Chapter 1: Basic notions of PDEs

1.1 PDEs of 2nd order

let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $\Omega_1, \Omega_2 \subseteq \mathbb{R}^n$,

$L: \underset{\text{e.g. } C^2(\Omega_1)}{\mathcal{F}_1(\Omega_1)} \longrightarrow \underset{\text{e.g. } C^0(\Omega_2)}{\mathcal{F}_2(\Omega_2)}$ differential operator,

$u \in \mathcal{F}_1(\Omega_1): \Omega_1 \longrightarrow \Omega_2$

Notation: $u_{x_i} \equiv \partial_i u = \frac{\partial u}{\partial x^i}$, $u_{x_i x_j} = \partial_i \partial_j u = \frac{\partial^2 u}{\partial x^i \partial x^j}$

<u>Def (classification)</u>:

(i) <u>non-linear PDEs</u> of 2nd order:

$$Lu = 0, \text{ with } L = L(x, u, \{u_{x_i}\}, \{u_{x_i x_j}\})$$

and $\frac{\partial L}{\partial u_{x_i x_j}} \neq 0$ for at least one pair of indizes $i, j \in \{1, \dots, n\}$

<u>Example</u>: consider $\Omega_1 = \Omega_2 = \mathbb{R}^2$ and

$$Lu = u_{x_1 x_1} u_{x_2 x_2} - u_{x_1 x_2}^2 = 0$$

(ii) Quasi-linear PDEs of 2nd order:

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x, u, \{u_{x_i}\})\, u_{x_i x_j} + b(x, u, \{u_{x_i}\}) = 0$$

Example: $\Omega_1 = \Omega_2 = \mathbb{R}^n$

$$\sum_{i,j=1}^{n} \left(\delta_{ij} - \frac{u_{x_i} u_{x_j}}{1+|\nabla u|^2} \right) u_{x_i x_j} = 0$$ minimal surfaces

(iii) Semi-linear PDEs of 2nd order:

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x)\, u_{x_i x_j} + b(x, u, \{u_{x_i}\}) = 0$$

Example:

$$-\Delta u = |\nabla u|^2 u$$

(iv) Linear PDEs of 2nd order

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x)\, u_{x_i x_j} + \sum_{i=1}^{n} a_i(x)\, u_{x_i} + a(x) u + f(x)$$

$$= 0$$

Example: $\Delta u = 0$

Remark: if we are looking for a solution $u \in C^2(\Omega_1)$:

$$a_{ij} = a_{ji} \quad \text{(derivatives commute)}$$

$$\left(A = (a_{ij})_{i,j=1,\dots,n} \text{ is symmetric} \Rightarrow \text{real eigenvalues} \right)$$

Def: Let $x_0 \in \mathbb{R}^n$, $\lambda_1, \dots, \lambda_n$ the (real) eigenvalues of $A = (a_{ij}(x_0))_{i,j=1,\dots,n}$, and

$$t = \#\{\lambda_i < 0\}$$

$$d_0 = \#\{\lambda_i = 0\}.$$

At $x = x_0$ the linear PDE / differential operator

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i x_j} + \sum_{i=1}^{n} a_i(x) u_{x_i} + a(x) u + f(x) = 0$$

is called

- elliptic $\Leftrightarrow d_0 = 0$ and $(t = 0$ or $t = n)$
- hyperbolic $\Leftrightarrow d_0 = 0$ and $(t = 1$ or $t = n-1)$
- ultra-hyperbolic $\Leftrightarrow d_0 = 0$ and $t \in \{2, \dots, n-2\}$
- parabolic $\Leftrightarrow d_0 > 0$

Remarks:

1) classification depends on the location $x = x_0$
(for quasi-linear PDEs classification also depends on the solution itself)

2) classification only depends on 2nd order terms (principal part)

Geometric interpretation:

Diagonalize $A = CDC^*$, $D = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$
$$CC^* = C^*C = \mathbb{1}$$

and consider quadratic form

$$Q(\xi) = \xi^T A \xi = \sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j \quad \left(= CDC^* \xi \cdot \xi\right)$$

(principle axis theorem)

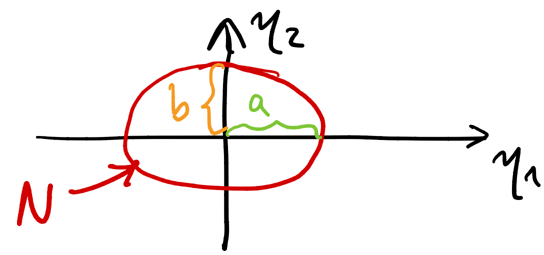
$$= \xi^T \overbrace{CC^*}^{\mathbb{1}} \overbrace{CDC^*}^{A} \xi = DC^*\xi \cdot C^*\xi = \eta^T D \eta$$

$$= \sum_{i=1}^{n} \lambda_i \eta_i^2 \equiv \tilde{Q}(\eta) \quad , \eta \equiv C^* \xi$$

Consider level sets $N = \{\eta \in \mathbb{R}^n \mid \tilde{Q}(\eta) = \text{const.}\}$

Example: $n = 2$ $\leadsto$ $N = \{\eta \in \mathbb{R}^2 \mid \lambda_1 \eta_1^2 + \lambda_2 \eta_2^2 = \text{const}\}$

elliptic case: $\lambda_1, \lambda_2 > 0$

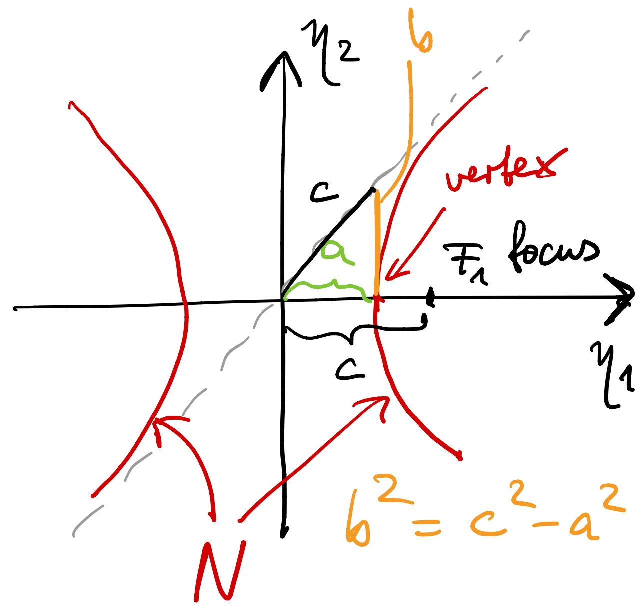


$$\frac{\eta_1^2}{\left(\text{const}/\lambda_1\right)} + \frac{\eta_2^2}{\left(\text{const}/\lambda_2\right)} \equiv \frac{\eta_1^2}{a^2} + \frac{\eta_2^2}{b^2} = 1$$

<u>hyperbolic case:</u> $\lambda_1 > 0\,,\ \lambda_2 < 0$

$$\frac{\eta_1^2}{\text{const.}/\lambda_1} + \frac{\eta_2^2}{\text{const.}/\lambda_2}$$

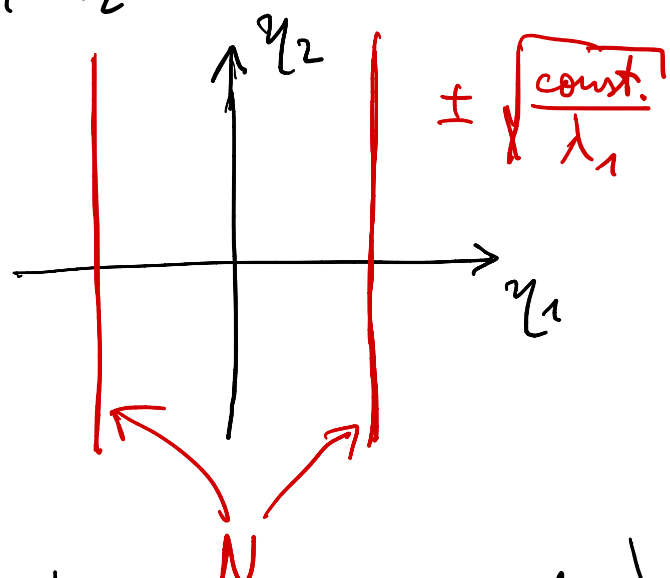
$$\equiv \frac{\eta_1^2}{a^2} - \frac{\eta_2^2}{b^2} = 1$$



<u>parabolic case:</u> $\lambda_1 > 0\,,\ \lambda_2 = 0$

$$\lambda_1 \eta_1^2 = \text{const.}$$

degenerate parabola



(if lower order terms are taken into account may recover actual parabola, e.g. $\rightsquigarrow \lambda\,\eta_1^2 + \eta_2 = \text{const}$)

Typical examples for principal types:

assume $a_{ij} = \text{const.}$ and consider

$$Lu = \sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} = \sum_{i,j=1}^{n} \sum_{k=1}^{n} C_{ik} \lambda_k C^*_{kj} u_{x_i x_j}$$

$$= \sum_{k=1}^{n} \lambda_k \left(\sum_{i,j=1}^{n} u_{x_i x_j} C_{ik} C_{jk} \right)$$

$$\left(\begin{aligned} &\text{define} \quad v(y) \equiv u(Cy), \quad x = Cy \\ &\leadsto \quad v_{y_i}(y) = \sum_{k=1}^{n} u_{x_k}(Cy) C_{ki} \\ &\qquad v_{y_k y_k} = \sum_{i,j=1}^{n} u_{x_i x_j}(Cy) C_{jk} C_{ik} \end{aligned} \right)$$

$$= \sum_{k=1}^{n} \lambda_k \, v_{y_k y_k}(y)$$

<u>elliptic case:</u> $\lambda_1, \dots, \lambda_n > 0$ $\qquad y = \sqrt{D}\,\tilde{y}$

set $\tilde{v}(\tilde{y}) \equiv v(\sqrt{\lambda_1}\,\tilde{y}_1, \dots, \sqrt{\lambda_n}\,\tilde{y}_n)$, $\quad y_i = \sqrt{\lambda_i}\,\tilde{y}_i$

$$\leadsto \sum_{k=1}^{n} \tilde{v}_{\tilde{y}_k \tilde{y}_k}(\tilde{y}) = \sum_{k=1}^{n} \lambda_k v_{y_k y_k}(\sqrt{\lambda_1}\,\tilde{y}_1, \dots, \sqrt{\lambda_n}\,\tilde{y}_n) = 0$$

$$\Longleftrightarrow \boxed{\tilde{\Delta} \tilde{v} = 0} \leadsto -\Delta u = f$$

<u>Poisson equation</u>

hyperbolic case: $\lambda_1 < 0, \ \lambda_j > 0 \ (j = 2, \ldots, n)$

set $\tilde{v}(\tilde{y}) \equiv v(\sqrt{-\lambda_1}\,\tilde{y}_1, \sqrt{\lambda_2}\,\tilde{y}_2, \ldots, \sqrt{\lambda_n}\,\tilde{y}_n)$

$$\rightsquigarrow \quad \tilde{v}_{\tilde{y}_1\tilde{y}_1} - \sum_{k=2}^{n} \tilde{v}_{\tilde{y}_k\tilde{y}_k} = 0$$

rename $x \Rightarrow (t, x)$

$$\rightsquigarrow \quad \boxed{\Box u = u_{tt} - \Delta u = 0}$$

linear wave equation

parabolic case: $\lambda_1 = 0$, assume $\lambda_2, \ldots, \lambda_n > 0$

$$\rightsquigarrow \quad \sum_{j=2}^{n} \lambda_j v_{y_j y_j} \equiv \sum_{j=2}^{n} \tilde{v}_{\tilde{y}_j\tilde{y}_j} = 0$$

elliptic in $\mathbb{R}^{n-1}$!

take 1st-order terms into account as well

$\rightsquigarrow$ can obtain equ of the form

$$\boxed{u_t - \Delta_{n-1} u = 0}$$

$i \in \{2, \ldots, n\}$

heat conduction equation

(diffusion equation)

1.2 First-order PDEs

Def (classification):

(i) non-linear

$$Lu = L(x, u, \{u_{x_i}\}) = 0$$

and $\frac{\partial L}{\partial u_i} \neq 0$ for at least one $i \in \{1, \dots, n\}$

Example: $u_{x_1} - u_{x_2}^2 = 0$

(ii) quasi-linear:

$$Lu = \sum_{k=1}^{n} a_k(x, u)\, u_{x_k} + b(x, u) = 0$$

Example: $u_{x_1} + u u_{x_2} = 0$

$\partial_t u + u \partial_x u = 0$

$\big(x = (x_1, x_2) \to (t, x)\big)$

inviscid Burgers equation

(iii) semi-linear:

$$Lu = \sum_{k=1}^{n} a_k(x)\, u_{x_k} + b(x,u) = 0$$

Example: $u_{x_1} = u$

(iv) linear:

$$Lu = \sum_{k=1}^{n} a_k(x)\, u_{x_k} + b(x) = 0$$

Example: $u_{x_1} + a\, u_{x_2} = 0$ linear

$\partial_t u + a\, \partial_x u = 0$ advection equation

We are mostly interested in systems of quasi-linear 1st order equs $(i = 1, \dots, m)$

Def: Let $B, U = (u_1, \dots, u_m) : I \subseteq \mathbb{R} \times \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $x, \xi \in \Omega$, $A_i(t,x,U)$ $m \times m$ matrices $\forall\, i \in \{1, \dots, n\}$, and

$$\mathcal{A}(t,x,U;\xi) \equiv \sum_{i=1}^{n} A_i(t,x,U)\, \xi_i \,.$$

The quasi-linear system

$$(*) \quad U_t + \sum_{i=1}^{n} A_i(t,x,U) U_{x_i} + B(t,x,U) = 0$$

is called <u>hyperbolic</u> at (t,x) if $\mathcal{A}(t,x,U;\xi)$ is diagonalizable for all $x,\xi \in \Omega$, $t \in I$. In particular, $(*)$ is hyperbolic if $\forall x,\xi \in \Omega, t \in I$ $\mathcal{A}$ has m <u>real</u> eigenvalues

$$\lambda_1(t,x,U;\xi) \leq \lambda_2(t,x,U;\xi) \leq \dots \leq \lambda_m(t,x,U;\xi)$$

and corresponding eigenvectors

$$\{r_i(x,t,U;\xi)\}_{i=1,\dots,m}$$

which form a basis of $\mathbb{R}^m$. Two <u>special cases</u>:

1) $A_i(t,x,U)$ symmetric $\forall i \in \{1,\dots,n\}$
   $\Rightarrow \mathcal{A}$ symmetric $\forall \xi \in \Omega$
   $\Rightarrow \mathcal{A}$ diagonalizable
   " $(*)$ is <u>symmetric hyperbolic</u> "

2) $\mathcal{A}(t,x,U;\xi)$ has m real, <u>distinct</u> eigenvalues

$$\lambda_1(t,x,U;\xi) < \lambda_2(\dots) \dots < \lambda_m(\dots)$$

$\forall x,\xi \in \Omega, t \in I$

$\Rightarrow \mathcal{A}(t,x,u;\xi)$ is diagonalizable

"$(\ast)$ is strictly hyperbolic"

Examples:

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t + \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{strictly hyperbolic}$$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t + \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_x = \begin{pmatrix} b_1(x,t) \\ b_2(x,t) \end{pmatrix} \quad \text{symmetric hyperbolic}$$

Remarks: 1) The system $(\ast)$ is elliptic at (t,x) if none of the eigenvalues $\lambda_i(t,x,u;\xi)$ are real.

2) The term "hyperbolic" is related to the fact that 2nd order hyperbolic PDEs can be recast as systems of 1st order hyperbolic PDEs (Exercise)

<u>Note:</u> distinguishing feature of wave equation is the existence of <u>plane wave solutions</u>

$$u(x,t) = f(\xi\cdot x - \sigma t), \text{ with } \sigma = \pm \underset{\uparrow \atop (=1)}{c}|\xi|$$

for any $\xi \in \Omega$

$\leadsto$ make this a defining feature of 1st order systems:

require <u>m distinct plane wave solutions</u> for each $\xi \in \Omega$.

Ansatz: $U(x,t;\xi) = V(\xi\cdot x - \sigma t)$

$\downarrow$ in $(\ast)$ with $B \equiv 0$

$$-\sigma V' + \sum_{i=1}^{n} \xi_i A_i V' = 0$$

$$\Longleftrightarrow \quad \boxed{\mathcal{A} V' = \sigma V'}$$

$\leadsto$ if $\mathcal{A} = \sum_{i=1}^{n} \xi_i A_i$ is diagonalizable ($m\times m$)

$\Rightarrow$ there are m distinct plane wave solns (the eigenvectors of $\mathcal{A}$ corresponding to the eigenvalues $\sigma_1,\dots,\sigma_m$)

Def (Conservation laws): Conservation laws are systems of 1st order PDEs that can be written in the form

$$U_t + \sum_{i=1}^{n} \underbrace{F^i(U)_{x_i}}_{= \partial_{x_i} F^i(U)} = S(t,x,U)$$

$U = (u_1, \dots, u_m)$: "conserved variables"

$F^i(U) = (f_1^i(U), \dots, f_m^i(U))$, $i = 1, \dots, n$
"vector of fluxes"

$S(t,x,U)$: "source terms"

The <u>Jacobians</u> of the flux functions $F^i(U)$ is

$$A_i(U) = \frac{\partial F^i}{\partial U} = \begin{pmatrix} \frac{\partial f_1^i}{\partial u_1} & \cdots & \frac{\partial f_1^i}{\partial u_m} \\ \vdots & & \\ \frac{\partial f_m^i}{\partial u_1} & \cdots & \frac{\partial f_m^i}{\partial u_m} \end{pmatrix}$$

<u>Remarks</u>: 1) conservation laws can be written as a system (*) of quasi-linear PDEs of 1st order by applying the chain rule:

$$\partial_{x_i} F^i(U) = \frac{\partial F^i}{\partial U} \frac{\partial U}{\partial x_i} = A_i(U)\, U_{x_i}$$

2) the source terms $S = B(t, x, U)$ can arise

- due to physical sources or sinks for otherwise conserved quantities
- due to change of coordinates "geometric source terms"

3) Motivation: consider integral form

$$\frac{d}{dt}\int U \, dV = -\int \partial_{x_i} F^i(U)\, dV$$

V: control volume with surface S

$$\overset{\text{divergence theorem}}{=} -\oint_S F^i(U)\, n_i \, dS$$

n_i: outward pointing normal

time-rate change of U inside V depends only on the total flux through the surface S

$\rightsquigarrow$ "<u>U is conserved</u>"

<u>Examples:</u> The inviscid Burgers equ and the linear advection equ are conservation laws:

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \quad f(u) = au \qquad \text{advection equ}$$

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \quad f(u) = \frac{1}{2}u^2 \qquad \text{inviscid Burgers equ}$$

1.3 Some properties of 1st order hyperbolic systems

1.3.1 Characteristics

1) Linear hyperbolic systems

Consider hyperbolic system of the form

$$U_t + \underbrace{A}_{\text{constant coefficients}} U_x = 0$$

hyperbolicity $\Rightarrow \exists\, Q$ with $A = Q \Lambda Q^{-1}$

where $\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_m \end{pmatrix}$ eigenvalues of A

$Q = \underbrace{(r_1, \dots, r_m)}_{\text{columns}}$ r_i: eigenvectors of A corresponding to λ_i

Note: if system symmetric hyperbolic

$\Rightarrow r_i$ orthonormal and $Q^{-1} = Q^T$

$$\rightsquigarrow \; U_t + Q \Lambda Q^{-1} U_x = 0 \qquad \Big| \times Q^{-1}$$

define $V = Q^{-1}U$ characteristic variables

$$\rightsquigarrow \boxed{V_t + \Lambda V_x = 0}$$

decoupled system!

(Note: $Q^{-1}U_x \to V_x$ requires A to have const. coefficients)

Characteristics: curves $\gamma(t)$ along which the PDE becomes an ODE

Consider $x = \gamma(t)$, $\quad v_i = v_i(\gamma(t), t)$

$$\rightsquigarrow \frac{dv_i}{dt} = \frac{\partial v_i}{\partial t} + \frac{dx}{dt}\frac{\partial v_i}{\partial x} = \frac{\partial v_i}{\partial t} + \gamma'(t)\frac{\partial v_i}{\partial x}$$

choose: $\gamma'(t) = \lambda_i$ characteristic speeds

$$\Rightarrow \frac{dv_i}{dt} = 0$$ i.e. v_i are constant along characteristics

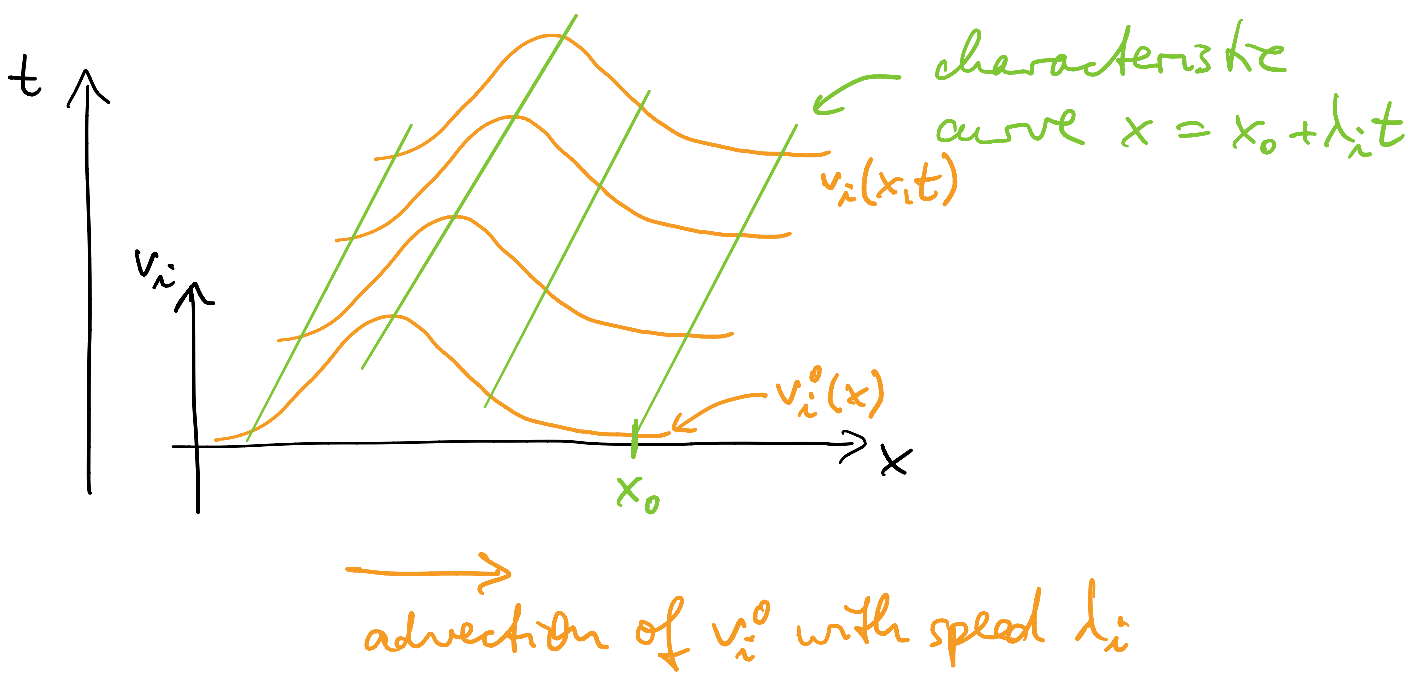
Therefore: given $v_i(x,0) = v_i^0(x)$ at $t=0$

$$v_i(x,t) = v_i^0(x_0) = v_i^0(x - \lambda_i t)$$

where $x(t) = x_0 + \lambda_i t$ is the characteristic $\gamma(t)$ that passes through x at t.

$\rightsquigarrow$ v_i at (x,t) is entirely determined by initial data v_i^0 at x_0, i.e. the PDE will translate the profile v_i^0 with velocity λ_i (the shape remains unchanged)

$\rightsquigarrow$ "advection equation"



General initial value problem:

Given
$$\begin{cases} U_t + AU_x = 0 \\ U^0 = (u_1^0, \dots, u_m^0) \end{cases} \qquad (I)$$

$$\Big\downarrow \text{ introduce characteristic variables } V = Q^{-1} U$$

$$\begin{cases} V_t + \Delta V_x = 0 \\ \qquad\qquad V^0 = Q^{-1} U^0 = (v_1^0, \dots, v_m^0) \end{cases}$$

$$\Big\downarrow$$

$$V = \{v_i(x,t)\} = \{v_i^0(x - \lambda_i t)\}, \quad i = 1, \dots, m$$

$$\Big\downarrow \text{ transform back}$$

$$\boxed{\begin{aligned} U(x,t) &= Q V(x,t) \\ &= \sum_{i=1}^{m} v_i(x,t)\, r_i \\ &= \sum_{i=1}^{m} v_i^0(x - \lambda_i t)\, r_i \end{aligned}} \quad \text{solution to } \boxed{\mathrm{I}})$$

Remarks:
- $U(x,t)$ is superposition of eigenvectors, i.e. waves propagating at finite speeds $\lambda_i \neq 0$.
- $U(x,t)$ entirely determined by initial data v_i^0 at points $x_0^i = x - \lambda_i t$.

2) Scalar conservation laws (quasi-linear case)

$$u_t + f(u)_x = u_t + \lambda(u)\, u_x = 0$$

$$\lambda(u) = \frac{df}{du} = f'(u)$$

consider characteristics $\Gamma_{x_0} = (\gamma(t), t)$ satisfying

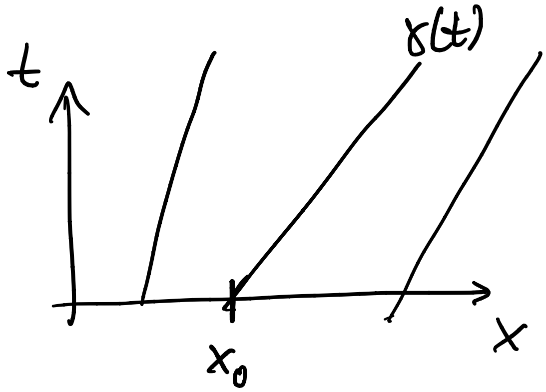
$$\gamma'(t) = \frac{dx}{dt} = \lambda(u), \quad \gamma(0) = x(0) = x_0$$

characteristic speed

If we assume $u(t,x)$ is smooth

$$\leadsto \frac{du}{dt} = u_t + \underbrace{\frac{dx}{dt}}_{\lambda(u)} u_x = 0$$

$\Rightarrow$ $u(t,x)$ is constant on characteristics, which are straight lines ($u = \text{const} \Rightarrow x' = \lambda = f'(u) = \text{const.}$), as long as $u(t,x)$ is / remains smooth



$$X = x_0 + \lambda(u_0(x_0))t$$

$$\downarrow$$

$$u(x,t) = u_0(x - \lambda(u_0(x_0))t)$$

Remark: properties of $f(u)$ influence properties of the solution $u(x,t)$, in particular the monotonicity properties:

(⤳ wave steepening etc., see Chap. 4)

- $\lambda(u)$ is monotone increasing $\Leftrightarrow$ convex flux

$$\frac{d\lambda(u)}{du} = \lambda'(u) = f''(u) > 0$$

- $\lambda(u)$ is monotone decreasing $\Leftrightarrow$ concave flux

$$\frac{d\lambda(u)}{du} = \lambda'(u) = f''(u) < 0$$

- $\lambda(u)$ has extrema for some u ⤳ non-convex, non-concave flux

$$\frac{d\lambda(u)}{du} = \lambda'(u) = f''(u) = 0$$

Exercise: characterize flux of inviscid Burger's equ, traffic flow equ, Buckley-Leverett equ

3) System of conservation laws

Consider hyperbolic system

$$U_t + F(U)_x = 0$$

$\leadsto$ can locally transform into decoupled system

hyperbolicity $\Rightarrow \exists\, Q = Q(x,t,U)$, $A = \frac{\partial F}{\partial U} = Q \Lambda Q^{-1}$

where $\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_m \end{pmatrix}$, $\lambda_i = \lambda_i(x,t,U)$

$Q = (r_1, \dots, r_m)$, eigenvectors r_i depend on x,t,U

columns

($\leadsto$ <u>locally</u>: $V_t + \Lambda V_x = 0$

meaning of V depends on U)

Λ and characteristics depend on state vector U

$\leadsto$ <u>the state/solution is "self-propagating"</u>

1.3.2 Domain of dependence & range of influence

<u>Def</u>: The domain of dependence $\mathcal{D}(x,t) \subseteq \mathbb{R}^n$ is the domain of the initial data $u^0(x)$ that entirely determines the solution $u(x,t)$ of a hyperbolic system $u_t + \sum_{i=1}^n A_i u_{x_i} = 0$.

<u>Theorem</u>: Let u be solution of symmetric hyperbolic system

$$u_t + \sum_{i=1}^n A_i u_{x_i} = 0,$$

with A_i having constant coefficients.

Let $(x_0, t_0) \in \mathbb{R}^n \times (0,\infty)$, $0 \leq t_1 \leq t_0$, and

$$B \equiv \{ x \in \mathbb{R}^n \mid |x - x_0| \leq M t_0 \}$$

$$C \equiv \{ x \in \mathbb{R}^n \mid |x - x_0| \leq M (t_0 - t_1) \}$$

with

$$M \equiv \max_{\substack{i=1,\dots,m \\ \xi \in \mathbb{R}^n,\ |\xi| = 1}} \{ \lambda_i^\xi \},$$

where λ_i^ξ are the eigenvalues of $A(\xi) = \sum_{i=1}^n A_i \xi_i$.

If $|u| \equiv 0$ on B, then $|u| \equiv 0$ on C.

Proof: Evans.

Remarks: 1) $\mathcal{D}$ is determined by fastest and slowest characteristic speed

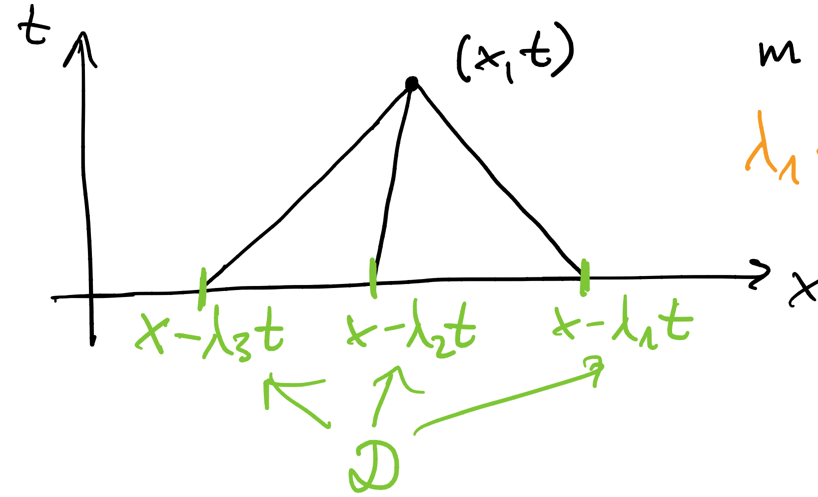
2) $\mathcal{D}$ is bounded, which is because characteristic speeds are finite $\lambda_i^\xi \neq 0$, $\lambda_i^\xi < \infty$.
Boundedness also applies to non-linear systems.

"information propagates at finite speed"

Example: Linear 1D system $U_t + A U_x = 0$ with const. coefficients

$$\rightsquigarrow \quad U(x,t) = \sum_{i=1}^{m} v_i^0(x - \lambda_i t)\, r_i \quad \text{(see above)}$$

$$\Rightarrow \quad \mathcal{D}(x,t) = \{ x - \lambda_i t \mid i = 1, \dots, m \}$$



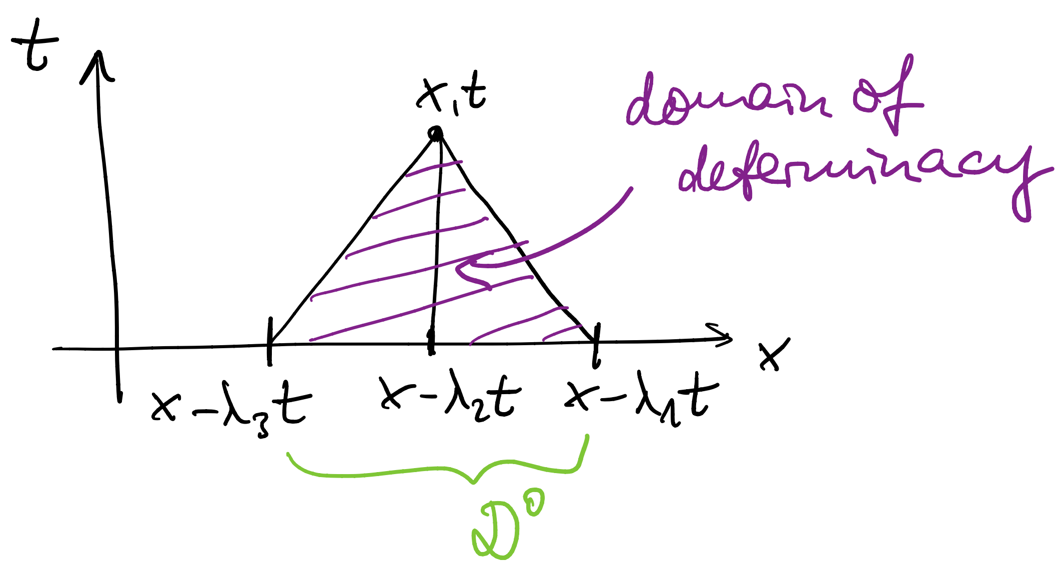
$m = 3$ system

$\lambda_1 < 0 < \lambda_2 < \lambda_3$

(for non-linear systems may be entire interval $[x-\lambda_3 t, x-\lambda_1 t] \subseteq \mathbb{R}$)

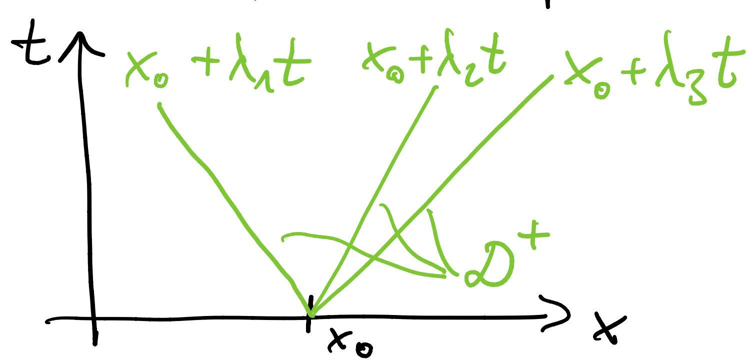
Def: Given a domain $\mathcal{D} \subseteq \mathbb{R}^n$ of the initial data, the <u>domain of determinacy</u> $\mathcal{D}^-$ is the set of points $(x,t) \in \mathcal{D}^- \subseteq \mathbb{R}^n \times (0,\infty)$, within the domain of existence of $U(x,t)$, in which $U(x,t)$ is solely determined by initial data on $\mathcal{D}$.

<u>Example:</u>



Def: Let $x_0 \in \mathbb{R}^n$. The range of influence $\mathcal{D}^+ \subseteq \mathbb{R}^n \times (0,\infty)$ is the set of points (x,t) in which the solution $U(x,t)$ is influenced by initial data $U^0(x_0)$ at the point $(x_0, 0)$.

<u>Example:</u>



$m=3$ system

$\lambda_1 < 0 < \lambda_2 < \lambda_3$