Chap. 4: properties of conservation laws (theoretical background)

consider system of conservation laws (1D):

$$\text{(IVP)} \qquad \begin{aligned} u_t + f(u)_x &= 0 \quad \text{in } \mathbb{R} \times (0,\infty) \\ u(x,0) &= u_0(x) \quad \text{on } \mathbb{R} \times \{t=0\} \end{aligned}$$

$$u = (u^1, \dots, u^m) : \mathbb{R} \times [0,\infty) \to \mathbb{R}^m$$

- show that in general classical solutions (smooth $\sim$ C^1) do not exist (globally)
- define weak solutions & entropy condition to guarantee uniqueness of weak solution
- Riemann problem: discontinuous initial data & non-linear waves
  $\to$ centrepiece of numerical algorithms

4.1 Local existence of classical solutions

Even in the case of smooth data f and u_0 there are in general no continuous solutions that exist <u>globally</u> in time.

<u>Example:</u> <u>Inviscid Burger's equ</u>

Consider scalar conservation law

$$u_t + f(u)_x = 0 \quad \text{with } f(u) = \frac{1}{2}u^2$$

and $u_0 \in C^\infty(\mathbb{R})$, $u_0(x) = \begin{cases} 1, & x \in (-\infty, -1] \\ 0, & x \in [1, \infty) \\ u_0' \leq 0 \end{cases}$

Remember (Sec. 1.3.1): characteristic $\Gamma_{x_0}^t = (\gamma(t), t)$

defined by $\quad \gamma'(t) = \frac{d\gamma}{dt} = \lambda(u) = f'(u).$

$$\gamma(0) = x_0$$

$$\leadsto \begin{cases} \frac{d}{dt} u(\gamma(t), t) = \partial_t u + \gamma' \partial_x u = u_t + f(u)_x = 0 \\ \gamma'(t) = f'(u(\gamma(t), t)) = f'(u(\gamma(0), 0)) = f'(u_0(x_0)) \\ \qquad = \text{const.} \end{cases}$$

$\Longrightarrow$ u const. along $\Gamma_{x_0}^t$

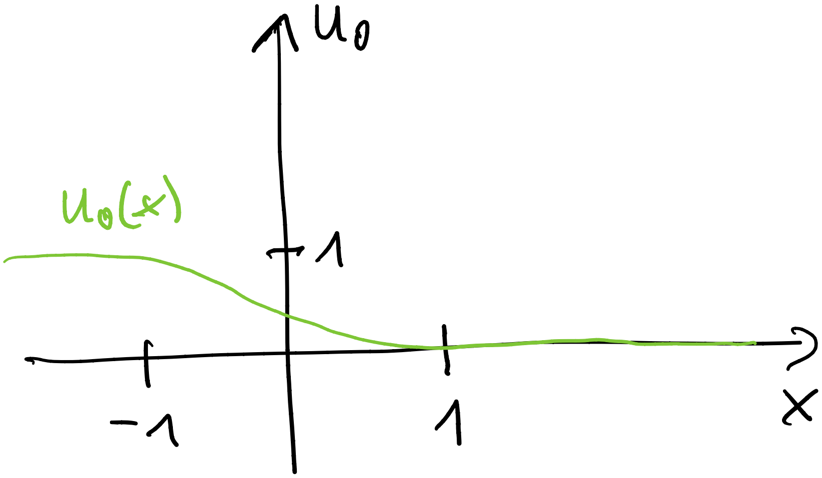
$\Gamma_{x_0}^t$ is a straight line

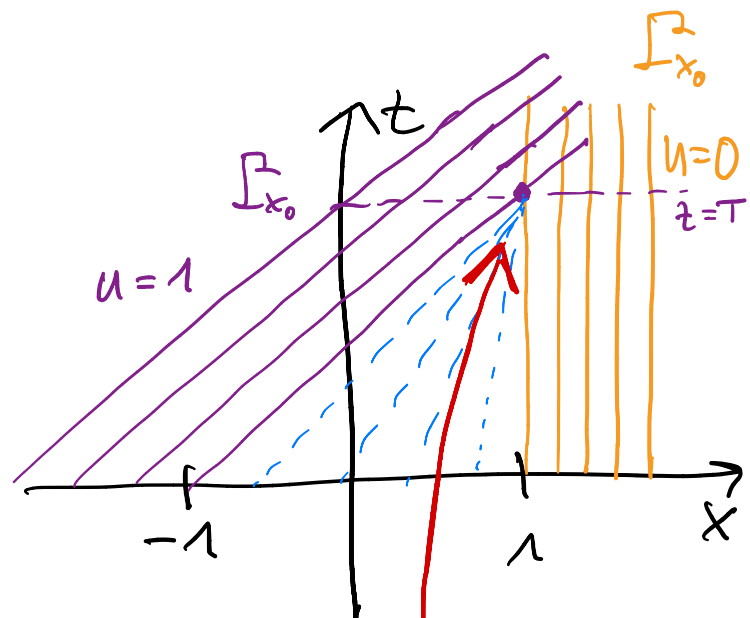
Here: $\gamma'(t) = f'(u_0(\gamma(0))) = u_0(\gamma(0))$

$$u(\gamma(t),t) = u(\gamma(0),0) = u_0(\gamma(0))$$

$\Gamma^2_{x_0}$ starting at $\left\{\begin{matrix} x_0 \in (-\infty,-1] \\ x_0 \in [1,\infty) \end{matrix}\right\}$ have slope $\left\{\begin{matrix} 1 \\ 0 \end{matrix}\right\}$,

and $u = \left\{\begin{matrix} 1 \\ 0 \end{matrix}\right\}$ along these lines.



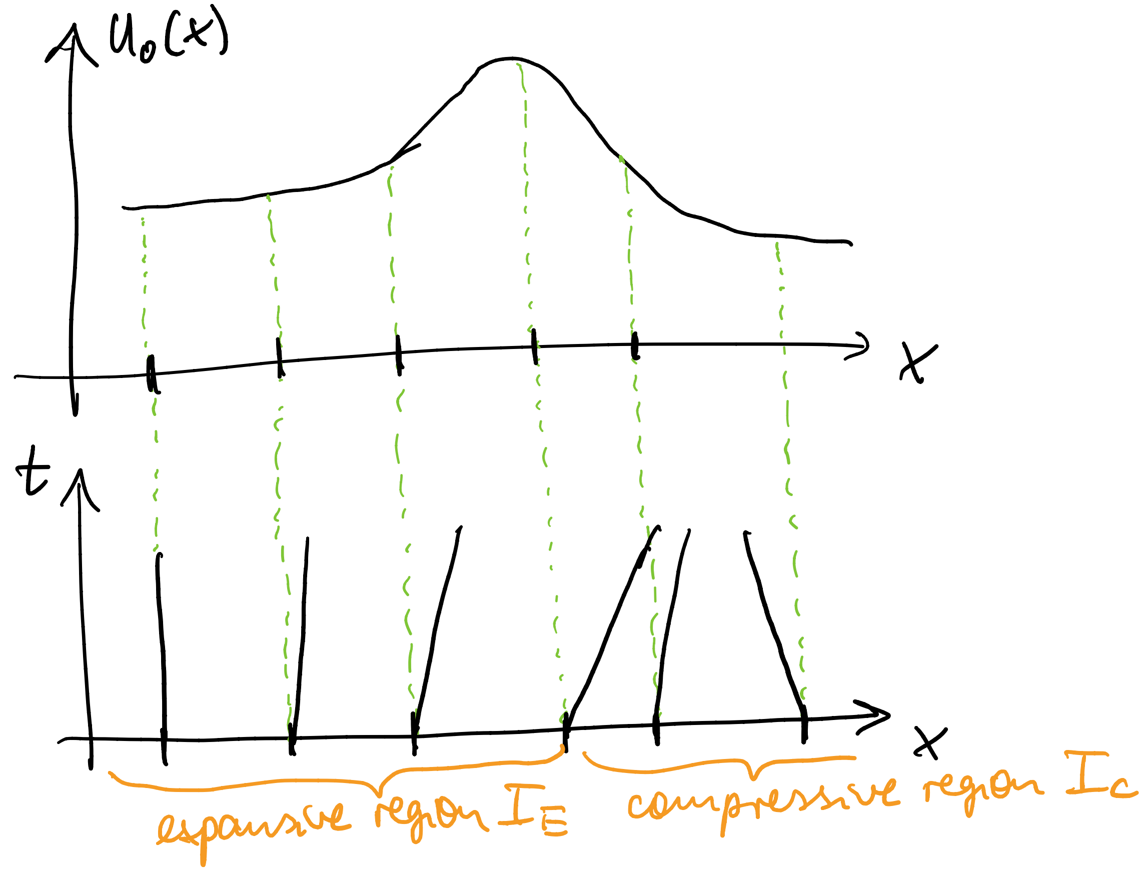


characteristics meet at finite time T

$\Rightarrow$ u cannot be continuous

i.e. $u \notin C^0(\mathbb{R}\times[0,\infty))$!

Example: wave steepening Consider scalar CL with smooth initial data $u_0(x)$ and convex flux function $\lambda'(u) = f''(u) > 0$



- "larger values of $u_0(x)$ will travel faster than smaller values of $u_0(x)$"
- I_E & I_C reversed for concave flux
- existence of I_E & I_C leads to crossing of characteristics and multi-valued solutions

$\rightsquigarrow$ "the wave breaks" ($u_x \to \infty$) at first crossing

write $u(x,t)$ in terms of characteristics $\gamma(t)$
(see existence result below)

$$u(x,t) = u(\gamma(t),t) = u_0\big(\underbrace{x - \lambda(u_0(x_0))t}_{=x_0}\big)$$

$$\rightsquigarrow \quad u_x = u_0'(x_0)\,\frac{\partial x_0}{\partial x} \quad , \quad x = x_0 + \lambda(u_0(x_0))t$$

$$\downarrow$$

$$\frac{\partial x}{\partial x_0} = 1 + \lambda'(u_0(x_0))\,u_0'(x_0)\,t = 1 + \lambda_x t$$

$$\frac{1}{u_x} = 0 \iff \frac{\partial x}{\partial x_0} = 0$$

$$\Updownarrow$$

$$\boxed{t_{break} = -\frac{1}{\lambda_x(u_0(x_0))}}$$

breaking first occurs for characteristic Γ_{x_0} for which $\lambda_x(u_0(x_0)) < 0$ and $|\lambda_x(u_0(x_0))|$ is maximal.

Theorem (local existence of classical solution):

Assume $f \in C^2(\mathbb{R})$, $u_0 \in C^1(\mathbb{R})$, $|f''|, |u_0'| \leq \text{const}$. Then there exists $T > 0$ such that (IVP) for scalar conservation law has a classical solution $u \in C^1(\mathbb{R} \times [0,T))$.

Proof: For $(x,t) \in \mathbb{R} \times [0,\infty)$ consider characteristic $\Gamma_{x_0} = (\gamma(t), t)$ passing through (x,t):

(i) $$x_0 = x - \lambda(u_0(x_0))t = x - f'(u_0(x_0))t$$
$$= x - f'(u(x,t))t$$

(Note: u const. along Γ_{x_0})

(ii) $$u(x,t) = u(x_0, 0) = u_0(x_0) = u_0(x - f'(u(x,t))t)$$

Define $F(v,x,t) \equiv v - u_0(x - f'(v)t)$, then

$$F(u_0(x), x, 0) = 0 \quad \text{and} \quad \partial_v F = 1 + u_0' f''(v)t \neq 0$$

Implicit function theorem $\Rightarrow$ for $t < T$ suff. small $\exists$ function u satisfying (ii)

differentiate (i):

$$\frac{\partial x}{\partial t} = 0 = \frac{\partial x_0}{\partial t} + f'(u_0(x_0)) + f''(u_0(x_0))\, u_0'(x_0) \frac{\partial x_0}{\partial t} t$$

$$= f'(u_0(x_0)) + \left[1 + f''(u_0(x_0))\, u_0'(x_0)\, t\right] \frac{\partial x_0}{\partial t}$$

$$\Leftrightarrow \frac{\partial x_0}{\partial t} = - \frac{f'(u_0(x_0))}{1 + f''(u_0(x_0))\, u_0'(x_0)\, t}$$

$$\frac{\partial x}{\partial x} = 1 = \frac{\partial x_0}{\partial x} + f''(u_0(x_0))\, u_0'(x_0)\, t \frac{\partial x_0}{\partial x}$$

$$\Leftrightarrow \frac{\partial x_0}{\partial x} = \frac{1}{1 + f''(u_0(x_0))\, u_0'(x_0)\, t}$$

differentiate (ii): (u const. along Γ)

$$u_t = u_0'(x) \frac{\partial x_0}{\partial t} = - \frac{u_0' f'(u)}{1 + f''(u)\, u_0'\, t}$$

$$u_x = u_0'(x) \frac{\partial x_0}{\partial x} = \frac{u_0'}{1 + f''(u)\, u_0'\, t}$$

$\Rightarrow$ $u_t + f'(u)\, u_x = 0$ and $u \in C^1(\mathbb{R})$

for $(x,t) \in \mathbb{R} \times [0,T)$ and T sufficiently small.

$\square$

<u>Remark:</u> <u>Systems of CLs in 1D:</u>

<u>Theorem</u>: Let $u_0 \in C_0^1(\mathbb{R};\mathbb{R}^m)$ and $u_t + f(u)_x = 0$ strictly hyperbolic. Then $\exists$ $T > 0$ such that the IVP admits a classical solution $u : [0,T) \times \mathbb{R} \to \mathbb{R}^m$.

<u>Proof</u>: Bressan pp. 67–70.

4.2 Weak solutions

4.1 Definition

Previous examples suggest to allow discontinuous (classically non-differentiable) solutions.

<u>Idea:</u> consider integral form of conservation law, then functions must only be integrable, e.g. $u \in L^\infty(\mathbb{R}\times[0,\infty);\mathbb{R}^m)$ or L^p

$$\|u\|_p = \left[\int_0^\infty \int_{\mathbb{R}} |u|^p \, dx\, dt\right]^{1/p} < \infty, \quad \|u\|_\infty = \inf\left\{ C \geq 0 \,\middle|\, \begin{array}{l} |u(x)| \leq C \\ \text{almost everywhere} \end{array} \right\} < \infty$$

$\leadsto$ consider for now u smooth solution of (IVP) and $v \in \underset{\substack{\uparrow \\ \text{compact support}}}{C_0^\infty}(\mathbb{R}\times[0,\infty);\mathbb{R}^m)$:

$$\Rightarrow \quad v \cdot u_t + v \cdot f(u)_x = 0$$

$\downarrow$ integrate

$$\int_0^\infty \int_{\mathbb{R}} v \cdot u_t \, dx\, dt + \int_0^\infty \int_{\mathbb{R}} v \cdot f(u)_x = 0$$

$\downarrow$ integrate by parts

$$\int_0^\infty\int_{\mathbb{R}} v_t \cdot u \, dx\,dt - \int_{\mathbb{R}} \Big[v\cdot u\Big]_{t=0}^{t=\infty} dx$$

$$+ \int_0^\infty\int_{\mathbb{R}} v_x \cdot f(u)\,dx\,dt - \int_0^\infty \underbrace{\Big[v\cdot f(u)\Big]_{x=-\infty}^{x=\infty}}_{=0} dt = 0$$

$$\Leftrightarrow \int_0^\infty\int_{\mathbb{R}} \big(v_t \cdot u + v_x \cdot f(u)\big)\,dx\,dt + \int_{\mathbb{R}} v(x,0)\, u_0(x)\,dt = 0$$

<u>Def</u>: Consider (IVP) with $u_0 \in L^\infty(\mathbb{R};\mathbb{R}^m)$.
Then $u \in L^\infty(\mathbb{R}\times[0,\infty);\mathbb{R}^m)$ is called a <u>weak solution</u> (integral solution, solution in the distributional sense) if and only if

$$\int_0^\infty \int_{\mathbb{R}} \big(u\cdot v_t + f(u)\cdot v_x\big)\,dx\,dt + \int_{\mathbb{R}} u_0 \cdot v(x,0)\,dx = 0$$

for all $v \in C_0^\infty(\mathbb{R}\times[0,\infty);\mathbb{R}^m)$.

compact support

Remark: if u is a weak solution that happens to be $C^1(\mathbb{R}\times[0,\infty);\mathbb{R}^m)$, then u is a classical solution of the (IVP).

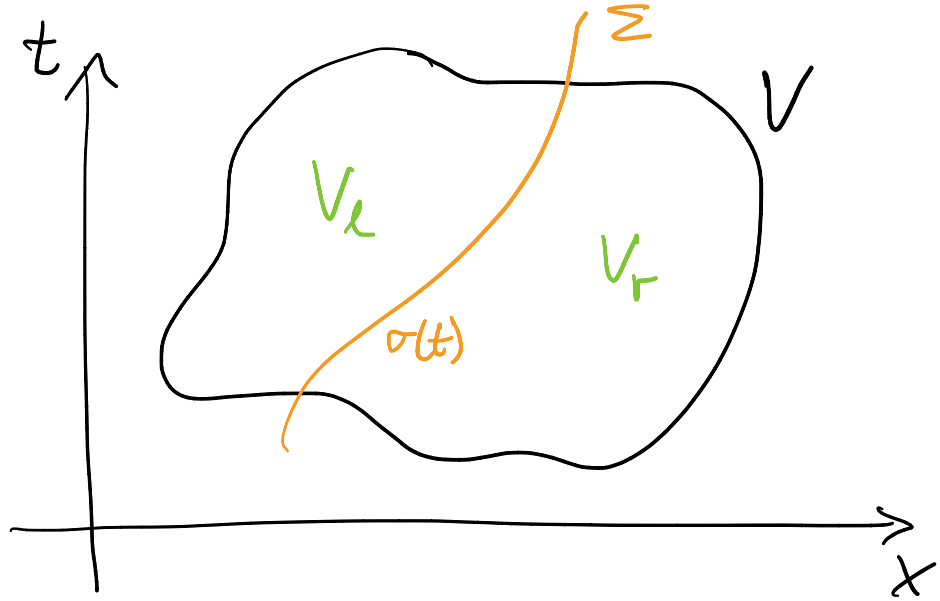
4.2 Behaviour near discontinuities (Jump conditions)

Theorem (Rankine-Hugoniot):

Consider $V \subset \mathbb{R}\times(0,\infty)$ separated by a smooth curve $\Sigma: t \mapsto (\sigma(t),t)$ in two parts V_l and V_r. Let $u \in L^1(\mathbb{R}\times(0,\infty))$ such that $u_l \equiv u|_{V_l} \in C^1(\overline{V_l})$ and $u_r \equiv u|_{V_r} \in C^1(\overline{V_r})$ and u_l, u_r locally satisfy the IVP on V_l, V_r in the classical sense. Then u is a weak solution of the IVP if and only if

$$(RH) \quad f(u_l(\sigma(t),t)) - f(u_r(\sigma(t),t)) = [u_l(\sigma(t),t) - u_r(\sigma(t),t)]\,\sigma'(t)$$

for all $t>0$.



<u>Notation:</u> (RH) is often written

$$f_\ell - f_r = \sigma'(u_\ell - u_r) \quad \text{along } \Sigma$$

or

$$[\![f]\!] = \sigma' [\![u]\!]$$

where $[\![\ ]\!]$ means "jump across the curve Σ"

<u>Proof:</u> Take $v \in C_0^\infty(V)$ (otherwise can always enlarge V)

compact support in V
does not necessarily vanish along Σ

Let $\nu = (\nu^1, \nu^2)$ denote the outer unit normal of V_ℓ ⟹ $\nu(t) = \frac{1}{\sqrt{1+\sigma'(t)^2}}(1, -\sigma'(t))$. Then:

$$0 \overset{\text{Def}}{=} \int_0^\infty \int_{\mathbb{R}} [u \cdot v_t + f(u) \cdot v_x] \, dx\,dt + \overbrace{\int_{\mathbb{R}} u_0 \cdot v(x,0)\, dx}^{=0,\ v \in C_0^\infty(V)}$$

$$= \iint_{V_\ell} [u \cdot v_t + f(u) \cdot v_x] \, dx\,dt + \iint_{V_r} [u \cdot v_t + f(u) \cdot v_x] \, dx\,dt$$

$u \in C^1$ on V_ℓ, V_r

$$= -\int_{V_\ell} \underbrace{[u_t + f(x)_x]}_{=0 \text{ on } V_\ell} \cdot v \, dx\,dt + \int_{\partial V_\ell} (u \cdot v\, \nu^2 + f(u) \cdot v\, \nu^1) \, d\ell$$

$\nu_r = -\nu_\ell = -\nu$

$$- \int_{V_r} \underbrace{[u_t + f(u)_x]}_{=0 \text{ on } V_r} \cdot v \, dx\,dt - \int_{\partial V_r} (u \cdot v\, \nu^2 + f(u) \cdot v\, \nu^1) \, d\ell$$

v vanishes on $\partial V_\ell \setminus \Sigma$, $\partial V_r \setminus \Sigma$

$$= \int_\Sigma \left[(u_\ell - u_r) \nu^2 + (f(u_\ell) - f(u_r)) \nu^1 \right] \cdot v \, d\ell$$

Since v arbitrary

$$\Rightarrow \quad f(u_\ell) - f(u_r) = \sigma' (u_\ell - u_r)$$

$\square$

4.3 Entropy condition

In general, the (IVP) has no unique weak solution, i.e. the (IVP) for weak solutions is not well-posed.

→ we will define a "selection criterion" (entropy condition) to pick the correct physical solution and restore well-posedness.

Example (non-uniqueness): Consider 1D CL:

inviscid Burger's equ.

$$u_t + \partial_x\left(\frac{u^2}{2}\right) = 0, \quad x \in \mathbb{R}, \ t > 0$$

$$u_0(x) = \begin{cases} 0, & x \le 0 \\ 1, & x > 0 \end{cases}$$

Define

$$u_1(x,t) \equiv \begin{cases} 0, & x < t/2 \\ 1, & x > t/2 \end{cases}$$

$$u_2(x,t) \equiv \begin{cases} 0, & x < 0 \\ x/t, & 0 \le x \le t \\ 1, & t < x \end{cases}$$

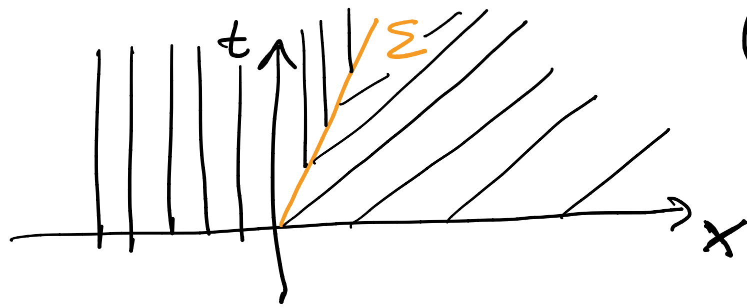
Both u_1, u_2 are piecewise C^1 and satisfy initial condition.

- u_1 satisfies RH conditions along $\Sigma = (\sigma(t), t)$ with $\sigma(t) = \frac{1}{2}t$:

$$\sigma'(t) = \frac{1}{2} \quad \text{and} \quad \frac{f(u_l) - f(u_r)}{u_l - u_r} = \frac{\frac{1}{2}u_l^2 - \frac{1}{2}u_r^2}{u_l - u_r}$$

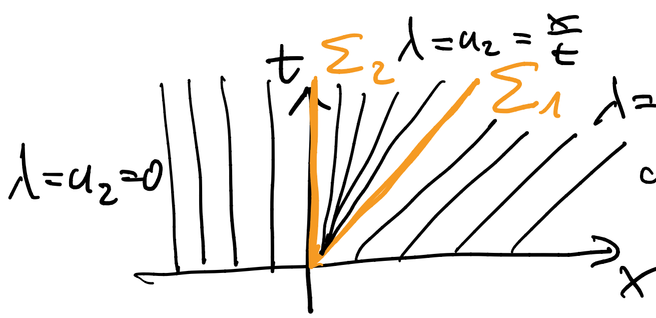
$$= \frac{1}{2}(u_l + u_r) = \frac{1}{2} \quad \checkmark$$

$\Rightarrow$ u_1 weak solution



(rarefaction shock, entropy violating)

- can check: u_2 satisfies RH conditions along $\Sigma_1 = (\sigma_1(t), t) = (t, t)$ and $\Sigma_2 = (\sigma_2(t), t) = (0, t)$



characteristics $\lambda = u_2$

$\Rightarrow$ u_2 also weak solution

In reality, discontinuities are never arbitrarily sharp, but are rather "smeared out" by some intrinsic viscosity of the fluid. Physically correct solutions should arise as the limit of solutions to the "regularized system"

$$u_t^\varepsilon + f(u^\varepsilon)_x - \underbrace{\varepsilon u_{xx}^\varepsilon}_{\text{"small viscosity effect"}} = 0 \quad \text{in } \mathbb{R}\times(0,\infty)$$

as $\varepsilon \to 0$, i.e. as the problem approaches the inviscid problem.

Following Theorem provides the "viscosity method" of how to select the correct weak solution:

Theorem: Let $u_0 \in L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and $f_i, B \in C^2(\mathbb{R}^n \times (0,\infty) \times \mathbb{R})$ with bounded derivatives. Then for any $\varepsilon > 0$ there exists a uniquely defined classical solution u^ε of

$$u_t + \sum_{i=1}^{n} \partial_i f_i(x,t,u) + B(x,t,u) = \varepsilon \Delta u \quad \text{in } \mathbb{R}^n \times (0,\infty)$$

$$u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^n$$

such that $u^\varepsilon \xrightarrow{\varepsilon \to 0} u$ almost everywhere in $\mathbb{R}^n \times (0,\infty)$ for some u that is a weak solution of ("viscosity limit")

$$\partial_t u + \sum_{i=1}^n \partial_i f_i(x,t,u) + B(x,t,u) = 0 \text{ in } \mathbb{R}^n \times (0,\infty)$$
$$u(x,0) = u_0(x) \text{ in } \mathbb{R}^n$$

Proof: omitted, see references in Kroener (p. 22).

Remark: For general systems the previous theorem is still an open problem.

Now: want necessary criterion for a weak soln. u to be the viscosity limit defined in the theorem.

Definition: Two smooth functions $\Phi, \Psi \in C^2(\mathbb{R}^m; \mathbb{R})$ are called an entropy / entropy-flux pair for the system of CLs $u_t + f(u)_x = 0$ if

(i) Φ is convex $(D^2\Phi(z) y \cdot y > 0 \ \forall z, y \in \mathbb{R}^m)$

(ii) $D\Phi(z)\, Df(z) = D\Psi(z), \quad z \in \mathbb{R}^m$

Definition: A weak solution u of the (IVP) is called an entropy solution if for any entropy pair Φ, Ψ it satisfies the inequality

$$\boxed{\Phi(u)_t + \Psi(u)_x \leq 0 \quad \text{in } \mathbb{R}\times(0,\infty)}$$

in the distributional sense, i.e.

$$\int_0^\infty \int_{\mathbb{R}} \Phi(u) v_t + \Psi(u) v_x \, dx\, dt \geq 0 \quad \forall \underset{v \geq 0}{v \in C_0^\infty(\mathbb{R}\times(0,\infty))}.$$

(Note: the latter expression is obtained by assuming u smooth, multiplying by v and integrating by parts)

Theorem: The viscosity limit $u = \lim_{\varepsilon \to 0} u^\varepsilon$ of

$$u^\varepsilon_t + f(u^\varepsilon)_x - \varepsilon u^\varepsilon_{xx} = 0 \quad \text{in } \mathbb{R}\times(0,\infty)$$

$$u^\varepsilon = u_0 \quad \text{on } \mathbb{R}\times\{t=0\}$$

is an entropy solution of

$$u_t + f(u)_x = 0 \quad \text{on } \mathbb{R}\times(0,\infty)$$

$$u = u_0 \quad \text{on } \mathbb{R}\times\{t=0\}$$

provided that u^ε is uniformly bounded in L^∞ and $u^\varepsilon \to 0$ as $|x| \to \infty$ sufficiently rapidly.

Proof: 1) choose entropy pair Φ, Ψ.

$$D\Phi(u^\varepsilon)\cdot\left(u_t^\varepsilon + \underbrace{f(u^\varepsilon)_x}_{=Df(u^\varepsilon)u_x^\varepsilon} - \varepsilon u_{xx}^\varepsilon\right) = 0$$

$$\Leftrightarrow \quad \Phi(u^\varepsilon)_t + \Psi(u^\varepsilon)_x = \varepsilon D\Phi(u^\varepsilon)u_{xx}^\varepsilon$$

$$= \varepsilon \Phi(u^\varepsilon)_{xx} - \varepsilon \underbrace{\left(D^2\Phi(u^\varepsilon)u_x^\varepsilon\right)\cdot u_x^\varepsilon}_{\substack{\geq 0 \\ (\Phi \text{ convex})}}$$

multiply by $\begin{cases} v \in C_0^\infty(\mathbb{R}\times(0,\infty)) \\ v \geq 0 \end{cases}$

integrate

- $$\int_0^\infty \int_{\mathbb{R}} \Phi(u^\varepsilon)_t v + \Psi(u^\varepsilon)_x v \, dx\, dt$$

$$\overset{\substack{\text{integrate} \\ \text{by parts}}}{=} \underbrace{\int_{\mathbb{R}} \left[\Phi(u^\varepsilon)v\right]_0^\infty dx + \int_0^\infty \left[\Psi(u^\varepsilon)v\right]_{-\infty}^\infty dx}_{=0,\ v \in C_0^\infty(\mathbb{R}\times(0,\infty))} - \int_0^\infty\int_{\mathbb{R}} \left[\Phi(u^\varepsilon)v_t + \Psi(u^\varepsilon)v_x\right] dx\,dt$$

$$\Rightarrow \int_0^\infty\int_{\mathbb{R}} \left[\Phi(u^\varepsilon)v_t + \Psi(u^\varepsilon)v_x\right] dx\,dt \overset{\substack{\Phi \text{ convex} \\ v \geq 0}}{\geq} - \int_0^\infty\int_{\mathbb{R}} \varepsilon \Phi(u^\varepsilon)_{xx} v \, dx\,dt$$

$$\overset{\substack{2\times \text{ integrate} \\ \text{by parts}}}{=} - \int_0^\infty\int_{\mathbb{R}} \varepsilon \Phi(u^\varepsilon) v_{xx} \, dx\,dt$$

$\varepsilon \to 0$

$u^\varepsilon \to u$

$$\int_0^\infty \int_{\mathbb{R}} \left[\Phi(u)\, v_t + \Psi(u)\, v_x\right] dx\, dt \geq 0$$

$\Rightarrow \quad u = \lim_{\varepsilon \to 0} u^\varepsilon$ <u>satisfies the entropy condition</u>

2) choose $v \in C_0^\infty(\mathbb{R} \times [0,\infty); \mathbb{R}^m)$:

$$\int_0^\infty \int_{\mathbb{R}} \left[u_t^\varepsilon + f(u^\varepsilon)_x - \varepsilon\, u_{xx}^\varepsilon\right] \cdot v \; dx\, dt = 0$$

$$\Big\downarrow \text{ integrate by parts}$$

$$-\int_0^\infty \int_{\mathbb{R}} u^\varepsilon \cdot v_t \, dx\, dt + \int_{\mathbb{R}} \underbrace{\left[u^\varepsilon \cdot v\right]_0^\infty}_{= -\int_{\mathbb{R}} u^\varepsilon(x,0)\cdot v\, dx \;\; (u^\varepsilon \text{ falls of } |x| \to \infty)} dx$$

$$-\int_0^\infty \int_{\mathbb{R}} f(u^\varepsilon)\, v_x \, dx\, dt - \int_0^\infty \int_{\mathbb{R}} \varepsilon\, u^\varepsilon\, v_{xx}\, dx\, dt = 0$$

$$\Leftrightarrow \int_0^\infty \int_{\mathbb{R}} \left[u^\varepsilon \cdot v_t + f(u^\varepsilon) \cdot v_x + \varepsilon\, u^\varepsilon \cdot v_{xx}\right] dx\, dt + \int_{\mathbb{R}} u_0^\varepsilon \cdot v\, dx = 0$$

$$\Big\downarrow \begin{array}{l} \varepsilon \to 0 \\ u^\varepsilon \to u \end{array}$$

$$\int_0^\infty \int_{\mathbb{R}} \left[u \cdot v_t + f(u) \cdot v_x \right] dx\,dt + \int_{\mathbb{R}} u_0 \cdot v \, dx = 0$$

$\Rightarrow$ u is weak solution.

$\square$

<u>Theorem (Uniqueness of entropy solution for scalar conservation laws):</u>

There exists – up to a measure zero – at most one entropy solution for the scalar conservation law

$$\begin{aligned} u_t + f(u)_x &= 0 \quad \text{on } \mathbb{R} \times (0,\infty) \\ u &= u_0(x) \quad \text{on } \mathbb{R} \times \{t=0\} \end{aligned}$$

Proof: Evans Sec. 11.4.3, p. 652

<u>Remarks:</u> 1) <u>Well-posedness for weak soln.</u>

Note that the previous theorem restores well-posedness, given the entropy selection/admissibility criterion

2) Interpretation of entropy criterion

$$\Phi(u)_t + \Psi(u)_x \leq 0$$

In physical applications ($\to$ Euler eqns)

$\Phi(u)$ will be the (thermodynamic) entropy and $\Psi(u)$ the entropy flux.

$\rightsquigarrow$ entropy evolves according to its flux, it can sharply increase ($\to$ across shocks) but it cannot decrease

$\rightsquigarrow$ entropy criterion serves as additional constraint the conservation law "does not know about" and that selects the thermodynamically/physically admissible mathematical solution to the CL.

4.4 The Riemann problem & Lax entropy condition

In this section we shall discuss the <u>Riemann problem</u> for hyperbolic systems of conservation laws, which is the IVP:

$$u_t + f(u)_x = 0$$
$$u(x,0) = u_0(x) = \begin{cases} u_l, & x<0 \\ u_r, & x>0 \end{cases}$$

with u_l, u_r constants. The solution of this IVP plays a central role in constructing numerical schemes to find weak solutions of systems of CLs.

4.4.1 First examples

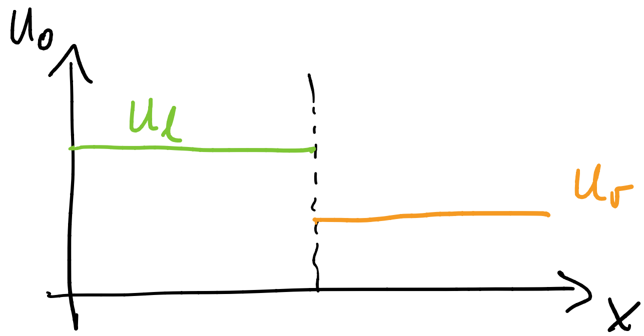
Consider simple example (inviscid Burgers equ.)

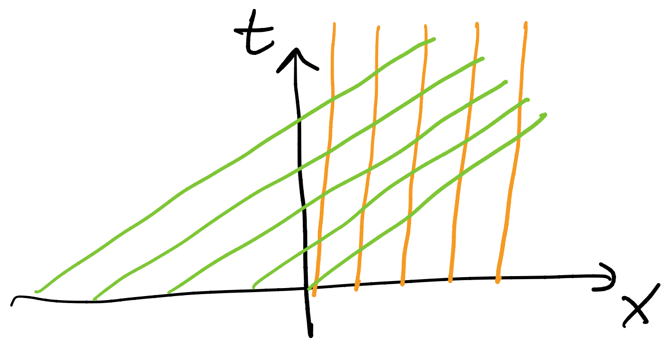
$$u_t + f(u)_x = 0, \quad f(u) = \frac{u^2}{2} \quad \text{convex}$$
$$u(x,0) = u_0(x) = \begin{cases} u_l, & x<0 \\ u_r, & x>0 \end{cases}$$

$\rightsquigarrow$ const. characteristic speeds: $\lambda'(u) = f''(u) = 1 > 0$

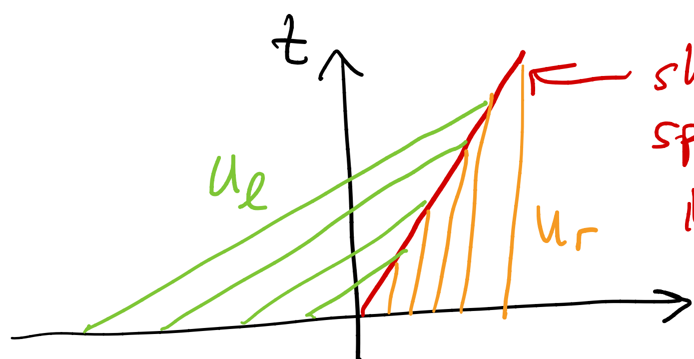
① assume $u_\ell > u_r$: "compressive data"

$$\lambda_\ell = \lambda(u_\ell) = u_\ell > u_r = \lambda(u_r) = \lambda_r$$





characteristics



RH jump conditions:

$$s = \sigma'(t) = \frac{f(u_\ell) - f(u_r)}{u_\ell - u_r} = \frac{1}{2} \frac{u_\ell^2 - u_r^2}{u_\ell - u_r}$$

$$= \frac{1}{2}(u_\ell + u_r)$$

weak solution:

$$u(x,t) = \begin{cases} u_\ell, & x - st < 0 \\ u_r, & x - st > 0 \end{cases}$$

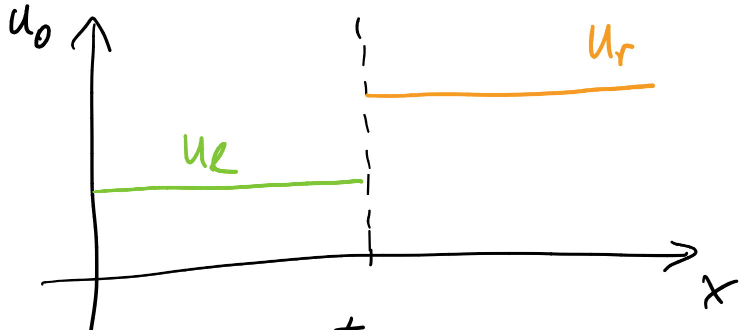
$$\boxed{\lambda(u_\ell) > s > \lambda(u_r)}$$

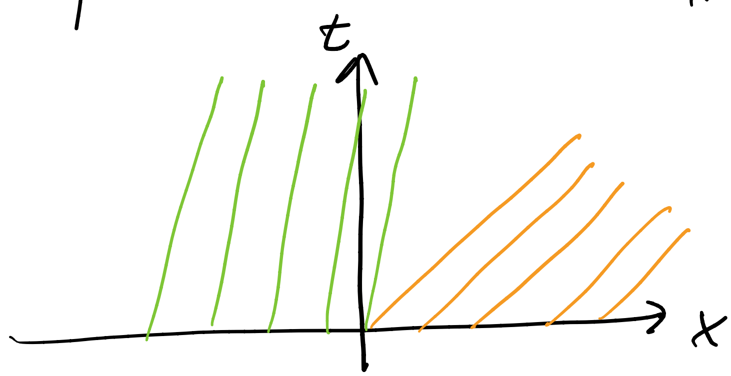
Lax entropy condition

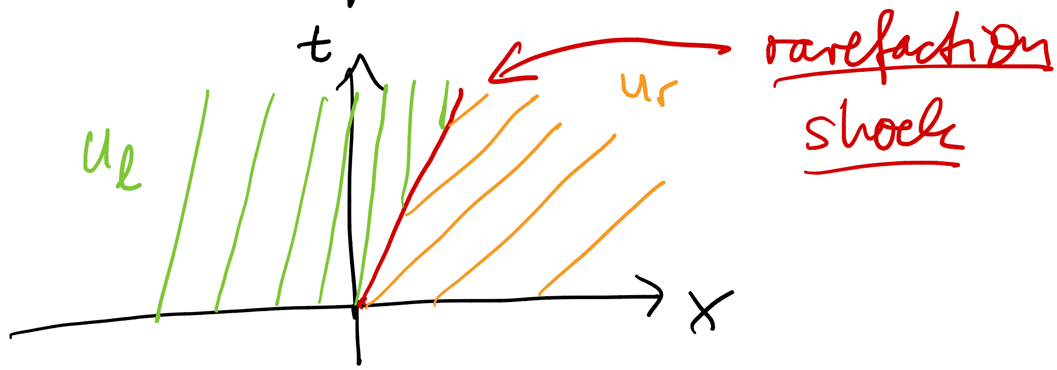
($\to$ will see satisfies entropy condition)

② assume $u_\ell < u_r$: "expansive data"

$$\lambda_\ell = \lambda(u_\ell) \qquad \lambda_r = \lambda(u_r)$$







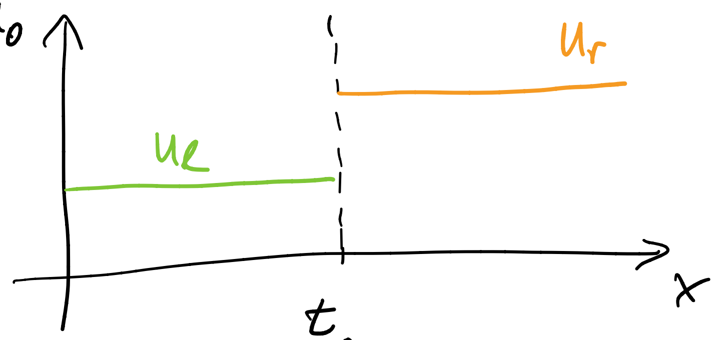
Similar to above, one weak solution is:

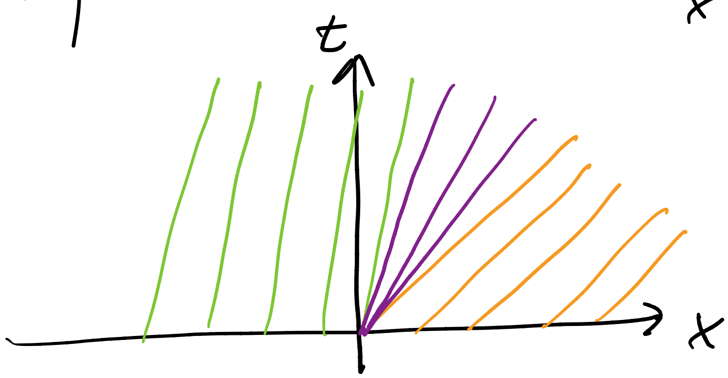
$$s = \frac{1}{2}(u_\ell + u_r), \quad u(x,t) = \begin{cases} u_\ell, & x - st < 0 \\ u_r, & x - st > 0 \end{cases}$$

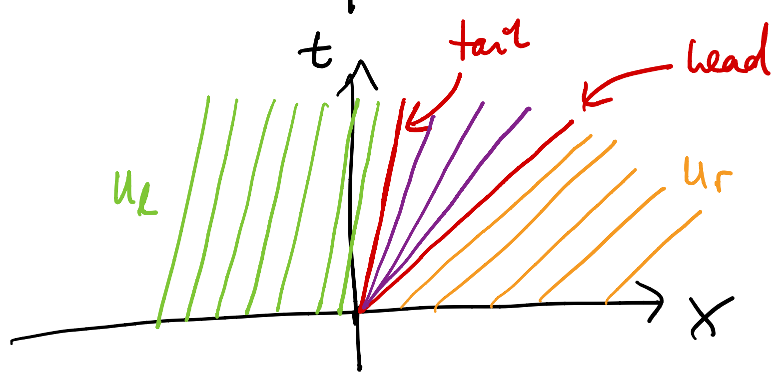
But: $\boxed{\lambda_\ell < s < \lambda_r}$ entropy-violating shock

discontinuity has not arisen from compression, characteristics diverge from the discontinuity

Another possibility:







all $\gamma(t)$ of wave emanate from same point

$$\left\{\begin{array}{lll} u(x,t) = u_\ell , & \frac{x}{t} \le \lambda_\ell = u_\ell \\ u(x,t) = \frac{x}{t} , & \lambda_\ell < \frac{x}{t} < \lambda_r \\ u(x,t) = u_r , & \frac{x}{t} \ge \lambda_r = u_r \end{array}\right\}$$

centered rarefaction wave

larger values of $u_0(x)$ propagate faster than smaller values $\rightarrow$ wave spreads and flattens

$\rightsquigarrow$ "rarefaction" (non-linear phenomenon such as shocks)

③ Complete solution:

$$u_\ell > u_r : \; u(x,t) = \begin{cases} u_\ell, & x < st \\ u_r, & x > st \end{cases}, \quad s = \frac{1}{2}(u_\ell + u_r)$$

$$u_r \leq u_\ell : \; u(x,t) = \begin{cases} u_\ell, & \frac{x}{t} \leq u_\ell \\ \frac{x}{t}, & u_\ell < \frac{x}{t} < u_r \\ u_r, & \frac{x}{t} \geq u_r \end{cases}$$