Module 3: Flow of Fluids and Bernoulli's Equation (CIVL 318)

Volume Flow Rate:	$Q = Av$
Mass Flow Rate:	$M = \rho Q$
Weight Flow Rate:	$W = \gamma Q$
Continuity Equation:	$A_1 v_1 = A_2 v_2$
Bernoulli's Equation:	$\frac{p_A}{\gamma} + z_A + \frac{v_A^2}{2g} = \frac{p_B}{\gamma} + z_B + \frac{v_B^2}{2g}$

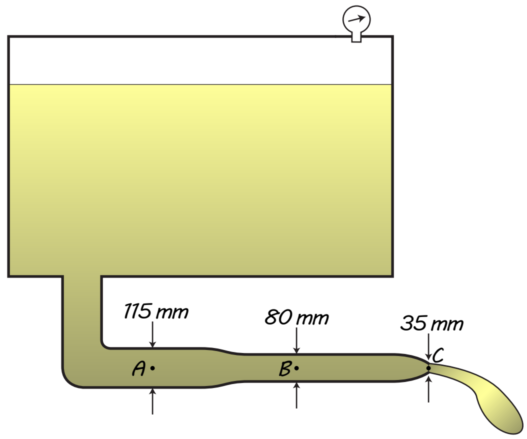
Table G: Dimensions of Steel Tubing

Outside Diameter (in)	Outside Diameter (mm)	Wall Thickness (mm)	Inside Diameter (mm)
$\frac{1}{8}$	3.18	0.813	1.549
		0.889	1.397
$\frac{3}{16}$	4.76	0.813	3.137
		0.889	2.985
$\frac{1}{4}$	6.35	0.889	4.572
		1.24	3.861
$\frac{5}{16}$	7.94	0.889	6.160
		1.24	5.448
$\frac{3}{8}$	9.53	0.889	7.747
		1.24	7.036
$\frac{1}{2}$	12.70	1.24	10.21
		1.65	9.46
$\frac{5}{8}$	15.88	1.24	13.39
		1.65	12.57
$\frac{3}{4}$	19.05	1.24	16.56
		1.65	15.75
$\frac{7}{8}$	22.23	1.24	19.74
		1.65	18.92
1	25.40	1.65	22.10
		2.11	21.18
$1\frac{1}{4}$	31.75	1.65	28.45
		2.11	27.53
$1\frac{1}{2}$	38.10	1.65	34.80
		2.11	33.88
$1\frac{3}{4}$	44.45	1.65	41.15
		2.11	40.23
2	50.80	1.65	47.50
		2.11	46.587

Table F: Schedule 40 Steel Pipe

Nominal Size (in)	Inside Diameter (mm)	Nominal Size (in)	Inside Diameter (mm)
$\frac{1}{8}$	6.8	4	102.3
$\frac{1}{4}$	9.2	5	128.2
$\frac{3}{8}$	12.5	6	154.1
$\frac{1}{2}$	15.8	8	202.7
$\frac{3}{4}$	20.9	10	254.5
1	26.6	12	303.2
$1\frac{1}{4}$	35.1	14	333.4
$1\frac{1}{2}$	40.9	16	381.0
2	52.5	18	428.7
$2\frac{1}{2}$	62.7	20	477.9
3	77.9	24	574.7
$3\frac{1}{2}$	90.1		

Example 1:



The average velocity of the flow at the nozzle C is 4.7 m/s. Determine:

- the average flow velocity at A
- the average flow velocity at B
- the volume flow rate, Q, through the system in L/s.

Solution:

$$A_A v_A = A_C v_C \Rightarrow v_A = \frac{A_C}{A_A} v_C$$

$$v_A = \frac{\pi(0.035\,\text{m})^2/4}{\pi(0.115\,\text{m})^2/4}(4.7\,\text{m/s})$$

$$= \frac{(0.035\,\text{m})^2}{(0.115\,\text{m})^2}(4.7\,\text{m/s})$$

$$= 0.43535\,\text{m/s}$$

$$= 0.435\,\text{m/s}$$

$$A_B v_B = A_C v_C \Rightarrow v_A = \frac{A_B}{A_A} v_B$$

$$v_B = \frac{\pi(0.035\,\text{m})^2/4}{\pi(0.080\,\text{m})^2/4}(4.7\,\text{m/s})$$

$$= \left(\frac{0.035\,\text{m}}{0.080\,\text{m}}\right)^2 (4.7\,\text{m/s})$$

$$= 0.89961\,\text{m/s}$$

$$= 0.900\,\text{m/s}$$

$$Q = A_C v_C = \pi(0.035\,\text{m})^2/4 \times (4.7\,\text{m/s})$$

$$= 0.0045219\,\text{m}^3/\text{s}$$

$$= 4.52\,\text{L/s}$$

Example 2:

Water, at 70°C flows through $\frac{7}{8}$-in. steel tubing, with 1.65 mm wall thickness, at an average velocity of 5.7 m/s. Determine:

- the volume flow rate, Q
- the mass flow rate, M
- the weight flow rate, W

Solution:

$$Q = Av$$

$$= \frac{\pi(0.01892)^2}{4}(5.7\,\text{m/s})$$

$$= 0.0016025\,\text{m}^3/\text{s}$$

$$= 1.603\,\text{L/s}$$

$$M = \rho_{70°\text{C}}(0.0016025\,\text{m}^3/\text{s}$$

$$= \left(978\,\text{kg/m}^3\right)\left(0.0016025\,\text{m}^3/\text{s}\right)$$

$$= 1.5673\,\text{kg/s}$$

$$= 1.567\,\text{kg/s}$$

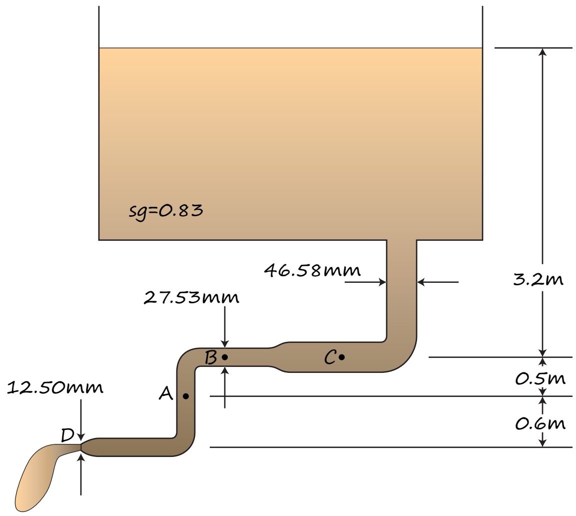
$$W = \gamma_{70°\text{C}}(0.0016025\,\text{m}^3/\text{s}$$

$$= \left(9.59\,\text{kN/m}^3\right)\left(0.016025\,\text{m}^3/\text{s}\right)$$

$$= 0.015368\,\text{kN/s}$$

$$= 15.37\,\text{N/s}$$

Example 3:



Oil, with a specific gravity of 0.83, flows under gravity from a tank, through a pipe system as shown, before entering the atmosphere through a nozzle at D. Determine:

- the pressure at A
- the pressure at B
- the pressure at C
- the volume flow rate through the system

Solution:

We begin by applying Bernoulli's Equation between a point S on the surface of the tank and the nozzle D. (These two points are chosen since the pressure and elevations at each are known. We also know the velocity head at S so the only unknown is the velocity at D.)

$$\begin{aligned}
\frac{p_S}{\gamma} + z_S + \frac{v_S^2}{2g} &= \frac{p_D}{\gamma} + z_D + \frac{v_D^2}{2g} \\
0 + 4.3 + 0 &= 0 + 0 + \frac{v_D^2}{2g} \\
v_D^2 &= 4.3(2 \times 9.81) \\
v_D &= \sqrt{84.366} = 9.1851\,\mathrm{m/s}
\end{aligned}$$

Then, using the continuity equation, we can find the velocities at A and B, and at C.
(Velocities at A and B are the same since the diameter of the tubing is the same.)

$$\begin{aligned}
A_D v_D &= A_A v_A \\
\Rightarrow v_A &= \frac{A_D v_D}{A_A} \\
&= \frac{\pi(0.0125\,\mathrm{mm})^2/4}{\pi(0.0273\,\mathrm{mm})^2/4} \cdot (9.1851\,\mathrm{m/s}) \\
&= 1.8936\,\mathrm{m/s} \\
\Rightarrow v_B &= 1.8936\,\mathrm{m/s}
\end{aligned}$$

$$\begin{aligned}
v_C &= \frac{A_D v_D}{A_C} \\
&= \frac{\pi(0.0125\,\mathrm{mm})^2/4}{\pi(0.04658\,\mathrm{mm})^2/4} \cdot (9.1851\,\mathrm{m/s}) \\
&= 0.66146\,\mathrm{m/s}
\end{aligned}$$

Now, we can use Bernoulli's Equation between the surface and A or between A and the nozzle D to find p_A. I use the surface simply because it has no velocity head:

$$\begin{aligned}
\frac{p_S}{\gamma} + z_S + \frac{v_S^2}{2g} &= \frac{p_A}{\gamma} + z_A + \frac{v_A^2}{2g} \\
0 + 3.7 + 0 &= \frac{p_A}{0.83 \times 9.81} + 0 + \frac{(1.8936)^2}{2 \times 9.81} \\
p_A &= 0.83 \times 9.81 \left(3.7 - \frac{(1.8936)^2}{2 \times 9.81}\right) \\
p_A &= 28.638\,\mathrm{kPa} \\
\boldsymbol{p_A} &= \mathbf{28.6\,kPa}
\end{aligned}$$

This procedure, using the surface and B, can be repeated to find p_B. Alternatively, using A and B is a good choice since they have the same velocity and the velocity head terms cancel out of Bernoulli's Equation:

$$\begin{aligned}
\frac{p_B}{\gamma} + z_B + \cancel{\frac{v_B^2}{2g}} &= \frac{p_A}{\gamma} + z_A + \cancel{\frac{v_A^2}{2g}} \\
\frac{p_B}{0.83 \times 9.81} + 0.5 &= \frac{p_A}{0.83 \times 9.81} + 0 \\
p_B &= p_A - 0.5\gamma \\
&= 28.638 - 0.5(0.83 \times 9.81) \\
p_B &= 24.567\,\mathrm{kPa} \\
\boldsymbol{p_B} &= \mathbf{24.6\,kPa}
\end{aligned}$$

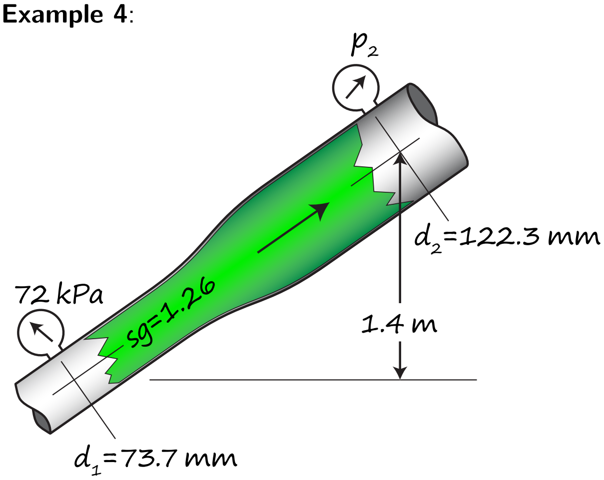
Using the surface and C, we get:

$$\begin{aligned}
\frac{p_S}{\gamma} + z_S + \frac{v_S^2}{2g} &= \frac{p_C}{\gamma} + z_C + \frac{v_C^2}{2g} \\
0 + 3.2 + 0 &= \frac{p_C}{0.83 \times 9.81} + 0 + \frac{(0.66146)^2}{2 \times 9.81} \\
p_C &= 0.83 \times 9.81 \left(3.2 - \frac{(0.66146)^2}{2 \times 9.81} \right) \\
p_C &= 25.874\,\mathrm{kPa} \\
\boldsymbol{p_C} &\boldsymbol{= 25.9\,\mathrm{kPa}}
\end{aligned}$$

To find the volume flow rate, use $Q = Av$ at any of the points where A and v are known:

$$\begin{aligned}
Q &= A_D v_D \\
&= \frac{\pi (0.0125\,\mathrm{m})^2}{4} \cdot (9.1851\,\mathrm{m/s}) \\
&= 0.0011272\,\mathrm{m}^3/\mathrm{s} \\
\boldsymbol{Q} &\boldsymbol{= 1.127\,\mathrm{L/s}}
\end{aligned}$$

Example 4:



Determine the pressure reading p_2 if $Q = 25\,\mathrm{L/s}$

Solution:

Find the cross-sectional areas at 1 and 2:

$$\begin{aligned} A_1 &= \frac{\pi(0.0737\,\mathrm{m})^2}{4} \\ &= 0.0042660\,\mathrm{m}^2 \end{aligned}$$

$$\begin{aligned} A_2 &= \frac{\pi(0.1223\,\mathrm{m})^2}{4} \\ &= 0.011747\,\mathrm{m}^2 \end{aligned}$$

Find the velocities at 1 and 2 for $Q = 25\,\mathrm{L/s}$:

$$\begin{aligned} v_1 &= \frac{Q}{A_1} \\ &= \frac{0.025\,\mathrm{m}^3/\mathrm{s}}{0.0042660\,\mathrm{m}^2} \\ &= 5.8603\,\mathrm{m/s} \end{aligned}$$

$$\begin{aligned} v_2 &= \frac{Q}{A_2} \\ &= \frac{0.025\,\mathrm{m}^3/\mathrm{s}}{0.011747\,\mathrm{m}^2} \\ &= 2.1282\,\mathrm{m/s} \end{aligned}$$

Calculate the velocity heads at 1 and 2 for $Q = 25\,\mathrm{L/s}$:

$$\begin{aligned} \frac{v_1^2}{2g} &= \frac{(5.8603)^2}{2 \times 9.81} \\ &= 1.7504\,\mathrm{m} \end{aligned}$$

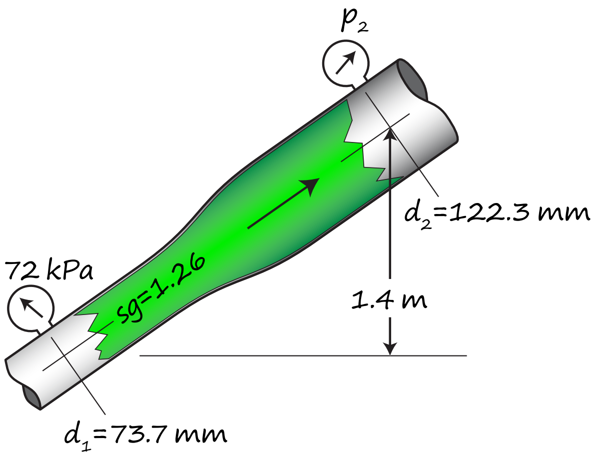
$$\begin{aligned} \frac{v_2^2}{2g} &= \frac{(2.1282)^2}{2 \times 9.81} \\ &= 0.23085\,\mathrm{m} \end{aligned}$$

Applying Bernoulli's Equation between 1 and 2:

$$\begin{aligned} \frac{p_1}{\gamma} + z_1 + \frac{v_1^2}{2g} &= \frac{p_2}{\gamma} + z_2 + \frac{v_2^2}{2g} \\ \frac{72}{1.26 \times 9.81} + 0 + 1.7504 &= \frac{p_2}{1.26 \times 9.81} + 1.4 + 0.23085 \\ 7.5754 &= \frac{p_2}{12.361} + 1.6309 \\ p_2 &= 73.480\,\mathrm{kPa} \\ \boldsymbol{p_2} &\boldsymbol{= 73.5\,\mathrm{kPa}} \end{aligned}$$

(So, in this case, $\boldsymbol{p_2 > p_1}$.)

Exercise 1:



Determine the pressure reading p_2 if $Q = 20\,\mathrm{L/s}$

Solution:

Find the cross-sectional areas at 1 and 2:

$$A_1 = \frac{\pi(0.0737\,\mathrm{m})^2}{4} = 0.0042660\,\mathrm{m}^2$$

$$A_2 = \frac{\pi(0.1223\,\mathrm{m})^2}{4} = 0.011747\,\mathrm{m}^2$$

Find the velocities at 1 and 2 for $Q = 20\,\mathrm{L/s}$:

$$v_1 = \frac{Q}{A_1} = \frac{0.020\,\mathrm{m^3/s}}{0.0042660\,\mathrm{m}^2} = 4.6882\,\mathrm{m/s}$$

$$v_2 = \frac{Q}{A_2} = \frac{0.020\,\mathrm{m^3/s}}{0.011747\,\mathrm{m}^2} = 1.7026\,\mathrm{m/s}$$

Calculate the velocity heads at 1 and 2 for $Q = 25\,\mathrm{L/s}$:

$$\frac{v_1^2}{2g} = \frac{(4.6882)^2}{2 \times 9.81} = 1.1203\,\mathrm{m}$$

$$\frac{v_2^2}{2g} = \frac{(1.7026)^2}{2 \times 9.81} = 0.14774\,\mathrm{m}$$

Applying Bernoulli's Equation between 1 and 2:

$$\frac{p_1}{\gamma} + z_1 + \frac{v_1^2}{2g} = \frac{p_2}{\gamma} + z_2 + \frac{v_2^2}{2g}$$

$$\frac{72}{1.26 \times 9.81} + 0 + 1.1203 = \frac{p_2}{1.26 \times 9.81} + 1.4 + 0.14774$$

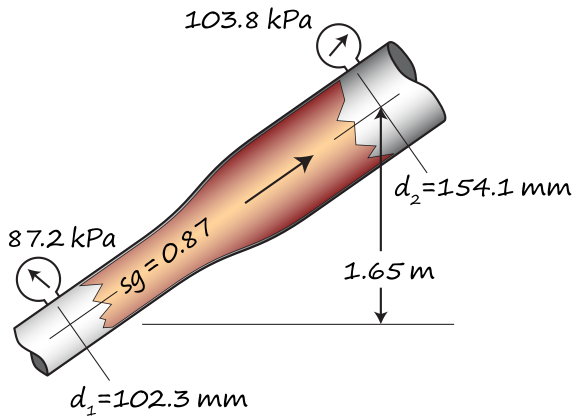
$$6.9453 = \frac{p_2}{12.361} + 1.5477$$

$$p_2 = 66.719\,\mathrm{kPa}$$

$$\boldsymbol{p_2 = 66.7\,\mathrm{kPa}}$$

(So, in this case, $\boldsymbol{p_2 < p_1}$.)

Example 5:



Determine Q, the volume flow rate.

Solution:

This question is subtly different from the previous one. We can solve directly for Q or solve for one of either v_1 or v_2. Let us solve for Q directly here.

First, find the velocity head values at 1 and at 2 in terms of Q:

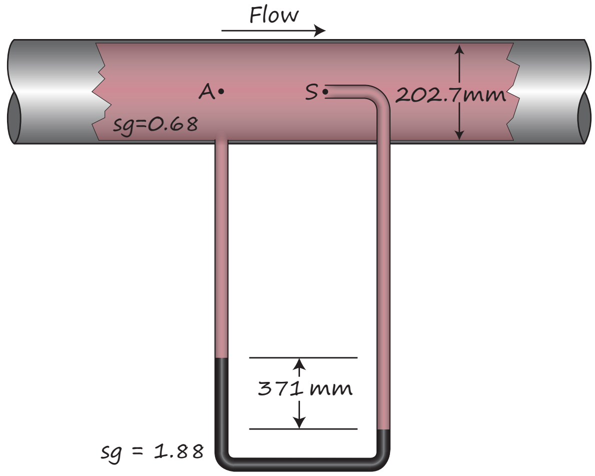
$$\begin{aligned}
A_1 &= \frac{\pi(0.1023\,\mathrm{m})^2}{4} \\
&= 0.0082194\,\mathrm{m}^2 \\
v_1 &= \frac{Q}{A_1} \\
&= \frac{Q}{0.0082194}\,\mathrm{m/s} \\
&= 121.66Q \\
\frac{v_1^2}{2g} &= \frac{\left(\frac{Q}{0.0082194}\right)^2}{2\times 9.81} \\
&= 754.43Q^2\,\mathrm{m}
\end{aligned}$$

$$\begin{aligned}
A_2 &= \frac{\pi(0.1541\,\mathrm{m})^2}{4} \\
&= 0.018651\,\mathrm{m}^2 \\
&= 53.617Q \\
v_2 &= \frac{Q}{0.018651}\,\mathrm{m/s} \\
\frac{v_2^2}{2g} &= \frac{\left(\frac{Q}{0.018651}\right)^2}{2\times 9.81} \\
&= 146.52Q^2\,\mathrm{m}
\end{aligned}$$

Applying Bernoulli's Equation between 1 and 2:

$$\begin{aligned}
\frac{p_1}{\gamma} + z_1 + \frac{v_1^2}{2g} &= \frac{p_2}{\gamma} + z_2 + \frac{v_2^2}{2g} \\
\frac{87.2}{0.87\times 9.81} + 0 + 754.43Q^2 &= \frac{103.8}{0.87\times 9.81} + 1.65 + 146.52Q^2 \\
10.217 + 754.43Q^2 &= 13.812 + 146.52Q^2 \\
Q^2 &= \frac{13.812 - 10.217}{754.43 - 146.52} \\
\boldsymbol{Q} &\approx \mathbf{76.9}\,\mathrm{L/s}
\end{aligned}$$

Example 6:



Determine Q, the volume flow rate.

Solution:

$$\frac{p_A}{\gamma} + z_A + \frac{v_A^2}{2g} = \frac{p_S}{\gamma} + z_S + \frac{v_S^2}{2g}$$

$$\frac{p_A}{0.68 \times 9.81} + \cancel{z_A} + \frac{v_A^2}{2g} = \frac{p_S}{0.68 \times 9.81} + \cancel{z_S} + \cancelto{0}{\frac{v_S^2}{2g}}$$

$$p_S - p_A = (0.68 \times 9.81)\frac{v_A^2}{2g}$$

Now look at the manometer:

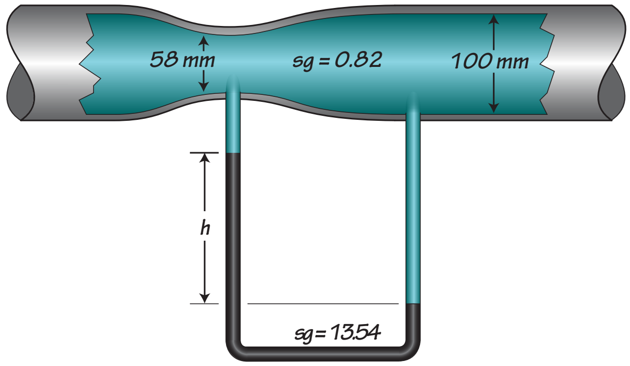
$$\begin{aligned} p_A + \cancel{(0.68 \times 9.81)d} + (1.88 \times 9.81)(0.371) &= p_S + (0.68 \times 9.81)(\cancel{d} + 0.371) \\ \Rightarrow p_S - p_A &= 1.88 \times 9.81 \times 0.371 - 0.68 \times 9.81 \times 0.371 \\ &= 4.3674 \end{aligned}$$

Substituting in the result above:

$$\begin{aligned} 4.3674 &= (0.68 \times 9.81)\frac{v_A^2}{2g} \\ v_A^2 &= 12.845 \\ v_A &= 3.584\,\text{m/s} \end{aligned}$$

$$\begin{aligned} Q &= \frac{\pi d^2}{4} \cdot 3.584\,\text{m}^3/\text{s} \\ &= 0.11566\,\text{m}^3/\text{s} \\ \boldsymbol{Q} &\approx \mathbf{116}\,\text{L/s} \end{aligned}$$

Example 7:



Determine Q, the volume flow rate, if $h = 210$ mm.

Solution:

Use the manometer to find the difference in pressure between the flow through the pipe and through the constriction:
(Calculate the pressure at the join of the pipe fluid and the manometer gauge fluid on the right hand side in the diagram, and let t be the vertical distance between the centre of the pipe and the pipe-fluid to gauge-fluid on the left hand side of the diagram.)

$$\begin{aligned} p_1 + \gamma_{\text{pipe}} \cdot t + \gamma_{\text{gauge}} \cdot h &= p_2 + \gamma_{\text{pipe}} \cdot t + \gamma_{\text{pipe}} \cdot h \\ p_1 + \gamma_{\text{gauge}} \cdot h &= p_2 + \gamma_{\text{pipe}} \cdot h \\ p_1 + 13.54 \times 9.81 \times 0.210 &= p_2 + 0.82 \times 9.81 \times 0.210 \\ p_2 - p_1 &= (13.54 - 0.82) \times 9.81 \times 0.210 \\ &= 26.204 \text{ kPa} \end{aligned}$$

Find the velocity heads at p_1 and p_2 in terms of Q:

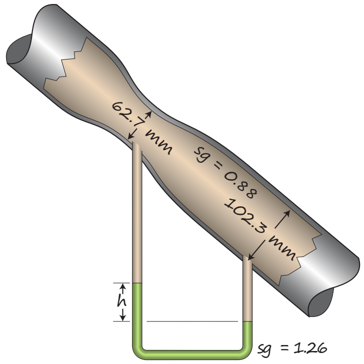
$$\begin{aligned} v_1 &= \frac{Q_1}{\pi (0.058)^2/4} \text{ m/s} \\ &= 378.49Q \text{ m/s} \\ \frac{v_1^2}{2g} &= 7301.5Q^2 \text{ m} \end{aligned}$$

$$\begin{aligned} v_2 &= \frac{Q_2}{\pi (0.100)^2/4} \text{ m/s} \\ &= 127.32Q \text{ m/s} \\ \frac{v_2^2}{2g} &= 826.22Q^2 \text{ m} \end{aligned}$$

Apply Bernoulli's Equation:

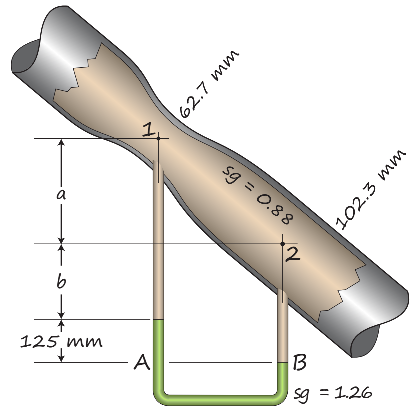
$$\begin{aligned} \frac{p_1}{\gamma} + \cancel{z_1} + \frac{v_1^2}{2g} &= \frac{p_2}{\gamma} + \cancel{z_2} + \frac{v_2^2}{2g} \\ \frac{p_2 - p_1}{\gamma} &= \frac{v_1^2}{2g} - \frac{v_2^2}{2g} \\ p_2 - p_1 &= \gamma \left[7301.5Q^2 - 826.22Q^2 \right] \\ 26.204 &= 0.82 \times 9.81 \times 6475.3Q^2 \\ Q^2 &= 0.00050307 \\ Q &= 0.022429 \text{ m}^3/\text{s} \\ &= 22.4 \text{ L/s} \end{aligned}$$

Exercise 2:



Determine Q, the volume flow rate, if $h = 125$ mm.

Solution:



Since $P_A = P_B$,

$$\begin{aligned}
p_1 + \gamma_{\text{pipe}}(a + \cancel{b}) + \gamma_{\text{gauge}} \cdot h &= p_2 + \gamma_{\text{pipe}}(\cancel{b} + h) \\
p_1 + (0.88 \times 9.81)(a) + (1.26 \times 9.81)(0.125) &= p_2 + (0.88 \times 9.81)(0.125) \\
p_2 - p_1 &= (0.88 \times 9.81)(a - 0.125) + (1.26 \times 9.81)(0.125) \\
&= 8.6328a + 0.46598 \text{ kPa}
\end{aligned}$$

Find the velocities at 1 and 2 relative to the flow Q:

$$\begin{aligned}
v_1 &= \frac{Q}{A_1} = \frac{Q}{\pi(0.0627)^2/4} = \frac{Q}{0.0030876} = 323.87Q \\
v_2 &= \frac{Q}{A_2} = \frac{Q}{\pi(0.1023)^2/4} = \frac{Q}{0.0082194} = 121.66Q
\end{aligned}$$

The associated velocity heads are:

$$\begin{aligned}
\frac{v_1^2}{2g} &= \frac{(328.87Q)^2}{2g} = 5346.3Q^2 \\
\frac{v_2^2}{2g} &= \frac{(121.66Q)^2}{2g} = 754.43Q^2
\end{aligned}$$

Apply Bernoulli's Equation:

$$\begin{aligned}
\frac{p_1}{\gamma} + z_1 + \frac{v_1^2}{2g} &= \frac{p_2}{\gamma} + z_2 + \frac{v_2^2}{2g} \\
\frac{p_1}{\gamma} + a + 5346.3Q^2 &= \frac{p_2}{\gamma} + 0 + 754.43Q^2 \\
\frac{p_2 - p_1}{\gamma} &= 5346.3Q^2 - 754.43Q^2 + a \\
p_2 - p_1 &= 0.88 \times 9.81 \left(4591.9Q^2 + a\right)
\end{aligned}$$

Incorporating our previous manometer result,

$$\begin{aligned}
8.6328a + 0.46598 &= 0.88 \times 9.81 \left(4591.9Q^2 + a\right) \\
\cancel{8.6328a} + 0.46598 &= 39641Q^2 + \cancel{8.6328a}
\end{aligned}$$

$$\begin{aligned}
Q &= 0.0034286 \text{ m}^3/\text{s} \\
&\approx 3.43 \text{ L/s}
\end{aligned}$$

Note that the answer is independent of a: it does not matter what angle the venturi meter is inclined at, the answer will be the same (including vertical or horizontal). This was not the case in the previous examples where the pressure was measured directly in the pipe with pressure meters.