

Chapter 6: Numerical Schemes for

hyperbolic systems of CLs

Consider hyperbolic system of CLs:

$$(I) \quad u_t + f(u)_x = 0 \quad \text{in } \mathbb{R} \times (0, \infty)$$

$$(II) \quad u(x, 0) = u_0(x) \quad \text{in } \mathbb{R} \times \{0\}$$

6.1 Conservative schemes

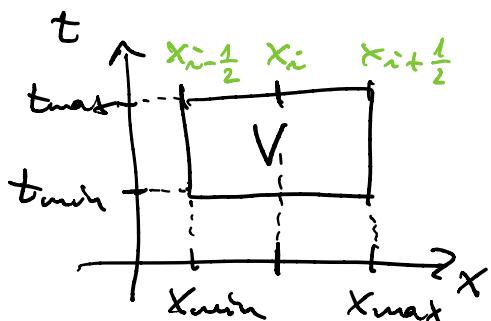
Motivation:

Integrate (I) over a rectangular control volume
 $V = (x_{min}, x_{max}) \times (t_{min}, t_{max}) \subseteq \mathbb{R} \times (0, \infty)$:

$$\iint [u_t + f(u)_x] dt dx$$

$$= \int [u(x, t_{max}) - u(x, t_{min})] dx$$

$$+ \int [f(u(x_{max}, t)) - f(u(x_{min}, t))] dt = 0$$



↓ set V to volume of
↓ one grid cell $[x_{i-1/2}, x_{i+1/2}] \times [t^n, t^{n+1}]$

$$(4) \quad \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} [u(x_i, t^{n+1}) - u(x_i, t^n)] dx + \int_{t^n}^{t^{n+1}} [f(u(x_{i+\frac{1}{2}}, t)) - f(u(x_{i-\frac{1}{2}}, t))] dt = 0$$

change in u in
 the volume $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$
 during $[t^n, t^{n+1}]$

- flow difference through
 cell interfaces during $[t^n, t^{n+1}]$

Assume there is a function $g: \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ ("numerical flux") such that (\simeq : "approximates"):

average
conserved
variable
in cell i

$$u_i^n \simeq \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x_i, t^n) dx$$

average
flux
through
interface
in Δt

$$g_{i+\frac{1}{2}}^n = g(u_i^n, u_{i+1}^n) \simeq \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} f(u(x_{i+\frac{1}{2}}, t)) dt$$

$$g_{i-\frac{1}{2}}^n = g(u_{i-1}^n, u_i^n) \simeq \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} f(u(x_{i-\frac{1}{2}}, t)) dt$$

$$\text{and } (4) \Leftrightarrow u_i^{n+1} - u_i^n = -\frac{\Delta t}{\Delta x} (g_{i+\frac{1}{2}}^n - g_{i-\frac{1}{2}}^n)$$

Consistency:

- We need to ensure that g is consistent with physical flux:

$$g(v, v) = f(v) \quad \forall v \in \mathbb{R}^m$$

This is because of the following example:

Consider 1D $u_0(x) = \text{const.}$

$\Rightarrow u = u_0$ is solution of CL and

$$g_{i+\frac{1}{2}}^0 = g(u_i^0, u_{i+1}^0) = g(u_0, u_0)$$

$$g_{i+\frac{1}{2}}^0 = \frac{1}{\Delta t} \int_0^{\Delta t} f(u(x_{i+\frac{1}{2}}, t)) dt = \frac{1}{\Delta t} \int_0^{\Delta t} f(u_0) dt = f(u_0)$$

- Also need to expect continuity as u_{i+1}, u_i vary

i.e. $g(u_i, u_{i+1}) \rightarrow f(v)$ as $u_{i+1}, u_i \rightarrow v$

no require Lipschitz continuity: there exist constants L_1, L_2 such that

$$\|g(u_{i+1}, u_i) - f(v)\| \leq L_1 \|u_{i+1} - v\|^{\alpha} + L_2 \|u_i - v\|^{\beta}$$

Def (Conservative scheme): let $f \in C^1(\mathbb{R}^m; \mathbb{R}^m)$

and $g \in C^{0,1}(\mathbb{R}^m \times \mathbb{R}^m; \mathbb{R}^m)$ be consistent with the system of CLs (I), i.e., $g(v, v) = f(v) \quad \forall v \in \mathbb{R}^m$.

Assume we have a numerical grid $\mathcal{G} = \{(i\Delta x, n\Delta t)\}$ $i \in \mathbb{Z}, n \in \mathbb{N}\}$ and discretized initial data $u_i^0 \in \mathbb{R}^m$. A scheme of the form

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} [g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, u_i^n)]$$

β said to be in conservation form with
numerical flux g.

Notation: $C^{0,1}$: Lipschitz continuous ($h \in C^{k,\alpha}$; k continuous derivatives)
(and $|h(x) - h(y)| \leq C \|x - y\|^\alpha$)

Remarks:

1) The above definition is an "abstract" definition
in the sense that the exact way u_i^n and $g_{i+\frac{1}{2}}^n$
approximate volume/time averages of u and $f(u)$
differ from scheme to scheme and, in particular, do
not exactly equal the integral expressions used
above to motivate them.

2) The definition can be generalized to

$$u_i^{n+1} - u_i^n = -\frac{\Delta t}{\Delta x} \left\{ \theta \left[g(u_i^{n+1}, u_{i+1}^{n+1}) + g(u_{i-1}^{n+1}, u_i^{n+1}) \right] \right. \\ \left. - (1-\theta) \left[g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, g_i^n) \right] \right\}$$

for $0 \leq \theta \leq 1$. For $\theta = 0$: explicit scheme (\Rightarrow Def)
 $\theta = 1$: implicit scheme

3) The important property of a scheme in conservative
form is that it guarantees the conservation
property of the solution on the discretized

level:

$$\sum_i \left(g_{i+\frac{1}{2}}^n - g_{i-\frac{1}{2}}^n \right) = \sum_i \left[g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, u_i^n) \right] = 0$$

↑
all counterparts cancel

$$\Rightarrow \boxed{\sum_i u_i^{n+1} = \sum_i u_i^n}$$

conservation on
discretized level

4) Consistency of $g \Rightarrow$ local truncation error is of order one (1st-order scheme)
Lemma 2.2.4 Kroener p. 45

6.2 Convergence: Lax-Wendroff

Theorem & entropy condition

Theorem (Lax-Wendroff 1960): (Comm. Pure Appl. Math. 13 (1960), 217-237)

Let $(u_e)_{e \in \mathbb{N}}$ be a sequence of discrete solutions of a scheme in conservation form with respect to $(h_e = \Delta x)_e$ and $(k_e = \Delta t)_e$, where $h_e, k_e \xrightarrow{e \rightarrow \infty} 0$ with $\frac{k_e}{h_e} = \text{const.}$

Assume that there exists $C \in \mathbb{R}$ such that

$$\sup_e \sup_{\mathbb{R} \times (0, \infty)} |u_e(x, t)| \leq C \quad (u_e \text{ uniformly bounded})$$

and $u_e \xrightarrow{e \rightarrow \infty} u$ almost everywhere in $\mathbb{R} \times (0, \infty)$.

Then u is a weak solution of the system of CLs.

Proof: Consider explicit case ($\theta = 0$), see above.
Implicit case is analogous.

Choose $\varphi \in C_0^\infty(\mathbb{R}^m \times (0, \infty))$ and multiply scheme by φ :

φ compact support

$$(*) \quad \underline{\Delta x} \underline{(u_i^{n+1} - u_i^n) \varphi(x_i, t^n)} = -\Delta t \underline{(g_{i+\frac{1}{2}}^{n+1} - g_{i-\frac{1}{2}}^n) \varphi(x_i, t)}$$

① Analyze convergence of LHS:

$$(*) \quad \sum_{n=0}^{\infty} (u_i^{n+1} - u_i^n) \varphi(x_i, t^n) = \sum_{n=1}^{\infty} u_i^n \left[\varphi(x_i, t^{n-1}) - \varphi(x_i, t^n) \right]$$

↓ $- u_i^0 \varphi(x_i, 0)$

(i) partial summation:

$$\sum_{k=m}^n f_k (g_{k+1} - g_k) = \underbrace{(f_m g_{m+1} - f_n g_n)}_{=0} - \sum_{k=m+1}^n g_k (f_k - f_{k-1})$$

as φ has
compact support

(ii) Abel's formula

$$A(t) = \sum_{0 \leq n \leq t} a_n \quad \text{partial sum function}$$

$$\sum_{x < n \leq y} a_n \phi(n) = A(y) \phi(y) - A(x) \phi(x) - \int_x^y A(z) \phi'(z) dz$$

for ϕ differentiable function on $[x, y]$

set $x = -1$

$$\text{MD} \sum_{n=0}^{\infty} a_n \phi(n) = A(x) \phi(x) - \int_0^x A(z) \phi'(z) dz$$

$$\text{MD}(\phi) = - \int_0^\infty u_\ell(x_i, t) \partial_t \phi(x_i, t) dt - u_i^0 \phi(x_i, 0)$$

for any $\ell \in \mathbb{N}$

Add sum over x_i :

$$\begin{aligned} \Delta x \sum_i \sum_{n=0}^{\infty} (u_i^{n+1} - u_i^n) \phi(x_i, t^n) \\ = - \sum_i \Delta x \left\{ \int_0^\infty u_\ell(x_i, t) \partial_t \phi(x_i, t) dt - u_i^0 \phi(x_i, 0) \right\} \\ = - \int_{\mathbb{R}} \int_0^\infty u_\ell(x, t) \partial_t \phi(x, t) dt dx - \int_{\mathbb{R}} u^0(x) \phi(x, 0) dx \end{aligned}$$

(note: both sums are finite due to $\phi \in C^\infty$) $+ O(\Delta x)$

$$\begin{array}{c} \downarrow \\ \ell \rightarrow \infty \end{array}$$
$$= - \int_{\mathbb{R}} \int_0^\infty u(x, t) \partial_t \phi(x, t) dt dx - \int_{\mathbb{R}} u^0(x) \phi(x, 0) dx$$

② Analyze convergence of RHF:

$$-\Delta t \sum_i \sum_{n=0}^{\infty} \left(g_{i+\frac{1}{2}}^n - g_{i-\frac{1}{2}}^n \right) \varphi(x_i, t^n)$$

partial summation

$$= -\Delta t \sum_i \sum_{n=0}^{\infty} g_{i+\frac{1}{2}}^n \left[\varphi(x_i, t^n) - \varphi(x_{i+1}, t^n) \right]$$

$$= \Delta x \Delta t \sum_i \sum_{n=0}^{\infty} g_{i+\frac{1}{2}}^n \partial_x \varphi(x_i, t^n) + O(\Delta t)$$



Set: $g_\ell(x, t) \equiv g_{i+\frac{1}{2}}^n \quad x_i \leq x \leq x_{i+1}$
 $t^n < t \leq t^{n+1}$

$$= g(u_i^n, u_{i+1}^n), \quad u_i^n = (u_\ell)_i^n$$

$$= \sum_i \sum_{n=0}^{\infty} \int_{t^n}^{t^{n+1}} \int_{x_i}^{x_{i+1}} g_\ell(x, t) \partial_x \varphi(x, t) dx dt + O(\Delta x)$$

$$= \iint_0^\infty g_\ell(x, t) \partial_x \varphi(x, t) dt dx + O(\Delta x)$$

$\downarrow l \rightarrow \infty$ *g consistent &
Lipschitz continuous*

$$\iint_0^\infty \underbrace{g(u(x, t), u(x+t))}_{=f(u)} \partial_x \varphi(x, t) dt dx$$

To see how this convergence arises:

extend good function u_i^n to $\mathbb{R} \times (0, \infty)$:

$$u_\ell(x, t) = u_i^n \text{ for } t^n < t \leq t^{n+1}$$

$$x_i - \frac{\Delta x}{2} < x \leq x_i + \frac{\Delta x}{2}$$

$$\Rightarrow g_\ell(x, t) = g(u_\ell(x - \frac{\Delta x}{2}), u_\ell(x + \frac{\Delta x}{2}))$$

$$\text{and } |g_\ell(x, t) - g(u(x, t), u(x, t))|$$

$$\begin{aligned} & \stackrel{g \text{ Lipschitz continuous}}{=} |g(u_\ell(x - \frac{\Delta x}{2}), u_\ell(x + \frac{\Delta x}{2})) - g(u(x, t), u(x, t))| \\ & \leq L_1 |u_\ell(x - \frac{\Delta x}{2}) - u(x, t)| + L_2 |u_\ell(x + \frac{\Delta x}{2}) - u(x, t)| \end{aligned}$$

(*) ~~if $L_1, L_2 < \infty$~~ $\downarrow l \rightarrow \infty$

$$0 \Rightarrow g_\ell(x, t) \rightarrow g(u(x, t), u(x, t)) = f(u)$$

This is because (dropping t):

$$\iint_0^\infty u_\ell(x + \frac{\Delta x}{2}) \varphi(x) dx dt = \iint_0^\infty u_\ell(x) \varphi(x - \frac{\Delta x}{2}) dx dt$$

$x \rightarrow x - \frac{\Delta x}{2}$

$$\xrightarrow{l \rightarrow \infty} \iint_0^\infty u(x) \varphi(x, t) dx dt$$

$$\Rightarrow u_\ell(x + \frac{\Delta x}{2}) \rightarrow u(x) \text{ weak}$$

$$\Rightarrow u_\ell^2(x + \frac{\Delta x}{2}) \rightarrow u^2(x) \text{ weak}$$

L^2_{loc} convergence:

$$\iint_0^\infty \int_{\mathbb{R}} |u_\ell(x + \frac{\sigma t}{2}) - u|^2 \varphi(x, t) dx dt = \iint_0^\infty \int_{\mathbb{R}} u_\ell(x - \frac{\sigma t}{2})^2 \varphi$$

$$- 2 \iint_{\mathbb{R}} u_\ell(x + \frac{\sigma t}{2}) u(x) \varphi + \iint_{\mathbb{R}} u^2 \varphi \xrightarrow{l \rightarrow \infty} 0$$

$\Rightarrow (\star\star\star)$

$$\underline{\textcircled{1} + \textcircled{2}} \Rightarrow (\star)$$

$$\downarrow l \rightarrow \infty$$

$$\iint_{\mathbb{R}^0} (u \cdot \varphi_t + f(u) \cdot \varphi_x) dt dx + \int_{\mathbb{R}} u_0 \cdot \varphi(x, 0) dx = 0$$

$\Rightarrow u$ weak solution

□

Examples: Some simple straightforward choices for the numerical flux (scalar CLs)

g	$\frac{1}{\Delta x} (g_{i+\frac{1}{2}} - g_{i-\frac{1}{2}})$
-----	--

$$(i) g(u, v) = f(u)$$

$$\frac{f(u_i) - f(u_{i-1})}{\Delta x}$$

$$(ii) g(u, v) = f(v)$$

$$\frac{f(u_{i+1}) - f(u_i)}{\Delta x}$$

$$(iii) g(u, v) = \frac{1}{2}(f(u) + f(v))$$

$$\frac{f(u_{i+1}) - f(u_{i-1})}{2 \Delta x}$$

Note: g consistent & Lipschitz continuous if f Lipschitz continuous

- can interpret conservative scheme as backward, forward and central differences with u_i approximating volume averages
- depending on f' (i) or (ii) can be unstable (violation of CFL condition), also (iii) is generally unstable
- Lax-Friedrichs method: can make (iii) a stable conservative method

$$u_i^n \rightarrow \frac{1}{2} (u_{i-1}^n + u_{i+1}^n)$$

$$u_i^{n+1} = \frac{1}{2} (u_{i-1}^n + u_{i+1}^n) - \frac{\Delta t}{2 \Delta x} [f(u_{i+1}^n) - f(u_{i-1}^n)]$$

$$= u_i^n + \frac{1}{2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n) \approx u_i^n + \frac{1}{2}(\Delta x)^2 u_{xx}$$

\Rightarrow numerical dissipation

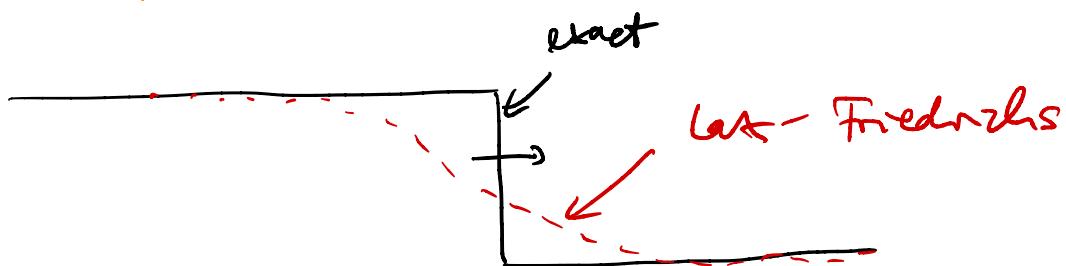
$\xrightarrow{\Delta x \rightarrow 0}$
 $\frac{\Delta t}{\Delta x} = \text{const.}$

write in conservative form:

$$g(u_{i-1}^n, u_i^n) = \frac{1}{2} [f(u_{i-1}^n) + f(u_i^n)] - \frac{\Delta x}{2\Delta t} (u_i^n - u_{i-1}^n)$$

\rightarrow damping term makes scheme stable,
(scheme also satisfies entropy condition)

but also "smears out" discontinuities



Remark: Importance of Conservation Form

Consider Burgers' equation, assume $u > 0$

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0$$

Conservative upwind:

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} \left(\frac{1}{2} (u_i^n)^2 - \frac{1}{2} (u_{i-1}^n)^2 \right)$$

Quasi-linear (non-conservative): $u_t + uu_x = 0$

'upwind':

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} u_i^n (u_i^n - u_{i-1}^n)$$

$$= u_i^n - \frac{\Delta t}{\Delta x} \left(\frac{1}{2} (u_i^n)^2 - \frac{1}{2} (u_{i-1}^n)^2 \right) + \frac{1}{2} \Delta t \Delta x \left(\frac{u_i^n - u_{i-1}^n}{\Delta x} \right)^2 \underbrace{\qquad\qquad\qquad}_{\simeq \frac{1}{2} \Delta x (u_x)^2}$$

does not vanish as $\Delta x \rightarrow 0$
 for shocks (u_x not bounded)
 \rightarrow different shock speed

Remark: Weak solutions under coordinate transformations

Let $u = u(x, t)$ be a classical (C^1) solution to a system of PDEs $u_t + f(u)_x = 0$. (i)

Consider: $u \xrightarrow{h^{-1}} \tilde{u} = h^{-1}(u)$ change of conserved variables

\Rightarrow then \tilde{u} is a C^1 solution of

$$(ii) \quad \tilde{u}_t + B(\tilde{u}) \tilde{u}_x = 0, \quad B(v) = [Dh(v)]^{-1} Df(h(v)) Dh(v) \quad v \in \mathbb{R}^n$$

assume there exists a function g with

$$Dg(v) = B(v) \quad \forall v \in \mathbb{R}^n$$

(ii) $\Leftrightarrow \tilde{u}_t + g(\tilde{u})_x = 0$ in conservative form

- Note:
- \tilde{u} C^1 solution of (ii) iff $u = h(\tilde{u})$ is a C^1 solution to $u_t + f(u)_x = 0$
 - this still holds if u, \tilde{u} are Lipschitz continuous
 - this equivalence of solutions does not hold for weak solutions

This is because the derivation of (ii) from (i) requires the existence of derivatives, which is only the case if u is smooth.

\Rightarrow need to look at the "right" (physical) conservation laws

Example: Burgers' equation $u_t + \left(\frac{1}{2}u^2\right)_x = 0$ (I)

$$\text{Consider } \tilde{u} = u^3 \Rightarrow g(\tilde{u}) = \frac{3}{4}\tilde{u}^{4/3}$$

$$\begin{cases} u_t + uu_x = 0 \\ 3u^2 u_t + 3u^3 u_x = 0 \end{cases} \quad | \cdot 3u^2$$

$$3u^2 u_t + 3u^3 u_x = 0$$

$$\begin{cases} u^3 + \left(\frac{3}{4}u^4\right)_x = 0 \end{cases}$$

$\Rightarrow \tilde{u}_t + g(\tilde{u})_x = 0$ (II) no conservation law
for u^3

Consider Riemann problem with $u_e > u_r$

no the unique weak solution is a shock traveling
at the speed given by the Rankine-Hugoniot
jump conditions:

$$S_I = \frac{\frac{1}{2}u_e^2 - \frac{1}{2}u_r^2}{u_e - u_r} = \frac{1}{2}(u_e + u_r)$$

$$S_{II} = \frac{\frac{3}{4}u_e^4 - \frac{3}{4}u_r^4}{u_e^3 - u_r^3} = \frac{3}{4} \frac{u_e^4 - u_r^4}{u_e^3 - u_r^3} \neq S_I \text{ if } u_e \neq u_r$$

\Rightarrow different weak solutions

Remark: The Lax-Wendroff theorem only
guarantees that a conservative scheme
converges to a weak solution (if it converges)
no how do we ensure to obtain the entropy
solution?

Def (Discrete entropy condition): let Φ, Ψ be an entropy pair for the system of CLs $u_t + f(u)_x = 0$ and let $G \in C^{0,1}(\mathbb{R}^m \times \mathbb{R}^m; \mathbb{R}^m)$ such that $G(v, v) = \Psi(v) \quad \forall v \in \mathbb{R}^m$. Let u_i^n be the solution of a numerical scheme in conservation form. Then u_i^n is said to satisfy a discrete entropy condition

if

$$\begin{aligned} \Phi(u_i^{n+1}) - \Phi(u_i^n) &\leq -\Theta \frac{\Delta t}{\Delta x} [G(u_i^n, u_{i+1}^n) - G(u_{i-1}^n, u_i^n)] \\ (\dagger) \quad &- (1-\Theta) \frac{\Delta t}{\Delta x} [G(u_i^{n+1}, u_{i+1}^{n+1}) - G(u_{i-1}^{n+1}, u_i^{n+1})]. \end{aligned}$$

G is called numerical entropy flux. The scheme is said to be consistent with the entropy condition if (\dagger) holds uniformly for $\Delta t, \Delta x \rightarrow 0$ for any Θ and G .

Theorem (Convergence to entropy solution):

Consider situation of lax-Wendroff theorem and assume that the scheme is consistent with the entropy condition. Then the scheme converges

to a weak solution that satisfies the entropy condition.

Proof: Exercise.

Remark (Absence of sufficient criterion for convergence):

A sufficient criterion for convergence requires some form of stability. For scalar CLs conservative schemes that are total variation diminishing (TVD) or whose total variation is uniformly bounded

$$\boxed{|\mathbf{U}_e(\cdot, t^n)| \leq C_0 + \epsilon_n, \quad TV_{[a,b]}(u_e(\cdot, t^n)) \leq C_1 + \epsilon_n,}$$

where $TV_{[a,b]}(f) = \sup_{\substack{a = x_0 < x_1 < \dots < x_n = b \\ n \in \mathbb{N}}} \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)|$

f only
discrete values $\Rightarrow \sum_{i=0}^{\infty} |f_{i+1} - f_i|$

guarantee a convergent (sub)sequence

(cf. Theorem 2.3.9, Corollary 2.3.11, 2.3.12 Kroener)

However, for nonlinear systems of CLs the true solution is in general not TVD in any reasonable sense, and there is no general sufficient criterion for convergence

6.3 Godunov's method for nonlinear systems of CLs

Motivation: find conservative scheme that does not try to eliminate discontinuities (see Lax for many examples) but rather exploits them

- Godunov methods are built around local Riemann problems as the central ingredient
- capture shocks ("shock-capturing methods")
- can be extended to higher-order schemes
- state-of-the-art in relativistic hydrodynamics

Idea: REA: reconstruct - evolve - average

1. Reconstruct a piece-wise polynomial function $u^n(x, t^n)$ from given cell-averages u_i^n
 - simplest case piece-wise constant function
 - $$u^n(x, t^n) = u_i^n \quad x_{i-\frac{1}{2}} \leq x \leq x_{i+\frac{1}{2}}$$

2. Evolve the system of CLs exactly (or approximately) with this initial data to obtain $u^{n+1}(x, t^{n+1})$

at $t^n + \Delta t$ (series of local Riemann problems)

3. Average function over grid cell to obtain new cell averages

$$u_i^{n+1} = \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u^{n+1}(x, t^{n+1}) dx$$

and start over with Step 1.

In detail: Consider the hyperbolic system

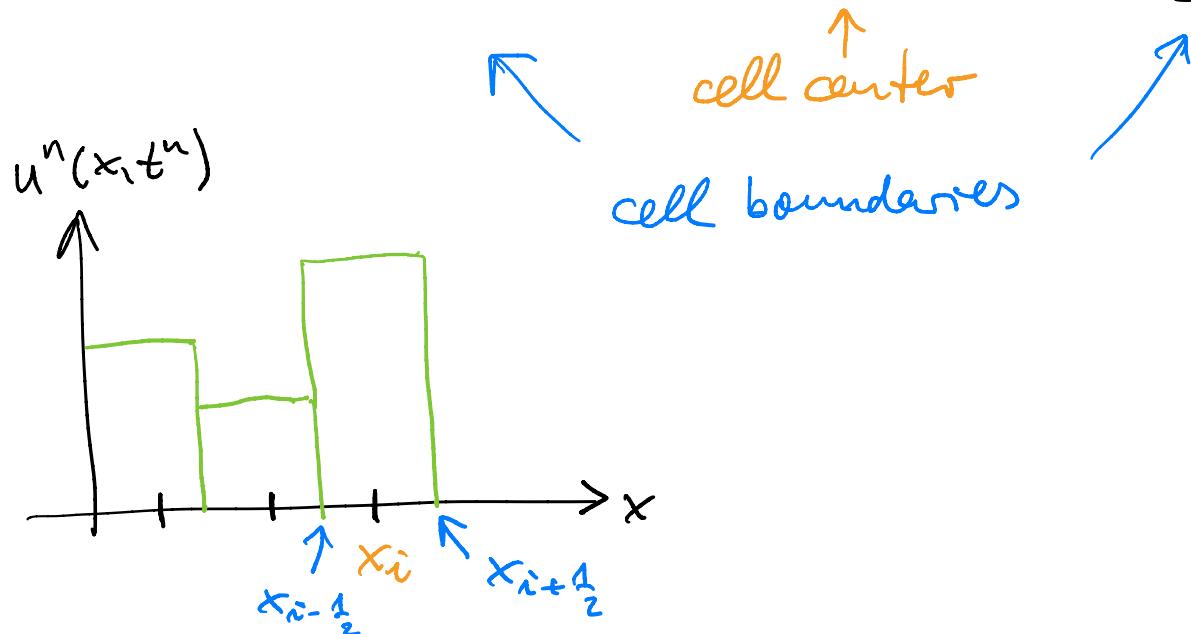
$$u_t + f(u)_x = 0 \quad \text{in } \mathbb{R} \times (0, \infty)$$

$$u(x, 0) = u_0(x) \quad \text{in } \mathbb{R}$$

and spatial grid of finite volumes $I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$

of size $\Delta x = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$ and

$$x_{i+\frac{1}{2}} = (i-1)\Delta x \quad x_i = \left(i - \frac{1}{2}\right)\Delta x \quad x_{i-\frac{1}{2}} = i\Delta x$$



- Assume $u_0(x) \in L^1(\mathbb{R})$ given and define

$$u_i^0 = \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u_0(x) dx$$

and assume that u_i^n is known

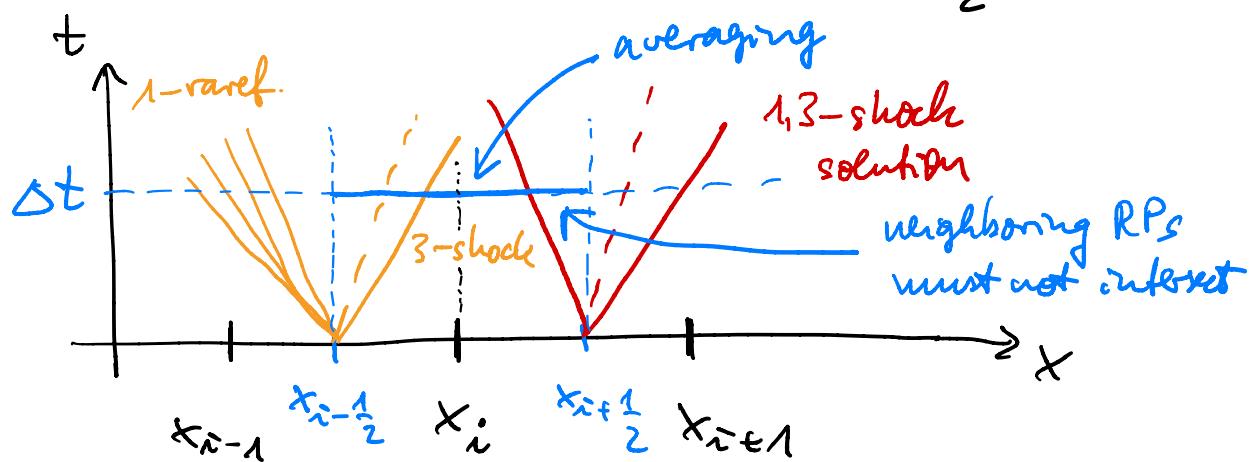
- Reconstruct piecewise constant function

$$u(x, t^n) = u_i^n \text{ for } x \in [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$$

- For each cell (x_i, x_{i+1}) solve the local Riemann problem

$$u_t + f(u)_x = 0 \text{ on } [x_i, x_{i+1}] \times [t^n, t^{n+1}]$$

$$u(x, t=t^n) = \begin{cases} u_i^n, & x < x_{i+\frac{1}{2}} \\ u_{i+1}^n, & x > x_{i+\frac{1}{2}} \end{cases}$$



→ obtain exact (or approximate) solution

$$w_i^n(x, t)$$

→ define global solution

$$v^n(x, t) \equiv w_i^n(x, t) \quad \begin{aligned} t^n &\leq t \leq t^{n+1} \\ x_i &\leq x \leq x_{i+1} \end{aligned}$$

must ensure that neighboring solutions w_i^n do not influence each other, i.e. that shocks with speeds

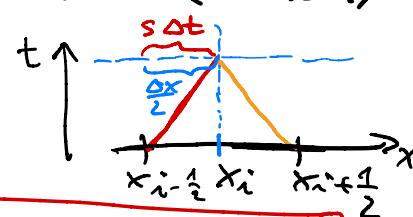
$$\lambda_k(u_i^n) < s_{i+\frac{1}{2}} < \lambda_k(u_{i+1}^n)$$

or rarefaction waves with speeds $\lambda_k(u_i^n)$ originating from I_i do not intersect with shocks with speeds

$$\lambda_k(u_{i-1}^n) < s_{i-\frac{1}{2}} < \lambda_k(u_i^n)$$

or rarefaction waves with speeds $\lambda_k(u_{i-1}^n)$

originating from I_{i-1}



CFL-like condition

$$\Rightarrow \boxed{\max\{\lambda_k \mid k \in \{1, \dots, m\}\} \Delta t \leq \frac{\Delta x}{2}}$$

maximum wave velocity throughout the domain

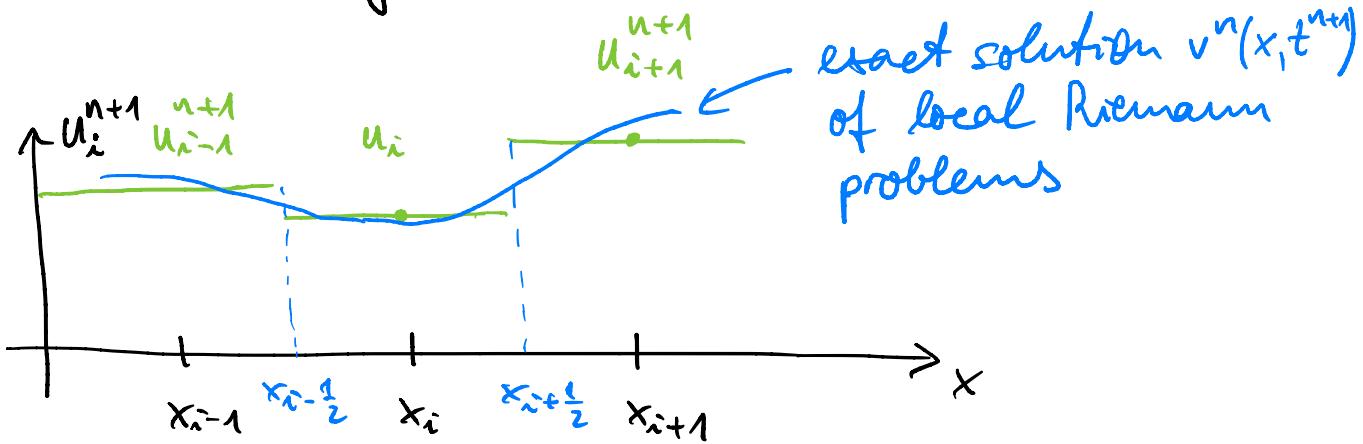
where λ_k are the eigenvalues of $Df(u)$.

- Compute new cell averages

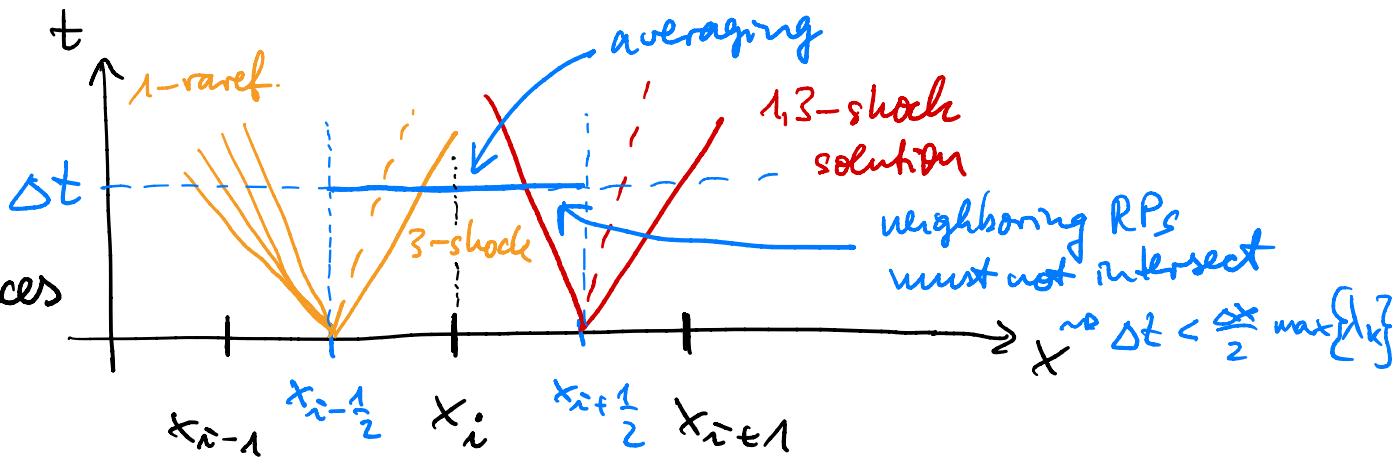
$$\begin{aligned}
 u_i^{n+1} &= \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} v^n(x, t^{n+1}) dx \\
 &= \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_i} w_{i-1}^n(x, t^{n+1}) dx + \frac{1}{\Delta x} \int_{x_i}^{x_{i+\frac{1}{2}}} w_i^n(x, t^{n+1}) dx
 \end{aligned}$$

Graphical Summary:

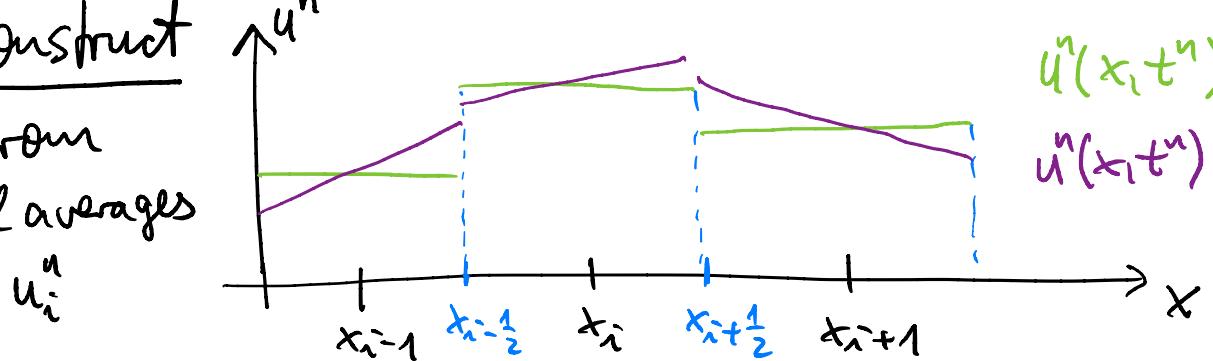
③
Average solution over cells



②
Evolve RPs at cell interfaces



①
Reconstruct from cell averages



Proposition: The Godunov method can be written in conservative form

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} [g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, u_i^n)]$$

with intercell numerical flux

$$g(u_i^n, u_{i+1}^n) \equiv f(v^n(x_{i+\frac{1}{2}}, t^n))$$

If the timestep Δt satisfies the condition

$$\Delta t < \frac{\Delta x}{2} \max \{ |\lambda_k| \mid k \in \{1, \dots, m\} \}$$

where λ_k are the eigenvalues of $Df(u)$. The Godunov method is consistent and consistent with the entropy condition.

Proof: ① v^n is (locally in time) the exact solution

see intro
in 6.1

$$\Rightarrow \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} [v^n(x, t^{n+1}) - v^n(x, t^n)] dx = - \int_{t^n}^{t^{n+1}} [f(v^n(x_{i+\frac{1}{2}}, t)) - f(v^n(x_{i-\frac{1}{2}}, t))] dt$$

$\underbrace{\hspace{10em}}$ $\underbrace{\hspace{10em}}$

$\stackrel{\text{def}}{=} u_i^{n+1} \Delta x$ $\stackrel{\text{def}}{=} u_i^n \Delta x$

Note $v^n(x_{i \pm \frac{1}{2}}, t)$ is constant on $[t^n, t^{n+1}]$
 (solution of RP on ray $\frac{x}{t} = 0$ through $x_{i \pm \frac{1}{2}}$
 wrt. u_i^n, u_{i+1}^n)

$$v^n(x_{i+\frac{1}{2}}, t^n) \stackrel{\text{def}}{=} w_i^n(x_{i+\frac{1}{2}}, t^n)$$

$$v^n(x_{i-\frac{1}{2}}, t^n) \stackrel{\text{def}}{=} w_{i-1}^n(x_{i-\frac{1}{2}}, t^n)$$

$$\Rightarrow \int_{t^n}^{t^{n+1}} f(v^n(x_{i \pm \frac{1}{2}}, t)) dt = \Delta t f(v^n(x_{i \pm \frac{1}{2}}, t^n))$$

Therefore,

$$(*) \Leftrightarrow u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} [f(v^n(x_{i+\frac{1}{2}}, t^n)) - f(v^n(x_{i-\frac{1}{2}}, t^n))]$$

Define numerical flux

$$g(u_i^n, u_{i+1}^n) = f(v^n(x_{i+\frac{1}{2}}, t^n))$$

Note: this makes sense since $v^n(x_{i \pm \frac{1}{2}}, t)$ is const.
 on $[t^n, t^{n+1}]$

$$\text{and } u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} [g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, u_i^n)]$$

\Rightarrow scheme in conservative form

- Also:
- if $u_i^n = u_{i+1}^n = \tilde{u}$, then $v^n(x_{i+\frac{1}{2}}, t^n) = \tilde{u}$
(constant state, not influenced by neighbouring RP)
 $\Rightarrow g(\tilde{u}, \tilde{u}) = f(\tilde{u}) \Rightarrow g$ is consistent.
 - g Lipschitz continuous if f Lipschitz continuous.

② entropy condition:

$v^n(x, t)$ as exact solution to the RP satisfies the entropy condition: let (Φ, Ψ) be an entropy pair
 $(\text{and } \Phi(v^n)_t + \Psi(v^n)_x \leq 0)$

$$\begin{aligned} & \Rightarrow \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \Phi(v^n(x, t^{n+1})) dx - \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \Phi(v^n(x, t^n)) dx \\ & + \int_{t^n}^{t^{n+1}} \Psi(v^n(x_{i+\frac{1}{2}}, t)) dt - \int_{t^n}^{t^{n+1}} \Psi(v^n(x_{i-\frac{1}{2}}, t)) dt \leq 0 \end{aligned}$$

- Since $v^n(x, t^n)$ is constant on $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$

$$\begin{aligned} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \Phi(v^n(x, t^n)) dx &= \Delta x \Phi(v^n(x, t^n)) \\ &= \Delta x \Phi\left(\frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} v^n(x, t^n) dx\right) = \Delta x \Phi(u^n) \end{aligned}$$

- Define

$$G(u_i^n, u_{i+1}^n) \equiv \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \Psi(v^n(x_{i+\frac{1}{2}}, t)) dt$$

$$= \Psi(v^n(x_{i+\frac{1}{2}}, t^n))$$

v^n constant on $x_{i+\frac{1}{2}}$ and $[t^n, t^{n+1}]$

For $u_i^n = u_{i+1}^n \equiv \tilde{u} \Rightarrow G(\tilde{u}, \tilde{u}) = \Psi(\tilde{u})$

no numerical entropy flux

Then:

$$\Phi(u_i^{n+1}) = \Phi\left(\frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} v^n(x, t^{n+1}) dx\right)$$

Φ convex

$$\leq \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \Phi(v^n(x, t^{n+1})) dx \quad (\text{Jensen's inequality})$$

(*)

$$\leq \Phi(u_i^n) - \frac{\Delta t}{\Delta x} [G(u_i^n, u_{i+1}^n) - G(u_{i+1}^n, u_i^n)]$$

\Rightarrow scheme is consistent with the entropy condition.

□