CS578 Statistical Machine Learning Lecture 6

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(based on slides by Tommi Jaakkola, MIT CSAIL)

Today's topics

- Rating (ordinal regression)
  - reduction to binary problems
  - SVM solution, on-line solution
- Ranking
  - ranking SVM

Rating problems

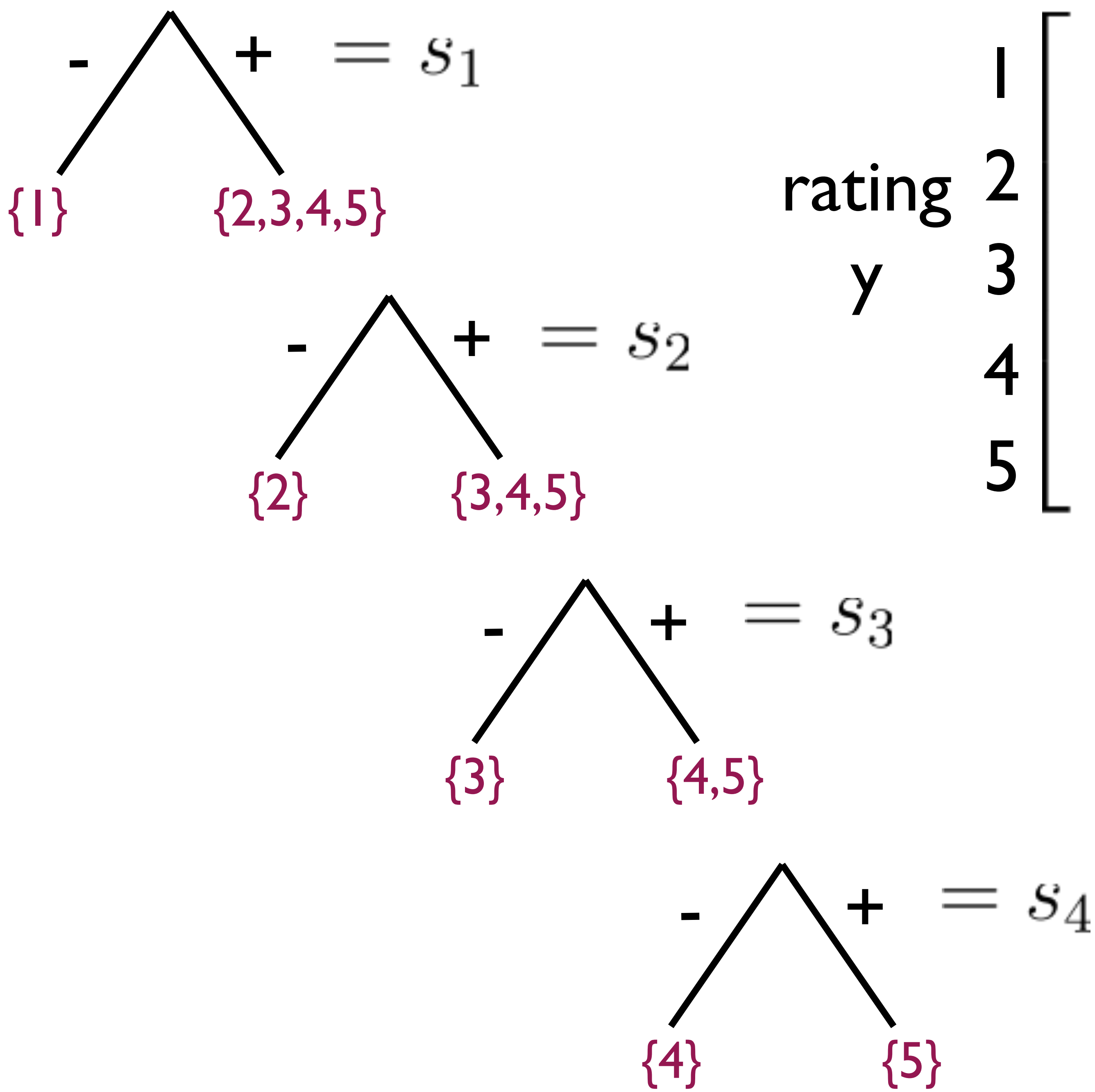
- A common prediction problem in recommender systems involves rating items (movies, products) on the basis some known features about such objects
- The rating scale is often 1-5 stars assigned to the object
- The key difference between rating problems and multi-way classification problems is that the rating scale is ordinal (e.g., 1<2<3<4<5) while class labels in multi-way classification problems are category symbols

Ordinal regression: setup

- Each item x_i is associated with a feature vector $\phi(x_i)$
  - e.g., product description, movie features, etc.
- We wish to predict an ordinal label $y_i \in \{1, \ldots, k\}$ for each item (reflecting views of one user)
- As in the multi-class setting, we translate each rating into a set of binary labels

Binary translation

- There are many ways to translate ratings into binary labels...



binary tasks

$$
\text{rating } y \quad
\begin{array}{c}
1 \\ 2 \\ 3 \\ 4 \\ 5
\end{array}
\begin{bmatrix}
-1 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 \\
1 & 1 & -1 & 0 \\
1 & 1 & 1 & -1 \\
1 & 1 & 1 & 1
\end{bmatrix}
$$

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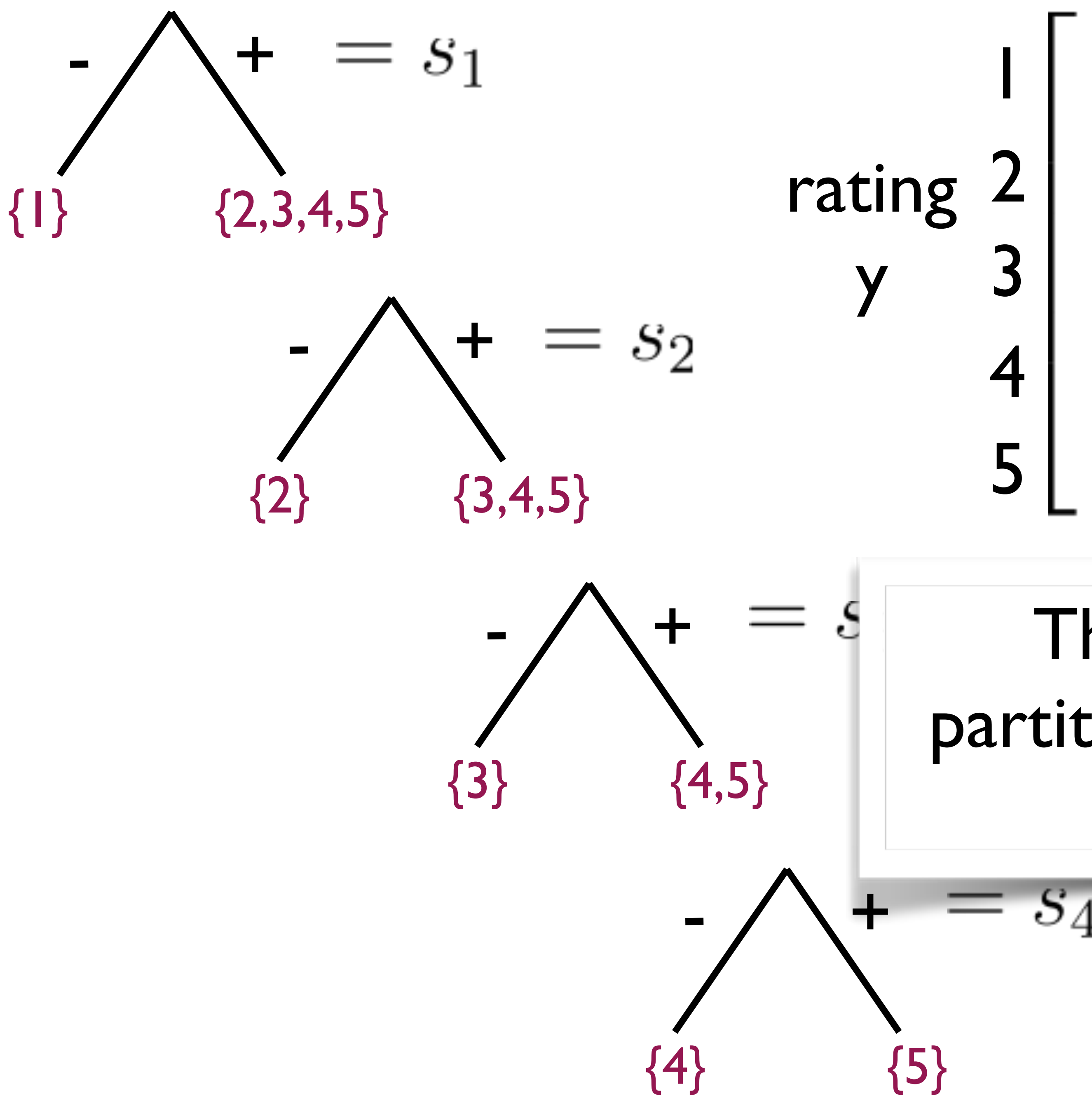


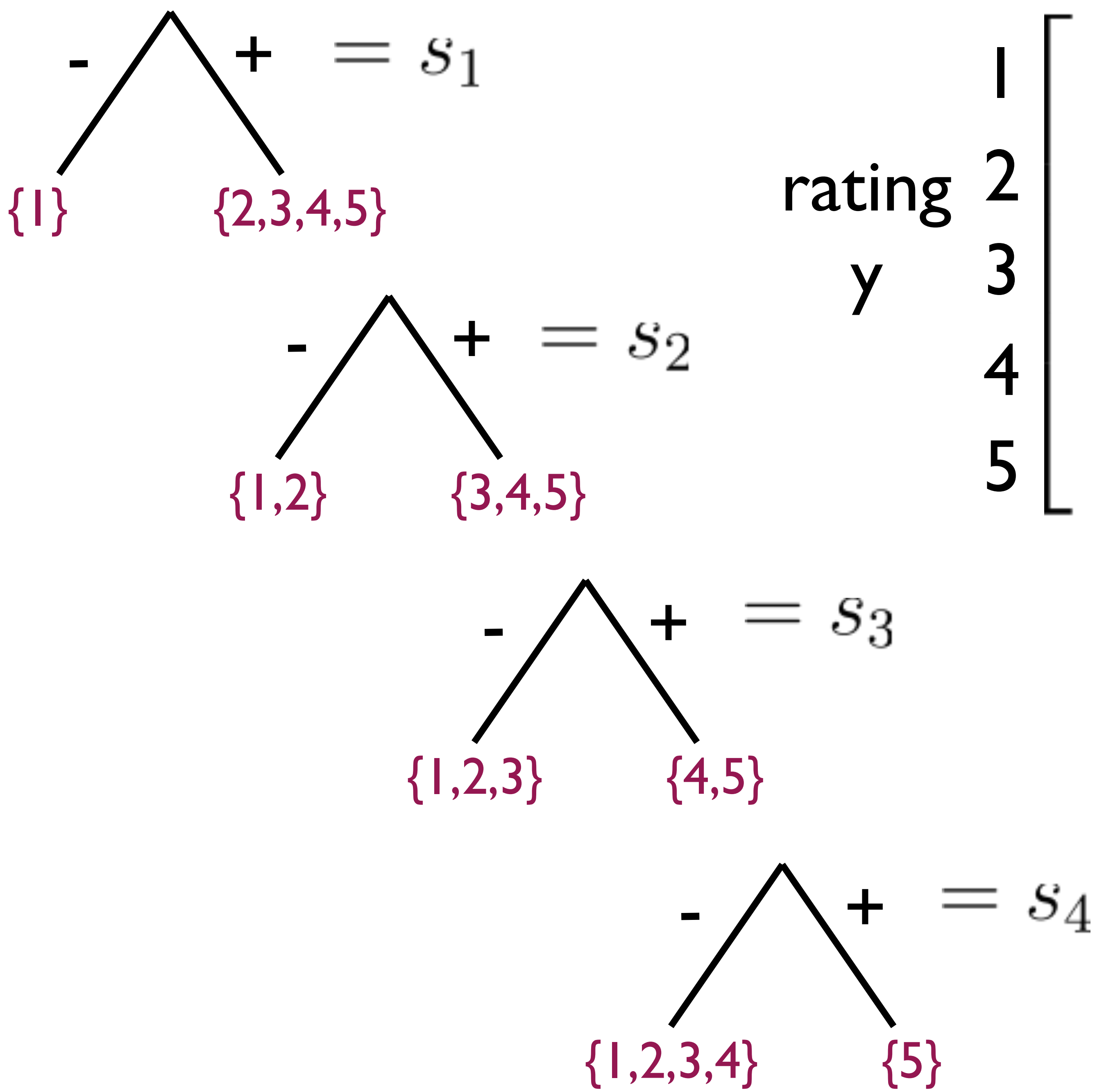
binary tasks

$$\text{rating } y \quad \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

These are independent partitions that do not enforce the ordinal scale

Binary translation

- We can create more relevant partitions by “sliding” across the ordinal scale



binary tasks

$$
\begin{array}{cc}
 & \\
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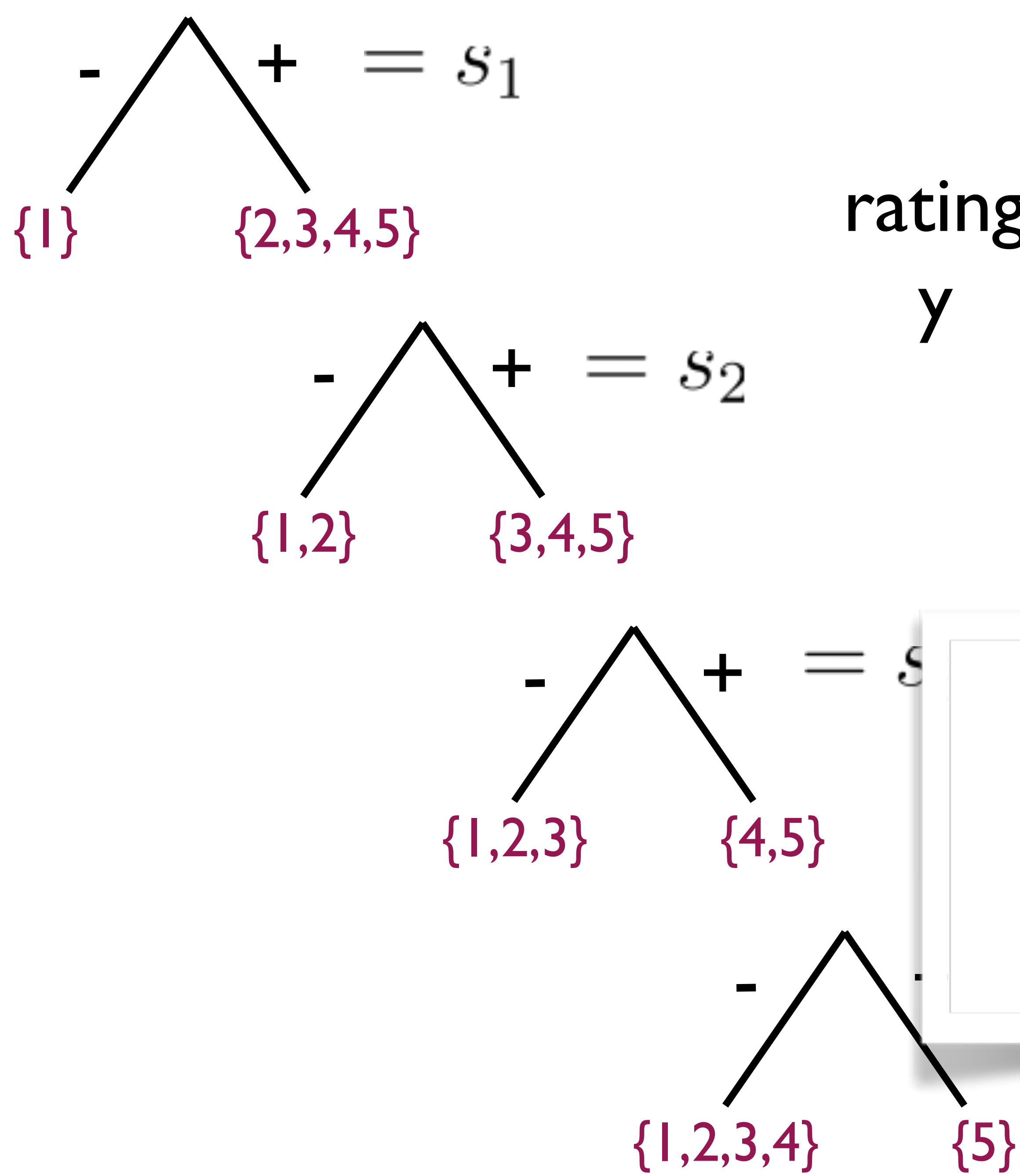


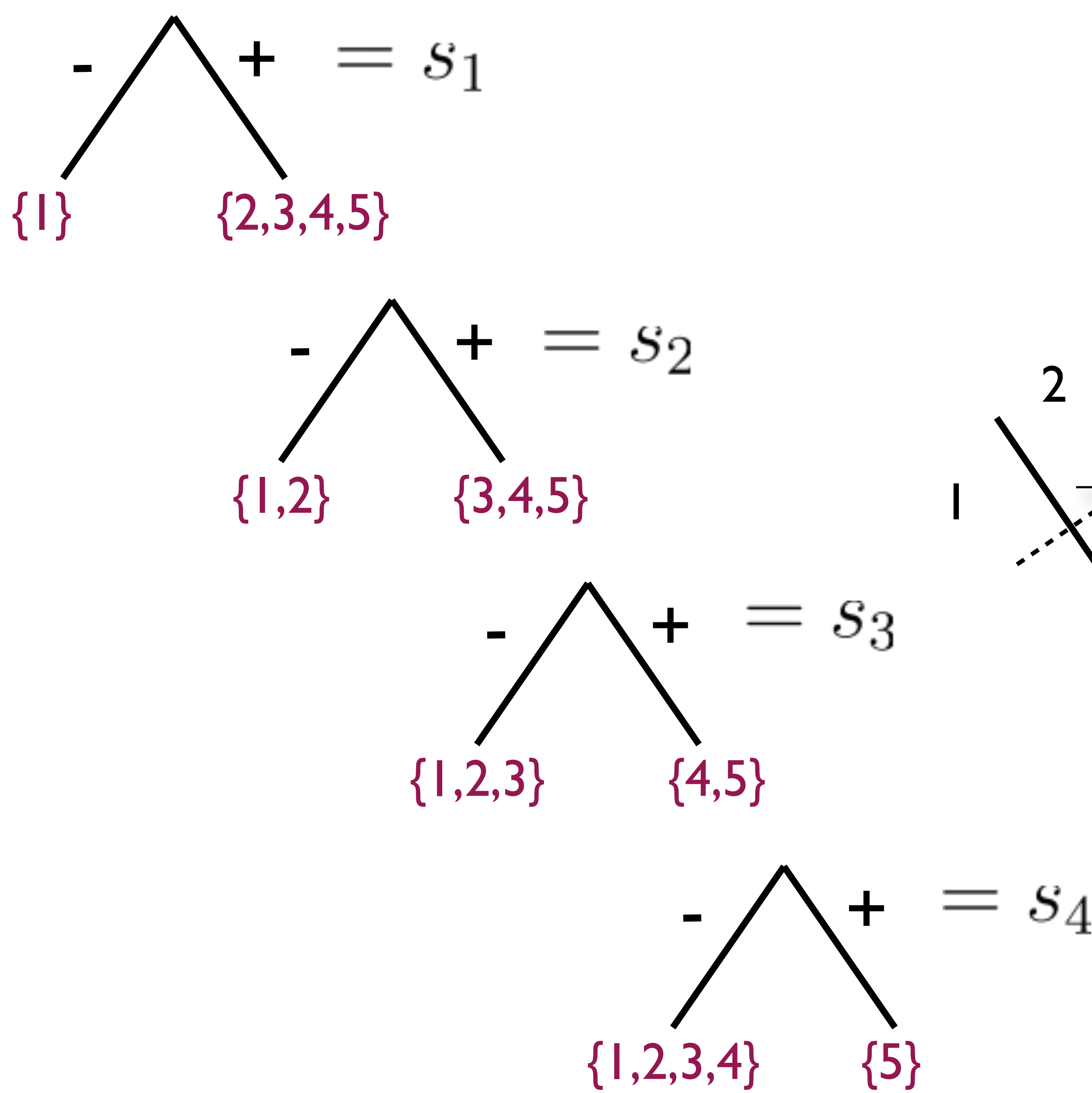
binary tasks

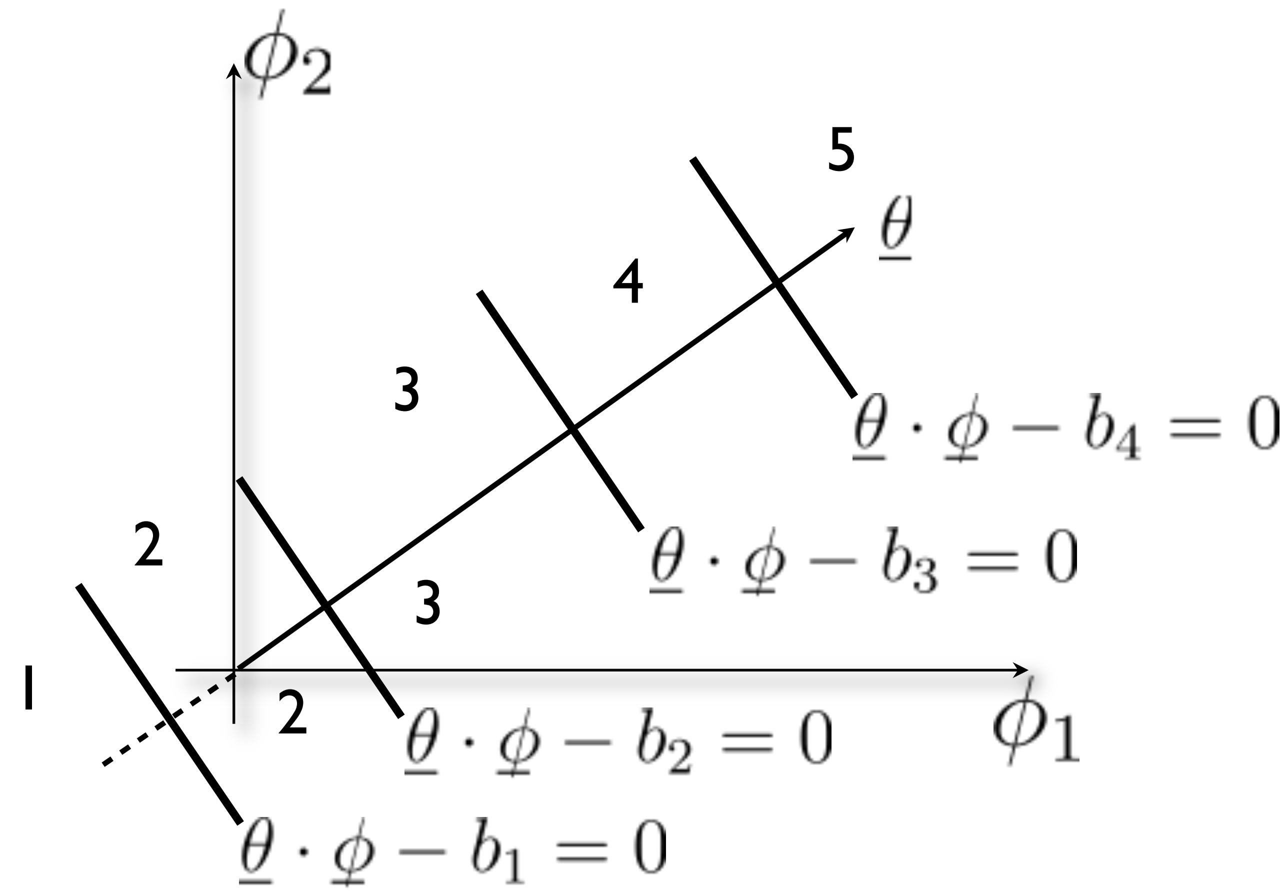
$$\text{rating } y \quad \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{bmatrix} -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

The partitions are now dependent ... need to enforce consistency of the binary labels across partitions

Ordinal regression

- We can specify a set of classifiers with shared parameters that always produce consistent binary labels





$$b_1 \leq b_2 \leq b_3 \leq b_4$$

Ordinal regression

- We can specify a set of classifiers with shared parameters that always produce consistent binary labels

$$b_1 \leq b_2 \leq b_3 \leq b_4$$

thresholds are different but ordered

- / + : {1} vs {2,3,4,5}

$$= s_1 = \text{sign}(\underbrace{\underline{\theta} \cdot \underline{\phi}(x_i)}_{\text{common prediction}} - b_1)$$

- / + : {1,2} vs {3,4,5}

$$= s_2 = \text{sign}(\underline{\theta} \cdot \underline{\phi}(x_i) - b_2)$$

- / + : {1,2,3} vs {4,5}

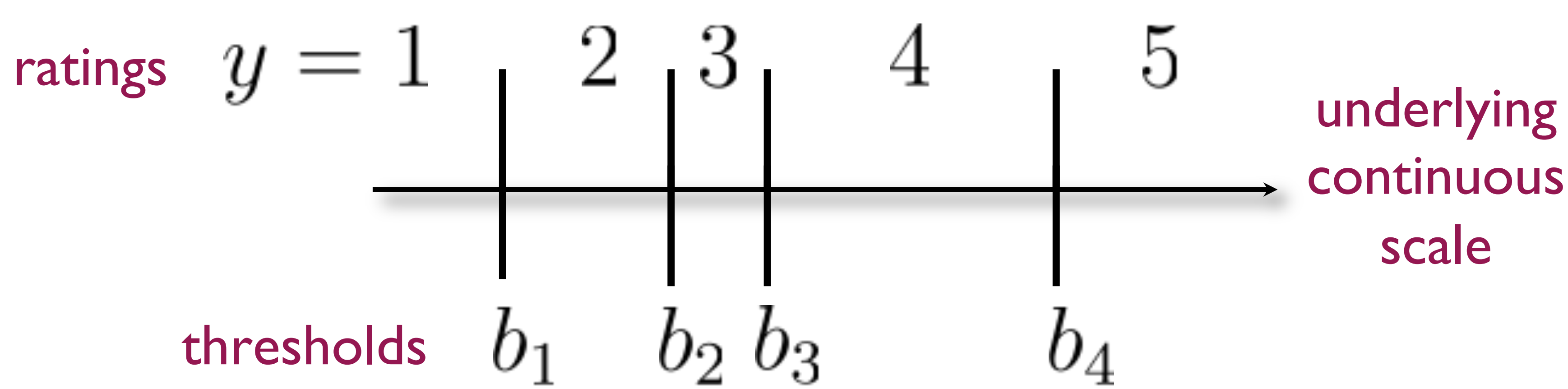
$$= s_3 = \text{sign}(\underline{\theta} \cdot \underline{\phi}(x_i) - b_3)$$

- / + : {1,2,3,4} vs {5}

$$= s_4 = \text{sign}(\underline{\theta} \cdot \underline{\phi}(x_i) - b_4)$$

Ordinal regression, 2nd view

- Each item x_i is associated with a feature vector $\phi(x_i)$
  - e.g., product description, movie features, etc.
- We wish to predict an ordinal label $y_i \in \{1, \ldots, k\}$ for each item (reflecting views of one user)
- We assume that there exists an underlying continuous scale from which ratings are obtained via thresholding



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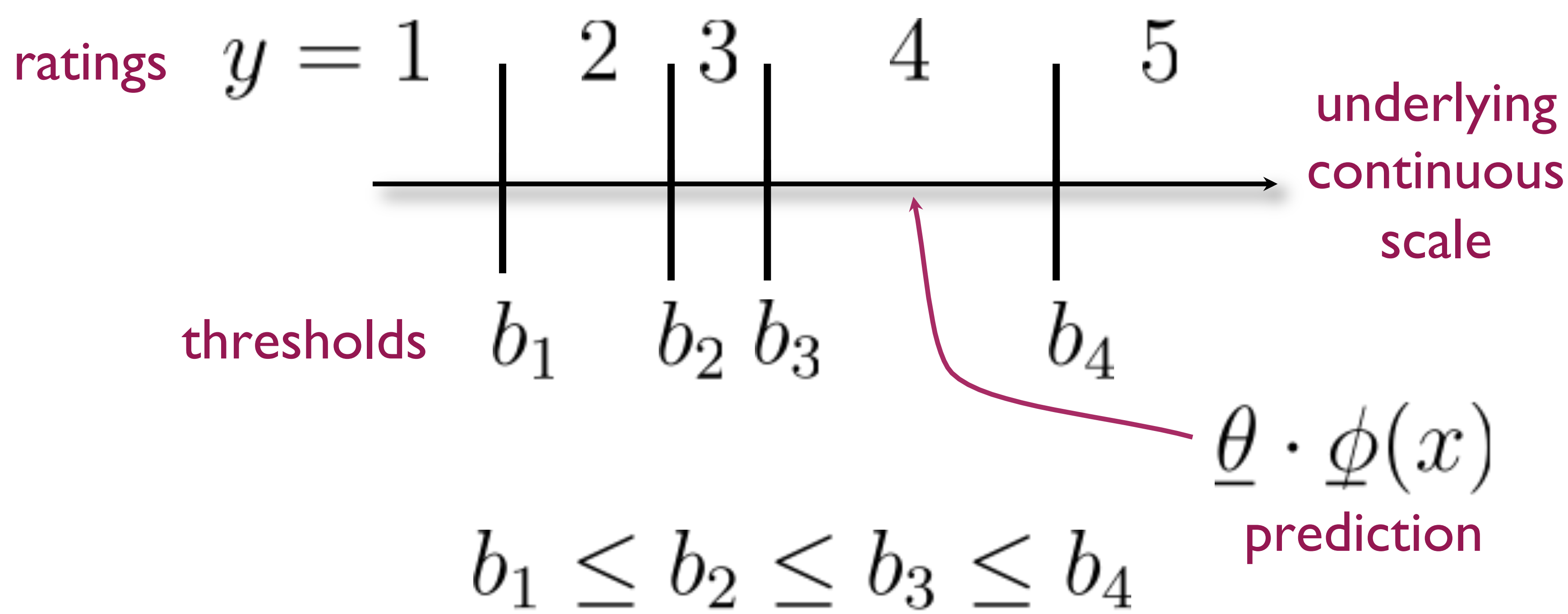


Ordinal regression, SVM style

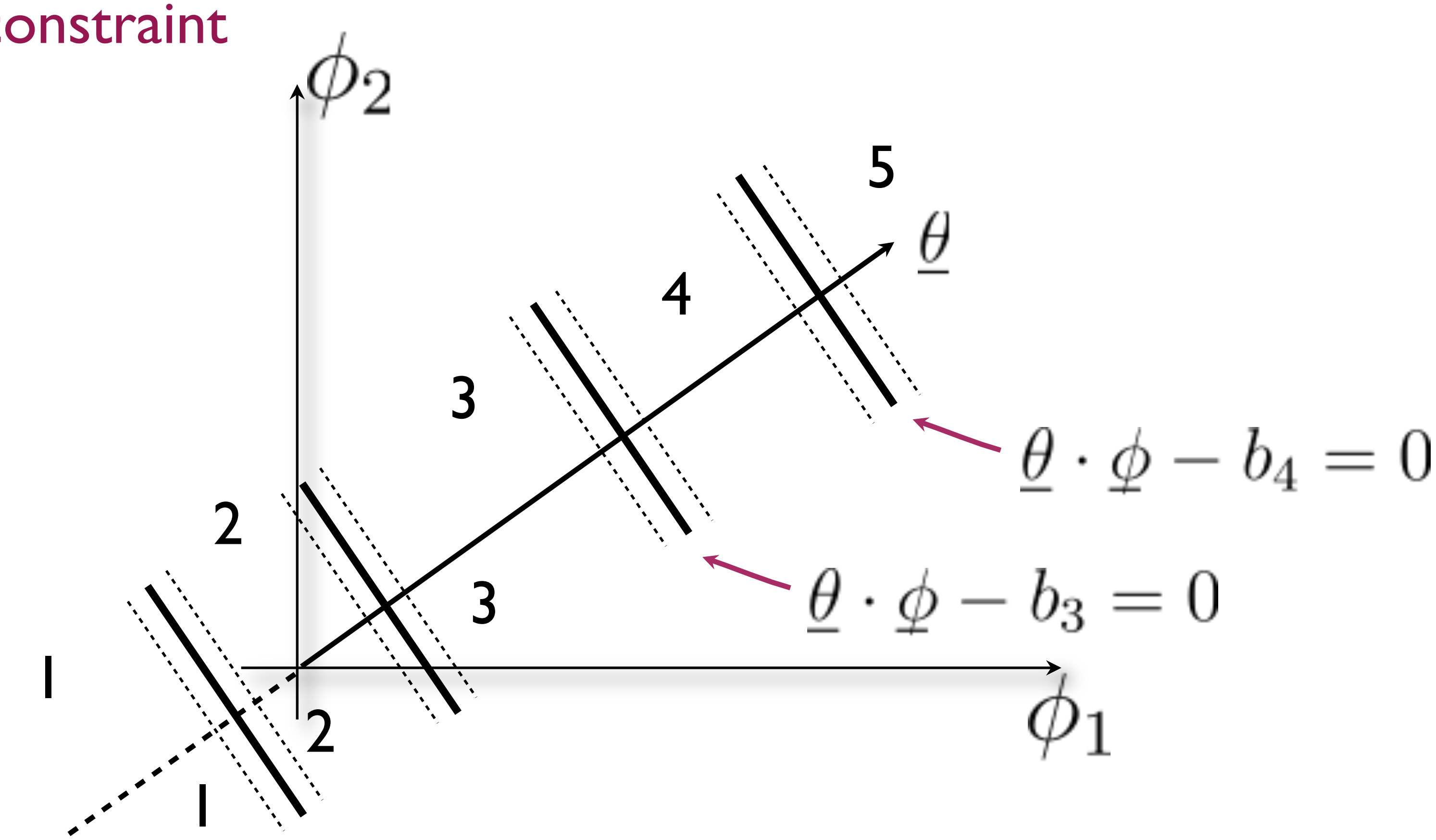
- Given a training set $D = \{(x_i, y_i)\}_{i=1,\ldots,n}$

$$\text{minimize} \quad \frac{1}{2}\|\underline{\theta}\|^2 \quad \text{with respect to } \underline{\theta},\, b_1, \ldots, b_{k-1}$$

$$\text{such that } b_1 \le b_2 \le \ldots \le b_{k-1} \text{ and}$$

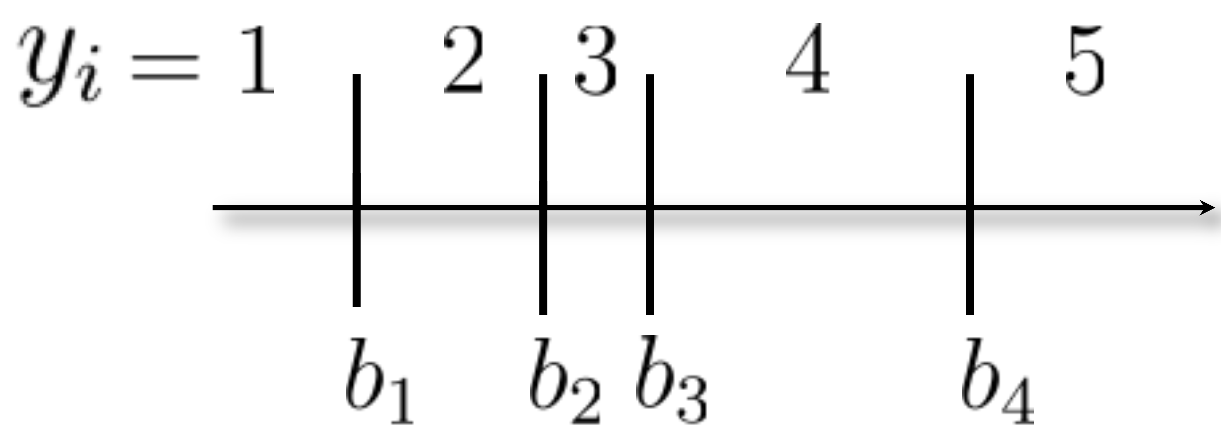
$$s_{il}\underbrace{(\underline{\theta} \cdot \underline{\phi}(x_i) - b_l) \ge 1}_{\text{binary classification constraint}},\; l = 1, \ldots, k-1, \quad i = 1, \ldots, n$$

k-1 binary labels obtained from each observed rating



Ordinal regression, SVM style

- Given a training set $D = \{(x_i, y_i)\}_{i=1,\ldots,n}$
- For instance, assume $k = 5$



- For sample i :

	$l = 1$	$l = 2$	$l = 3$	$l = 4$
$y_i = 3$	$s_{il} = +1$	$s_{il} = +1$	$s_{il} = -1$	$s_{il} = -1$
$y_i = 1$	$s_{il} = -1$	$s_{il} = -1$	$s_{il} = -1$	$s_{il} = -1$

Ordinal regression, PRank

- We can also define a mistake driven perceptron algorithm for solving ordinal regression problems
- The updates are modified slightly due to shared parameters

cycle through the training set $i = 1, \ldots, n$

for each example i

$$E_i = \{l : \; s_{il}(\underline{\theta} \cdot \underline{\phi}(x_i) - b_l) \leq 0\}$$

identify all binary mistakes

$$\underline{\theta} \leftarrow \underline{\theta} + \Big(\sum_{l \in E_i} s_{il}\Big)\, \underline{\phi}(x_i)$$

perform a collective update based on the mistakes

$$b_l \leftarrow b_l - s_{il}, \; l \in E_i$$

update thresholds of each classifier

Note: having a threshold is equivalent to having an extra feature, in which all samples have -1. Thus, the update rule for b_l is not surprising.

Ordinal regression, PRank

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for each example i

$$E_i = \{l : \; s_{il}(\underline{\theta} \cdot \underline{\phi}(x_i) - b_l) \leq 0\}$$ identify all binary mistakes

$$\underline{\theta} \leftarrow \underline{\theta} + \Big(\sum_{l \in E_i} s_{il}\Big)\, \underline{\phi}(x_i)$$ perform a collective update based on the mistakes

$$b_l \leftarrow b_l - s_{il}, \; l \in E_i$$ update thresholds of each classifier

- **Lemma**: if the thresholds are set to zero initially, they will maintain the correct ordering in the course of the algorithm

(See Lemma 1 in [1] if interested in the proof.)

PRank, mistake bound

- **Theorem:** Assume that there exists $\underline{\theta}^*, b_1^*, \ldots, b_{k-1}^*$

$$\|\underline{\theta}^*\|^2 + \sum_{l=1}^{k-1} b_l^{*2} = 1$$

such that

$$s_{il}(\underline{\theta}^* \cdot \underline{\phi}(x_i) - b_l^*) \geq \gamma, \quad l = 1, \ldots, k-1, \quad i = 1, \ldots, n$$

then the algorithm makes at most

$$(k-1)\frac{R^2 + 1}{\gamma^2}$$

binary mistakes on the training set.

(See Theorem 2 in [1] if interested in the proof.)

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Ranking

- Rating products, movies, etc. using a few values (e.g., 1-5 stars) results in a partial ranking of the items
- Many rating / classification problems are better viewed as ranking problems
  - suggest movies in the order of user interest in them,
  - rank websites to display in response to a query,
  - suggest genes relevant to a particular disease condition, etc.
- By casting the learning problem as a ranking problem we can also incorporate other types of data / feedback
  - e.g., click through data from users

Ranking example

- We would like to rank n websites (find top sites to display) in response to a few query words

$$x = \text{context (set of query words)}$$
$$y = \text{website}$$

$$\begin{array}{lll} (x_1, y_2) & & (x_2, y_7) \\ (x_1, y_{10}) & & (x_2, y_2) \\ (x_1, y_3) & & (x_2, y_1) \\ \hline \cdots & & \cdots \\ (x_1, y_n) & & (x_2, y_4) \end{array}$$

$x_1 = \{ \text{ ranking applications } \}$ $\quad x_2 = \{ \text{ ranking SVM code } \}$

Ranking example

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(x_1, y_2)	(x_2, y_7) **x**
(x_1, y_{10})	(x_2, y_2)
(x_1, y_3) **x**	(x_2, y_1)
$\dots$	$\dots$
(x_1, y_n)	(x_2, y_4)

$x_1 = \{ \text{ ranking applications } \}$ $\qquad$ $x_2 = \{ \text{ ranking SVM code } \}$

- The available data contain user selections (clicks) of websites out of those displayed to them

From selections to preferences

- We can interpret a user click as a statement that they prefer the selected link over others displayed in the context of the query

$$(x_1, y_2)$$
$$(x_1, y_{10})$$
$$(x_1, y_3) \quad \mathbf{x}$$
$$\cdots$$
$$(x_1, y_n)$$

$$x_1 = \{ \text{ ranking applications } \}$$

$$(x_1, y_3) > \{(x_1, y_{10}), (x_1, y_2)\}$$

$$(x_2, y_7) \quad \mathbf{x}$$
$$(x_2, y_2)$$
$$(x_2, y_1)$$
$$\cdots$$
$$(x_2, y_4)$$

$$x_2 = \{ \text{ ranking SVM code } \}$$

$$(x_2, y_7) > \{(x_2, y_2), (x_2, y_1)\}$$

Ranking function

- Our goal is to estimate a ranking function over pairs f(x,y) such that its values are consistent with the observed preferences.

$$(x_2, y_7) > \{(x_2, y_2), (x_2, y_1)\}$$

$$\Rightarrow \quad f(x_2, y_7) > f(x_2, y_2), \;\; f(x_2, y_7) > f(x_2, y_1)$$

- We can parameterize this function in terms of feature vectors extracted from each pair (context,website)

$$f(x, y; \underline{\theta}) = \underline{\theta} \cdot \underline{\phi}(x, y)$$

Ranking function

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- We can parameterize this function in terms of feature vectors extracted from each pair (context,website)

$$f(x, y; \underline{\theta}) = \underline{\theta} \cdot \underline{\phi}(x, y)$$

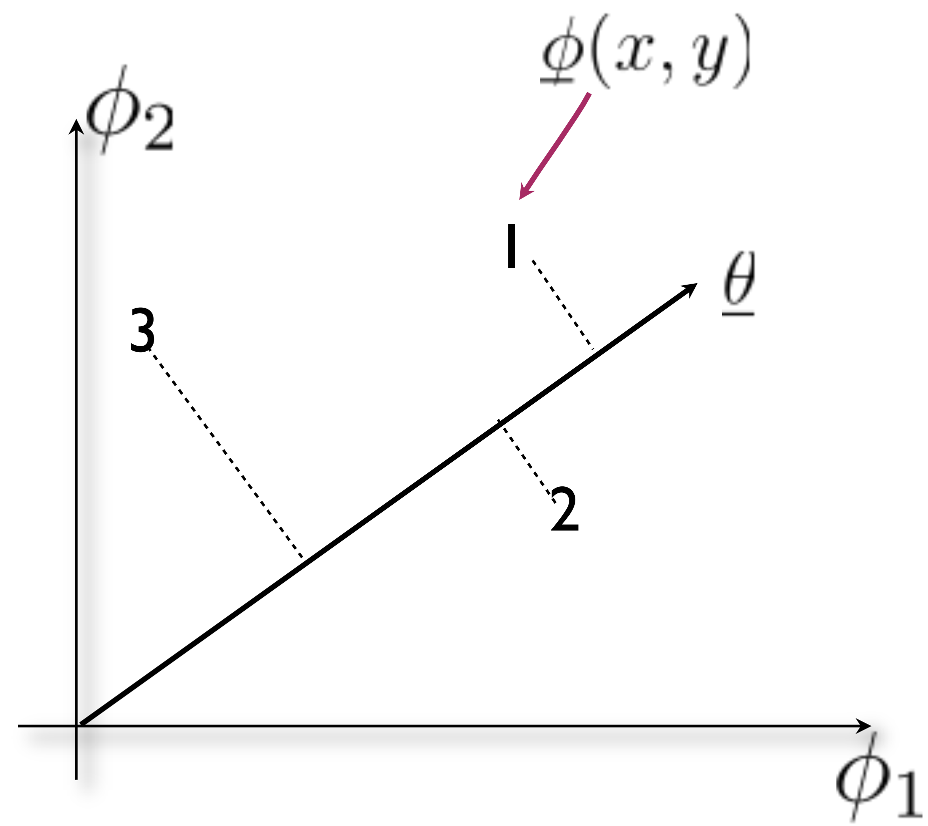
where the features could be, e.g.,

$$\phi_w(x, y) = \left\{ \begin{array}{ll} 1, & \text{if word } w \text{ appears in } x \text{ and } y \\ 0, & \text{otherwise} \end{array} \right\}$$

for all $w \in \mathcal{W}$

Ranking function

- The ranking function gives rise to a total ordering of the pairs via projection to the parameter vector



1 > 2
2 > 3

$$f(x, y; \underline{\theta}) = \underline{\theta} \cdot \underline{\phi}(x, y)$$

SVM rank

- A training set of order relations between pairs

$$D = \{ \{(x_i, y_j) > (x_k, y_l)\} \}$$

- An SVM style algorithm for finding a consistent ranking function

$$\text{minimize } \frac{1}{2}||\underline{\theta}||^2 \text{ with respect to } \underline{\theta} \text{ such that}$$

$$\underline{\theta} \cdot \underline{\phi}(x_i, y_j) \geq \underline{\theta} \cdot \underline{\phi}(x_k, y_l) + 1,$$

$$\forall \{(x_i, y_j) > (x_k, y_l)\} \text{ in } D$$

SVM rank

- A training set of order relations between pairs

$$D = \{\, \{(x_i, y_j) > (x_k, y_l)\} \,\}$$

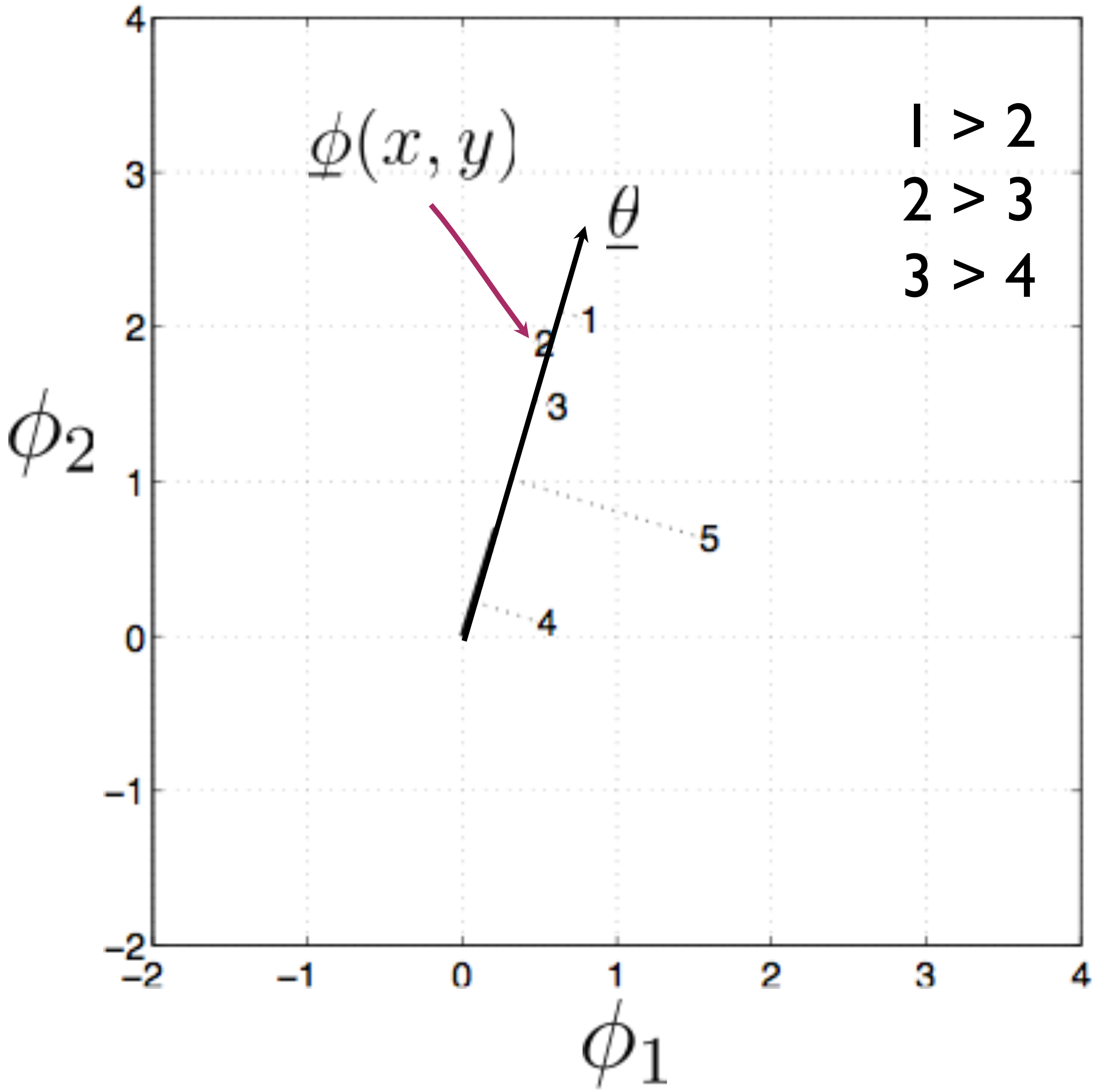
- An SVM style algorithm for finding a consistent ranking function

$$\text{minimize} \quad \frac{1}{2}\|\underline{\theta}\|^2 + C \sum_{ij;kl} \xi_{ij;kl} \quad \text{subject to}$$

$$\underline{\theta} \cdot \underline{\phi}(x_i, y_j) \geq \underline{\theta} \cdot \underline{\phi}(x_k, y_l) + 1 - \xi_{ij;kl}, \quad \xi_{ij;kl} \geq 0$$
$$\forall \, \{(x_i, y_j) > (x_k, y_l)\} \text{ in } D$$

- It is important to appropriately weight or choose which constraints to include

The effect of ranking constraints



adding a single constraint can have a large effect on the ranking solution