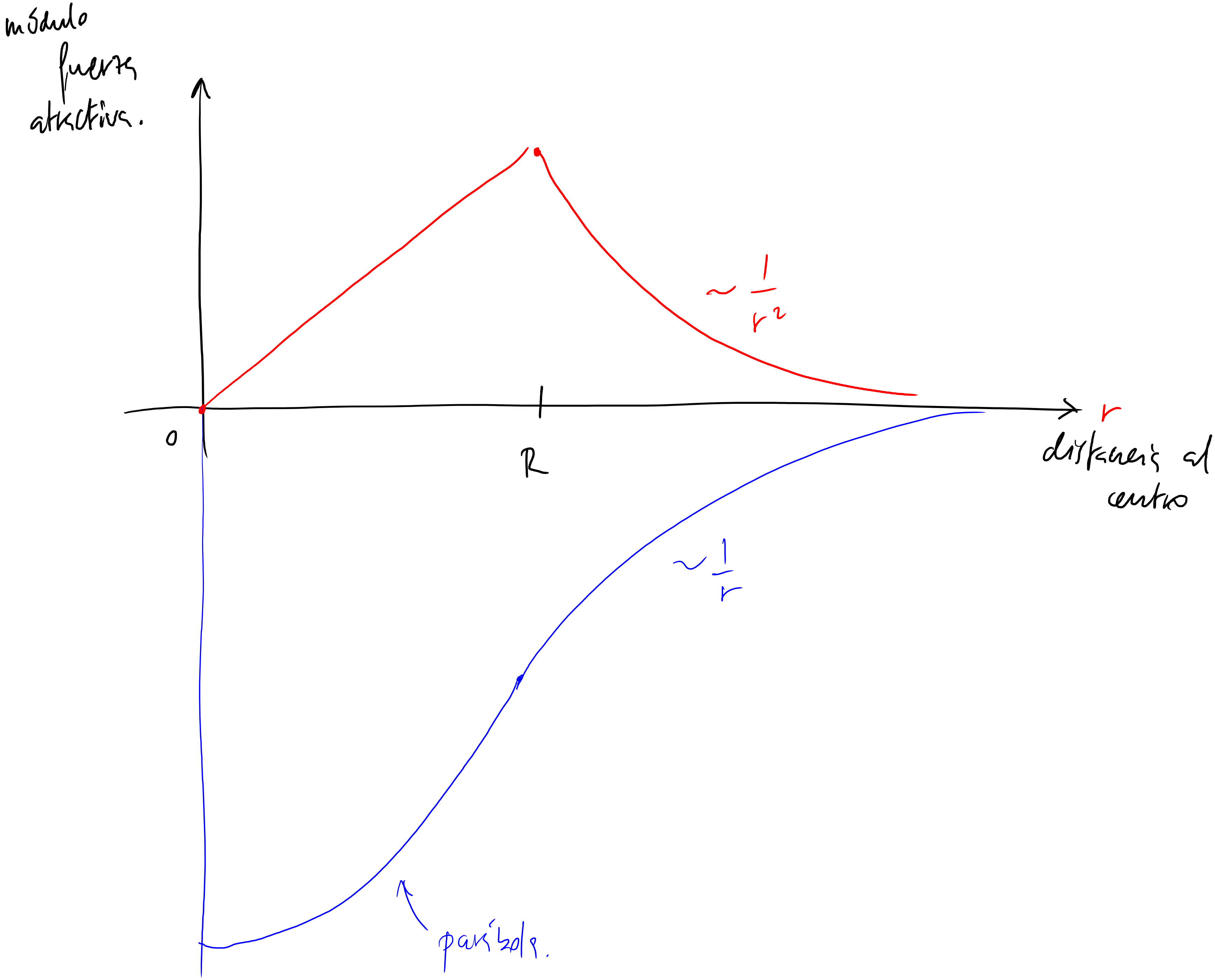
* $$V(x) = -G \int_{\mathbb{R}^3} \frac{\rho(y)}{|x-y|}\, dy = -G \int_{|y|<R} \frac{D}{|x-y|}\, dy$$

* $$\Delta V(x) = \begin{cases} 4\pi G D & |x| < R \\ 0 & |x| > R \end{cases}, \quad V \text{ es continua en } \mathbb{R}^3.$$

Intentamos:

$$V(x) = \begin{cases} C_3 + C_4 |x|^2 & , \ |x| < R. \\ \cancelto{0}{C_1} + C_2 \dfrac{1}{|x|} & , \ |x| > R \end{cases}$$

(Falta encontrar las constantes).



módulo
fuerza
atractiva.
$\sim \frac{1}{r^2}$
0
R
r
distancia al
centro
$\sim \frac{1}{r}$
parábola.

Cosas que sabemos resolver:

* Ec. Laplace en el cuadrado. ($\Rightarrow$ ec. Poisson en el cuadrado).

* Ec. Poisson en $\mathbb{R}^d$.

* Ω dominio abierto, acotado?

Ec. Poisson:

$$\left.\begin{array}{ll} \Delta u = f & \text{en } \Omega \\ u = 0 & \text{en } \partial\Omega \end{array}\right\}$$

"Forma débil" de esta ec:

$\varphi \in \mathcal{C}_c^\infty(\Omega)$, supongo Ω frontera $\mathcal{C}^1$, acotado.

$$\int_\Omega \varphi \Delta u = \int_\Omega f \cdot \varphi \quad , \quad \forall \varphi \in \mathcal{C}_c^\infty(\Omega).$$

$\Delta u = \operatorname{div}(\nabla u)$

$$\Leftrightarrow - \int_\Omega \nabla \varphi \cdot \nabla u = \int_\Omega f \cdot \varphi \quad , \quad \forall \varphi \in \mathcal{C}_c^\infty(\Omega).$$

Teo. (Lax-Milgram)

H Hilbert real. $\ell : H \to \mathbb{R}$ lineal, continua.

$a : H \times H \to \mathbb{R}$ bilineal, continua.

y además coercivo: $\exists\, \lambda > 0$ tal que $a(v,v) \geq \lambda \|v\|^2$, $\forall v \in H$.

Entonces $\exists!\; u \in H$ tal que

$$a(u,v) = \ell(v) \qquad \forall\; v \in H.$$

Si a es además simétrica, entonces el funcional

$$\mathcal{F}(v) := \frac{1}{2} a(v,v) - \ell(v) \quad , \quad v \in H$$

$$\mathcal{F} : H \to \mathbb{R}$$

tiene un único punto crítico, que es u, y además u es el único mínimo.

Caso particular: Si $a(u,v) := \langle u,v \rangle$, entonces esto es el teo representación de Riesz.

Teo. (representación de Riesz)

H Hilbert real, $\ell : H \to \mathbb{R}$ lineal, continua.

$$\Rightarrow \exists!\, u \in H \;/\; \langle u,v \rangle = \ell(v) \quad \forall\, v \in H.$$

Caso particular: a es simétrica. Entonces también puedo obtenerlo a partir del t. de Riesz:

$$\langle v,w \rangle_a := a(v,w) \qquad , \; v,w \in H$$

es un producto escalar.

(simétrico, bilineal, $\langle v,v \rangle_a = a(v,v) \geq \lambda \|v\|^2$

$\langle v,v \rangle_a = 0 \iff v = 0$)

Además, $\|\cdot\|_a$ es equivalente a $\|\cdot\|$:

$$\|v\|_a^2 = a(v,v) \geq \lambda \|v\|^2$$

$$\|v\|_a^2 = a(v,v) \leq C \|v\|^2$$

a es continua.

ℓ continua en $\|\cdot\|_a \Rightarrow \exists!\, u \in H$ tal que

$$\langle u,v \rangle_a = \ell(v)$$

$$a(u,v) = \ell(v).$$

11 de mayo

Teo. (Lax-Milgram)

H Hilbert real. $\ell : H \to \mathbb{R}$ lineal, continua.

$a : H \times H \to \mathbb{R}$ bilineal, continua.

y además coercive: $\exists \lambda > 0$ tal que $a(v,v) \geq \lambda \|v\|^2$, $\forall v \in H$.

Entonces $\exists!$ $u \in H$ tal que

$$a(u,v) = \ell(v) \qquad \forall \; v \in H.$$

Si a es además simétrica, entonces el funcional

$$F(v) := \frac{1}{2} a(v,v) - \ell(v) \quad , \quad v \in H$$

$$F : H \to \mathbb{R}$$

tiene un único punto crítico, que es u, y además u es el único mínimo.

Resultados previos

T^a (punto fijo de Banach)

$\mathbb{X}$ esp. Banach $\varphi : \mathbb{X} \to \mathbb{X}$ contractiva:

$$\exists \; L \in (0,1) \; / \quad \| \varphi(x) - \varphi(y) \| \leq L \, \| x - y \| .$$

Entonces $\exists ! \; u \in \mathbb{X}$ punto fijo de φ.

$$\varphi(u) = u$$

T^a representación Riesz — lo vimos ayer.

Consecuencias: H esp. Hilbert.

* $\langle u, w \rangle = \langle v, w \rangle \quad \forall \; w \in H$

$$\Rightarrow \quad u = v.$$

* $\langle u, w \rangle = 0 \quad \forall \; w \in H \quad \Rightarrow \quad u = 0.$

Lema. X Banach, $T: X \to X$ lineal, continua, con $\| T - I \| = \lambda < 1$
↑ identidad

Entonces, T es invertible con inversa continua.
(biyectiva).

Dem. Fijo $x \in X$. ¿$\exists! \, y \in X$ con $Ty = x$?

$$Ty = x \quad \Big\| \quad Ty - x - y = -y$$

$$Ty - x = 0 \quad \Big\| \quad \underbrace{x + y - Ty}_{\varphi(y)} = y$$

$$\| \varphi(y) - \varphi(z) \| = \| y - Ty - z + Tz \| = \| (I - T)(y - z) \| \leq$$

$$\leq \lambda \| y - z \| \Rightarrow \varphi \text{ contractiva.}$$

Tiene único pto. fijo ✓ T biyectiva.

Inversa continua?

$$\| Ty \| = \| Ty - y + y \| = \| y + (T - I) y \| \geq$$

$$\geq \| y \| - \| (T - I) y \| \geq \| y \| - \lambda \| y \| = (1 - \lambda) \| y \|$$

$$\| Ty \| \geq (1 - \lambda) \| y \|$$

$$y = T^{-1} x \Rightarrow \quad \| x \| \geq (1 - \lambda) \| T^{-1} x \|$$

$$\| T^{-1} x \| \leq \frac{1}{1 - \lambda} \| x \| \quad ✓.$$

Dem. T^a Lax-Milgram.

$\ell : H \to \mathbb{R}$ lineal, $\quad |\ell v| \leq C \|v\|$

$a : H \times H \to \mathbb{R}$ bilineal, $\quad |a(v,w)| \leq C \|v\| \|w\|$

a coerciva: $\quad a(v,v) \geq \lambda \|v\|^2$

$\exists!\, u \in H \,/\, \quad a(u,v) = \ell(v) \quad \forall\, v.$

"Traducción":

Riesz $\Rightarrow \exists!\, f \in H \,/\, \quad \ell(v) = \langle v, f \rangle \quad , \; v \in H.$

Riesz $\Rightarrow$ Dado $u \in H$, $\exists!\, A(u) \in H \,/\, \quad a(u,v) = \langle A(u), v \rangle \; , v \in H.$

$\star \quad A : H \to H$ es lineal:

$\star \; A(u) + A(w) = A(u+w)$?

$\langle A(u+w), v \rangle = a(u+w, v) = a(u,v) + a(w,v)$

$= \langle A(u), v \rangle + \langle A(w), v \rangle =$

$= \langle A(u) + A(w), v \rangle \Rightarrow A(u+w) = A(u) + A(w).$

$\star \; A(\alpha u) = \alpha A(u) \longrightarrow$ (igual).

* A es continua:

$$\|A(u)\|^2 = \langle A(u), A(u)\rangle = a(u, A(u)) \leq$$

$$\leq C\, \|u\|\, \|A(u)\|$$

$$\Rightarrow \quad \|A(u)\| \leq C\, \|u\| \quad \checkmark$$

Quiero encontrar u tal que

$$a(u,v) = \ell(v) \quad \forall\ v \in H.$$

es lo mismo que

$$\langle A(u), v\rangle = \langle f, v\rangle \quad \forall\ v \in H$$

$$\Leftrightarrow \quad A(u) = f$$

Necesito ver que A es invertible

$\Leftrightarrow$ rA invertible para algún $r > 0$.

$\| rA - I \|$?

$$\|(rA-I)(v)\|^2 = \langle (rA-I)(v), (rA-I)(v)\rangle =$$

$$= r^2\|A(v)\|^2 + \|v\|^2 - 2r\overbrace{\langle A(v),v\rangle}^{a(v,v)} \le$$

$$\le r^2 C^2\|v\|^2 + \|v\|^2 - 2r\lambda\|v\|^2 =$$

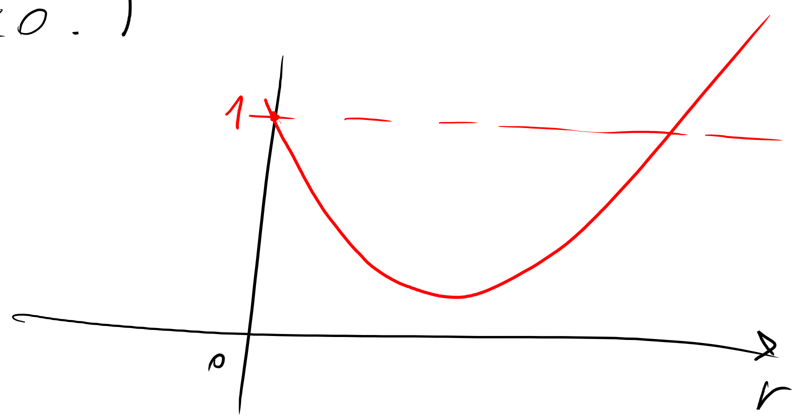
$$= \|v\|^2\big(\underbrace{C^2r^2 - 2\lambda r + 1}_{p(r)}\big) \overset{?}{<} L\|v\|^2$$

con $L<1$?

$p'(r) = 2C^2 r - 2\lambda$

$\exists\, r>0 \,/\, p(r) < 1$?

$$\left.\begin{array}{l} p(0) = 1 \\ p'(0) = -2\lambda < 0. \end{array}\right\}$$



$\Rightarrow rA$ invertible

$\Rightarrow A$ invertible ✓.

2ª parte. $\quad \mathcal{F}(v) := \frac{1}{2} a(v,v) - \ell(v)$

Supongo a simétrica.

¿Puntos críticos?

$$\delta_v \mathcal{F}(u) := \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{F}(u+\varepsilon v) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \left[\frac{1}{2} a(u+\varepsilon v, u+\varepsilon v) - \ell(u+\varepsilon v) \right]$$

$$= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \left[\frac{1}{2} a(u,u) + \varepsilon^2 \frac{1}{2} a(v,v) + \varepsilon a(u,v) - \ell(u) - \varepsilon \ell(v) \right]$$

$$= \varepsilon a(v,v) + a(u,v) - \ell(v) \Big|_{\varepsilon=0} = a(u,v) - \ell(v).$$

$\Rightarrow$ hay un único punto crítico.

Es un un mínimo porque $\mathcal{F}$ es convexa. ✓.

12 de mayo

Def. (Derivada débil). $\Omega \subseteq \mathbb{R}^d$ abierto, $f : \Omega \to \mathbb{R}$ loc. integrable. $n \in \{1, \dots, d\}$. Decimos que $g \in L^1_{loc}(\Omega)$ es derivada débil de f respecto x_n si:

$$\int_\Omega f \, \partial_{x_n} \varphi = - \int_\Omega g \, \varphi \qquad \forall \; \varphi \in \mathcal{C}_c^\infty(\Omega).$$

Si esto pasa, escribimos $\partial_{x_n} f = g$ (débil).

* Razonable: si $f \in \mathcal{C}^1(\Omega)$, $\partial_{x_n} f$ es una derivada débil de f resp. x_n.
* $\partial_{x_n} f$ (débil) no cambia si cambio f en un conjunto de medida nula.
* "La der. débil está definida en c.t.p.":

① Si g es der. débil de f
$h(x) = g(x)$ p.c.t. $x \in \Omega$ $\Bigg\}$ $\Rightarrow$ h también es der. débil de f.

② Si $g = \partial_{x_n} f$ (débil)
$h = \partial_{x_n} f$ (débil) $\Bigg\}$ $\Rightarrow$ $g = h$ c.t. $x \in \Omega$.

<u>Dem.</u> ①: claro.

②:
$$\left.\begin{array}{l} \displaystyle\int f \, \partial_{x_n} \varphi = -\int g \varphi \\[2ex] \displaystyle\int f \, \partial_{x_n} \varphi = -\int h \varphi \end{array}\right\} \forall \varphi \in \mathcal{C}_c^\infty(\Omega)$$

$$\Rightarrow \int g\varphi = \int h\varphi \quad \forall \varphi \in \mathcal{C}_c^\infty(\Omega). \quad (*)$$

Lema fundamental C.V. (versión L^1)

$$f : \Omega \to \mathbb{R} \quad L^1_{loc}(\Omega)$$

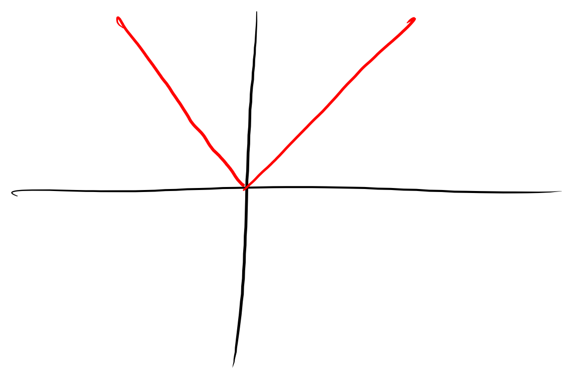
$$\int_\Omega f \varphi = 0 \quad \forall \varphi \in \mathcal{C}^\infty_c(\Omega) \Rightarrow f = 0 \text{ c.t. } x \in \Omega$$

$\otimes \Rightarrow g = h$ c.t. $x \in \Omega$ ✓

Ejemplos

* $f \in \mathcal{C}^1(-1,1) \Rightarrow f'$ es der. débil.

* $f : (-1,1) \to \mathbb{R}, \quad f(x) := |x|.$



$$g(x) = \begin{cases} -1 & x \in (-1,0) \\ 1 & x \in (0,1) \end{cases}$$

$f' = g$ (débil)

<u>Dem.</u> $\displaystyle\int_{-1}^{1} f\varphi' = -\int_{-1}^{1} g\varphi \qquad \forall\, \varphi \in C_c^\infty(-1,1).$

$$\parallel$$

$$\int_{-1}^{0} (-x)\varphi'(x)\,dx + \int_{0}^{1} x\varphi'(x)\,dx =$$

$$= \int_{-1}^{0} \varphi(x)\,dx + \underbrace{(-x)\varphi(x)\Big|_{x=-1}^{x=0}}_{=0} - \int_{0}^{1} \varphi(x)\,dx + \underbrace{x\varphi(x)\Big|_{x=0}^{x=1}}_{=0.}$$

$$= -\int_{-1}^{1} g(x)\varphi(x)\,dx \quad \checkmark.$$

* $f:(-1,1)\to\mathbb{R}$, $\quad f(x)=\begin{cases}-1 & x<0\\ 1 & x>0\end{cases}$

No tiene derivada débil:

Si tuviera, $f'=g$ (débil). $g\in L^1_{loc}(-1,1)$.

$$\int_{-1}^{1} f\varphi' = -\int_{-1}^{1} g\varphi \qquad \forall\ \varphi\in\mathcal{C}_c^\infty(\Omega).$$

$$\parallel$$

$$-\int_{-1}^{0}\varphi' + \int_{0}^{1}\varphi' = \cancel{\varphi(-1)}-\varphi(0)+\cancel{\varphi(1)}-\varphi(0) = -2\varphi(0)$$

$$\int_{-1}^{1} g\varphi = 2\varphi(0) \qquad \forall\ \varphi\in\mathcal{C}_c^\infty(\Omega).$$

No hay ninguna $g\in L^1_{loc}(-1,1)$ que cumpla esto:

Elijo $\varphi\in\underbrace{\mathcal{C}_c^\infty(-1,0)}_{\subseteq\,\mathcal{C}_c^\infty(-1,1)} \Rightarrow g=0$ c.t. $x\in(-1,0)$.

Elijo $\varphi\in\mathcal{C}_c^\infty(0,1) \Rightarrow g=0$ c.t. $x\in(0,1)$.

$\Rightarrow g=0$ c.t. $x\in(-1,1)$. $\times$.

Lema. $f: I \to \mathbb{R}$, $I \subseteq \mathbb{R}$ intervalo abierto, $f \in L^1_{loc}(I)$.

f tiene derivada débil $\iff$ $\exists$ h localmente AC. en I tal que $h = f$ c.t. $x \in I$.

En ese caso: f' (débil) $= h'$ (clásica).

Ejemplo: h, f tienen derivada débil $=0$.

$h: (-1,1) \to \mathbb{R}$
$h(x) = 1 \quad \forall x$

$f: (-1,1) \to \mathbb{R}$

$$f(x) = \begin{cases} 1 & \text{si } x \in (-1,1) \setminus \mathbb{Q} \\ 0 & \text{si } x \in \mathbb{Q}. \end{cases}$$

Ejemplo: $f(x): (-1,1) \to \mathbb{R}$, $f(x) = \dfrac{1}{|x|}$.

$$f(x) = \sqrt{|x|} \quad ?$$

17 mayo

Ejemplo: $f(x):(-1,1)\to\mathbb{R}$, $f(x)=\dfrac{1}{|x|}$.

$\star$ $f(x)=\sqrt{|x|}$?

$$f'(x)=\begin{cases}\dfrac{1}{2\sqrt{x}} & x>0\\ -\dfrac{1}{2\sqrt{-x}} & x<0\end{cases}$$

En general: $f:(-1,1)\to\mathbb{R}$, $f(x)=|x|^{-k}$, $k\in\mathbb{R}$.

f tiene derivada débil $\Leftrightarrow$ $k\leq 0$

Dem. (1ª dem.) f es esencialmente AC $\Leftrightarrow$ $k \leq 0$.

(2ª dem.)

* Si $k = 0 \Rightarrow f(x) = 1 \quad \forall\ x \neq 0 \Rightarrow f' = 0$ (débil).

* Si $k > 0$. Si hay derivada débil g, debe pasar:

$$\int_{-1}^{1} f \cdot \varphi' = - \int_{-1}^{1} g \cdot \varphi \qquad \forall\ \varphi \in \mathcal{C}_c^\infty(-1,1).$$

* Si $\varphi \in \mathcal{C}_c^\infty(-1,0)$,

$$\left.\begin{array}{l} \displaystyle\int_{-1}^{0} (-x)^{-k} \varphi' = - \int_{-1}^{0} g\, \varphi \\ \qquad \| \qquad\qquad\qquad \| \\ \displaystyle\int_{-1}^{1} (-k)(-x)^{-k-1} \varphi(x)\,dx \end{array}\right\} \Rightarrow g(x) = k(-x)^{-k-1}$$

c.t. $x \in (-1,0)$.

* Si $\varphi \in \mathcal{C}_c^\infty(0,1) \Rightarrow \cdots \Rightarrow f(x) = -k\, x^{-k-1}$

c.t. $x \in (0,1)$.

Entonces $g \notin L^1_{loc}(-1,1) \Rightarrow$ contradicción $\Rightarrow$ no hay derivada débil.

* Si $K<0$: tengo que probar que

$$\underbrace{\int_{-1}^{1} f\varphi'}\ =-\int_{-1}^{1} g\cdot\varphi \quad \forall\ \varphi\in \mathcal{C}_c^\infty(-1,1)$$

con $g(x):=\begin{cases} -K\,x^{-K-1} & x>0 \\ K\,|x|^{-K-1} & x<0 \end{cases}$

$$\int_{-1}^{1} f\varphi' = \underbrace{\int_{-1}^{-\varepsilon} f\varphi'}_{I_1} + \underbrace{\int_{\varepsilon}^{1} f\varphi'}_{I_2} + \underbrace{\int_{-\varepsilon}^{\varepsilon} f\varphi'}_{\to 0\ (\varepsilon\to 0)}$$

$f(x)=\begin{cases} x^K & x>0 \\ (-x)^K & x<0. \end{cases}$

$$\left(|x|^{-K}\right)' = -K\,|x|^{-K-1}\,\mathrm{sgn}(x)$$

$$I_1 = -k \int_{-1}^{-\varepsilon} (-x)^{-k-1} \varphi(x)\, dx + (-x)^{-k} \varphi(x) \Big|_{x=-1}^{x=-\varepsilon} =$$

$$= -\int_{-1}^{-\varepsilon} g(x) \varphi(x)\, dx + \underbrace{\varepsilon^{-k} \varphi(-\varepsilon)}_{\to 0 \quad (\varepsilon \to 0)}$$

$$I_2 = k \int_{\varepsilon}^{1} x^{-k-1} \varphi(x)\, dx + x^{-k} \varphi(x) \Big|_{x=\varepsilon}^{=1} =$$

$$= -\int_{\varepsilon}^{1} g(x) \varphi(x)\, dx - \underbrace{\varepsilon^{-k} \varphi(\varepsilon)}_{\to 0 \quad (\varepsilon \to 0)}$$

$$g(x) = -k\, x^{-k-1} \quad (x > 0)$$

$$I_1 + I_2 \xrightarrow{\varepsilon \to 0} -\int_{-1}^{1} g(x) \varphi(x)\, dx.$$

Ejemplo: $f: B \to \mathbb{R}$, $f(x) := |x|^{-K}$.

$B \subseteq \mathbb{R}^d$ bola de centro 0 radio 1.

f tiene der. débil $\iff K < d-1$ $(d \geq 2)$.

$\iff K \leq 0$ $(d = 1)$.

$*$ $|x|^{-K}$ integrable en $B \iff K < d$

$$\int_B |x|^{-K} dx = \int_0^1 \int_{S^{d-1}} r^{-K} r^{d-1} \, d\sigma \, dr =$$

$$= \omega_d \int_0^1 r^{-K+d-1} dr < \infty$$

$$\iff -K + d - 1 > -1$$

$$K < d$$

Espacios de Sobolev

Def. u sol. débil de (incompleta).

$$\left.\begin{array}{l} \Delta u = f \text{ en } \Omega \\ u = 0 \text{ en } \partial\Omega \end{array}\right\}$$

si "$u=0$" en $\partial\Omega$ y se cumple

$$\int_\Omega \nabla u \, \nabla \varphi = -\int_\Omega f \cdot \varphi \quad \forall \, \varphi \in \mathcal{C}_c^\infty(\Omega)$$

Notación: $f : \mathbb{R}^d \to \mathbb{R}$, $\quad f = f(x) = f(x_1, \dots, x_d)$

$$\underline{\underline{\partial_{x_1} \partial_{x_2} f}}, \quad \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} f \quad \frac{\partial^2}{\partial x_1 \partial x_2} f.$$

$$\partial_{x_1}^{n_1} \partial_{x_2}^{n_2} \cdots \partial_{x_d}^{n_d} f \equiv \partial^\alpha f, \quad \alpha = (n_1, n_2, \dots, n_d).$$

orden de $\alpha \equiv |\alpha| = n_1 + n_2 + \dots + n_d$.

Def: $k \geq 0$, $\Omega \subseteq \mathbb{R}^d$ abierto. El espacio $H^k(\Omega)$:

$$H^k(\Omega) = \left\{ f \in L^2(\Omega) \,/\, \begin{array}{l} \forall \alpha \text{ con } |\alpha| \leq k \text{ existe } \partial^\alpha f \text{ (débil)} \\ \text{con } \partial^\alpha f \in L^2(\Omega) \end{array} \right\}$$

Si $u, v \in H^k(\Omega)$,

$$\langle u, v \rangle_{H^k(\Omega)} := \sum_{|\alpha| \leq k} \int_\Omega \partial^\alpha u \cdot \partial^\alpha v \quad \text{prod. escalar.}$$

$$\|u\|^2_{H^k(\Omega)} = \sum_{|\alpha| \leq k} \int_\Omega |\partial^\alpha u|^2 \quad \text{norma asociada.}$$

Ejemplos:

* $K=0$: $\quad H^0(\Omega) = L^2(\Omega)$.

* $K=1$: $\quad H^1(\Omega) = \{ u \in L^2(\Omega) / \partial_{x_i} u \in L^2(\Omega) \ \forall i=1,\dots,d \}$

$$= \{ u \in L^2(\Omega) / \nabla u \in L^2(\Omega) \}.$$

$$\langle u,v \rangle_{H^1(\Omega)} = \int_\Omega u \cdot v + \int_\Omega \partial_{x_1} u \, \partial_{x_1} v + \dots + \int_\Omega \partial_{x_d} u \, \partial_{x_d} v$$

$$= \int_\Omega u \cdot v + \int_\Omega \nabla u \cdot \nabla v$$

$$\|u\|^2_{H^1(\Omega)} = \int_\Omega u^2 + \int_\Omega |\nabla u|^2$$

Ejemplos:

* En dim. 1: $(k \geq 1)$

$$H^k(a,b) = \left\{ u \in L^2(a,b) \,/\, \begin{array}{l} \exists\, v \in \mathcal{C}^{k-1}(a,b), \quad v^{(k-1)} \in AC(a,b) \\ v^{(k)} \in L^2(a,b), \quad \text{con} \quad u = v \text{ c.t. } x \in (a,b) \end{array} \right\}$$

* $u : (-1,1) \to \mathbb{R}, \quad u(x) = |x|^{-k}.$

$$u \in H^0(-1,1) = L^2(-1,1) \iff k < \frac{1}{2}.$$

$$u \in H^1(-1,1) \qquad \iff \qquad k < -\frac{1}{2} \quad \text{o bien} \quad k = 0.$$

$$\left[\begin{array}{l} u(x) = |x|^{-k} \\ k \leq 0 \Rightarrow u'(x) = -k\,|x|^{-k-1} \operatorname{sgn}(x). \\ \qquad \in L^2 \iff -2k-2 > -1 \\ \qquad\qquad -2k > 1 \quad k < -\frac{1}{2} \end{array} \right]$$

Teo. $H^k(\Omega)$ con el prod escalar $\langle \cdot , \cdot \rangle_{H^k(\Omega)}$ es un esp. de Hilbert.

Def. $H^k_0(\Omega) :=$ el cierre de $\mathcal{C}^\infty_c(\Omega)$ en la norma de $H^k(\Omega)$.

$$H^k_0(\Omega) \subseteq H^k(\Omega).$$

$H^k_0(\Omega) =$ "funciones de $H^k(\Omega)$ que valen 0 en el borde".

$H^k_0(\Omega)$ es esp. de Hilbert con $\langle \cdot , \cdot \rangle_{H^k(\Omega)}$.

Def. $\Omega \subseteq \mathbb{R}^d$ abierto, $f \in L^2(\Omega)$.
u es sol. débil de $\Delta u = f$ en Ω, $u = 0$ en $\partial\Omega$
cuando $u \in H^1_0(\Omega)$, y se cumple:

$$\int_\Omega \nabla u \nabla \varphi = -\int_\Omega f \cdot \varphi \qquad \forall\, \varphi \in \mathcal{C}^\infty_c(\Omega).$$

Equivalentemente,

$$\int_\Omega \nabla u \nabla v = -\int_\Omega f \cdot v \qquad \forall\, v \in H^1_0(\Omega)$$

$$\mathscr{C}_c^\infty(\Omega) \subseteq H_0^k(\Omega) \subseteq H^k(\Omega)$$

$$H^k(\Omega) \subseteq H^{k-1}(\Omega)$$

$$u \in H^k(a,b) \Rightarrow \quad u \in \mathscr{C}^{k-1}$$

$$u' \in \mathscr{C}^{k-2}$$

$$\vdots$$

$$u^{(k-1)} \in AC$$

$$u^{(k)} \in L^2$$

Soluciones débiles de ec. Poisson

Queremos usar Lax-Milgram:

$$\ell : H^1_0(\Omega) \to \mathbb{R}, \quad \ell(v) = -\int_\Omega f \cdot v \qquad \text{lineal} \checkmark$$

$$a : H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{R}, \quad a(u,v) = \int_\Omega \nabla u \nabla v \qquad \text{bilineal.}$$

* ℓ continua?

$$|\ell(v)| = \left| \int_\Omega f \cdot v \right| \leq \underbrace{\left(\int_\Omega f^2\right)^{1/2}}_{=:C} \underbrace{\left(\int_\Omega v^2\right)^{1/2}}_{\leq \|v\|_{H^1(\Omega)}} \leq C \|v\|_{H^1(\Omega)}$$

$$\|v\|^2_{H^1(\Omega)} = \int_\Omega v^2 + \int_\Omega |\nabla v|^2 \geq \int_\Omega v^2 = \|v\|_2^2$$

★ a continua?

$$|a(u,v)| \leq \int_\Omega |\nabla u||\nabla v| \leq \left(\int_\Omega |\nabla u|^2\right)^{1/2} \left(\int_\Omega |\nabla v|^2\right)^{1/2} =$$

$$= \|\nabla u\|_2 \, \|\nabla v\|_2 \leq \|u\|_{H^1(\Omega)} \, \|v\|_{H^1(\Omega)} \quad \checkmark$$

★ a coerciva?

$$a(v,v) = \int_\Omega |\nabla v|^2 \geq \lambda \|v\|^2_{H^1(\Omega)} \quad ?$$

$$\int_\Omega |\nabla v|^2 \geq \lambda \int_\Omega v^2 + \lambda \int_\Omega |\nabla v|^2$$

(Si es cierto tiene que ser $\lambda < 1$)

19 mayo

$\Omega \subseteq \mathbb{R}^d$ abto, acotado, $f \in L^2(\Omega)$, $\lambda \in \mathbb{R}$.

Ec. Helmholtz

$$\left.\begin{array}{ll} \Delta u + \lambda u = f & \text{en } \Omega \\ u = 0 & \text{en } \partial\Omega \end{array}\right\} \quad (H)$$

Def: u es sol. débil de (H) si $u \in H_0^1(\Omega)$,

$$\underbrace{\int_\Omega \nabla u \cdot \nabla v - \lambda \int_\Omega u v}_{} = \underbrace{- \int_\Omega f \cdot v}_{} \quad , v \in H_0^1(\Omega).$$

Para usar Lax-Milgram:

$$a(u,v) = \int_\Omega \nabla u \nabla v - \lambda \int u v \quad , \quad \ell(v) := - \int f v.$$

Condicions: a bilineal, continua, ℓ lineal contínua — fàcil

a coerciva:

$$a(v,v) = \int_\Omega |\nabla v|^2 - \lambda \int_\Omega v^2 \geq \int_\Omega |\nabla v|^2 - \frac{\lambda}{\lambda_\Omega} \int_\Omega |\nabla v|^2 =$$

$$= \left(1 - \frac{\lambda}{\lambda_\Omega}\right) \int_\Omega |\nabla v|^2$$

si $1 - \dfrac{\lambda}{\lambda_\Omega} > 0$

$\Rightarrow a$ és coerciva.

$\boxed{\lambda < \lambda_\Omega}$ ← si això ocorre, (P) té una sol. dèbil única

Ejemplo: $\mathcal{F}(u) := \int_0^1 (u'(x) + \operatorname{sen} x)^2 \, dx$, $\mathcal{F} : H_0^1(0,1) \to \mathbb{R}$.

¿Tiene extremos?

$$\mathcal{F}(u) = \int_0^1 (u')^2 + \underbrace{\int_0^1 (\operatorname{sen} x)^2}_{C} + \int_0^1 u'(x) \operatorname{sen} x \, dx$$

para usar Lax-Milgram

$$\tilde{\mathcal{F}}(u) = \int_0^1 (u')^2 + \int_0^1 u'(x) \operatorname{sen} x \, dx = \frac{1}{2} a(u,u) - \ell(u)$$

$$a(u,v) = 2 \int_0^1 u' v', \qquad \ell(u) = -\int_0^1 u'(x) \operatorname{sen} x \, dx.$$

Lax-Milgram $\Rightarrow$ $\exists !$ mínimo de $\mathcal{F}$ en $H_0^1(0,1)$,

y es el único punto crítico de $\mathcal{F}$.

$*$ $\mathcal{F}(u) := \int_0^1 (u'(x) + \operatorname{sen} x)^2 dx$, $\mathcal{F} : D \to \mathbb{R}$

$D = \{ u \in \mathcal{C}^2[0,1] \,/\, u(0) = 0 = u(1) \}$

$F(x, u, z) = (z + \operatorname{sen} x)^2$

$\partial_u F = 0$, $\partial_z F = 2(z + \operatorname{sen} x)$

$0 = (u' + \operatorname{sen} x)'$, $\left.\begin{array}{l} u''(x) = -\cos x \\ u(0) = u(1) = 0 \end{array}\right\}$

Tiene sol única (en $\mathcal{C}^2[0,1]$).

$\Rightarrow$ $\mathcal{F}$ tiene un único pto. crítico. $\left.\right\}$ $\Rightarrow$ tiene un único mínimo.

$\mathcal{F}$ convexa

Transformada de Fourier

Def. $f \in L^1(\mathbb{R}^d;\mathbb{C})$, $f:\mathbb{R}^d \to \mathbb{C}$

prod. escalar

$$\mathcal{F}(f)(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x)\, e^{-ix\xi}\, dx \quad , \quad \xi \in \mathbb{R}^d .$$

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi)$$

$$\mathcal{F}(f) : \mathbb{R}^d \to \mathbb{C}.$$

Serie de Fourier de $f \in L^1(-\pi,\pi)$:

$$\hat{f}_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx \quad , \quad n \in \mathbb{Z}.$$

(Otras definiciones

$$\mathcal{F}(f)(\xi) := \int_{\mathbb{R}^d} f(x)\, e^{-ix\xi}\, dx$$

$$\mathcal{F}(f)(\xi) := \int_{\mathbb{R}^d} f(x)\, e^{-2\pi i x\xi}\, dx$$

)

Def. $\mathcal{F}^{-1}(f)(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x)\, e^{ix\xi}\, dx \quad , \quad \xi \in \mathbb{R}^d.$

$\shortparallel$ $\check{f}(\xi)$ "transformada inversa".

(serie de Fourier:

$$f = \sum_{n\in\mathbb{Z}} \hat{f}_n e^{inx} \quad)$$

Analogía:

$$\left\{ \begin{array}{l} \mathcal{F}(f)(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x)\, e^{-ix\xi}\, dx \quad , \quad \xi \in \mathbb{R}^d. \\ \hat{f}_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx \quad , \quad n \in \mathbb{Z}. \end{array} \right.$$

(no lo hemos demostrado.)

$$\left\{ \begin{array}{l} f(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \hat{f}(\xi)\, e^{ix\xi}\, d\xi \quad , \quad x \in \mathbb{R}^d. \\ f = \sum_{n\in\mathbb{Z}} \hat{f}_n e^{inx} \end{array} \right.$$

Propiedades

* Lineal: $\widehat{f+g} = \hat{f} + \hat{g}$, $\widehat{\lambda f} = \lambda \hat{f}$. $(f, g \in L^1(\mathbb{R}^d))$.

* $f \in L^1(\mathbb{R})$, simétrica $(f(x) = f(-x) \ \forall x) \Rightarrow \hat{f}$ es real.

$$\sqrt{2\pi}\,\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-ix\xi}\, dx = \underbrace{\int_{-\infty}^{\infty} f(x) \cos(\xi x)\, dx}_{} - i \underbrace{\int_{-\infty}^{+\infty} \underbrace{f(x) \operatorname{sen}(\xi x)}_{\text{impar}}\, dx}_{=0.}$$

24 mayo

$$f \in L^1(\mathbb{R}^d), \qquad \hat{f}(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x)\, e^{-ix\xi}\, dx \quad , \ \xi \in \mathbb{R}^d.$$

$$\check{f}(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(\xi)\, e^{ix\xi}\, d\xi \quad , \ x \in \mathbb{R}^d$$

Propiedades

* Lineal
* Traslaciones: $g(x) := f(x-\tau)$, $\tau \in \mathbb{R}^d$ fijo.

$$(2\pi)^{d/2}\, \hat{g}(\xi) = \int_{\mathbb{R}^d} f(x-\tau)\, e^{-ix\xi}\, dx = \int_{\mathbb{R}^d} f(y)\, e^{-i(y+\tau)\xi}\, dy =$$

$$= e^{-i\xi\tau} \int_{\mathbb{R}^d} f(y) e^{-iy\xi} dy = (2\pi)^{d/2}\, e^{-i\xi\tau}\, \hat{f}(\xi)$$

$$\widehat{f(x-\tau)}(\xi) = e^{-i\xi\tau}\, \hat{f}(\xi)$$

★ <u>Cambios de escala</u> $g(x) := f\left(\frac{x}{\lambda}\right)$, $\lambda > 0$.

$$(2\pi)^{d/2}\ \hat{g}(\xi) = \int f\left(\frac{x}{\lambda}\right) e^{-ix\xi}\, dx = \left[\frac{x}{\lambda} = y,\quad dx = \lambda^d dy\right]$$

$$= \lambda^d \int f(y) e^{-i\lambda y\xi}\, dy = \lambda^d (2\pi)^{d/2}\ \hat{f}(\lambda\xi)$$

$$\widehat{f\left(\frac{x}{\lambda}\right)}(\xi) = \lambda^d\ \hat{f}(\lambda\xi)$$

★ $\hat{f}(0) = \int_{\mathbb{R}^d} f(x)\,dx \cdot (2\pi)^{-d/2}$.

$\star$ Derivadas: (Serie de Fourier: $\widehat{f'}_n = in \hat{f}_n$)

$$f \in L^1(\mathbb{R}^d) \cap \mathcal{C}^1(\mathbb{R}^d), \quad \partial_{x_j} f \in L^1(\mathbb{R}^d)$$

$$(2\pi)^{d/2} \widehat{\partial_{x_j} f}(\xi) = \int \partial_{x_j} f(x) e^{-ix\xi} dx = -\int f(x) e^{-ix\xi} \cdot (-i\xi_j) dx =$$

$$= i\xi_j (2\pi)^{d/2} \hat{f}(\xi)$$

$$\widehat{\partial_{x_j} f}(\xi) = i\xi_j \hat{f}(\xi) \qquad j = 1, \dots, d$$

Consecuencias:

$$\widehat{\nabla f}(\xi) = i \underbrace{\xi}_{\in \mathbb{R}^d} \cdot \underbrace{\hat{f}(\xi)}_{\in \mathbb{C}}$$

$$\widehat{\partial^2_{x_j} f}(\xi) = -\xi_j^2 \hat{f}(\xi) \quad \Big\| \quad \widehat{\Delta f}(\xi) = -|\xi|^2 \hat{f}(\xi)$$

★ Convolución:

$$f, g \in L^1(\mathbb{R}^d), \qquad (f*g)(x) := \int_{\mathbb{R}^d} f(y)\, g(x-y)\, dy = \int_{\mathbb{R}^d} f(x-y)\, g(y)\, dy.$$

(definido en casi todo $x \in \mathbb{R}^d$).

$$\widehat{f*g}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} (f*g)(x)\, e^{-ix\xi}\, dx = (2\pi)^{-d/2} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(y)\, g(x-y)\, e^{-ix\xi}\, dy\, dx$$

$$= (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(y) \int_{\mathbb{R}^d} g(x-y)\, e^{-ix\xi}\, dx\, dy = \left[\begin{array}{ll} x-y = z & (\text{en } x) \\ dx = dz & \end{array}\right]$$

$$= (2\pi)^{-d/2} \int f(y) \int g(z)\, e^{-i(z+y)\xi}\, dz\, dy =$$

$$= (2\pi)^{-d/2} \left(\int f(y)\, e^{-iy\xi}\, dy\right)\left(\int g(z)\, e^{-iz\xi}\, dz\right)$$

$$= (2\pi)^{d/2}\, \hat{f}(\xi)\, \hat{g}(\xi)$$

$$\widehat{f*g} = (2\pi)^{d/2}\, \hat{f} \cdot \hat{g}$$

* Inversa: $f \in L^1(\mathbb{R}^d)$, $\hat{f} \in L^1(\mathbb{R}^d)$, $\check{f} \in L^1(\mathbb{R}^d)$.

$$\mathcal{F}(\mathcal{F}^{-1}(f)) = f \quad , \quad \mathcal{F}^{-1}(\mathcal{F}(f)) = f$$

$$\hat{\check{f}} = f \qquad \check{\hat{f}} = f$$

Dem. formal:

$$(2\pi)^d \mathcal{F}^{-1}(\mathcal{F}(f)) = \int_{\mathbb{R}^d} \mathcal{F}(f)(\xi) e^{+ix\xi} d\xi \cdot (2\pi)^{d/2} =$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \overbrace{f(y) e^{-iy\xi} e^{+ix\xi}} dy \, d\xi =$$

$$\text{"} = \int_{\mathbb{R}^d} f(y) \underbrace{\int_{\mathbb{R}^d} e^{-i\xi(y-x)} d\xi}_{\substack{\text{no tiene sentido} \\ \text{pero es } (2\pi)^d \delta_{x-y}}} dy = (2\pi)^d f(x) \text{"}$$

* Consecuencia:

sabemos que: $\widecheck{f * g} = (2\pi)^{d/2}\, \check{f} \cdot \check{g}$ (misma cuenta que antes).

lo aplico a $\hat{f}, \hat{g}$: $(\hat{f} * \hat{g})^{\vee} = (2\pi)^{d/2}\, f \cdot g$

$$\hat{f} * \hat{g} = (2\pi)^{d/2}\, \widehat{f \cdot g}$$

* $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$

$\|f\|_2 = \|\hat{f}\|_2$. (isometría en $L^2(\mathbb{R}^d)$).

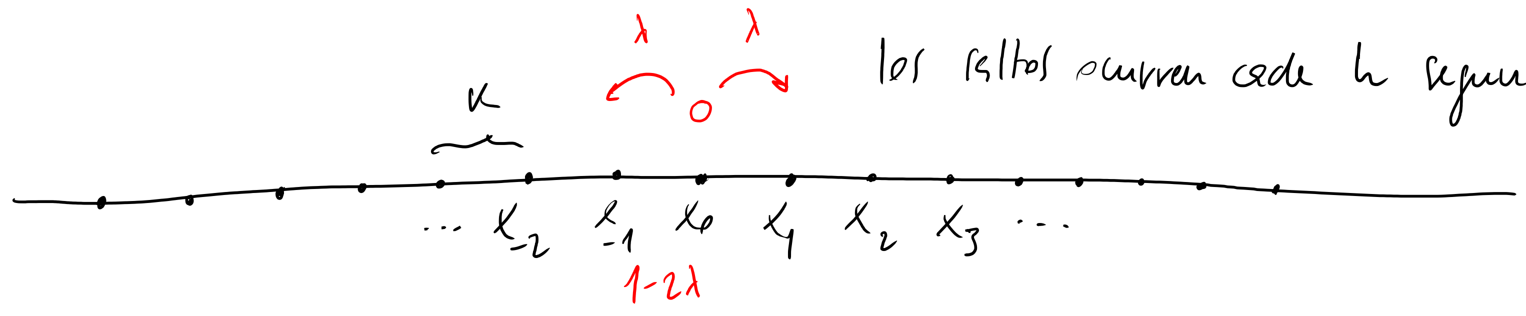
$$\int_{\mathbb{R}^d} \hat{f}(\xi)\, \hat{g}(\xi)\, d\xi = \int_{\mathbb{R}^d} f(x)\, g(x)\, dx.$$

($\Rightarrow$ puedo definir $\mathcal{F} : L^2(\mathbb{R}^d; \mathbb{C}) \longrightarrow L^2(\mathbb{R}^d; \mathbb{C})$.)

Ecuación de difusión (del calor)

$$\partial_t u = \Delta u, \qquad u = u(t,x), \qquad \begin{array}{l} t \geq 0 \text{ tiempo} \\ x \in \mathbb{R}^d \text{ espacio.} \end{array}$$

$$\Delta u = \partial_{x_1}^2 u + \dots + \partial_{x_d}^2 u$$



los saltos ocurren cada h segundos.

$u_j(t) \equiv$ probabilidad de que esté en x_j en tiempo $t = \underbrace{i}_{\text{entero} \geq 0} \cdot h$

$x_j := \kappa j$

$$u_j(t+h) = (1-2\lambda)\, u_j(t) + \lambda\, u_{j+1}(t) + \lambda\, u_{j-1}(t)$$

$$\frac{u_j(t+h) - u_j(t)}{h} = \frac{\lambda}{h}\left(u_{j+1}(t) + u_{j-1}(t) - 2u_j(t)\right)$$

Elijo $h \to 0$, $\kappa \to 0$, $\dfrac{\lambda}{h} = \dfrac{1}{2\kappa^2}$ (elijo esta escala).

$$\frac{u_j(t+h) - u_j(t)}{h} = \frac{1}{2\kappa^2}\left(u_{j+1}(t) + u_{j-1}(t) - 2u_j(t)\right)$$

Suponemos $u_j(t) \approx u(t, x_j)$

$$\partial_t u(t, x_j) = \underbrace{\partial_x^2 u(t, x_j)}$$

porque: $$f''(x) = \lim_{\kappa \to 0} \frac{1}{2\kappa}\left(f(x+\kappa) + f(x-\kappa) - 2f(x)\right)$$

$$f'(x) = \lim_{\kappa \to 0} \frac{1}{\kappa}\left(f(x+\kappa) - f(x)\right)$$

$$f''(x) = \lim_{\kappa \to 0}\left(\frac{1}{2\kappa}\left(f'(x+\kappa) - f'(x)\right) + \frac{1}{2\kappa}\left(f'(x) - f'(x-\kappa)\right)\right)$$

$$= \lim_{\kappa \to 0} \frac{1}{2\kappa}\left(f'(x+\kappa) - f'(x-\kappa)\right)$$

Propiedades ec. calor

Def. $u:(0,\infty)\times\mathbb{R}^d\to\mathbb{R}$ es sol. de la ec. del calor

$$\partial_t u = \Delta u \quad \text{en} \quad t>0,\ x\in\mathbb{R}^d \qquad \circledast$$

cuando: $\star$ u es $\mathcal{C}^1$ en t, $\mathcal{C}^2$ en x.

$\star$ se cumple $\circledast$.

$u:[0,\infty)\times\mathbb{R}^d\to\mathbb{R}$ es sol. del PVI:

$$\left.\begin{aligned} \partial_t u &= \Delta u && \text{en} \quad t>0,\ x\in\mathbb{R}^d \\ u(0,x) &= f(x) && \text{en} \quad x\in\mathbb{R}^d \end{aligned}\right\} \circledast\circledast$$

cuando: $\star$ u es continua en $[0,\infty)\times\mathbb{R}^d$.

$\star$ u es $\mathcal{C}^1$ en t, $\mathcal{C}^2$ en x. en $(0,\infty)\times\mathbb{R}^d$

$\star$ se cumple $\circledast\circledast$.

* Linealidad: u, v soluciones de $\partial_t u = \Delta u$

$\Rightarrow u+v$ también, λu también. $(\lambda \in \mathbb{R})$.

* Invariante por movimientos rígidos: $R: \mathbb{R}^d \to \mathbb{R}^d$ mov. rígido.

$$\left.\begin{array}{l} \text{Si } u \text{ es sol.} \\ v(t,x) := u(t, Rx) \end{array}\right\} \Rightarrow v \text{ también es sol.}$$

* Cambio de escala: $\lambda, \mu > 0$

$$\left.\begin{array}{l} u \text{ es sol.} \\ v(t,x) := u(\lambda t, \mu x) \end{array}\right\} \quad \left.\begin{array}{l} \partial_t v = \lambda \partial_t u \\ \partial_{x_j} v = \mu \partial_{x_j} u \end{array}\right\} \quad \frac{1}{\lambda}\partial_t v = \frac{1}{\mu^2}\Delta v$$

$$\Delta v = \mu^2 \Delta u$$

$$\partial_t v = \frac{\lambda}{\mu^2} \Delta v$$

* No invariante per inversió en t. (no es reversible):

$\left.\begin{array}{l} u \text{ solució en } t\in\mathbb{R},\ x\in\mathbb{R}^d. \\ v(t,x) := u(-t,x) \end{array}\right\}$ $\partial_t u = -\Delta u$

Solució per Fourier

$u = u(t,x)$ es solució del PVI. $\left.\begin{array}{l} \partial_t u = \Delta u \\ u(0,x) = u_0(x) \end{array}\right\}$

$$\varphi(t,\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} u(t,x)\, e^{-ix\xi}\, dx \equiv \hat{u}(t,\xi)$$

$$\partial_t \varphi(t,\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \partial_t u(t,x)\, e^{-ix\xi}\, dx =$$

$$= (2\pi)^{-d/2} \int_{\mathbb{R}^d} \Delta u(t,x)\, e^{-ix\xi}\, dx = \widehat{\Delta u}(t,\xi) = -|\xi|^2 \varphi(t,\xi)$$

$$\partial_t \varphi(t,\xi) = -|\xi|^2 \varphi(t,\xi)$$

$$\varphi(t,\xi) = \varphi(0,\xi)\, e^{-|\xi|^2 t} = \hat{u}_0(\xi)\, e^{-|\xi|^2 t}$$

$$\varphi(0,\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} u(0,x)\, e^{-i\xi x}\, dx = \hat{u}_0(\xi)$$

Sol. ec. calor:

$$\left.\begin{aligned} \partial_t u &= \Delta u \quad \text{en} \quad t>0,\ x \in \mathbb{R}^d \\ u\big|_{t=0} &= u_0 \end{aligned}\right\}$$

$$\hat{u}(t,\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} u(t,x) e^{-ix\xi}\, dx \equiv \varphi(t,\xi)$$

$$\left.\begin{aligned} \partial_t \varphi &= -|\xi|^2 \varphi \\ \varphi\big|_{t=0} &= \hat{u}_0 \end{aligned}\right\} \quad \varphi(t,\xi) = \hat{u}_0(\xi)\, e^{-t|\xi|^2}, \qquad u?$$

$$u(t,x) = \mathcal{F}_\xi^{-1}\left[\hat{u}_0(\xi)\, e^{-t|\xi|^2}\right](x)$$

Uso que: $\mathcal{F}^{-1}(f \cdot g) = (2\pi)^{-d/2} \mathcal{F}^{-1}(f) * \mathcal{F}^{-1}(g)$

$$u(t,x) = (2\pi)^{-d/2} u_0 * \underbrace{\mathcal{F}_{\xi}^{-1}\left(e^{-t|\xi|^2}\right)}_{?}$$

Luego veremos que:

$$\left[\mathcal{F}^{-1}\left(e^{-t|\xi|}\right) = (2t)^{-d/2} e^{-\frac{|x|}{4t}} \right] \quad (\text{falta}).$$

Luego:

$$u(t,x) = (4\pi t)^{-d/2} u_0 * e^{-\frac{|x|^2}{4t}}$$

$$u(t,x) = (4\pi t)^{-d/2} \int_{\mathbb{R}^d} u_0(y) e^{-\frac{|x-y|^2}{4t}} dy$$

<u>Teo.</u> $u_0 : \mathbb{R}^d \to \mathbb{R}$ continua, acotada. Definimos

$$u(t,x) = (4\pi t)^{-d/2} \int_{\mathbb{R}^d} u_0(y)\, e^{-\frac{|x-y|^2}{4t}}\, dy \qquad \begin{array}{l} t>0 \\ x \in \mathbb{R}^d \end{array}$$

Entonces u es solución de la ec. del calor $\partial_t u = \Delta u$ en $(0,\infty)\times\mathbb{R}^d$,

* u se puede extender a una función continua en $[0,\infty)\times\mathbb{R}^d$, que es solución del PVI para la ec. calor

(dicho de otra forma:

$$\lim_{t\to 0} u(t,x) = u_0(x) \qquad \forall\ x\in\mathbb{R}^d \quad).$$

Dem. (idea): $\partial_t u = \Delta u$ directo.

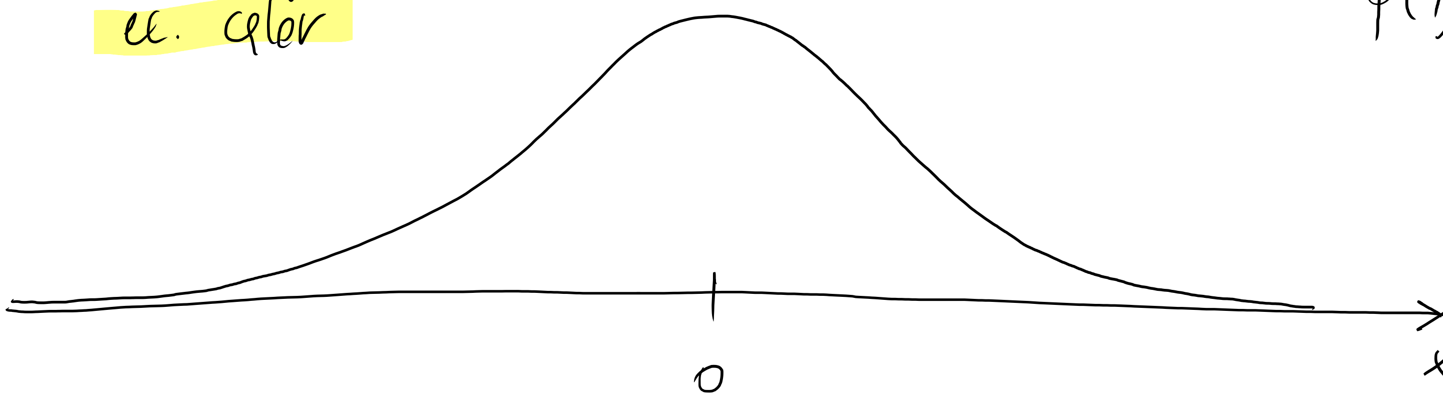
$$\lim_{t\to 0} u(t,x) = u_0(x) \ ?$$

en la variable x.

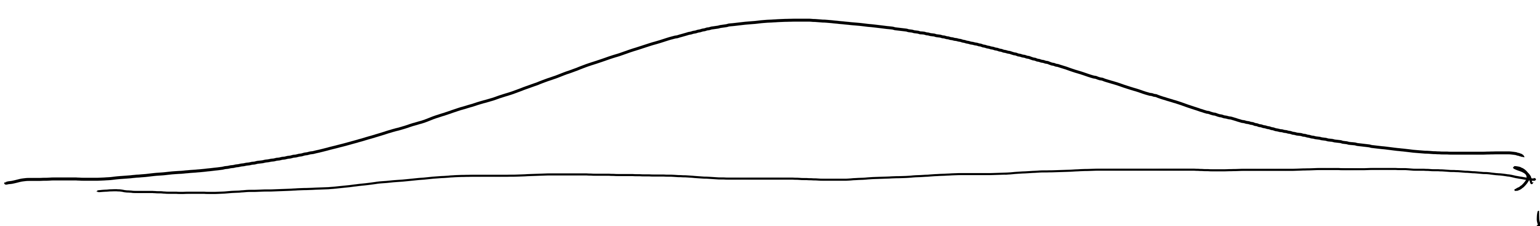
$$\left\{ \begin{array}{l} \phi(t,x) := (4\pi t)^{-d/2} e^{-\frac{|x|^2}{4t}} \\ (t>0,\ x\in\mathbb{R}^d). \end{array} \right. , \qquad u = u_0 * \phi$$

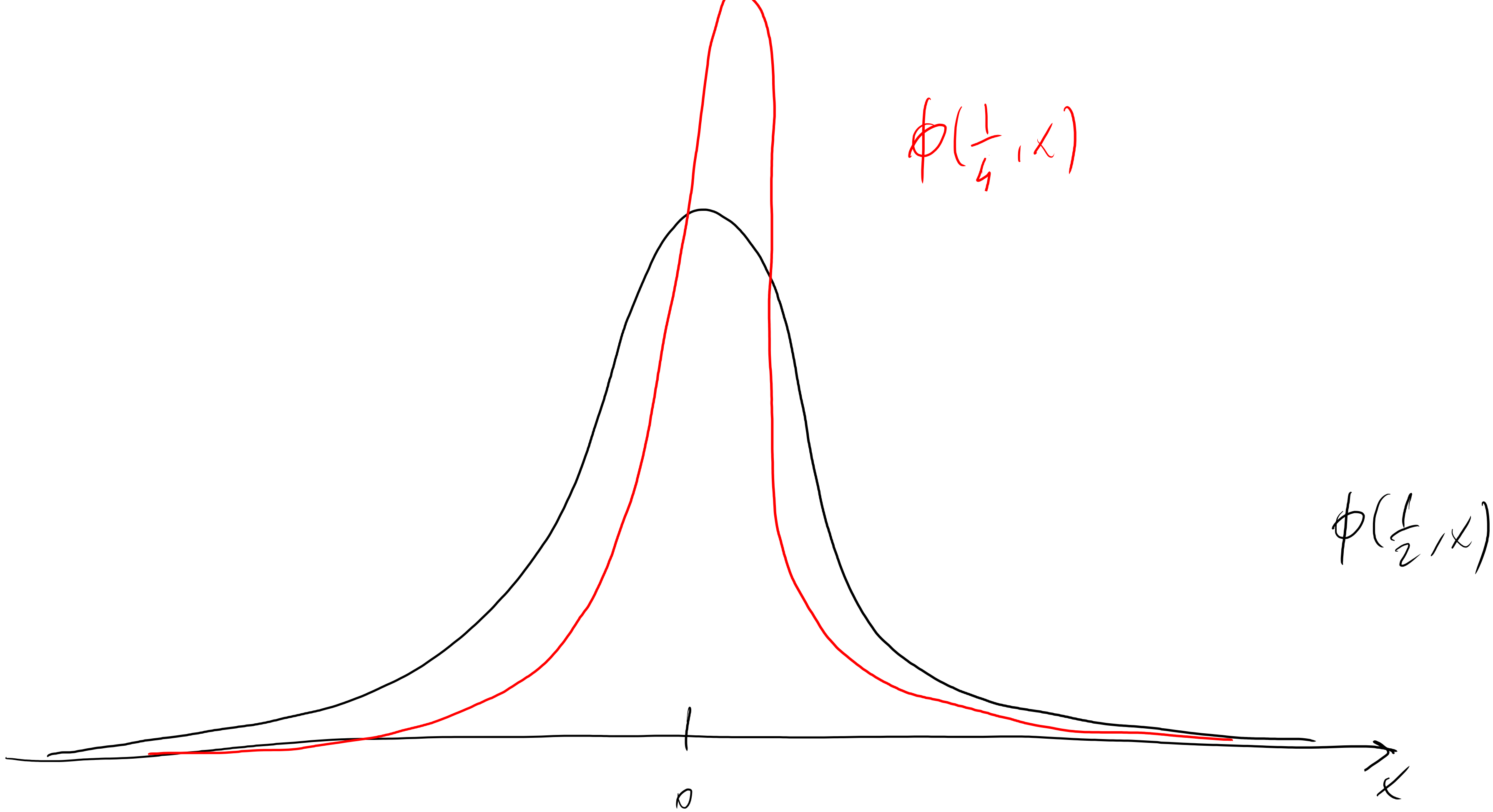
solución fundamental ec. calor

$\phi(1,x)$



$\phi(2,x)$





$\phi(\frac{1}{4}, x)$
$\phi(\frac{1}{2}, x)$
0
x

Ejemplo:

$$\star \quad \mathcal{F}\left(e^{-t|x|^2}\right) = (2t)^{-d/2} e^{-\frac{|\xi|^2}{4t}}$$

Es suficiente probarlo para $t = \frac{1}{2}$:

$$\mathcal{F}\left(e^{-\frac{|x|^2}{2}}\right) = e^{-\frac{|\xi|^2}{2}}$$

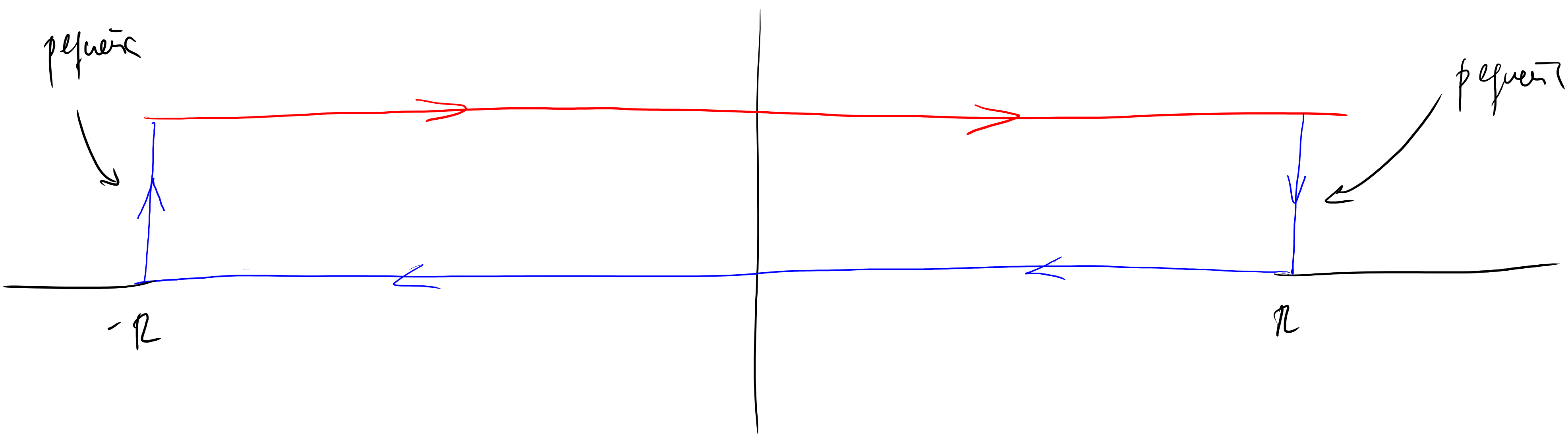
Primero en $d = 1$

$$(2\pi)^{1/2}\, \mathcal{F}\left(e^{-\frac{|x|^2}{2}}\right)(\xi) = \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} e^{-ix\xi}\, dx = e^{-\frac{1}{2}\xi^2} \underbrace{\int_{-\infty}^{+\infty} e^{-\frac{1}{2}(x+i\xi)^2}\, dx}_{\checkmark}$$

$$\frac{x^2}{2} + ix\xi = \frac{1}{2}\left(x^2 + 2ix\xi\right) =$$

$$= \frac{1}{2}\left[(x+i\xi)^2 + \xi^2\right]$$

$$\int_{-\infty}^{+\infty} e^{-\frac{1}{2}(x+i\xi)^2}\,dx = \int_{-\infty}^{+\infty} e^{-\frac{1}{2}x^2}\,dx = \sqrt{2\pi}$$



$$\int_{-\infty}^{+\infty} e^{-\frac{1}{2}x^2}\,dx \ ?$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}(x^2+y^2)}\,dx\,dy = \int_0^{\infty} e^{-\frac{1}{2}r^2}\,r\cdot 2\pi\,dr =$$

$$= -2\pi\, e^{-\frac{1}{2}r^2}\Big|_{r=0}^{r=\infty} = 2\pi$$

$$= \left(\int_{-\infty}^{+\infty} e^{-\frac{1}{2}x^2}dx\right)\left(\int_{-\infty}^{+\infty} e^{-\frac{1}{2}y^2}dy\right) = \left(\int_{-\infty}^{+\infty} e^{-\frac{1}{2}x^2}dx\right)^2$$

Conclusión: $\mathcal{F}(e^{-\frac{1}{2}|x|^2}) = e^{-\frac{1}{2}|\xi|^2}$ en $d=1$.

Para cualquier d:

$$\mathcal{F}(e^{-\frac{1}{2}|x|^2}) = (2\pi)^{-d/2} \int \cdots \int e^{-\frac{1}{2}x_1^2} e^{-\frac{1}{2}x_2^2} \cdots e^{-\frac{1}{2}x_d^2} e^{-ix_1\xi_1} \cdots e^{-ix_d\xi_d} \, dx_1 \cdots dx_d$$

$$= \prod_{j=1}^{d} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}x_j^2} e^{-ix_j\xi_j} \, dx_j = \prod_{j=1}^{d} e^{-\frac{1}{2}\xi_j^2} \quad \checkmark$$

Propiedades de la solución:

$$u(t,x) = (4\pi t)^{-d/2} \int_{\mathbb{R}^d} u_0(y)\, e^{-\frac{|x-y|^2}{4t}}\, dy \qquad \begin{array}{l} t>0 \\ x\in\mathbb{R}^d \end{array}$$

(u_0 continua, acotada)

* Si u_0 integrable,

$$\int_{\mathbb{R}^d} u(t,x)\, dx = \int_{\mathbb{R}^d} u_0(x)\, dx \qquad \forall\ t>0.$$

(conservación de la masa).

Dem: $$\int_{\mathbb{R}^d} u\, dx = (4\pi t)^{-d/2} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u_0(y)\, e^{-\frac{|x-y|^2}{4t}}\, dy\, dx =$$

$$= (4\pi t)^{-d/2} \int_{\mathbb{R}^d} u_0(y) \underbrace{\int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}}\, dx}_{(4\pi t)^{d/2}}\, dy = \int u_0 \ \checkmark.$$

* Si $u_0 \geq 0$ en $\mathbb{R}^d$, $u_0 > 0$ en algún punto

$$\Rightarrow \quad u(t,x) > 0 \quad \forall\, t>0,\ \forall\, x \in \mathbb{R}^d.$$

26 de mayo

$$\left.\begin{array}{ll} \partial_t u = \Delta u & \text{en } t>0,\ x\in\mathbb{R}^d \\ u|_{t=0} = u_0 & \text{en } x\in\mathbb{R}^d \end{array}\right\} \quad u_0 \text{ continua, acotada.}$$

(una) solución: $u(t,x) = u_0 * \phi\,(t,x) = (4\pi t)^{-d/2} \displaystyle\int_{\mathbb{R}^d} u_0(y)\, e^{-\frac{|x-y|^2}{4t}}\, dy$

* $u_0 \geq 0$ en $\mathbb{R}^d \Rightarrow u(t,x) \geq 0$ en $\mathbb{R}^d$.

* $$\left.\begin{array}{l} u_0 \text{ continua,} \\ u_0 \geq 0 \text{ en } \mathbb{R}^d \\ u_0(x_0) > 0 \text{ para algún } x_0\in\mathbb{R}^d \end{array}\right\} \Rightarrow u(t,x) > 0 \quad \forall\, t>0,\ x\in\mathbb{R}^d$$

<u>Dem</u> u_0 continua $\Rightarrow \exists\, r>0,\ \varepsilon>0 \ /\ u_0(x) > \varepsilon \quad \forall\, x\in B(x_0,r)$.

$$u(t,x) = (4\pi t)^{-d/2} \int_{\mathbb{R}^d} u_0(y)\, e^{-\frac{|x-y|^2}{4t}}\, dy \geq (4\pi t)^{-d/2} \int_{B(x_0,r)} \varepsilon\, e^{-\frac{|x-y|^2}{4t}}\, dy > 0.$$

Conservación de la masa (en general):

¿es cierta para todas las soluciones?

$$\frac{d}{dt}\int_{\mathbb{R}^d} u(t,x)\,dx = \int_{\mathbb{R}^d} \partial_t u(t,x)\,dx = \int_{\mathbb{R}^d} \Delta u(t,x)\,dx = 0.$$

$$\int_{B_R} \Delta u(t,x)\,dx = \int_{B_R} \operatorname{div}(\nabla u(t,x))\,dx = \int_{\partial B_R} \underbrace{\nabla u(t,x)}\cdot N(x)\,dS(x)$$

→ necesito que $|\nabla u(t,x)| \to 0$ $|x| \to \infty$.

Cierta para soluciones "que decaigan bien cuando $x \to \infty$".

Unicidad de solución?

* En general la solución clásica del PVI no es única.

* <u>Teo:</u> u_0 continua, acotada, integrable.

  $\exists!$ solución clásica del PVI $u = u(t,x)$ tal que

  $x \mapsto u(t,x)$ es integrable $\forall t \geq 0$.

Idea:

$$\frac{d}{dt}\int_{\mathbb{R}^d} u(t,x)^2\,dx = \int_{\mathbb{R}^d} 2u(t,x)\,\partial_t u(t,x)\,dx = 2\int_{\mathbb{R}^d} u(t,x)\,\overbrace{\Delta u(t,x)}^{div(\nabla u)}\,dx$$

$$= -2\int_{\mathbb{R}^d} \nabla u(t,x)\cdot\nabla u(t,x)\,dx = -2\int_{\mathbb{R}^d} |\nabla u(t,x)|^2\,dx \leq 0$$

Si esto funciona, tomo dos soluciones u, v del mismo PVI.

$w := u - v.$ $\quad \dfrac{d}{dt}\displaystyle\int w^2 = -2\int |\nabla w|^2 \leq 0.$

$$\int w(t,x)^2\,dx \leq \underbrace{\int w(0,x)^2\,dx}_{= 0} \quad \forall t \geq 0.$$

$\Rightarrow w(t,x) = 0 \;\forall t, x \Rightarrow u = v$ ✓.

Ec. calor en un dominio

$\Omega \subseteq \mathbb{R}^d$ abierto,

$$\left.\begin{array}{ll} \partial_t u = \Delta u & \text{en } t>0,\ x \in \Omega \\ u|_{t=0} = u_0 & \text{en } x \in \overline{\Omega}. \\ \star\ u(t,x) = 0 & \text{en } t>0,\ x \in \partial\Omega \end{array}\right\} \text{ problema de contorno.}$$

$\longrightarrow$ Dirichlet

[$\star$ $\nabla u(t,x) \cdot \underbrace{N(x)}_{\text{normal}} = 0$ en $t>0$, $x \in \partial\Omega$ $\longrightarrow$ Neumann]

Solución en $\Omega = (0,L)$:

Sep. variables: buscamos soluciones del tipo:

$$u(t,x) = \varphi(t)\,\psi(x).$$

$$\varphi'(t)\,\psi(x) = \varphi(t)\,\psi''(x), \qquad \psi(0) = \psi(L) = 0.$$

$$\frac{\varphi'(t)}{\varphi(t)} = \frac{\psi''(x)}{\psi(x)}$$

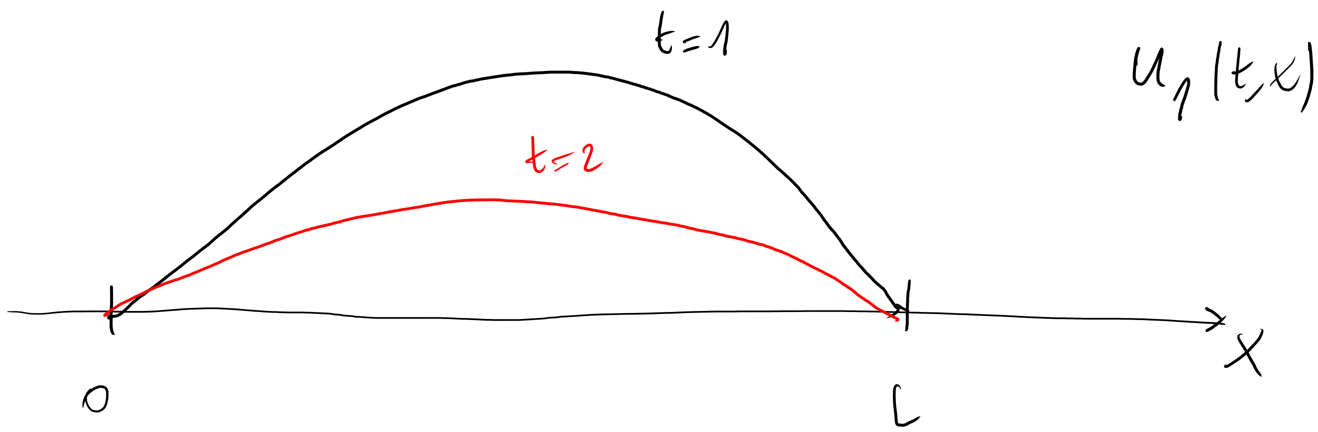
$$\varphi'(t) = -\lambda \varphi(t)$$

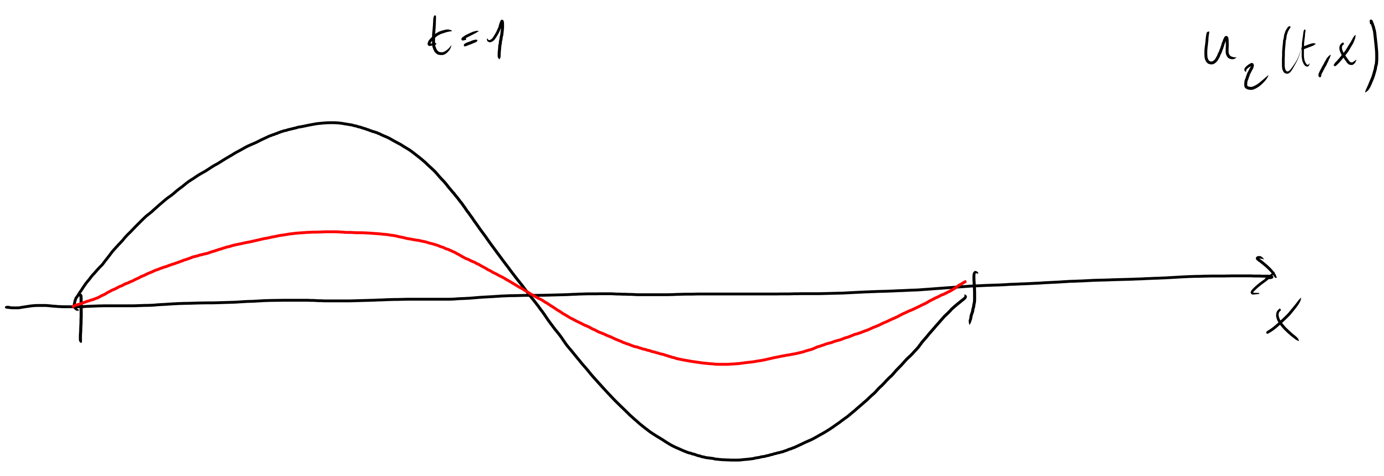
$$\psi''(x) = -\lambda \psi(x), \quad \psi(0) = \psi(L) = 0.$$

$$\Rightarrow \quad \psi(x) = B \operatorname{sen}\left(\frac{n\pi x}{L}\right), \quad \lambda = \frac{n^2\pi^2}{L^2}.$$

$$\varphi(t) = A e^{-\lambda t} = A e^{-\frac{n^2\pi^2}{L^2} t}$$

$$u(t,x) = e^{-\frac{n^2\pi^2}{L^2} t} \operatorname{sen}\left(\frac{n\pi x}{L}\right) = u_n(t,x).$$





Para cumplir cond. inicial $u|_{t=0} = u_0$, buscamos una sol. de la forma:

$$u(t,x) = \sum_{n=1}^{\infty} A_n \, u_n(t,x)$$

$$u(0,x) = \sum_{n=1}^{\infty} A_n \operatorname{sen}\left(\frac{n\pi x}{L}\right) = u_0(x) \quad ?$$

Elijo $A_n \equiv$ coeficientes de serie de Fourier en senos de u_0.

<u>Teo</u>. $u_0 \in \mathcal{C}^3[0,L]$, $u_0(0) = u_0(L) = u''(0) = u''(L) = 0$.

Entonces u, definida arriba, es solución del PVI/de contorno para la ec. del calor en $[0,L]$.

Propiedades:

* La sol. es única.

Este cálculo es riguroso:

$$\frac{d}{dt}\int_\Omega u(t,x)^2\,dx = \int_\Omega 2u(t,x)\,\partial_t u(t,x)\,dx = 2\int_\Omega u(t,x)\,\overbrace{\Delta u(t,x)}^{\operatorname{div}(\nabla u)}$$

$$= -2\int_\Omega \nabla u(t,x)\cdot\nabla u(t,x)\,dx = -2\int_\Omega |\nabla u(t,x)|^2\,dx \le 0$$

→ termino como antes.

1 junio

Ec. calor:

* Sol. explícita en $\mathbb{R}^d$.
* Sol. explícita en $[0,L]$ (serie trigonométrica).

* Ejemplo:

$$\left.\begin{array}{ll} \partial_t u = \partial_x^2 u & \text{en } t>0,\ x\in(0,L). \\ u|_{t=0} = u_0 & x\in[0,L] \\ \underbrace{\partial_x u(t,0) = \partial_x u(t,L) = 0}_{\text{tipo Neumann}} & (t>0) \end{array}\right\}$$

En general, si pedimos:

$$u|_{\partial\Omega} = 0 \equiv \text{Dirichlet}$$

$$\nabla u \cdot N|_{\partial\Omega} = 0 \equiv \text{Neumann}$$

N: normal a $\partial\Omega$.

La ec. del calor en Ω acotado condiciones Neumann:

$$\left.\begin{aligned} &\partial_t u = \Delta u \quad \text{en } t>0,\ x\in\Omega \\ &u|_{t=0} = u_0 \quad x\in\overline{\Omega} \\ &\nabla u(t,x)\cdot N(x) = 0 \quad \forall\, t>0,\ x\in\partial\Omega \end{aligned}\right\}$$

Propiedad: las soluciones clásicas de este problema conservan $\int_\Omega u$:

$$\frac{d}{dt}\int_\Omega u(t,x)\,dx = \int_\Omega \partial_t u(t,x) = \int_\Omega \Delta u(t,x)\,dx = \int_\Omega \operatorname{div}(\nabla u(t,x))\,dx =$$

$$= \int_{\partial\Omega} \overbrace{\nabla u(t,x)\cdot \underbrace{N(x)}_{\text{normal exterior}}}^{=0.}\,dS(x) = 0.$$

$$\left.\begin{array}{lll} \partial_t u = \partial_x^2 u & \text{en} & t>0,\ x\in(0,L). \\ u|_{t=0} = u_0 & & x\in[0,L] \\ \partial_x u(t,0) = \partial_x u(t,L) = 0 & & (t>0) \end{array}\right\}$$

Resolvemos por variables separadas:

tipo Neumann

$$u(t,x) = \varphi(t)\psi(x), \qquad \varphi'(t)\psi(x) = \varphi(t)\psi''(x)$$

$$\frac{\varphi'(t)}{\varphi(t)} = \frac{\psi''(x)}{\psi(x)} \Rightarrow \varphi' = -\lambda\varphi, \quad \psi'' = -\lambda\psi$$

Borde: $\varphi(t)\psi'(0) = \varphi(t)\psi'(L) = 0 \quad \forall\, t>0$

(no trivial) $\Rightarrow \psi'(0) = \psi'(L) = 0.$

$$\left.\begin{array}{l} \psi'' = -\lambda\psi \\ \psi'(0) = \psi'(L) = 0 \end{array}\right\} \qquad \begin{array}{l} \psi(x) = A\cos(\sqrt{\lambda}\cdot x) + B\,\text{sen}(\sqrt{\lambda}\cdot x) \\ \psi'(x) = -\sqrt{\lambda}A\,\text{sen}(\sqrt{\lambda}\cdot x) + B\sqrt{\lambda}\cos(\sqrt{\lambda}\cdot x). \end{array}$$

$$\psi'(0) = B\sqrt{\lambda} = 0 \Rightarrow B = 0.$$

$$\psi'(L) = -\sqrt{\lambda}A\,\text{sen}(L\sqrt{\lambda}) = 0 \Rightarrow \text{sen}(L\sqrt{\lambda}) = 0$$

$$\psi(x) = A\cos\left(\frac{n\pi x}{L}\right) \qquad \lambda = \frac{n^2\pi^2}{L^2}$$

$$\varphi(t) = C\cdot e^{-\frac{n^2\pi^2}{L^2}t}$$

$$u_n(t,x) = e^{-\frac{n^2\pi^2}{L^2}t} \cos\left(\frac{n\pi x}{L}\right), \quad t \geq 0, \; x \in [0,L].$$

Para resolver cond. inicial, busco u de la forma:

$$u(t,x) = \sum_{n=1}^{\infty} A_n u_n(t,x).$$

$$u(0,x) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \rightsquigarrow$$

Elijo $A_n \equiv$ coeficientes de la serie de Fourier en cosenos de u_0.

Teo. $u_0 \in \mathcal{C}^3[0,L]$ con $u_0'(0) = u_0'(L) = 0$.

$\Rightarrow$ la serie anterior converge a una función $\mathcal{C}^1$ en t, $\mathcal{C}^2$ en x, sol. del PVI para la ec. del calor (condiciones Neumann).

Dem. sabemos que $\sum_{n=1}^{\infty} |A_n|^2 n^6 < \infty$.

— resto de la dem. igual.

Ec. reacción química

Queremos escribir modelos para la dependencia en tiempo de las concentraciones de sustancias químicas.

$$A + B + C + \cdots \longrightarrow A' + B' + C' + \cdots$$

$$2A + B \longrightarrow C$$

$$\underbrace{A + 3B}_{\text{reactivos}} \longrightarrow \underbrace{B + C}_{\text{productos.}}$$

Ley de acción de masas

La velocidad de una reacción es proporcional a la concentración de cada uno de los reactivos.

$A, B, C, \ldots \equiv$ especies químicas

$a, b, c, \ldots \equiv$ concentraciones

$a = a(t), \quad b = b(t), \quad c = c(t).$

Ejemplos

$A \xrightarrow{k} B.$

$$\frac{d}{dt} a = -k a \qquad \left[a(t) = a(0)\, e^{-kt} \right]$$

$$\frac{d}{dt} b = k a$$

2 de junio

Ley de acción de masas

La velocidad de una reacción es proporcional a la concentración de cada uno de los reactivos.

* $A \xrightarrow{K} B$ $\qquad \frac{d}{dt}a = -Ka \quad , \quad \frac{d}{dt}b = Ka$

* $A + B \xrightarrow{K} C$ $\qquad \frac{d}{dt}a = -Kab \qquad \frac{d}{dt}b = -Kab \qquad \frac{d}{dt}c = Kab$

* $2A \xrightarrow{K} B$ $\qquad \frac{d}{dt}a = \underbrace{-2Ka^2}_{\text{velocidad de la reacción}} \qquad \frac{d}{dt}b = Ka^2$

* $A \xrightarrow{K} B + C$ $\qquad \frac{d}{dt}a = -Ka \quad , \quad \frac{d}{dt}b = Ka , \quad \frac{d}{dt}c = Ka$

* $A \xrightarrow{k_1} B \xrightarrow{k_2} C$

$$\frac{d}{dt} a = -k_1 a \qquad \frac{d}{dt} b = k_1 a - k_2 b$$

$$\frac{d}{dt} c = k_2 b$$

* $2A + B \xrightarrow{k} C$

$$\left\{ \begin{array}{l} \dfrac{d}{dt} a = -2k a^2 b \\[2ex] \dfrac{d}{dt} c = k a^2 b \end{array} \right. \qquad \frac{d}{dt} b = -k a^2 b$$

* $A \underset{k_2}{\overset{k_1}{\rightleftharpoons}} B + C$

$$\left[H_2O \rightleftharpoons H^+ + OH^- \right]$$

$$\left[H_2O + H_2O \rightleftharpoons H_3O^+ + OH^- \right]$$

$$\left[\begin{array}{l} A \xrightarrow[k_1]{} B + C \\ B + C \xrightarrow[k_2]{} A \end{array} \right]$$

$$\left. \begin{array}{l} \dfrac{d}{dt} a = -k_1 a + k_2 bc \\[2ex] \dfrac{d}{dt} b = k_1 a - k_2 bc \\[2ex] \dfrac{d}{dt} c = k_1 a - k_2 bc \end{array} \right\}$$

* Michaelis-Menten :

$$\underbrace{S}_{\text{"sustrato"}} + \underbrace{E}_{\text{"enzima"}} \underset{K_2}{\overset{K_1}{\rightleftharpoons}} \underbrace{C}_{\text{"compuesto"}} \xrightarrow{K_3} \underbrace{P}_{\text{producto.}} + E$$

$$\frac{d}{dt} s = -K_1 s e + K_2 c$$

$$\frac{d}{dt} e = -K_1 s e + K_2 c + K_3 c$$

$$\frac{d}{dt} c = K_1 s e - K_2 c - K_3 c$$

$$\frac{d}{dt} p = K_3 c$$

* $2A + B \xrightarrow{K} 3A$

$$\frac{d}{dt} a = -2Ka^2b + 3Ka^2b$$

$$= \underbrace{Ka^2b}$$

$$\frac{d}{dt} b = -Ka^2b$$

Propiedades a estudiar:

* Conservación
* Positividad
* Existencia / unicidad
* Asintótico. $(t \to \infty)$

Conservación

* $A \xrightarrow{K} B$ $\qquad \frac{d}{dt} a = -Ka \quad , \quad \frac{d}{dt} b = Ka$

$$\frac{d}{dt}(a+b) = 0 \quad \checkmark$$

* $A + B \xrightarrow{K} C$ $\qquad \frac{d}{dt} a = -Kab \qquad \frac{d}{dt} b = -Kab \qquad \frac{d}{dt} c = Kab$

$$\frac{d}{dt}(a+b+2c) = 0$$

* $2A \xrightarrow{K} B$ $\qquad \frac{d}{dt} a = -2Ka^2 \qquad \frac{d}{dt} b = Ka^2$

$$\frac{d}{dt}(a+2b) = 0.$$

* $A \xrightarrow{K} B + C$ $\qquad \frac{d}{dt} a = -Ka \quad , \quad \frac{d}{dt} b = Ka, \quad \frac{d}{dt} c = Ka$

$$\frac{d}{dt}(2a+b+c) = 0$$

* $A+B \underset{K_2}{\overset{K_1}{\rightleftharpoons}} C+D$

$$\left\{\begin{array}{l} \frac{d}{dt}a = -K_1 ab + K_2 cd \\ \frac{d}{dt}b = -K_1 ab + K_2 cd \\ \frac{d}{dt}c = K_1 ab - K_2 cd \\ \frac{d}{dt}d = K_1 ab - K_2 cd \end{array}\right.$$

$$\frac{d}{dt}(a+b+c+d) = 0.$$

$$\frac{d}{dt}(a+c) = 0$$

$$\frac{d}{dt}(b+d) = 0$$

$\vdots$

Equilibrios

$a(0)=a_0, \quad b(0)=b_0.$

* $A \xrightarrow{K} B$ $\qquad \frac{d}{dt}a = -Ka, \quad \frac{d}{dt}b = Ka$

Eq: $0 = Ka \Rightarrow (a=0, \ b_\infty = \text{const.})$

$a(t)+b(t) = \underbrace{a_0+b_0}_{\text{cond. inicial}}$, espero que $(a(t), b(t)) \to (0, a_0+b_0)$ tal que $a_0+b_0 = b_\infty$.

* $A+B \xrightarrow{K} C$ $\qquad \frac{d}{dt}a = -Kab \qquad \frac{d}{dt}b = -Kab \qquad \frac{d}{dt}c = Kab$

Eq: $Kab = 0 \Leftrightarrow$ $\begin{cases} (a=0, \ b=b_\infty, \ c=c_\infty) \\ (a=a_\infty, \ b=0, \ c=c_\infty) \end{cases}$

Espero que $(a(t), b(t), c(t)) \to (a_\infty, b_\infty, c_\infty)$ equilibrio

con $a_0+b_0+2c_0 = a_\infty+b_\infty+2c_\infty$ $\qquad$ (porque $\frac{d}{dt}(a+b+2c)=0$)

$a_0+c_0 = a_\infty+c_\infty$ $\qquad$ $\left(\frac{d}{dt}(a+c)=0\right)$

$b_0+c_0 = b_\infty+c_\infty$ $\qquad$ $\left(\frac{d}{dt}(b+c)=0\right)$

* si $a_\infty = 0$: $\quad a_0+b_0+2c_0 = b_\infty+2c_\infty$

$a_0+c_0 = c_\infty \qquad a_\infty = 0$

$b_0+c_0 = b_\infty+c_\infty \Rightarrow b_\infty = b_0+c_0-c_\infty = b_0-a_0$

OK cuando $b_0 \geq a_0$.

* si $b_\infty = 0$: OK cuando $a_0 \geq b_0$,

$b_\infty = 0, \quad a_\infty = a_0-b_0, \quad c_\infty = b_0+c_0$

* $2A \xrightarrow{K} B$ $\qquad \frac{d}{dt}a = \underbrace{-2Ka^2}_{\text{velocidad de la reacción}} \qquad \frac{d}{dt}b = Ka^2$

* $A \xrightarrow{K} B+C$ $\qquad \frac{d}{dt}a = -Ka, \quad \frac{d}{dt}b = Ka, \quad \frac{d}{dt}c = Ka$

7 junio

Conservación positividad

Para todos los sistemas basados en ley de acción de masas:

$$A_1 + A_2 + \cdots + A_n \longrightarrow B_1 + B_2 + \cdots + B_m$$

se cumple que

$$\left.\begin{array}{ll} a_i^0 \geq 0 & \forall\, i=1,\dots,n \\ b_j^0 \geq 0 & \forall\, j=1,\dots,m \end{array}\right\} \Rightarrow \begin{array}{lll} a_i(t) \geq 0 & \forall\, i=1,\dots,n, & \forall\, t \geq 0 \\ b_j(t) \geq 0 & \forall\, j=1,\dots,m, & \forall\, t \geq 0. \end{array}$$

¿Por qué?

$$A + B \xrightarrow{K} C \qquad \frac{d}{dt}a = \underbrace{-Kab}_{\text{contiene } a} \qquad \frac{d}{dt}b = \underbrace{-Kab}_{\text{contiene } b}, \quad \frac{d}{dt}c = \underbrace{Kab}_{\geq 0.}.$$

Lema. $x' = f(x)$, $x : [0,T) \to \mathbb{R}^d$, $f : \mathbb{R}^d \to \mathbb{R}^d$

Suponemos que: $x = (x_1, \dots, x_d)$.

$$\left.\begin{array}{l} x_i \geq 0 \quad \forall\, i = 1, \dots, d \\ \exists\, j \,/\, x_j = 0 \end{array}\right\} \Rightarrow f(x) \geq 0$$ (todas sus coordenadas sean ≥ 0).

Entonces:

$$x_i(0) \geq 0 \quad \forall\, i = 1, \dots, d \Rightarrow x_i(t) \geq 0 \quad \forall\, i = 1, \dots, d.$$

Dem (idea). Considero:

$$\left.\begin{array}{l} \frac{d}{dt} x^\varepsilon = f(x^\varepsilon) + \varepsilon \\ x^\varepsilon(0) = x(0) + \varepsilon \end{array}\right\}$$ (sumo ε en cada coordenada).

Estabilidad resp. ecuación y resp. condiciones iniciales:

$$\Rightarrow \quad x^\varepsilon \xrightarrow[\varepsilon \to 0]{} x$$ uniformemente en compactos.

Contradicción: supongo que $\exists$ $t'>0$, $\varepsilon>0$, $j\in\{1,\dots,d\}$ con $\left(X^{\varepsilon}(t')\right)_j < 0$

$\Rightarrow$ $\exists$ $t^*>0$ con:

$$\left.\begin{aligned} &\left(X^{\varepsilon}(t^*)\right)_K = 0 \quad \text{para cierto } K\in\{1,\dots,d\} \\ &\left(X^{\varepsilon}(t)\right)_i \geq 0 \quad \forall\ t\in[0,t^*),\ \forall\ i=1,\dots,d \\ &\left(X^{\varepsilon}(t)\right)_K < 0 \quad \text{para } t\in(t^*,t^*+\delta) \end{aligned}\right\}$$

$$\Rightarrow \frac{d}{dt} X^{\varepsilon}_K(t^*) = \left[f(X^{\varepsilon}(t^*))\right]_K + \varepsilon \geq \varepsilon$$

contradicción.

Existencia / unicidad soluciones

Teo. (Picard-Lindelöf)

$x' = f(x)$, $x : [0,T) \to \mathbb{R}^d$, $f : \mathbb{R}^d \to \mathbb{R}^d$ loc. Lipschitz.

$\Rightarrow \exists!$ solución del PVI $\left.\begin{array}{l} x' = f(x) \\ x(0) = x_0 \end{array}\right\}$ para cada $x_0 \in \mathbb{R}^d$,
máximal

(que no se puede extender — definida en (T_*, T^*) $\left.\begin{array}{l} T_* \in [-\infty, 0) \\ T^* \in (0, +\infty] \end{array}\right\}$

$\star$ Adicionalmente si $T^* < \infty \Rightarrow |x(t)| \to \infty$
cuando $t \to T^*$.

$x' = x$ en $\mathbb{R}$ $\Rightarrow x(t) = Ce^t$

$x' = x^2$ en $\mathbb{R}$ $\Rightarrow \int \frac{1}{x^2}\,dx = t - C$

$-\frac{1}{x} = t - C$

$$\boxed{x = \frac{1}{C-t}}$$

Ejemplos:

* $A \xrightarrow{K} B+C$, $\frac{d}{dt} a = -Ka$, $\frac{d}{dt} b = Ka$, $\frac{d}{dt} c = Ka$.

(fácil porque es lineal).

* $A+B \xrightarrow{K} C$ $\frac{d}{dt} a = -Kab$ $\frac{d}{dt} b = -Kab$, $\frac{d}{dt} c = Kab$.

loc. Lipschitz $\Rightarrow$ existencia, unicidad local en $[0, T^*)$.

Suponemos que $a_0, b_0, c_0 \geq 0$. $\Rightarrow$ $a(t), b(t), c(t) \geq 0$ en $[0, T^*)$.

$$\frac{d}{dt}\left(a + b + 2c\right) = 0$$

$$a(t) + b(t) + 2c(t) = a_0 + b_0 + 2c_0 \quad \forall\ t \in [0, T^*).$$

$\Rightarrow$ $\lim_{t\to T^*} a(t)$, $\lim_{t\to T^*} b(t)$, $\lim_{t\to T^*} c(t)$ no puede ser $+\infty$.

$\lim_{t\to T^*}\left(|a(t)|^2 + |b(t)|^2 + |c(t)|^2\right)$ no puede ser $+\infty$.

$\Rightarrow$ $T^* = +\infty$. ✓

Ejemplos:

$$A \underset{k_2}{\overset{k_1}{\rightleftharpoons}} B + C$$

$\boxed{EC}$

$$\frac{d}{dt} a = -k_1 a + k_2 bc$$

$$\frac{d}{dt} b = k_1 \cdot a - k_2 bc$$

$$\frac{d}{dt} c = k_1 a - k_2 bc$$

$\boxed{\text{Conser.}}$

$$\frac{d}{dt}(2a+b+c) = 0 \qquad \frac{d}{dt}(a+b) = 0 \qquad \frac{d}{dt}(a+c) = 0$$

$\boxed{\text{Conser. positividad}}$

si todas las cantidades son positivas y una de ellas es $0 \Rightarrow$ la derivada de esa, es ≥ 0

suponemos todas no negat.

$$\frac{d}{dt} a = k_2 bc \geq 0$$

$$\frac{d}{dt} b = k_1 a \geq 0$$

$$\frac{d}{dt} c = k_1 a \geq 0$$

conserva positividad porque los unicos terminos negativos estan multiplicados por la propia densidad.

$\boxed{\text{Existencia global en } [0, \infty)}$

Tengo que ver la existencia local

¿es loc. Lipschitz? → las func. C^1 son Lip.
Sí C^∞ también

pero no es lineal → tengo que explotarlo

$\boxed{\text{SI LA SOLUCIÓN NO ES GLOBAL} \Rightarrow \text{CONV} \to +\infty}$

porque conservación masa } ⇒ la solución no
positividad } puede explotar en tiempo finito

$\boxed{\text{Equilibrio}}$

$$\left.\begin{aligned} 0 &= -k_1 a + k_2 bc \\ 0 &= k_1 \cdot a - k_2 bc \\ 0 &= k_1 a - k_2 bc \end{aligned}\right\}$$

$$\boxed{a_\infty = \frac{k_2}{k_1} b_\infty c_\infty} \leftarrow \text{EQUILIBRIO}$$

(Familia con 2 parámetros)

Suponemos $a(0) = a_0$, $b(0) = b_0$, $c(0) = c_0$

$$\left.\begin{aligned} a_\infty &= \frac{k_2}{k_1} b_\infty c_\infty \\ a_\infty + b_\infty &= a_0 + b_0 \\ a_\infty + c_\infty &= a_0 + c_0 \end{aligned}\right\}$$