Review of the Basics

What is a scattering cross section? What is a differential cross section?

A cross section describes the effective area of a target for interacting with a particle, and is measured in units of barns ($10^{-28}m^2 = 1\ Barn$). The differential cross section requires the definition of the solid angle:

$$d\Omega = \sin\theta d\theta d\phi \tag{1}$$

The differential cross section is the effective area of a target for interacting with a particle and scattering into the solid angle $d\Omega$. The cross section can be obtained by integrating the differential cross section over the solid angle.

$$\sigma = \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} \frac{d\sigma}{d\Omega} d\phi \tag{2}$$

Describe electron scattering from protons at various energy scales

The nature of electron proton scattering depends on the wavelength of the virtual photon in comparison with the radius of the proton. At very low energies ($\lambda \gg r_p$) the scattering is effectively described by elastic scattering of the electron in a static potential of an effectively point-like proton. If the electron is non-relativistic the differential cross section is described by Rutherford.

$$\left(\frac{d\sigma}{d\Omega}\right)_{Rutherford} = \frac{\alpha^2}{16E_K^2 \sin^4(\theta/2)} \tag{3}$$

The energy of the electron is the non-relativistic kinematic energy. When the electron is relativistic, but $\lambda \gg r_p$ still applies, the Mott cross section describes the reaction properly.

$$\left(\frac{d\sigma}{d\Omega}\right)_{Mott} = \frac{\alpha^2}{4E^2\sin^4(\theta/2)}\cos^2(\theta/2) \tag{4}$$

Both of these formulas can be derived using a point-like proton and never making mention of spin, implying that at these energies the *possible* spin-spin magnetic interaction is negligable. As the electron energy increases $\lambda \approx r_p$, *form factors* are introduced to describe the finite spatial extent of the proton's charge distribution.

$$F(\boldsymbol{q}^2) = \int \rho(\boldsymbol{r}) e^{i\boldsymbol{q}\cdot\boldsymbol{r}} d^3\boldsymbol{r} \tag{5}$$

The form factor is simply the three dimensional Fourier transform of the charge distribution of the proton, and the Mott cross section 4 can be modified to reflect.

$$\left(\frac{d\sigma}{d\Omega}\right)_{Mott} = \frac{\alpha^2}{4E^2\sin^4(\theta/2)}\cos^2(\theta/2)|F(\boldsymbol{q}^2)|^2 \tag{6}$$

Equation 6 can also be used for modeling scattering from heavier nuclei. It can be shown that the most general lorentz invariant form of elastic electron-proton scattering is described in terms of two form factors; $G_E(Q^2)$ describes the proton's charge distribution, while $G_M(Q^2)$ describes the magnetic moment distribution. The cross section is given by the Rosenbluth formula.

$$\left(\frac{d\sigma}{d\Omega}\right)_{Rosenbluth} = \frac{\alpha^2}{4E_1^2\sin^4(\theta/2)}\frac{E_3}{E_1}\left[\frac{G_E^2+\tau G_M^2}{1+\tau}\cos^2(\theta/2) + 2\tau G_M^2\sin^2(\theta/2)\right] \tag{7}$$

Where E_1 is the incident electron energy, and E_3 is the scattered electron energy and the term $\tau = \frac{Q^2}{4m_p^2}$. In general, a good model for the proton charge distribution is given by $\rho(r) = \rho_0 e^{-r/a}$, with the parameters being determined experimentally.

When the wavelength of the virtual photon becomes small $\lambda < r_p$, the elastic scattering cross section also becomes small. The dominant process is inelastic scattering where the virtual photon interacts with a consituent quark inside the proton and the proton subsequently breaks up. At very high electron energies $\lambda \ll r_p$, the virtual photon resolves the detailed dynamic structure of the proton, which appears to be a sea of strongly interacting quarks and gluons. The cross section for this deeply inelastic scattering (DIS) is described in terms of two structure functions $F_1(x, Q^2)$ and $F_2(x, Q^2)$.

$$\frac{d\sigma}{dxdQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[\left(1 - y - \frac{m_p^2 y^2}{Q^2}\right)\frac{F_2(x, Q^2)}{x} + y^2 F_1(x, Q^2)\right] \tag{8}$$

The electron kinematics are described by the four momentum transfer $q^\mu = l^\mu - l'^\mu$, the negative momentum transfer $Q^2 = -q^\mu q_\mu$ and x and y are given below.

$$x = \frac{Q^2}{2P \cdot q} \qquad\qquad y = \frac{P \cdot q}{P \cdot l} \tag{9}$$

For DIS kinematics $Q^2 \gg m_p^2 y^2$ and the cross section 8 reduces to a more compact form.

$$\frac{d\sigma}{dxdQ^2} \approx \frac{4\pi\alpha^2}{Q^4}\left[(1 - y)\frac{F_2(x, Q^2)}{x} + y^2 F_1(x, Q^2)\right] \tag{10}$$

There is much more that can be said on the topic of DIS, including the Bjorken scaling of the structure functions $F(x, Q^2) \to F(x)$, but I don't find the need to dive deep into those details in this whirlwind overview of electron scattering.

What is a magnetic dipole moment? What do different theories predict for the magnetic moment of electrons and protons?

The magnetic dipole moment of a spinning charge is related to its (spin) angular momentum. The magnetic dipole moment can be used as in interaction term in the Hamiltonian according to the following relation, where $\boldsymbol{B}$ is the magnetic field vector.

$$H = -\boldsymbol{\mu} \cdot \boldsymbol{B} \tag{11}$$

The classical calculation for a charged particle with spin gives,

$$\boldsymbol{\mu} = \frac{q}{2m}\boldsymbol{S} \tag{12}$$

where m is the particle mass and q is the particle charge. Dirac showed (following the conclusions of the Dirac equation) that for point-like Dirac Fermions, the magnetic moment is actually twice the classical value.

$$\boldsymbol{\mu} = \frac{q}{m}\boldsymbol{S} \tag{13}$$

Radiative corrections in QED lead to further slight refinement of this value,

$$g = 2 + \frac{\alpha}{\pi} + \mathcal{O}(\alpha^2) \tag{14}$$

where the g value is mutliplied by the classical formula to predict the magnetic moment (Dirac theory says $g = 2$), which is in precise agreement with experimental measurements for the electron. The proton however, is composed of quarks and therefore its magnetic moment is not that of a Dirac Fermion, but the combination of the magnetic moments of the sea of constituents. The observation of the proton's magnetic moment is strong evidence that it is not a point-like particle. In electron scattering experiments at low energy where $Q^2 \ll 4m_p^2$ the form factor G_M can be seen as the Fourier transform of magnetic moment of the proton.

$$G_M(Q^2) \approx G_M(\boldsymbol{q}^2) = \int \mu(\boldsymbol{r}) e^{i\boldsymbol{q}\cdot\boldsymbol{r}} d^3\boldsymbol{r} \tag{15}$$