RESEARCH PROPOSAL

What is the initial state that may lead to cooperation? Is cooperation path-dependent? The idea of this research is to derive a solution concept in the class of correlated extensive games. For that purpose, we will consider three structures of the game and one extension (in networks, but needs to be more thought). In the first case we will study cooperation given an exogenous/correlated initial state of the two-players binary action simultaneous-move game and how this different configurations may change equilibrium predictions. The second game extends the first to two stages, and considers a natural "entanglement" between players after the first simultaneous-move step. Given the result in the first step, player 1 and player 2 need to reach an agreement with players 3 and 4, respectively, in "independent" two-players binary action simultaneous-move games. Finally, the third game is an extension of this two-step game to a finitely-long extensive game to study Poincarè recurrences. That means that after a sufficiently long but finite time, return to a state arbitrarily close to (for continuous state systems), or exactly the same as (for discrete state systems), their initial state.

I would like to clarify that I am open to suggestions, alternative approaches, and different research topics within the areas of game theory and economic theory. While this particular project is one that I find intriguing, it does not fully capture the scope of my interests.

1. BACKGROUND

Quantum game theory extends classical game theory by incorporating the principles of quantum mechanics, enabling the use of quantum randomization and communication devices. This novel approach significantly broadens the strategy space for players, as quantum mechanics allows for observables and correlations that defy classical probability constraints (Dahl and Landsburg, 2011).

In classical games, strategies are conditioned on realizations of classical random variables. However, quantum strategies condition actions on quantum mechanical observables, which are governed by the principles of ambiguity and relationship. This non-classical expansion introduces strategic possibilities that cannot be achieved with classical tools alone.

For example, classical probability follows certain constraints, such as the triangle inequality for probabilities of differences between sequences of variables. Quantum mechanics, however, predicts and demonstrates through laboratory experiments the existence of observables that violate these constraints [1]. Such phenomena enable quantum strategies to achieve outcomes that are otherwise unattainable in classical systems.

In games of complete information, equilibria resulting from quantum strategies correspond to correlated equilibria in the sense of Aumann (Aumann, 1974), though not all correlated equilibria are sustainable as quantum equilibria. The divergence between classical and quantum theories becomes even more pronounced in games of private information (Eisert et al., 1999). In such games, the equivalence between mixed and behavioral strategies, established in classical settings, breaks down in the quantum context (Brandenburger, 2010). This breakdown allows quantum technology to achieve outcomes that are Pareto superior to any achievable classical correlated equilibrium, though not necessarily Pareto optimal (Bostanci and Watrous, 2022).

The potential of quantum strategies lies in their ability to exploit entanglement/relationships and interference/order effects, providing players with unprecedented coordination without direct communication. This capacity, recently established by the results of quantum mechanics in explaining neural systems in humans (see Georgiev, 2021), has profound implications for economics, decision-making, and computational systems, motivating further exploration into the theoretical and practical applications of quantum game theory.

Our premise is that both the degree of relationship an agent has with their interactive counterpart and the sequence of prior actions or situations significantly influence an individual's present state and subsequent behavior. Consequently, the order of events plays a crucial role in shaping how we act in the future. We extend the concept of order effects to a strategic setting where learning is excluded—that is, the game is not played repeatedly until players "learn how to play" it.

[1] See Pothos and Busemeyer, 2022 and Kahneman and Tversky, 1984 for a detailed explanation of the violation of the law of total probability in real life.

The goal of our framework is to explore how past decisions influence present choices and to examine the extent of correlation between players. To achieve this, we transition from epistemic uncertainty—arising from a lack of knowledge and modeled using classical probability—to ontic uncertainty, which reflects the absence of an inherent feature within the system itself. Ontic uncertainty pertains to the fundamental nature of the system, independent of our knowledge about it.

2. METHODOLOGY

There are two important clarifications to make. First, this approach serves as a generalization of the existing game-theoretic framework; it is neither intended to replace nor to improve upon it. Second, the objective of this project is twofold: (1) to explore how an alternative mathematical framework can be integrated into game theory, and (2) to facilitate the study of order effects and define the initial conditions between players prior to the start of the game. These initial conditions are, in my view, critical in determining how the game unfolds and evolves over time.

Consider a 2x2 simultaneous-move prisoner's dilemma game where each player either cooperates (C) or defects (D). Each possible pure strategy is represented by basis vectors (kets):

$$|C\rangle = |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |D\rangle = |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Therefore, $|C\rangle$ and $|D\rangle$ are what we will call states [2]. These unit vectors are called qubits, and they simply encode a player's strategies.

Suppose a player wants to randomize over her strategies. That implies both strategies are in a superposition/linear state:

$$|\psi\rangle = \alpha |C\rangle + \beta |D\rangle \tag{2.1}$$

where $\alpha, \beta \in \mathbb{C}$ and $|C\rangle, |D\rangle \in \mathcal{H}$. These coefficients α and β satisfy $|\alpha|^2 + |\beta|^2 = 1$. These are the probability amplitudes and $|\alpha|^2$ represents the probability of player j cooperating and $|\beta|^2$ the probability of player j defecting. These type of vectors are normalized so the Euclidean norm of is equal to 1.

Suppose we want to compute the probability of player j playing C. In this case, the state is $|\psi\rangle = \alpha |C\rangle + \beta |D\rangle$ and the measurement (what we observe once the player has taken an action) is performed in an orthonormal basis $|b_j\rangle$ where b_j is each action player j can take. The probability amplitude to observe player j playing $C \in b_j$ is given by $\langle C| |\psi\rangle = \alpha$ [3], and thus, the probability of observing C is $\mathbb{P}(C) = |\alpha|^2 = a_j$. As we are going to work with finite set of actions, equation 3.1. can be written as:

$$|\psi\rangle = \sum_i c_i |b_i\rangle \tag{2.2}$$

[2] There exist other possible states such as $|CC\rangle$ or $|CD\rangle$, among others.

[3] The vector $\langle C|$ is the row vector of $|C\rangle$ and it is called bra. Because we deal with complex numbers, we need to apply the complex conjugate of the column vector: $\langle C| = |\psi\rangle^\dagger = (|\psi\rangle^T)^*$. First transpose and then complex conjugate the imaginary part.

where i indicates the actions player j has at her disposal, that is, $\{C, D\}$; $c_i = \{\alpha, \beta\}$ and $|b_i\rangle$ the basis vectors of each possible action.

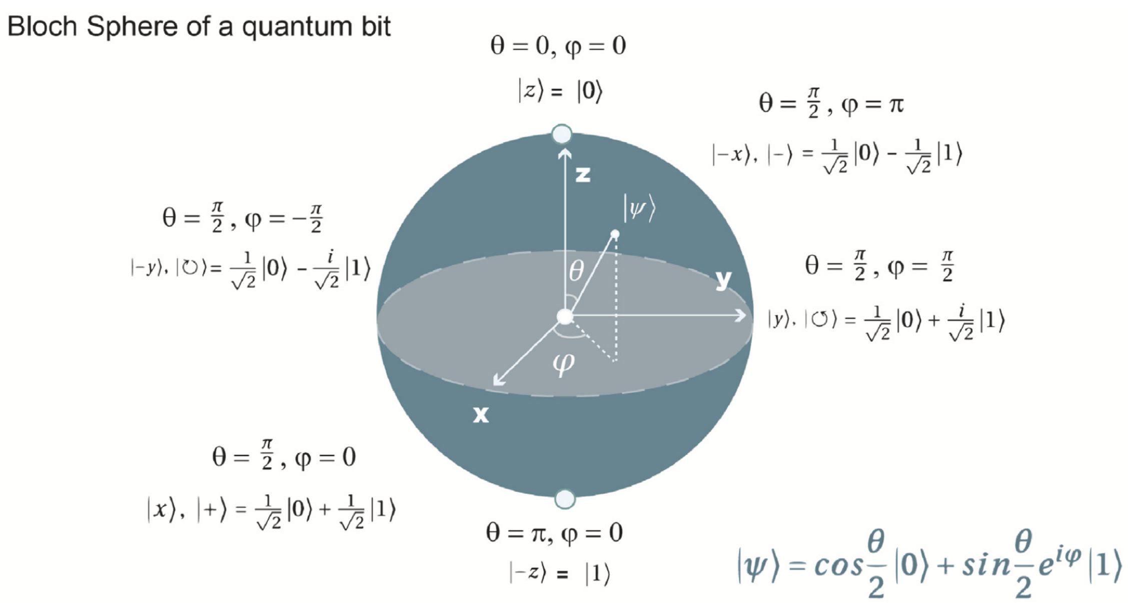
Each decision, or set of decisions, are encoded by matrices called unitary operators satisfying: $U^\dagger U = \mathbb{I} = UU^\dagger$. The reason for them is twofold: (1) the eigenvalues of these matrices are real, and thus, have intuitive meaning, and (2) total probability is preserved. Suppose the state/situation is: $|\psi\rangle = \alpha\,|C\rangle + \beta\,|D\rangle$ and after applying a unitary operator U, the new state is $|\psi'\rangle$. Then, total probability is given by:

$$||\psi'||^2 = \langle\psi'|\,|\psi'\rangle = \langle\psi|\,U^\dagger U\,|\psi\rangle = \langle\psi|\,|\psi\rangle \tag{2.3}$$

These operators have an intuitive interpretation depending on what the player(s) wants to do [4]. For binary-action cases, we can use a bloch spehere in order to understand the meaning of the different unitary operators (commonly referred to as gates in information theory terminology)). We know the player has two possible pure actions which fit in the poles of the sphere (see Figure 3.1 for illustrative purposes). This space gives us all the possible future states that can be reached by a player after applying a unitary operator to his strategy vector.

Figure 2.1

Bloch sphere: state space of a single qubit (binary-action case)



Source: Kharsa, Bouridane & Amira (2022).
Note: Remember $|C\rangle = |0\rangle$ and $|D\rangle = |1\rangle$.

[4]See Appendix in LINK GITHUB for an exhaustive table explaining all these matrices, their rationale and applications.

2.1. Approach

2.1.1. Game 1

Consider a game with two players $i = \{1, 2\}$ and binary-action sets $A^1 = \{C, D\} = A^2$. That is, players are given these "qubits" they can manipulate. The stage-game payoffs are:

		Player 2	
		C	D
Player 1	C	$3, 3$	$-1, 5$
	D	$5, -1$	$6, 6$

The idea of this game is to consider different initial situations/states and see how equilibrium predictions change. For instance, consider a case in which both players want to cooperate and so before the game starts, they "manipulate" their qubits in that way. Then, each player can apply a rotation gate like $R_y(\theta)$ to their respective qubits to create a weighted uncertainty between cooperation $|0\rangle$ and defection $|1\rangle$ where the weight reflects their preference for cooperation [5]. For each player:

$$R_y(\theta_i)\,|0\rangle = \cos\frac{\theta_i}{2}\,|0\rangle + \sin\frac{\theta_i}{2}\,|1\rangle \tag{2.4}$$

However, to ensure that both players are correlated in their cooperation preference, strategies may be entangled using a CNOT gate after applying a superposition (randomization) gate like Hadamard to one of the qubits. That is, we get an initial Bell State:

1. $$H\,|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \Longrightarrow |\psi^1_{initial}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle \tag{2.5}$$

2. $$CNOT\,\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |\psi^2_{initial}\rangle \tag{2.6}$$

where if one player cooperates $|0\rangle$, the other is guaranteed to cooperate $|0\rangle$, and vice versa. Players can now rationally fine-tune their preferences for cooperation or defection by applying phase shifts or additional rotations.

2.1.2. Game 2

Extends the previous game but now player 1 and 2, in a second stage, have to reach cooperation with other players 3 and 4, respectively. The idea is exactly the same, but now the initial state in the second stage is defined by how player 1 and 2 played before, and is unknown by players

[5] If $\theta_1 = \theta_2 = \pi/4$, each qubit is 70% likely to be in $|0\rangle$ (cooperation) and 30% likely to be in $|1\rangle$ (defect).

3 and 4. Players 3 and 4 know players 1 and 2 played before, but they do not know how the game ended. They may infer, by rationality, how it finished, but they do not know each player's initial state in the first stage.

Suppose the first stage is not about cooperation-defection but a zero-sum game where one player "losses" and the other "wins":

		Player 2: E	Player 2: F
Player 1	A	$-1, 1$	$2, -2$
	B	$-2, 2$	$-4, 4$

And the subsequent games:

		Player 3: C	Player 3: D
Player 1	C	$3, 3$	$-1, 5$
	D	$5, -1$	$6, 6$

		Player 4: C	Player 4: D
Player 2	C	$3, 3$	$-1, 5$
	D	$5, -1$	$6, 6$

2.1.3. Game 3

This is an extension of Game 1 but played long but finitely many times to study Poincarè recurrences. This procedure requires simulation and the idea is to predefine an algorithm given the initial state. In the process, we will apply different types of punishments together with the allowed unitary operators to find consistent equilibria. The steps to study Poincarè recurrencies are:

1. Diagonalize unitary operator to find its eigenvalues and if all are rational multiples of 2π recurrence is guaranteed.
2. Simulate the game and track $| \, |\psi\rangle_{initial} \, | \, |\psi(t)\rangle\rangle|^2$.
3. Verify recurrence: $(| \, |\psi\rangle_{initial} \, | \, |\psi(T)\rangle\rangle|^2) \to 1$ for some T.

3. EXPECTED RESULTS AND CONTRIBUTIONS

A preliminary hypothesis is that the solution concept for this game involves equilibrium strategies being dependent on the application of unitary operators (the allowed actions, often referred to as time evolution operators) to the game. Specifically, the strategies derived from the time evolution operators correspond to Nash equilibria.

In extending this idea to a finitely-long correlated extensive-form game, an important observation is that if this hypothesis holds, players, at each stage, may perceive their actions as conscious choices, yet the equilibrium itself exhibits path dependence. This means that what might be considered "deviations" from rationality are integral to reaching the equilibrium. Furthermore, these path dependencies are influenced by the initial state, and for any initial state leading to an equilibrium, a Poincarè recurrence emerges.

Finally, another expected result is that this approach can also work as a refinement of the forward induction requirement to select subgame perfect Nash equilibria. The Nash equilibrium that could be obtained as a result of applying a unitary operation is indeed an equilibrium.

This project is intended as a generalization of the existing game-theoretic framework, rather than a replacement or improvement. Its primary objectives are twofold: (1) to investigate how an alternative mathematical framework can be incorporated into game theory, and (2) to facilitate the analysis of order effects and the definition of initial conditions between players prior to the game. These initial conditions, I argue, play a crucial role in shaping how the game progresses and evolves over time.

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APPENDIX

This section aims to present a rationale for considering an alternative mathematical approach when studying specific situations.

3.1. Law of total probability

What we call "interference" is merely a breakdown of the law of total probability. Let's consider the example in Pothos & Busemeyer (2022). Suppose we disclose to a decision maker some preliminary information and then we compare two conditions. In the first condition we simply measure event B whereas in the second condition we first measure event A and then event B. Given this situation, according to the classical/bayesian probability theory (CPT), the total probability is given by:

$$P(B) = P(A\&B) + P(\sim A\&B) \tag{3.1}$$

Now suppose we study this situation under a quantum probability theory (QPT). The decision maker, after being given the preliminary information, her state is represented by the vector $|\psi\rangle$ [6]. In this first case, a positive outcome from the measurement [7] of B is represented by a projector P_B which projects the state vector $|\psi\rangle$ onto the subspace representing B by the matrix product $P_B \cdot |\psi\rangle$. The probability of a positive outcome to action B then equals the squared length (probability amplitude) $\| P_B \cdot |\psi\rangle \|^2$. Now consider the positive outcome of observing A. As we defined in our law of total probability the event "not A" $\sim A$, let's consider $I - P_A$ the orthogonal projector that projects onto the negative outcome for A. Thus, rewriting the projector for B we get $P_B = P_B P_A + P_B(I - P_A)$ and then:

$$\begin{aligned} \| P_B |\psi\rangle \|^2 &= \| P_B P_A |\psi\rangle + P_B(I - P_A) |\psi\rangle \|^2 \\ &= \| P_B P_A |\psi\rangle \|^2 + \| P_B(I - P_A) |\psi\rangle \|^2 + \Delta \end{aligned} \tag{3.2}$$

where Δ is the sum of cross products produced by squaring the sum, and this is what we called the interference term. It takes 0 if the measures/observations are compatible and the QPT satisfies the CPT law of total probability. If Δ is different from 0 (either positive or negative), that means the measures are incompatible [8], and the CPT law of total probability is violated. To

[6] These vectors are called kets and are simply column vectors. The word state simply means either A or B so the unit vector/ket representing each one can be denoted as $|A\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|B\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

[7] Positive outcome means that B has been observed, that is, a player j has played action $a^j = B$.

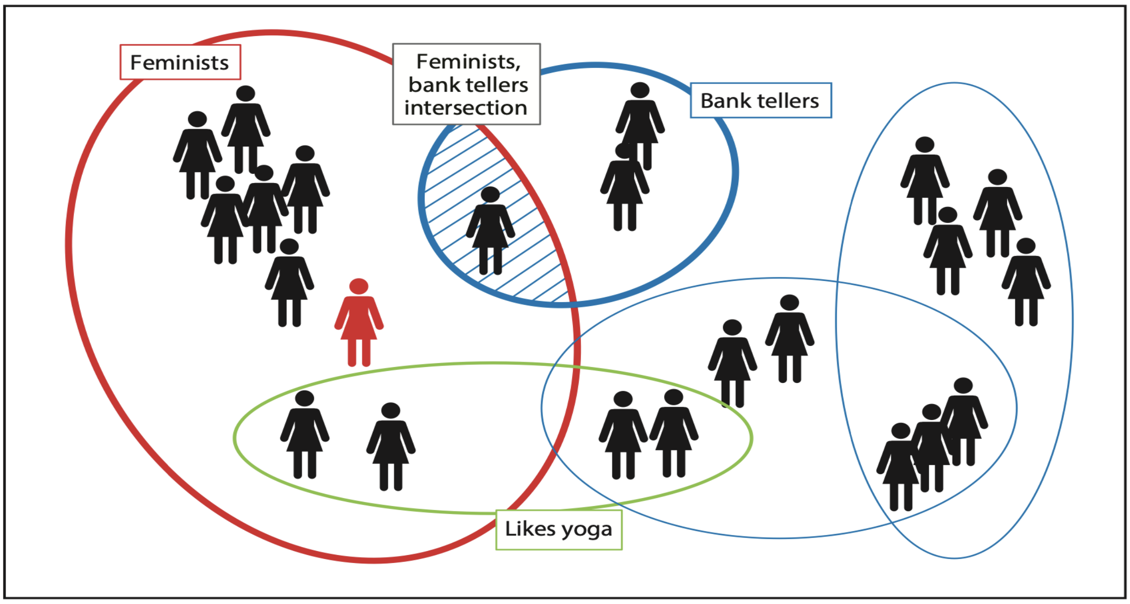
[8] Decisions (or measures) are incompatible if the presence of one decision problem alters our perception of a subsequent decision problem.

emphasize the importance of the order, we will usually write $P(A\&thenB)$ for $P(A\&B)$. Thus, conjunction is no longer conmutative as it is in the CPT. For the sake of clarity, $\| P_B P_A |\psi\rangle \|^2$ is $P(A\&thenB)$, and $\| P_B(I - P_A) |\psi\rangle \|^2$ is $P(\sim A\&B)$.

3.1.1. Linda Problem

To motivate the use of the QPT, let's use the famous Linda problem stated in Tversky & Kahneman (1983) and extrapolated to the QPT context by Pothos & Busemeyer (2022). For this situation, CPT begins with a sample space such as:

Figure 3.1
Linda problem



Source: Pothos & Busemeyer (2022)

which contains all the various possible realizations for Linda. In the experiment, participants were told of a hypothetical person, Linda, who was described as looking like a feminist but not a bank teller. Participants were asked to rank-order the likelihood of several statements about Linda. The critical statements were that Linda is a feminist (F), Linda is a bank teller (BT), and Linda is a feminist and a bank teller. The controversy arose because the results concluded the following:

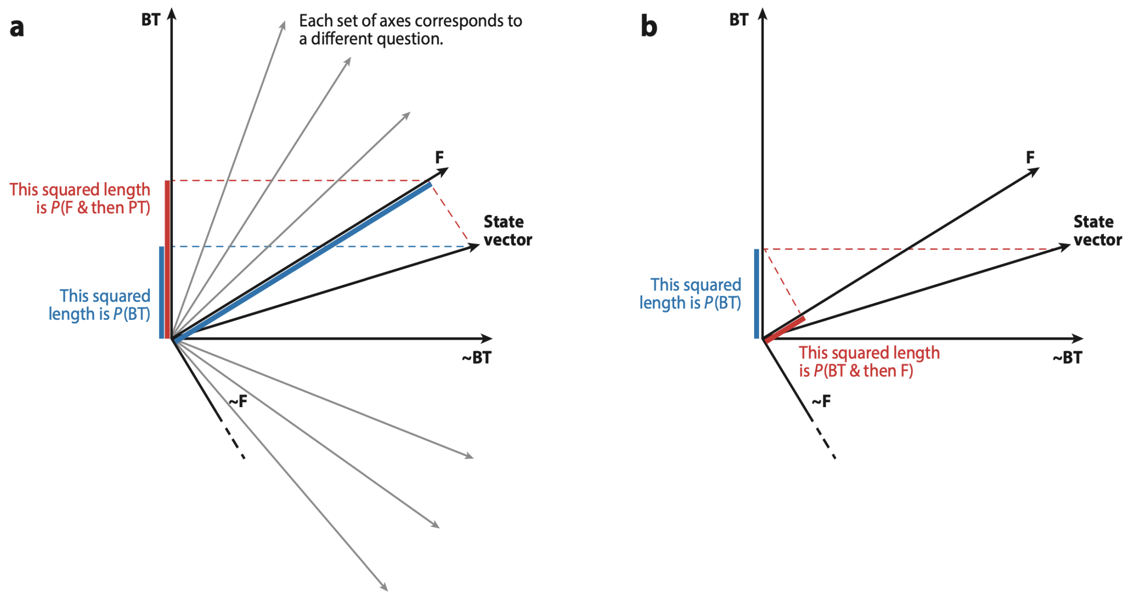
$$P(F) > P(F\&BT) > P(BT) \tag{3.3}$$

A possible outcome of a question, such as whether or not Linda is a feminist, is represented by a subset of the sample space. The outcome from a pair of questions, such as whether Linda is a feminist and bank teller, is represented by the intersection of subsets, as in the shaded region of overlap between the red and blue ellipses in Figure 2.1. The beliefs a person has about these questions are represented by a probability function that assigns a probability to each subset. For example, the probability that Linda is a feminist is the probability assigned to the red ellipse. The probabilities assigned to the union of mutually exclusive events must add. The larger the subset for a question outcome is, the more possible Lindas we can imagine consistent with this question outcome, and the more likely this question outcome will be—that is, the probability of a question outcome depends on the size of the corresponding subset.

On the other hand, QPT begins with a vector space, such as the following two-dimensional space:

Figure 3.2

Projection in a vector space



Source: Pothos & Busemeyer (2022)

Such a vector space contains all possible answers for questions about Linda. For example, for the question "Is Linda a bank teller?" there are two potential answers (yes or no) [9], represented by two unit-length vectors at a 90° angle to each other [10], forming the usual two-dimensional Cartesian plane. The answers to a different question, like "Is Linda feminist?", can be represented by a different pair -yes or no question- of orthogonal vectors rotated by some angle. Basically, **QPT is a way to assign probabilities to subspaces whereas CPT assigns probabilities to subsets of elements**.

The set of beliefs a person has about these questions is represented by a (unit length) state vector $|\psi\rangle$. The probability of a question outcome is obtained by projecting the state vector onto the subspace representing the answer, and then computing the squared length. For the purpose of this project, to compute the conjunction of two question outcomes (or decision-problems), we typically have to employ a sequential projection, which corresponds to resolving one question/decision-problem after the other. Suppose we are interested in $P(F\&BT)$. In this environment, we need to compute $P(F\&thenBT)$ which requires projecting the state vector onto the F ray and then, projecting this previous projection onto the BT ray. The probability

[9] Where yes is BT in the y-axis and no is $\sim BT$ in the x-axis.

[10] These answers are mathematically and intuitively orthogonal to each other.

amplitude of this last projection represents $P(F\&thenBT) = P(F\&BT)$. Working in the other direction, we conclude the nonconmutativity property of QPT (sequence of projections):

$$P(F\&thenBT) \neq P(BT\&thenF) \tag{3.4}$$

which subsequently leads to non-zero interference effects. Moreover, every time we resolve a question/decision-problem, the state vector has to change in a specific way; the vector collapses.

3.2. Decision-making

This table captures mostly all the possible actions a player can take and their intuition. These actions are called "gates".

Table 3.1

Quantum Gates and Their Applications: Strategic Decision-Making Process

Quantum Gates (Strategies)	Application	Rationale	Intuition	Mathematical Representation
Identity (I)	Single-qubit gate	"Let's do nothing" (status quo; default choice)	Preserve my initial state. If at the beginning I was inclined to cooperate, let's stick to C.	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
Pauli Gate X - NOT gate (X)	Single-qubit gate	"Switch my choice: If intended to cooperate, I'll defect instead, and vice versa."	When a player believes that doing the opposite of what was intended yields higher payoffs or as a counter to the opponent's expected cooperation/defection.	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Pauli Gate Z (Z)	Single-qubit gate	"I am unsure what to play at the beginning (initial state in mixed strategies/superposition) so let's introduce a phase in defection and change how it interferes with cooperation when the other player moves".	The phase of β determines how $\|1\rangle$ (defection) interferes with $\|0\rangle$ (cooperation). The player leaves conditional on the other player's strategy whether she will play defection or not.	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Pauli Gate Y (Y)	Single-qubit gate	"Switch my choice between cooperation and defection, but also introduce a phase to influence how my strategy interacts with the opponent's strategy in superpositions or entangled states".	If I expect my opponent to be in mixed strategies/superposition, I'll introduce a complex phase that changes how interference occurs.	$\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$

Quantum Gates (Strategies)	Application	Rationale	Intuition	Mathematical Representation
Hadamard (H)	Single-qubit gate	“Let me hedge my bets and mix cooperation and defection equally.”	The player is unsure whether the opponent will cooperate or defect and chooses a balanced strategy.	$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$
Rotation Rx (Rx)	Single-qubit gate	"I want to test a strategy favoring defection (see bloch sphere where x-axis is located and defection $\lvert 1\rangle = \lvert D\rangle$) but keep cooperation plausible so I apply a small rotation angle".	Allows for gradual shifts in probabilities between two strategies along the x-axis.	$\begin{bmatrix} cos(\frac{\theta}{2}) & -isin(\frac{\theta}{2}) \\ -isin(\frac{\theta}{2}) & cos(\frac{\theta}{2}) \end{bmatrix}$
Rotation Ry (Ry)	Single-qubit gate	"I want to randomize (keep both options open), but I favor more cooperation rather than defection".	A player applies $R_y(\pi/4)$ to create a 70-30 split in favor of cooperation.	$\begin{bmatrix} cos(\frac{\theta}{2}) & sin(\frac{\theta}{2}) \\ -sin(\frac{\theta}{2}) & cos(\frac{\theta}{2}) \end{bmatrix}$
Rotation Rz (Rz)	Single-qubit gate	"I do not want to change the probabilities with which I am willing to play cooperation-defection, but I want to adjust interference patterns when interacting with my opponent".	If the initial state is entangled, a phase shift can realign the strategy correlations between players.	$\begin{bmatrix} exp(\frac{-i\theta}{2}) & 0 \\ 0 & exp(\frac{i\theta}{2}) \end{bmatrix}$
CNOT	Multi-qubit gate	“If my choice is this, force my opponent to follow.”	Introduces a dependence between two players’ strategies. Models retaliation strategies.	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Quantum Gates (Strategies)	Application	Rationale	Intuition	Mathematical Representation
SWAP	Multi-qubit gate	"Exchange strategies between two players."	Useful for interchanging roles or payoffs.	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
Controlled-phase	Multi-qubit gate	"I want to subtly influence how both of us defecting contributes to the overall outcome.".	Creates strategic entanglement based on mutual choices.	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\phi} \end{bmatrix}$
Toffoli	Multi-qubit gate	Conditional on two players cooperating, force a third player to act.	Three-way strategic dependence.	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$
Entangling Gates	Multi-qubit gate	"Correlate my strategy with my opponent's to ensure we cooperate or defect together".	Generate correlations between players' strategies. Used when players' payoffs are interdependent.	Not a single matrix that entangle.

These strategies represent what each player can do given their allowed actions. Single-qubit gates refer to binary-action situations whereas multi-qubit gates refer to strategies applied in situations in which for instance, both players are inclined to cooperate: $|CC\rangle = |C\rangle \otimes |C\rangle$. That is, multi-qubit gates are applied to "entangle" qubits, create conditional effects, or encode shared strategies.