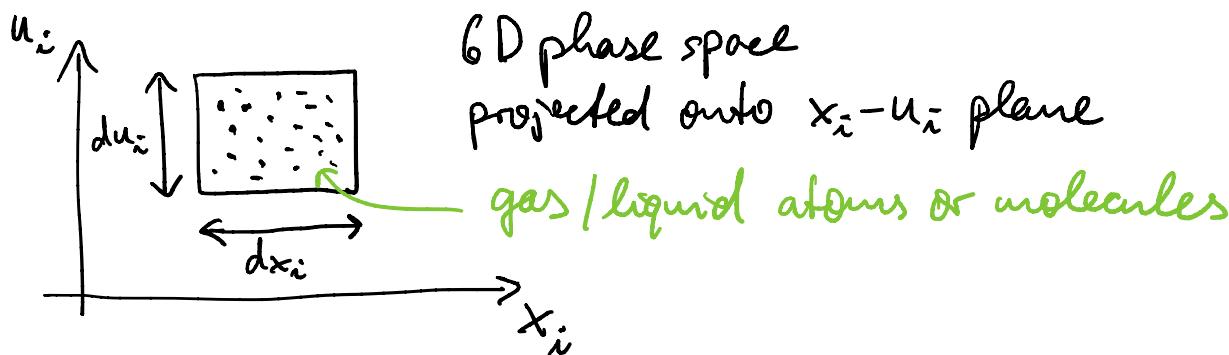


# Chapter 2: Basic Equations

## 2.1 Continuous media: Boltzmann Eq



In principle evolution of system fully determined by Newtonian mechanics (N-body equations):

$$\frac{d\vec{x}_i}{dt} = \vec{u}_i \quad , \quad \frac{d\vec{u}_i}{dt} = \vec{F}_i(x_j, u_j, t) \quad \forall \text{ particles } i, j$$

$\uparrow$  force on  $i$

But: 1 mole  $\hat{=} N = 6 \times 10^{23}$  particles  $\rightarrow$  computationally prohibitive

$\rightarrow$  statistical approach (continuous media)  
 $\rightarrow$  hydrodynamics

$$dN = f(\vec{x}, \vec{u}, t) d\vec{x} d\vec{u}$$

$\uparrow$  number of particles in phase-space control volume  
 $d\vec{x} d\vec{u}$

$\nwarrow$  distribution function

- ignore internal degrees of freedom  $\rightarrow$  vibration, rotation
- identical particles of mass  $m$
- ignore QM effects

Assume particles subject to external force per unit mass  $\vec{F}$  ( $\approx$  constant over typical inter-particle separation)

$\Rightarrow$  particle number  $f(\vec{x}, \vec{u}, t) d\vec{x} d\vec{u}$  conserved along their trajectories in phase space (Liouville's theorem) in absence of interactions:

(\*)

$$\Delta f = f(\vec{x} + \underbrace{\vec{u} dt}_{\text{change in position}}, \vec{u} + \underbrace{\vec{F} dt}_{\text{change in velocity due to } \vec{F}}, t + dt) - f(\vec{x}, \vec{u}, t) = [\Delta f]_{\text{coll}}$$

change in position

change in velocity  
due to  $\vec{F}$

$$\begin{aligned} dt &\rightarrow 0 \\ \Delta f &\rightarrow \frac{df}{dt} \end{aligned}$$

↑ change in  $f$  in  $dt$  due to collisions

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + u_i \frac{\partial f}{\partial x_i} + F_i \frac{\partial f}{\partial u_i} = \left[ \frac{\partial f}{\partial t} \right]_{\text{coll}}$$

Boltzmann's transport equation

→ evolution of distribution function in 6-D phase space  $(\vec{x}, \vec{u})$

→ change  $dN$  in number of particles within a fixed volume  $d\vec{x} d\vec{u}$  equals net number of particles entering or leaving this volume

( → Liouville's theorem )

Some definitions:

$$n(\vec{x}, t) \equiv \int f(\vec{x}, \vec{u}, t) d\vec{u}$$

$\underbrace{\qquad\qquad\qquad}_{\text{# particles per unit volume in } \vec{u}, \vec{u} + d\vec{u}}$

total number of particles per unit volume

- $\ell \equiv n^{1/3}$  mean particle separation

Note: any physical length scale  $d\vec{x}$  we are interested in must be  $\gg \ell$  for statistical approach to be valid

- $\rho(\vec{x}, t) = \int m f(\vec{x}, \vec{u}, t) d\vec{u}$  mass density
- $\vec{v}(\vec{x}, t) = \frac{1}{S} \int \vec{u} m f(\vec{x}, \vec{u}, t) d\vec{u}$  bulk velocity
- $\rho E(\vec{x}, t) = \frac{1}{2} \int m \vec{u}^2 f(\vec{x}, \vec{u}, t) d\vec{u}$  specific internal energy  $E$   
 $\uparrow \vec{u} = \vec{v} - \vec{v}$  peculiar velocity

Special case:

- elastic collisions (energy & momentum conserved)
- low density (collisions of 3 & more particles)  
 (can be neglected)
- absence of external forces  $\vec{F} \equiv 0$

unique solution  $\downarrow$  statistical mechanics

$$f(\vec{x}, \vec{u}, t) d\vec{u} = n(\vec{x}, t) \left[ \frac{m}{2\pi k_B T(\vec{x}, t)} \right]^{3/2} \exp \left[ -\frac{m(\vec{u} - \vec{v})^2}{2k_B T(\vec{x}, t)} \right] d\vec{u}$$

Maxwellian velocity

distribution  
 (equilibrium state  $t \rightarrow \infty$ ,  
 $\frac{\partial f}{\partial t} = 0$ )

## 2.2 From Boltzmann to Euler

Def: k-th moment of the Boltzmann equation

$$\int u_k \left[ \frac{\partial f}{\partial t} + u_i \frac{\partial f}{\partial x_i} + F_i \frac{\partial f}{\partial u_i} \right] du = \int u_k \left[ \frac{\partial f}{\partial t} \right]_{\text{coll}} du,$$

where  $u_k = \vec{u}^k$ , i.e.  $u_0 = 1$ ,  $u_1 = \vec{u}$ ,  $u_2 = u^2$  etc.

General properties of collision term:

If collisions are elastic and neither create nor destroy particles, then

$$\int \left[ \frac{\partial f}{\partial t} \right]_{\text{coll}} d\vec{u} = 0 \quad \begin{matrix} \text{number of particles} \\ \text{conserved} \end{matrix}$$

$$\int \left[ \frac{\partial f}{\partial t} \right]_{\text{coll}} u_i d\vec{u} = 0 \quad \begin{matrix} \text{total momentum} \\ \text{conserved} \end{matrix}$$

$$\int \left[ \frac{\partial f}{\partial t} \right]_{\text{coll}} u^2 d\vec{u} = 0 \quad \begin{matrix} \text{total energy} \\ \text{conserved} \end{matrix}$$

$$\lim_{u \rightarrow \infty} u^k f = 0 \quad \begin{matrix} \text{total number, momentum} \\ \text{energy must be finite} \end{matrix}$$

$$\text{Also: } \left( \int_{\Omega} v \frac{\partial u}{\partial x_i} = \int_{\Omega} uv n_i d\Gamma - \int_{\Omega} u \frac{\partial v}{\partial x_i} \right)$$

$$\int \frac{\partial f}{\partial u_i} d\vec{u} \stackrel{P.i.}{=} \int f n_i d\Gamma - \int \underbrace{\frac{\partial(1)}{\partial u_i} f}_{=0} d\vec{u} = 0$$

$$\int u_j \frac{\partial f}{\partial u_i} d\vec{u} \stackrel{P.i.}{=} \underbrace{\int u_j f n_i d\Gamma}_{|u|= \pm \infty} - \int \underbrace{\frac{\partial u_j}{\partial u_i} f}_{=\delta_{ij}} d\vec{u}$$

$$= -\delta_{ij} \frac{g}{m}$$

$$\frac{1}{2} \int u^2 \frac{\partial f}{\partial u_i} d\vec{u} = \frac{1}{2} \int u^2 f n_i d\Gamma - \frac{1}{2} \int \underbrace{\frac{\partial u^2}{\partial u_i} f}_{|u|= \pm \infty} d\vec{u} \stackrel{\frac{\partial(\sum_j u_j^2)}{\partial u_i} = 2u_i}{=} 2u_i$$

$$= - \int u_i f d\vec{u} = - \frac{g}{m} v_i$$

① From  $\theta$ -moment of Boltzmann eqn:

$$m \int \frac{\partial f}{\partial t} d\vec{u} + m \int u_i \frac{\partial f}{\partial x_i} d\vec{u} + m \overline{v}_i \underbrace{\int \frac{\partial f}{\partial u_i} d\vec{u}}_{=0} = \int \left[ \frac{\partial f}{\partial t} \right]_{\text{coll}} d\vec{u} = 0$$

$u_i, x_i$  independent

$$\Downarrow \Leftrightarrow \frac{\partial}{\partial t} \int m f d\vec{u} + \frac{\partial}{\partial x_i} \int u_i m f d\vec{u} = 0$$

$\Leftrightarrow$ 

$$\frac{\partial \mathbf{S}}{\partial t} + \nabla \cdot (\mathbf{S} \vec{v}) = 0$$

continuity  
equation

Note:  $V$  large enough

$$\Rightarrow \int \frac{\partial \mathbf{S}}{\partial t} dV + \underbrace{\int \nabla \cdot (\mathbf{S} \vec{v}) dV}_{\text{Gauss}}$$

$$= \int_V \mathbf{S} v_i n_i d\Omega = 0 \quad \vec{v} = 0 \text{ on } \partial V$$

↑  
unit normal

$$= \frac{\partial}{\partial t} \int_V \mathbf{S} dV = \frac{\partial M_V}{\partial t} = 0 \quad \text{mass conservation}$$

② From 1st-moment of Boltzmann eqn:

$$m \int u_i \frac{\partial f}{\partial t} d\vec{u} + m \int u_i u_j \frac{\partial f}{\partial x_i} d\vec{u} + m F_j \underbrace{\int u_i \frac{\partial f}{\partial u_j} d\vec{u}}_{= -\delta_{ij} \frac{f}{m}} = 0$$

$$= \int u_i \left[ \frac{\partial f}{\partial t} \right]_{\text{coll}} d\vec{u} = 0$$

$$\Leftrightarrow \frac{\partial}{\partial t} (S v_i) + \frac{\partial}{\partial x_i} \int m u_i u_j f d\vec{u} - S F_i = 0$$

$$\int m \tilde{u}_i \tilde{u}_j f d\vec{u} = \int m (\tilde{u}_i + v_i)(\tilde{u}_j + v_j) f d\vec{u}$$

$\xrightarrow{\text{cross terms vanish}}$

$$\int \tilde{u}_i f d\vec{u} = 0 \quad \xrightarrow{\text{peculiar velocities}} \quad \equiv s v_i v_j + P_{ij}$$

$$P_{ij} = \int m \tilde{u}_i \tilde{u}_j f d\vec{u} \quad \begin{matrix} \text{pressure} \\ \text{tensor} \end{matrix}$$

Most astrophysical cases:

$$\text{isotropic pressure} \Rightarrow P_{ij} = P \delta_{ij}$$

$$P = \frac{1}{3} \int m \tilde{u}^2 f d\vec{u}$$

$$(P = \frac{2}{3} s \epsilon)$$

$$\Leftrightarrow \frac{\partial}{\partial t} (s v_i) + \frac{\partial}{\partial x_j} (s v_i v_j) = - \frac{\partial P}{\partial x_i} + s \vec{F}_i$$

$$\Leftrightarrow \boxed{\frac{\partial}{\partial t} (s \vec{v}) + \nabla \cdot \Pi = s \vec{F}}$$

momentum  
equation

$$\Pi = s v_i v_j + P \delta_{ij} \quad \begin{matrix} \text{momentum} \\ \text{flux tensor} \end{matrix}$$

Note: • pressure gradient and external forces act as source (or sink) terms

- force due to pressure gradient results from exchange of energy between bulk flow and peculiar motions

③ From 2nd-moment of Boltzmann eqn:  
 (Exercise):

$$\frac{\partial}{\partial t} \left[ S \left( \frac{v^2}{2} + \epsilon \right) \right] + \frac{\partial}{\partial x_i} \left[ S v_i \left( \frac{v^2}{2} + \epsilon \right) \right] = - \frac{\partial h_i}{\partial x_i} - \frac{\partial}{\partial x_i} (P v_i) \\ + S v_i F_i$$

where  $h_i = \int \frac{m}{2} \tilde{u}_i \tilde{u}^2 f d\tilde{u}$  conduction heat flux

If  $\vec{h}$  can be neglected (typical for astrophys. systems):

$$\rightsquigarrow \boxed{\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_i} [(E+P)v_i] = S v_i F_i}$$

total  
energy  
equation

$$E \equiv \frac{1}{2} S v^2 + S \epsilon$$

total  
energy  
density

Remarks: 1) ①, ②, ③ are known as the Euler equations. They can be rewritten as a system of conservation laws:

$$\boxed{\begin{aligned} u_t + f^i(u)_{x_i} &= S \\ \partial_t u + \frac{\partial}{\partial x_i} f^i(u) &= S \end{aligned}}$$

where

$$u = \begin{pmatrix} s \\ s \vec{v} \\ E \end{pmatrix} \quad \text{"conserved variables"}$$

$$f^i = \begin{pmatrix} s v^i \\ s v^i v^j + \delta_{ij} p \\ v^i (E + p) \end{pmatrix} \quad \begin{array}{l} j = 1, 2, 3 \\ (\text{momentum components}) \end{array} \quad \text{"fluxes"}$$

$$S = \begin{pmatrix} 0 \\ s \vec{F} \\ s \vec{J} \cdot \vec{F} \end{pmatrix} \quad \begin{array}{l} i = 1, 2, 3 \\ (\text{flux directions}) \end{array} \quad \text{"sources"}$$

2) Equation of state: The Euler eqns. are five equations for six unknowns  $s, \{v^i\}, P, e = sE$

and require relation  $P = P(s, e)$  to close the system  $\rightarrow$  "equation of state"

Ideal gas:  $p = (\gamma - 1)e = (\gamma - 1) \beta E$   
 with  $\gamma = \frac{c_p}{c_v}$  the ratio of specific heats  
 (adiabatic constant)

In general:  $p = p(S, e, X_i, \dots)$  can be very  
 complicated ( $\rightarrow$  different chemical  
 elements, ionization states, compl.  
 reactions etc.)

Exercise: Show strict hyperbolicity of the Euler equations.

3) Validity of continuous medium approximation  
 require (see above):

$$(i) \quad \lambda \ll dx \ll l_{sys} \quad \begin{matrix} \text{mean free path of particle} \\ \uparrow \end{matrix} \quad \begin{matrix} \text{finite size element of medium} \\ \uparrow \end{matrix} \quad \begin{matrix} \text{characteristic size} \\ \text{of physical system} \end{matrix}$$

$f$  should not vary over  $dx$  &  $dN$   
 should be large enough for averaging  
 to be meaningful

(ii) interparticle forces must be short-range  
 i.e. on distances  $l_{force} \ll dx$ , otherwise

energy & momentum exchanged with  
fluid elements far away

example for long-range: external grav.  
field  
→ external force

Note: self-gravity cannot be obtained  
from Boltzmann eqn due to  
long-range nature  
→ require additional Poisson eqn.

- (iii) peculiar motion close to boundary of  
element  $d\mathbf{x}$  carries particles into  
adjacent elements with different  $\mathbf{s}(\mathbf{z})$ ,  $\mathbf{v}(\mathbf{z})$ ,  
 $T(\mathbf{z})$  and "diffusion"  
and friction forces appear  $\Rightarrow$   
microscopic exchange of  
momentum & energy  
neglected in Euler equation  
→ need to take conduction heat flux  
 $\vec{h}$  and other terms into account  
(see below)

## 2.3 Viscosity: Navier - Stokes equations

When interchange ("diffusion") of particles between adjacent fluid elements cannot be neglected  
 → internal friction or viscosity

no Modify momentum flux density:

$$(\star) \quad \pi \rightarrow \bar{\pi} = \pi - \sigma, \quad \bar{\pi}_{ij} = \delta v_i v_j + \underbrace{p \delta_{ij}}_{\text{stress tensor}} - \underline{\sigma_{ij}}$$

viscous stress tensor

- friction requires differences in velocity between particles →  $\sigma$  should depend on  $\frac{\partial v_i}{\partial x_j}$   
 If velocity gradients small, 1st order derivatives OK, and  $\sigma$  is linear in  $\frac{\partial v_i}{\partial x_j}$
- $\sigma = 0$  when fluid rotates rigidly or  $v \equiv \text{const.}$

↓ Landau & Lifshitz Fluid Mechanics §15

$$\sigma_{ij} = \eta \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3} \frac{\partial v_k}{\partial x_k} \delta_{ij} \right) + \eta \frac{\partial v_k}{\partial x_k} \delta_{ij}$$

↑ shear viscosity coefficient  $\eta$ ,  $\eta$  independent of velocity,  $\geq 0$

↑ bulk viscosity coefficient

- bulk: energy transfer b/w translational and internal motions of fluid particles
- not possible for ideal mono-atomic gases for example → no internal DOF
  - not for incompressible fluids  $\frac{\partial \rho}{\partial t} = 0$   
( $\nabla \cdot \mathbf{v} = 0 \Rightarrow \nabla \cdot \vec{v} = 0$ )
  - if present frictional force opposing changes of the volume of fluid elements generated
  - usually not relevant in astrophysics

- shear: "dynamic viscosity coefficient"
- momentum diffusion through shear motion: faster fluid elements decelerate, slower elements accelerate

With the replacement ( $\star$ ) we obtain the

Navier - Stokes equations:

Navier (1827)  
Stokes (1845)

$$\frac{\partial \mathbf{s}}{\partial t} + \frac{\partial}{\partial x_i} (\mathbf{s} v_i) = 0$$

$$\frac{\partial}{\partial t} (\mathbf{s} v_i) + \frac{\partial}{\partial x_j} (\mathbf{s} v_i v_j - \sigma_{ij}) = - \frac{\partial P}{\partial x_i} + \mathbf{s} \vec{f}_i \quad (i=1,2,3)$$

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_j} [(E+P) v_j - \sigma_{jk} v_k] = \mathbf{s} v_j \vec{f}_j$$

# Properties of viscous effects:

- Viscous timescale:

Assume that viscous effects dominate & consider incompressible fluid for simplicity ( $\Rightarrow \sigma_{ij} = \eta \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$ )

momentum eq.

$$\text{no } \frac{\partial}{\partial t} v_i \approx \frac{1}{\rho} \frac{\partial \sigma_{ij}}{\partial x_j} \Rightarrow \frac{|\vec{v}|}{t_v} \sim \sqrt{\frac{1}{l_{sys}} \frac{|\vec{\nabla}|}{l_{sys}}}$$

characteristic time & length scales

$$v = \frac{\eta}{\rho} \quad \text{kinematic viscosity coefficient}$$

$$t_v = \frac{l_{sys}^2}{v} \quad \text{viscous timescale}$$

- Magnitude of effects:

$\rightarrow$  molecular viscosity (peculiar motions of particles) wrt to bulk motion

$$v_{mol} = \lambda \tilde{u}_{sys}$$

mean  
free path

characteristic  
velocity of peculiar motions

$$t_v = \frac{l_{sys}^2}{\lambda \tilde{u}_{sys}}$$

HII cloud  
(interstellar)  
 $\sim 3 \times 10^{12} \text{ yr}$

$$\begin{cases} l_{sys} \sim 10^{19} \text{ cm} \\ \lambda \sim 10^{14} \text{ cm} \\ \tilde{u}_{sys} \sim 10^4 \frac{\text{cm}}{\text{s}} \end{cases}$$

protostellar  
disk (star formation)  
 $\sim 3 \times 10^{14} \text{ yr}$

$$\begin{cases} l_{sys} \sim 10^{14} \text{ cm} \\ \lambda \sim 10^{10} \text{ cm} \\ \tilde{u}_{sys} \sim 10^5 \frac{\text{cm}}{\text{s}} \end{cases}$$

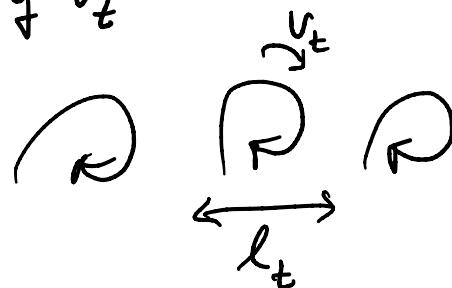
$> \sim 10^2 - 10^4 \times \text{age of universe!}$

$\Rightarrow$  negligible for all practical purposes

$\rightarrow$  turbulent viscosity (turbulent flows within)  
a bulk flow

$\rightsquigarrow$  random motions of eddies with characteristic size  $l_t$  and velocity  $v_t$

$$v_t = l_t v_t$$



Consider accretion disk:



$$l_t \sim h$$

$$v_t \sim \propto c_s$$

$\in (0,1)$   $\uparrow$  sound speed

(if  $v_t > c_s \Rightarrow$  shock  $\Rightarrow$  dissipation  
of turbulent motions until  
 $v_t < c_s$ )

$$\rightsquigarrow v_t = \propto c_s h$$

$\alpha$ : " $\alpha$ -viscosity" (Shakura & Sunyaev 1973)

$\rightarrow$  source for such disk turbulence

is the magneto-rotational instability (MRI)

(Balbus & Hawley 1991)

## 2.4 Magneto hydrodynamics

Ionized medium (plasma) can conduct electric currents  $\Rightarrow$  interacts with EM fields  
 $\rightarrow$  complex interplay b/w plasma motions & fields

Simplest possible approximation of magnetized plasmas:  
magnetohydrodynamics (MHD):

- (i) plasma treated as continuous medium, described by Navier-Stokes Eqns in absence of EM fields
- (ii) Positive & negative charges are locally and globally balanced at all times  
( $\rightarrow$  fluid elements are neutral)
- (iii) electrons are in statistical equilibrium with ions  
(same temperature)
- (iv) interparticle collisions frequent enough for all effects of magnetic forces to be instantaneously transferred from  $e^-$  to ions and neutral particles (if present)

Note: for currents to flow,  $e^-$  & ions cannot move at the same velocity. However, the relative drift

velocity is so small that it can be neglected  
(see below)

## Vacuum Maxwell eqns:

$$\text{I} \quad \nabla \cdot \vec{E} = 4\pi q$$

$$\text{II} \quad \nabla \cdot \vec{B} = 0$$

$$\text{III} \quad \nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

$$\text{IV} \quad \nabla \times \vec{B} = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \simeq \frac{4\pi}{c} \vec{j}$$

Gauss (cgs)  
units!

Plasma motion approximations:

$$1) \text{ III} \Rightarrow \frac{E_{\text{sys}}}{l_{\text{sys}}} \sim \frac{B_{\text{sys}}}{c t_{\text{sys}}}$$

$E_{\text{sys}}, B_{\text{sys}}$ : characteristic field strengths

$l_{\text{sys}}, t_{\text{sys}}$ : characteristic length & time scales

$v_{\text{sys}} = \frac{l_{\text{sys}}}{t_{\text{sys}}}$  characteristic plasma velocity

$$\text{IV} \quad \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \sim \frac{E_{\text{sys}}}{c t_{\text{sys}}} \sim \frac{B_{\text{sys}} l_{\text{sys}}}{c^2 t_{\text{sys}}} \sim \frac{v_{\text{sys}}^2}{c^2} |\nabla \times \vec{B}| \ll |\nabla \times \vec{B}|$$

assuming  
non-relativistic plasma

Drift velocity: consider  $e^-$ -p plasma (H-plasma)

different velocities of  $e^-$ , p cause current

$$\vec{j} = q_e (n_e \vec{v}_e - n_p \vec{v}_p) = q_e n_e (\vec{v}_e - \vec{v}_p) = q_e n_e \vec{v}_{\text{drift}}$$

number densities

$$|\nabla \times \vec{B}| \approx \frac{B_{\text{sys}}}{l_{\text{sys}}} \stackrel{\text{IV}}{\approx} \frac{4\pi}{c} q n_e |v_{\text{drift}}|$$

$$B_{\text{sys}} \sim 10^3 \text{ G}$$

$$n_e \sim 10^{23.1} \frac{1}{\text{cm}^3}$$

$$\Rightarrow v_{\text{drift}} \sim \begin{cases} 10^{-12} \frac{\text{cm}}{\text{s}} & \text{solar plasma} \\ & (\text{convection zone}) \quad l_{\text{sys}} \sim 10^{10} \text{ cm} \\ 10^{-2} \frac{\text{cm}}{\text{s}} & \text{interstellar medium} \\ (\text{compare turbulent}) \\ (\text{velocities} \sim 1 \frac{\text{km}}{\text{s}}) & l_{\text{sys}} \sim 1 \text{ pc} \\ & B_{\text{sys}} \sim 10^{-5} \text{ G} \\ & n_e \sim 10^{-3} \frac{1}{\text{cm}^3} \end{cases}$$

$$2) q|\vec{E}| \stackrel{\text{I}}{\sim} \frac{E_{\text{sys}}^2}{l_{\text{sys}}} \stackrel{\text{I)}}{\sim} \frac{v_{\text{sys}}^2}{c^2} \frac{B_{\text{sys}}^2}{l_{\text{sys}}} \stackrel{\text{IV}}{\sim} \frac{v_{\text{sys}}^2}{c^2} \frac{1}{c} |\vec{j} \times \vec{B}| \ll \frac{1}{c} |\vec{j} \times \vec{B}| \sim |\vec{J}_L|$$

$\frac{1}{c} |\vec{j}| \sim B_{\text{sys}}/l_{\text{sys}}$

↑ non-relativistic plasma

→ neglect effects of space charge → Gauss' law I can be dropped

## MHD equations:

- Ohm's law:  $\vec{j} = \sigma_e \vec{E}$  (plasma at rest)
  - non-relat. motion ↓
  - $\sigma_e = \frac{n_e e^2}{m_e f_c}$  electric conductivity
  - $f_c$  collision frequency b/w ions and  $e^-$
$$\text{V} \quad \vec{j} = \sigma_e \left( \vec{E} + \frac{1}{c} \vec{j} \times \vec{B} \right)$$
- Ohmic dissipation (current decay due to finite conductivity):
 
$$\frac{\partial e}{\partial t} = \frac{1}{\sigma_e} |\vec{j}|^2$$
 conversion of EM energy into internal energy

- reaction of B-field to motion of plasma:

$\text{IV} \text{ in } \text{IV}:$

$$\vec{E} = \frac{c}{4\pi\sigma_e} \vec{\nabla} \times \vec{B} - \frac{\vec{v}}{c} \times \vec{B}$$

electric field

in III:

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{\nabla} \times \vec{B}) - \vec{\nabla} \times (\gamma_e \vec{\nabla} \times \vec{B})$$

induction equation

$$\gamma_e = \frac{c^2}{4\pi\sigma_e} \quad \underline{\text{electric resistivity}}$$

at rest:  $\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times (\gamma_e \vec{\nabla} \times \vec{B}) \Rightarrow \frac{B_{\text{sys}}}{t_{\text{res}}} \sim -\gamma_e \frac{B_{\text{sys}}}{l_{\text{sys}}^2}$

$$\Rightarrow t_{\text{res}} \sim \frac{l_{\text{sys}}^2}{\gamma_e} \quad \begin{array}{l} \text{decay timescale of} \\ \text{magnetic field due} \\ \text{to finite resistivity} \end{array}$$

Note:  $[\gamma_e] = [\text{length}] \times [\text{velocity}]$

and  $\gamma_e$  also "magnetic diffusivity"

- EM fields affect motion of plasma:

Lorentz force  $\vec{F}_L = \frac{1}{c} \vec{j} \times \vec{B} \stackrel{\text{IV}}{=} \frac{1}{4\pi} (\vec{\nabla} \times \vec{B}) \times \vec{B}$

$$= \underbrace{\frac{1}{4\pi} (\vec{B} \cdot \vec{\nabla}) \vec{B}}_{\text{magnetic tension}} - \underbrace{\frac{1}{8\pi} \vec{\nabla} (\vec{B}^2)}_{\text{magnetic pressure}}$$

magnetic tension

$\rightarrow$  curvature of  $\vec{B}$ -field lines

magnetic pressure force  $P_m = \frac{\vec{B}^2}{8\pi}$

(vanishes for  $\vec{B}$  const. on straight lines)

In principle:  $\vec{F}_{EM} = \vec{F}_L + q\vec{E}$ , but  $q|\vec{E}| \ll |\vec{F}_L|$  (see above)

MP

$$\frac{\partial \vec{s}}{\partial t} + \nabla \cdot (\vec{s} \vec{v}) = 0$$

$$\frac{\partial}{\partial t}(\vec{s} \vec{v}) + \nabla \cdot \vec{\pi} = \frac{1}{4\pi} (\vec{B} \cdot \nabla) \vec{B} - \frac{1}{8\pi} \nabla \cdot (\vec{B}^2) + \vec{s} \vec{f}$$

$$\frac{\partial \vec{E}}{\partial t} + \nabla \cdot [(\vec{E} + \vec{P}) \vec{v}] = \frac{4\pi}{c^2} \gamma_e |\vec{j}|^2 + \vec{s} \vec{v} \cdot \vec{f}$$

$$\frac{\partial \vec{B}}{\partial t} = -\nabla \times (\gamma_e \nabla \times \vec{B}) + \nabla \times (\vec{v} \times \vec{B})$$

Ideal MHD approximation: assume  $\boxed{\gamma_e = 0}$

Exercise: Show that the MHD equations can be written in conservation form:

$$\frac{\partial \vec{s}}{\partial t} + \nabla \cdot (\vec{s} \vec{v}) = 0$$

(conservation of mass)

$$\frac{\partial \vec{s}}{\partial t} + \nabla \cdot \vec{T} = 0$$

(conservation of momentum)

$$\frac{\partial \vec{e}}{\partial t} + \nabla \cdot \vec{u} = 0$$

(conservation of energy)

$$\frac{\partial \vec{B}}{\partial t} + \nabla \cdot \vec{y} = \vec{s}_{res}$$

((non) conservation of magnetic flux)

$$\text{or } \partial_t u + \partial_{x_i} f^i(u) = S$$

with  $u = \begin{pmatrix} \vec{s} \\ \vec{S} \\ \tau \\ \vec{B} \end{pmatrix}$

- $\vec{s}$  ← conserved momentum
- $\tau$  ← conserved energy

$$f(u) = \begin{pmatrix} \vec{s} \vec{v} \\ T \\ U \\ Y \end{pmatrix}, \quad S = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vec{S}_{\text{res}} \end{pmatrix}$$

where:

$$\vec{S} \equiv s \vec{v} \quad (\text{momentum density})$$

$$T_{ij} \equiv \underbrace{s v_i v_j}_{\text{Reynolds stress tensor}} + \delta_{ij} \left( P + \frac{B^2}{8\pi} \right) - \frac{1}{4\pi} B_i B_j \quad (\text{stress tensor})$$

↑  
 isotropic pressure

magnetic part  
 of Maxwell stress tensor

$$\begin{aligned} \tau &\equiv \frac{1}{2} s v^2 + e + \underbrace{\frac{B^2}{8\pi}}_{\text{potential}} \quad (\text{total energy density}) \\ &= E + \frac{B^2}{8\pi} \end{aligned}$$

$$\vec{U} = (E + P) \vec{v} + \frac{1}{4\pi} \left( -\frac{v}{c} \times \vec{B} + \gamma_e \vec{j} \right) \times \vec{B} \quad (\text{energy flow vector})$$

$$\gamma_{ij} \equiv v_i B_j - v_j B_i$$

$$\vec{S}_{\text{res}} = \frac{\gamma_e}{4\pi} \nabla^2 \vec{B} + \vec{j} \times \nabla \gamma_e \quad (\text{resistive source term})$$

## Limits of MHD approximation:

Collisionless regime:

Drift velocity only small as long as EM forces due to large-scale fields balanced by frictional forces due to interparticle collisions

→ breaks down at low density when  $e^-$ -ions decouple:

$$f_e^L = f_e^{\text{coll}} \quad \begin{matrix} \text{"Coulomb} \\ \text{"logarithm", weakly} \\ \text{dependent on } T, n_e \end{matrix}$$

$\nearrow$

$$\propto 3.7 \ln \Delta T^{-3/2} n_e$$

$\frac{q_e B}{\omega_{ce}}$  Larmor (cyclotron) frequency of collisions b/w free  $e^-$

Zone C frequency of gyration

Example: star-forming cloud:  $T \sim 10K$ ,  $B \sim 10^{-5} G$

$$\Rightarrow n_e^{\text{crit}} \approx 10 \frac{1}{\text{cm}^3}$$

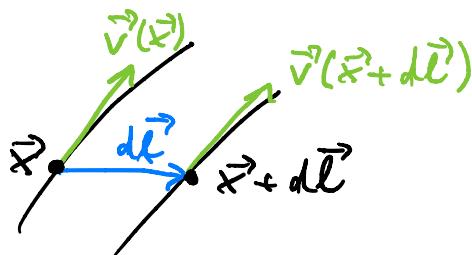
is rather high, especially bc of low ionization degree

But: often MHD approximation works well even in collisionless regime (e.g. solar wind past the Earth)

→ non-linear phenomena (not captured by MHD model) cause collisionless plasma to behave as continuous medium

# Flux conservation & field freezing:

Assume ideal MHD:  $\gamma_e \equiv 0$ .

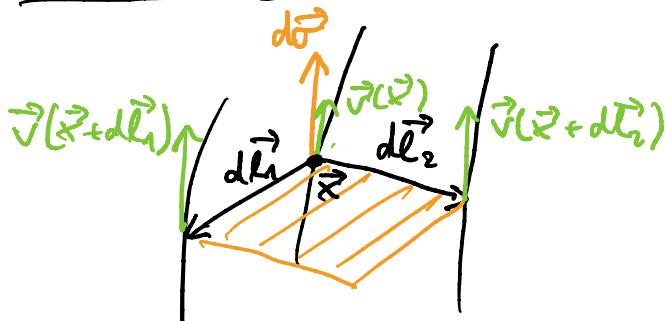


Co-moving line element (Lagrangian derivative):  
Taylor

$$\frac{d}{dt}(d\vec{l}) = \frac{d(\vec{x} + d\vec{l})}{dt} - \frac{d\vec{x}}{dt} \stackrel{\text{def}}{=} \vec{v}(\vec{x} + d\vec{l}) - \vec{v}(\vec{x}) \stackrel{*}{=} (\nabla \vec{v}) \cdot d\vec{l}$$

(change of  $d\vec{l}$  comoving with flow)

Co-moving surface element  $d\vec{\sigma} = d\vec{l}_1 \times d\vec{l}_2$ :



$$\frac{d(d\vec{\sigma})}{dt} = \frac{d(d\vec{l}_1)}{dt} \times d\vec{l}_2$$

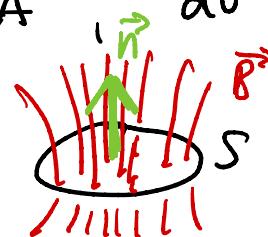
$$+ d\vec{l}_1 \times \frac{d(d\vec{l}_2)}{dt}$$

$$= d\vec{l}_1 \cdot \nabla \vec{v} \times d\vec{l}_2 - d\vec{l}_2 \cdot \nabla \vec{v} \times d\vec{l}_1$$

vector identity  $\nabla \times (A \times B) = (A \cdot \nabla)B - (B \cdot \nabla)A - A \times (\nabla \times B) + B \times (\nabla \times A)$

$$= -(\vec{d}\vec{\sigma} \times \nabla) \times \vec{v} = -\nabla \vec{v} \cdot \vec{d}\vec{\sigma} + \nabla \cdot \vec{v} \vec{d}\vec{\sigma}$$

Magnetic flux:  $\Phi_m = \int_S \vec{B} \cdot \vec{n} dA$ ,  $d\vec{\sigma} = \vec{n} dA$



$$\text{and } \frac{d}{dt}(\Phi_m) = \frac{d}{dt}(\vec{B} \cdot d\vec{\sigma}) = \frac{d\vec{B}}{dt} \cdot d\vec{\sigma} + \vec{B} \cdot \frac{d(d\vec{\sigma})}{dt}$$

Note:  $\frac{\partial \vec{B}}{\partial t} \stackrel{\eta_c=0}{=} \nabla \times (\vec{v} \times \vec{B})$  vector identity  $= \vec{B} \cdot \nabla \vec{v} - \vec{B} \nabla \cdot \vec{v} - \vec{v} \cdot \nabla \vec{B}$

$$\Rightarrow \frac{d\vec{B}}{dt} \stackrel{\text{def}}{=} \frac{\partial \vec{B}}{\partial t} + \vec{v} \cdot \nabla \vec{B} = \vec{B} \cdot \nabla \vec{v} - \vec{B} \nabla \cdot \vec{v}$$

Therefore:

$$\begin{aligned} \frac{d}{dt}(\Phi_m) &= (\vec{B} \cdot \nabla \vec{v} - \vec{B} \nabla \cdot \vec{v}) \cdot d\vec{\sigma} + \vec{B} \cdot (-\nabla \vec{v} \cdot d\vec{\sigma} + \nabla \cdot \vec{v} d\vec{\sigma}) \\ &= 0 \end{aligned}$$

co-moving surface element  $d\vec{\sigma}$  arbitrary

$\Rightarrow$  flux through any surface  $S$  moving with the fluid is conserved:

$$\Phi_m = \int_S \vec{B} \cdot \vec{n} dA = \text{const.}$$

field freezing (consider limit  $d\vec{\sigma} \rightarrow 0$ , i.e. "single field line". Since  $\Phi_m \rightarrow 0$  divide by  $\epsilon \rightarrow 0$ )

$$\begin{aligned} \frac{d}{dt}\left(\frac{\vec{B}}{S}\right) &= \frac{1}{S}(\vec{B} \cdot \nabla \vec{v} - \vec{B} \nabla \cdot \vec{v}) + \frac{\vec{B}}{S} \nabla \cdot \vec{v} \\ &= \left(\frac{\vec{B}}{S}\right) \cdot \nabla \vec{v} \end{aligned}$$

$\Rightarrow$  line element  $d\vec{l} \parallel \vec{B}$  moves as  $\frac{\vec{B}}{c}$  (see (1))  
 i.e. plasma along  $d\vec{l}$  and  $\vec{B}$  move together  
 and  $\vec{B}$ -field is "frozen in" the fluid

Note: motion along the field lines is preserved  
 however!

### Limits of ideal MHD:

- $\eta_e = 6.5 \times 10^{12} \frac{\ln \Delta}{T^{3/2}} [\text{cm}^2/\text{s}]$  (fully ionized hydrogen plasma)  
 $\sim 10^8$  for HII region  $\left( T \sim 10^4 \text{K}, \ln \Delta \approx 20, \right.$   
 $n_e = 10 - 100 \frac{1}{\text{cm}^3}$   
 $l_{\text{sys}} \sim 1 \text{pc} \left. \right)$
- $\rightarrow$  diffusivity timescale  $t_{\text{res}} \sim \frac{l_{\text{sys}}^2}{\eta_e} \sim 10^{21} \text{yr}$   
 $\sim 10^{11}$  age of universe  
 $\rightarrow$  ideal MHD very good!
- But:  $\eta_e$  becomes large /  $t_{\text{res}}$  small if
  - 1) where field becomes very tangled or when oppositely directed fields come in close contact (small  $l_{\text{sys}}$ )

## 2) plasma strongly turbulent

$\Rightarrow$  turbulent magnetic diffusivity

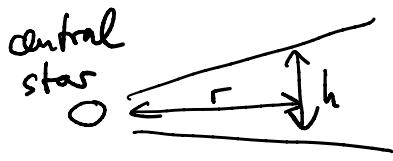
$$\eta_e^t \equiv v_t l_t = \underbrace{v_t}_{\text{Sec. 2.3}}$$

## 3) plasma weakly ionized

If  $n_e \ll n_n$ ,  $n_n$ : number density of neutral particles

$$\eta_e = 300 \frac{n_n}{n_e} \sqrt{T} \left[ \frac{\text{cm}^2}{\text{s}} \right] \quad \text{valid up to } T \approx 10^4 \text{ K}$$

Example: protoplanetary disk (neglect turbulence)



$$\frac{h}{r} \approx 0.1, T \approx 10^3 \text{ K}$$

$$r_{\text{sys}} \approx 1.5 \times 10^{12} \text{ cm} \quad \begin{matrix} \text{(height at Earth's)} \\ \text{radius} \end{matrix}$$

$$t_{\text{disk}} \approx 10^7 \text{ yr} \quad (\text{lifetime of disk})$$

$$\frac{n_e}{n_n} \approx 10^{-6} \quad (t_{\text{disk}} \stackrel{!}{=} t_{\text{res}})$$

$$\text{For } t_{\text{res}} < t_{\text{disk}} \text{ one requires } \frac{n_e}{n_n} < 10^{-6}$$

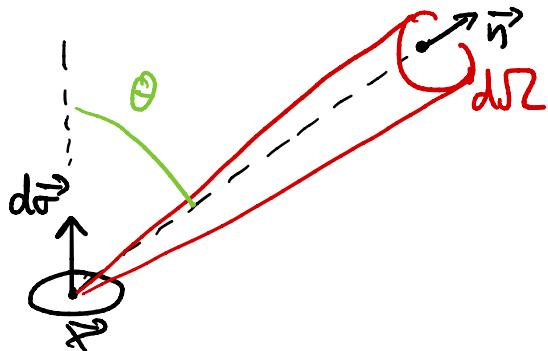
$\Rightarrow$  even at these low ionization degrees  
ideal MHD is still valid!

## 2.5 Radiation transfer

Def: radiation intensity

$$I_r(\vec{x}, \vec{n}, v, t) = \frac{dE_r}{\vec{n} \cdot d\vec{\sigma} d\Omega dv dt}$$

energy of photons that pass through an area  $d\vec{\sigma}$  at point  $\vec{x}$  in direction  $\vec{n}$ , within a solid angle  $d\Omega$  in the frequency interval  $(v, v+dv)$  in time  $dt$



Phase space for photons: use  $\vec{u} \mapsto \vec{q}$  (momentum) as photons are massless and propagate with velocity  $c$

$$\left. \begin{aligned} \text{total number} & \quad dN = f(\vec{x}, \vec{q}, t) d\vec{x} d\vec{q} \\ \text{total radiation energy} & \quad dE_r = h\nu f(\vec{x}, \vec{q}, t) d\vec{x} d\vec{q} \end{aligned} \right\} \begin{matrix} \downarrow \\ \text{within phase space volume } d\vec{x} d\vec{q} \end{matrix}$$

Note: consider a beam of photons as in previous definition.



$$\text{so } d\vec{x} = c dt (\vec{n} \cdot d\vec{\sigma})$$

spatial volume swept out by the beam in time  $dt$

$$\begin{aligned} d\vec{q} &= q^2 dq d\Omega \\ &= \left(\frac{hv}{c}\right)^2 \frac{h}{c} dv d\Omega \end{aligned}$$

spherical polar coord.  
in momentum space  
 $q = |\vec{q}|$ ,  $\vec{q} = \frac{hv}{c} \vec{n}$

$$\Rightarrow dE_v = \frac{h^4 v^3}{c^2} f(\vec{x}, v, \vec{n}, t) \vec{n} \cdot d\vec{\sigma} dv d\Omega dt$$

$$\Rightarrow I_v = \frac{h^4 v^3}{c^2} f$$

Boltzmann equation:

$$\vec{f} = 0 \text{ for photons}$$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + cn_i \frac{\partial f}{\partial x_i} = \left[ \frac{\partial f}{\partial t} \right]_{\text{coll}}$$

$u_i = cn_i$

$$\frac{1}{c} \frac{\partial I_v}{\partial t} + (\vec{n} \cdot \vec{\nabla}) I_v = \left[ \frac{\partial I_v}{\partial t} \right]_{\text{coll}}$$

$$= \frac{\partial}{\partial s}$$

equation of radiation transfer  
interaction of photons with matter  
→ absorption, emission & scattering

For a time-independent problem:

$$\frac{dI_r}{ds} = \left[ \frac{\partial I_r}{\partial t} \right]_{\text{coll}}$$

Computational challenge:

$$I_r(\vec{x}, v, \vec{n}, t) = I_r(r, \theta, \phi, v, \theta', \phi', t)$$

6+1 D function!

momentum space

take 100 grid points per variable

→ need to store  $(10^2)^6 = 10^{12}$  values per time step!

→ direct numerical solution of radiation transfer equation computationally prohibitive, need additional simplifications (reduce number of spatial dimensions etc.)

or solve set of approximate equations (moment schemes etc.)

Exercise: M1 moment scheme

## 2.6 Relativistic Hydrodynamics

In special relativity the Euler equations can be written as

$$\partial_\mu (su^\mu) = 0$$

$$\partial_\mu T^{\mu\nu} = 0$$

For an ideal/perfect fluid (in local rest frame no energy fluxes & anisotropic stresses):

$$T^{\mu\nu} = \underbrace{(E + p)}_{\substack{\text{total energy} \\ \text{density}}} u^\mu u^\nu + p \gamma^{\mu\nu}$$

Minkowski metric

energy-momentum tensor

specific internal energy measured in local rest frame

$$E = sc^2 + e = s(1+\epsilon)$$

$\epsilon$  henceforth set  $c=1$

General relativity: invoke the equivalence principle and use the substitution rules

$$\partial_\mu (su^\mu) = 0$$

$$\partial_\mu T^{\mu\nu} = 0$$



equivalence  
principle, substitution rules

$$g_{\mu\nu} \rightarrow g_{\mu\nu}$$

$$\partial_\mu \rightarrow \nabla_\mu$$

$$\nabla_\mu (s u^\mu) = 0$$

$$\nabla_\mu T^{\mu\nu} = 0$$

$$S_{SR}, X_{SR}^M T_{\mu\cdots\mu}^{i_1\cdots i_r} \xrightarrow{\text{in...jse}} S_{GR}, X_{GR}^M$$

$T_{\mu\cdots\mu}^{i_1\cdots i_r}$   
in...jse  
is GR

$$T^{\mu\nu} = (E + p) u^\mu u^\nu + p g^{\mu\nu}$$

(alternatively: Bianchi identities imply  $\nabla_\mu T^{\mu\nu} = 0$ )

Goal: want to rewrite these equations as an initial boundary value problem in conservation form so that one can use similar techniques as in the Newtonian case

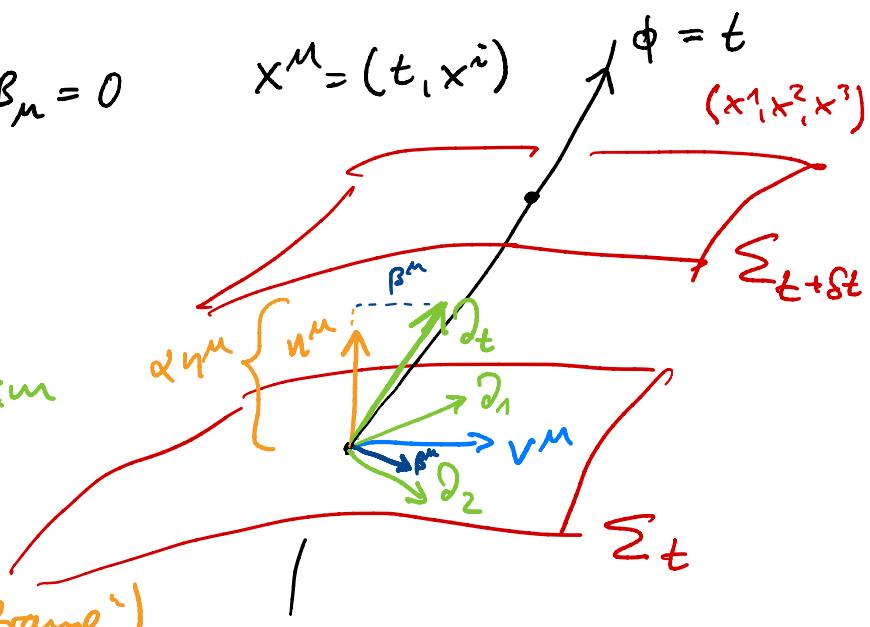
→ 3+1 decomposition of spacetime

$$(\partial_t)^M = \alpha u^M + \beta^M, \quad n^M \beta_M = 0$$

↑ "lapse" ↑ "shift"

represent 4 DOF to specify coordinates (diffeomorphism invariance of GR)

$u^M = n^M$ : Eulerian observer ("lab frame")



no motion wrt spatial coordinates

$$\|\partial_t\|^2 = (\partial_t)^\mu (\partial_t)_\mu = -\alpha^2 + \beta_\mu \beta^\mu = -\alpha^2 + \beta^2$$

so  $(\partial_t)^\mu$  timelike  $\Leftrightarrow \beta^2 < \alpha^2$

$(\partial_t)^\mu$  null  $\Leftrightarrow \beta^2 = \alpha^2$

$(\partial_t)^\mu$  spacelike  $\Leftrightarrow \beta^2 > \alpha^2$

Adapted coordinates:

$$(\partial_t)^\mu = (1, 0, 0, 0)$$

$$\beta^\mu = (0, \beta^i)$$

$$\text{so } n^\mu = \frac{1}{\alpha} (\partial_t)^\mu - \frac{1}{\alpha} \beta^\mu = \left( \frac{1}{\alpha}, -\frac{1}{\alpha} \beta^i \right)$$

$$n_\mu = (-\alpha, 0, 0, 0)$$

$$g_{00} = g(\partial_t, \partial_t) = -\alpha^2 - \beta^2$$

$$g_{0i} = g(\partial_t, \partial_i) = (\alpha n^\mu + \beta^\mu)(\partial_i)_\mu$$

$$= \underbrace{\alpha n^\mu (\partial_i)_\mu}_= 0 + \beta^\mu (\partial_i)_\mu = \beta_i$$

$$g_{ij} = g(\partial_i, \partial_j) \equiv \delta_{ij}$$

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 - \beta^2 & \beta_i \\ \beta_i & \delta_{ij} \end{pmatrix}$$

$$\text{or: } ds^2 = -\alpha^2 dt^2 + \gamma_{ij} (dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

↳ measures proper time that elapses going from one time slice to the next along  $ds$  no "lapse"

Note: a specific choice of  $\alpha$  &  $\beta^i$  represents a specific choice of coordinates (gauge)

Example:  $\alpha \equiv 1$  (Geodesic gauge)  
 $\beta^i \equiv 0$

•  $ds^2 = \alpha dt^2 = dt^2$  for Eulerian observers

$\Rightarrow t = \tau$  represents proper time of Eulerian observers

- curves  $c$  of  $\{x^i\} = \text{const.}$  are orthogonal to  $\Sigma_t$ , i.e. Eulerian observers move on geodesics:

$$a_\mu = \nabla_c \dot{c} = \nabla_\mu u = n^\nu \nabla_\nu n_\mu$$

$$\begin{cases} n^\nu \nabla_\nu \alpha = 0 & \mu = 0 \\ 0 & \mu = 1, 2, 3 \end{cases}$$

"Gaussian normal coordinates"

## Adapted equations of hydrodynamics:

Split fluid velocity

$$u^\mu = W(n^\mu + v^\mu)$$

$$\text{and } v^i = \frac{ds^i}{d\tau_{\text{Eul}}} = \frac{u^i}{W} + \frac{1}{\alpha} \beta^i \quad \begin{array}{l} \text{fluid velocity} \\ \text{relative to Eulerian obs.} \end{array}$$

$$W = -u^\mu n_\mu \quad \begin{array}{l} \text{Lorentz factor b/w Eulerian} \\ \text{\& comoving (fluid) frame} \end{array}$$

and define

$$E_E \equiv n^\mu n^\nu T_{\mu\nu} = (E+p)W^2 - p = hsW^2 - p$$

$$S_i \equiv -g_i{}^\mu n^\nu T_{\mu\nu} = (E+p)W^2 v_i = hsW^2 v_i$$

$$S_{ij} \equiv g_i{}^\mu g_j{}^\nu T_{\mu\nu} = hsW^2 v_i v_j + p \delta_{ij}$$

$$(h \equiv 1 + \epsilon + p \quad \text{specific enthalpy})$$

(total energy density, momentum density,  
stress-energy as measured by Eulerian observer)

then one obtains:

$$\bullet 0 = \nabla_\mu (g^{\mu\nu} u^\nu) \rightarrow \boxed{\partial_t (\sqrt{g} D) + \partial_i (\sqrt{g} D \tilde{v}^i) = 0}$$

$D = \gamma W$  rest mass measured by Eulerian obs.

$$\tilde{v}^i = \alpha v^i - \beta^i$$

- $\partial = n_\mu \nabla_\nu T^{\mu\nu}$  (projection of conservation law)  
(along Eulerian observer)

$$\rightarrow \partial_t (\sqrt{g} E_E) + \partial_j [\sqrt{g} (\alpha S^j - E_E \beta^j)] = \alpha \sqrt{g} (K_{ij} S^{ij} - S^k \partial_k \ln \alpha)$$

extrinsic curvature  
of  $\Sigma_t$

- $\partial = \gamma_i{}^\mu \nabla_\mu T^\nu{}_\nu$  (spatial projection)

$$\rightarrow \partial_t (\sqrt{g} S_i) + \partial_j [\sqrt{g} (S_j \tilde{v}^i + \rho \delta^i{}_j)] = \sqrt{g} \left[ \frac{\alpha}{2} S^{kl} \partial_i \delta_{kl} \right. \\ \left. + S_\ell \partial_i \beta^\ell - S \partial_i \alpha \right]$$

no conservative formulation:  $\partial_t u + \partial_i f^i(u) = S(u)$

$$u = \sqrt{g} (D, S_i, \tilde{v})$$

$$f^i = \sqrt{g} (D \tilde{v}^i, S_j \tilde{v}^i + \rho \delta^i{}_j, \tilde{v} \tilde{v}^i + \rho v^i)$$

$$S(u) = (0, \alpha \sqrt{g} (K_{ij} S^{ij} - S^k \partial_k \ln \alpha), \sqrt{g} (\frac{\alpha}{2} S^{kl} \partial_i \delta_{kl} + S_\ell \partial_i \beta^\ell - S \partial_i \alpha))$$

$$(T = E_E - D)$$

Remarks: 1) The energy equation is written in terms of  $\tau = E_E - D$ , in order to obtain the "correct" Newtonian limit:

$$\tau = E_E - D = \gamma h w^2 - p - \gamma w$$

$$\frac{p}{\gamma c^2} \stackrel{\ll 1}{=} \underbrace{\gamma w(w-1)}_{\approx 1 + \left(\frac{w}{c}\right)^2} + \gamma \varepsilon w^2 \approx \frac{1}{2} \gamma v^2 + \gamma \varepsilon$$

$$\approx \left[1 + \frac{1}{2} \left(\frac{w}{c}\right)^2\right] \left[\frac{1}{2} \left(\frac{w}{c}\right)^2 + \dots\right]$$

2)  $s(u)$ : "geometric source terms"

Note: do not depend on time derivatives of space-time quantities!

3) EOS: As in Newtonian hydrodynamics, the system is closed by an equation of state  
 $p = p(\gamma, \varepsilon, \dots)$

4) Conservatives & primitives:

$$u \equiv \sqrt{f}(D, S_i, \tau) \quad \text{"conservatives"}$$

$$w \equiv (\gamma, v^i, \varepsilon) \quad \text{"primitives"}$$

In order to compute flux terms during evolution, one must compute  $w$  from  $u$  numerically (no need  $p = p(\gamma, \varepsilon)$ )

typically: obtain these via non-linear  
root-finding in some variable  $z$

$$f(z) = z(\xi, \epsilon, v^i) - z$$

where  $z(\xi, \epsilon, v^i)$  is computed from  
 $\xi, \epsilon, v^i$  obtained from conservatives

Example:  $z=p$

$$\text{and } v^i = \frac{\xi^i}{z+p}, \quad w = \sqrt{1-v_i \cdot v^i}$$

$$\epsilon = \frac{D}{W}$$

$$\epsilon = \frac{z - DW + p(1-w^2)}{DW}$$

See Siegel et al. ApJ 859, 71 (2018) for a discussion of recovery schemes.

5) Hyperbolicity: the eqns in conservative form are strongly hyperbolic if the EOS is causal,

$$c_s^2 = \frac{\partial P}{\partial \epsilon} < 1 \quad (\text{sound speed} < \text{speed of light})$$

see Ante 1990, Anton et al. 2006 ApJ 637, 296 for hydrodynamic & magnetohydrodynamic case

6) Special-relativistic limit:

$$\alpha \rightarrow 1, \beta^i \rightarrow 0, \gamma_{ij} \rightarrow \eta_{ij}$$

$$(\Rightarrow v^i \rightarrow v^i)$$

and  $u = (1, S_i, \varepsilon)$

$$f^i(u) = (Dv^i, S_j v^i + p \delta^i_j, \varepsilon v^i + p v^i)$$

$$S(u) = 0$$

7) Newtonian limit: SR limit plus

$$\frac{v}{c} \ll 1, \omega \rightarrow 1 + \frac{1}{2} \left( \frac{v}{c} \right)^2$$

$$\frac{p}{sc^2} \ll 1, h \rightarrow 1, \varepsilon \rightarrow E = s\varepsilon + \frac{1}{2} sv^2$$

and  $u = (s, sv_j, E)$

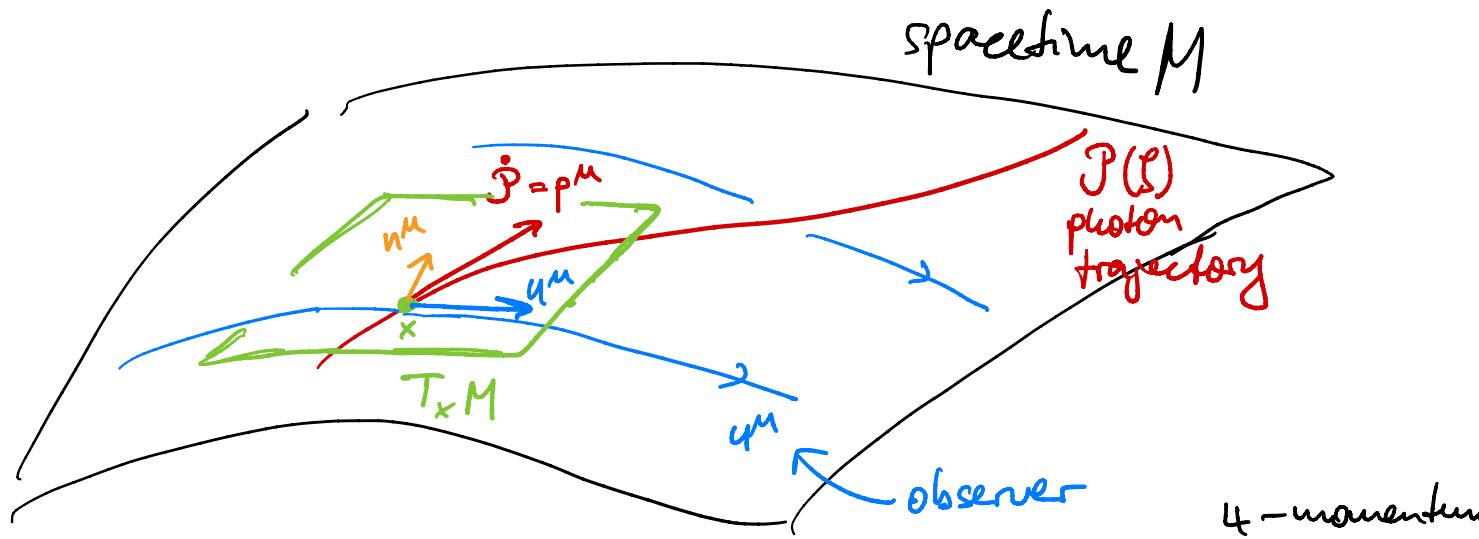
$$f^i(u) = (Dv^i, sv^i v_j + p \delta^i_j, (E+p)v^i)$$

$$S(u) = 0$$

(cf. Sec. 2.2)

## 2.7 Relativistic radiation transfer

Reading: Thorne 1981, MNRAS 194, 439  
 Straumann Chap. 3.11



- Consider light-like (massless) particle  $p_\mu p^\mu = 0$  moving along trajectory  $P(s)$ , parametrized such that  $\dot{P}(s) = p^\mu$

Decompose:  $p^\mu = -p^\nu u_\nu$  ( $u^\mu + n^\mu$ )

$\nwarrow$  unit normal  
 $\nwarrow$  in orthogonal space to  $u^\mu$

$$n^\mu u_\mu = 1, \quad n^\mu u_\mu = 0$$

- Reparametrize by proper spatial distance  $l$  travelled by photon as seen by observer  $u^\mu$ :

$$l \equiv \int (-p_\mu u^\mu) ds, \quad dl = (-p_\mu u^\mu) ds$$

$$\text{as } P = P(l), \quad \dot{P}_l = \frac{dP}{dl} = \frac{dP}{ds} \frac{ds}{dl}$$

$$= (-\rho^{\mu} u_{\mu})(u^{\mu} + n^{\mu}) \frac{1}{(-\rho u^{\mu} n^{\mu})} = u^{\mu} + n^{\mu}$$

- Consider particle distribution function

$$f(x^{\mu}, p^{\mu}) \quad (\text{number density of light-like particles in phase space})$$

↓      ↓  
 point in spacetime      4-momentum  
 $P$       on light cone in  $T_p M$

actually:  $f = f(t, x^i, p^i)$  as  $p^0$  determined by

$$p_{\mu} p^{\mu} = 0 \quad (\text{light cone particles})$$

- Note: along each photon trajectory the number density is conserved (in absence of interactions with medium)

(Liouville's theorem):  $\frac{df}{dl} = 0$

$$\text{as } \boxed{\frac{df}{dl} = \frac{\partial f}{\partial x^{\mu}} \frac{dx^{\mu}}{dl} + \frac{\partial f}{\partial p^i} \frac{dp^i}{dl} = S(x^{\mu}, p^{\mu}, f)}$$

GR radiation transfer equation

## (Boltzmann equation in GR)

$S$ : source term ("collision terms") that accounts for emission, absorption & scattering

Note:  $\frac{dx^\mu}{dl} = \dot{\mathcal{P}}_l = n^\mu + u^\mu$  (see above)

$$\frac{dp^i}{dl} = \dot{p}^i = (\dot{\mathcal{P}}_l)^i = -\Gamma_{\mu\nu}^i p^\mu \dot{x}^\nu$$

$$= -\Gamma_{\mu\nu}^i p^\mu (n^\nu + u^\nu) \quad (\text{null geodesic!})$$

Moment formalism: (analogous to Newtonian case)

Thorne 1981, MNRAS 194, 439

Define  $k$ -th moment  
of distribution function

$$M_v^{\alpha_1 \dots \alpha_k} = \int \frac{f(x^\mu, p^\mu) \delta(p^\mu u_\mu + v)}{(-p^\mu u_\mu)^{k-2}} p^{\alpha_1} \dots p^{\alpha_k} dV_p$$

↑ frequency

$v = -p^\mu u_\mu$

↙ restrict integration to light cone

→ can derive evolution equations for

the moments in conservation form,  
analogous to the Newtonian case,  
closed by Eddington factors  
(→ "M1 scheme")

see Thorne 1981 & Shibata et al. Prog. Theo. Phys.  
125, 1255 (2011)