Chap. 4: properties of conservation laws (theoretical background)

consider system of conservation laws (1D):

$$\text{(IVP)} \qquad \begin{aligned} u_t + f(u)_x &= 0 \quad \text{in } \mathbb{R} \times (0,\infty) \\ u(x,0) &= u_0(x) \quad \text{on } \mathbb{R} \times \{t=0\} \end{aligned}$$

$$u = (u^1, \dots, u^m) : \mathbb{R} \times [0,\infty) \to \mathbb{R}^m$$

- show that in general classical solutions (smooth $\rightsquigarrow C^1$) do not exist (globally)
- define weak solutions & entropy condition to guarantee uniqueness of weak solution
- Riemann problem: discontinuous initial data & non-linear waves
  $\rightarrow$ centrepiece of numerical algorithms

4.1 Local existence of classical solutions

Even in the case of smooth data f and u_0 there are in general no continuous solutions that exist <u>globally</u> in time.

<u>Example:</u> <u>Inviscid Burger's equ</u>

Consider scalar conservation law

$$u_t + f(u)_x = 0 \quad \text{with } f(u) = \frac{1}{2}u^2$$

and $u_0 \in C^\infty(\mathbb{R})$, $u_0(x) = \begin{cases} 1, & x \in (-\infty, -1] \\ 0, & x \in [1, \infty) \\ u_0' \le 0 \end{cases}$

Remember (Sec. 1.3.1): characteristic $\Gamma^t_{x_0} = (\gamma(t), t)$ defined by

$$\gamma'(t) = \frac{d\gamma}{dt} = \lambda(u) = f'(u).$$
$$\gamma(0) = x_0$$

$$\leadsto \begin{cases} \frac{d}{dt} u(\gamma(t), t) = \partial_t u + \gamma' \partial_x u = u_t + f(u)_x = 0 \\ \gamma'(t) = f'(u(\gamma(t), t)) = f'(u(\gamma(0), 0)) = f'(u_0(x_0)) \\ \qquad = \text{const.} \end{cases}$$

$\Rightarrow$ u const. along $\Gamma^t_{x_0}$

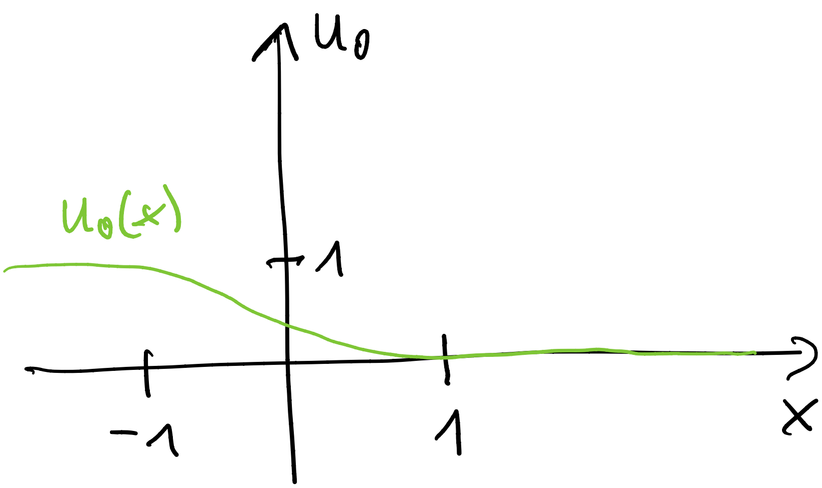
$\Gamma^t_{x_0}$ is a straight line

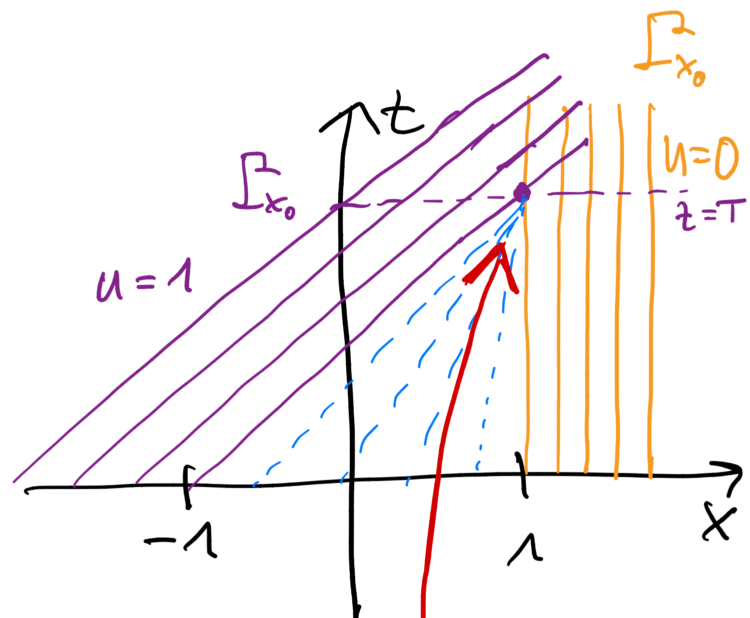
Here: $\gamma'(t) = f'(u_0(\gamma(0))) = u_0(\gamma(0))$

$$u(\gamma(t),t) = u(\gamma(0),0) = u_0(\gamma(0))$$

Γ_{x_0} starting at $\left\{\begin{matrix} x_0 \in (-\infty,-1] \\ x_0 \in [1,\infty) \end{matrix}\right\}$ have slope $\left\{\begin{matrix} 1 \\ 0 \end{matrix}\right\}$,

and $u = \left\{\begin{matrix} 1 \\ 0 \end{matrix}\right\}$ along these lines.



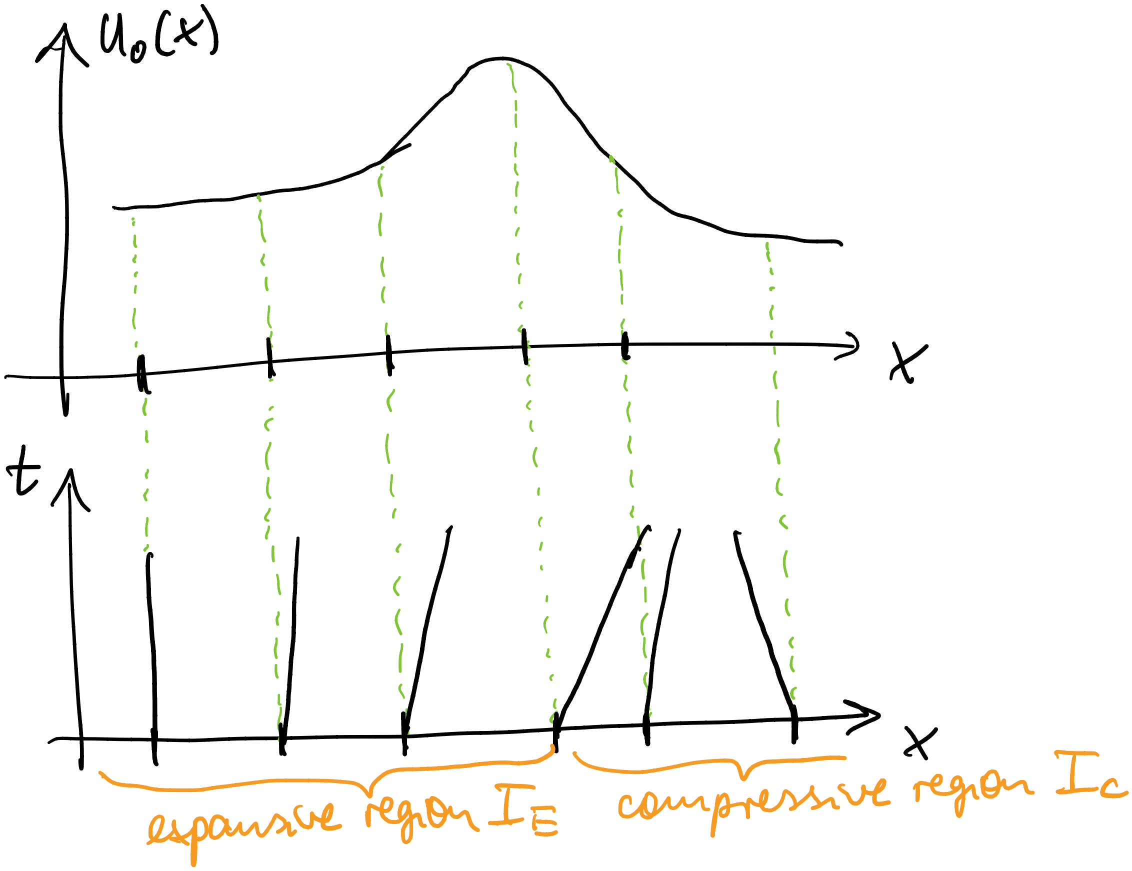


characteristics meet at finite time T

$\Rightarrow$ u cannot be continuous

i.e. $u \notin C^0(\mathbb{R}\times[0,\infty))$!

Example: wave steepening. Consider scalar CL with smooth initial data $u_0(x)$ and convex flux function $\lambda'(u) = f''(u) > 0$



- "larger values of $u_0(x)$ will travel faster than smaller values of $u_0(x)$"
- I_E & I_C reversed for concave flux
- existence of I_E & I_C leads to crossing of characteristics and multi-valued solutions

$\rightsquigarrow$ "the wave breaks" $(u_x \to \infty)$ at first crossing write $u(x,t)$ in terms of characteristics $\gamma(t)$ (see existence result below)

$$u(x,t) = u(\gamma(t),t) = u_0(\underbrace{x - \lambda(u_0(x_0))t}_{=x_0})$$

$$\leadsto u_x = u_0'(x_0)\frac{\partial x_0}{\partial x} \quad , \quad x = x_0 + \lambda(u_0(x_0))t$$

$$\downarrow$$

$$\frac{1}{u_x} = 0 \Leftrightarrow \frac{\partial x}{\partial x_0} = 0 \qquad \frac{\partial x}{\partial x_0} = 1 + \lambda'(u_0(x_0))\, u_0'(x_0)t$$

$$\Updownarrow \qquad = 1 + \lambda_x t$$

$$\boxed{t_{break} = -\frac{1}{\lambda_x(u_0(x_0))}}$$

breaking <u>first</u> occurs for characteristic Γ_{x_0} for which $\lambda_x(u_0(x_0)) < 0$ and $|\lambda_x(u_0(x_0))|$ is maximal.

<u>Theorem (local existence of classical solution):</u>

Assume $f \in C^2(\mathbb{R})$, $u_0 \in C^1(\mathbb{R})$, $|f''|, |u_0'| \leq$ const. Then there exists $T > 0$ such that (IVP) for <u>scalar conservation law</u> has a classical solution $u \in C^1(\mathbb{R} \times [0,T))$.

<u>Proof:</u> For $(x,t) \in \mathbb{R} \times [0,\infty)$ consider characteristic $\Gamma_{x_0} = (\gamma(t),t)$ passing through (x,t):

(i) $\quad x_0 = x - \lambda(u_0(x_0))t = x - f'(u_0(x_0))t$

$\quad (\text{Note: } u \text{ const. along } \Gamma_{x_0}) \qquad = x - f'(u(x,t))t$

(ii) $u(x,t) = u(x_0,0) = u_0(x_0) = u_0(x - f'(u(x,t))t)$

Define $F(v,x,t) \equiv v - u_0(x - f'(v)t)$, then

$F(u_0(x),x,0) = 0$ and $\partial_v F = 1 + u_0' f''(v) t \neq 0$

Implicit function theorem $\Rightarrow$ for $t < T$ suff. small $\exists$ function u satisfying (ii)

differentiate (i):

$$\frac{\partial x}{\partial t} = 0 = \frac{\partial x_0}{\partial t} + f'(u_0(x_0)) + f''(u_0(x_0))\, u_0'(x_0) \frac{\partial x_0}{\partial t} t$$

$$= f'(u_0(x_0)) + \left[1 + f''(u_0(x_0))\, u_0'(x_0)\, t\right] \frac{\partial x_0}{\partial t}$$

$$\Leftrightarrow \frac{\partial x_0}{\partial t} = - \frac{f'(u_0(x_0))}{1 + f''(u_0(x_0))\, u_0'(x_0)\, t}$$

$$\frac{\partial x}{\partial x} = 1 = \frac{\partial x_0}{\partial x} + f''(u_0(x_0))\, u_0'(x_0)\, t \frac{\partial x_0}{\partial x}$$

$$\Leftrightarrow \frac{\partial x_0}{\partial x} = \frac{1}{1 + f''(u_0(x_0))\, u_0'(x_0)\, t}$$

differentiate (ii): (u const. along Γ)

$$u_t = u_0'(x) \frac{\partial x_0}{\partial t} = - \frac{u_0' f'(u)}{1 + f''(u)\, u_0'\, t}$$

$$u_x = u_0'(x) \frac{\partial x_0}{\partial x} = \frac{u_0'}{1 + f''(u)\, u_0'\, t}$$

$\Rightarrow$ $u_t + f'(u) u_x = 0$ and $u \in C^1(\mathbb{R})$ for $(x,t) \in \mathbb{R} \times [0,T)$ and T sufficiently small.

$\square$

Remark: Systems of CLs in 1D:

Theorem: Let $u_0 \in C_0^1(\mathbb{R};\mathbb{R}^m)$ and $u_t + f(u)_x = 0$ strictly hyperbolic. Then $\exists\ T > 0$ such that the IVP admits a classical solution $u : [0,T) \times \mathbb{R} \to \mathbb{R}^m$.

Proof: Bressan pp. 67–70.

4.2 Weak solutions

4.2.1 Definition

Previous examples suggest to allow discontinuous (classically non-differentiable) solutions.

<u>Idea:</u> consider integral form of conservation law, then functions must only be integrable, e.g. $u \in L^\infty(\mathbb{R}\times[0,\infty);\mathbb{R}^m)$ or L^p

$$\|u\|_p = \left[\int_0^\infty \int_{\mathbb{R}} |u|^p \, dx\, dt\right]^{1/p} < \infty \,, \quad \|u\|_\infty = \inf\left\{ C \geq 0 \;\middle|\; |u(x)| \leq C \text{ almost everywhere}\right\} < \infty$$

$\rightsquigarrow$ consider for now u smooth solution of (IVP) and $v \in C_0^\infty(\mathbb{R}\times[0,\infty);\mathbb{R}^m)$:

$\uparrow$ compact support

$$\Rightarrow \quad v\cdot u_t + v\cdot f(u)_x = 0$$

$\downarrow$ integrate

$$\int_0^\infty\int_{\mathbb{R}} v\cdot u_t \, dx\,dt + \int_0^\infty\int_{\mathbb{R}} v\cdot f(u)_x = 0$$

$\downarrow$ integrate by parts

$$\int_0^\infty\int_{\mathbb{R}} v_t \cdot u \, dx\, dt - \int_{\mathbb{R}} \Big[v \cdot u \Big]_{t=0}^{t=\infty} dx$$

$$+ \int_0^\infty\int_{\mathbb{R}} v_x \cdot f(u)\, dx\, dt - \int_0^\infty \underbrace{\Big[v \cdot f(u) \Big]_{x=-\infty}^{x=\infty}}_{=0} dt = 0$$

$$\Leftrightarrow \int_0^\infty\int_{\mathbb{R}} \big(v_t \cdot u + v_x \cdot f(u)\big) dx\, dt + \int_{\mathbb{R}} v(x,0)\, u_0(t)\, dt = 0$$

<u>Def</u>: Consider (IVP) with $u_0 \in L^\infty(\mathbb{R}; \mathbb{R}^m)$.
Then $u \in L^\infty(\mathbb{R} \times [0,\infty); \mathbb{R}^m)$ is called a <u>weak solution</u> (integral solution, solution in the distributional sense) if and only if

$$\int_0^\infty \int_{\mathbb{R}} \big(u \cdot v_t + f(u) \cdot v_x\big)\, dx\, dt + \int_{\mathbb{R}} u_0 \cdot v(x,0)\, dx = 0$$

for all $v \in C_0^\infty(\mathbb{R} \times [0,\infty); \mathbb{R}^m)$.

compact support

Remark: if u is a weak solution that happens to be $C^1(\mathbb{R}\times[0,\infty);\mathbb{R}^m)$, then u is a classical solution of the (IVP).

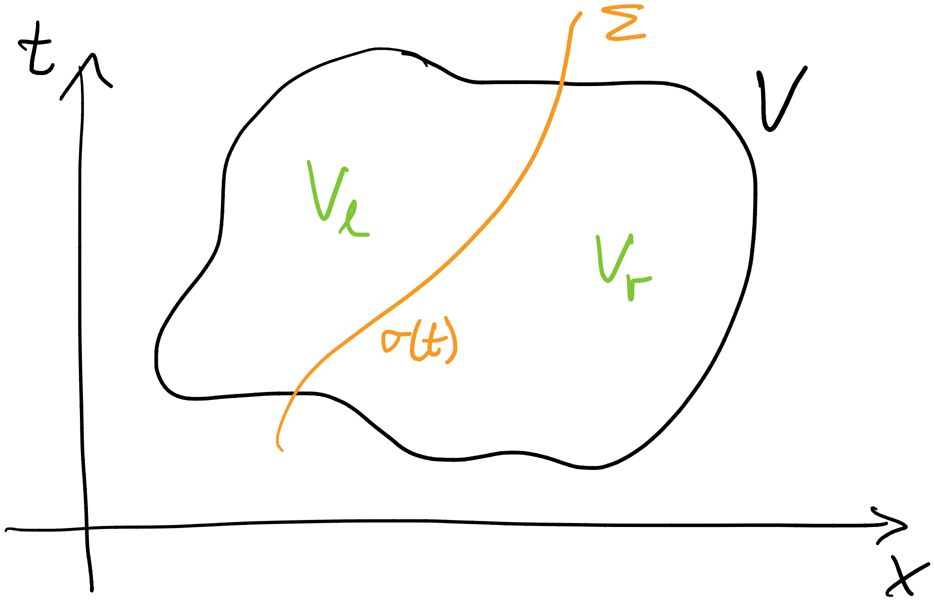
4.2.2 Behaviour near discontinuities (Jump conditions)

Theorem (Rankine – Hugoniot):

Consider $V \subset \mathbb{R}\times(0,\infty)$ separated by a smooth curve $\Sigma: t \mapsto (\sigma(t),t)$ in two parts V_ℓ and V_r. Let $u \in L^1(\mathbb{R}\times(0,\infty))$ such that $u_\ell \equiv u|_{V_\ell} \in C^1(\overline{V_\ell})$ and $u_r \equiv u|_{V_r} \in C^1(\overline{V_r})$ and u_ℓ, u_r locally satisfy the IVP on V_ℓ, V_r in the classical sense. Then u is a weak solution of the IVP if and only if

$$\text{(RH)}\quad f(u_\ell(\sigma(t),t)) - f(u_r(\sigma(t),t)) = [u_\ell(\sigma(t),t) - u_r(\sigma(t),t)]\,\sigma'(t)$$

for all $t > 0$.



<u>Notation:</u> (RH) is often written

$$f_l - f_r = \sigma' (u_l - u_r) \quad \text{along } \Sigma$$

or

$$[\![f]\!] = \sigma' [\![u]\!]$$

where $[\![\ ]\!]$ means "jump across the curve Σ"

<u>Proof:</u> Take $v \in C_0^\infty(V)$ (otherwise can always enlarge V)
compact support in V
does not necessarily vanish along Σ

Let $\nu = (\nu^1, \nu^2)$ denote the outer unit normal of V_l $\Rightarrow$ $\nu(t) = \frac{1}{\sqrt{1+\sigma'(t)^2}} \left(1, -\sigma'(t)\right)$. Then:

$$\overset{\text{Def}}{0 =} \int_0^\infty \int_{\mathbb{R}} [u \cdot v_t + f(u) \cdot v_x] \, dx\,dt + \underbrace{\int_{\mathbb{R}} u_0 \cdot v(x,0) \, dx}_{=0,\ v \in C_0^\infty(V)}$$

$$= \iint_{V_\ell} [u \cdot v_t + f(u) \cdot v_x] \, dx\,dt + \iint_{V_r} [u \cdot v_t + f(u) \cdot v_x] \, dx\,dt$$

$u \in C^1$ on V_ℓ, V_r

$$= -\int_{V_\ell} \underbrace{[u_t + f(x)_x]}_{=0 \text{ on } V_\ell} \cdot v \, dx\,dt + \int_{\partial V_\ell} (u \cdot v\, \nu^2 + f(u) \cdot v\, \nu^1) \, d\ell$$

$\nu_r = -\nu_\ell = -\nu$

$$- \int_{V_r} \underbrace{[u_t + f(u)_x]}_{=0 \text{ on } V_r} \cdot v \, dx\,dt - \int_{\partial V_r} (u \cdot v\, \nu^2 + f(u) \cdot v\, \nu^1) \, d\ell$$

v vanishes on $\partial V_\ell \setminus \Sigma$, $\partial V_r \setminus \Sigma$

$$= \int_\Sigma \left[(u_\ell - u_r) \nu^2 + (f(u_\ell) - f(u_r)) \nu^1 \right] \cdot v \, d\ell$$

Since v arbitrary

$$\Rightarrow \quad f(u_\ell) - f(u_r) = \sigma' (u_\ell - u_r)$$

$\square$

4.3 Entropy condition

In general, the (IVP) has no unique weak solution, i.e. the (IVP) for weak solutions is not well-posed.

→ we will define a "selection criterion" (entropy condition) to pick the correct physical solution and restore well-posedness.

Example (non-uniqueness): Consider 1D CL:

inviscid Burger's equ.

$$u_t + \partial_x\left(\frac{u^2}{2}\right) = 0, \quad x \in \mathbb{R}, \quad t > 0$$

$$u_0(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0 \end{cases}$$

Define

$$u_1(x,t) \equiv \begin{cases} 0, & x < t/2 \\ 1, & x > t/2 \end{cases}$$

$$u_2(x,t) \equiv \begin{cases} 0, & x < 0 \\ x/t, & 0 \leq x \leq t \\ 1, & t < x \end{cases}$$

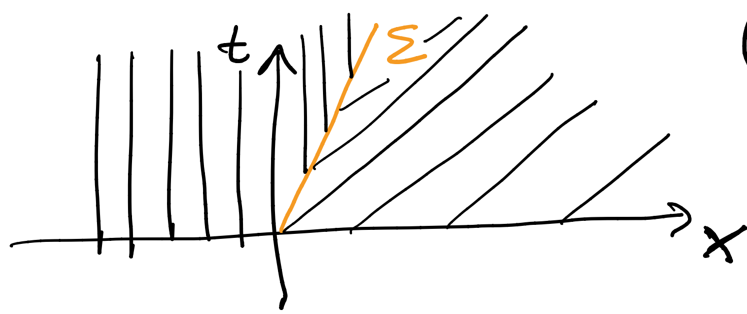
Both u_1, u_2 are piecewise C^1 and satisfy initial condition.

- u_1 satisfies RH conditions along $\Sigma = (\sigma(t), t)$ with $\sigma(t) = \frac{1}{2}t$:

$$\sigma'(t) = \frac{1}{2} \quad \text{and} \quad \frac{f(u_\ell) - f(u_r)}{u_\ell - u_r} = \frac{\frac{1}{2}u_\ell^2 - \frac{1}{2}u_r^2}{u_\ell - u_r}$$

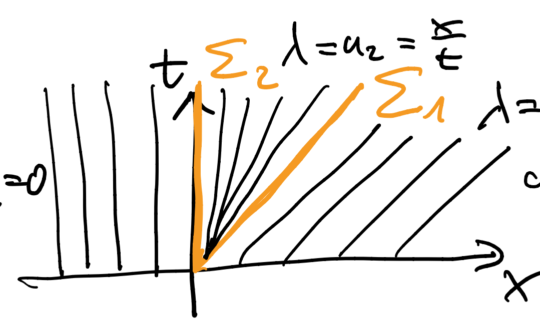
$$= \frac{1}{2}(u_\ell + u_r) = \frac{1}{2} \quad \checkmark$$

$\Rightarrow u_1$ weak solution



(rarefaction shock, entropy violating)

- can check: u_2 satisfies RH conditions along $\Sigma_1 = (\sigma_1(t), t) = (t, t)$ and $\Sigma_2 = (\sigma_2(t), t) = (0, t)$



characteristics $\lambda = u_2$

$\Rightarrow u_2$ also weak solution

In reality, discontinuities are never arbitrarily sharp, but are rather "smeared out" by some intrinsic viscosity of the fluid. Physically correct solutions should arise as the limit of solutions to the "regularized system"

$$u^\varepsilon_t + f(u^\varepsilon)_x - \underbrace{\varepsilon u^\varepsilon_{xx}}_{\text{"small viscosity effect"}} = 0 \quad \text{in } \mathbb{R}\times(0,\infty)$$

as $\varepsilon \to 0$, i.e. as the problem approaches the inviscid problem.

Following Theorem provides the "viscosity method" of how to select the correct weak solution:

Theorem: Let $u_0 \in L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and $f_i, B \in C^2(\mathbb{R}^n \times (0,\infty) \times \mathbb{R})$ with bounded derivatives. Then for any $\varepsilon > 0$ there exists a uniquely defined classical solution u^ε of

$$u_t + \sum_{i=1}^{n} \partial_i f_i(x,t,u) + B(x,t,u) = \varepsilon \Delta u \text{ in } \mathbb{R}^n \times (0,\infty)$$

$$u(x,0) = u_0(x) \text{ in } \mathbb{R}^n$$

such that $u^\varepsilon \xrightarrow{\varepsilon\to 0} u$ almost everywhere in $\mathbb{R}^n \times (0,\infty)$ for some u that is a weak solution of ("viscosity limit")

$$\partial_t u + \sum_{i=1}^n \partial_i f_i(x,t,u) + B(x,t,u) = 0 \text{ in } \mathbb{R}^n\times(0,\infty)$$

$$u(x,0) = u_0(x) \text{ in } \mathbb{R}^n$$

Proof: omitted, see references in Kroener (p. 22).

Remark: For general systems the previous theorem is still an open problem.

Now: want necessary criterion for a weak soln. u to be the viscosity limit defined in the theorem

Idea: want an "entropy" function $\Phi(u)$ (motivated by thermodynamics) that is conserved in smooth parts of a weak solution, but that changes across a discontinuity in u in a way that it excludes solutions that do not correspond to a viscosity limit

(Note: the thermodynamic entropy is conserved in smooth flows but increases across shocks)

$\rightsquigarrow$ want that

entropy flux

$$\Phi(u)_t + \Psi(u)_x = 0 \quad (u \text{ smooth})$$

$$\Phi(u)_t + \Psi(u)_x < 0 \quad \text{across a discontinuity}$$

$\rightarrow$ observe that in smooth parts:

$$\Phi(u)_t + \Psi(u)_x = D\Phi(u)\cdot u_t + D\Psi(u)\cdot u_x$$

$$\overset{u_t+f(u)_x=0}{=} \Big[\underbrace{-D\Phi(u)\,Df(u) + D\Psi(u)}_{\overset{!}{=}0}\Big]\, u_x \overset{!}{=} 0$$

$\rightarrow$ Trivial choice: $\Phi(u) = u$, but then also conserved across discontinuities. Therefore require $D^2\Phi$ to be positive definite (Φ convex).

<u>Definition:</u> Two smooth functions $\Phi, \Psi \in C^2(\mathbb{R}^m;\mathbb{R})$ are called an <u>entropy/entropy-flux pair</u> for the system of CLs $u_t + f(u)_x = 0$ if

(i) Φ is convex ($D^2\Phi(z)y\cdot y > 0 \ \forall z, y \in \mathbb{R}^m$)

(ii) $D\Phi(z)\,Df(z) = D\Psi(z), \quad z \in \mathbb{R}^m$

Definition: A weak solution u of the (IVP) is called an entropy solution if for any entropy pair Φ, Ψ it satisfies the inequality

$$\boxed{\Phi(u)_t + \Psi(u)_x \leq 0 \quad \text{in } \mathbb{R}\times(0,\infty)}$$

in the distributional sense, i.e.

$$\int_0^\infty \int_{\mathbb{R}} \Phi(u) v_t + \Psi(u) v_x \, dx\, dt \geq 0 \quad \forall \underset{v \geq 0}{v \in C_0^\infty(\mathbb{R}\times(0,\infty))}.$$

(Note: the latter expression is obtained by assuming u smooth, multiplying by v and integrating by parts)

Theorem: The viscosity limit $u = \lim_{\varepsilon \to 0} u^\varepsilon$ of

$$u^\varepsilon_t + f(u^\varepsilon)_x - \varepsilon u^\varepsilon_{xx} = 0 \quad \text{in } \mathbb{R}\times(0,\infty)$$
$$u^\varepsilon = u_0 \quad \text{on } \mathbb{R}\times\{t=0\}$$

is an entropy solution of

$$u_t + f(u)_x = 0 \quad \text{on } \mathbb{R}\times(0,\infty)$$
$$u = u_0 \quad \text{on } \mathbb{R}\times\{t=0\}$$

provided that u^ε is uniformly bounded in L^∞ and $u^\varepsilon \to 0$ as $|x| \to \infty$ sufficiently rapidly.

Proof: 1) choose entropy pair Φ, Ψ.

$$D\Phi(u^\varepsilon)\cdot\left(u^\varepsilon_t + \underbrace{f(u^\varepsilon)_x}_{=Df(u^\varepsilon)u^\varepsilon_x} - \varepsilon u^\varepsilon_{xx}\right) = 0$$

$$\Leftrightarrow \quad \Phi(u^\varepsilon)_t + \Psi(u^\varepsilon)_x = \varepsilon D\Phi(u^\varepsilon) u^\varepsilon_{xx}$$

$$= \varepsilon\,\Phi(u^\varepsilon)_{xx} - \varepsilon \underbrace{\left(D^2\Phi(u^\varepsilon)u^\varepsilon_x\right)\cdot u^\varepsilon_x}_{\substack{\geq 0 \\ (\Phi \text{ convex})}}$$

$$\Big\downarrow \text{ multiply by } \begin{cases} v \in C_0^\infty(\mathbb{R}\times(0,\infty)) \\ v \geq 0 \end{cases} \text{ integrate}$$

- $$\int_0^\infty\int_{\mathbb{R}} \Phi(u^\varepsilon)_t v + \Psi(u^\varepsilon)_x v \,dx\,dt$$

$$\overset{\substack{\text{integrate} \\ \text{by parts}}}{=} \underbrace{\int_{\mathbb{R}}\left[\Phi(u^\varepsilon)v\right]_0^\infty dx + \int_0^\infty\left[\Psi(u^\varepsilon)v\right]_{-\infty}^\infty dx}_{=0,\ v\in C_0^\infty(\mathbb{R}\times(0,\infty))} - \int_0^\infty\int_{\mathbb{R}}\left[\Phi(u^\varepsilon)v_t + \Psi(u^\varepsilon)v_x\right]dx\,dt$$

$$\Rightarrow \int_0^\infty\int_{\mathbb{R}}\left[\Phi(u^\varepsilon)v_t + \Psi(u^\varepsilon)v_x\right]dx\,dt \overset{\substack{\Phi \text{ convex} \\ v\geq 0}}{\geq} -\int_0^\infty\int_{\mathbb{R}} \varepsilon\,\Phi(u^\varepsilon)_{xx}\,v\,dx\,dt$$

$$\overset{\substack{2\times \text{ integrate} \\ \text{by parts}}}{=} -\int_0^\infty\int_{\mathbb{R}} \varepsilon\,\Phi(u^\varepsilon)v_{xx}\,dx\,dt$$

$$\Big\downarrow \begin{matrix} \varepsilon \to 0 \\ u^\varepsilon \to u \end{matrix}$$

$$\int_0^\infty \int_{\mathbb{R}} \left[\Phi(u)\, v_t + \Psi(u)\, v_x \right] dx\, dt \geq 0$$

$\Rightarrow \quad u = \lim_{\varepsilon \to 0} u^\varepsilon$ <u>satisfies the entropy condition</u>

2) choose $v \in C_0^\infty(\mathbb{R} \times [0,\infty); \mathbb{R}^m)$:

$$\int_0^\infty \int_{\mathbb{R}} \left[u_t^\varepsilon + f(u^\varepsilon)_x - \varepsilon u_{xx}^\varepsilon \right] \cdot v \; dx\, dt = 0$$

$$\Big\downarrow \text{ integrate by parts}$$

$$-\int_0^\infty \int_{\mathbb{R}} u^\varepsilon \cdot v_t \, dx\, dt + \int_{\mathbb{R}} \underbrace{\left[u^\varepsilon \cdot v \right]_0^\infty dx}_{= -\int_{\mathbb{R}} u^\varepsilon(x,0) \cdot v \, dx \ \left(u^\varepsilon \text{ falls of } |x| \to \infty\right)}$$

$$-\int_0^\infty \int_{\mathbb{R}} f(u^\varepsilon)\, v_x \, dx\, dt - \int_0^\infty \int_{\mathbb{R}} \varepsilon u^\varepsilon v_{xx} \, dx\, dt = 0$$

$$\Leftrightarrow \int_0^\infty \int_{\mathbb{R}} \left[u^\varepsilon \cdot v_t + f(u^\varepsilon) \cdot v_x + \varepsilon u^\varepsilon \cdot v_{xx} \right] dx\, dt + \int_{\mathbb{R}} u_0^\varepsilon \cdot v \, dx = 0$$

$$\Big\downarrow \begin{matrix} \varepsilon \to 0 \\ u^\varepsilon \to u \end{matrix}$$

$$\int_0^\infty \int_{\mathbb{R}} \left[u \cdot v_t + f(u) \cdot v_x \right] dx\,dt + \int_{\mathbb{R}} u_0 \cdot v \, dx = 0$$

$\Rightarrow$ u is weak solution.

$\square$

<u>Theorem (Uniqueness of entropy solution for scalar conservation laws):</u>

There exists – up to a measure zero – at most one entropy solution for the scalar conservation law

$$\begin{aligned} u_t + f(u)_x &= 0 \quad &&\text{on } \mathbb{R} \times (0,\infty) \\ u &= u_0(x) \quad &&\text{on } \mathbb{R} \times \{t=0\} \end{aligned}$$

Proof: Evans Sec. 11.4.3, p. 652

<u>Remarks:</u> 1) <u>Well-posedness for weak soln.</u>

Note that the previous theorem restores well-posedness, given the entropy selection/admissibility criterion

2) <u>Uniqueness:</u>

- Note entropy solution is a weak solution that satisfies the entropy inequalities for <u>any</u> entropy pair Φ, Ψ of the CL.

- 1D scalar equations:

  There exists at most one weak solution that satisfies the entropy inequalities for ALL entropy pairs of the CL.

  Note that ONE entropy pair may be sufficient to rule out a given weak solution as entropy solution.

- 1D systems of CLs: similar uniqueness theorem (but more involved to prove)

- multi-D system: uniqueness still an open problem

3) <u>Existence:</u>

- <u>1D scalar</u> CL ($m=1$): any Φ convex

  $\leadsto$ obtain corresponding flux: $\Psi(z) \equiv \int_{z_0}^{z} \Phi'(y) f'(y) dy$ $\quad z \in \mathbb{R}$

- $\underline{m=2}$: find Φ, Ψ, with Φ convex, and

$$(*) \quad \underbrace{\left(\Phi_{z_1}, \Phi_{z_2}\right)}_{D\Phi} Df(z) = \underbrace{\begin{pmatrix} \Psi_{z_1} \\ \Psi_{z_2} \end{pmatrix}}_{D\Psi}, \quad z \in \mathbb{R}^2$$

- $\underline{m>2}$: $(*)$ over-determined (2 unknowns Φ, Ψ but $m>2$ equations)
  $\longrightarrow$ no solution in general

- <u>symmetric systems</u>: there always exists an entropy function $\Phi(u) \equiv u^T u = u \cdot u$ (Godunov 1962)

- <u>hydrodynamics</u>: thanks to 2nd law of thermodynamics, one can write down an entropy pair using the thermodynamic entropy (modulo a minus sign)

  $\Phi(u) \equiv -S$

  $\Psi(u)$: -entropy flux

  Note: mathematical entropy can only <u>decrease</u>:

$$\Phi(u)_t + \Psi(u)_x \leq 0$$

$\rightsquigarrow$ physical entropy evolves according to its flux and can never decrease, but increase across discontinuities ($\rightarrow$ shocks)

$\rightsquigarrow$ entropy inequalities serve as additional constraint the conservation law "does not know about" and that select the thermodynamically / physically admissible solution to the CL

4) Physical interpretation

- entropy pair $\rightarrow$ see 3)
- viscosity limit & entropy solution

Note that the entropy solution for the 1D Euler equs is the unique solution to the 1D Navier-Stokes equs in the limit viscosity $\varepsilon \to 0$

Navier-Stokes equs (incompressible fluid)		Euler equs
$u_t + f(u)_x = -\varepsilon u_{xx}$		$u_t + f(u)_x = 0$
u^{ε}	$\xrightarrow{\varepsilon \to 0}$	u
classical solution (unique)		entropy solution (unique)