# STA 2503/MMF 1928 Project 1 - American Options

**Siyu Jia, Dixin Mou, Zixun Zhai**

University of Toronto
Toronto, Ontario, Canada
October $1^{st}$, 2022

# Question 1

*Proof.* We start the proof by considering

$$X^{(N)} = \log(\frac{S_T}{S_0}) = \sum_{n=1}^{N}(r\Delta t + \sigma\sqrt{\Delta t}\epsilon_n)$$

The m.g.f of $X^{(N)}$ is:

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}[e^{\mu X^{(N)}}] &= \mathbb{E}^{\mathbb{P}}[e^{\sum_{n=1}^{N}(\mu r\Delta t + \mu\sigma\sqrt{\Delta t}\epsilon_n)}] \\
&= \mathbb{E}^{\mathbb{P}}[\prod_{n=1}^{N} e^{\mu r\Delta t + \mu\sigma\sqrt{\Delta t}\epsilon_n}] \\
&= (\mathbb{E}^{\mathbb{P}}[e^{\mu r\Delta t + \mu\sigma\sqrt{\Delta t}\epsilon_n}])^N \qquad \text{as } \epsilon_n \text{ is i.i.d}
\end{aligned}$$

We then investigate the inner term $\mathbb{E}^{\mathbb{P}}[e^{\mu r\Delta t + \mu\sigma\sqrt{\Delta t}\epsilon_n}]$:

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}[e^{\mu r\Delta t + \mu\sigma\sqrt{\Delta t}\epsilon_n}] &= e^{\mu r\Delta t + \mu\sigma\sqrt{\Delta t}} \cdot \frac{1}{2}(1 + \frac{(\mu - r) - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}) \\
&\quad + e^{\mu r\Delta t - \mu\sigma\sqrt{\Delta t}} \cdot \frac{1}{2}(1 - \frac{(\mu - r) - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}) \\
&= [1 + \mu r\Delta t + \mu\sigma\sqrt{\Delta t} + \frac{1}{2}\mu^2\sigma^2\Delta t + o(\Delta t)] \cdot \frac{1}{2}(1 + \frac{(\mu - r) - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}) \\
&\quad + [1 + \mu r\Delta t - \mu\sigma\sqrt{\Delta t} + \frac{1}{2}\mu^2\sigma^2\Delta t + o(\Delta t)] \cdot \frac{1}{2}(1 - \frac{(\mu - r) - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t})
\end{aligned}$$

Note that we collect all the terms containing $\Delta t$ with order 2 or higher to be $o(\Delta t)$ because as $N \to \infty$ and $\Delta t \to 0$, $o(\Delta t)$ will converge to 0 faster than $\Delta t$ or $\Delta t$ with lower orders.

We then continue the algebra manipulation:

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}}[e^{\mu r\Delta t+\mu\sigma\sqrt{\Delta t}\epsilon_n}] &= [1+\mu r\Delta t+\mu\sigma\sqrt{\Delta t}+\frac{1}{2}\mu^2\sigma^2\Delta t+o(\Delta t)]\cdot\frac{1}{2}(1+\frac{(\mu-r)-\frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}) \\
&\quad+[1+\mu r\Delta t-\mu\sigma\sqrt{\Delta t}+\frac{1}{2}\mu^2\sigma^2\Delta t+o(\Delta t)]\cdot\frac{1}{2}(1-\frac{(\mu-r)-\frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}) \\
&= \frac{1}{2}[1+\frac{(\mu-r)-\frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}+\mu r\Delta t+\mu\sigma\sqrt{\Delta t} \\
&\quad+\mu((\mu-r)-\frac{1}{2}\sigma^2)\Delta t+\frac{1}{2}\mu^2\sigma^2\Delta t \\
&\quad+1-\frac{(\mu-r)-\frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}+\mu r\Delta t-\mu\sigma\sqrt{\Delta t} \\
&\quad+\mu((\mu-r)-\frac{1}{2}\sigma^2)\Delta t+\frac{1}{2}\mu^2\sigma^2\Delta t+o(\Delta t)] \\
&= 1+\mu r\Delta t+\mu((\mu-r)-\frac{1}{2}\sigma^2)\Delta t+\frac{1}{2}\mu^2\sigma^2\Delta t+o(\Delta t) \\
&= 1+(\mu r+\mu^2-\mu r-\frac{1}{2}\mu\sigma^2+\frac{1}{2}\mu^2\sigma^2)\Delta t+o(\Delta t) \\
&= 1+(\mu(\mu-\frac{1}{2}\sigma^2)+\frac{1}{2}\mu^2\sigma^2)\Delta t+o(\Delta t) \\
&= e^{(\mu(\mu-\frac{1}{2}\sigma^2)+\frac{1}{2}\mu^2\sigma^2)\Delta t}+o(\Delta t)
\end{aligned}
$$

As a result:

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}}[e^{\mu X^{(N)}}] &= (e^{(\mu(\mu-\frac{1}{2}\sigma^2)+\frac{1}{2}\mu^2\sigma^2)\Delta t}+o(\Delta t))^N \\
&= e^{(\mu(\mu-\frac{1}{2}\sigma^2)T+\frac{1}{2}\mu^2\sigma^2T)} \qquad \text{as } N\to\infty \text{ and } \Delta t=\tfrac{T}{N}
\end{aligned}
$$

We notice that the m.g.f of the $X^{(N)}$ is equal to the m.g.f of a random variable $Y$ whcih follows the normal distribution with mean to be $(\mu-\frac{1}{2}\sigma^2)T$ and variance to be $\sigma^2T$.

Thus, we prove that:

$$
X^{(N)} \xrightarrow[N\to\inf]{d} (\mu-\frac{1}{2}\sigma^2)T+\sigma^2TZ
$$

where

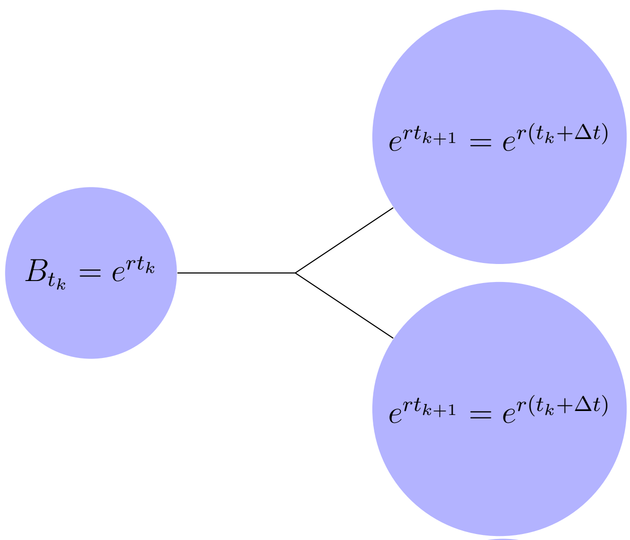
$$
Z \overset{\mathbb{P}}{\sim} \mathcal{N}(0,1)
$$

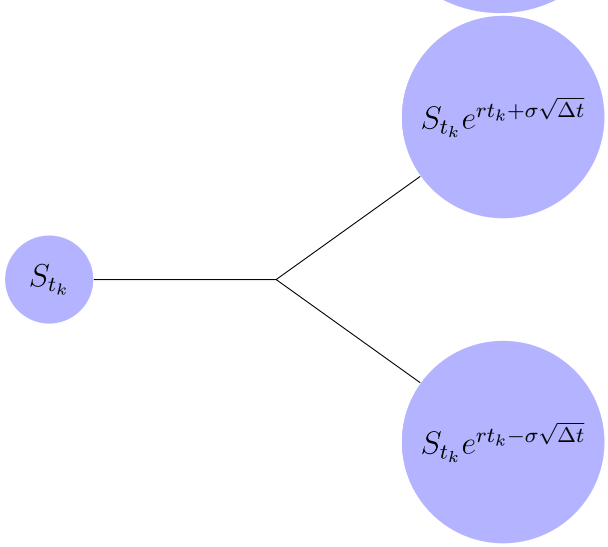
$\square$

# Question 2

## Part 1

We firstly find the martingale measure $\mathbb{Q}(\epsilon_k = \pm 1)$ where $\mathbb{Q}(\epsilon_k = 1) = q$ and $\mathbb{Q}(\epsilon_k = -1) = 1 - q$. Consider time $t_k$ and $t_{k_1}$, we have the following for $B$ and $S$.



We construct the following $\mathbb{Q}$-martingale:

$$
\begin{aligned}
\frac{S_{t_k}}{e^{rt_k}} &= q\frac{S_{t_k}e^{r\Delta t+\sigma\sqrt{\Delta t}}}{e^{r(t_k+\Delta t)}} + (1-q)\frac{S_{t_k}e^{r\Delta t-\sigma\sqrt{\Delta t}}}{e^{r(t_k+\Delta t)}} \\
\Rightarrow 1 &= q\frac{e^{r\Delta t+\sigma\sqrt{\Delta t}}}{e^{r\Delta t}} + (1-q)\frac{e^{r\Delta t-\sigma\sqrt{\Delta t}}}{e^{r\Delta t}} \\
\Rightarrow q(e^{r\Delta t+\sigma\sqrt{\Delta t}} &- e^{r\Delta t-\sigma\sqrt{\Delta t}}) = e^{r\Delta t} - e^{r\Delta t-\sigma\sqrt{\Delta t}} \\
\Rightarrow q &= \frac{e^{r\Delta t} - e^{r\Delta t-\sigma\sqrt{\Delta t}}}{e^{r\Delta t+\sigma\sqrt{\Delta t}} - e^{r\Delta t-\sigma\sqrt{\Delta t}}}
\end{aligned}
$$

We then Taylor-expand the exponential terms and collect all all the terms containing $\Delta t$ with order 2 or higher to be $o(\Delta t)$:

$$
\begin{aligned}
q &= \frac{e^{r\Delta t} - e^{r\Delta t-\sigma\sqrt{\Delta t}}}{e^{r\Delta t+\sigma\sqrt{\Delta t}} - e^{r\Delta t-\sigma\sqrt{\Delta t}}} \\
&= \frac{1 + r\Delta t - (1 + r\Delta t - \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t) + o(\Delta t)}{1 + r\Delta t + \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t - (1 + r\Delta t - \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t) + o(\Delta t)} \\
&= \frac{\sigma\sqrt{\Delta t} - \frac{1}{2}\sigma^2\Delta t + o(\Delta t)}{2\sigma\sqrt{\Delta t} + o(\Delta t)} \\
&= \frac{1}{2} - \frac{1}{4}\sigma\sqrt{\Delta t} + o(\sqrt{\Delta t}) \\
&= \frac{1}{2}(1 - \frac{1}{2}\sigma\sqrt{\Delta t}) + o(\sqrt{\Delta t})
\end{aligned}
$$

## Part 2

Similarly, we want to find the martingale measure $\mathbb{Q}^S(\epsilon_k = \pm 1)$ where $\mathbb{Q}^S(\epsilon_k = 1) = h$ and $\mathbb{Q}^S(\epsilon_k = -1) = 1 - h$. Consider time $t_k$ and $t_{k_1}$, this time we use $S$ as the numeraire.

We construct the following $\mathbb{Q}^S$-martingale:

$$\begin{aligned}
\frac{e^{rt_k}}{S_{t_k}} &= h\frac{e^{r(t_k+\Delta t)}}{S_{t_k}e^{r\Delta t+\sigma\sqrt{\Delta t}}} + (1-h)\frac{e^{r(t_k+\Delta t)}}{S_{t_k}e^{r\Delta t-\sigma\sqrt{\Delta t}}} \\
\Rightarrow 1 &= h\frac{e^{r\Delta t}}{e^{r\Delta t+\sigma\sqrt{\Delta t}}} + (1-h)\frac{e^{r\Delta t}}{e^{r\Delta t-\sigma\sqrt{\Delta t}}} \\
\Rightarrow e^{2r\Delta t} &= he^{2r\Delta t-\sigma\sqrt{\Delta t}} + (1-h)e^{2r\Delta t+\sigma\sqrt{\Delta t}} \\
\Rightarrow e^{2r\Delta t} - e^{2r\Delta t+\sigma\sqrt{\Delta t}} &= h(e^{2r\Delta t-\sigma\sqrt{\Delta t}} - e^{2r\Delta t+\sigma\sqrt{\Delta t}}) \\
\Rightarrow h &= \frac{e^{2r\Delta t} - e^{2r\Delta t+\sigma\sqrt{\Delta t}}}{e^{2r\Delta t-\sigma\sqrt{\Delta t}} - e^{2r\Delta t+\sigma\sqrt{\Delta t}}}
\end{aligned}$$

We then Taylor-expand the exponential terms and collect all all the terms containing $\Delta t$ with order 2 or higher to be $o(\Delta t)$:

$$\begin{aligned}
h &= \frac{e^{2r\Delta t} - e^{2r\Delta t+\sigma\sqrt{\Delta t}}}{e^{2r\Delta t-\sigma\sqrt{\Delta t}} - e^{2r\Delta t+\sigma\sqrt{\Delta t}}} \\
&= \frac{1 + 2r\Delta t - (1 + 2r\Delta t + \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t) + o(\Delta t)}{1 + 2r\Delta t - \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t - (1 + 2r\Delta t + \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t) + o(\Delta t)} \\
&= \frac{-\sigma\sqrt{\Delta t} - \frac{1}{2}\sigma^2\Delta t + o(\Delta t)}{-2\sigma\sqrt{\Delta t} + o(\Delta t)} \\
&= \frac{\sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t + o(\Delta t)}{2\sigma\sqrt{\Delta t} + o(\Delta t)} \\
&= \frac{1}{2}(1 + \frac{1}{2}\sigma\sqrt{\Delta t}) + o(\sqrt{\Delta t})
\end{aligned}$$