Beyond Divine Proportions

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Super-Rational Computational Geometry
an Unfinished Work

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Beyond Divine Proportions: Super-Rational Computational Geometry—an Unfinished Work

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10 9 8 7 6 5 4 3 2

“God invented the integers. . . ”
—Leopold Kronecker

Preface

This *unfinished work* is being released on the occasion of the 2021 South Florida GIS Expo as a means of soliciting feedback from the GIS community.

The less developed a project, the easier it is to incorporate suggested changes.

Summary

This collection of articles elaborates on the consequences of a design choice in which geometric objects are built up without using floating-point arithmetic, but with integers only.

We begin with "Motivation," whose title is sufficient to summarize its content.

The main arc of the collection is to establish storage requirements (i.e., how many bits and words are needed) in order to achieve or surpass the precision many GIS professionals have grown accustomed to.

In "Storage Requrements," we show that 64-bit integers are more than sufficient to surpass the precision of conventional typical floating-point point numbers. Additionally, they can be used efficiently to build up more complex structures, such as lines.

So as not interrupt the flow of "Storage Requirements," the subtopics of "Line Amid Two Points," "Line through a Point and the Centroid of a P-Parallelogram" (a p-parallelogram is a special kind of parallelogram), and "Line through the Centroids of Two P-Parallelograms" are presented in separate articles.

Introduction

In this collection of papers, we begin to develop a GIS system that is is more comprehensible and more accessible than what presently dominates the industry.

One goal is that a wide range of GIS applications—with emphasis on computational geometry—be comprehensible to the very young. Young people should have tools to *precisely quantify* concepts such as “near,” “within,” “in the direction of,” and so forth.

Although conventional geometry and trigonometry, as taught in many school systems, take a steps in this direction, they go off on a tangent whith respect to the *central* question of GIS:

Where?

The number system, we believe, is the natural starting point for a journey toward a more comprehensible, more accessible GIS. Numbers and quantity go hand in hand. To the extent we are able, we will favor

- integers rather than fractions or real numbers,
- multiplication rather than division, and
- squares rather than square roots

We acknowledge N J Wildberger for a great deal of inspiration as well as content. Much of what is presented herein flows from his work, especially his 2005 book *Divine Proportions: Rational Trigonometry to Universal Geometry* (Wild Egg Pty Ltd; Australia. ISBN 097574920X). The author also has gained much from Wilberger’s numerous lectures (search: “wildtrig”) on various topics in pure and applied mathematics.

Motivation

July 30, 2021

We assert that the computational power of GIS can be made accessible to a wider range of people. In particular, people of younger age or with less *formal* education than seem to populate the current roster of “GIS professionals.”

GIS has inherent *complexity*. However, aspects of GIS are unnecessarily *complicated*.

In particular, some popular GIS systems exhibit unnecessary complexity in at least two ways:

1. the application of floating point number systems

2. the obfuscation of a fundamental building block of geometry: the line.

We also assert that precision matters. It is of little use to communicate a location without also communicating the precision with which that location is known.

Storage Requirements

August 7, 2021

THIS ARTICLE establishes storage requirements for the structures

- point
- line
- quadrance[1] between points

1 Notation

1.1 The Symbol N

Unless otherwise specified, the symbol N represents an arbitrary—constant—power of two. A secondary aim of this article is to establish additional bands, if not constraints, on N.

1.2 The Symbols a–z

Unless otherwise specified, the symbols

$$a\ b\ c\ d\ e\ f\ g\ h\ i\ j\ k\ l\ m\ n\ o\ p\ q\ r\ s\ t\ u\ v\ w\ x\ y\ z$$

each represent an arbitrary integer.

1.3 Concatenation and the Symbol $\times$

$$ab \equiv a \times b$$

That is, the expression ab, where the symbols a and b are concatenated, has the same meaning as the expression $a \times b$.

If both a and b are integers, ab is the product of a and b.

For example, if $a = 6$ and $b = 7$, $ab = a \times b = 6 \times 7 = 42$.

[1] Quadrance can be thought of as the square of distance.

1.4 Whitespace and Lists

Two or more operands separated by whitespace form a list. The expression

$$a\ b$$

denotes a list containing the items a and b—in that order. Blank lines distingish the above list from what is not the list.

Lists may be preceeded by an open parenthesis and proceeded by a close parenthesis, indicating the boundaries of the list. For example,

$$()\ (a)\ (a\ b)$$

denotes a list of three items: an empty list, a list of one item, and list of two items.

2 Cardinality of a List

The cardinality of a list is the number of items in that list. The cardinality operator is $\#$. For example:

$$\#() = 0$$

$$\#(0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) = 9$$

$$\#(f\ o\ u\ r) = 4$$

3 List Items

Let l be a list. Provided that $0 < k \leq \#l$, l_k is the k^{th} item of l. For example, $(11\ 12\ 13)_2 = 12$.

4 Types of Lists

A list may be declared to be of a certain type using a "type name." Operations on particular types of lists may be defined.

5 Vectors

Let "vector" be a type name. Let

$$\text{vec} = \text{vector}$$

then the expression

$$\text{vec}(a\ b)$$

denotes a vector (i.e., a list of type vector).

6 Untyped Lists

We might wish to define a list in terms of a typed list:

$$\text{list } t(a\ b) = (a\ b)$$

where t is a type name. For example,

$$\text{list vec}(a\ b) = (a\ b)$$

7 Max and Mix

The "max" of a list of integers is an integer among the list items whose value is not less than any other integer in the list. Similarly, The "min" of a list of integers is an integer among the list items whose value is not greater than any other integer in the list. For example:

$$\max(0\ 1) = 1$$

$$\max(0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) = 0$$

$$\min(0\ 1) = 0$$

$$\min(0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) = 0$$

8 Dot Product of Two Vectors

Let the symbol $\cdot$ denote the binary operation "dot product."

$$\text{vec } u \cdot \text{vec } v \equiv \sum_{i=1}^{n} u_i v_i$$

where u and v are lists of integers, $0 < \#u = \#v = n$. For example,

$$\text{vec } (2\ 3) \cdot \text{vec } (5\ 7) = 2 \times 5 + 3 \times 7 = 10 + 21 = 31$$

9 Intervals

Let "interval" be a type name. Let itv = interval. Provided that $a \leq b$,

$$k \in \text{itv } (a\ b) \Leftrightarrow a \leq k \leq b$$

$$n \notin \text{itv } (a\ b) \Leftrightarrow n < a \vee b < n$$

In other words, itv $(a\ b)$ indicates an arbitrary integer k, between a and b inclusive. And n represents an arbitrary integer beyond itv $(a\ b)$.

For example,

$$\text{itv } (0\ 1)$$

represents an unknown integer that could be either 0 or 1, and that can be only 0 or 1.

9.1 Intervals in Intervals

$$\text{itv }(a\ b) \in \text{itv }(c\ d) \Leftrightarrow a \in \text{itv }(c\ d) \wedge b \in \text{itv }(c\ d)$$

In other words, itv $(a\ b)$ is in itv $(c\ d)$ if and only if $c \leq a \leq d$ and $c \leq b \leq d$. For example,

$$\text{itv }(1 - 2^{60} \quad 2^{60} - 1) \in \text{itv }(1 - 2^{61} \quad 2^{61} - 1)$$

9.2 Special Intervals

$$\begin{aligned}
\mathbf{N_0} &\equiv \text{itv}(1 - N \quad N - 1) \\
\mathbf{N_1} &\equiv \text{itv}(1 - 2N \quad 2N - 1) \\
\mathbf{N_2} &\equiv \text{itv}(1 - 4N \quad 4N - 1) \\
\mathbf{N_k} &\equiv \text{itv}(1 - 2^k N \quad 2^k N - 1)
\end{aligned}$$

$$\begin{aligned}
\mathbf{N_0^2} &\equiv \text{itv}(1 - N^2 \quad N^2 - 1) \\
\mathbf{N_1^2} &\equiv \text{itv}(1 - 2N^2 \quad 2N^2 - 1) \\
\mathbf{N_2^2} &\equiv \text{itv}(1 - 4N^2 \quad 4N^2 - 1) \\
\mathbf{N_k^2} &\equiv \text{itv}(1 - 2^k N^2 \quad 2^k N^2 - 1)
\end{aligned}$$

9.3 Interval Addition

$$\text{itv}(a\ b) + \text{itv}(c\ d) \equiv \text{itv}(a + c \quad b + d)$$

In particular,

$$\mathbf{N_0} + \mathbf{N_0} \in \mathbf{N_1}$$

since

$$\begin{gathered}
\mathbf{N_0} + \mathbf{N_0} = \text{itv}(1 - N \quad N - 1) + \text{itv}(1 - N \quad N - 1) \\
= \\
\text{itv}(2 - 2N \quad 2N - 2) \in \mathbf{N_1}
\end{gathered}$$

9.4 Interval Subtraction

$$\text{itv}(a\ b) - \text{itv}(c\ d) \equiv \text{itv}(a - d \quad b - c)$$

In particular,

$$\mathbf{N_0} - \mathbf{N_0} \in \mathbf{N_1}$$

since

$$\begin{gathered}
\mathbf{N_0} - \mathbf{N_0} = \text{itv}(1 - N \quad N - 1) - \text{itv}(1 - N \quad N - 1) \\
= \\
\text{itv}(2 - 2N \quad 2N - 2) \in \mathbf{N_1}
\end{gathered}$$

9.5 Interval Multiplication

$$\mathrm{itv}(a\ b) \times \mathrm{itv}(c\ d) \equiv \mathrm{itv}(\min l \quad \max l)$$

where $l = (ac\ ad\ bc\ bd)$. In particular,

$$\mathbf{N_0} \times \mathbf{N_0} \in \mathbf{N_0^2}$$

since

$$\mathbf{N_0} \times \mathbf{N_0} = \mathrm{itv}(1-N \quad N-1) \times \mathrm{itv}(1-N \quad N-1)$$
$$=$$
$$\mathrm{itv}(-N^2+2N-1 \quad N^2-2N+1) \in \mathrm{itv}(1-N^2 \quad N^2-1) \in \mathbf{N_0^2}$$

10 Points

Let "point" be a type name.

Unless otherwise specified,

$$\#\ \mathrm{list}\ p = 2$$

where p is a point. I.e., points are two-dimensional.

For point$(a\ b)$,

$$a \in \mathbf{N_0}, b \in \mathbf{N_0}$$

In other words, each coordinate value, a and b is taken from a finite range of integers between $1-N$ and $N-1$.

11 Definition of Vector Quadrance

Let q = quadrance, a unary operator. The quadrance of a vector v of integers is the integer:

$$\mathrm{q}v \equiv v \cdot v$$

In particular, q vec$(a\ b)$ = vec$(a\ b) \cdot$ vec$(a\ b)$. And, provided that $a, b \in \mathbf{N_0}$:

$$\text{q vec(a b)} \in \mathbf{N_0 N_0} + \mathbf{N_0 N_0} \in \mathbf{N_0^2} + \mathbf{N_0^2} \in \mathbf{N_1^2}$$

12 Proportions

Let "proportion" be a type name. Let prp = proportion. Provided that l is a list of integers; $1 < \#l$, q vec $l \neq 0$:

$$\mathrm{prp}\ l$$

is a proportion. An expression equivalent to prp$(0,0)$ is undefined. However, both prp$(1,0)$ and prp$(0,1)$ are defined.

12.1 Equivalent Proportions

$$\text{prp}(a\ b) = \text{prp}(c\ d) \Leftrightarrow ad = bc$$

13 Vector Difference

Provided that u anv v are vectors of integers, $0 < \#u = \#v = n$,

$$u - v \equiv \text{vec}(u_1 - v_1 \quad u_2 - v_2 \quad \dots \quad u_n - v_n)$$

Provided that $u_k \in \mathbf{N_0}$ and $v_k \in \mathbf{N_0}$, $w_k \in \mathbf{N_1}$, where $w = u - v$.

14 Vector Sum

Provided that u anv v are vectors of integers, $0 < \#u = \#v = n$,

$$u + v \equiv \text{vec}(u_1 + v_1 \quad u_2 + v_2 \quad \dots \quad u_n + v_n)$$

Provided that $u_k \in \mathbf{N_0}$ and $v_k \in \mathbf{N_0}$, $w_k \in \mathbf{N_1}$, where $w = u + v$.

15 Differecnce between Points

$$\text{point}(a\ b) - \text{point}(c\ d) \equiv \text{vec}(a - c \quad b - d)$$

Provided that $a, b, c, d \in \mathbf{N_0}$; $a - c \in \mathbf{N_1}$, $b - d \in \mathbf{N_1}$.

Again, for emphasis, the difference between points is a vector, not a point.

16 Sum of Two Points

$$\text{point}(a\ b) + \text{point}(c\ d) \equiv \text{vec}(a + c \quad b + d)$$

Provided that $a, b, c, d \in \mathbf{N_0}$; $a + c \in \mathbf{N_1}$, $b + d \in \mathbf{N_1}$.

Again, for emphasis, the sum of points is a vector, not a point.

17 Array

Let "array" be a type name. Let aa = array[2].

17.1 Array Product

Let u and v be arrays, $0 < \#u = \#v = n$.

$$w = uv \Leftrightarrow w_i = u_i v_i, \forall\ 1 \leq i \leq n$$

[2] Mnemonic: "aa" is an array of a's.

18 Pitch

Let "pitch" be a type name.

18.1 Pitch of two Points

Let p and r be points.

$$\text{pitch}(p\ r) \equiv \text{pitch}(r_1 - p_1 \quad p_2 - r_2)$$

In particular, pitch(0 1) is defined, whereas the hypothetical "slope" $\frac{1}{0}$ is not.

Also, in particular, provided that $p_1, p_2, r_1, r_2 \in \mathbf{N_0}$; $m = \text{pitch}(p\ r)$: $m_1, m_2 \in \mathbf{N_1}$.

18.2 Pitch-Quadrance

Let p be a pitch. Then

$$\text{q}\ p \equiv \text{q vec list}\ p$$

19 Distinct Points

Two points p and r are distinct if and only if

$$\text{q}\ s \neq 0$$

where $s = r - p$.

20 P-Parallelograms

Let "p-parallelogram" be a type name. Let ppara = p-parallelogram. Let p, r, s, and t be points distinct from one another. Provided that

$$\text{pitch}(p\ r) = \text{pitch}(t\ s) \text{ and}$$

$$\text{pitch}(r\ s) = \text{pitch}(p\ t)$$

pppara($p\ r\ s\ t$) is defined.

21 Quadrance between Two Points

Let p and r be points. Let[3] $d = p - r$.

$$\text{q}(p\ r) \equiv \text{q}\ d$$

Provided that $p, r, s, t \in \mathbf{N_0} : d \in \mathbf{N_1}, \text{q}\ d \in \mathbf{N_2^2}$

[3] Recall that d is a vector.

22 Lines and Half-Planes I

Let "line," "nplane," (inclusive half-plane) and "xplane" (exclusive half plane) be type names.

Let $l = (c\ a\ b)$, $0 < a^2 + b^2$.

22.1 Pitch of Lines and Half-planes

$$\text{pitch line } l \equiv \text{pitch}(a\ b)$$

$$\text{pitch nplane } l \equiv \text{pitch}(a\ b)$$

$$\text{pitch xplane } l \equiv \text{pitch}(a\ b)$$

22.2 Dot Product of Vector with Line or Half-Plane

Let v be a vector, $\#v = 3$. Let h be any one of line l, nplane l, or xplane l.

$$v \cdot h \equiv v \cdot \text{vec list } h$$

22.3 Point on a Line

$$\text{point}(x\ y) \in \text{line } l \Leftrightarrow u \cdot l = 0$$

where

$$u = \text{vec}(1\ x\ y)$$

22.4 Point in a Half-Plane

$$\text{point}(x\ y) \in \text{xplane } l \Leftrightarrow u \cdot l < 0$$

$$\text{point}(x\ y) \in \text{nplane } l \Leftrightarrow u \cdot l \leq 0$$

where

$$u = \text{vec}(1\ x\ y)$$

23 Product of a Scalar and a Point

$$k \times \text{point}(a\ b) \equiv \text{vec}(ka \quad kb)$$

24 Protocentroid of a P-Parallelogram

$$\text{protocentroid ppara}(p\ r\ s\ t) \equiv p + s = r + t,$$

where p, r, s, and t are points.[4]

Provided that $p_1, s_1 \in \mathbf{N_0}$; $(p + s)_1 \in \mathbf{N_1}$.

[4] Recall that the sum of two points is a vector.

25 Determinant of Two Points

Let "determinant" be a type name. Let det = determinant. Let $p = (a\ \ b)$ and $r = (c\ \ d)$. Then

$$\det(p\ \ r) \equiv \det(ad - bc).$$

Provided that $a, b, c, d \in \mathbf{N_0}; ad \in \mathbf{N_0^2}, bc \in \mathbf{N_0^2}; \det(p\ \ r) \in \mathbf{N_1^2}$.

26 Line through Two Points

Let p and r be distinct points. Let l be the line passing through p and r. Then

$$l = \text{line}(p_1r_2 - p_2r_1 \quad p_2 - r_2 \quad r_1 - p_1) =$$

$$= \text{line}(\det(\text{list } p \quad \text{list } r) \quad p_2 - r_2 \quad r_1 - p_1)$$

Provided that $p_1, p_2, r_1, r_2 \in \mathbf{N_0}, l_1 \in \mathbf{N_1^2}, l_2 \in \mathbf{N_1}, l_3 \in \mathbf{N_1}$.

27 Line Amid Two P-Parallelograms

(See an accompanying paper for the definition of "amid.")

Provided that x and y are lists of eight integers each and satisfy

$$P \equiv \text{ppara}(\text{point}(x_1 \quad y_1)\ \text{point}(x_2 \quad y_2)\ \text{point}(x_3 \quad y_3)\ \text{point}(x_4 \quad y_4))$$

and

$$R \equiv \text{ppara}(\text{point}(x_5 \quad y_5)\ \text{point}(x_6 \quad y_6)\ \text{point}(x_7 \quad y_7)\ \text{point}(x_8 \quad y_8))$$

The line l amid each pair of diagonal-points of two p-parallelograms is

$$l = \text{line}(ad - bc \quad 2b - 2d \quad 2c - 2a)$$

where $a = x_1 + x_3, b = y_1 + y_3, c = x_5 + x_7, d = y_5 + y_7$.

Provided that $x_k, y_k \in \mathbf{N_0}$:

$$l_1 \in \mathbf{N_1N_1} - \mathbf{N_1N_1} \in \mathbf{N_2^2} - \mathbf{N_2^2} \in \mathbf{N_3^2}$$

$$l_2, l_3 \in 2\mathbf{N_1} - 2\mathbf{N_1} \in \mathbf{N_2} - \mathbf{N_2} \in \mathbf{N_3}$$

28 Storage Requirements

The above considerations tolerate points with items $\in \mathbf{N_0}$ and lines with zero-order terms $\in \mathbf{N_3}$ and first-order terms $\in \mathbf{N_3^2}$.

Suppose our computing machinery accomodates words of $n + 4$ bits[5], in which we store any value $\in \mathbf{N_3}$. Then[6] n corresponds with values in $\mathbf{N_0}$. Two such words would have $2n + 8$ bits, which is enough to store any value $\in \mathbf{N_8^2}$ and therefore any value $\in \mathbf{N_3^2}$.

[5] perhaps 64 bits per word

[6] we might choose $n = 60$.

Line Amid Two Points

July 31, 2021

1 Definition of "Amid"

The follwing defines "amid" with respect to a line and two points. In particular, the existence of a centroid of the two points plays no role.

$$l \equiv \text{line}(c \;\; a \;\; b), 0 < a^2 + b^2$$

$$v \equiv \text{vec list } l$$

$$\begin{aligned} p_1 &\equiv \text{point}(x_1 \quad y_1) \\ p_2 &\equiv \text{point}(x_2 \quad y_2) \end{aligned}$$

$$\begin{aligned} u_1 &\equiv \text{vec}(1 \quad x_1 \quad y_1) \\ u_2 &\equiv \text{vec}(1 \quad x_2 \quad y_2) \end{aligned}$$

$$\begin{aligned} k_1 &\equiv v \cdot u_1 \\ k_2 &\equiv v \cdot u_2 \end{aligned}$$

line l is "amid" points p_1 and $p_2 \Leftrightarrow k_1 = -k_2$.

2 Line Amid Two Points

The line l, $\text{line}(-ax_1 - ax_2 - by_1 - by_2 \quad 2a \quad 2b), 0 < a^2 + b^2$ is amid $\text{point}(x_1 \quad y_1)$ and $\text{point}(x_2 \quad y_2)$.

Proof. Let $k_1 =$

$$\text{vec}(-ax_1 - ax_2 - by_1 - by_2 \quad 2a \quad 2b) \cdot \text{vec}(1 \quad x_1 \quad y_1)$$

$$=$$

$$-ax_1 - ax_2 - by_1 - by_2 + 2ax_1 + 2by_1$$

$$=$$

$$ax_1 - ax_2 + by_1 - by_2$$

And let $k_2 =$

$$\text{vec}(-ax_1 - ax_2 - by_1 - by_2 \quad 2a \quad 2b) \cdot \text{vec}(1 \quad x_2 \quad y_2)$$

$$=$$

$$-ax_1 - ax_2 - by_1 - by_2 + 2ax_2 + 2by_2$$

$$=$$

$$-ax_1 + ax_2 - by_1 + by_2$$

$k_1 = -k_2$. QED

Partial Quadrance to a Line

August 18, 2021

Given $p = \text{point}(x\ y)$ and $l = \text{line}(c\ a\ b)$, the partial quadrance, pq, of p to l is

$$\text{pq}(p\ l) \equiv \text{vec}(1\ x\ y) \cdot \text{vec}\ l = c + ax + by$$

Consequently,

$$\text{pq}(p\ l) = \text{pq}(l\ p)$$

As expressed by Wildberger, the quadrance from p to l is

$$\frac{(c + ax + by)^2}{a^2 + b^2}$$

In terms of the quadrance and partial quadrance then, the quadrance from p to l may be expressed

$$\frac{\text{q pq}(p\ l)}{\text{q vec}(l_2\ l_3)}$$

where the quadrance of an integer n is

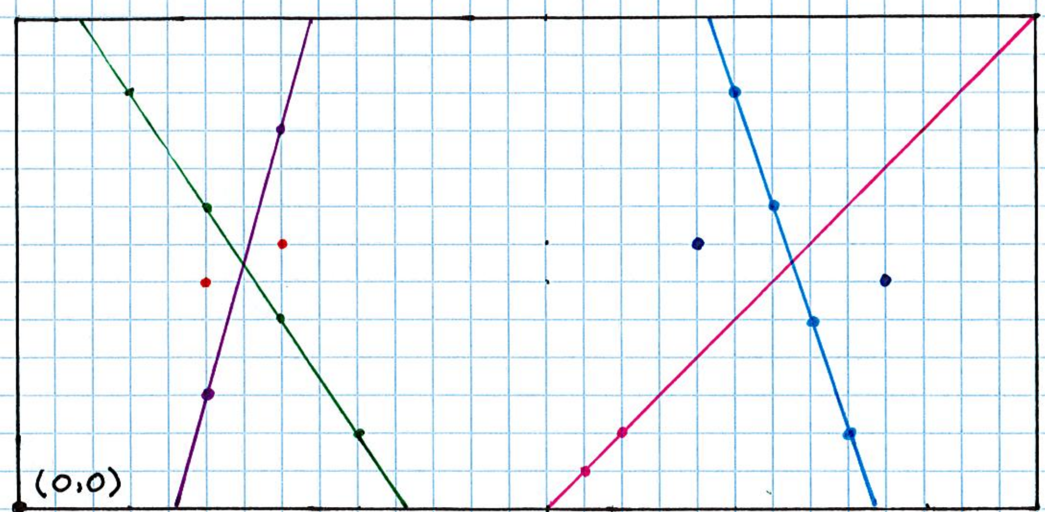
$$\text{q}\ n \equiv nn$$

Line l is amid[1] two points p_1 and p_2 if and only if

$$\text{pq}(l\ p_1) = -\text{pq}(l\ p_2)$$

[1]See also the separate article "Line Amid Two Points"

LINES AMID POINTS
EXAMPLE CALCULATIONS



	line	A	B	C	D	C-D	E	F
		x_1	x_2	$x_1 y_2$	$x_2 y_1$		Δx	Δy
		y_1	y_2					
(31 -3 -2)	GREEN	9	7	45	14	31	-2	3
		2	5					
(29 -7 2)	PURPLE	5	7	50	21	29	2	7
		3	10					
(68 -3 -1)	CYAN	22	21	110	42	68	-1	3
		2	5					
(14 -1 1)	PINK	15	16	30	16	14	1	1
		1	2					
	Points							
	RED	5	7					
		6	7					

	c	ax	by	partial quadrance
GREEN	31	(-3)(5)	(-2)(6)	
	31	-15	-12	4
	31	(-3)(7)	(-2)(7)	
	31	-21	-14	-4
PURPLE	29	(-7)(5)	(2)(6)	
	29	-35	12	6
	29	(-7)(7)	(2)(7)	
	29	-49	14	-6

	x_1	x_2
Points	y_1	y_2
BLUE	18	23
	7	6

	c	ax	by	partial quadrance
PINK	14	(-1)(18)	(1)(7)	
	14	-18	7	3
	14	(-1)(23)	(1)(6)	
	14	-23	6	-3
CYAN	68	(-3)(18)	(-1)(7)	
	68	-54	-7	7
	68	(-3)(23)	(-1)(6)	
	68	-69	-6	-7

Author's Note: It is my intent to rework this article, wherein there would be no reference to the rational numbers, exceping the integers. Additionally, it would reference the line amid the diagonals of a p-parallelogram rather than the centroid, since the centroid may not exist among the ordered pairs of integers. Additionally, the notation would be made consistent with the foregoing articles.

Line through a Point and the Centroid of a P-Parallelogram

July 20, 2021

Consider $x_k, y_k \in \mathbf{Q}$. Provided that quadrilateral

$$P \equiv ([x_1, y_1], [x_2, y_2], [x_3, y_3], [x_4, y_4])$$

is a parallelogram, the centroid of P is[1]

$$C_P = \left[\frac{x_{13}}{2}, \frac{y_{13}}{2}\right] = \left[\frac{x_{24}}{2}, \frac{y_{24}}{2}\right].$$

where $x_{13} \equiv x_1 + x_3,\ y_{13} \equiv y_1 + y_3,\ x_{24} \equiv x_2 + x_4,\ y_{24} \equiv y_2 + y_4$.

Provided that $C_P \neq [x_5, y_5] = R$, the line l_{PR} passing through C_P and R is

$$\mathrm{vec}(\frac{x_{13}}{2} y_5 - x_5 \frac{y_{13}}{2} \quad \frac{y_{13}}{2} - y_5 \quad x_5 - \frac{x_{13}}{2}) \cdot \mathrm{vec}(1\ x\ y) = 0$$

or[2]

$$\mathrm{vec}(x_{13} y_5 - x_5 y_{13} \quad 2y_{13} - y_5 \quad 2x_5 - x_{13}) \cdot \mathrm{vec}(1\ x\ y) = 0$$

[1]Recall that the centroid of the parallelogram is the centroid of the diagonals.

[2]If $x_k, y_k \in \mathbf{Z}$, all elements of the left matrix are in $\mathbf{Z}$.

Author's Note: I intend a reworking along the same lines denoted on "Line through a Point and the Centroid of a P-Parallelogram"

Line through the Centroids of Two P-Parallelograms

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Let $P = \text{ppara}(\text{point}(x_1 \quad y_1)\ \text{point}(x_2 \quad y_2)\ \text{point}(x_3 \quad y_3)\ \text{point}(x_4 \quad y_4))$. Let[1]

$$2C_P = \text{vec}(x_{13} \quad y_{13}) = \text{vec}(x_{24} \quad y_{24}).$$

where $x_{13} \equiv x_1 + x_3,\ y_{13} \equiv y_1 + y_3,\ x_{24} \equiv x_2 + x_4,\ y_{24} \equiv y_2 + y_4$. Similarly, let $R = \text{ppara}(\text{point}(x_5 \quad y_5)\ \text{point}(x_6 \quad y_6)\ \text{point}(x_7 \quad y_7)\ \text{point}(x_8 \quad y_8))$. Let

$$2C_R = \text{vec}(x_{57} \quad y_{57}) = \text{vec}(x_{68} \quad y_{68}).$$

where $x_{57} \equiv x_5 + x_7,\ y_{57} \equiv y_5 + y_7,\ x_{68} \equiv x_6 + x_8,\ y_{68} \equiv y_6 + y_8$.

Provided that $C_P \neq C_R$, the line through the centroids of P and R satisfies

$$\text{vec}(x_{13}y_{57} - x_{57}y_{13} \quad 2y_{13} - 2y_{57} \quad 2x_{57} - 2x_{13}) \cdot \text{vec}(1\ x\ y)$$

[1] Recall that the centroid of a parallelogram is the midpoint of either diagonal.