4.4 The Riemann problem & Lax entropy condition

In this section we shall discuss the Riemann problem for hyperbolic systems of conservation laws, which is the IVP:

(Riemann 1860)

$$u_t + f(u)_x = 0$$

$$u(x,0) = u_0(x) = \begin{cases} u_l , & x<0 \\ u_r , & x>0 \end{cases}$$

with u_l, u_r constants. The solution of this IVP plays a central role in constructing numerical schemes to find weak solutions of systems of CLs.

4.4.1 First examples

Consider simple ~~example~~ example (inviscid Burgers equ.)

$$u_t + f(u)_x = 0, \quad f(u) = \frac{u^2}{2}$$

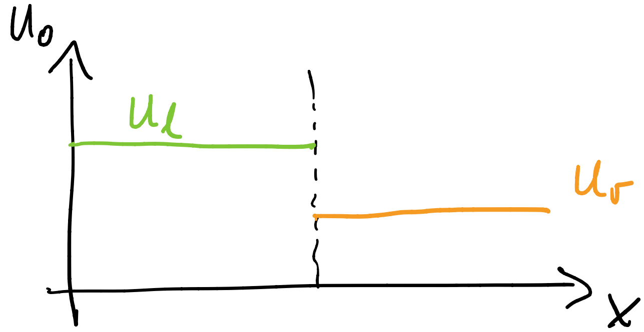
convex

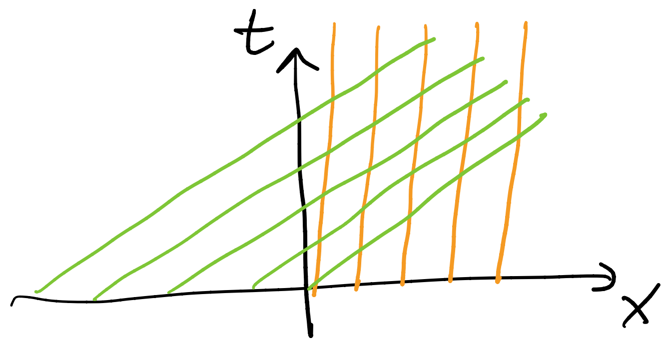
$$u(x,0) = u_0(x) = \begin{cases} u_l , & x<0 \\ u_r , & x>0 \end{cases}$$

$\rightsquigarrow$ const. characteristic speeds: $\lambda'(u) = f''(u) = 1 > 0$

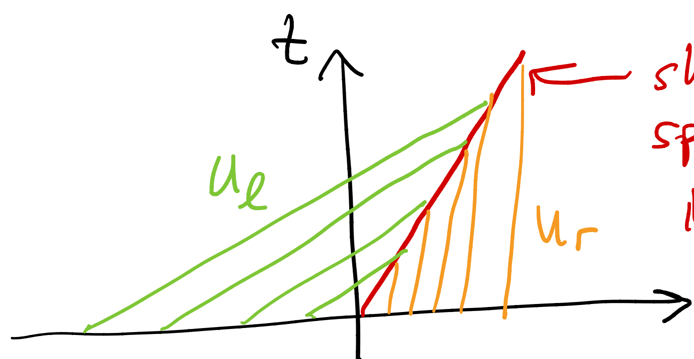
① assume $u_\ell > u_r$: "compressive data"

$$\lambda_\ell = \lambda(u_\ell) = u_\ell > u_r = \lambda(u_r) = \lambda_r$$





characteristics



shock Σ with speed $\sigma'(t) = s$

immediate crossing of characteristics!

RH jump conditions:

$$s = \sigma'(t) = \frac{f(u_\ell) - f(u_r)}{u_\ell - u_r} = \frac{1}{2}\,\frac{u_\ell^2 - u_r^2}{u_\ell - u_r}$$

$$= \frac{1}{2}(u_\ell + u_r)$$

weak solution:

$$u(x,t) = \begin{cases} u_\ell, & x - st < 0 \\ u_r, & x - st > 0 \end{cases}$$

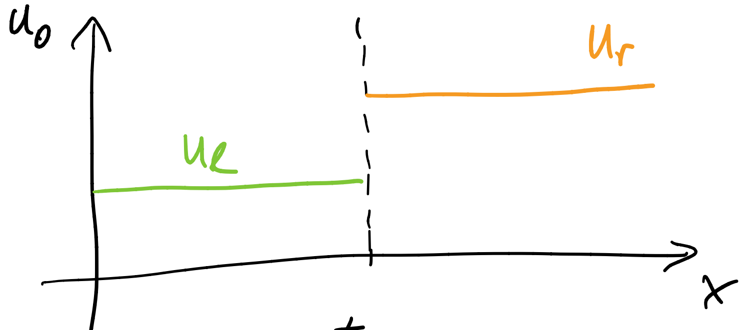
$$\boxed{\lambda(u_\ell) > s > \lambda(u_r)}$$

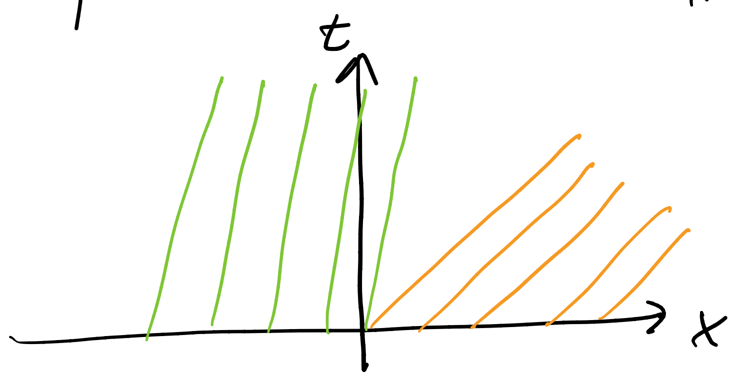
<u>Lax entropy condition</u>

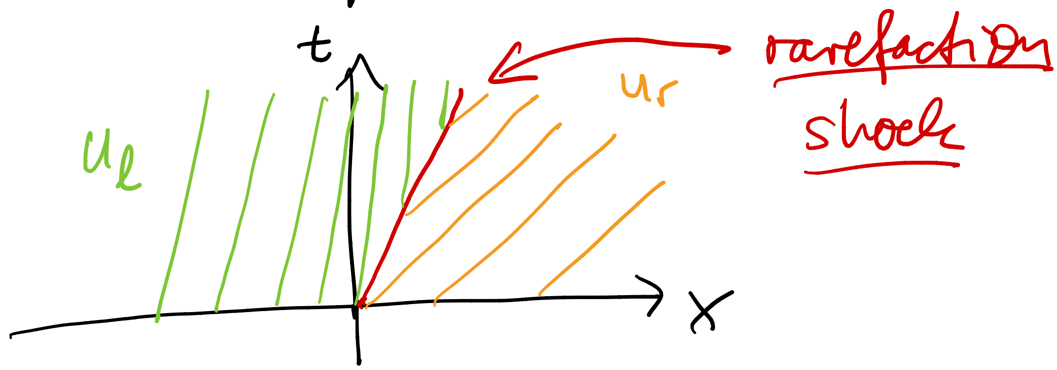
(→ will see satisfies entropy condition)

② <u>assume $u_\ell < u_r$:</u> "expansive data"

$$\lambda_\ell = \lambda(u_\ell) \qquad \lambda_r = \lambda(u_r)$$







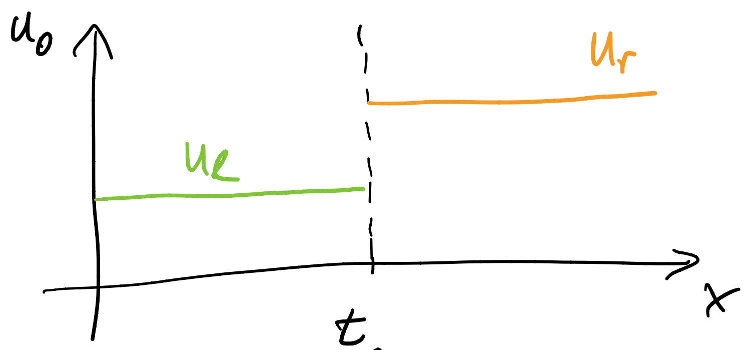
Similar to above, one weak solution is:

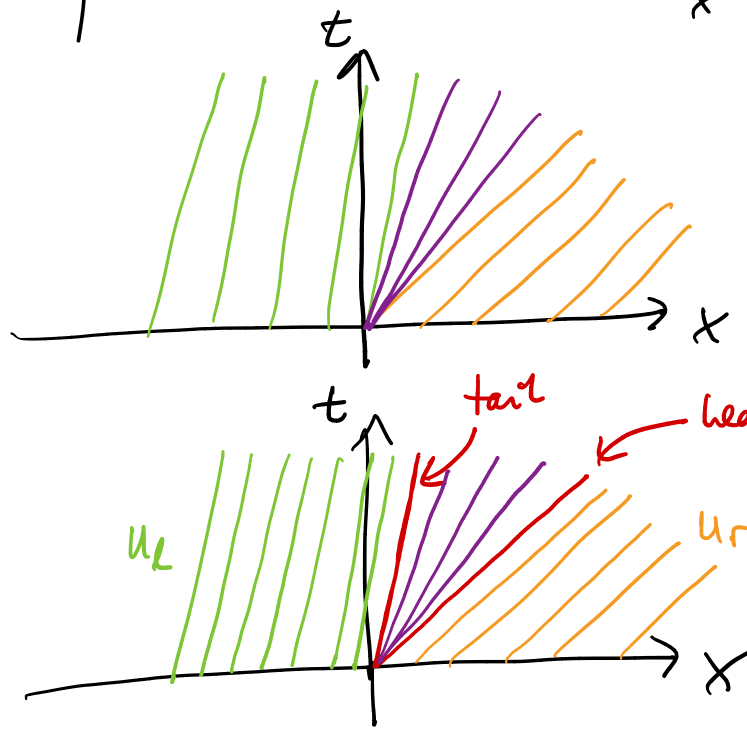
$$s = \frac{1}{2}(u_\ell + u_r), \quad u(x,t) = \begin{cases} u_\ell, & x - st < 0 \\ u_r, & x - st > 0 \end{cases}$$

But: $\boxed{\lambda_l < s < \lambda_r}$ entropy-violating shock

discontinuity has not arisen from compression, characteristics diverge from the discontinuity

Another possibility:





all $\gamma(t)$ of wave emanate from same point

$$\begin{cases} u(x,t) = u_l, & \frac{x}{t} \le \lambda_l = u_l \\ u(x,t) = \frac{x}{t}, & \lambda_l < \frac{x}{t} < \lambda_r \\ u(x,t) = u_r, & \frac{x}{t} \ge \lambda_r = u_r \end{cases}$$

centered rarefaction wave

larger values of $u_0(x)$ propagate faster than smaller values → wave spreads and flattens

⤳ "rarefaction" (non-linear phenomenon such as shocks)

③ <u>Complete solution:</u>

$$u_\ell > u_r : \; u(x,t) = \begin{cases} u_\ell , & x < st \\ u_r , & x > st \end{cases} , \quad s = \frac{1}{2}(u_\ell + u_r)$$

$$u_r \leq u_\ell : \; u(x,t) = \begin{cases} u_\ell , & \frac{x}{t} \leq u_\ell \\ \frac{x}{t} , & u_\ell < \frac{x}{t} < u_r \\ u_r , & \frac{x}{t} \geq u_r \end{cases}$$

<u>Now:</u> general Riemann problem for systems of CLs

$$u_t + f(u)_x = 0$$ (Riemann 1860)

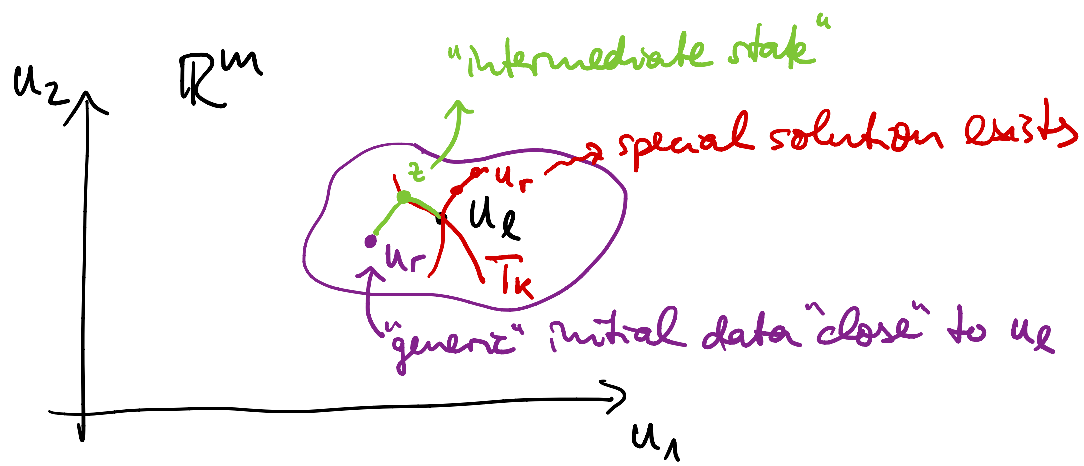
$$u(x,0) = u_0(x) = \begin{cases} u_\ell , & x<0 \\ u_r , & x>0 \end{cases}$$

with u_ℓ, u_r constants.

<u>Idea:</u> multi-step approach to the problem

→ find special solutions first (rarefaction waves, shocks, contact discontinuities)

→ represent general solution as a combination of special solutions



4.4.2 Riemann invariants & characteristic fields

First we'll need to define characteristics for systems of CLs:

<u>Def (k-characteristic):</u> Let $u = u(x,t) \in \mathbb{R}^m$ be a smooth solution of the hyperbolic system

$$u_t + f(u)_x = 0 \quad , \quad x \in \mathbb{R},\ t \in [0,T]$$

and let $\lambda_1(u) \leq \lambda_2(u) \leq \dots \leq \lambda_m(u)$ be the eigenvalues of $Df(u)$. A curve $\Gamma_k = (\gamma_k(t), t) \in C^1([0,\tau])$ with

$$\gamma'(t) = \lambda_k(u(\gamma(t),t)), \quad 0 \leq t \leq \tau$$

is called a <u>k-characteristic</u>.

In general $u(x,t)$ is not constant along a k-characteristic ($\rightarrow$ k-characteristics are not straight lines) but there are other suitable functions $w(u)$:

Lemma & Def (Riemann invariant):

Let $u = u(x,t)$ be a smooth solution of the system $u_t + f(u)_x = 0$ and let $\Gamma_k = (\gamma_k(t), t)$ be a k-characteristic wrt. λ_k.

Let $w \in C^1(\mathbb{R}^m, \mathbb{R})$ with

(*) $$Df(u)^T \cdot \nabla w = \lambda_k \nabla w,$$

then $w(u)$ is <u>constant along Γ_k</u>

$$\frac{d}{dt} w\big(u(\gamma_k(t), t)\big) = 0.$$

A function w with the property $Df(v)^T \cdot \nabla w(v) = \lambda_k \nabla w(v)$ for all $v \in \mathbb{R}^m$ is called a <u>Riemann invariant.</u>

<u>Proof:</u>
$$\frac{d}{dt} w\big(u(\gamma(t), t)\big) = \sum_{l=1}^{m} \Big[\partial_x u_l \, \gamma'(t) + \underbrace{\partial_t u_l}_{= -[f(u)_x]_l}\Big] \partial_l w$$

γ k-charact.

$$= \sum_{l=1}^{m} \big[\lambda_k \partial_x u_l - (Df(u)\partial_x u)_l\big] \partial_l w$$

$$= \sum_{j=1}^{m} \lambda_k \partial_x u_j \, \partial_j w - \sum_{l=1}^{m} \Big[\sum_{j=1}^{m} \partial_j f_l(u) \partial_x u_j\Big] \partial_l w$$

$$= \sum_{j=1}^{m} \left[\lambda_k \partial_j w - \sum_{\ell=1}^{m} \partial_j f_\ell(u) \partial_\ell w \right] \partial_x u_j$$

$$= \underbrace{\left[\lambda_k \nabla w - Df(u)^T \cdot \nabla w \right]}_{=0} \cdot \partial_x u$$

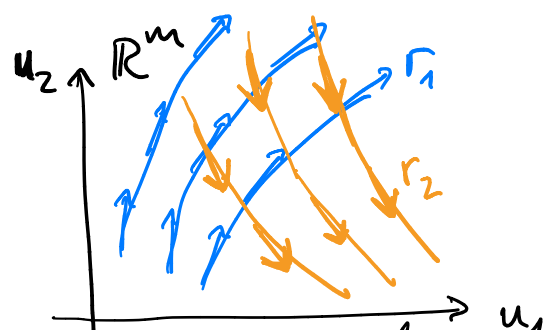
$$= 0$$

$\square$

Def & Lemma (k-Riemann invariant):

Consider a system $u_t + f(u)_x = 0$, $u = u(x,t) \in \mathbb{R}^m$ and let $r_k(u)$ denote the eigenvectors of $Df(u)$ corresponding to the eigenvalues $\lambda_k(u)$.

Consider an integral curve ($v \in C^1(\mathbb{R};\mathbb{R}^m): \xi \mapsto v(\xi)$) of r_k in state space:



$$v'(\xi) = \mu(\xi)\, r_k(v(\xi))$$

for some $\mu \in C^0(\mathbb{R};\mathbb{R})$ that depends on the parametrization ξ of the curve v.

A function $w \in C^1(\mathbb{R}^m;\mathbb{R})$ with

$$\nabla w(v(\xi)) \cdot r_k(v(\xi)) = 0$$

for an integral curve v of r_k is constant along the integral curve,

$$\frac{d}{dt} w(v(\xi)) = 0.$$

Such a w is called a *k-Riemann invariant*.

Proof: $$\frac{d}{dt} w(v(\xi)) = \nabla w(v(\xi)) \cdot v'(\xi)$$
$$= \nabla w(v(\xi)) \cdot \mu(\xi)\, r_k(v(\xi))$$
$$= 0$$

$\square$

Remarks: 1) Existence

One can show (e.g. Kroener Theorem 4.1.12): There are $m-1$ k-Riemann invariants $w_1, \dots, w_{m-1} \in C^1(\mathbb{R}^m, \mathbb{R})$ such that their gradients $\nabla w_1, \dots, \nabla w_{m-1}$ are linearly independent.

2) Riemann invariant $\Rightarrow$ k-Riemann invariant

Let w be a Riemann invariant with

respect to $\lambda_j(u)$. Then w is a k-Riemann invariant for all $k \neq j$:

Let $v \in \mathbb{R}^m$, then: $\lambda_j(v) \cdot r_k(v)^T \nabla w(v)$

$$= r_k(v)^T \cdot Df(v)^T \cdot \nabla w(v)$$

dropping v

$$= \left[Df \cdot r_k \right]^T \cdot \nabla w$$

$$= \lambda_k \, r_k^T \cdot \nabla w$$

$$\Rightarrow \quad (\lambda_j - \lambda_k) \, \nabla w \cdot r_k = 0$$

$$j \neq k: \quad \nabla w \cdot r_k = 0$$

Now: more on behaviour of characteristics

$\rightsquigarrow$ note that the change of λ_k along an integral curve $v(\xi)$ of r_k can be computed as:

$$\frac{d}{d\xi}\lambda_k(v(\xi)) = \nabla\lambda_k(v(\xi))\cdot v'(\xi)$$
$$= \mu(\xi)\,\nabla\lambda_k(v(\xi))\cdot r_k(v(\xi))$$

Def: Let $D \subseteq \mathbb{R}^m$. A k-characteristic is called

- linearly degenerate in D if and only if $\nabla\lambda_k \cdot r_k = 0$ in D
- genuinely non-linear in D if and only if $\nabla\lambda_k \cdot r_k \neq 0$ in D
  (λ_k varies monotonically along $v(\xi)$)

A system $u_t + f(u)_x = 0$, $u(x,t) \in \mathbb{R}^m$ is called linearly degenerate or genuinely non-linear in D if the property holds for all k-characteristics in D.

<u>Remarks:</u> 1) for genuinely non-linear k-characteristics we can always use some scaled r_k, such that $\nabla\lambda_k \cdot r_k = 1$.

2) For scalar equs, $\lambda^1(v) = f'(v)$ and $r^1(v) = 1$

$\leadsto$ gen. non-linear $\Leftrightarrow$ $f''(v) \neq 0$

(convexity requirement cf. Sec. 1.3.1)

3) For constant coeff. linear hyperbolic systems $u_t + A u_x = 0$

$\nabla \lambda_k(v) \equiv 0$ everywhere by construction

$\Rightarrow$ linearly degenerate

For non-linear linearly degenerate systems λ_k constant along a given integral curve, but takes different values along different integral curves

4.4.3 Special solutions: rarefaction waves

Motivation: are there <u>continuous</u> solutions to the Riemann problem?

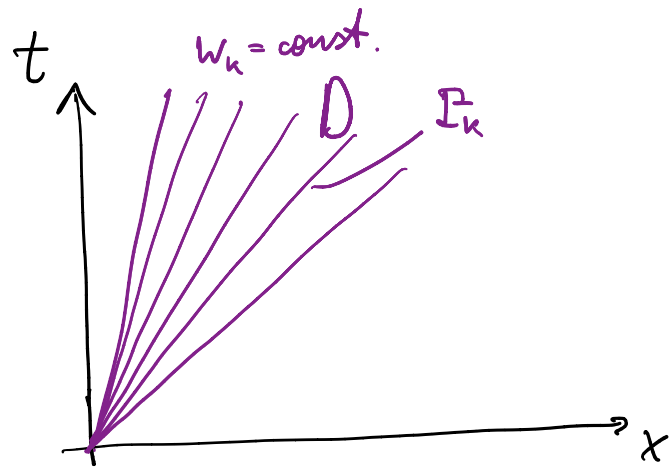
<u>Def (k-rarefaction waves):</u> Let $D \subset \mathbb{R} \times (0,\infty)$ and $u \in C^1(D; \mathbb{R}^m)$ be a solution of the strictly hyperbolic system $\partial_t u + f(u)_x = 0$ with $\lambda_k(u)$ being the eigenvalues of $Df(u)$.
If all k-Riemann invariants w_j are constant in D,

$$\frac{d}{dt} w_j(u(\gamma(t),t)) = 0 \quad \forall j = 1, \dots, m-1$$

with $\gamma'(t) = \lambda_k$ for $0 \le t \le \tau$ and $(\gamma(t),t) \in D$, the solution u is called a <u>k-rarefaction wave</u>.

<u>Remark:</u> k-rarefaction waves have the same property as solutions for scalar conservation laws (u const. along characteristics)

Theorem: Let the system $u_t + f(u)_x = 0$ be strictly hyperbolic and let u be a k-rarefaction wave in $D \subseteq \mathbb{R} \times (0,\infty)$. Then the k-characteristics are straight lines in D along which u is constant.



Proof: Consider k-characteristic γ & corresp eigenvalue λ_k with left eigenvector l_k, $l_k Df(u) = \lambda_k l_k$. Set $n \equiv \begin{pmatrix} \lambda_k \\ 1 \end{pmatrix}$, then

$$l_k \frac{d}{dt} u(\gamma(t),t) = l_k \underbrace{(\underbrace{\gamma'(t)}_{=\lambda_k} u_x + u_t)}_{n\cdot\nabla u = \partial_n u}$$

$$= l_k(\underbrace{u_t + Df(u)u_x}_{=0}) = 0$$

$$\Rightarrow \partial_n u = 0$$

Also: $0 = \frac{d}{dt} w_j(u(\gamma(t),t)) = \nabla w_j (\underbrace{u_x \gamma' + u_t}_{n\cdot\nabla u = \partial_n u})$

$$= \nabla w_j \, \partial_n u$$

$$\Rightarrow \underbrace{\begin{pmatrix} l_k \\ \nabla w_1 \\ \vdots \\ \nabla w_{m-1} \end{pmatrix}}_{\substack{\text{linearly} \\ \text{independent} \\ \text{(Sec. 4.4.2)}}} \partial_n u = 0 \qquad (*)$$

$$\alpha_0 l_k + \sum_{j=1}^{m-1} \alpha_j \nabla w_j \overset{!}{=} 0 \qquad | \cdot r_k$$

$$\nabla w_j \cdot r_k \overset{\text{Def}}{=} 0 \downarrow$$

$$\alpha_0 \, l_k \cdot r_k = 0$$

Observe: $\lambda_k l_k r_\ell = l_k Df r_\ell = l_k \lambda_\ell r_\ell$

$$\Rightarrow (\lambda_k - \lambda_\ell) \, l_k \cdot r_\ell = 0$$

$$\Rightarrow l_k \cdot r_\ell = 0 \qquad k \neq \ell$$

$$\Rightarrow l_k r_k \neq 0$$

$\Rightarrow \{l_k, \nabla w_j\}$ linearly independent

$$\overset{(*)}{\Rightarrow} \partial_n u = \frac{d}{dt} u(\gamma(t),t) = 0$$

$$\Rightarrow \gamma'(t) = \lambda_k(\underbrace{u(\gamma(t),t)}_{=\text{const.}}) = \text{const.}$$

$\square$

Theorem (existence and structure of rarefaction waves):

Consider the strictly hyperbolic system

$$u_t + f(u)_x = 0 \quad \text{in } \mathbb{R}\times(0,\infty)$$

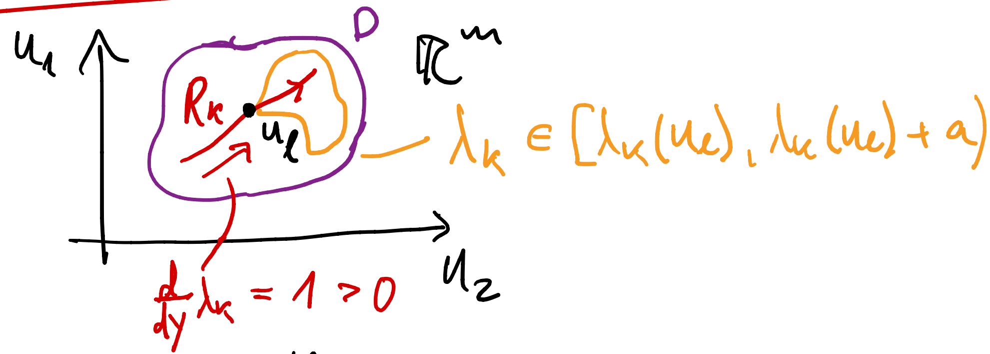
with $Df(u) \in C^1(\mathbb{R}^m, \mathbb{R}^{m\times m})$ and let λ_k be a genuinely non-linear characteristic field on $D \subset \mathbb{R}^m$ with $\nabla\lambda_k(v)\cdot r_k(v) > 0 \ (\equiv 1)\ \forall v \in D$ and let $u_e \in D$. Then there exists $a > 0$ and a function $v \in C^1([\lambda_k(u_e), \lambda(u_e)+a]; \mathbb{R}^m)$ such that $v(\lambda_k(u_e)) = u_e$ and such that

the Riemann problem with initial values

$$u(x,0) = u_0(x) = \begin{cases} u_e, & x<0 \\ \underbrace{v(y)}_{\text{"}u_r\text{"}}, & x>0 \end{cases}$$

for all $y \in [\lambda_k(u_e), \lambda_k(u_e)+a)$ can be solved by a continuous weak solution, which we call a k-rarefaction wave. One has $\frac{d}{dy}\lambda_k(v(y)) > 0$, i.e. $\lambda_k(u_e) < \lambda_k(u_r)$, and

$$(\ast)\quad \boxed{u(x,t) = \begin{cases} u_\ell\,, & \frac{x}{t} < \lambda_k(u_\ell) \\ v(x/t)\,, & \lambda_k(u_\ell) < \frac{x}{t} < y \\ v(y)\,, & y < \frac{x}{t}\,. \end{cases}}\qquad \begin{matrix} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III} \end{matrix}$$



Proof: $\exists\, a > 0$ sufficiently small such that the IVP

$$\begin{cases} v'(y) = r_k(v(y))\,, & \lambda_k(u_\ell) < y < \lambda_k(u_\ell)+a \\ v(\lambda_k(u_\ell)) = u_\ell \end{cases}$$

has a unique solution (ODE!).

Also: $\frac{d}{dy}\lambda_k(v(y)) = \nabla\lambda_k(v(y))\, v'(y) = \nabla\lambda_k \cdot r_k \overset{\text{assumption}}{=} 1$ slope: +1

$\Rightarrow$ since $\lambda_k(\underbrace{v(\lambda_k(u_\ell))}_{u_\ell}) = \lambda_k(u_\ell)$

$$(\ast\ast)\quad \lambda_k(v(y)) = y \text{ all along } \lambda_k(u_\ell) < y < \lambda_k(u_\ell)+a$$

For ansatz $(\ast)$ we have $\partial_t u + Df(u)\, u_x = 0$ in the distributional sense:

In I and III $u = \text{const.} \Rightarrow u_t + Df(u)u_x = 0$ ✓

In II: $v'(x/t)\left(-\frac{x}{t^2}\right) + Df(v(x/t))\, v'(x/t)\frac{1}{t}$

$$= r_k(v(x/t))\left(-\frac{x}{t^2}\right) + \frac{1}{t}\underbrace{\lambda_k(v(x/t))}_{\overset{(\ast\ast)}{=}\frac{x}{t}} r_k(v(x/t))$$

$= 0$ ✓

Let w be a k-Riemann invariant in D.

Then: $\frac{d}{dt} w(\underbrace{u(\gamma(t),t)}_{\substack{=\text{const}\\ \text{in I, III}}}) \overset{\text{I, III}}{=} 0$

$$\frac{d}{dt} w(u(\gamma(t),t)) = \nabla w(\underbrace{u_x\gamma' + u_t}_{\substack{=0 \text{ in II}\\ \text{see above}}}) = 0$$

where γ is the corresponding k-characteristic with $\gamma'(t) = \lambda_k(u(\gamma(t),t))$.

$\Rightarrow$ any w is constant in D

$\Rightarrow$ u is rarefaction wave. □

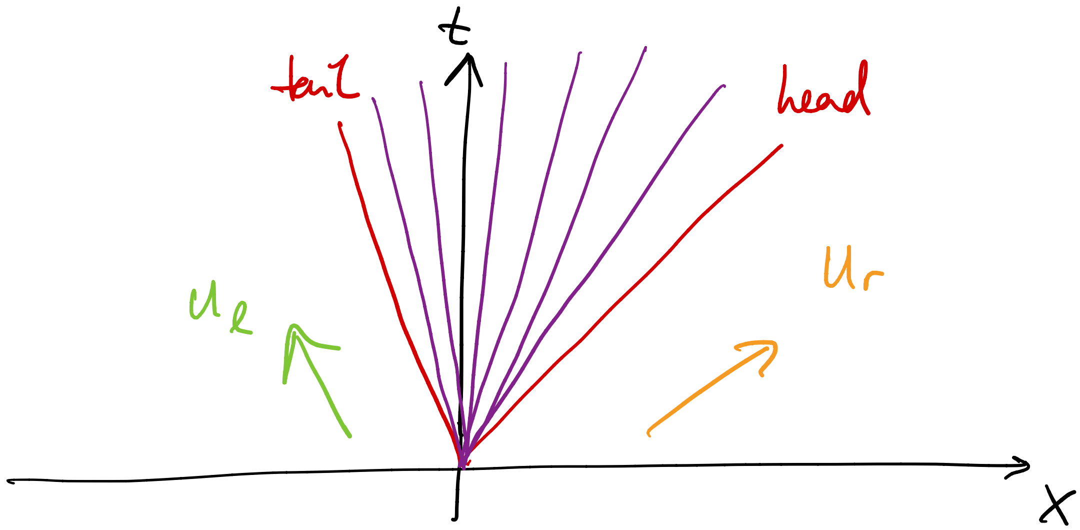
Remarks: 1) This rarefaction wave solution is the uniquely determined weak solution satisfying the entropy condition for $u_l < u_r$. (satisfies entropy-pair jump conditions, cf. (4.1.57) in Kroener)

2) This rarefaction wave solution is called a centered rarefaction wave as all variation in the "solution flow" arises from $x = t = 0$. Note that it is a similarity solution (only depends on $\frac{x}{t}$ $\rightsquigarrow$ looks similar on all scales) and is constant along every $\frac{x}{t} = \text{const.}$ ray.

3) Note that from $\frac{d}{dy} \lambda_k(v(y)) > 0$ it follows that

$$\lambda_k(u_\ell) < \lambda_k(u_r)$$

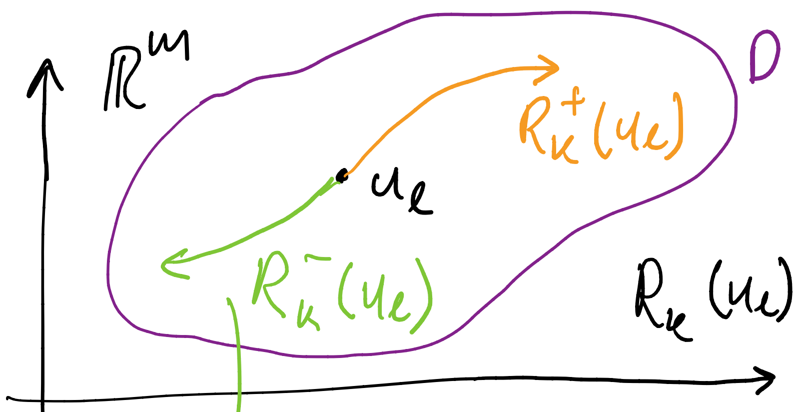
$\rightsquigarrow$ divergence of characteristics



A rarefaction wave is a continuous solution to the Riemann problem that "smooths out" the initial discontinuity!

Def: Given a fixed state $u_l \in \mathbb{R}^m$ we define the integral curve $v'(\xi) = r_k(v(\xi))$, $v(0) = u_l$, $\xi \in (-a, a)$ and $a > 0$ sufficiently small, considered in the previous theorem as the <u>k-rarefaction curve</u> $R_k(u_l)$. If $(\lambda_k(u_l), r_k(u_l))$ are genuinly non-linear, we write:

$$R_k^+(u_l) = \{ z \in R_k(u_l) \mid \lambda_k(z) > \lambda_k(u_l) \}$$
$$R_k^-(u_l) = \{ z \in R_k(u_l) \mid \lambda_k(z) < \lambda_k(u_l) \}$$



$R_k(u_l) = R_k^+(u_l) \cup \{u_l\} \cup R_k^-(u_l)$

compressive data $\Rightarrow$ no single rarefaction solution possible

4.4.4 Special solutions: shocks and the Lax entropy condition

Now: look for solution that "keeps the discontinuity"!
Reconsider behaviour of weak solutions across a discontinuity: Consider weak solution of strictly hyperbolic system

$$u_t + f(u)_x = 0 \quad \text{in } \mathbb{R}\times(0,\infty)$$

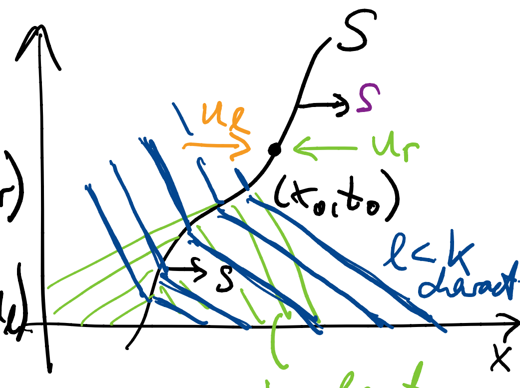
that is piecewise smooth with a jump along $S = (\sigma(t), t)$ for a curve σ with $\sigma' \equiv s$ the shock velocity. Let $(x_0, t_0) \in S$ and u_l, u_r the limit values of u from left and right at (x_0, t_0):

$$u_l = \lim_{x \nearrow x_0} u(x, t_0), \quad u_r = \lim_{x \searrow x_0} u(x, t_0).$$

Assume

$$\lambda_1(u_r) < \dots < \lambda_k(u_r) < s < \lambda_{k+1}(u_r) < \dots < \lambda_n(u_r)$$

$$\lambda_1(u_l) < \dots < \lambda_j(u_l) < s < \lambda_{j+1}(u_l) < \dots < \lambda_n(u_l)$$



where $\lambda_1(u), \dots, \lambda_m(u)$ are the eigenvalues of $Df(u)$.

$\rightsquigarrow$ k-conditions on u_r at (x_0, t_0) from initial data at $t=0$ on right side due to <u>impinging</u> ℓ-characteristics $\lambda_1, \dots, \lambda_k < s$

$m-j$ conditions on u_ℓ from initial data on left side due to impinging ℓ-characteristics $\lambda_{j+1}, \dots, \lambda_m > s$

m conditions relating u_r & u_ℓ (jump cond.)

$$s(u_r - u_\ell) = f(u_r) - f(u_\ell)$$

$\rightsquigarrow$ $2m+1$ unknowns:

$$u_\ell = \begin{pmatrix} u_{\ell 1} \\ \vdots \\ u_{\ell m} \end{pmatrix}, \quad u_r = \begin{pmatrix} u_{r1} \\ \vdots \\ u_{rm} \end{pmatrix}, \quad s$$

Necessary condition for computing the unknowns for arbitrary given initial data at $t=0$ is that

$$k + m - j + m = 2m + k - j \overset{!}{=} 2m + 1$$

$$\Leftrightarrow \quad j = k - 1$$

$$\Leftrightarrow \quad \begin{aligned} \lambda_k(u_r) < s < \lambda_{k+1}(u_r) \\ \lambda_{k-1}(u_\ell) < s < \lambda_k(u_\ell) \end{aligned}$$

(Lax Comm. Pure and Applied Math. 10 (1957) 537, Sec. 7)

or $(*)$ $$\boxed{\lambda_k(u_r) < s < \lambda_k(u_\ell)}$$

Lax entropy condition

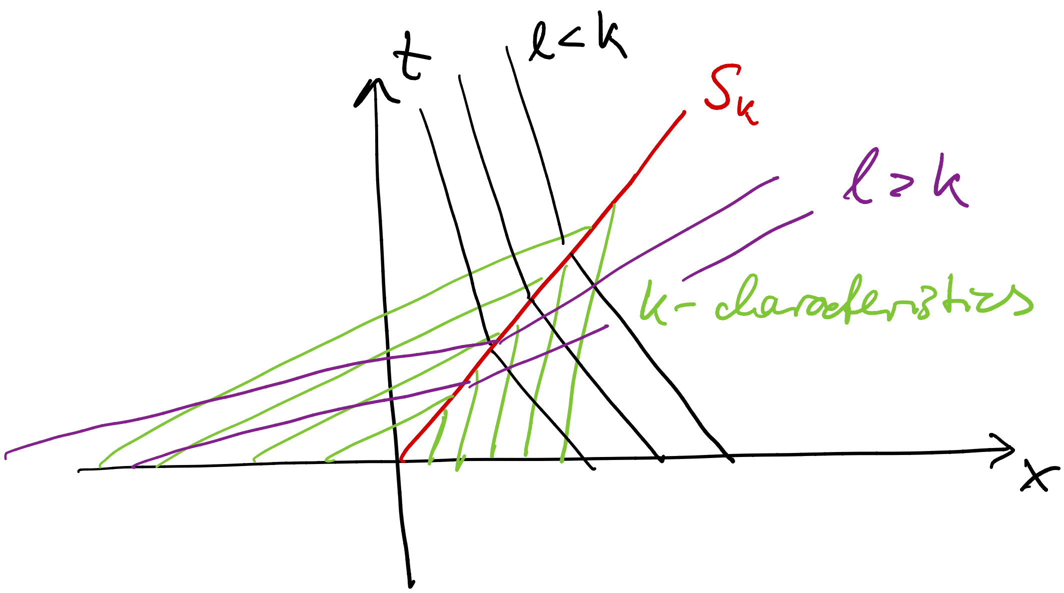
Def (Lax entropy condition, k-shock):

A discontinuous weak solution $u(x,t)$ satisfying the RH jump conditions and the Lax entropy condition (*) is called a k-shock.

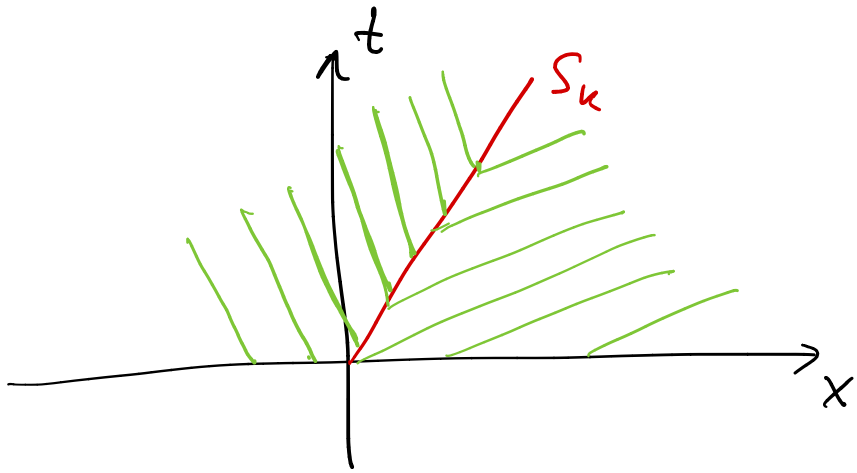
Remarks: 1) The Lax entropy condition states that the k-characteristics impinge on the discontinuity while the other characteristics cross it:

$$\lambda_\ell(u_\ell) < s \quad \& \quad \lambda_\ell(u_r) < s \quad \text{for } \ell < k$$

$$\lambda_\ell(u_\ell) > s \quad \& \quad \lambda_\ell(u_r) > s \quad \text{for } \ell > k$$



2) A <u>rarefaction shock</u> $\lambda_k(u_r) > s > \lambda_k(u_l)$ cannot be uniquely determined by above conditions due to <u>expansive</u> nature, need additional conditions on the discontinuity



$\rightsquigarrow$ Lax entropy condition excludes rarefaction shocks

Existence of k-shock solutions:

Def: Given state $u_l \in \mathbb{R}^m$, define <u>shock set</u>

$$S(u_l) \equiv \left\{ z \in \mathbb{R}^m \;\middle|\; f(z) - f(u_l) = s(z - u_l) \text{ for a constant } s = s(z, u_l) \right\}$$

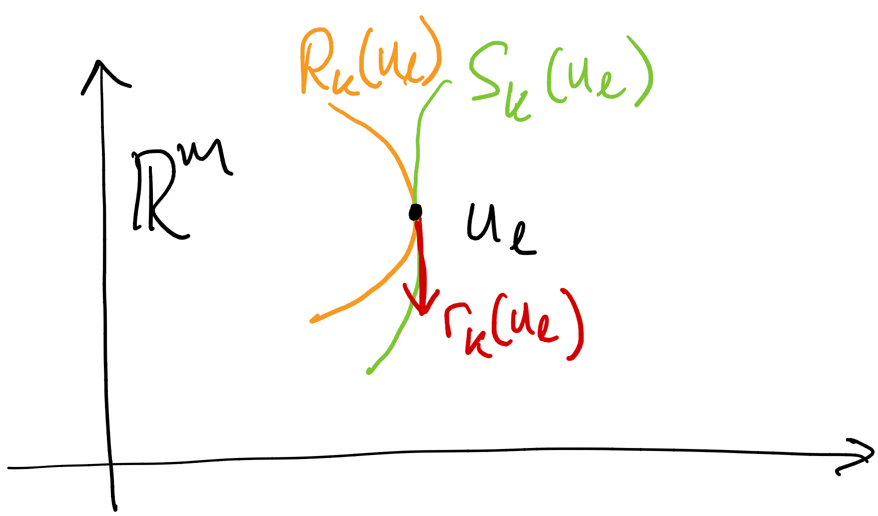
where f is the flux of a system of conservation laws.

Theorem (properties of shock set): Consider the RP for a strictly hyperbolic system of CLs $u_t + f(u)_x = 0$, and let $u_\ell \in \mathbb{R}^m$ be a given left state of the RP. Then there exists a neighborhood of u_ℓ in which $S(u_\ell)$ is the union of m smooth curves $S_k(u_\ell)$ $(k = 1, \dots, m)$ with

(i) $S_k(u_\ell)$ passes through u_ℓ with tangent $r_k(u_\ell)$, i.e. $s(0) = u_\ell$, $s'(0) = r_k(u_\ell)$

(ii) $$\lim_{\substack{z \to u_\ell \\ z \in S_k(u_\ell)}} s(z, u_\ell) = \lambda_k(u_\ell)$$

(iii) $$s(z, u_\ell) = \frac{\lambda_k(z) + \lambda_k(u_\ell)}{2} + \mathcal{O}(|z - u_\ell|^2)$$
as $z \in S_k(u_\ell) \to u_\ell$



<u>Proof:</u> Evans Theorem 2, Sec. 11.2.3 p627.

Idea: use strict hyperbolicity & implicit function theorem to show existence of m curves $\{s_k(t)\}_{k=1,\dots,m}$ with $s_k(0)=u_e$, $s_k'(0) = r_k(u_e)$ ($\Rightarrow$ (i)) and of a function $s: \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$, such that

$$(*) \quad f(s_k(t)) - f(u_e) = s(s_k(t), u_e)(s_k(t) - u_e)$$

$$\Big\downarrow \left.\frac{d}{dt}\right|_{t=0}$$

$$\lambda_k(u_e)\, r_k(u_e) = Df(u_e)\, s_k'(0) = \sigma(u_e, u_e)\, s_k'(0) = s(u_e, u_e)\, r_k(u_e)$$

$$\Longrightarrow \text{(ii)}$$

$$\left.\frac{d^2}{dt^2}\right|_{t=0} (*) \rightsquigarrow 2s(s_k(t), u_e) - \lambda_k(u_e) - \lambda_k(s_k(t)) = \mathcal{O}(t^2) \text{ as } t \to 0$$

$$\Longrightarrow \text{(iii)}$$

$\square$

Remarks: 1) $R_k(u_l)$ vs. $S_k(u_l)$:

Note the curves $R_k(u_l)$ and $S_k(u_l)$ are different, but share the same tangent vector at u_l and agree to first order in $|z - u_l|$ at u_l.

2) Structure of k-shocks:

Let (λ_k, r_k) be genuinly non-linear pair in $D \subseteq \mathbb{R}^m$ with $u_r \in S_k(u_l)$, then

$$\boxed{u(x,t) = \begin{cases} u_l & x < st \\ u_r & x > st \end{cases}}$$

is a weak solution to the RP with $s = s(u_r, u_l)$.

Note: u satisfies $u_t + f(u)_x = 0$ since piecewise smooth and $u_t = u_x = 0$; u satisfies RH jump condition by construction through choice of u_r (and s).

$$\frac{d}{dt}\Big|_{t=0} \lambda_k(S_k(t)) = D\lambda_k(S_k(t))\, S_k'(t)\Big|_{t=0}$$
$$= \nabla\lambda_k(u_l)\, r_k(u_l) \neq 0$$

(genuinly non-linear)

$\Rightarrow \lambda_k(S_k(t))$ cannot be constant

(iii) theorem
$\Rightarrow$
$$\lambda_k(u_r) < s(u_r, u_l) < \lambda_k(u_l)$$
or
$$\lambda_k(u_l) < s(u_r, u_l) < \lambda_k(u_r)$$
Lax entropy violating

$\leadsto$ can only admit u_r satisfying first relation!

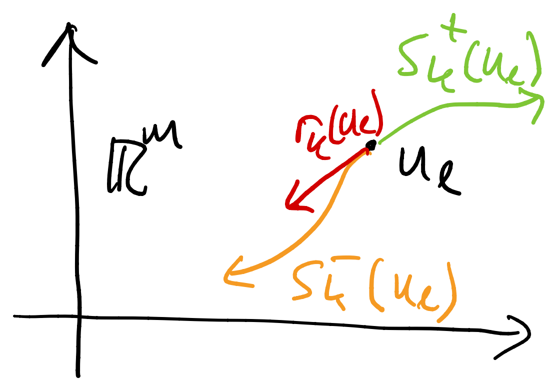
<u>Def:</u> Let $u_l \in D \subseteq \mathbb{R}^m$ and (λ_k, r_k) be a genuinely nonlinear pair (in D) of a strictly hyperbolic system of CLs with shock set $S(u_l)$. Then define:

$$S_k^+(u_l) \equiv \{ z \in S_k(u_l) \mid \lambda_k(u_l) < \sigma(z, u_l) < \lambda_k(z) \}$$

$$S_k^-(u_l) \equiv \{ z \in S_k(u_l) \mid \lambda_k(z) < \sigma(z, u_l) < \lambda_k(u_l) \}$$

<u>Remarks:</u>

- $S_k(u_l) = S_k^+(u_l) \cup \{u_l\} \cup S^-(u_l)$
- (u_r, u_l) admissible if and only if $u_r \in S_k^-(u_l)$

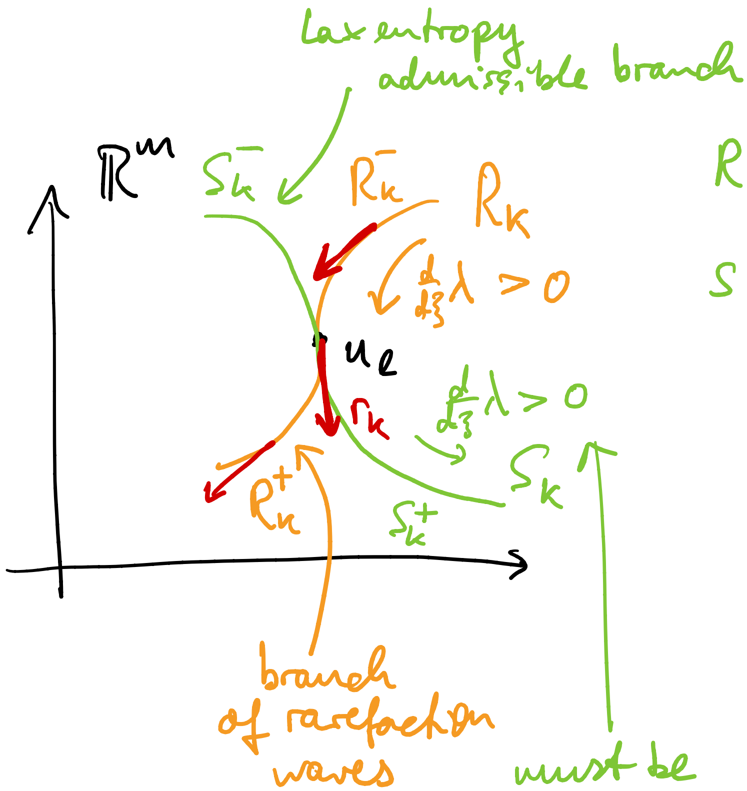


Theorem (equivalence of entropy conditions):

Consider the hyperbolic and genuinely non-linear system $u_t + f(u)_x = 0$ in $\mathbb{R} \times (0,\infty)$ and assume $u = u(x,t)$ is a solution with a weak shock, i.e. with sufficiently small jump $u_l - u_r$. Then the entropy condition along the discontinuity is equivalent to the Lax entropy condition.

Proof: see Kroener Theorem 4.1.25 p. 298.

Brief summary of special solutions so far:



RH jump conditions

$$s(u_\ell - u_r) = f(u_\ell) - f(u_r)$$

m conditions for $2m+1$ unknowns

fix u_ℓ, vary u_r

$\rightsquigarrow$ 1-parameter family of curves

must be increasing in this direction as dictated by R_k and common behavior at u_ℓ $\rightsquigarrow$ $v' = s' = r_k$

$$S_k^+(u_\ell) \equiv \{ z \in S_k(u_\ell) \mid \lambda_k(u_\ell) < \sigma(z, u_\ell) < \lambda_k(z) \}$$

$$S_k^-(u_\ell) \equiv \{ z \in S_k(u_\ell) \mid \lambda_k(z) < \sigma(z, u_\ell) < \lambda_k(u_\ell) \}$$

4.4.5 Contact Discontinuities

Theorem: Consider a strictly hyperbolic system of CLs $u_t + f(u)_x = 0$ and suppose the pair (λ_k, r_k) is linearly degenerate in $D \subseteq \mathbb{R}^m$ for some $k \in \{1, \dots, m\}$. Then for all $u_\ell \in D$:

(i) $R_k(u_\ell) = S_k(u_\ell)$

(ii) $s(z, u_\ell) = \lambda_k(z) = \lambda_k(u_\ell) \;\; \forall z \in S_k(u_\ell)$

(discontinuity is along the k-characteristic $\gamma' = \lambda_k = s$)

Proof: Consider the unique solution of the IVP

$$\begin{cases} v'(\xi) = r_k(v(\xi)) & \xi \in (-a, a) \subseteq \mathbb{R} \\ v(0) = u_\ell \end{cases}$$

with $a > 0$ sufficiently small.

Then
$$\frac{d}{d\xi} \lambda_k(v(\xi)) = \nabla \lambda_k(v(\xi)) \cdot v'(\xi) = \nabla \lambda_k(v(\xi)) \cdot r_k(v(\xi)) \overset{\text{linearly degenerate}}{=} 0$$

$\Rightarrow \lambda_k$ constant along $R_k(u_e)$

$$\rightsquigarrow \quad f(v(\xi)) - f(u_e) = \int_0^{\xi} Df(v(t))v'(t)\,dt$$

$$= \int_0^{\xi} \lambda_k(v(t))\, r_k(v(t))\,dt$$

$$= \lambda_k(u_e) \int_0^{\xi} v'(t)\,dt = \lambda_k(u_e)(v(\xi) - u_e)$$

$\Rightarrow R_k(u_e) = S_k(u_e)$ where they exist, and

$s(z, u_e) = \lambda_k(u_e)$ for $z \in R_k(u_e) = S_k(u_e)$

$\square$

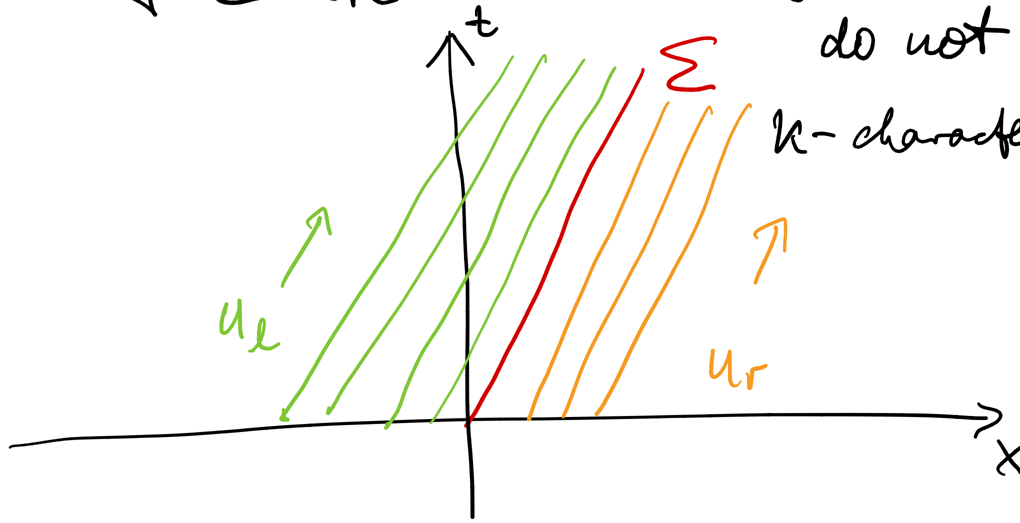
<u>Def (contact discontinuity):</u>

Let u be a weak solution of $u_t + f(u)_x = 0$ with a discontinuity along $\Sigma = \{(\sigma(t), t)\} \subseteq \mathbb{R} \times (0,\infty)$. Let $\lambda_k(u^{\pm}(\sigma(t), t)) = s(t) = \sigma'(t)$ for some characteristic field λ_k of $Df(u)$, $k \in \{1, \dots, m\}$, and

$$u^{\pm}(\sigma(t), t) = \lim_{\varepsilon \to 0} u(s(t) \pm \varepsilon, t).$$

Then Σ is called a <u>contact discontinuity</u> wrt. λ_k.

<u>Remarks:</u> 1) The <u>propagation velocity</u> $s = \sigma'$ of Σ and the characteristic speed λ_k on both sides of Σ are the same ("fluid elements do not cross Σ")



2) <u>Structure of contact discontinuity:</u>

Suppose (λ_k, r_k) is linearly degenerate in $D \subseteq \mathbb{R}^m$ and $u_r \in S_k(u_l)$, then

$$\boxed{u(x,t) = \begin{cases} u_l, & x < st \\ u_r, & x > st \end{cases}}$$

with $s = s(u_r, u_l) = \lambda_k(u_r) = \lambda_k(u_l)$ is a weak solution of the RP.

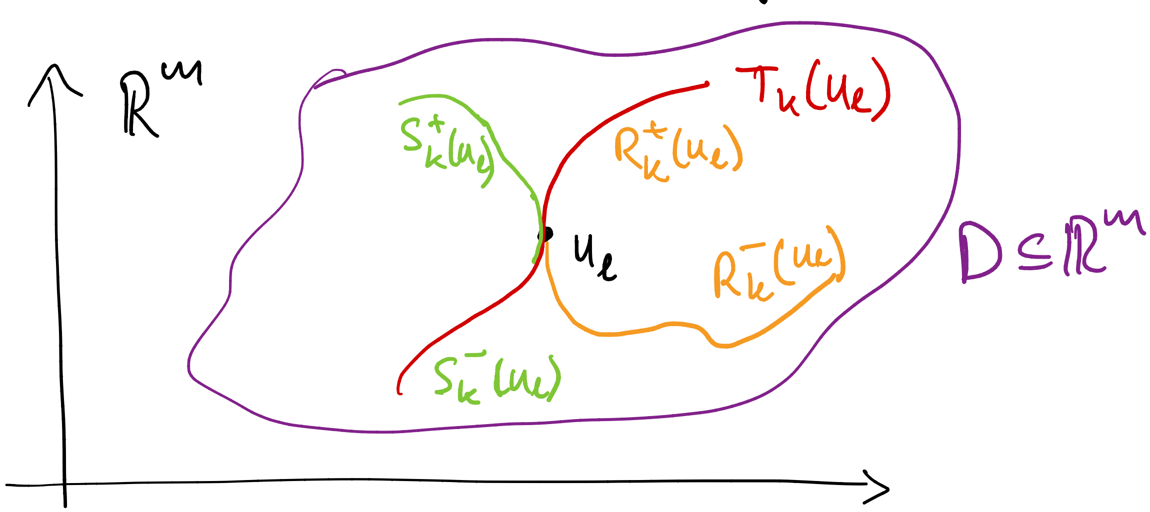
4.4.6 Local solution of Riemann's problem

Def: Let $u_l \in D \subseteq \mathbb{R}^m$ and (λ_k, r_k) be an eigenvalue-eigenvector pair of a strictly hyperbolic system of CLs with shock set $S(u_l)$ and k-rarefaction curve $R_k(u_l)$. Then define the curve

existence of k-rarefaction wave

existence of k-shock satisfying Lax entropy condition

$$T_k(u_l) \equiv \begin{cases} R_k^+(u_l) \cup \{u_l\} \cup S_k^-(u_l) & \text{if } (\lambda_k, r_k) \text{ is genuinely non-linear in } D \\ R_k(u_l) = S_k(u_l) & \text{if } (\lambda_k, r_k) \text{ is linearly degenerate in } D \end{cases}$$



Remarks: 1) $T_k(u_\ell)$ is C^1 (cf. (ii) of theorem on shock set) and $\dot{v}(0) = v(u_\ell, u_\ell) = r_k$

2) u_ℓ can be joined with u_r by a k-rarefaction wave, a k-shock, or a k-contact discontinuity if $u_r \in T_k(u_\ell)$.

Now: consider general case $u_r \notin T_k(u_\ell)$ for some $k \in \{1, \dots, m\}$.

Theorem (Lax 1957):

(Comm. Pure and Applied Math. 10 (1957) 537)

Let $u_\ell \in D \subseteq \mathbb{R}^m$. Consider the strictly hyperbolic system

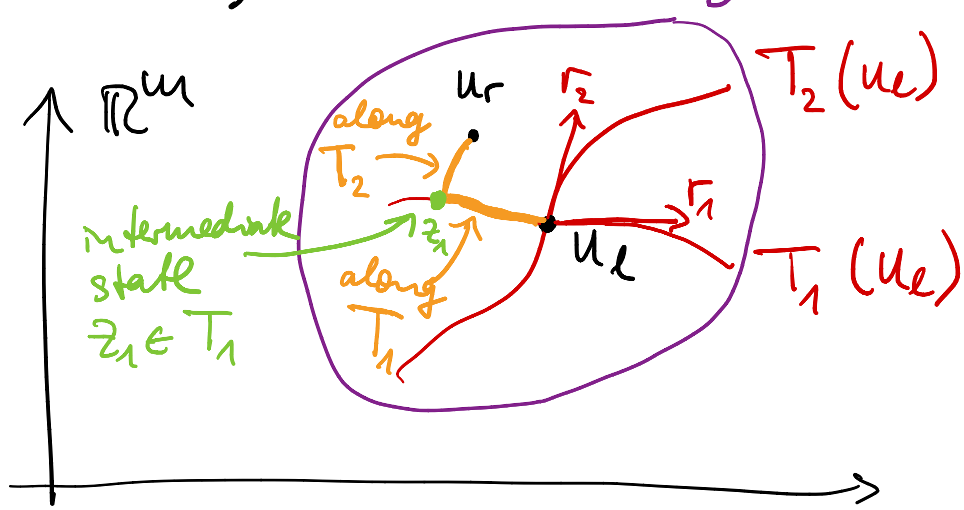
$$u_t + f(u)_x = 0$$

and assume that every pair $(\lambda_k, r_k)_{k \in \{1, \dots, m\}}$ is either genuinely non-linear or linearly degenerate in D. Then there exists a neighborhood $\tilde{D} \subseteq D$ of u_ℓ such that for all $u_r \in \tilde{D}$ there exists a unique weak solution $u(x,t)$ to the Riemann problem

$$u(x,0) = \begin{cases} u_\ell(x) & \text{if } x < 0 \\ u_r(x) & \text{if } x \geq 0 \end{cases}$$

which is constant along lines through the origin (at most $m+1$ constant states connected by shocks, rarefaction waves & contact discontinuities).

Idea:



connect (u_r, u_ℓ) by introducing intermediate states following local "coordinate lines" given by the curves $\{T_k(u_\ell)\}$ in $\tilde{D}$.
Such coordinates exist locally
$\leadsto$ tangent vectors of $T_k(u_\ell)$ at u_ℓ are $\{r_k\}$, which form a basis of $\mathbb{R}^m$.

Infinitesimal version:
(linearized Riemann Problem)

$$\boxed{u_r - u_\ell = \sum_{k=1}^{m} \varepsilon_k r_k(u_\ell)}$$

Proof (as in Evans Sec. 11.2.4 p635):

Denote $\tau_k(z) - \tau_k(u_\ell) =$ (signed) distance along curve $T_k(u_\ell)$

> 0 if $z \in R_k^+(u_\ell)$

< 0 if $z \in S_k^-(u_\ell)$

Define a curve $\Phi : \mathbb{R}^m \to \mathbb{R}^m$

$\varepsilon \mapsto z$

For given $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m)$, $0 < |\varepsilon|$ small,

chose states $z_0 = u_\ell, z_1, \dots, z_m = z$ such that

$$\begin{cases} z_1 \in T_1(z_0), \; \tau_1(z_1) - \tau_1(z_0) = \varepsilon_1 \\ \vdots \\ z_m \in T_m(z_{m-1}), \; \tau_m(z_m) - \tau(z_{m-1}) = \varepsilon_m . \end{cases}$$

Note that $\Phi \in C^1$ (T_k are C^1) and $\Phi(0) = z_0 = u_\ell$.

Also: $\left.\frac{\partial \Phi}{\partial \varepsilon_k}\right|_{\varepsilon=0} = \lim_{\varepsilon_k \to 0} \Phi(0, \dots, \varepsilon_k, \dots, 0) - \Phi(0) = r_k(z_0 = u_\ell)$

$\Rightarrow D\Phi(0) = (r_1(u_\ell), \dots, r_m(u_\ell))$ is non-singular

as $\{r_k\}_{k=1,\dots,m}$ are a basis

Inverse function theorem $\Rightarrow$ there exists a neighborhood $\tilde{D}$ of u_ℓ (of $\varepsilon = 0$) such that $\forall\, u_r \in \tilde{D}$ there exists a unique parameter $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m)$ such that $\Phi(\varepsilon) = u_r$.

- If $z_k \in R_k^+(z_{k-1})$ [(z_k, z_{k-1}) joined by rarefaction wave
note: ε is signed]

$$u_k(x,t) = \begin{cases} z_{k-1} & \text{if } \frac{x}{t} < \lambda_k(z_{k-1}) \\ v(x/t) & \text{if } \lambda_k(z_{k-1}) < \frac{x}{t} < \lambda_k(z_k) \\ z_k & \text{if } \lambda_k(z_k) < \frac{x}{t} \end{cases}$$

where $v(y)$ is defined by the IVP

$$\begin{cases} v'(y) = r_k(v(y)) \,, & \lambda_k(z_{k-1}) < y < \lambda_k(z_{k-1}) + a \\ v(\lambda_k(z_{k-1})) = z_{k-1} \end{cases}$$

for $a > 0$.

- If $z_k \in S_k^-(z_{k-1})$ [(z_k, z_{k-1}) joined by a k-shock]

$$u_k(x,t) = \begin{cases} z_{k-1} & \text{if } \frac{x}{t} < s(z_k, z_{k-1}) \\ z_k & \text{if } s(z_k, z_{k-1}) < \frac{x}{t} \end{cases}$$

with $\lambda_k(z_k) < s(z_k, z_{k-1}) < \lambda_k(z_{k-1})$

Both cases: $u_k = \text{const.} = z_{k-1}$ for

$$\frac{x}{t} < \begin{cases} \lambda_k(z_{k-1}) < \lambda_k(z_0) \pm \varepsilon \\ s(z_k, z_{k-1}) < \lambda_k(z_{k-1}) < \lambda_k(z_0) \pm \varepsilon \end{cases} \quad \text{for some } \varepsilon > 0$$

$u_k = \text{const.} = z_k$ for

$$\frac{x}{t} > \begin{cases} \lambda_k(z_k) > \lambda_k(z_0) \pm \varepsilon \\ s(z_k, z_{k-1}) > \lambda_k(z_k) > \lambda_k(z_0) \pm \varepsilon \end{cases} \quad \text{for some } \varepsilon > 0$$

$\Rightarrow$ u_k const. outside $\lambda_k(u_l) - \varepsilon < \frac{x}{t} < \lambda_k(u_l) + \varepsilon$ for some small ε, provided z_k, z_{k-1} close enough to u_l.

Since $\lambda_1(u_l) < \dots < \lambda_m(u_l)$ (strictly hyperbolic) all waves connecting $u_l, z_1, \dots, z_{m-1}, u_r$ do not intersect!

$\Rightarrow$ global solution u is constant along lines through the origin ($\frac{x}{t} = \text{const.}$)

The global solution u is also unique as it satisfies the entropy condition by construction. $\square$

Summary: structure of solution