Chapter 7: Approximate Riemann Solvers

<u>Idea</u>: • Godunov-type methods are computationally expensive, as one must solve a Riemann problem at every cell interface at each time step.

→ Riemann problem is most demanding task in the method

However, all that is needed about the local solution to the Riemann problem is the solution along the ray $\frac{x}{t} = 0$ ($v^n(x_{i+\frac{1}{2}}, t^n)$ = const. along the ray, i.e. on $[t^n, t^{n+1}]$), in order to compute the flux $f(v^n(x_{i+\frac{1}{2}}, t^n))$.

→ complicated iterative procedure for little information!

• replace "<u>exact</u>" Riemann solver by an "approximate" (cheaper) Riemann solver.

→ gives rise to class of <u>Godunov-type methods</u>

6.1 Roe's Riemann solver

<u>Idea:</u> replace actual Riemann problem by an (approximate) linearized problem defined locally at each cell interface

<u>Def (Riemann solver of Roe):</u>

Replace the RP
$$u_t + f(u)_x = 0 \quad \text{in } \mathbb{R}\times(0,\infty)$$
$$u(x,0) = \begin{cases} u_l & x<0 \\ u_r & x\geq 0 \end{cases}$$

by the linear problem

$$(*) \qquad w_t + A_{lr} w_x = 0 \quad \text{in } \mathbb{R}\times(0,\infty)$$
$$w(x,0) = \begin{cases} w_l & x<0 \\ w_r & x\geq 0 \end{cases}$$

where $A_{lr} = A(u_l, u_r) \in Mat(\mathbb{R}^{m\times m})$ satisfies the conditions

(i) $f(v) - f(w) = A(v,w)\,(v-w)$
(conservation across discontinuities)

(ii) $\|A(v,w) - Df(v)\| \to 0$ as $\|w-v\| \to 0$
(consistency with exact Jacobian)

(iii) $A(v,w)$ has only real eigenvalues and has a complete set of eigenvectors (system is hyperbolic and RP thus solvable)

The <u>Roe scheme</u> is defined by replacing the exact solution of the local RP in Godunov's method by the exact solution to (*).

Motivation:

- linearized problem justified <u>at most</u> cell interfaces

$$\| u_{i+1}^n - u_i^n \| \sim \mathcal{O}(\Delta x)$$

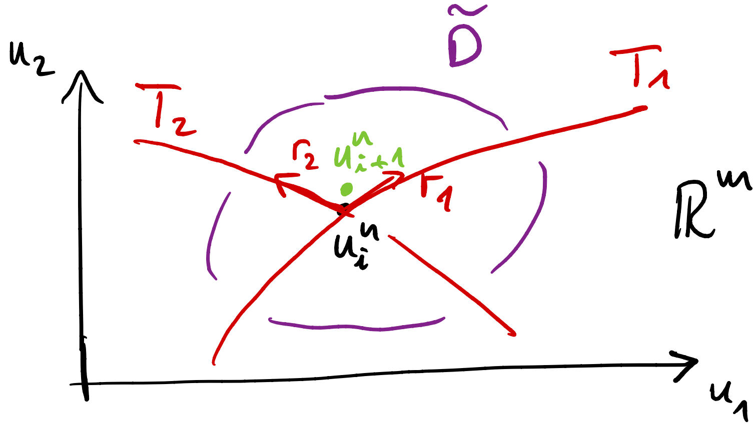
$$Df(u_{i+1}) \approx Df(u_i)$$

constant matrix

appropriate average state $\bar{u} = \bar{u}(u_{i+1}, u_i)$

$$u_t + f(u)_x = 0 \rightsquigarrow u_t + Df(\bar{u}) u_x = 0$$

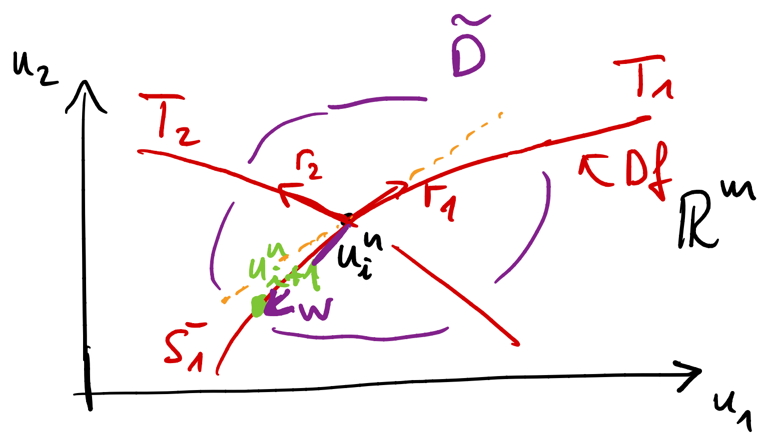
State space: integral curves & shock sets needed to connect u_{i+1}^n, u_i^n are nearly straight lines



$$u_{i+1}^n - u_i^n = \sum_{k=1}^m \alpha_k r_k$$

infinitesimal (linearized) Riemann problem

- near shocks: u_{i+1}^n, u_i^n may be separated far in state space at least in one direction $k = k_s$ along a shock set $S_{k_s}^-$.



$\rightsquigarrow$ make $w \equiv u_{i+1}^n - u_i^n$ an eigenvalue of $A(u_i, u_{i+1})$ instead of r_1

Note: u_{i+1}^n cannot be on more than one shock set at a time. For all other families $k \neq k_s$ u_{i+1}^n is "close" to u_i^n $\longrightarrow$ "there is at most one strong wave"

$\longrightarrow$ need to make sure that the corresponding single shock wave

$w \equiv u^n_{i+1} - u^n_i$ is an eigenvector of $A(u^n_{i+1}, u^n_i)$, so that the linearized problem "captures" the shock "outside of the linear domain"

RH jump conditions: $f(v) - f(w) = s(v-w)$

Require: $A(v,w)(v-w) \overset{!}{=} s(v-w) = f(v) - f(w)$

$\Rightarrow$ requirement (i)

<u>Construction of $A(v,w)$:</u>

- Consider the line integral along the path

$$u(\xi) = u^n_i + (u^n_{i+1} - u^n_i)\xi, \quad 0 \le \xi \le 1$$

$$f(u_{i+1}) - f(u_i) = \int_0^1 \frac{df(u(\xi))}{d\xi}\, d\xi = \int_0^1 Df(u(\xi))\, u'(\xi)\, d\xi$$

$$= (u^n_{i+1} - u^n_i) \underbrace{\int_0^1 Df(u(\xi))\, d\xi}_{A(u_i, u_{i+1})}$$

$\rightarrow A(u_i, u_{i+1})$ satisfies (i) & (ii), but (iii) is not guaranteed.

Also: integral difficult to evaluate in general (closed form may not exist)

- Roe (1981): introduce coordinate transformation $z(u)$

path: $z(\xi) = z_i + (z_{i+1} - z_i)\xi$, $\quad z_i = z(u_i)$

$$\rightsquigarrow \quad f(u_{i+1}) - f(u_i) = \int_0^1 \frac{d f(z(\xi))}{d\xi}\, d\xi$$

$$= (z_{i+1} - z_i) \underbrace{\int_0^1 Df_z(z(\xi))\, d\xi}_{\equiv\, C(u_i,\, u_{i+1})}$$

Jacobian w.r.t. $z = z(u)$

$$u_{i+1} - u_i = \int_0^1 \frac{d u(z(\xi))}{d\xi}\, d\xi$$

$$= \int_0^1 \frac{d u(z(\xi))}{dz}\, z'(\xi)\, d\xi$$

$$= (z_{i+1} - z_i) \underbrace{\int_0^1 \frac{d u(z(\xi))}{dz}\, d\xi}_{\equiv\, B(u_i,\, u_{i+1})}$$

$$\rightsquigarrow \quad A(u_i, u_{i+1}) = C(u_i, u_{i+1})\, B(u_i, u_{i+1})$$

Harten, Lax, van Leer (1983): this procedure guarantees (iii) if the system has a (convex) entropy function Φ $\leadsto$ choose $z(u) = \nabla\Phi(u)$

wrt. to u

$\leadsto$ $A = A_{HLL}$ is similar to a symmetric matrix and hence has real eigenvalues $\to$ symmetric hyperbolic

<u>Proposition:</u> Assume that there exists a matrix $A(v,w)$ as in the previous definition. Then the Roe scheme as defined above can be written in conservative form

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left[g(w_i^n, w_{i+1}^n) - g(w_{i-1}^n, w_i^n)\right]$$

where the numerical flux g is given by

$$g(v,w) = \frac{1}{2}\left[f(v) - f(w)\right] - \frac{1}{2}\sum_{k=1}^{m} |\lambda_k|\,\alpha_k r_k$$

with (λ_k, r_k) being the eigenvalues and eigenvectors of $A(v,w)$. The coefficients ("wave strengths") α_k are determined by

$$w - v = \sum_{k=1}^{m} \alpha_k r_k$$

Proof: see Kroener Lemma 4.4.8 p.342
Toro Sec. 11.1.3

Remarks: 1) Note that the exact solution to the linearized RP can be easily written down:

$$w(x,t) = \begin{cases} u_l & \frac{x}{t} < \lambda_1 \\ u_k & \lambda_k \leq \frac{x}{t} < \lambda_{k+1}, \; k \in \{1,\dots,m-1\} \\ u_r & \lambda_m \leq \frac{x}{t} \end{cases}$$

where $u_k \equiv u_l + \sum_{j=1}^{k} \alpha_j r_j$ and α_j are given by $u_r - u_l = \sum_{j=1}^{m} \alpha_j r_j$.

2) In general, the Roe scheme does not approximate the entropy solution (see counter example Remark 4.4.9 Kroener p.345)

→ <u>additional entropy "fixes" required</u>

see Toro Sec 11.4, LeVeque Sec. 15.3

3) Construction of $A(v,w)$ is complicated and expensive in general

→ Roe-Pike method (Toro Sec. 11.3)

offers approach to compute all quantities needed for flux computation etc. <u>without</u> actually constructing $A(v,w)$

<u>Roe's scheme for the Euler equations</u>:

Consider Euler's equs in conservative form

$$u_t + f(u)_x = 0$$

Roe (1981): $z = \frac{u}{\sqrt{\rho}}$ ⇝ find that

$$A(u_l, u_r) = Df(\bar{u}), \text{ where}$$

wrt. to u

$$\bar{u} = \begin{pmatrix} \bar{\rho} \\ \bar{\rho}\bar{u} \\ \bar{e} \end{pmatrix} \quad \text{and}$$

$$\left.\begin{aligned} \bar{\rho} &= \sqrt{\rho_r \rho_l} \\ \bar{u} &= \frac{\sqrt{\rho_l}\, u_l + \sqrt{\rho_r}\, u_r}{\sqrt{\rho_l} + \sqrt{\rho_r}} \\ \bar{H} &= \frac{\sqrt{\rho_l}\, H_l + \sqrt{\rho_r}\, H_r}{\sqrt{\rho_l} + \sqrt{\rho_r}}, \end{aligned}\right\} \text{"Roe mean values"} \qquad H \equiv \frac{e+p}{\rho} \text{ specific enthalpy}$$

then $A(u_l, u_r)$ satisfies conditions (i)–(iii).

<u>Remark:</u> For the Roe-scheme with mean values as defined above the density and pressure may become negative $\rho < 0$, $p < 0$ (see Kroener lemma 4.4.14 p. 350) disadvantageous for problems where low densities are expected!

$\rightarrow$ <u>additional fixes are required</u>

see Einfeldt et al. J. Comp. Phys. 92, 273 (1991)

$\rightarrow$ Einfeldt et al. show that for certain Riemann problems there is <u>no linearization that preserves positivity of ρ, e</u> !