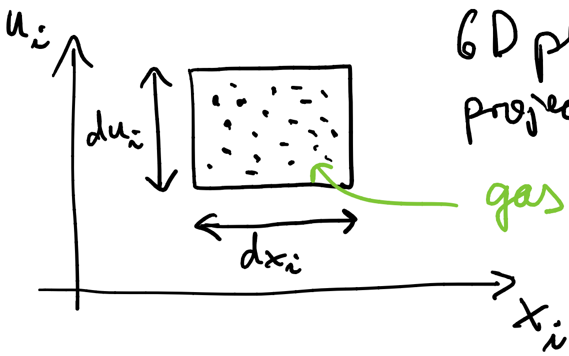
# Chapter 2: Basic Equations

## 2.1 Continuous media: Boltzmann Eq




6D phase space projected onto $x_i$–$u_i$ plane

gas / liquid atoms or molecules

In principle evolution of system fully determined by Newtonian mechanics (<u>N-body equations</u>):

$$\frac{d\vec{x}_i}{dt} = u_i \ , \ \frac{d\vec{u}_i}{dt} = \vec{F}_i(x_j, u_j, t) \quad \forall \text{ particles } i, j$$

force on $i$

<u>But</u>: 1 mole $\hat{=}$ $N = 6 \times 10^{23}$ particles → <u>computationally prohibitive</u>

→ <u>statistical approach</u> (continuous media → <u>hydrodynamics</u>)

$$dN = f(\vec{x}, \vec{u}, t)\, d\vec{x}\, d\vec{u}$$

number of particles in phase-space control volume $d\vec{x}\, d\vec{u}$

distribution function

- ignore internal degrees of freedom → vibration, rotation
- identical particles of mass $m$
- ignore QM effects

Assume particles subject to external force per unit mass $\vec{F}$ ( $\approx$ constant over typical inter-particle separation)

$\rightsquigarrow$ particle number $f(\vec{x}, \vec{u}, t)\, d\vec{x}\, d\vec{u}$ conserved along their trajectories in phase space (Liouville's theorem) in absence of interactions:

(*)

$$\Delta f = f(\vec{x} + \underbrace{\vec{u}\,dt}_{\text{change in position}}, \vec{u} + \underbrace{\vec{F}\,dt}_{\text{change in velocity due to } \vec{F}}, t + dt) - f(\vec{x}, \vec{u}, t) = [\Delta f]_{coll}$$

$\uparrow$ change in $f$ in $dt$ due to collisions

$dt \to 0$

$\Delta f \to \dfrac{df}{dt}$

$$\boxed{\frac{df}{dt} = \frac{\partial f}{\partial t} + u_i \frac{\partial f}{\partial x_i} + F_i \frac{\partial f}{\partial u_i} = \left[\frac{\partial f}{\partial t}\right]_{coll}}$$

Boltzmann's transport equation

$\rightarrow$ evolution of distribution function in 6-D phase space $(\vec{x}, \vec{u})$

$\rightarrow$ change $dN$ in number of particles within a fixed volume $d\vec{x}\, d\vec{u}$ equals net number of particles entering or leaving this volume

( $\rightarrow$ Liouville's theorem)

Some definitions:

- $n(\vec{x}, t) \equiv \int \underbrace{f(\vec{x}, \vec{u}, t)\, d\vec{u}}_{\#\text{particles per unit volume in } \vec{u}, \vec{u}+d\vec{u}}$ total number of particles per unit volume

- $l \equiv n^{1/3}$ mean particle separation

  Note: any physical length scale $dx$ we are interested in must be $\gg l$ for statistical approach to be valid

- $\rho(\vec{x},t) \equiv \int m \, f(\vec{x},\vec{u},t)\, d\vec{u}$ mass density

  mass of particle

- $\vec{v}(\vec{x},t) \equiv \frac{1}{\rho} \int \vec{u}\, m\, f(\vec{x},\vec{u},t)\, d\vec{u}$ bulk velocity

- $\rho \varepsilon(\vec{x},t) \equiv \frac{1}{2} \int m\, \tilde{u}^2 f(\vec{x},\vec{u},t)\, d\vec{u}$ specific internal energy $\varepsilon$

  $\vec{\tilde{u}} \equiv \vec{u} - \vec{v}$ peculiar velocity

<u>Special case</u>:
- elastic collisions (energy & momentum conserved)
- low density (collisions of 3 & more particles can be neglected)
- absence of external forces $\vec{F} \equiv 0$

unique solution $\downarrow$ statistical mechanics

$$f(\vec{x},\vec{u},t)\, d\vec{u} = n(\vec{x},t) \left[ \frac{m}{2\pi k_B T(\vec{x},t)} \right]^{3/2} \exp\left[ - \frac{m(\underbrace{\vec{u}-\vec{v}}_{\equiv \tilde{u}})^2}{2 k_B T(\vec{x},t)} \right] d\vec{u}$$

peculiar velocity

Maxwellian velocity distribution

(equilibrium state $t \to \infty$, $\frac{\partial f}{\partial t} = 0$)

# 2.2 From Boltzmann to Euler

Def: k-th moment of the Boltzmann equation

$$\int U_k \left[\frac{\partial f}{\partial t} + u_i \frac{\partial f}{\partial x_i} + F_i \frac{\partial f}{\partial u_i}\right] du = \int U_k \left[\frac{\partial f}{\partial t}\right]_{coll} d\vec{u},$$

where $U_k = \vec{u}^k$, i.e. $U_0 = 1$, $U_1 = \vec{u}$, $U_2 = u^2$ etc.

General properties of collision term:

If collisions are elastic and neither create nor destroy particles, then

$$\int \left[\frac{\partial f}{\partial t}\right]_{coll} d\vec{u} = 0$$ number of particles conserved

$$\int \left[\frac{\partial f}{\partial t}\right]_{coll} u_i \, d\vec{u} = 0$$ total momentum conserved

$$\int \left[\frac{\partial f}{\partial t}\right]_{coll} u^2 \, d\vec{u} = 0$$ total energy conserved

$$\lim_{u \to \infty} u^k f = 0$$ total number, momentum energy must be finite

Also: $\left( \int_\Omega v \frac{\partial u}{\partial x_i} = \int_{\partial\Omega} u v n_i \, d\Gamma - \int_\Omega u \frac{\partial v}{\partial x_i} \right)$

$$\int \frac{\partial f}{\partial u_i} \, d\vec{u} \overset{p.i.}{=} \int_{|u|=\pm\infty} f \, n_i \, d\Gamma - \int \underbrace{\frac{\partial(1)}{\partial u_i}}_{=0} f \, d\vec{u} = 0$$

$$\int u_j \frac{\partial f}{\partial u_i} \, d\vec{u} \overset{p.i.}{=} \underbrace{\int_{|u|=\pm\infty} u_j f \, n_i \, d\Gamma}_{=0} - \int \underbrace{\frac{\partial u_j}{\partial u_i}}_{=\delta_{ij}} f \, d\vec{u}$$

$$= -\delta_{ij} \frac{\rho}{m}$$

$$\frac{1}{2} \int u^2 \frac{\partial f}{\partial u_i} \, d\vec{u} = \underbrace{\frac{1}{2} \int_{|u|=\pm\infty} u^2 f \, n_i \, d\Gamma}_{=0} - \frac{1}{2} \int \underbrace{\frac{\partial u^2}{\partial u_i}}_{\frac{\partial (\sum_j u_j^2)}{\partial u_i} = 2u_i} f \, d\vec{u}$$

$$= - \int u_i f \, d\vec{u} = -\frac{\rho}{m} v_i$$

(1) From <u>0-moment</u> of Boltzmann equ:

$$m \int \frac{\partial f}{\partial t} \, d\vec{u} + m \int u_i \frac{\partial f}{\partial x_i} \, d\vec{u} + m F_i \underbrace{\int \frac{\partial f}{\partial u_i} \, d\vec{u}}_{=0} = \underbrace{\int \left[\frac{\partial f}{\partial t}\right]_{coll} d\vec{u}}_{=0}$$

$u_i, x_i$ independent

$$\downarrow$$

$$\Longleftrightarrow \frac{\partial}{\partial t} \int m f \, d\vec{u} + \frac{\partial}{\partial x_i} \int u_i m f \, d\vec{u} = 0$$

$$\Leftrightarrow \quad \boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0}$$ continuity equation

<u>Note:</u> V large enough

$$\rightsquigarrow \int \frac{\partial \rho}{\partial t} dV + \underbrace{\int \nabla \cdot (\rho \vec{v})\, dV}_{\overset{\text{Gauss}}{=} \int_{\partial V} \rho v^i n_i \, d\Omega \;=\; 0}$$

$n_i$: unit normal; $= 0$ since $\vec{v} = 0$ on $\partial V$

$$= \frac{\partial}{\partial t} \int_V \rho \, dV = \frac{\partial M_V}{\partial t} = 0 \quad \text{mass conservation}$$

② ~~From~~ <u>1st-moment</u> of Boltzmann equ:

$$m \int u_i \frac{\partial f}{\partial t} d\vec{u} + m \int u_i u_j \frac{\partial f}{\partial x_i} d\vec{u} + m F_j \underbrace{\int u_i \frac{\partial f}{\partial u_j} d\vec{u}}_{= -\delta_{ij} \frac{\rho}{m}}$$

$$= \int u_i \left[\frac{\partial f}{\partial t}\right]_{coll} d\vec{u} = 0$$

$$\Leftrightarrow \frac{\partial}{\partial t}(\rho V_i) + \frac{\partial}{\partial x_i} \int m u_i u_j f \, d\vec{u} - \rho F_i = 0$$

$$\int m u_i u_j f \, d\vec{u} = \int m (\tilde{u}_i + v_i)(\tilde{u}_j + v_j) f \, d\vec{u}$$

cross terms vanish

$\int \tilde{u}_i f \, d\vec{u} = 0$

peculiar velocities

$$= \int m v_i v_j f \, d\vec{u} + \int m \tilde{u}_i \tilde{u}_j f \, d\vec{u}$$

$$\equiv \rho v_i v_j + P_{ij}$$

$$P_{ij} \equiv \int m \tilde{u}_i \tilde{u}_j f \, d\vec{u}$$ pressure tensor

Most astrophysical cases:

isotropic pressure $\Rightarrow$ $P_{ij} = P \delta_{ij}$

$$P \equiv \frac{1}{3} \int m \tilde{u}^2 f \, d\vec{u}$$

$$\left( P = \frac{2}{3} \rho \varepsilon \right)$$

$$\Leftrightarrow \quad \frac{\partial}{\partial t} (\rho v_i) + \frac{\partial}{\partial x_j} (\rho v_i v_j) = -\frac{\partial P}{\partial x_i} + \rho F_i$$

$$\Leftrightarrow \quad \boxed{\begin{array}{c} \dfrac{\partial}{\partial t} (\rho \vec{v}) + \nabla \cdot \Pi = \rho \vec{F} \\[2ex] \Pi \equiv \rho v_i v_j + P \delta_{ij} \quad \text{momentum flux tensor} \end{array}}$$ momentum equation

Note: • pressure gradient and external forces act as <u>source (or sink) terms</u>

- force due to pressure gradient results from exchange of energy between bulk flow and peculiar motions

③ From 2nd-moment of Boltzmann equ:

(Exercise):

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{v^2}{2}+\varepsilon\right)\right]+\frac{\partial}{\partial x_i}\left[\rho v_i\left(\frac{v^2}{2}+\varepsilon\right)\right]=-\frac{\partial h_i}{\partial x_i}-\frac{\partial}{\partial x_i}(P v_i)$$

$$+\rho v_i F_i$$

where $h_i \equiv \int \frac{m}{2}\tilde{u}_i \tilde{u}^2 f \, d\vec{u}$ conduction heat flux

If $\vec{h}$ can be neglected (typical for astrophys. systems):

$$\leadsto \boxed{\frac{\partial E}{\partial t}+\frac{\partial}{\partial x_i}\left[(E+P)v_i\right]=\rho v_i F_i}$$ total energy equation

$$E \equiv \frac{1}{2}\rho v^2+\rho\varepsilon$$ total energy density

Remarks: 1) ①, ②, ③ are known as the Euler equations. They can be rewritten as a system of conservation laws:

$$\boxed{\begin{aligned} u_t + f^i(u)_{x_i} &= S \\ \partial_t u + \frac{\partial}{\partial x_i} f^i(u) &= S \end{aligned}}$$

where

$$u = \begin{pmatrix} \rho \\ \rho \vec{v} \\ E \end{pmatrix}$$ "conserved variables"

$$f^i = \begin{pmatrix} \rho v^i \\ \rho v^i v^j + \delta_{ij} p \\ v^i (E + p) \end{pmatrix}$$ $j = 1,2,3$ (momentum components), $i = 1,2,3$ (flux directions) "fluxes"

$$S = \begin{pmatrix} 0 \\ \rho \vec{F} \\ \rho \vec{v} \cdot \vec{F} \end{pmatrix}$$ "sources"

2) Equation of state: The Euler equs. are five equations for six unknowns $\rho, \{v^i\}, P, e \equiv \rho\varepsilon$

$\rightsquigarrow$ require relation $P = P(\rho, e)$ to close the system $\rightarrow$ "equation of state"

Ideal gas: $p = (\gamma - 1) e = (\gamma - 1) \rho \varepsilon$

internal energy density

with $\gamma = \frac{c_p}{c_v}$ the ratio of specific heats (adiabatic constant)

In general: $p = p(\rho, e, X_i, \dots)$ can be very

composition

complicated ($\hookrightarrow$ different chemical elements, ionization states, compl. reactions etc.)

<u>Exercise:</u> Show strict hyperbolicity of the Euler equations.

## 3) <u>Validity of</u> continuous medium approximation

require (see above):

(i) $\lambda \ll dx \ll l_{sys}$

mean free path of particle

finite size element of medium

characteristic size of physical system

$f$ should not vary over $dx$ & $dN$ should be large enough for averaging to be meaningful

(ii) interparticle forces must be short-range i.e. on distances $l_{force} \ll dx$, otherwise energy & momentum exchanged with fluid elements far away

example for long-range: external grav. field
$\rightarrow$ external force

Note: self-gravity cannot be obtained from Boltzmann equ due to long-range nature
$\rightarrow$ require additional Poisson equ.

(iii) peculiar motion close to boundary of element $dx$ carries particles into adjacent elements with different $\rho(\vec{x})$, $v(\vec{x})$, $T(\vec{x})$ $\rightsquigarrow$ "diffusion"

$\rightsquigarrow$ friction forces appear $\Rightarrow$ microscopic exchange of momentum & energy neglected in Euler equation

$\rightarrow$ need to take conduction heat flux $\vec{h}$ and other terms into account (see below)

## 2.3 Viscosity: Navier-Stokes equations

When interchange ("diffusion") of particles between adjacent fluid elements cannot be neglected

→ internal friction or viscosity

⇝ Modify momentum flux density:

$$(*) \quad \pi \longrightarrow \Pi \equiv \pi - \underset{\substack{\uparrow \\ \text{viscous stress tensor}}}{\sigma}, \quad \Pi_{ij} = \rho v_i v_j + \underbrace{P\delta_{ij} - \sigma_{ij}}_{\substack{\text{stress tensor} \\ -\bar{\sigma}_{ij}}}$$

- friction requires differences in velocity between particles → $\sigma$ should depend on $\frac{\partial v_i}{\partial x_j}$
  If velocity gradients small, 1st order derivatives OK, and $\sigma$ is linear in $\frac{\partial v_i}{\partial x_j}$
- $\sigma = 0$ when fluid rotates rigidly or $v \equiv$ const.

↓ Landau & Lifshitz Fluid Mechanics §15

$$\sigma_{ij} = \underset{\substack{\uparrow \\ \text{shear viscosity coefficient} \\ \text{("dynamic viscosity coefficient")}}}{\eta} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3} \frac{\partial v_k}{\partial x_k} \delta_{ij} \right) + \underset{\substack{\uparrow \\ \text{bulk viscosity coefficient}}}{\zeta} \frac{\partial v_k}{\partial x_k} \delta_{ij}$$

$\eta, \zeta$ independent of velocity, $\geq 0$

**bulk:** energy transfer btw translational and internal motions of fluid particles

→ not possible for ideal mono-atomic gases for example → no internal DOF

→ not for incompressible fluids $\frac{\partial \rho}{\partial t} = 0$ ($\nabla \cdot v = 0 \Rightarrow \rho \nabla \cdot \vec{v} = 0$)

→ if present frictional force opposing changes of the volume of fluid elements generated

→ usually not relevant in astrophysics

**shear:** "dynamic viscosity coefficient"

→ momentum diffusion through shear motion: faster fluid elements decelerate, slower elements accelerate

With the replacement (*) we obtain the

**Navier-Stokes equations:**

Navier (1827)
Stokes (1845)

$$\boxed{\begin{aligned} &\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i) = 0 \\ &\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j - \sigma_{ij}) = -\frac{\partial P}{\partial x_i} + \rho F_i \qquad (i = 1,2,3) \\ &\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_j}\left[(E+P)v_j - \sigma_{jk} v_k\right] = \rho v_j F_j \end{aligned}}$$

# Properties of viscous effects:

- Viscous timescale:

  Assume that viscous effects dominate & consider incompressible fluid for simplicity $\left(\Rightarrow \sigma_{ij} = \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right)$

momentum eq. $\rightsquigarrow$

$$\frac{\partial}{\partial t} v_i \simeq \frac{1}{\rho} \frac{\partial \sigma_{ij}}{\partial x_j} \Rightarrow \frac{|\vec{v}|}{t_v} \sim \nu \frac{1}{l_{sys}} \frac{|\vec{v}|}{l_{sys}}$$

characteristic time & length scales

$$\nu \equiv \frac{\eta}{\rho} \quad \underline{\text{kinematic viscosity coefficient}}$$

$$t_\nu = \frac{l_{sys}^2}{\nu} \quad \underline{\text{viscous timescale}}$$

- Magnitude of effects:

  $\rightarrow$ <u>molecular viscosity</u> (peculiar motions of particles wrt to bulk motion)

$$\nu_{mol} = \lambda \, \tilde{u}_{sys}$$

mean free path

characteristic velocity of peculiar motions

$$t_\nu = \frac{l_{sys}^2}{\lambda \tilde{u}_{sys}}$$

HI cloud (interstellar) $\sim 3 \times 10^{12}\,$yr $\quad \left(l_{sys} \sim 10^{19}\,\text{cm},\ \lambda \sim 10^{14}\,\text{cm},\ \tilde{u}_{sys} \sim 10^4\,\frac{\text{cm}}{\text{s}}\right)$

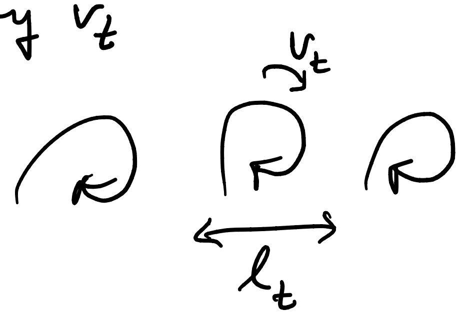
protostellar disk (star formation) $\sim 3 \times 10^{14}\,$yr $\quad \left(l_{sys} \sim 10^{14}\,\text{cm},\ \lambda \sim 10^{10}\,\text{cm},\ \tilde{u}_{sys} \sim 10^5\,\frac{\text{cm}}{\text{s}}\right)$

$> \sim 10^2 - 10^4 \times$ age of universe!

$\Rightarrow$ negligible for all practical purposes

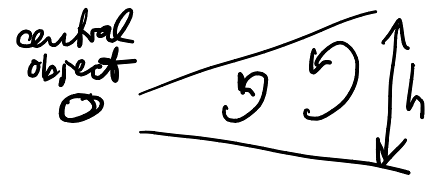
$\rightarrow$ <u>turbulent viscosity</u> (turbulent flows within a bulk flow)

$\rightsquigarrow$ random motions of eddies with characteristic size $l_t$ and velocity $v_t$




$$\nu_t = l_t \, v_t$$

Consider <u>accretion disk</u>:




$l_t \sim h$

$v_t \sim \alpha \, c_s$

$\alpha \in (0,1)$ ; $c_s$: sound speed

(if $v_t > c_s \Rightarrow$ shocks $\Rightarrow$ dissipation of turbulent motions until $v_t < c_s$)

$\rightsquigarrow \nu_t = \alpha \, c_s \, h$

$\alpha$: "$\alpha$-viscosity" (Shakura & Sunyaev 1973)

$\rightarrow$ source for such disk turbulence is the <u>magnetorotational instability</u> (MRI) (Balbus & Hawley 1991)

## 2.4 Magnetohydrodynamics

Ionized medium (plasma) can conduct electric currents $\Rightarrow$ interacts with EM fields

$\rightarrow$ complex interplay btw plasma motions & fields

Simplest possible approximation of magnetized plasmas: magnetohydrodynamics (MHD):

(i) plasma treated as continuous medium, described by Navier-Stokes Eqns in absence of EM fields

(ii) Positive & negative charges are locally and globally balanced at all times (→ fluid elements are neutral)

(iii) electrons are in statistical equilibrium with ions (same temperature)

(iv) Interparticle collisions frequent enough for all effects of magnetic forces to be instantaneously transferred from $e^-$ to ions and neutral particles (if present)

Note: for currents to flow, $e^-$ & ions cannot move at the same velocity. However, the relative drift velocity is so small that it can be neglected (see below)

## Vacuum Maxwell eqns:

Gauss (cgs) units!

I $\nabla \cdot \vec{E} = 4\pi q$

II $\nabla \cdot \vec{B} = 0$

III $\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$

IV $\nabla \times \vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \simeq \frac{4\pi}{c}\vec{j}$

Plasma motion approximations:

1) III $\Longrightarrow \frac{E_{sys}}{l_{sys}} \sim \frac{B_{sys}}{c\, t_{sys}}$

$E_{sys}, B_{sys}$ : characteristic field strengths

$l_{sys}, t_{sys}$ : characteristic length & time scales

$v_{sys} \equiv \frac{l_{sys}}{t_{sys}}$ characteristic plasma velocity

$$\leadsto \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \sim \frac{E_{sys}}{c\, t_{sys}} \sim \frac{B_{sys}\, l_{sys}}{c^2 t_{sys}^2} \sim \frac{v_{sys}^2}{c^2}|\nabla \times \vec{B}| \ll |\nabla \times \vec{B}|$$

assuming non-relativistic plasma

Drift velocity: consider $e^-$-p plasma (H-plasma)

different velocities of $e^-$, p cause current

$$\vec{j} = q_e(n_e \vec{v}_e - n_p \vec{v}_p) = q_e n_e (\vec{v}_e - \vec{v}_p) = q_e n_e \vec{v}_{drift}$$

number densities

$$|\nabla \times \vec{B}| \approx \frac{B_{sys}}{l_{sys}} \overset{\text{IV}}{\approx} \frac{4\pi}{c} q_e n_e |v_{drift}|$$

$B_{sys} \sim 10^3\,G$
$n_e \sim 10^{23} \frac{1}{cm^3}$
$l_{sys} \sim 10^{10}\,cm$

$$\Rightarrow \quad v_{drift} \sim \begin{cases} 10^{-12} \frac{cm}{s} & \text{solar plasma (convection zone)} \\ 10^{-2} \frac{cm}{s} & \text{interstellar medium} \end{cases}$$

(compare turbulent velocities $\sim 1 \frac{km}{s}$)

$l_{sys} \sim 1\,pc$
$B_{sys} \sim 10^{-5}\,G$
$n_e \sim 10^{-3} \frac{1}{cm^3}$

$$2) \quad q|\vec{E}| \overset{\text{I}}{\sim} \frac{E_{sys}^2}{l_{sys}} \overset{1)}{\sim} \frac{v_{sys}^2}{c^2} \frac{B_{sys}^2}{l_{sys}} \overset{\text{IV}}{\sim} \frac{v_{sys}^2}{c^2} \frac{1}{c} |\vec{j} \times \vec{B}| \ll \frac{1}{c} |\vec{j} \times \vec{B}| \sim |\vec{F}_L|$$

$\frac{1}{c}|\vec{j}| \sim B_{sys}/l_{sys}$

non-relativistic plasma

$\rightarrow$ neglect effects of space charge $\rightarrow$ Gauss' law I can be dropped

## MHD equations:

- Ohm's law: $\vec{j} = \sigma_e \vec{E}$ (plasma at rest)

$\sigma_e = \frac{n_e e^2}{m_e f_c}$ electric conductivity

$f_c$: collision frequency btw ions and $e^-$

non-relat. motion

$$\text{V} \qquad \vec{j} = \sigma_e \left( \vec{E} + \frac{\vec{v}}{c} \times \vec{B} \right)$$

- Ohmic dissipation (current decay due to finite conductivity):

$$\frac{\partial e}{\partial t} = \frac{1}{\sigma_e} |\vec{j}|^2$$

conversion of EM energy into internal energy

- reaction of B-field to motion of plasma:

IV in V:
$$\vec{E} = \frac{c}{4\pi\sigma_e} \vec{\nabla} \times \vec{B} - \frac{\vec{v}}{c} \times \vec{B}$$
electric field

in III:
$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) - \nabla \times (\eta_e \nabla \times \vec{B})$$
induction equation

$$\eta_e \equiv \frac{c^2}{4\pi\sigma_e}$$
electric resistivity

at rest:
$$\frac{\partial \vec{B}}{\partial t} = -\nabla \times (\eta_e \nabla \times \vec{B}) \Rightarrow \frac{B_{sys}}{t_{res}} \sim -\eta_e \frac{B_{sys}}{l_{sys}^2}$$

$$\Rightarrow t_{res} \sim \frac{l_{sys}^2}{\eta_e}$$
decay timescale of magnetic field due to finite resistivity

Note: $[\eta_e] = [\text{length}] \times [\text{velocity}]$

$\rightsquigarrow$ $\eta_e$ also "magnetic diffusivity"

- EM fields affect motion of plasma:

Lorentz force
$$\vec{F}_L = \frac{1}{c} \vec{j} \times \vec{B} \overset{IV}{=} \frac{1}{4\pi} (\nabla \times \vec{B}) \times \vec{B}$$
$$= \underbrace{\frac{1}{4\pi} (\vec{B} \cdot \nabla) \vec{B}}_{\text{magnetic tension}} - \underbrace{\frac{1}{8\pi} \nabla (\vec{B}^2)}_{\text{magnetic pressure force}}$$

$$P_m = \frac{B^2}{8\pi}$$

$\rightarrow$ curvature of $\vec{B}$-field lines (vanishes for $\vec{B}$ const. on straight lines)

In principle: $\vec{F}_{EM} = \vec{F}_L + q\vec{E}$, but $q|\vec{E}| \ll |\vec{F}_L|$ (see above)

$$\leadsto \boxed{\begin{aligned} &\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0 \\ &\frac{\partial}{\partial t}(\rho\vec{v}) + \nabla\cdot\Pi = \frac{1}{4\pi}(\vec{B}\cdot\nabla)\vec{B} - \frac{1}{8\pi}\nabla(\vec{B}^2) + \rho\vec{F} \\ &\frac{\partial E}{\partial t} + \nabla\cdot[(E+P)\vec{v}] = \frac{4\pi}{c^2}\eta_e|\vec{j}|^2 + \rho\,\vec{v}\cdot\vec{F} \\ &\frac{\partial \vec{B}}{\partial t} = -\nabla\times(\eta_e\nabla\times\vec{B}) + \nabla\times(\vec{v}\times\vec{B}) \end{aligned}}$$

<u>Ideal MHD approximation:</u> assume $\boxed{\eta_e \equiv 0}$

<u>Exercise:</u> Show that the MHD equations can be written in <u>conservation form</u>:

$$\boxed{\begin{aligned} &\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0 \\ &\frac{\partial \vec{S}}{\partial t} + \nabla\cdot T = 0 \\ &\frac{\partial \tau}{\partial t} + \nabla\cdot U = 0 \\ &\frac{\partial \vec{B}}{\partial t} + \nabla\cdot Y = \vec{S}_{res} \end{aligned}}$$

(conservation of mass)

(conservation of momentum)

(conservation of energy)

((non) conservation of magnetic flux)

or $\partial_t u + \partial_{x_i} f^i(u) = S$

with
$$u = \begin{pmatrix} \rho \\ \vec{S} \\ \tau \\ \vec{B} \end{pmatrix} \begin{array}{l} \\ \leftarrow \text{conserved momentum} \\ \leftarrow \text{conserved energy} \\ \\ \end{array}$$

$$f(u) = \begin{pmatrix} \rho \vec{v} \\ T \\ U \\ Y \end{pmatrix}, \qquad S = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vec{S}_{res} \end{pmatrix}$$

where:

$$\vec{S} \equiv \rho \vec{v} \qquad \text{(momentum density)}$$

$$T_{ij} \equiv \underbrace{\rho v_i v_j}_{\substack{\text{Reynolds} \\ \text{stress tensor}}} + \delta_{ij}\Big(\underset{\substack{\uparrow \\ \text{isotropic} \\ \text{pressure}}}{P} + \underbrace{\frac{B^2}{8\pi}\Big) - \frac{1}{4\pi} B_i B_j}_{\substack{\text{magnetic part} \\ \text{of Maxwell} \\ \text{stress tensor}}} \qquad \text{(stress tensor)}$$

$$\tau \equiv \underbrace{\frac{1}{2}\rho v^2}_{\text{kinetic}} + \underbrace{e + \frac{B^2}{8\pi}}_{\text{potential}} \qquad \text{(total energy density)}$$

$$= E + \frac{B^2}{8\pi}$$

$$\vec{U} = (E+P)\vec{v} + \frac{1}{4\pi}\left(-\frac{\vec{v}}{c} \times \vec{B} + \eta_e \vec{j}\right) \times \vec{B} \qquad \text{(energy flow vector)}$$

$$Y_{ij} \equiv v_i B_j - v_j B_i$$

$$\vec{S}_{res} = \frac{\eta_e}{4\pi} \nabla^2 \vec{B} + \vec{j} \times \nabla \eta_e \qquad \text{(resistive source term)}$$

# Limits of MHD approximation:

## Collisionless regime:

Drift velocity only small as long as EM forces due to large-scale fields balanced by frictional forces due to interparticle collisions

→ breaks down at low density when $e^-$ & ions decouple:

"Coulomb logarithm", weakly dependent on $T$, $n_e$

$$f_e^L = f_e^{coll}$$

$\frac{q_e B}{2\pi m_e c}$ Larmor (cyclotron) frequency of gyration

$3.7 \ln \Lambda \, T^{-3/2} n_e$ frequency of collisions btw free $e^-$

<u>Example</u>: star-forming cloud: $T \sim 10\,K$, $B \sim 10^{-5}\,G$

$\Rightarrow n_e^{crit} \approx 10 \frac{1}{cm^3}$

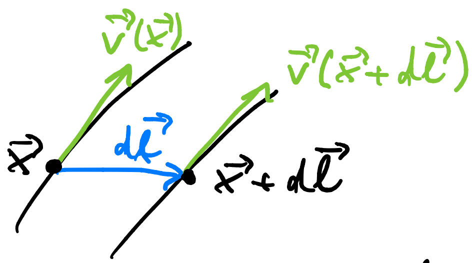
⇝ rather high, especially bc of low ionization degree

<u>But</u>: often MHD approximation works well even in <u>collisionless regime</u> (e.g. solar wind past the Earth)

→ non-linear phenomena (not captured by MHD model) cause collisionless plasma to behave as continuous medium

# Flux conservation & field freezing:

Assume ideal MHD: $\eta_e \equiv 0$.

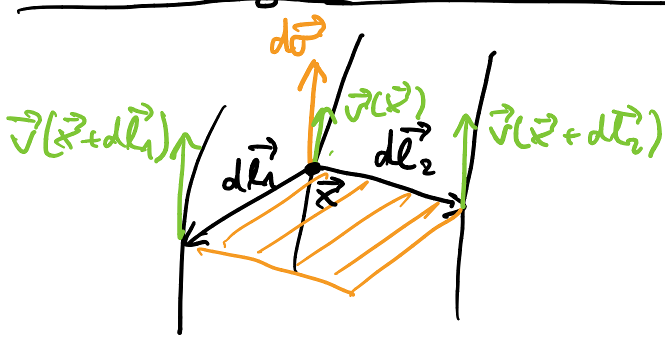



Co-moving line element (Lagrangian derivative):

$$\frac{d}{dt}(d\vec{\ell}) = \frac{d(\vec{x}+d\vec{\ell})}{dt} - \frac{d\vec{x}}{dt} \overset{\text{def}}{=} \vec{v}(\vec{x}+d\vec{\ell}) - \vec{v}(\vec{x}) \overset{\text{Taylor}}{=} (\nabla\vec{v})\cdot d\vec{\ell} \quad (*)$$

(change of $d\vec{\ell}$ comoving with flow)

Co-moving surface element $d\vec{\sigma} \equiv d\vec{\ell}_1 \times d\vec{\ell}_2$:




$$\frac{d(d\vec{\sigma})}{dt} = \frac{d(d\vec{\ell}_1)}{dt} \times d\vec{\ell}_2 + d\vec{\ell}_1 \times \frac{d(d\vec{\ell}_2)}{dt}$$

$$= d\vec{\ell}_1\cdot\nabla\vec{v} \times d\vec{\ell}_2 - d\vec{\ell}_2\cdot\nabla\vec{v} \times d\vec{\ell}_1$$

vector identity

$$= -(d\vec{\sigma}\times\nabla)\times\vec{v} = -\nabla\vec{v}\cdot d\vec{\sigma} + \nabla\cdot\vec{v}\, d\vec{\sigma}$$

Magnetic flux: $\Phi_m \equiv \int_S \vec{B}\cdot\vec{n}\, dA$, $\quad d\vec{\sigma} = \vec{n}\, dA$

$$\leadsto \frac{d}{dt}(d\Phi_m) = \frac{d}{dt}(\vec{B}\cdot d\vec{\sigma}) = \frac{d\vec{B}}{dt}\cdot d\vec{\sigma} + \vec{B}\cdot\frac{d(d\vec{\sigma})}{dt}$$

Note: $$\frac{\partial \vec{B}}{\partial t} \overset{\eta_e=0}{=} \nabla\times(\vec{v}\times\vec{B}) \overset{\substack{\text{vector}\\ \text{identity}}}{=} \vec{B}\cdot\nabla\vec{v} - \vec{B}\nabla\cdot\vec{v} - \vec{v}\cdot\nabla\vec{B}$$

$$\Rightarrow \frac{d\vec{B}}{dt} \overset{\text{def}}{=} \frac{\partial\vec{B}}{\partial t} + \vec{v}\cdot\nabla\vec{B} = \vec{B}\cdot\nabla\vec{v} - \vec{B}\nabla\cdot\vec{v}$$

Therefore:

$$\frac{d}{dt}(d\Phi_m) = (\vec{B}\cdot\nabla\vec{v} - \vec{B}\nabla\cdot\vec{v})\cdot d\vec{\sigma} + \vec{B}\cdot(-\nabla\vec{v}\cdot d\vec{\sigma} + \nabla\cdot\vec{v}\, d\vec{\sigma})$$
$$= 0$$

co-moving surface element $d\vec{\sigma}$ arbitrary

$\Rightarrow$ $\boxed{\text{flux through any surface } S \text{ moving with the fluid is conserved: } \quad \Phi_m = \int_S \vec{B}\cdot\vec{n}\, dA = \text{const.}}$

field freezing (consider limit $d\vec{\sigma}\to 0$, i.e. "single field line". Since $\Phi_m \to 0$ divide by $\rho\to 0$)

$$\frac{d}{dt}\left(\frac{\vec{B}}{\rho}\right) = \frac{1}{\rho}(\vec{B}\cdot\nabla\vec{v} - \vec{B}\nabla\cdot\vec{v}) + \frac{\vec{B}}{\rho}\nabla\cdot\vec{v}$$
$$= \left(\frac{\vec{B}}{\rho}\right)\cdot\nabla\vec{v}$$

$\Rightarrow$ line element $d\vec{l} \parallel \vec{B}$ moves as $\frac{\vec{B}}{\rho}$ (see (*))

i.e. plasma along $d\vec{l}$ and $\vec{B}$ move together

$\rightsquigarrow$ $\vec{B}$-field is "frozen in" the fluid

Note: motion along the field lines is preserved however!

## Limits of ideal MHD:

- $\eta_e = 6.5 \times 10^{12} \frac{\ln \Lambda}{T^{3/2}}$ [$cm^2/s$] (fully ionized hydrogen plasma)

  $\sim 10^8$ for H II region $\left( \begin{array}{l} T \sim 10^4 K, \ \ln\Lambda \simeq 20, \\ n_e = 10-100 \frac{1}{cm^3} \\ l_{sys} \sim 1pc \end{array} \right)$

  $\rightsquigarrow$ diffusivity timescale $t_{res} \sim \frac{l_{sys}^2}{\eta_e} \sim 10^{21}$ yr

  $\sim 10^{11}$ age of universe

  $\rightarrow$ ideal MHD very good!

- But: $\eta_e$ becomes large / $t_{res}$ small if

  1) where field becomes very tangled or when oppositely directed fields come in close contact (small $l_{sys}$)

2) plasma <u>strongly turbulent</u>

⇝ turbulent magnetic diffusivity

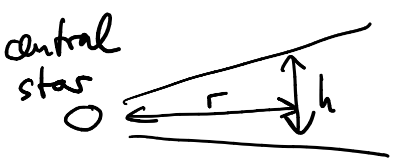
$$\eta_e^t \equiv v_t \, l_t = \nu_t$$ Sec. 2.3

3) plasma <u>weakly ionized</u>

If $n_e \ll n_n$, $n_n$: number density of neutral particles

$$\eta_e = 300 \, \frac{n_n}{n_e} \sqrt{T} \, \left[\frac{\mathrm{cm}^2}{\mathrm{s}}\right]$$ valid up to $T \sim 10^4\,\mathrm{K}$

Example: protoplanetary disk (neglect turbulence)




$\frac{h}{r} \approx 0.1$, $T \sim 10^3\,\mathrm{K}$

$l_{sys} \approx 1.5 \times 10^{12}\,\mathrm{cm}$ (height at Earth's radius)

$t_{disk} \sim 10^7\,\mathrm{yr}$ (lifetime of disk)

$\frac{n_e}{n_n} \sim 10^{-6}$ ($t_{disk} \overset{!}{=} t_{res}$)

For $t_{res} < t_{disk}$ one requires $\frac{n_e}{n_n} < 10^{-6}$

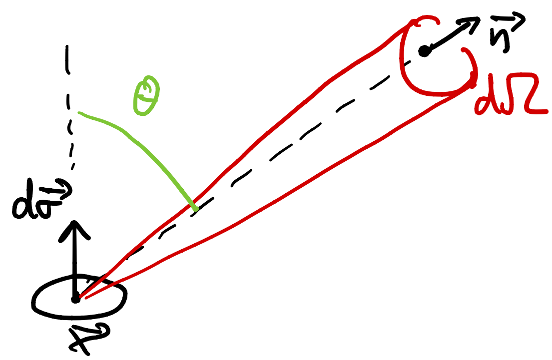
$\Rightarrow$ even at these low ionization degrees ideal MHD is still valid!

# 2.5 Radiation transfer

**Def:** radiation intensity

$$I_\nu(\vec{x}, \vec{n}, \nu, t) \equiv \frac{dE_\nu}{\vec{n} \cdot d\vec{\sigma}\, d\Omega\, d\nu\, dt}$$

energy of photons that pass through an area $d\vec{\sigma}$ at point $\vec{x}$ in direction $\vec{n}$, within a solid angle $d\Omega$ in the frequency interval $(\nu, \nu + d\nu)$ in time $dt$




**Phase space for photons:** use $\vec{u} \mapsto \vec{q}$ (momentum) as photons are massless and propagate with velocity $c$

$$\left.\begin{aligned} &\text{total number} && dN = f(\vec{x}, \vec{q}, t)\, d\vec{x}\, d\vec{q} \\ & && \quad\downarrow \\ &\text{total radiation energy} && dE_\nu = h\nu\, f(\vec{x}, \vec{q}, t)\, d\vec{x}\, d\vec{q} \end{aligned}\right\} \text{within phase space volume } d\vec{x}\, d\vec{q}$$

Note: consider a beam of photons as in previous definition.


$$\rightsquigarrow \; d\vec{x} = c\,dt\,\underbrace{(\vec{n}\cdot d\vec{\sigma})}_{\text{effective area}}$$

spatial volume swept out by the beam in time $dt$

$$d\vec{q} = q^2\,dq\,d\Omega = \left(\frac{h\nu}{c}\right)^2 \frac{h}{c}\,d\nu\,d\Omega$$

spherical polar coord. in momentum space

$q = |\vec{q}|, \quad \vec{q} = \frac{h\nu}{c}\vec{n}$

$$\Rightarrow \; dE_\nu = \frac{h^4\nu^3}{c^2} f(\vec{x},\nu,\vec{n},t)\;\vec{n}\cdot d\vec{\sigma}\,d\nu\,d\Omega\,dt$$

$$\Rightarrow \; \boxed{I_\nu = \frac{h^4\nu^3}{c^2} f}$$

## Boltzmann equation:

$\vec{F} = 0$ for photons

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \underbrace{c n_i}_{u_i = c n_i}\frac{\partial f}{\partial x_i} = \left[\frac{\partial f}{\partial t}\right]_{coll}$$

$$\boxed{\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \underbrace{(\vec{n}\cdot\vec{\nabla})}_{=\frac{\partial}{\partial s}} I_\nu = \underbrace{\left[\frac{\partial I_\nu}{\partial t}\right]_{coll}}_{\text{interaction of photons with matter}}}$$

equation of radiation transfer

$\rightarrow$ absorption, emission & scattering

For a time-independent problem:

$$\boxed{\frac{dI_\nu}{ds} = \left[\frac{\partial I_\nu}{\partial t}\right]_{coll}}$$

Computational challenge:

$$I_\nu(\vec{x}, \nu, \vec{n}, t) = I_\nu(r, \theta, \phi, \underbrace{\nu, \theta', \phi'}_{\text{momentum space}}, t)$$

6+1 D function!

take 100 grid points per variable

→ need to store $(10^2)^6 = 10^{12}$ values per time step!

→ direct numerical solution of radiation transfer equation computationally prohibitive, need additional simplifications (reduce number of spatial dimensions etc.)

or solve set of approximate equations (moment schemes etc.)

Exercise: M1 moment scheme

## 2.6 Relativistic Hydrodynamics

In special relativity the Euler equations can be written as

$$\partial_\mu (\rho u^\mu) = 0$$

$$\partial_\mu T^{\mu\nu} = 0$$

For an ideal/perfect fluid (in local rest frame no energy fluxes & anisotropic stresses):

specific enthalpy

$$T^{\mu\nu} = \overbrace{(E+p)}^{\rho h} u^\mu u^\nu + p\,\eta^{\mu\nu}$$

energy-momentum tensor

total energy density

isotropic pressure

Minkowski metric

specific internal energy measured in local rest frame

$$E = \rho c^2 + e = \rho(1+\varepsilon)$$

henceforth set $c=1$

General relativity: invoke the equivalence principle and use the substitution rules

$$\partial_\mu (\rho u^\mu) = 0$$

$$\partial_\mu T^{\mu\nu} = 0$$

$\downarrow$ equivalence principle, substitution rules

$$\eta_{\mu\nu} \to g_{\mu\nu}$$
$$\partial_\mu \to \nabla_\mu$$
$$S_{SR}, X^\mu_{SR}, T^{i_1 \dots i_r}_{j_1 \dots j_s\, SR} \to S_{GR}, X^\mu_{GR}, T^{i_1 \dots i_r}_{j_1 \dots j_s\, GR}$$

$$\nabla_\mu (\rho u^\mu) = 0$$
$$\nabla_\mu T^{\mu\nu} = 0$$
$$T^{\mu\nu} = (E+p) u^\mu u^\nu + p g^{\mu\nu}$$

(alternatively: Bianchi identities imply $\nabla_\mu T^{\mu\nu} = 0$)

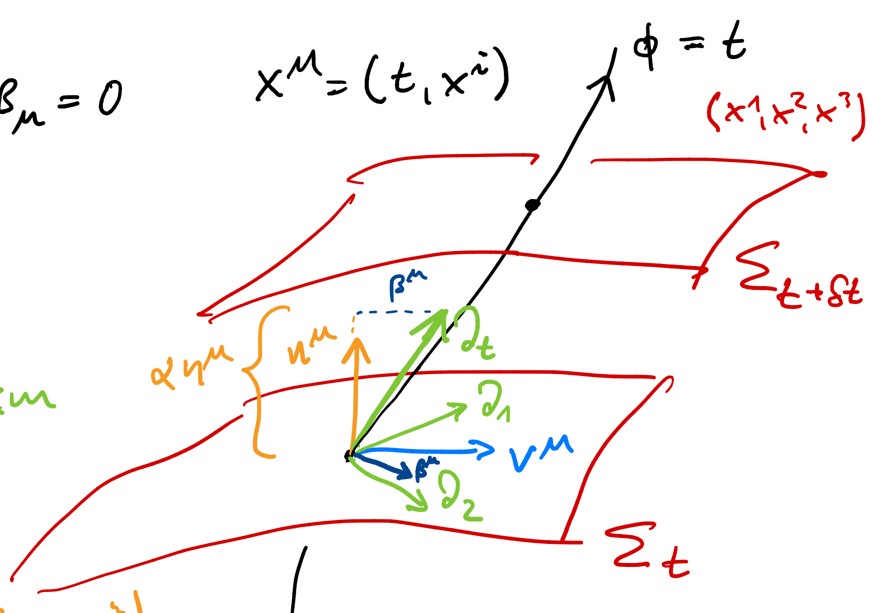
Goal: want to rewrite these equations as an initial boundary value problem in conservation form so that one can use similar techniques as in the Newtonian case

$\longrightarrow$ 3+1 decomposition of spacetime

$$(\partial_t)^\mu = \underset{\text{"lapse"}}{\alpha}\, n^\mu + \underset{\text{"shift"}}{\beta^\mu}, \quad n^\mu \beta_\mu = 0 \qquad x^\mu = (t, x^i)$$

represent 4 DOF to specify coordinates (diffeomorphism invariance of GR)

$u^\mu = n^\mu$: Eulerian observer ("lab frame")



no motion wrt spatial coordinates

$$\|\partial_t\|^2 = (\partial_t)^\mu (\partial_t)_\mu = -\alpha^2 + \beta_\mu \beta^\mu = -\alpha^2 + \beta^2$$

$\rightsquigarrow$ $(\partial_t)^\mu$ timelike $\Leftrightarrow \beta^2 < \alpha^2$

$(\partial_t)^\mu$ null $\Leftrightarrow \beta^2 = \alpha^2$

$(\partial_t)^\mu$ spacelike $\Leftrightarrow \beta^2 > \alpha^2$

Adapted coordinates:

$$(\partial_t)^\mu \equiv (1, 0, 0, 0)$$

$$\beta^\mu \equiv (0, \beta^i)$$

$$\rightsquigarrow \quad n^\mu = \frac{1}{\alpha} (\partial_t)^\mu - \frac{1}{\alpha} \beta^\mu = \left(\frac{1}{\alpha}, -\frac{1}{\alpha}\beta^i\right)$$

$$n_\mu = (-\alpha, 0, 0, 0)$$

$$g_{00} = g(\partial_t, \partial_t) = -\alpha^2 - \beta^2$$

$$g_{0i} = g(\partial_t, \partial_i) = (\alpha n^\mu + \beta^\mu)(\partial_i)_\mu$$

$$= \alpha \underbrace{n^\mu (\partial_i)_\mu}_{=0} + \beta^\mu (\partial_i)_\mu = \beta_i$$

$$g_{ij} = g(\partial_i, \partial_j) \equiv \gamma_{ij}$$

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 - \beta^2 & \beta_i \\ \beta_i & \gamma_{ij} \end{pmatrix}$$

or: $ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$

↳ measures proper time that elapses going from one time slice to the next along ds ⇝ "lapse"

**Note:** a specific choice of $\alpha$ & $\beta^i$ represents a specific choice of coordinates (gauge)

**Example:** $\alpha \equiv 1$, $\beta^i \equiv 0$ (Geodesic gauge)

- $ds^2 = \alpha\, dt^2 = dt^2$ for Euleran observers

  $\Rightarrow t = \tau$ represents proper time of Euleran observers

- curves $c$ of $\{x^i\} = \text{const.}$ are orthogonal to $\Sigma_t$, i.e. Euleran observers move on geodesics:

$$a_\mu = \nabla_{\dot{c}} \dot{c} = \nabla_u u = n^\nu \nabla_\nu n_\mu$$

$$\begin{cases} n^\nu \nabla_\nu \alpha = 0 & \mu = 0 \\ 0 & \mu = 1,2,3 \end{cases}$$

"Gaussian normal coordinates"

## Adapted equations of hydrodynamics:

Split fluid velocity

$$u^\mu = W(n^\mu + v^\mu)$$

$\leadsto$ $$v^i \equiv \frac{ds^i}{d\tau_{Eul}} = \frac{u^i}{W} + \frac{1}{\alpha}\beta^i$$ fluid velocity relative to Eulerian obs.

$$W \equiv -u^\mu n_\mu$$ Lorentz factor btw. Eulerian & comoving (fluid) frame

and define

$$E \equiv n^\mu n^\nu T_{\mu\nu} = (\mathcal{E}+p)W^2 - p = h\rho W^2 - p$$

$$S_i \equiv -\gamma_i{}^\mu n^\nu T_{\mu\nu} = (\mathcal{E}+p)W^2 v_i = h\rho W^2 v_i$$

$$S_{ij} \equiv \gamma_i{}^\mu \gamma_j{}^\nu T_{\mu\nu} = h\rho W^2 v_i v_j + p\gamma_{ij}$$

$\left(h \equiv 1 + \varepsilon + \frac{p}{\rho}\right.$ specific enthalpy$\left.\right)$

(total energy density, momentum density, stress-energy as measured by Eulerian observer)

then one obtains:

- $0 = \nabla_\mu(\rho u^\mu) \longrightarrow \boxed{\partial_t(\sqrt{\gamma}\, D) + \partial_i(\sqrt{\gamma}\, D\, \tilde{v}^i) = 0}$

$D \equiv \rho W$ rest mass measured by Eulerian obs.

$\tilde{v}^i \equiv \alpha v^i - \beta^i$

- $0 = n_\mu \nabla_\nu T^{\mu\nu}$ (projection of conservation law along Eulerian observer)

$$\rightarrow \boxed{\partial_t(\sqrt{\gamma}\, E_E) + \partial_j\left[\sqrt{\gamma}\,(\alpha S^j - E_E \beta^j)\right] = \alpha\sqrt{\gamma}\,(K_{ij} S^{ij} - S^k \partial_k \ln\alpha)}$$

$K_{ij}$: intrinsic curvature of $\Sigma_t$

- $0 = \gamma_i{}^\mu \nabla_\nu T^\nu{}_\mu$ (spatial projection)

$$\rightarrow \boxed{\partial_t(\sqrt{\gamma}\, S_i) + \partial_j\left[\sqrt{\gamma}\,(S_j \tilde{v}^j + p\,\delta^i{}_j)\right] = \sqrt{\gamma}\left[\frac{\alpha}{2} S^{kl}\partial_i \gamma_{kl} + S_l\,\partial_i \beta^l - S\,\partial_i \alpha\right]}$$

$\leadsto$ Conservative formulation: $\partial_t u + \partial_i f^i(u) = S(u)$

$$u = \sqrt{\gamma}\,(D, S_i, \tau)$$

$$f^i \equiv \sqrt{\gamma}\,(D\tilde{v}^i,\ S_j \tilde{v}^i + p\,\delta^i{}_j,\ \tau\tilde{v}^i + p v^i)$$

$$S(u) \equiv \left(0,\ \alpha\sqrt{\gamma}\,(K_{ij} S^{ij} - S^k\partial_k \ln\alpha),\ \sqrt{\gamma}\left(\frac{\alpha}{2} S^{kl}\partial_i\gamma_{kl} + S_l\,\partial_i\beta^l - S\,\partial_i\alpha\right)\right)$$

$$(\tau \equiv E_E - D)$$

Remarks: 1) The energy equation is written in terms of $\tau = E_E - D$, in order to obtain the "correct" Newtonian limit:

$$\tau = E_E - D = \rho h W^2 - p - \rho W$$

$$\overset{\frac{p}{\rho c^2} \ll 1}{=} \underbrace{\rho W(W-1)}_{\approx \left[1+\frac{1}{2}\left(\frac{v}{c}\right)^2 \dots\right]\left[\frac{1}{2}\left(\frac{v}{c}\right)^2 + \dots\right]} + \rho \varepsilon \underbrace{W^2}_{\approx 1 + \left(\frac{v}{c}\right)^2 \dots} \approx \frac{1}{2}\rho v^2 + \rho\varepsilon$$

2) $s(u)$: "geometric source terms"

Note: do not depend on time derivatives of space-time quantities!

3) EOS: As in Newtonian hydrodynamics, the system is closed by an equation of state $p = p(\rho, \varepsilon, \dots)$

4) Conservatives & primitives:

$$u \equiv \sqrt{\gamma}\,(D, S_i, \tau) \quad \text{"conservatives"}$$

$$w \equiv (\rho, v^i, \varepsilon) \quad \text{"primitives"}$$

In order to compute flux terms during evolution, one must compute $w$ from $u$ numerically (no need $p = p(\rho, \varepsilon)$)

typically: obtain these via <u>non-linear root-finding</u> in some variable $z$

$$f(z) = z(\rho, \varepsilon, v^i) - z$$

where $z(\rho, \varepsilon, v^i)$ is computed from $\rho, \varepsilon, v^i$ obtained from conservatives

Example: $z = p$

$$\rightsquigarrow \quad v^i = \frac{S^i}{\tau + p}, \quad W = \sqrt{1 - v_i v^i}$$

$$\rho = \frac{D}{W}$$

$$\varepsilon = \frac{\tau - DW + p(1 - W^2)}{DW}$$

See Siegel et al. ApJ 859, 71 (2018) for a discussion of recovery schemes.

5) <u>Hyperbolicity</u>: the eqns in conservative form are strongly hyperbolic if the EOS is causal,

$$c_s^2 = \frac{\partial p}{\partial e} < 1 \qquad \text{(sound speed < speed of light)}$$

see Anile 1990, Anton et al. 2006 ApJ 637, 296 for hydrodynamic & magnetohydrodynamic case

6) <u>Special-relativistic limit:</u>

$$\alpha \to 1, \; \beta^i \to 0, \; \gamma_{ij} \to \eta_{ij}$$

$$(\Rightarrow \tilde{v}^i \to v^i)$$

$\rightsquigarrow$ $u = (D, S_i, \tau)$

$$f^i(u) = (Dv^i, S_j v^i + p\delta^i{}_j, \tau v^i + pv^i)$$

$$s(u) = 0$$

7) <u>Newtonian limit:</u> SR limit plus

$$\frac{v}{c} \ll 1, \; W \to 1 + \frac{1}{2}\left(\frac{v}{c}\right)^2$$

$$\frac{p}{\rho c^2} \ll 1, \; h \to 1, \; \tau \to E = \rho\varepsilon + \frac{1}{2}\rho v^2$$

$\rightsquigarrow$ $u = (\rho, \rho v_j, E)$

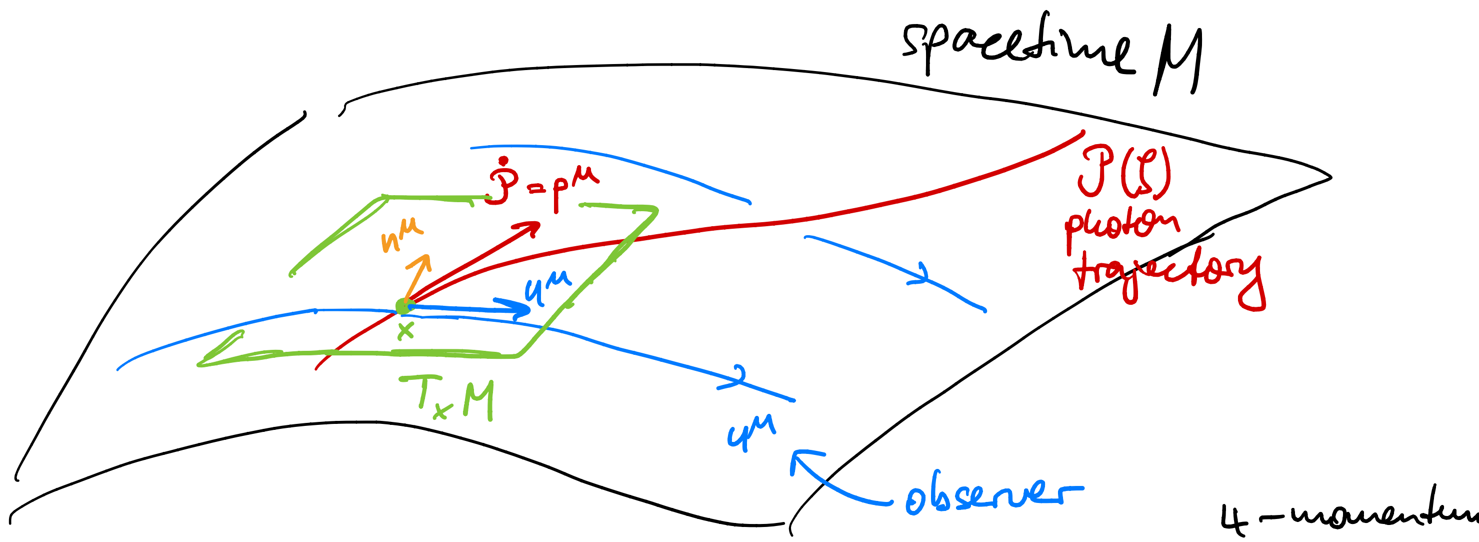
$$f^i(u) = (Dv^i, \rho v^i v_j + p\delta^i{}_j, (E+p)v^i)$$

$$s(u) = 0$$

(cf. Sec. 2.2)

# 2.7 Relativistic radiation transfer

Reading: Thorne 1981, MNRAS 194, 439
Straumann Chap. 3.11




- Consider light-like (massless) particle $p_\mu p^\mu = 0$ (4-momentum) moving along trajectory $\mathcal{P}(\varsigma)$, parametrized such that $\dot{\mathcal{P}}(\varsigma) = p^\mu$

Decompose: $$p^\mu = -p^\nu u_\nu \, (u^\mu + n^\mu)$$ unit normal in orthogonal space to $u^\mu$

$$n^\mu n_\mu = 1, \quad n^\mu u_\mu = 0$$

- Reparametrize by proper spatial distance $\ell$ travelled by photon as seen by observer $u^\mu$:

$$\ell \equiv \int (-p_\mu u^\mu)\, d\varsigma, \quad d\ell = (-p_\mu u^\mu)\, d\varsigma$$

$$\rightsquigarrow \quad \mathcal{P} = \mathcal{P}(\ell), \quad \dot{\mathcal{P}}_\ell = \frac{d\mathcal{P}}{d\ell} = \frac{d\mathcal{P}}{d\varrho}\frac{d\varrho}{d\ell}$$

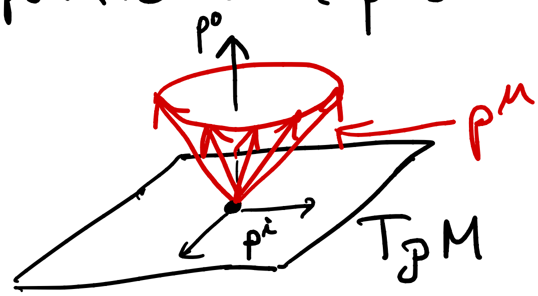
$$= (-p^\mu u_\mu)(u^\mu + n^\mu)\frac{1}{(-p_\mu u^\mu)} = u^\mu + n^\mu$$

- Consider particle distribution function

$$f(x^\mu, p^\mu)$$ (number density of light-like particles in phase space)

$x^\mu$: point in spacetime $\mathcal{P}$

$p^\mu$: 4-momentum on light cone in $T_\mathcal{P} M$




actually: $f = f(t, x^i, p^i)$ as $p^0$ determined by $p_\mu p^\mu = 0$ (light cone particles)

- Note: along each photon trajectory the number density is conserved (in absence of interactions with medium) (Liouville's theorem): $\frac{df}{d\ell} = 0$

$$\rightsquigarrow \quad \boxed{\frac{df}{d\ell} = \frac{\partial f}{\partial x^\mu}\frac{dx^\mu}{d\ell} + \frac{\partial f}{\partial p^i}\frac{dp^i}{d\ell} = S(x^\mu, p^\mu, f)}$$

GR radiation transfer equation

(Boltzmann equation in GR)

S: source term ("collision terms") that accounts for emission, absorption & scattering

<u>Note:</u> $\frac{dx^\mu}{d\ell} = \dot{\mathcal{P}}_\ell = n^\mu + u^\mu$ (see above)

$$\frac{dp^i}{d\ell} = \dot{p}^i = (\dot{\mathcal{P}}_\ell)^i = -\Gamma^i_{\mu\nu}\, p^\mu \dot{x}^\nu$$

$$= -\Gamma^i_{\mu\nu}\, p^\mu (n^\nu + u^\nu)$$ (null geodesic!)

## Moment formalism:

(analogous to Newtonian case)

Thorne 1981, MNRAS 194, 439

Define k-th moment of distribution function

restrict integration to light cone

$$M_\nu^{\alpha_1 \dots \alpha_k} \equiv \int \frac{f(x^\mu, p^\mu)\, \delta(p^\mu u_\mu + \nu)}{(-p^\mu u_\mu)^{k-2}}\, p^{\alpha_1} \cdots p^{\alpha_k}\, dV_p$$

frequency $\nu = -p^\mu u_\mu$

$\rightarrow$ can derive evolution equations for

the moments in <u>conservation form</u>,
analogous to the Newtonian case,
closed by Eddington factors
($\rightarrow$ "M1 scheme")
see Thorne 1981 & Shibata et al. Prog. Theo. Phys. 125, 1255 (2011)