STA 2503/MMF 1928 Project 1 - American Options

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Question 1

Proof. We start the proof by considering

$$X^{(N)} = \log(\frac{S_T}{S_0}) = \sum_{n=1}^{N}(r\Delta t + \sigma\sqrt{\Delta t}\epsilon_n)$$

The m.g.f of $X^{(N)}$ is:

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}[e^{\mu X^{(N)}}] &= \mathbb{E}^{\mathbb{P}}[e^{\sum_{n=1}^{N}(\mu r\Delta t + \mu\sigma\sqrt{\Delta t}\epsilon_n)}] \\
&= \mathbb{E}^{\mathbb{P}}[\prod_{n=1}^{N} e^{\mu r\Delta t + \mu\sigma\sqrt{\Delta t}\epsilon_n}] \\
&= (\mathbb{E}^{\mathbb{P}}[e^{\mu r\Delta t + \mu\sigma\sqrt{\Delta t}\epsilon_n}])^N \qquad \text{as } \epsilon_n \text{ is i.i.d}
\end{aligned}$$

We then investigate the inner term $\mathbb{E}^{\mathbb{P}}[e^{\mu r\Delta t + \mu\sigma\sqrt{\Delta t}\epsilon_n}]$:

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}[e^{\mu r\Delta t + \mu\sigma\sqrt{\Delta t}\epsilon_n}] &= e^{\mu r\Delta t + \mu\sigma\sqrt{\Delta t}} \cdot \frac{1}{2}(1 + \frac{(\mu - r) - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}) \\
&\quad + e^{\mu r\Delta t - \mu\sigma\sqrt{\Delta t}} \cdot \frac{1}{2}(1 - \frac{(\mu - r) - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}) \\
&= [1 + \mu r\Delta t + \mu\sigma\sqrt{\Delta t} + \frac{1}{2}\mu^2\sigma^2\Delta t + o(\Delta t)] \cdot \frac{1}{2}(1 + \frac{(\mu - r) - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}) \\
&\quad + [1 + \mu r\Delta t - \mu\sigma\sqrt{\Delta t} + \frac{1}{2}\mu^2\sigma^2\Delta t + o(\Delta t)] \cdot \frac{1}{2}(1 - \frac{(\mu - r) - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t})
\end{aligned}$$

Note that we collect all the terms containing Δt with order 2 or higher to be $o(\Delta t)$ because as $N \to \infty$ and $\Delta t \to 0$, $o(\Delta t)$ will converge to 0 faster than Δt or Δt with lower orders.

We then continue the algebra manipulation:

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}}[e^{\mu r\Delta t+\mu\sigma\sqrt{\Delta t}\epsilon_n}] &= [1+\mu r\Delta t+\mu\sigma\sqrt{\Delta t}+\frac{1}{2}\mu^2\sigma^2\Delta t+o(\Delta t)]\cdot\frac{1}{2}(1+\frac{(\mu-r)-\frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}) \\
&+[1+\mu r\Delta t-\mu\sigma\sqrt{\Delta t}+\frac{1}{2}\mu^2\sigma^2\Delta t+o(\Delta t)]\cdot\frac{1}{2}(1-\frac{(\mu-r)-\frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}) \\
&= \frac{1}{2}[1+\frac{(\mu-r)-\frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}+\mu r\Delta t+\mu\sigma\sqrt{\Delta t} \\
&+\mu((\mu-r)-\frac{1}{2}\sigma^2)\Delta t+\frac{1}{2}\mu^2\sigma^2\Delta t \\
&+1-\frac{(\mu-r)-\frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}+\mu r\Delta t-\mu\sigma\sqrt{\Delta t} \\
&+\mu((\mu-r)-\frac{1}{2}\sigma^2)\Delta t+\frac{1}{2}\mu^2\sigma^2\Delta t+o(\Delta t)] \\
&= 1+\mu r\Delta t+\mu((\mu-r)-\frac{1}{2}\sigma^2)\Delta t+\frac{1}{2}\mu^2\sigma^2\Delta t+o(\Delta t) \\
&= 1+(\mu r+\mu^2-\mu r-\frac{1}{2}\mu\sigma^2+\frac{1}{2}\mu^2\sigma^2)\Delta t+o(\Delta t) \\
&= 1+(\mu(\mu-\frac{1}{2}\sigma^2)+\frac{1}{2}\mu^2\sigma^2)\Delta t+o(\Delta t) \\
&= e^{(\mu(\mu-\frac{1}{2}\sigma^2)+\frac{1}{2}\mu^2\sigma^2)\Delta t}+o(\Delta t)
\end{aligned}
$$

As a result:

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}}[e^{\mu X^{(N)}}] &= (e^{(\mu(\mu-\frac{1}{2}\sigma^2)+\frac{1}{2}\mu^2\sigma^2)\Delta t}+o(\Delta t))^N \\
&= e^{(\mu(\mu-\frac{1}{2}\sigma^2)T+\frac{1}{2}\mu^2\sigma^2T)} \qquad \text{as } N\to\infty \text{ and } \Delta t=\tfrac{T}{N}
\end{aligned}
$$

We notice that the m.g.f of the $X^{(N)}$ is equal to the m.g.f of a random variable Y whcih follows the normal distribution with mean to be $(\mu-\frac{1}{2}\sigma^2)T$ and variance to be σ^2T.

Thus, we prove that:

$$X^{(N)} \xrightarrow[N\to\inf]{d} (\mu-\frac{1}{2}\sigma^2)T+\sigma^2TZ$$

where

$$Z \overset{\mathbb{P}}{\sim} \mathcal{N}(0,1)$$

□