

Chapter 5: The Riemann problem for the Euler Equations

5.1 General setup

Consider the Euler equations in conservative form:

$$u_t + f(u)_x = 0$$

$$\begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix}_t + \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (E + p)u \end{bmatrix}_x = 0$$

assume ideal gas EOS: $p = RST = (\gamma - 1)e_{int}$

$$E = \frac{p}{\gamma - 1} + \frac{1}{2} \rho u^2$$

Exercise: a) Show that

$$Df(u) = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2}(\gamma - 3)u^2 & (\gamma - \gamma)u & \gamma - 1 \\ \frac{1}{2}(\gamma - 1)u^3 - uH & H - (\gamma - 1)u^2 & \gamma u \end{pmatrix}$$

$$\text{where } H = \frac{E+p}{\rho} = h + \frac{1}{2}u^2 \quad (\text{total specific enthalpy}).$$

Furthermore, show that Df has eigenvalues and eigenvectors (λ_k, r_k) :

$$\lambda_1 = u - c, \quad \lambda_2 = u, \quad \lambda_3 = u + c$$

$$r_1 = \begin{pmatrix} 1 \\ u - c \\ H - uc \end{pmatrix}, \quad r_2 = \begin{pmatrix} 1 \\ u \\ \frac{1}{2}u^2 \end{pmatrix}, \quad r_3 = \begin{pmatrix} 1 \\ u + c \\ H + uc \end{pmatrix}$$

where $c = \sqrt{\left. \frac{\partial P}{\partial S} \right|_{S=\text{const}}} = \sqrt{\frac{\partial P}{\partial S}}$ (sound speed for ideal gas)

b) Show that (λ_1, r_1) & (λ_3, r_3) are genuinely non-linear, while (λ_2, r_2) is linearly degenerate.

c) Show that s (specific entropy) is a Riemann invariant and that the k-Riemann invariants are given by

$$1-RI : \quad s, \quad u + \frac{2c}{\gamma-1}$$

$$2-RI : \quad u, \quad p$$

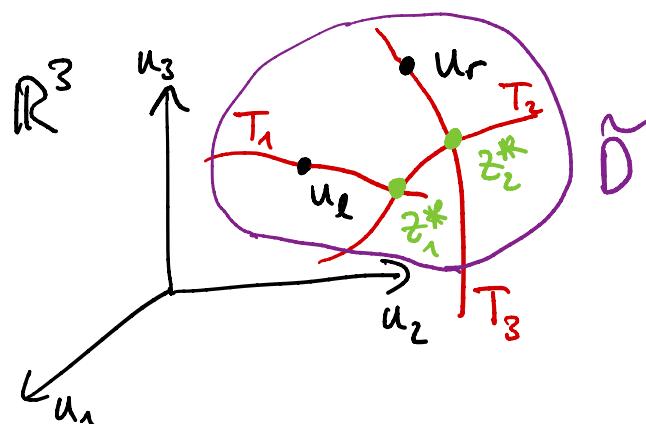
$$3-RI : \quad s, \quad u - \frac{2c}{\gamma-1}$$

and General structure of the RP for the Euler equations

Situation in state space:

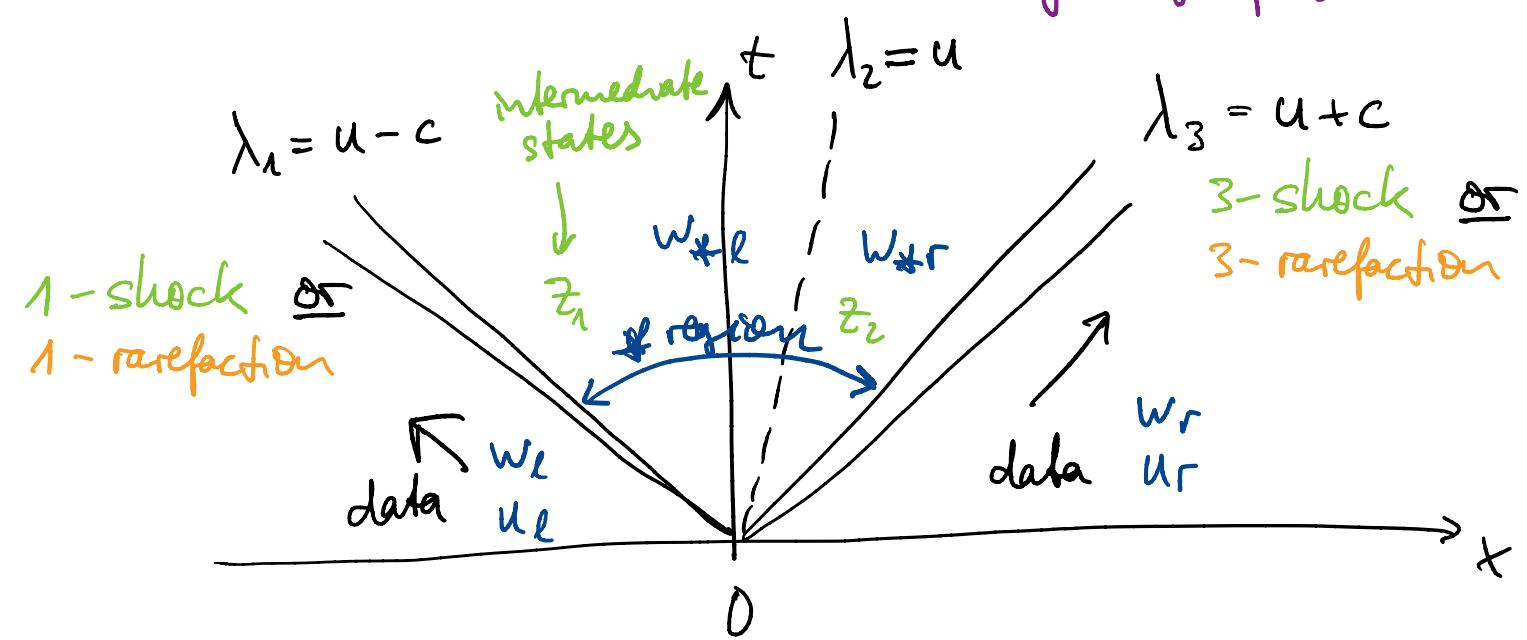
$m = 3$ equations

$$u = (u_1, u_2, u_3) = (s, su, E)$$



Situation in real space:

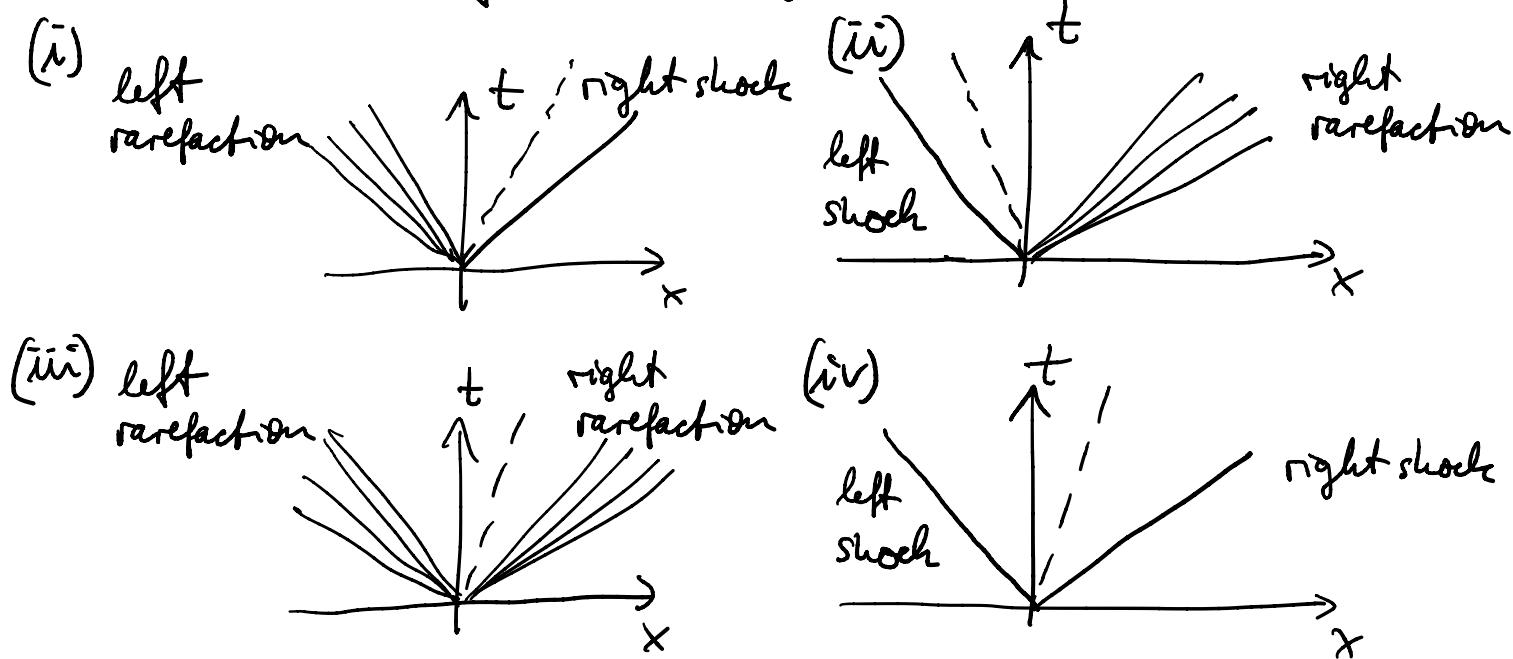
contact discontinuity ("entropy wave")
→ jump in s



Remarks: 1) Since both u & p are constant across contact discontinuities (\rightarrow constant in \mathbb{M} -region) it is easier to work with the primitive variables $w = (s, u, p)$ rather than the conserved variables $u = (s, su, E)$

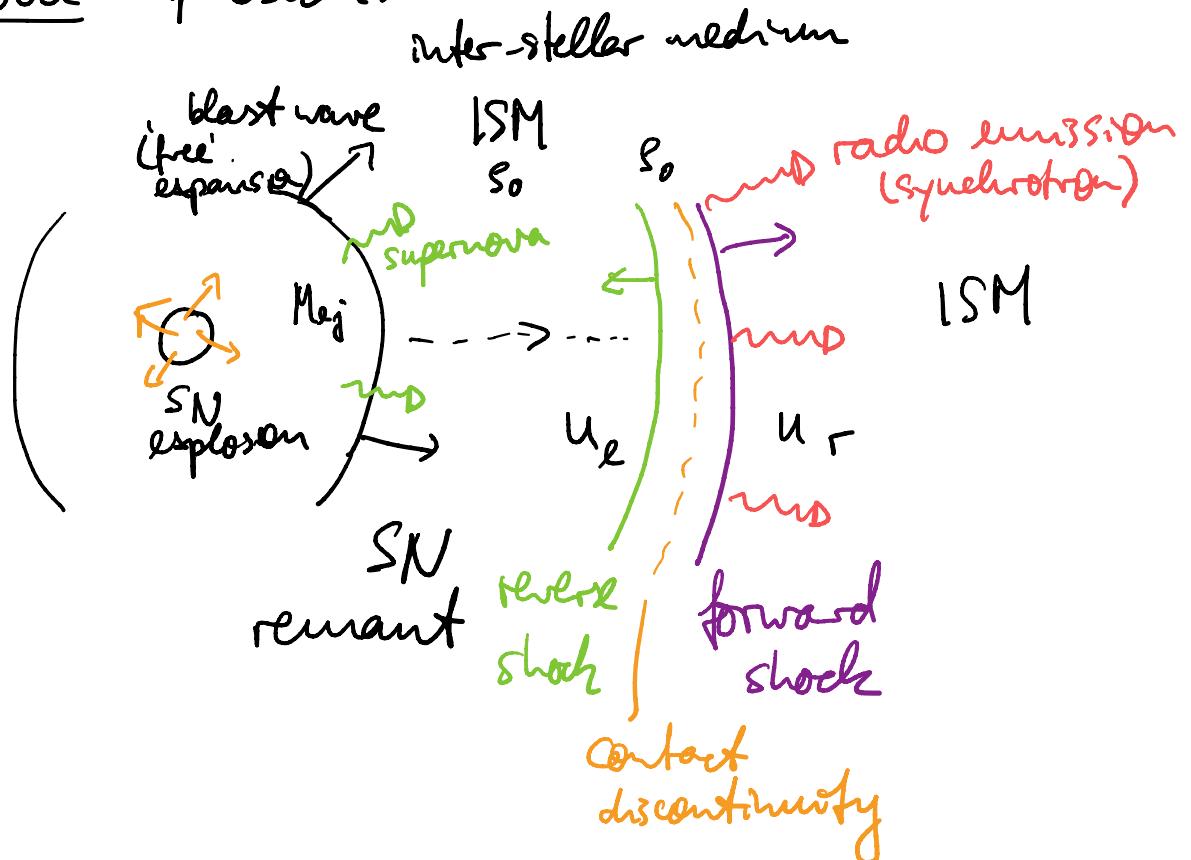
Recall: k-Riemann invariants are constant along integral curves of Γ_k , i.e. across rarefaction waves and contact discontinuities

2) There are 4 possible patterns for the solution of the RP:

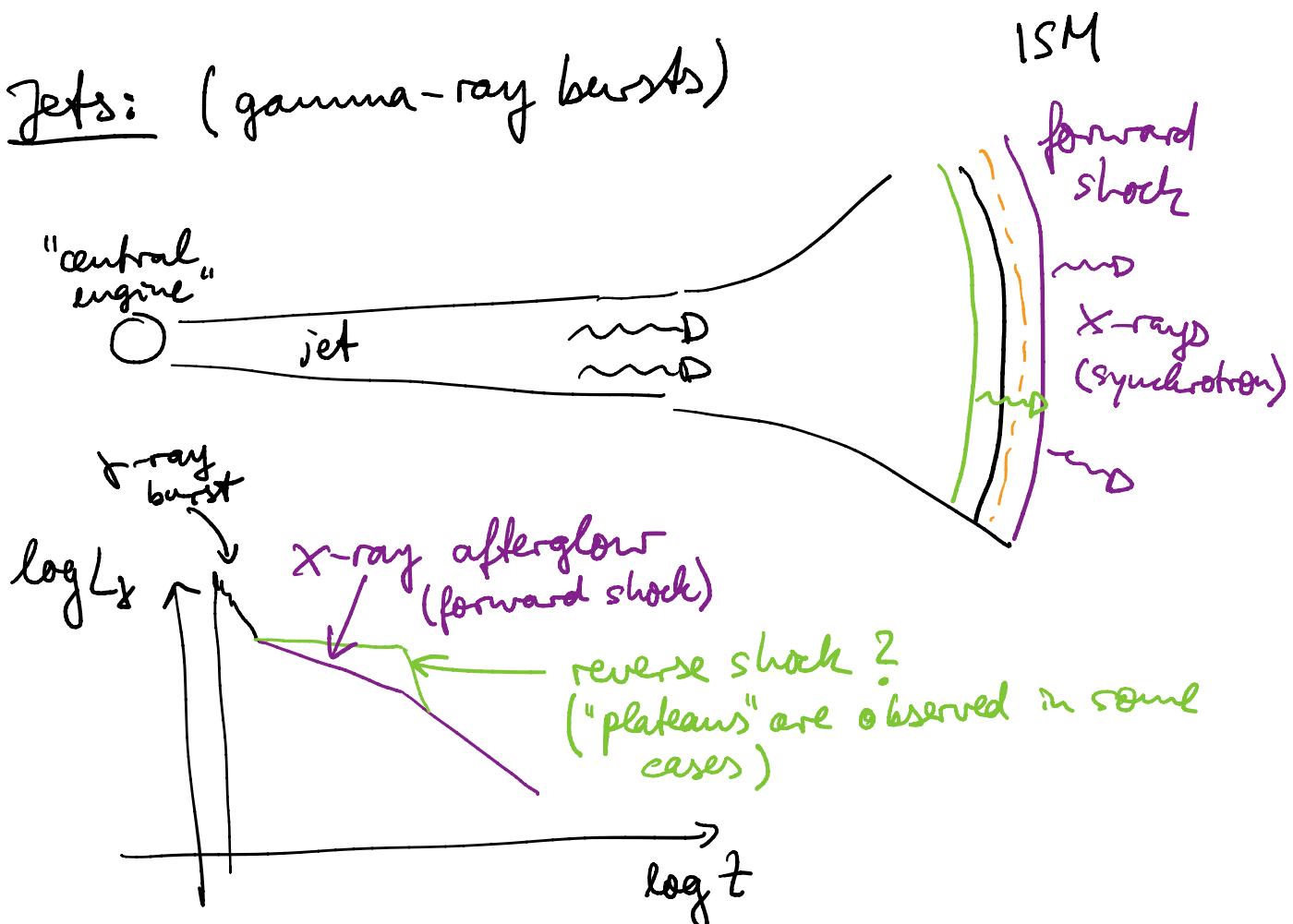


3) Some examples in astrophysics

Supernova explosion:



Jets: (gamma-ray bursts)



5.2 Solution strategy

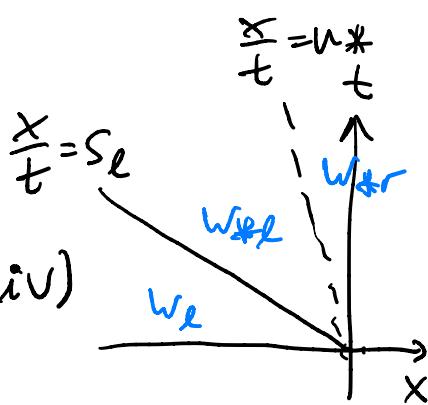
- ① Assume we know u_*, p_* ($= \text{const.}$ in $*\text{-region}$), evaluate the case (i) - (iv) that applies to this (u^*, p^*) configuration and write down the combined solution $w = (v, u, p)$, $w = w(x, t)$.
- ② Given a (reasonable) initial guess for (u_*^{in}, p_*^{in}) , find an iterative scheme to determine (u_*, p_*) from which the final solution will follow with ①.

Let's do it:

① Treat regions left and right of the contact discontinuity separately:

(1) $S \equiv \frac{x}{t} \leq u_* : \underline{\text{left side}}$

$P_* > P_L$ \Rightarrow left shock (ii) or (iv)



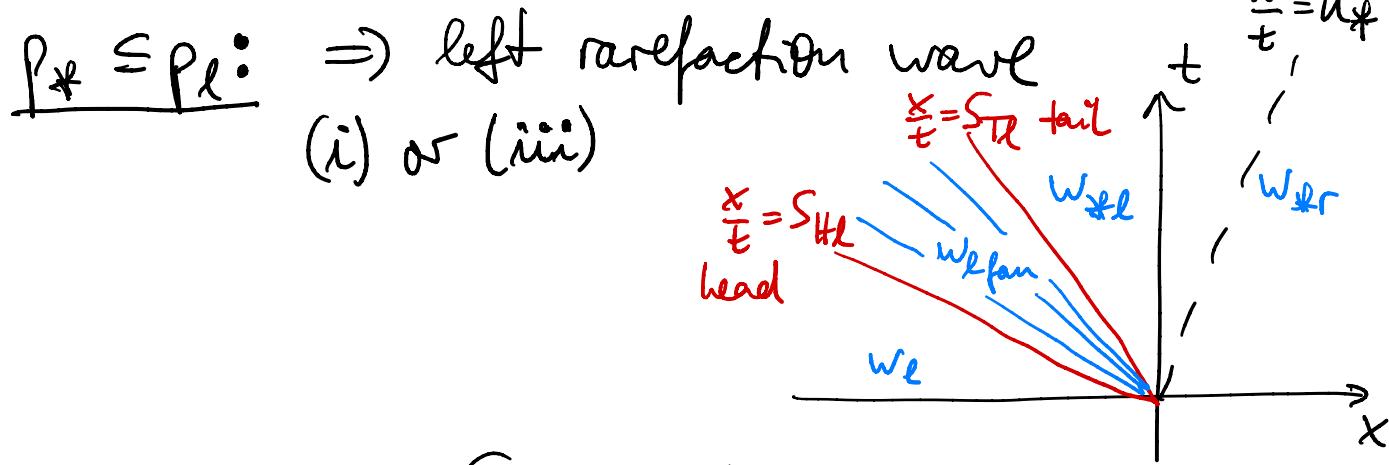
$$w(x,t)_{\text{left}} = \begin{cases} w_{*L}^{\text{sh}} & \text{if } S_L \leq \frac{x}{t} \leq u_* \\ w_L & \text{if } \frac{x}{t} \leq S_L \end{cases}$$

where $w_{*L}^{\text{sh}} = (S_{*L}^{\text{sh}}, u_*, P_*)$
 "known"

$$S_{*L} = S_L \frac{\frac{P_L}{P_*} + \frac{\gamma-1}{\gamma+1}}{\frac{\gamma-1}{\gamma+1} \frac{P_*}{P_L} + 1}$$

and $\underline{S_L} = u_L - c_L \sqrt{\frac{\gamma+1}{2\gamma} \frac{P_*}{P_L} + \frac{\gamma-1}{2\gamma}}$

Exercise: Derive S_{*L}, S_L from R-H jump conditions



$$w(x, t)_{\text{left}} = \begin{cases} w_e & , \frac{x}{t} \leq S_{HL} \\ w_{\text{fan}} & , S_{HL} \leq \frac{x}{t} \leq S_{TL} \\ w_{*r} & , S_{TL} \leq \frac{x}{t} \leq u_* \end{cases}$$

where $w_{*r} = (S_{*r}, u_*, P_*)$
 "known"

w_{fan} = state "inside" rarefaction wave

Find: S_{*r} , S_{HL} , S_{TL} , w_{fan}

$$\left. \begin{array}{l} I \quad S \\ II \quad U + \frac{2c}{\gamma - 1} \end{array} \right\} = 1 - \text{Riemann invariants}$$

MP can use isentropic law $P = k s^\gamma$ ↑ adiab. const. given
 across wave

$$P_L = k s_L^\gamma \Rightarrow k = \frac{P_L}{s_L^\gamma}$$

$$S_{*r} = S_L \left(\frac{P_*}{P_L} \right)^{\frac{1}{\gamma}}$$

then:

$$c = \sqrt{\frac{2P}{\gamma S}}_s = \sqrt{\gamma k s^{\gamma-1}} = \sqrt{\gamma \frac{P}{S}}$$

$$c_{*l} = c_l \left[\frac{P_{*l}}{P_l} \left(\frac{P_{*l}}{P_l} \right)^{-\frac{1}{\gamma}} \right]^{1/2} = c_l \left(\frac{P_{*l}}{P_l} \right)^{\frac{\gamma-1}{2\gamma}}$$

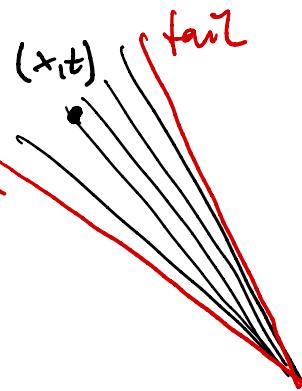
Head & tail speeds:

$$S_{Hl} = \lambda_1(u_l) = u_l - c_l$$

$$S_{Tl} = \lambda_1(u_*) = u_* - c_{*l}$$

Welfan:

note solution
constant on
lines (1-characteristics)
through the origin



$$\frac{dx}{dt} = \frac{x}{t} = \lambda_1(u) = u - c$$

relate to u_l, c_l via II:

$$u_l + \frac{2c_l}{\gamma-1} = u + \frac{2c}{\gamma-1}$$

combine to:

$$S = S_l \left[\frac{2}{\gamma+1} + \frac{\gamma-1}{(\gamma+1)c_l} (u_l - \frac{x}{t}) \right]^{\frac{2}{\gamma-1}}$$

$$u = \frac{2}{\gamma+1} \left[c_l + \frac{\gamma-1}{2} u_l + \frac{x}{t} \right]$$

$$P = P_l \left[\frac{2}{\gamma+1} + \frac{\gamma-1}{(\gamma+1)c_l} (u_l - \frac{x}{t}) \right]^{\frac{2\gamma}{\gamma-1}}$$

(2) $s = \frac{x}{t} \geq u_*$: right scroll analogous to above

$p_* > p_r$ \Rightarrow right shock (i) or (iv)

$$w(x,t)_{\text{right}} = \begin{cases} w_{*,r}^{\text{sh}}, & u_* \leq \frac{x}{t} \leq s_r \\ w_r, & \frac{x}{t} \geq s_r \end{cases}$$

with $w_{*,r}^{\text{sh}} = (s_{*,r}^{\text{sh}}, u_*, p_*)$

$$s_{*,r}^{\text{sh}} = s_r \frac{\frac{p_*}{p_r} + \frac{\gamma-1}{\gamma+1}}{\frac{\gamma-1}{\gamma+1} \frac{p_*}{p_r} + 1}$$

$$s_r = u_r + c_r \sqrt{\frac{\gamma+1}{2\gamma} \frac{p_*}{p_r} + \frac{\gamma-1}{2\gamma}}$$

$p_* \leq p_r$ \Rightarrow right rarefaction wave (ii) or (iii)

$$w(x,t)_{\text{right}} = \begin{cases} w_{*,r,\text{fan}}, u_* \leq \frac{x}{t} \leq s_{Tr} \\ w_{r,\text{fan}}, s_{Tr} \leq \frac{x}{t} \leq s_{Hr} \\ w_r, \quad \frac{x}{t} \geq s_{Hr} \end{cases}$$

with

$$w_{*,r,\text{fan}} = (s_{*,r}, u_*, p_*)$$

$$S_{\text{fr}} = S_r \left(\frac{P_{\text{fr}}}{P_r} \right)^{1/\gamma}$$

$$S_{Hr} = \lambda_3(u_r) = u_r + c_r$$

$$S_{Tr} = \lambda_3(u_*) = u_* + c_*$$

$$w_{rfan} = \begin{cases} S = S_r \left[\frac{2}{\gamma+1} - \frac{\gamma-1}{(\gamma+1)c_r} (u_r - \frac{x}{t}) \right]^{\frac{\gamma}{\gamma-1}} \\ u = \frac{2}{\gamma+1} \left[-c_r + \frac{\gamma-1}{2} u_r + \frac{x}{t} \right] \\ P = P_r \left[\frac{2}{\gamma+1} - \frac{\gamma-1}{(\gamma+1)c_r} (u_r - \frac{x}{t}) \right]^{\frac{\gamma}{\gamma-1}} \end{cases}$$

② Find (u_*, P_*) given initial guesses

Proposition: The solution P_* is given by the root of the function (assuming an ideal gas EOS):

$$f(p, w_e, w_r) = f_L(p, w_e) + f_R(p, w_r) + u_r - u_e$$

where

$$f_L(p, w_e) = \begin{cases} (p - p_e) \sqrt{\frac{A_e}{p + B_e}}, & p > p_e \text{ (shock)} \\ \frac{2c_e}{\gamma-1} \left[\left(\frac{p}{p_e} \right)^{\frac{\gamma-1}{2\gamma}} - 1 \right], & p \leq p_e \text{ (rarefaction)} \end{cases}$$

$$f_r(p, w_e) = \begin{cases} (p - p_r) \sqrt{\frac{A_r}{p + B_r}}, & p > p_r \text{ (shock)} \\ \frac{2c_r}{\gamma - 1} \left[\left(\frac{p}{p_r} \right)^{\frac{\gamma - 1}{2\gamma}} - 1 \right], & p \leq p_r \text{ (rarefaction)} \end{cases}$$

with given constants

$$A_e = \frac{2}{(\gamma + 1) S_e}, \quad B_e = \frac{\gamma - 1}{\gamma + 1} p_e$$

$$A_r = \frac{2}{(\gamma + 1) S_r}, \quad B_r = \frac{\gamma - 1}{\gamma + 1} p_r$$

The solution for the velocity u_* is given by

$$u_* = \frac{1}{2}(u_e + u_r) + \frac{1}{2}[f_r(p_*) - f_e(p_*)].$$

Proof: ① f_e for left shock ($p > p_e$):

transform R.H jump conditions into frame

moving with the shock front " $u \wedge u$:

$$\text{I} \quad \hat{u}_e = u_e - S_e, \quad \hat{u}_* = u_* - S_e$$

Shock velocity

$\leftarrow S_e$	$\hat{S}_e = 0$
S_e	S_{*e}
u_e	u_*
p_e	p_*

$$\Rightarrow \text{I} \quad S_e \hat{u}_e = S_{*e} \hat{u}_* \equiv Q_e$$

$$\text{II} \quad S_e \hat{u}_e^2 + p_e = S_{*e} \hat{u}_*^2 + p_*$$

$$\text{III} \quad \hat{u}_e (\hat{E}_e + p_e) = \hat{u}_* (\hat{E}_{*e} + p_*)$$

mass flux through
shock

$$\left(\hat{S}_e = 0 \Leftrightarrow f(\hat{u}_e) - f(\hat{u}_*) = 0 \right)$$

$$\text{II \& III} \Rightarrow Q_e = - \frac{P_* - P_e}{\hat{u}_* - \hat{u}_e} \quad \text{V}$$

$\downarrow \text{IV} \quad \hat{u}_e - \hat{u}_* = u_e - u_*$

$$Q_e = - \frac{P_* - P_e}{u_* - u_e}$$

$$\text{VI} \Leftrightarrow u_* = u_e - \frac{P_* - P_e}{Q_e}$$

alternatively: I \& IV: $Q_e^2 = - \frac{P_* - P_e}{\frac{1}{S_{*e}} - \frac{1}{S_e}}$

$S_{*e} = S_e \frac{\frac{\gamma-1}{\gamma+1} + \frac{P_e}{P_*}}{\frac{\gamma-1}{\gamma+1} \frac{P_e}{P_*} + 1} \quad (\text{see above})$

$Q_e = \sqrt{\frac{P_* + B_e}{A_e}}$

in VI:

$u_* = u_e - f_e(P_*, w_e)$

② f_e for left rarefaction wave ($P \leq P_e$)

1-Riemann invariant: $u_e + \frac{2c_e}{\gamma-1} = u_* + \frac{2c_{*e}}{\gamma-1}$

using sound speed $c_{*l} = c_l \left(\frac{P_*}{P_l} \right)^{\frac{k-1}{2\gamma}}$
 (see above)

$$\text{and } u_* = u_e - f_l(P_*, w_e)$$

③ Similarly for right shock & right rarefaction wave:

$$u_* = u_r + f_r(P_*, w_r)$$

④ eliminating u_* yields

$$u_e - f_l(P_*, w_e) = u_r + f_r(P_*, w_r)$$

$$\Leftrightarrow f(P_*, w_e, w_r) = f_l(P_*, w_e) + f_r(P_*, w_r) + u_r - u_e \\ \stackrel{!}{=} 0$$

⑤ with P_* solution of ④ u_* can be found by adding ② & ③:

$$u_* = \frac{1}{2}(u_e + u_r) + \frac{1}{2}[f_r(P_*) - f_l(P_*)]$$

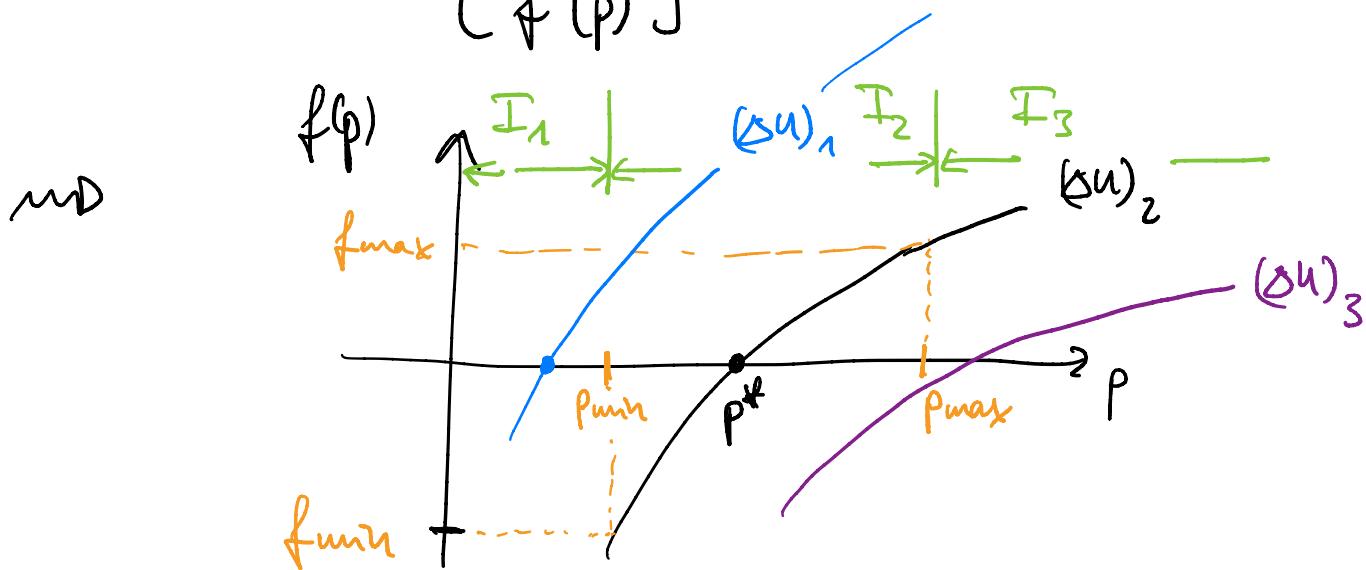
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Remarks: 1) Behavior of $f(p)$

Can show: • $f'(p) > 0 \rightarrow$ monotonically increasing
 (exercise)

• $f''(p) < 0 \rightarrow$ concave

• $\begin{bmatrix} f'(p) \\ f''(p) \end{bmatrix} \xrightarrow[p \rightarrow \infty]{} 0$



$$p_{\min} = \min(p_L, p_r), \quad f_{\min} = f(p_{\min})$$

$$p_{\max} = \max(p_L, p_r), \quad f_{\max} = f(p_{\max})$$

with s_L, p_L, s_r, p_r given the behavior of f
 depends on $\Delta u = u_r - u_L$

3 regimes: $f_{\min}, f_{\max} > 0$ $\Rightarrow p_* \in I_1 = (0, p_{\min})$

$f_{\min} \leq 0, f_{\max} \geq 0$ $\Rightarrow p_* \in I_2 = [p_{\min}, p_{\max}]$

$f_{\min}, f_{\max} < 0$ $\Rightarrow p_* \in I_3 = (p_{\max}, \infty)$

$I_1: P_* < P_L, P_R \Rightarrow 2$ rarefaction waves

$I_2:$ or $P_L < P_* < P_R \Rightarrow 1$ rarefaction wave
 $P_R < P_* < P_L \Rightarrow 1$ shock wave

$I_3: P_* > P_L, P_R \Rightarrow 2$ shock waves

2) Vacuum

A physical solution to $f(p)$ requires $f(0) < 0$
(monotonicity property)

For $P_* = P_{\text{crit}} = 0$ we have $P_* \leq P_L, P_R$
⇒ left & right rarefaction

and

$$(u_r - u_l)_{\text{crit}} \equiv -f_L(0) - f_r(0) \\ = \frac{2c_L}{\gamma-1} + \frac{2c_r}{\gamma-1} > u_r - u_l$$

If $\Delta u \equiv u_r - u_l > \Delta u_{\text{crit}}$ vacuum ($P_* < 0$) is created → unphysical

no explicit manifestation of
"smallness constraint" of local
Riemann problem (cf. Sec. 4.4.5)