STA 2503/MMF 1928 Project 1 - American Options

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Question 1

Proof. We start the proof by considering

$$X^{(N)} = \log(\frac{S_T}{S_0}) = \sum_{n=1}^{N}(r\Delta t + \sigma\sqrt{\Delta t}\epsilon_n)$$

The m.g.f of $X^{(N)}$ is:

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}[e^{uX^{(N)}}] &= \mathbb{E}^{\mathbb{P}}[e^{\sum_{n=1}^{N}(ur\Delta t + u\sigma\sqrt{\Delta t}\epsilon_n)}] \\
&= \mathbb{E}^{\mathbb{P}}[\prod_{n=1}^{N} e^{ur\Delta t + u\sigma\sqrt{\Delta t}\epsilon_n}] \\
&= (\mathbb{E}^{\mathbb{P}}[e^{ur\Delta t + u\sigma\sqrt{\Delta t}\epsilon_1}])^N && \text{as } \epsilon_n \text{ is i.i.d}
\end{aligned}$$

We then investigate the inner term $\mathbb{E}^{\mathbb{P}}[e^{ur\Delta t + u\sigma\sqrt{\Delta t}\epsilon_1}]$:

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}[e^{ur\Delta t + u\sigma\sqrt{\Delta t}\epsilon_1}] &= e^{ur\Delta t + u\sigma\sqrt{\Delta t}} \cdot \frac{1}{2}(1 + \frac{(\mu - r) - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}) \\
&\quad + e^{ur\Delta t - u\sigma\sqrt{\Delta t}} \cdot \frac{1}{2}(1 - \frac{(\mu - r) - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}) \\
&= [1 + ur\Delta t + u\sigma\sqrt{\Delta t} + \frac{1}{2}u^2\sigma^2\Delta t + o(\Delta t)] \cdot \frac{1}{2}(1 + \frac{(\mu - r) - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}) \\
&\quad + [1 + ur\Delta t - u\sigma\sqrt{\Delta t} + \frac{1}{2}u^2\sigma^2\Delta t + o(\Delta t)] \cdot \frac{1}{2}(1 - \frac{(\mu - r) - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t})
\end{aligned}$$

Note that we collect all the terms containing Δt with order 2 or higher to be $o(\Delta t)$ because as $N \to \infty$ and $\Delta t \to 0$, $o(\Delta t)$ will converge to 0 faster than Δt or Δt with lower orders.

We then continue the algebra manipulation:

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}}[e^{ur\Delta t+u\sigma\sqrt{\Delta t}\epsilon_n}] &= [1+ur\Delta t+u\sigma\sqrt{\Delta t}+\frac{1}{2}u^2\sigma^2\Delta t+o(\Delta t)]\cdot\frac{1}{2}(1+\frac{(\mu-r)-\frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}) \\
&\quad +[1+ur\Delta t-u\sigma\sqrt{\Delta t}+\frac{1}{2}u^2\sigma^2\Delta t+o(\Delta t)]\cdot\frac{1}{2}(1-\frac{(\mu-r)-\frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}) \\
&= \frac{1}{2}[1+\frac{(\mu-r)-\frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}+ur\Delta t+u\sigma\sqrt{\Delta t} \\
&\quad +u((\mu-r)-\frac{1}{2}\sigma^2)\Delta t+\frac{1}{2}u^2\sigma^2\Delta t \\
&\quad +1-\frac{(\mu-r)-\frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}+ur\Delta t-u\sigma\sqrt{\Delta t} \\
&\quad +u((\mu-r)-\frac{1}{2}\sigma^2)\Delta t+\frac{1}{2}u^2\sigma^2\Delta t+o(\Delta t)] \\
&= 1+ur\Delta t+u((\mu-r)-\frac{1}{2}\sigma^2)\Delta t+\frac{1}{2}u^2\sigma^2\Delta t+o(\Delta t) \\
&= 1+(ur+u\mu-ur-\frac{1}{2}u\sigma^2+\frac{1}{2}u^2\sigma^2)\Delta t+o(\Delta t) \\
&= 1+(u(\mu-\frac{1}{2}\sigma^2)+\frac{1}{2}u^2\sigma^2)\Delta t+o(\Delta t) \\
&= e^{(u(\mu-\frac{1}{2}\sigma^2)+\frac{1}{2}u^2\sigma^2)\Delta t}+o(\Delta t)
\end{aligned}
$$

As a result:

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}}[e^{uX^{(N)}}] &= (e^{(u(\mu-\frac{1}{2}\sigma^2)+\frac{1}{2}u^2\sigma^2)\Delta t}+o(\Delta t))^N \\
&= e^{(u(\mu-\frac{1}{2}\sigma^2)T+\frac{1}{2}u^2\sigma^2T)} \qquad \text{as } N\to\infty \text{ and } \Delta t=\tfrac{T}{N}
\end{aligned}
$$

We notice that the m.g.f of the $X^{(N)}$ is equal to the m.g.f of a random variable Y whcih follows the normal distribution with mean to be $(\mu-\frac{1}{2}\sigma^2)T$ and variance to be $\sigma^2 T$.

Thus, we prove that:

$$
X^{(N)} \xrightarrow[N\to\infty]{d} (\mu-\frac{1}{2}\sigma^2)T+\sigma^2 TZ
$$

where

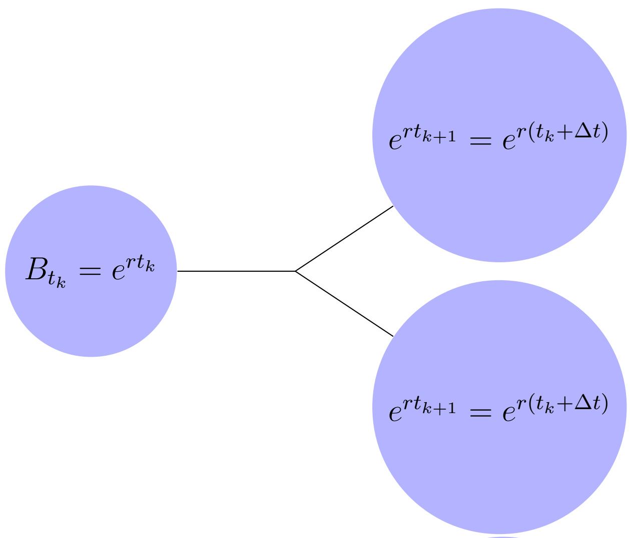
$$
Z \overset{\mathbb{P}}{\sim} \mathcal{N}(0,1)
$$

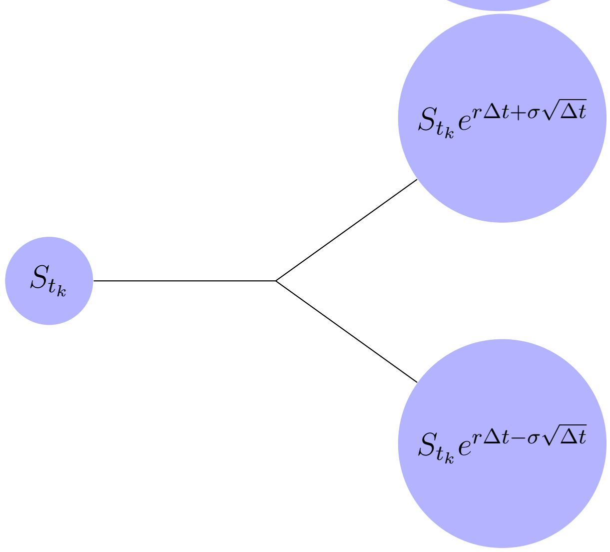
$\square$

Question 2

Part 1 – $\mathbb{Q}$

We firstly find the martingale measure $\mathbb{Q}(\epsilon_k = \pm 1)$ where $\mathbb{Q}(\epsilon_k = 1) = q_k$ and $\mathbb{Q}(\epsilon_k = -1) = 1 - q_k$. Consider time t_k and t_{k+1}, we have the following for B and S.





We construct the following $\mathbb{Q}$-martingale:

$$\begin{aligned}
\frac{S_{t_k}}{e^{rt_k}} &= q_k \frac{S_{t_k} e^{r\Delta t + \sigma\sqrt{\Delta t}}}{e^{r(t_k+\Delta t)}} + (1-q_k)\frac{S_{t_k} e^{r\Delta t - \sigma\sqrt{\Delta t}}}{e^{r(t_k+\Delta t)}} \\
\Rightarrow 1 &= q_k \frac{e^{r\Delta t + \sigma\sqrt{\Delta t}}}{e^{r\Delta t}} + (1-q_k)\frac{e^{r\Delta t - \sigma\sqrt{\Delta t}}}{e^{r\Delta t}} \\
\Rightarrow q_k(e^{r\Delta t + \sigma\sqrt{\Delta t}} &- e^{r\Delta t - \sigma\sqrt{\Delta t}}) = e^{r\Delta t} - e^{r\Delta t - \sigma\sqrt{\Delta t}} \\
\Rightarrow q_k = q &= \frac{e^{r\Delta t} - e^{r\Delta t - \sigma\sqrt{\Delta t}}}{e^{r\Delta t + \sigma\sqrt{\Delta t}} - e^{r\Delta t - \sigma\sqrt{\Delta t}}} \qquad \text{As } q_k \text{ does not depend on } k \\
\Rightarrow q &= \frac{1 - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}
\end{aligned}$$

We then Taylor-expand the exponential terms and collect all all the terms containing Δt with order 2 or higher to be $o(\Delta t)$:

$$\begin{aligned}
q &= \frac{e^{r\Delta t} - e^{r\Delta t - \sigma\sqrt{\Delta t}}}{e^{r\Delta t + \sigma\sqrt{\Delta t}} - e^{r\Delta t - \sigma\sqrt{\Delta t}}} \\
&= \frac{1 + r\Delta t - (1 + r\Delta t - \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t) + o(\Delta t)}{1 + r\Delta t + \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t - (1 + r\Delta t - \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t) + o(\Delta t)} \\
&= \frac{\sigma\sqrt{\Delta t} - \frac{1}{2}\sigma^2\Delta t + o(\Delta t)}{2\sigma\sqrt{\Delta t} + o(\Delta t)} \\
&= \frac{1}{2} - \frac{1}{4}\sigma\sqrt{\Delta t} + o(\sqrt{\Delta t}) \\
&= \frac{1}{2}(1 - \frac{1}{2}\sigma\sqrt{\Delta t}) + o(\sqrt{\Delta t})
\end{aligned}$$

Now let us find the $\mathbb{Q}$ distribution of S_T as $N \to \infty$. We first define $X_T = log(\frac{S_T}{S_0})$, then we have:

$$X_T = \log(\frac{S_T}{S_0}) = \log(e^{\sum_{n=1}^{N} r\Delta t + \sigma\sqrt{\Delta t}\epsilon_n}) = \sum_{n=1}^{N} r\Delta t + \sigma\sqrt{\Delta t}\epsilon_n$$

Consider the m.g.f of X_T:

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}[e^{uX_T}] &= \mathbb{E}^{\mathbb{P}}[e^{\sum_{n=1}^{N}(ur\Delta t+u\sigma\sqrt{\Delta t}\epsilon_n)}] \\
&= \mathbb{E}^{\mathbb{P}}[\prod_{n=1}^{N} e^{ur\Delta t+u\sigma\sqrt{\Delta t}\epsilon_n}] \\
&= (\mathbb{E}^{\mathbb{P}}[e^{ur\Delta t+u\sigma\sqrt{\Delta t}\epsilon_1}])^N \qquad \text{as } \epsilon_n \text{ is i.i.d}
\end{aligned}$$

We then investigate the inner term $\mathbb{E}^{\mathbb{P}}[e^{ur\Delta t+u\sigma\sqrt{\Delta t}\epsilon_1}]$:

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}[e^{ur\Delta t+u\sigma\sqrt{\Delta t}\epsilon_1}] &= e^{ur\Delta t+u\sigma\sqrt{\Delta t}}\cdot\frac{1}{2}(1-\frac{1}{2}\sigma\sqrt{\Delta t}+o(\sqrt{\Delta t})) \\
&\quad + e^{ur\Delta t-u\sigma\sqrt{\Delta t}}\cdot\frac{1}{2}(1+\frac{1}{2}\sigma\sqrt{\Delta t}+o(\sqrt{\Delta t})) \\
&= [1+ur\Delta t+u\sigma\sqrt{\Delta t}+\frac{1}{2}u^2\sigma^2\Delta t+o(\Delta t)]\cdot\frac{1}{2}(1-\frac{1}{2}\sigma\sqrt{\Delta t}+o(\sqrt{\Delta t})) \\
&\quad + [1+ur\Delta t-u\sigma\sqrt{\Delta t}+\frac{1}{2}u^2\sigma^2\Delta t+o(\Delta t)]\cdot\frac{1}{2}(1+\frac{1}{2}\sigma\sqrt{\Delta t}+o(\sqrt{\Delta t})) \\
&= \frac{1}{2}[1-\frac{1}{2}\sigma\sqrt{\Delta t}+ur\Delta t+u\sigma\sqrt{\Delta t}-\frac{1}{2}u\sigma^2\Delta t+\frac{1}{2}u^2\sigma^2\Delta t \\
&\quad +1+\frac{1}{2}\sigma\sqrt{\Delta t}+ur\Delta t-u\sigma\sqrt{\Delta t}-\frac{1}{2}u\sigma^2\Delta t+\frac{1}{2}u^2\sigma^2\Delta t+o(\Delta t)] \\
&= 1+ur\Delta t-\frac{1}{2}u\sigma^2\Delta t+\frac{1}{2}u^2\sigma^2\Delta t+o(\Delta t) \\
&= 1+(u(r-\frac{1}{2}\sigma^2)+\frac{1}{2}u^2\sigma^2)\Delta t+o(\Delta t) \\
&= e^{(u(r-\frac{1}{2}\sigma^2)+\frac{1}{2}u^2\sigma^2)\Delta t+o(\Delta t)}
\end{aligned}$$

Then we have the m.g.f of X_T to be:

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}[e^{uX_T}] &= (\mathbb{E}^{\mathbb{P}}[e^{ur\Delta t+u\sigma\sqrt{\Delta t}\epsilon_1}])^N \\
&= (e^{(u(r-\frac{1}{2}\sigma^2)+\frac{1}{2}u^2\sigma^2)\Delta t+o(\Delta t)})^N \\
&= e^{(u(r-\frac{1}{2}\sigma^2)T+\frac{1}{2}u^2\sigma^2T)} \qquad \text{as } N\to\infty \text{ and } \Delta t=\frac{T}{N}
\end{aligned}$$

We notice that the m.g.f of the X_T is equal to the m.g.f of a random variable Y whcih follows the normal distribution with mean to be $(r-\frac{1}{2}\sigma^2)T$ and variance to be σ^2T. Thus,

$$S_T \overset{\mathbb{Q}}{\sim} lognormal((r-\frac{1}{2}\sigma^2)T+\log(S_0),\sigma^2T)$$

Part 2 – $\mathbb{Q}^S$

Similarly, we want to find the martingale measure $\mathbb{Q}^S(\epsilon_k = \pm 1)$ where $\mathbb{Q}^S(\epsilon_k = 1) = h_k$ and $\mathbb{Q}^S(\epsilon_k = -1) = 1 - h_k$. Consider time t_k and t_{k+1}, this time we use S as the numeraire.

We construct the following $\mathbb{Q}^S$-martingale:

$$\begin{aligned}
\frac{e^{rt_k}}{S_{t_k}} &= h_k \frac{e^{r(t_k+\Delta t)}}{S_{t_k} e^{r\Delta t + \sigma\sqrt{\Delta t}}} + (1-h_k)\frac{e^{r(t_k+\Delta t)}}{S_{t_k} e^{r\Delta t - \sigma\sqrt{\Delta t}}} \\
\Rightarrow 1 &= h_k \frac{e^{r\Delta t}}{e^{r\Delta t + \sigma\sqrt{\Delta t}}} + (1-h_k)\frac{e^{r\Delta t}}{e^{r\Delta t - \sigma\sqrt{\Delta t}}} \\
\Rightarrow e^{2r\Delta t} &= h_k e^{2r\Delta t - \sigma\sqrt{\Delta t}} + (1-h_k) e^{2r\Delta t + \sigma\sqrt{\Delta t}} \\
\Rightarrow e^{2r\Delta t} - e^{2r\Delta t + \sigma\sqrt{\Delta t}} &= h_k (e^{2r\Delta t - \sigma\sqrt{\Delta t}} - e^{2r\Delta t + \sigma\sqrt{\Delta t}}) \\
\Rightarrow h_k = h &= \frac{e^{2r\Delta t} - e^{2r\Delta t + \sigma\sqrt{\Delta t}}}{e^{2r\Delta t - \sigma\sqrt{\Delta t}} - e^{2r\Delta t + \sigma\sqrt{\Delta t}}} \qquad \text{As } h_k \text{ does not depend on } k \\
\Rightarrow h &= \frac{1 - e^{\sigma\sqrt{\Delta t}}}{e^{-\sigma\sqrt{\Delta t}} - e^{\sigma\sqrt{\Delta t}}}
\end{aligned}$$

We then Taylor-expand the exponential terms and collect all all the terms containing Δt with order 2 or higher to be $o(\Delta t)$:

$$\begin{aligned}
h &= \frac{e^{2r\Delta t} - e^{2r\Delta t + \sigma\sqrt{\Delta t}}}{e^{2r\Delta t - \sigma\sqrt{\Delta t}} - e^{2r\Delta t + \sigma\sqrt{\Delta t}}} \\
&= \frac{1 + 2r\Delta t - (1 + 2r\Delta t + \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t) + o(\Delta t)}{1 + 2r\Delta t - \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t - (1 + 2r\Delta t + \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t) + o(\Delta t)} \\
&= \frac{-\sigma\sqrt{\Delta t} - \frac{1}{2}\sigma^2\Delta t + o(\Delta t)}{-2\sigma\sqrt{\Delta t} + o(\Delta t)} \\
&= \frac{\sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t + o(\Delta t)}{2\sigma\sqrt{\Delta t} + o(\Delta t)} \\
&= \frac{1}{2}(1 + \frac{1}{2}\sigma\sqrt{\Delta t}) + o(\sqrt{\Delta t})
\end{aligned}$$

Now let us find the $\mathbb{Q}^S$ distribution of S_T as $N \to \infty$.We first define $X_T =$

$log(\frac{S_T}{S_0})$, then we have:

$$X_T = \log(\frac{S_T}{S_0}) = \log(e^{\sum_{n=1}^{N} r\Delta t + \sigma\sqrt{\Delta t}\epsilon_n}) = \sum_{n=1}^{N} r\Delta t + \sigma\sqrt{\Delta t}\epsilon_n$$

Consider the m.g.f of X_T:

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}[e^{uX_T}] &= \mathbb{E}^{\mathbb{P}}[e^{\sum_{n=1}^{N}(ur\Delta t + u\sigma\sqrt{\Delta t}\epsilon_n)}] \\
&= \mathbb{E}^{\mathbb{P}}[\prod_{n=1}^{N} e^{ur\Delta t + u\sigma\sqrt{\Delta t}\epsilon_n}] \\
&= (\mathbb{E}^{\mathbb{P}}[e^{ur\Delta t + u\sigma\sqrt{\Delta t}\epsilon_1}])^N \qquad\qquad \text{as } \epsilon_n \text{ is i.i.d}
\end{aligned}$$

We then investigate the inner term $\mathbb{E}^{\mathbb{P}}[e^{ur\Delta t + u\sigma\sqrt{\Delta t}\epsilon_1}]$:

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}[e^{ur\Delta t + u\sigma\sqrt{\Delta t}\epsilon_1}] &= e^{ur\Delta t + u\sigma\sqrt{\Delta t}} \cdot \frac{1}{2}(1 + \frac{1}{2}\sigma\sqrt{\Delta t} + o(\sqrt{\Delta t})) \\
&\quad + e^{ur\Delta t - u\sigma\sqrt{\Delta t}} \cdot \frac{1}{2}(1 - \frac{1}{2}\sigma\sqrt{\Delta t} + o(\sqrt{\Delta t})) \\
&= [1 + ur\Delta t + u\sigma\sqrt{\Delta t} + \frac{1}{2}u^2\sigma^2\Delta t + o(\Delta t)] \cdot \frac{1}{2}(1 + \frac{1}{2}\sigma\sqrt{\Delta t} + o(\sqrt{\Delta t})) \\
&\quad + [1 + ur\Delta t - u\sigma\sqrt{\Delta t} + \frac{1}{2}u^2\sigma^2\Delta t + o(\Delta t)] \cdot \frac{1}{2}(1 - \frac{1}{2}\sigma\sqrt{\Delta t} + o(\sqrt{\Delta t})) \\
&= \frac{1}{2}[1 + \frac{1}{2}\sigma\sqrt{\Delta t} + ur\Delta t + u\sigma\sqrt{\Delta t} + \frac{1}{2}u\sigma^2\Delta t + \frac{1}{2}u^2\sigma^2\Delta t \\
&\quad + 1 - \frac{1}{2}\sigma\sqrt{\Delta t} + ur\Delta t - u\sigma\sqrt{\Delta t} + \frac{1}{2}u\sigma^2\Delta t + \frac{1}{2}u^2\sigma^2\Delta t + o(\Delta t)] \\
&= 1 + ur\Delta t + \frac{1}{2}u\sigma^2\Delta t + \frac{1}{2}u^2\sigma^2\Delta t + o(\Delta t) \\
&= 1 + (u(r + \frac{1}{2}\sigma^2) + \frac{1}{2}u^2\sigma^2)\Delta t + o(\Delta t) \\
&= e^{(u(r+\frac{1}{2}\sigma^2) + \frac{1}{2}u^2\sigma^2)\Delta t + o(\Delta t)}
\end{aligned}$$

Then we have the m.g.f of X_T to be:

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}[e^{uX_T}] &= (\mathbb{E}^{\mathbb{P}}[e^{ur\Delta t + u\sigma\sqrt{\Delta t}\epsilon_1}])^N \\
&= \left(e^{(u(r+\frac{1}{2}\sigma^2) + \frac{1}{2}u^2\sigma^2)\Delta t + o(\Delta t)}\right)^N \\
&= e^{(u(r+\frac{1}{2}\sigma^2)T + \frac{1}{2}u^2\sigma^2 T)} \qquad\qquad \text{as } N \to \infty \text{ and } \Delta t = \tfrac{T}{N}
\end{aligned}$$

We notice that the m.g.f of the X_T is equal to the m.g.f of a random variable Y whcih follows the normal distribution with mean to be $(r + \frac{1}{2}\sigma^2)T$ and variance to be $\sigma^2 T$. Thus,

$$S_T \overset{\mathbb{Q}^S}{\sim} lognormal((r + \frac{1}{2}\sigma^2)T + \log(S_0), \sigma^2 T)$$