Chapter 6: Numerical Schemes for hyperbolic systems of CLs

Consider hyperbolic system of CLs:

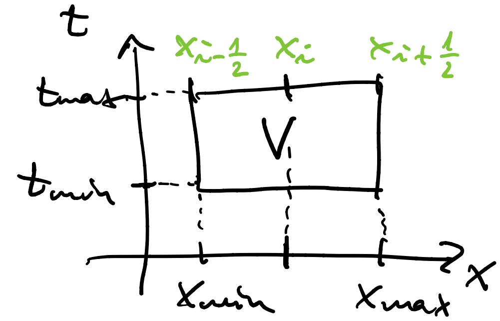
$$\text{(I)} \quad u_t + f(u)_x = 0 \quad \text{in } \mathbb{R}\times(0,\infty)$$

$$\text{(II)} \quad u(x,0) = u_0(x) \quad \text{in } \mathbb{R}\times\{0\}$$

6.1 Conservative schemes

Motivation:

Integrate (I) over a rectangular control volume $V = (x_{min}, x_{max})\times(t_{min}, t_{max}) \subseteq \mathbb{R}\times(0,\infty)$:



$$\iint \left[u_t + f(u)_x\right] dt\,dx$$

$$= \int \left[u(x,t_{max}) - u(x,t_{min})\right] dx$$

$$+ \int \left[f(u(x_{max},t)) - f(u(x_{min},t))\right] dt = 0$$

$\Big\downarrow$ set V to volume of one grid cell $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]\times[t^n, t^{n+1}]$

$$(\ast)\quad \underbrace{\int_{x_{i-\frac12}}^{x_{i+\frac12}} \left[u(x,t^{n+1}) - u(x,t^n)\right] dx}_{\substack{\text{change in } u \text{ in}\\ \text{the volume } [x_{i-\frac12}, x_{i+\frac12}]\\ \text{during } [t^n, t^{n+1}]}} + \underbrace{\int_{t^n}^{t^{n+1}} \left[f(u(x_{i+\frac12},t)) - f(u(x_{i-\frac12},t))\right] dt}_{\substack{-\text{ flow difference through}\\ \text{cell interfaces during } [t^n, t^{n+1}]}} = 0$$

Assume there is a function $g: \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ ("numerical flux") such that ($\simeq$: "approximates"):

average conserved variable in cell i

$$u_i^n \simeq \frac{1}{\Delta x} \int_{x_{i-\frac12}}^{x_{i+\frac12}} u(x,t^n)\, dx$$

average flux through interface in Δt

$$g_{i+\frac12}^n = g(u_i^n, u_{i+1}^n) \simeq \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} f(u(x_{i+\frac12},t))\, dt$$

$$g_{i-\frac12}^n = g(u_{i-1}^n, u_i^n) \simeq \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} f(u(x_{i-\frac12},t))\, dt$$

$$\rightsquigarrow \quad (\ast) \Leftrightarrow u_i^{n+1} - u_i^n = -\frac{\Delta t}{\Delta x}\left(g_{i+\frac12}^n - g_{i-\frac12}^n\right)$$

Consistency:

- We need to ensure that g is consistent with physical flux:

$$g(v,v) = f(v) \quad \forall\, v \in \mathbb{R}^m$$

This is because of the following example:

Consider 1D $u_0(x) = \text{const}$.

$\Rightarrow$ $u = u_0$ is solution of CL and

$$g^0_{i+\frac{1}{2}} = g(u^0_i, u^0_{i+1}) = g(u_0, u_0)$$

$$g^0_{i+\frac{1}{2}} = \frac{1}{\Delta t}\int_0^{\Delta t} f(u(x_{i+\frac{1}{2}}, t))\, dt = \frac{1}{\Delta t}\int_0^{\Delta t} f(u_0)\, dt = f(u_0)$$

- Also need to expect continuity as u_{i+1}, u_i vary

i.e. $g(u_i, u_{i+1}) \longrightarrow f(v)$ as $u_{i+1}, u_i \to v$

$\rightsquigarrow$ require Lipschitz continuity: there exist constants L_1, L_2 such that

$$\| g(u_{i+1}, u_i) - f(v) \| \leq L_1 \| u_{i+1} - v\|^\alpha + L_2 \| u_i - v\|^\alpha$$

<u>Def (Conservative scheme)</u>: Let $f \in C^1(\mathbb{R}^m; \mathbb{R}^m)$ and $g \in C^{0,1}(\mathbb{R}^m \times \mathbb{R}^m; \mathbb{R}^m)$ be consistent with the system of CLs (I), i.e., $g(v, v) = f(v)$ $\forall v \in \mathbb{R}^m$.
Assume we have a numerical grid $\mathcal{G} = \{(i\Delta x, n\Delta t) \mid i \in \mathbb{Z}, n \in \mathbb{N}\}$ and discretized initial data $u^0_i \in \mathbb{R}^m$.
A scheme of the form

$$u^{n+1}_i = u^n_i - \frac{\Delta t}{\Delta x}\left[g(u^n_i, u^n_{i+1}) - g(u^n_{i-1}, u^n_i) \right]$$

is said to be in conservation form with numerical flux g.

Notation: $C^{0,1}$: Lipschitz continuous ($h \in C^{k,\alpha}$: k continuous derivatives and $|h(x) - h(y)| \leq C \|x-y\|^{\alpha}$)

Remarks:

1) The above definition is an "abstract" definition in the sense that the exact way u_i^n and $g_{i+\frac{1}{2}}^n$ approximate volume / time averages of u and $f(u)$ differ from scheme to scheme and, in particular, do not exactly equal the integral expressions used above to motivate them.

2) The definition can be generalized to

$$u_i^{n+1} - u_i^n = -\frac{\Delta t}{\Delta x}\Big\{\theta\big[g(u_i^{n+1}, u_{i+1}^{n+1}) + g(u_{i-1}^{n+1}, u_i^{n+1})\big] - (1-\theta)\big[g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, g_i^n)\big]\Big\}$$

for $0 \leq \theta \leq 1$. For $\theta = 0$: explicit scheme ($\Leftrightarrow$ Def)
$\theta = 1$: implicit scheme

3) The important property of a scheme in conservative form is that it guarantees the conservation property of the solution on the discretized

level:

$$\sum_i \left(g^n_{i+\frac{1}{2}} - g^n_{i-\frac{1}{2}}\right) = \sum_i \left[g(u^n_i, u^n_{i+1}) - g(u^n_{i-1}, u^n_i)\right] = 0$$

all counterparts cancel

$$\Rightarrow \boxed{\sum_i u_i^{n+1} = \sum_i u_i^n}$$

conservation on discretized level

4) Consistency of g $\Rightarrow$ local truncation error is of order one (1st-order scheme)
Lemma 2.2.4 Kroener p. 45

6.2 Convergence: Lax-Wendroff theorem & entropy condition

<u>Theorem (Lax-Wendroff 1960):</u> (Comm. Pure Appl. Math. 13 (1960), 217-237)

Let $(u_\ell)_{\ell\in\mathbb{N}}$ be a sequence of discrete solutions of a scheme in conservation form with respect to $(h_\ell = \Delta x)_\ell$ and $(k_\ell = \Delta t)_\ell$, where $h_\ell, k_\ell \xrightarrow{\ell\to\infty} 0$ with $\frac{k_\ell}{h_\ell} = \text{const.}$

Assume that there exists $C \in \mathbb{R}$ such that

$$\sup_\ell \sup_{\mathbb{R}\times(0,\infty)} |u_\ell(x,t)| \leq C \qquad (u_\ell \text{ uniformly bounded})$$

and $u_\ell \xrightarrow{\ell\to\infty} u$ almost everywhere in $\mathbb{R}\times(0,\infty)$.

Then u is a weak solution of the system of CLs.

Proof: Consider explicit case ($\theta = 0$), see above.
Implicit case is analogous.

Choose $\varphi \in C_0^\infty(\mathbb{R}^m \times [0,\infty))$ and multiply scheme by φ:

compact support

overlap with initial data

$$(*) \quad \Delta x \left(u_i^{n+1} - u_i^n\right) \varphi(x_i, t^n) = -\Delta t \left(g^n_{i+\frac{1}{2}} - g^n_{i-\frac{1}{2}}\right) \varphi(x_i, t)$$

① Analyze convergence of LHS:

$$(**) \quad \sum_{n=0}^{\infty} \left(u_i^{n+1} - u_i^n\right) \varphi(x_i, t^n) = \sum_{n=1}^{\infty} u_i^n \left[\varphi(x_i, t^{n-1}) - \varphi(x_i, t^n)\right] - u_i^0 \varphi(x_i, 0)$$

(i) partial summation:

$$\sum_{k=m}^{n} f_k (g_{k+1} - g_k) = \left(\underbrace{f_n g_{n+1}}_{\substack{= 0 \\ \text{as } \varphi \text{ has} \\ \text{compact support}}} - f_m g_m\right) - \sum_{k=m+1}^{n} g_k (f_k - f_{k-1})$$

(ii) Abel's formula

$$A(t) = \sum_{0 \le n \le t} a_n \quad \text{partial sum function}$$

$$\sum_{x < n \le y} a_n \phi(n) = A(y)\phi(y) - A(x)\phi(x) - \int_x^y A(z) \phi'(z) \, dz$$

for ϕ differentiable function on $[x, y]$

set $x = -1$

$$\rightsquigarrow \sum_{n=0}^{x} a_n \phi(n) = A(x)\phi(x) - \int_0^x A(z)\phi'(z)\,dz$$

$$\rightsquigarrow (**) = -\int_0^\infty u_\ell(x_i,t)\,\partial_t \varphi(x_i,t)\,dt - u_i^0 \varphi(x_i,0)$$

for any $\ell \in \mathbb{N}$

Add sum over x_i:

$$\Delta x \sum_i \sum_{n=0}^{\infty} (u_i^{n+1} - u_i^n)\,\varphi(x_i,t^n)$$

$$= -\sum_i \Delta x \left\{ \int_0^\infty u_\ell(x_i,t)\,\partial_t\varphi(x_i,t)\,dt - u_i^0\varphi(x_i,0) \right\}$$

$$= -\int_{\mathbb{R}}\int_0^\infty u_\ell(x,t)\,\partial_t\varphi(x,t)\,dt\,dx - \int_{\mathbb{R}} u^0(x)\varphi(x,0)\,dx$$

$$+\mathcal{O}(\Delta x)$$

(note: both sums are finite due to $\varphi \in C_0^\infty$)

$$\Big\downarrow\; \ell \to \infty$$

$$= -\int_{\mathbb{R}}\int_0^\infty u(x,t)\,\partial_t\varphi(x,t)\,dt\,dx - \int_{\mathbb{R}} u^0(x)\varphi(x,0)\,dx$$

② Analyze convergence of RHS:

$$-\Delta t \sum_i \sum_{n=0}^{\infty} \left(g^n_{i+\frac{1}{2}} - g^n_{i-\frac{1}{2}}\right) \varphi(x_i, t^n)$$

partial summation

$$= -\Delta t \sum_i \sum_{n=0}^{\infty} g^n_{i+\frac{1}{2}} \left[\varphi(x_i, t^n) - \varphi(x_{i+1}, t^n)\right]$$

$$= \Delta x \Delta t \sum_i \sum_{n=0}^{\infty} g^n_{i+\frac{1}{2}} \, \partial_x \varphi(x_i, t^n) + \mathcal{O}(\Delta x)$$

Set: $g_\ell(x,t) \equiv g^n_{i+\frac{1}{2}}$ $\quad x_i \leq x < x_{i+1}$, $\quad t^n < t \leq t^{n+1}$

$= g(u_i^n, u_{i+1}^n)$, $\quad u_i^n = (u_\ell)_i^n$

$$= \sum_i \sum_{n=0}^{\infty} \int_{t^n}^{t^{n+1}} \int_{x_i}^{x_{i+1}} g_\ell(x,t) \, \partial_x \varphi(x,t) \, dx \, dt + \mathcal{O}(\Delta x)$$

$$= \int_{\mathbb{R}} \int_0^{\infty} g_\ell(x,t) \, \partial_x \varphi(x,t) \, dt \, dx + \mathcal{O}(\Delta x)$$

$\ell \to \infty$ $\quad$ g consistent & Lipschitz continuous

$$\int_{\mathbb{R}} \int_0^{\infty} \underbrace{g(u(x,t), u(x,t))}_{= f(u)} \, \partial_x \varphi(x,t) \, dt \, dx$$

To see how this convergence arises:

extend grid function u_i^n to $\mathbb{R}\times(0,\infty)$:

$$u_\ell(x,t) = u_i^n \quad \text{for} \quad t^n < t \le t^{n+1}$$
$$x_i - \frac{\Delta x}{2} < x \le x_i + \frac{\Delta x}{2}$$

$$\leadsto g_\ell(x,t) = g\left(u_\ell\left(x-\frac{\Delta x}{2}\right), u_\ell\left(x+\frac{\Delta x}{2}\right)\right)$$

and $\left|g_\ell(x,t) - g(u(x,t),u(x,t))\right|$

g lipschitz continuous

$$= \left|g\left(u_\ell\left(x-\frac{\Delta x}{2}\right), u_\ell\left(x+\frac{\Delta x}{2}\right)\right) - g(u(x,t),u(x,t))\right|$$
$$\le L_1\left|u_\ell\left(x-\frac{\Delta x}{2}\right) - u(x,t)\right| + L_2\left|u_\ell\left(x+\frac{\Delta x}{2}\right) - u(x,t)\right|$$

$(\cancel{**})\downarrow \ell\to\infty$

$$0 \quad \Rightarrow \quad g_\ell(x,t) \to g(u(x,t),u(x,t)) = f(u)$$

This is because (dropping t):

$$\int_0^\infty\int_{\mathbb{R}} u_\ell\left(x+\frac{\Delta x}{2}\right)\varphi(x)\,dx\,dt \overset{x\to x-\frac{\Delta x}{2}}{=} \int_0^\infty\int_{\mathbb{R}} u_\ell(x)\,\varphi\left(x-\frac{\Delta x}{2}\right)dx\,dt$$

$$\xrightarrow{\ell\to\infty} \int_0^\infty\int_{\mathbb{R}} u(x)\,\varphi(x,t)\,dx\,dt$$

$$\Rightarrow u_\ell\left(x+\frac{\Delta x}{2}\right) \to u(x) \quad \text{weak}$$

$$\Rightarrow u_\ell^2\left(x+\frac{\Delta x}{2}\right) \to u^2(x) \quad \text{weak}$$

L^2_{loc} convergence:

$$\int_0^\infty\int_{\mathbb{R}} \left|u_\ell\left(x+\frac{\Delta t}{2}\right)-u\right|^2 \varphi(x,t)\,dx\,dt = \int_0^\infty\int_{\mathbb{R}} u_\ell\left(x-\frac{\Delta t}{2}\right)^2 \varphi$$

$$-2\iint u_\ell\left(x+\frac{\Delta t}{2}\right) u(x)\,\varphi + \iint u^2 \varphi \xrightarrow{\ell\to\infty} 0$$

$\Rightarrow (***)$

$\underline{1+2} \Rightarrow (*)$

$$\Big\downarrow \ell\to\infty$$

$$\int_{\mathbb{R}}\int_0^\infty \left(u\cdot\varphi_t + f(u)\cdot\varphi_x\right) dt\,dx + \int_{\mathbb{R}} u_0\cdot\varphi(x,0)\,dx = 0$$

$\Rightarrow$ u weak solution

$\square$

<u>Examples:</u> Some simple straightforward choices for the numerical flux (scalar CLs)

g	$\frac{1}{\Delta x}\left(g_{i+\frac{1}{2}} - g_{i-\frac{1}{2}}\right)$

(i) $g(u,v) = f(u)$ | $\dfrac{f(u_i) - f(u_{i-1})}{\Delta x}$

(ii) $g(u,v) = f(v)$ | $\dfrac{f(u_{i+1}) - f(u_i)}{\Delta x}$

(iii) $g(u,v) = \frac{1}{2}(f(u) + f(v))$ | $\dfrac{f(u_{i+1}) - f(u_{i-1})}{2\Delta x}$

Note: g consistent & Lipschitz continuous if f Lipschitz continuous

$\leadsto$ • can interpret conservative scheme as backward, forward and central differences with u_i approximating volume averages

• depending on f' (i) or (ii) can be unstable (violation of CFL condition), also (iii) is generally unstable

• <u>Lax-Friedrichs method</u>: can make (iii) a stable conservative method

$$u_i^n \to \frac{1}{2}\left(u_{i-1}^n + u_{i+1}^n\right)$$

$$\leadsto u_i^{n+1} = \frac{1}{2}\left(u_{i-1}^n + u_{i+1}^n\right) - \frac{\Delta t}{2\Delta x}\left[f(u_{i+1}^n) - f(u_{i-1}^n)\right]$$

$$= u_i^n + \frac{1}{2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) \simeq u_i^n + \frac{1}{2}\underbrace{(\Delta x)^2}_{\substack{\to 0 \\ \text{as } \Delta x \to 0 \\ \frac{\Delta t}{\Delta x} = \text{const.}}} u_{xx}$$

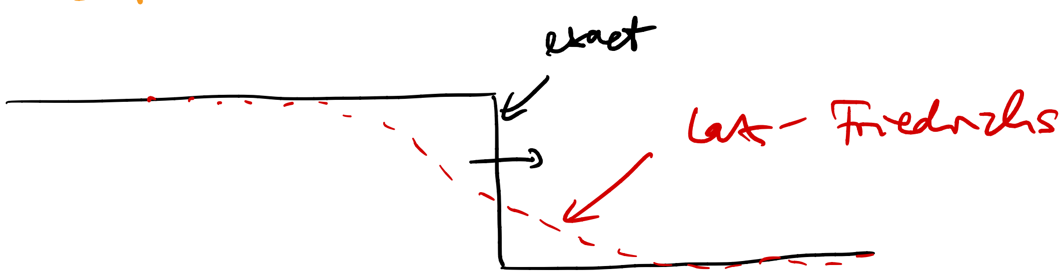
⇝ numerical dissipation

write in conservative form:

$$g(u_{i-1}^n, u_i^n) = \frac{1}{2}\left[f(u_{i-1}^n) + f(u_i^n)\right] - \frac{\Delta x}{2\Delta t}\left(u_i^n - u_{i-1}^n\right)$$

→ damping term makes scheme stable, (scheme also satisfies entropy condition)

but also "smears out" discontinuities



Remark: Importance of Conservation Form

Consider Burgers' equation, assume $u > 0$

$$u_t + \left(\tfrac{1}{2}u^2\right)_x = 0$$

Conservative upwind:

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(\frac{1}{2}(u_i^n)^2 - \frac{1}{2}(u_{i-1}^n)^2\right)$$

Quasi-linear (non-conservative): $\quad u_t + u u_x = 0$

'upwind':

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} u_i^n \left(u_i^n - u_{i-1}^n\right)$$

$$= u_i^n - \frac{\Delta t}{\Delta x}\left(\frac{1}{2}(u_i^n)^2 - \frac{1}{2}(u_{i-1}^n)^2\right)$$

$$+ \underbrace{\frac{1}{2}\Delta t\,\Delta x\left(\frac{u_i^n - u_{i-1}^n}{\Delta x}\right)^2}_{\simeq \frac{1}{2}\Delta x (u_x)^2}$$

does not vanish as $\Delta x \to 0$ for shocks (u_x not bounded)

$\to$ different shock speed

Remark: Weak solutions under coordinate transformations

Let $u = u(x,t)$ be a classical (C^1) solution to a system of CLs $u_t + f(u)_x = 0$. (i)

Consider: $u \overset{h^{-1}}{\longmapsto} \tilde{u} = h^{-1}(u)$ change of conserved variables

$\leadsto$ then $\tilde{u}$ is a C^1 solution of

$$(*) \qquad \tilde{u}_t + B(\tilde{u})\tilde{u}_x = 0, \quad B(v) \equiv [Dh(v)]^{-1} Df(h(v)) Dh(v), \quad v \in \mathbb{R}^n$$

assume there exists a function g with

$$Dg(v) = B(v) \quad \forall\, v \in \mathbb{R}^n$$

(ii) $\rightsquigarrow$ (*) $\Leftrightarrow$ $\tilde{u}_t + g(\tilde{u})_x = 0$ in conservative form

Note: • $\tilde{u}$ C^1 solution of (ii) iff $u = h(\tilde{u})$ is a C^1 solution to $u_t + f(u)_x = 0$

• this still holds if $u, \tilde{u}$ are Lipschitz continuous

• this equivalence of solutions <u>does not hold for weak solutions</u>

This is because the derivation of (ii) from (i) requires the existence of derivatives, which is only the case if u is smooth.

$\Rightarrow$ need to look at the "right" (physical) conservation laws

<u>Example:</u> Burgers' equation $u_t + \left(\frac{1}{2}u^2\right)_x = 0$ (I)

Consider $\tilde{u} = u^3$ $\rightsquigarrow$ $g(\tilde{u}) = \frac{3}{4}\tilde{u}^{4/3}$

$$u_t + u u_x = 0 \quad | \cdot 3u^2$$

$$3u^2 u_t + 3u^3 u_x = 0$$

$$(u^3)_t + \left(\frac{3}{4}u^4\right)_x = 0$$

$\rightsquigarrow \quad \tilde{u}_t + g(\tilde{u})_x = 0 \quad (\mathrm{II}) \quad \rightsquigarrow$ conservation law for u^3

Consider Riemann problem with $u_\ell > u_r$

$\rightsquigarrow$ the unique weak solution is a shock traveling at the speed given by the Rankine–Hugoniot jump conditions:

$$S_{\mathrm{I}} = \frac{\frac{1}{2}u_\ell^2 - \frac{1}{2}u_r^2}{u_\ell - u_r} = \frac{1}{2}(u_\ell + u_r)$$

$$S_{\mathrm{II}} = \frac{\frac{3}{4}u_\ell^4 - \frac{3}{4}u_r^4}{u_\ell^3 - u_r^3} = \frac{3}{4}\,\frac{u_\ell^4 - u_r^4}{u_\ell^3 - u_\ell^3} \neq S_{\mathrm{I}} \text{ if } u_\ell \neq u_r$$

$\Rightarrow$ different weak solutions

<u>Remark</u>: The Lax-Wendroff theorem only guarantees that a conservative scheme converges to a weak solution (if it converges)

$\rightsquigarrow$ how do we ensure to obtain the entropy solution?

Def (Discrete entropy condition): Let Φ, Ψ be an entropy pair for the system of CLs $u_t + f(u)_x = 0$ and let $G \in C^{0,1}(\mathbb{R}^m \times \mathbb{R}^m; \mathbb{R}^m)$ such that $G(v,v) = \Psi(v)$ $\forall v \in \mathbb{R}^m$. Let u_i^n be the solution of a numerical scheme in conservation form. Then u_i^n is said to satisfy a <u>discrete entropy condition</u> if

$$\Phi(u_i^{n+1}) - \Phi(u_i^n) \leq -\theta \frac{\Delta t}{\Delta x}\left[G(u_i^n, u_{i+1}^n) - G(u_{i-1}^n, u_i^n)\right]$$

$$(\ast) \qquad\qquad - (1-\theta)\frac{\Delta t}{\Delta x}\left[G(u_i^{n+1}, u_{i+1}^{n+1}) - G(u_{i-1}^{n+1}, u_i^{n+1})\right].$$

G is called <u>numerical entropy flux</u>. The scheme is said to be consistent with the entropy condition if $(\ast)$ holds uniformly for $\Delta t, \Delta x \to 0$ for any Φ and G.

Theorem (Convergence to entropy solution):

Consider situation of Lax-Wendroff theorem and assume that the scheme is consistent with the entropy condition. Then the scheme converges to a weak solution that satisfies the entropy condition.

<u>Proof</u>: Exercise.

Remark (Absence of sufficient criterion for convergence):

A sufficient criterion for convergence requires some form of stability. For scalar CLs conservative schemes that are total variation diminishing (TVD) or whose total variation is uniformly bounded

$$\left[\ |u_\ell(\cdot,t^n)| \le C_0 \quad \forall\, \ell, n, \quad TV_{[a,b]}(u_\ell(\cdot,t^n)) \le C_1 \quad \forall\, \ell, n, \right.$$

where
$$TV_{[a,b]}(f) = \sup_{\substack{a=x_0<x_1<\dots x_n=b \\ n\in\mathbb{N}}} \sum_{k=0}^{n} |f(x_{k+1}) - f(x_k)|$$

$$\left. \overset{\substack{f \text{ only} \\ \text{discrete} \\ \text{values}}}{=} \sum_{i=0}^{\infty} |f_{i+1} - f_i| \ \right]$$

guarantee a convergent (sub)sequence (cf. Theorem 2.3.9, Corollary 2.3.11, 2.3.12 Kroener)

However, for nonlinear systems of CLs the true solution is in general not TVD in any reasonable sense, and there is no general sufficient criterion for convergence

6.3 Godunov's method for nonlinear systems of CLs

Motivation: find conservative scheme that does not try to eliminate discontinuities (see Leveque for many examples) but rather exploits them

$\leadsto$ Godunov methods are built around local Riemann problems as the central ingredient

$\rightarrow$ capture shocks ("shock-capturing methods")

$\rightarrow$ can be extended to higher-order schemes

$\rightarrow$ state-of-the art in relativistic hydrodynamics

Idea: REA: reconstruct - evolve - average

1. Reconstruct a piece-wise polynomial function $u^n(x, t^n)$ from given cell-averages u_i^n

   $\rightarrow$ simplest case piece-wise constant function

$$u^n(x, t^n) \equiv u_i^n \qquad x_{i-\frac{1}{2}} < x \leq x_{i+\frac{1}{2}}$$

2. Evolve the system of CLs exactly (or approximately) with this initial data to obtain $u^{n+1}(x, t^{n+1})$

at $t^n + \Delta t$ (series of local Riemann problems)

3. Average function over grid cell to obtain new cell averages

$$u_i^{n+1} = \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u^{n+1}(x, t^{n+1})\, dx$$

and start over with Step 1.

<u>In detail:</u> Consider the hyperbolic system

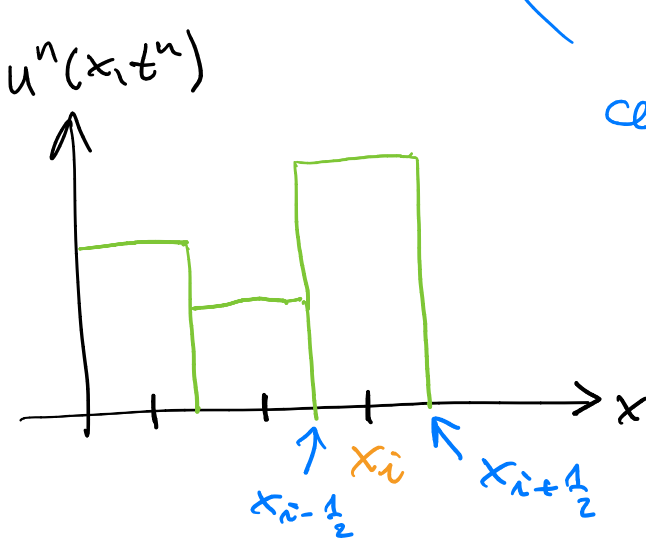
$$u_t + f(u)_x = 0 \qquad \text{in } \mathbb{R} \times (0, \infty)$$
$$u(x,0) = u_0(x) \qquad \text{in } \mathbb{R}$$

and spatial grid of finite volumes $I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ of size $\Delta x = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$ and

$$x_{i+\frac{1}{2}} = (i-1)\Delta x \qquad x_i = (i - \tfrac{1}{2})\Delta x \qquad x_{i+\frac{1}{2}} = i \Delta x$$

cell center

cell boundaries



- Assume $u_0(x) \in L^1(\mathbb{R})$ given and define

$$u_i^0 \equiv \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u_0(x)\, dx$$

$\rightsquigarrow$ assume that u_i^n is known

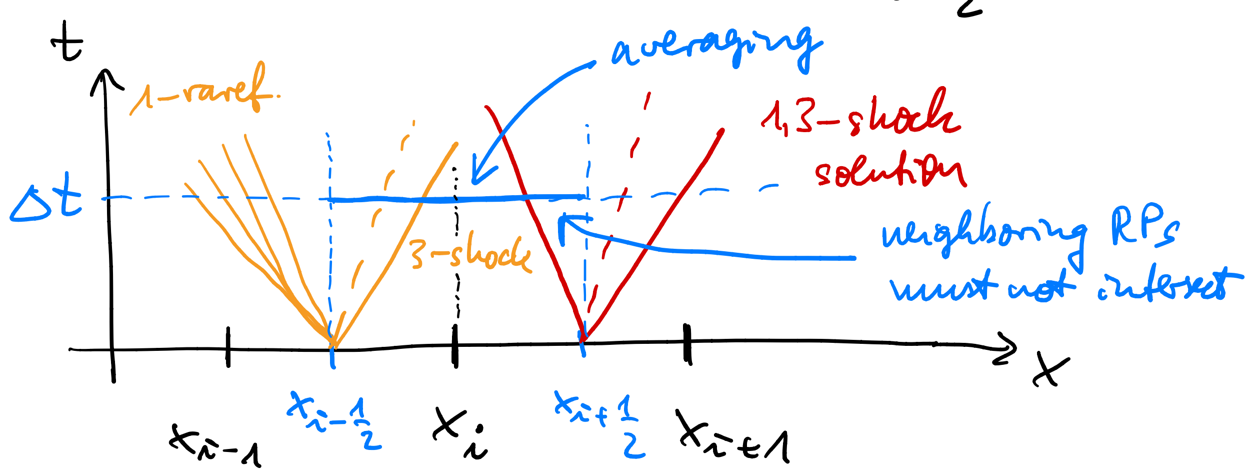
- Reconstruct piecewise constant function

$$u^n(x, t^n) \equiv u_i^n \quad \text{for } x \in [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$$

- For each cell (x_i, x_{i+1}) solve the local Riemann problem

$$u_t + f(u)_x = 0 \quad \text{on } [x_i, x_{i+1}] \times [t^n, t^{n+1}]$$

$$u(x, t = t^n) = \begin{cases} u_i^n, & x < x_{i+\frac{1}{2}} \\ u_{i+1}^n, & x > x_{i+\frac{1}{2}} \end{cases}$$



$\longrightarrow$ obtain exact (or approximate) solution

$w_i^n(x,t)$

$\rightarrow$ define global solution

$$V^n(x,t) \equiv w_i^n(x,t) \qquad \begin{array}{l} t^n \le t \le t^{n+1} \\ x_i \le x \le x_{i+1} \end{array}$$

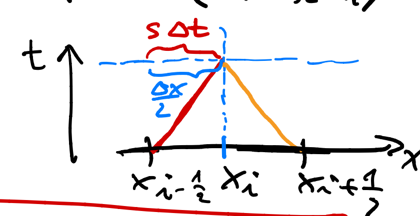
must ensure that neighboring solutions w_i^n do not influence each other, i.e. that shocks with speeds

$\lambda_k(u_i^n) < s_{i+\frac{1}{2}} < \lambda_k(u_{i+1}^n)$

or rarefaction waves with speeds $\lambda_k(u_i^n)$ originating from I_i do not intersect with shocks with speeds

$\lambda_k(u_{i-1}^n) < s_{i-\frac{1}{2}} < \lambda_k(u_i^n)$

or rarefaction waves with speeds $\lambda_k(u_{i-1})$ originating from I_{i-1}



CFL-like condition

$$\Rightarrow \boxed{\max\{|\lambda_k| \mid k \in \{1,\dots,m\}\}\,\Delta t \le \frac{\Delta x}{2}}$$

maximum wave velocity throughout the domain

where λ_k are the eigenvalues of $Df(u)$.

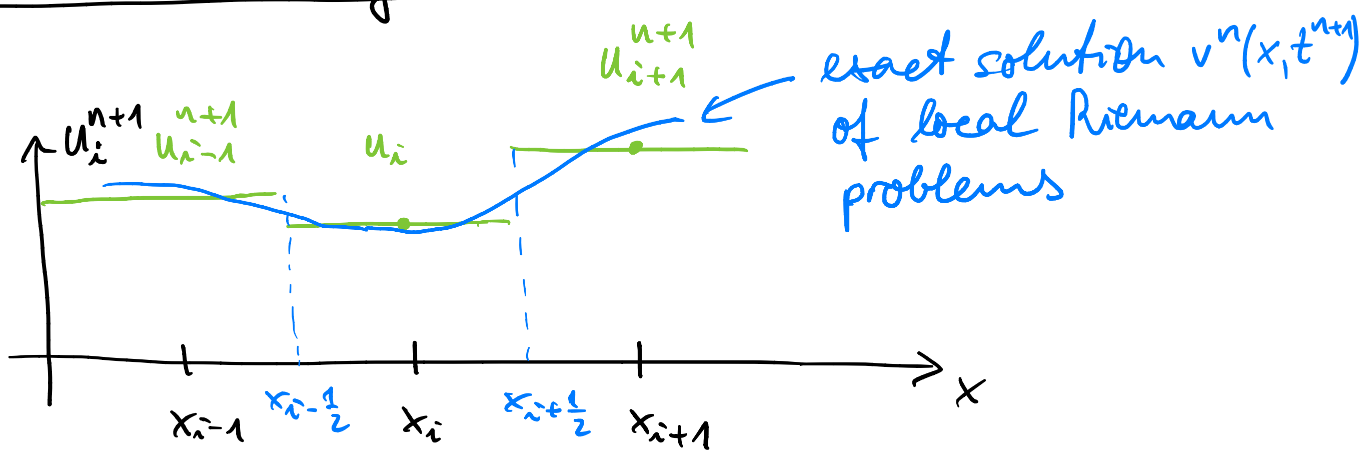
- Compute new cell averages

$$u_i^{n+1} \equiv \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} v^n(x, t^{n+1})\, dx$$

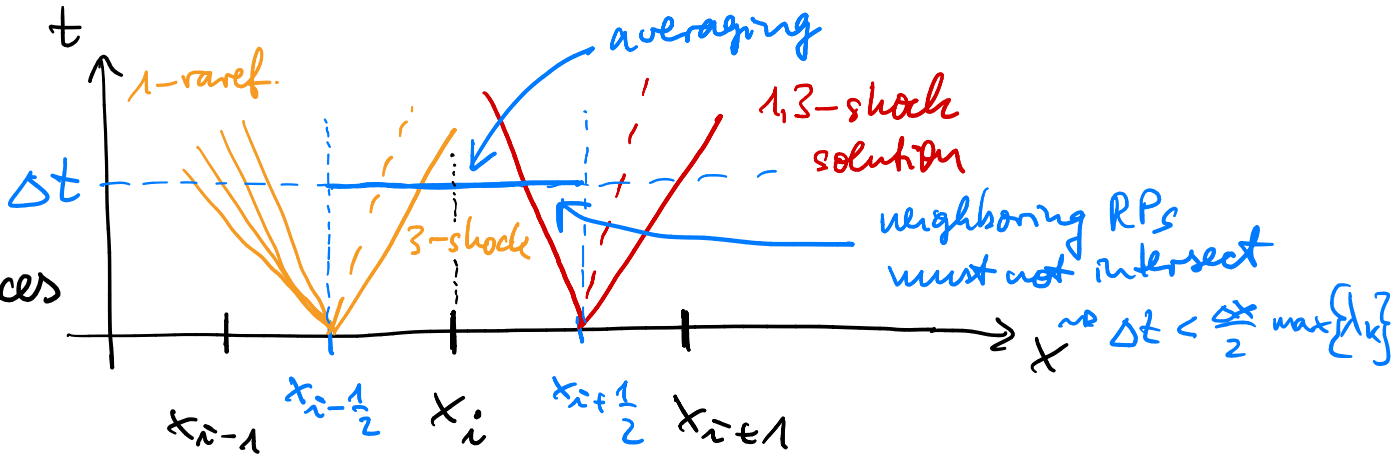
$$= \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_i} w_{i-1}^n(x, t^{n+1})\, dx + \frac{1}{\Delta x} \int_{x_i}^{x_{i+\frac{1}{2}}} w_i^n(x, t^{n+1})\, dx$$

Graphical summary:

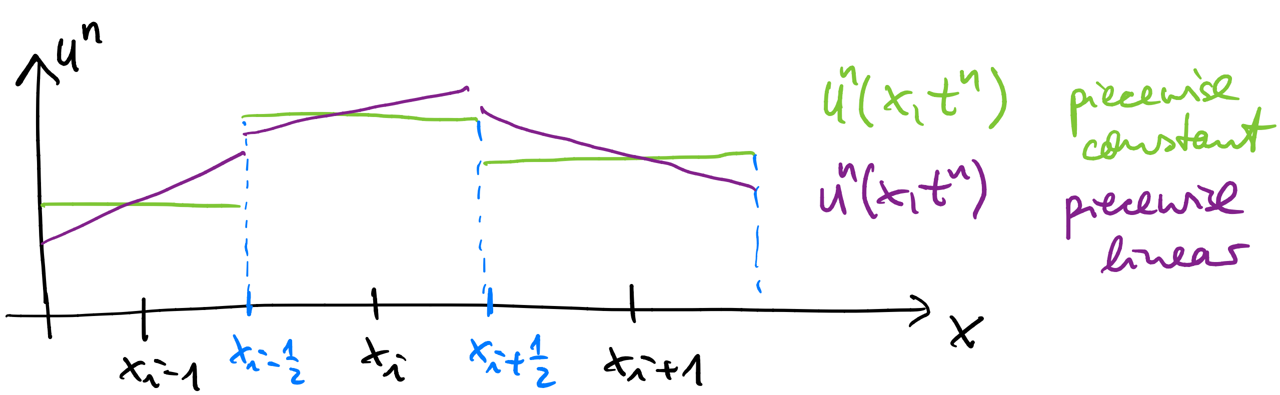
③ **Average** solution over cells



② **Evolve** RPs at cell interfaces



① **Reconstruct** from cell averages u_i^n



Proposition: The Godunov method can be written in conservative form

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left[g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, u_i^n)\right]$$

with intercell numerical flux

$$g(u_i^n, u_{i+1}^n) \equiv f\left(v^n(x_{i+\frac{1}{2}}, t^n)\right)$$

if the timestep Δt satisfies the condition

$$\Delta t < \frac{\Delta x}{2} \max\left\{ |\lambda_k| \;\middle|\; k \in \{1, \dots, m\} \right\}$$

where λ_k are the eigenvalues of $Df(u)$. The Godunov method is consistent and consistent with the entropy condition.

Proof: 1 v^n is (locally in time) the exact solution

see intro in 6.1

$$\Rightarrow \quad (*) \quad \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left[\underbrace{v^n(x, t^{n+1})}_{\overset{\text{def}}{=} u_i^{n+1}\Delta x} - \underbrace{v^n(x, t^n)}_{\overset{\text{def}}{=} u_i^n \Delta x}\right] dx = -\int_{t^n}^{t^{n+1}} \left[f\left(v^n(x_{i+\frac{1}{2}}, t)\right) - f\left(v^n(x_{i-\frac{1}{2}}, t)\right)\right] dt$$

Note $v^n(x_{i\pm\frac{1}{2}}, t)$ is constant on $[t^n, t^{n+1}]$

(solution of RP on ray $\frac{x}{t} = 0$ through $x_{i\pm\frac{1}{2}}$ wrt. u_i^n, u_{i+1}^n)

$$v^n(x_{i+\frac{1}{2}}, t^n) \stackrel{def}{=} w_i^n(x_{i+\frac{1}{2}}, t^n)$$

$$v^n(x_{i-\frac{1}{2}}, t^n) \stackrel{def}{=} w_{i-1}^n(x_{i-\frac{1}{2}}, t^n)$$

$$\Rightarrow \int_{t^n}^{t^{n+1}} f(v^n(x_{i\pm\frac{1}{2}}, t))\,dt = \Delta t\, f(v^n(x_{i\pm\frac{1}{2}}, t^n))$$

Therefore,

$$(*) \Leftrightarrow u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left[f(v^n(x_{i+\frac{1}{2}}, t^n)) - f(v^n(x_{i-\frac{1}{2}}, t^n))\right]$$

Define numerical flux

$$g(u_i^n, u_{i+1}^n) \equiv f(v^n(x_{i+\frac{1}{2}}, t^n))$$

Note: this makes sense since $v^n(x_{i+\frac{1}{2}}, t)$ is const. on $[t^n, t^{n+1}]$

$$\rightsquigarrow \quad u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left[g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, u_i^n)\right]$$

$\Rightarrow$ scheme in <u>conservative form</u>

Also: • if $u_i^n = u_{i+1}^n = \tilde{u}$, then $v^n(x_{i+\frac{1}{2}}, t^n) = \tilde{u}$
(constant state, not influenced by neighboring RP)
$\Rightarrow$ $g(\tilde{u}, \tilde{u}) = f(\tilde{u})$ $\Rightarrow$ g is <u>consistent</u>.

• g Lipschitz continuous if f Lipschitz continuous.

② entropy condition:

$v^n(x,t)$ as exact solution to the RP satisfies the entropy condition: let (Φ, Ψ) be an entropy pair
$\left(\rightsquigarrow \Phi(v^n)_t + \Psi(v^n)_x \leq 0\right)$

$$\Rightarrow \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \Phi(v^n(x, t^{n+1}))\, dx - \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \Phi(v^n(x, t^n))\, dx$$

$(\ast\ast)$
$$+ \int_{t^n}^{t^{n+1}} \Psi(v^n(x_{i+\frac{1}{2}}, t))\, dt - \int_{t^n}^{t^{n+1}} \Psi(v^n(x_{i-\frac{1}{2}}, t))\, dt \leq 0$$

• Since $v^n(x, t^n)$ is constant on $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \Phi(v^n(x, t^n))\, dx = \Delta x\, \Phi(v^n(x, t^n))$$
$$= \Delta x\, \Phi\left(\frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} v^n(x, t^n)\, dx\right) = \Delta x\, \Phi(u_i^n)$$

- Define

$$G(u_i^n, u_{i+1}^n) \equiv \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \Psi\left(v^n(x_{i+\frac{1}{2}}, t)\right) dt$$

$$= \Psi\left(v^n(x_{i+\frac{1}{2}}, t^n)\right)$$

v^n constant on $x_{i\pm\frac{1}{2}}$ and $[t^n, t^{n+1}]$

For $u_i^n = u_{i+1}^n \equiv \tilde{u} \Rightarrow G(\tilde{u}, \tilde{u}) = \Psi(\tilde{u})$

$\rightsquigarrow$ numerical entropy flux

Then:

$$\Phi(u_i^{n+1}) = \Phi\left(\frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} v^n(x, t^{n+1})\, dx\right)$$

Φ convex

$$\leq \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \Phi\left(v^n(x, t^{n+1})\right) dx \quad \text{(Jensen's inequality)}$$

$$\overset{(**)}{\leq} \Phi(u_i^n) - \frac{\Delta t}{\Delta x}\left[G(u_i^n, u_{i+1}^n) - G(u_{i-1}^n, u_i^n)\right]$$

$\Rightarrow$ scheme is consistent with the entropy condition.

$\square$