Storage Requirements

August 7, 2021

THIS ARTICLE establishes storage requirements for the structures

- point
- line
- quadrance[1] between points

1 Notation

1.1 The Symbol N

Unless otherwise specified, the symbol N represents an arbitrary—constant—power of two. A secondary aim of this article is to establish additional bands, if not constraints, on N.

1.2 The Symbols a–z

Unless otherwise specified, the symbols

$$a\ b\ c\ d\ e\ f\ g\ h\ i\ j\ k\ l\ m\ n\ o\ p\ q\ r\ s\ t\ u\ v\ w\ x\ y\ z$$

each represent an arbitrary integer.

1.3 Concatenation and the Symbol $\times$

$$ab \equiv a \times b$$

That is, the expression ab, where the symbols a and b are concatenated, has the same meaning as the expression $a \times b$.

If both a and b are integers, ab is the product of a and b.

For example, if $a = 6$ and $b = 7$, $ab = a \times b = 6 \times 7 = 42$.

[1] Quadrance can be thought of as the square of distance.

1.4 Whitespace and Lists

Two or more operands separated by whitespace form a list. The expression

$$a\ b$$

denotes a list containing the items a and b—in that order. Blank lines distingish the above list from what is not the list.

Lists may be preceeded by an open parenthesis and proceeded by a close parenthesis, indicating the boundaries of the list. For example,

$$()\ (a)\ (a\ b)$$

denotes a list of three items: an empty list, a list of one item, and list of two items.

2 Cardinality of a List

The cardinality of a list is the number of items in that list. The cardinality operator is $\#$. For example:

$$\#() = 0$$

$$\#(0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) = 9$$

$$\#(f\ o\ u\ r) = 4$$

3 List Items

Let l be a list. Provided that $0 < k \leq \#l$, l_k is the k^{th} item of l. For example, $(11\ 12\ 13)_2 = 12$.

4 Types of Lists

A list may be declared to be of a certain type using a "type name." Operations on particular types of lists may be defined.

5 Vectors

Let "vector" be a type name. Let

$$\text{vec} = \text{vector}$$

then the expression

$$\text{vec}(a\ b)$$

denotes a vector (i.e., a list of type vector).

6 Untyped Lists

We might wish to define a list in terms of a typed list:

$$\text{list } t(a\ b) = (a\ b)$$

where t is a type name. For example,

$$\text{list vec}(a\ b) = (a\ b)$$

7 Max and Mix

The "max" of a list of integers is an integer among the list items whose value is not less than any other integer in the list. Similarly, The "min" of a list of integers is an integer among the list items whose value is not greater than any other integer in the list. For example:

$$\max(0\ 1) = 1$$

$$\max(0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) = 0$$

$$\min(0\ 1) = 0$$

$$\min(0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) = 0$$

8 Dot Product of Two Vectors

Let the symbol $\cdot$ denote the binary operation "dot product."

$$\text{vec } u \cdot \text{vec } v \equiv \sum_{i=1}^{n} u_i v_i$$

where u and v are lists of integers, $0 < \#u = \#v = n$. For example,

$$\text{vec } (2\ 3) \cdot \text{vec } (5\ 7) = 2 \times 5 + 3 \times 7 = 10 + 21 = 31$$

9 Intervals

Let "interval" be a type name. Let itv = interval. Provided that $a \leq b$,

$$k \in \text{itv } (a\ b) \Leftrightarrow a \leq k \leq b$$

$$n \notin \text{itv } (a\ b) \Leftrightarrow n < a \vee b < n$$

In other words, itv $(a\ b)$ indicates an arbitrary integer k, between a and b inclusive. And n represents an arbitrary integer beyond itv $(a\ b)$.

For example,

$$\text{itv } (0\ 1)$$

represents an unknown integer that could be either 0 or 1, and that can be only 0 or 1.

9.1 Intervals in Intervals

$$\text{itv }(a\ b) \in \text{itv }(c\ d) \Leftrightarrow a \in \text{itv }(c\ d) \wedge b \in \text{itv }(c\ d)$$

In other words, itv $(a\ b)$ is in itv $(c\ d)$ if and only if $c \leq a \leq d$ and $c \leq b \leq d$. For example,

$$\text{itv }(1 - 2^{60} \quad 2^{60} - 1) \in \text{itv }(1 - 2^{61} \quad 2^{61} - 1)$$

9.2 Special Intervals

$$\mathbf{N_0} \equiv \text{itv}(1 - N \quad N - 1)$$
$$\mathbf{N_1} \equiv \text{itv}(1 - 2N \quad 2N - 1)$$
$$\mathbf{N_2} \equiv \text{itv}(1 - 4N \quad 4N - 1)$$
$$\mathbf{N_k} \equiv \text{itv}(1 - 2^k N \quad 2^k N - 1)$$

$$\mathbf{N_0^2} \equiv \text{itv}(1 - N^2 \quad N^2 - 1)$$
$$\mathbf{N_1^2} \equiv \text{itv}(1 - 2N^2 \quad 2N^2 - 1)$$
$$\mathbf{N_2^2} \equiv \text{itv}(1 - 4N^2 \quad 4N^2 - 1)$$
$$\mathbf{N_k^2} \equiv \text{itv}(1 - 2^k N^2 \quad 2^k N^2 - 1)$$

9.3 Interval Addition

$$\text{itv}(a\ b) + \text{itv}(c\ d) \equiv \text{itv}(a + c \quad b + d)$$

In particular,

$$\mathbf{N_0} + \mathbf{N_0} \in \mathbf{N_1}$$

since

$$\mathbf{N_0} + \mathbf{N_0} = \text{itv}(1 - N \quad N - 1) + \text{itv}(1 - N \quad N - 1)$$
$$=$$
$$\text{itv}(2 - 2N \quad 2N - 2) \in \mathbf{N_1}$$

9.4 Interval Subtraction

$$\text{itv}(a\ b) - \text{itv}(c\ d) \equiv \text{itv}(a - d \quad b - c)$$

In particular,

$$\mathbf{N_0} - \mathbf{N_0} \in \mathbf{N_1}$$

since

$$\mathbf{N_0} - \mathbf{N_0} = \text{itv}(1 - N \quad N - 1) - \text{itv}(1 - N \quad N - 1)$$
$$=$$
$$\text{itv}(2 - 2N \quad 2N - 2) \in \mathbf{N_1}$$

9.5 Interval Multiplication

$$\text{itv}(a\ b) \times \text{itv}(c\ d) \equiv \text{itv}(\min l \quad \max l)$$

where $l = (ac\ ad\ bc\ bd)$. In particular,

$$\mathbf{N_0} \times \mathbf{N_0} \in \mathbf{N_0^2}$$

since

$$\mathbf{N_0} \times \mathbf{N_0} = \text{itv}(1 - N \quad N - 1) \times \text{itv}(1 - N \quad N - 1)$$

$$=$$

$$\text{itv}(-N^2 + 2N - 1 \quad N^2 - 2N + 1) \in \text{itv}(1 - N^2 \quad N^2 - 1) \in \mathbf{N_0^2}$$

10 Points

Let "point" be a type name.

Unless otherwise specified,

$$\#\ \text{list}\ p = 2$$

where p is a point. I.e., points are two-dimensional.

For point$(a\ b)$,

$$a \in \mathbf{N_0}, b \in \mathbf{N_0}$$

In other words, each coordinate value, a and b is taken from a finite range of integers between $1 - N$ and $N - 1$.

11 Definition of Vector Quadrance

Let q = quadrance, a unary operator. The quadrance of a vector v of integers is the integer:

$$\text{q}v \equiv v \cdot v$$

In particular, q vec$(a\ b)$ = vec$(a\ b) \cdot$ vec$(a\ b)$. And, provided that $a, b \in \mathbf{N_0}$:

$$\text{q vec(a b)} \in \mathbf{N_0 N_0} + \mathbf{N_0 N_0} \in \mathbf{N_0^2} + \mathbf{N_0^2} \in \mathbf{N_1^2}$$

12 Proportions

Let "proportion" be a type name. Let prp = proportion. Provided that l is a list of integers; $1 < \#l$, q vec $l \neq 0$:

$$\text{prp}\ l$$

is a proportion. An expression equivalent to prp$(0, 0)$ is undefined. However, both prp$(1, 0)$ and prp$(0, 1)$ are defined.

12.1 Equivalent Proportions

$$\text{prp}(a\ b) = \text{prp}(c\ d) \Leftrightarrow ad = bc$$

13 Vector Difference

Provided that u anv v are vectors of integers, $0 < \#u = \#v = n$,

$$u - v \equiv \text{vec}(u_1 - v_1 \quad u_2 - v_2 \quad \dots \quad u_n - v_n)$$

Provided that $u_k \in \mathbf{N_0}$ and $v_k \in \mathbf{N_0}$, $w_k \in \mathbf{N_1}$, where $w = u - v$.

14 Vector Sum

Provided that u anv v are vectors of integers, $0 < \#u = \#v = n$,

$$u + v \equiv \text{vec}(u_1 + v_1 \quad u_2 + v_2 \quad \dots \quad u_n + v_n)$$

Provided that $u_k \in \mathbf{N_0}$ and $v_k \in \mathbf{N_0}$, $w_k \in \mathbf{N_1}$, where $w = u + v$.

15 Differecnce between Points

$$\text{point}(a\ b) - \text{point}(c\ d) \equiv \text{vec}(a - c \quad b - d)$$

Provided that $a, b, c, d \in \mathbf{N_0}$; $a - c \in \mathbf{N_1}$, $b - d \in \mathbf{N_1}$.

Again, for emphasis, the difference between points is a vector, not a point.

16 Sum of Two Points

$$\text{point}(a\ b) + \text{point}(c\ d) \equiv \text{vec}(a + c \quad b + d)$$

Provided that $a, b, c, d \in \mathbf{N_0}$; $a + c \in \mathbf{N_1}$, $b + d \in \mathbf{N_1}$.

Again, for emphasis, the sum of points is a vector, not a point.

17 Array

Let "array" be a type name. Let aa = array[2].

17.1 Array Product

Let u and v be arrays, $0 < \#u = \#v = n$.

$$w = uv \Leftrightarrow w_i = u_i v_i, \forall\ 1 \leq i \leq n$$

[2]Mnemonic: "aa" is an array of a's.

18 Pitch

Let "pitch" be a type name.

18.1 Pitch of two Points

Let p and r be points.

$$\text{pitch}(p\ r) \equiv \text{pitch}(r_1 - p_1 \quad p_2 - r_2)$$

In particular, pitch(0 1) is defined, whereas the hypothetical "slope" $\frac{1}{0}$ is not.

Also, in particular, provided that $p_1, p_2, r_1, r_2 \in \mathbf{N_0}$; $m = \text{pitch}(p\ r)$: $m_1, m_2 \in \mathbf{N_1}$.

18.2 Pitch-Quadrance

Let p be a pitch. Then

$$\text{q}\ p \equiv \text{q vec list}\ p$$

19 Distinct Points

Two points p and r are distinct if and only if

$$\text{q}\ s \neq 0$$

where $s = r - p$.

20 P-Parallelograms

Let "p-parallelogram" be a type name. Let ppara = p-parallelogram. Let p, r, s, and t be points distinct from one another. Provided that

$$\text{pitch}(p\ r) = \text{pitch}(t\ s) \text{ and}$$

$$\text{pitch}(r\ s) = \text{pitch}(p\ t)$$

pppara($p\ r\ s\ t$) is defined.

21 Quadrance between Two Points

Let p and r be points. Let[3] $d = p - r$.

$$\text{q}(p\ r) \equiv \text{q}\ d$$

Provided that $p, r, s, t \in \mathbf{N_0} : d \in \mathbf{N_1}, \text{q}\ d \in \mathbf{N_2^2}$

[3] Recall that d is a vector.

22 Lines and Half-Planes I

Let "line," "nplane," (inclusive half-plane) and "xplane" (exclusive half plane) be type names.

Let $l = (c\ a\ b)$, $0 < a^2 + b^2$.

22.1 Pitch of Lines and Half-planes

$$\text{pitch line } l \equiv \text{pitch}(a\ b)$$

$$\text{pitch nplane } l \equiv \text{pitch}(a\ b)$$

$$\text{pitch xplane } l \equiv \text{pitch}(a\ b)$$

22.2 Dot Product of Vector with Line or Half-Plane

Let v be a vector, $\#v = 3$. Let h be any one of line l, nplane l, or xplane l.

$$v \cdot h \equiv v \cdot \text{vec list } h$$

22.3 Point on a Line

$$\text{point}(x\ y) \in \text{line } l \Leftrightarrow u \cdot l = 0$$

where

$$u = \text{vec}(1\ x\ y)$$

22.4 Point in a Half-Plane

$$\text{point}(x\ y) \in \text{xplane } l \Leftrightarrow u \cdot l < 0$$

$$\text{point}(x\ y) \in \text{nplane } l \Leftrightarrow u \cdot l \leq 0$$

where

$$u = \text{vec}(1\ x\ y)$$

23 Product of a Scalar and a Point

$$k \times \text{point}(a\ b) \equiv \text{vec}(ka \quad kb)$$

24 Protocentroid of a P-Parallelogram

$$\text{protocentroid ppara}(p\ r\ s\ t) \equiv p + s = r + t,$$

where p, r, s, and t are points.[4]

Provided that $p_1, s_1 \in \mathbf{N_0}$; $(p + s)_1 \in \mathbf{N_1}$.

[4] Recall that the sum of two points is a vector.

25 Determinant of Two Points

Let "determinant" be a type name. Let det = determinant. Let $p = (a\ b)$ and $r = (c\ d)$. Then

$$\det(p\ r) \equiv \det(ad - bc).$$

Provided that $a, b, c, d \in \mathbf{N_0}; ad \in \mathbf{N_0^2}, bc \in \mathbf{N_0^2}; \det(p\ r) \in \mathbf{N_1^2}$.

26 Line through Two Points

Let p and r be distinct points. Let l be the line passing through p and r. Then

$$l = \mathrm{line}(p_1 r_2 - p_2 r_1 \quad p_2 - r_2 \quad r_1 - p_1) =$$

$$= \mathrm{line}(\det(\mathrm{list}\ p \quad \mathrm{list}\ r) \quad p_2 - r_2 \quad r_1 - p_1)$$

Provided that $p_1, p_2, r_1, r_2 \in \mathbf{N_0}, l_1 \in \mathbf{N_1^2}, l_2 \in \mathbf{N_1}, l_3 \in \mathbf{N_1}$.

27 Line Amid Two P-Parallelograms

(See an accompanying paper for the definition of "amid.")

Provided that x and y are lists of eight integers each and satisfy

$$P \equiv \mathrm{ppara}(\mathrm{point}(x_1 \quad y_1)\ \mathrm{point}(x_2 \quad y_2)\ \mathrm{point}(x_3 \quad y_3)\ \mathrm{point}(x_4 \quad y_4))$$

and

$$R \equiv \mathrm{ppara}(\mathrm{point}(x_5 \quad y_5)\ \mathrm{point}(x_6 \quad y_6)\ \mathrm{point}(x_7 \quad y_7)\ \mathrm{point}(x_8 \quad y_8))$$

The line l amid each pair of diagonal-points of two p-parallelograms is

$$l = \mathrm{line}(ad - bc \quad 2b - 2d \quad 2c - 2a)$$

where $a = x_1 + x_3, b = y_1 + y_3, c = x_5 + x_7, d = y_5 + y_7$.

Provided that $x_k, y_k \in \mathbf{N_0}$:

$$l_1 \in \mathbf{N_1 N_1} - \mathbf{N_1 N_1} \in \mathbf{N_2^2} - \mathbf{N_2^2} \in \mathbf{N_3^2}$$

$$l_2, l_3 \in 2\mathbf{N_1} - 2\mathbf{N_1} \in \mathbf{N_2} - \mathbf{N_2} \in \mathbf{N_3}$$

28 Storage Requirements

The above considerations tolerate points with items $\in \mathbf{N_0}$ and lines with zero-order terms $\in \mathbf{N_3}$ and first-order terms $\in \mathbf{N_3^2}$.

Suppose our computing machinery accomodates words of $n + 4$ bits[5], in which we store any value $\in \mathbf{N_3}$. Then[6] n corresponds with values in $\mathbf{N_0}$. Two such words would have $2n + 8$ bits, which is enough to store any value $\in \mathbf{N_8^2}$ and therefore any value $\in \mathbf{N_3^2}$.

[5] perhaps 64 bits per word

[6] we might choose $n = 60$.