Chapter 5: The Riemann problem for the Euler Equations

5.1 General setup

Consider the Euler equations in conservative form:

$$u_t + f(u)_x = 0$$

$$\begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix}_t + \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (E+p)u \end{bmatrix}_x = 0$$

assume <u>ideal gas EOS</u>: $p = R\rho T = (\gamma-1)e_{int}$

$$E = \frac{p}{\gamma-1} + \frac{1}{2}\rho u^2$$

<u>Exercise</u>: a) Show that

$$Df(u) = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2}(\gamma-3)u^2 & (3-\gamma)u & \gamma-1 \\ \frac{1}{2}(\gamma-1)u^3 - uH & H-(\gamma-1)u^2 & \gamma u \end{pmatrix}$$

where $H = \frac{E+p}{\rho} = h + \frac{1}{2}u^2$ (total specific enthalpy).

Furthermore, show that Df has eigenvalues and eigenvectors (λ_k, r_k):

$$\lambda_1 = u - c \;, \qquad \lambda_2 = u \;, \qquad \lambda_3 = u + c$$

$$r_1 = \begin{pmatrix} 1 \\ u-c \\ H-uc \end{pmatrix}, \quad r_2 = \begin{pmatrix} 1 \\ u \\ \frac{1}{2}u^2 \end{pmatrix}, \quad r_3 = \begin{pmatrix} 1 \\ u+c \\ H+uc \end{pmatrix}$$

where

$$c = \sqrt{\left.\frac{\partial p}{\partial \rho}\right|_{s=\text{const}}} = \sqrt{\frac{\gamma p}{\rho}} \quad \left(\text{sound speed for ideal gas}\right)$$

b) Show that (λ_1, r_1) & (λ_3, r_3) are genuinely non-linear, while (λ_2, r_2) is linearly degenerate.

c) Show that s (specific entropy) is a Riemann invariant and that the k-Riemann invariants are given by

1-RI: $\quad s, \quad u + \dfrac{2c}{\gamma - 1}$

2-RI: $\quad u, \quad p$

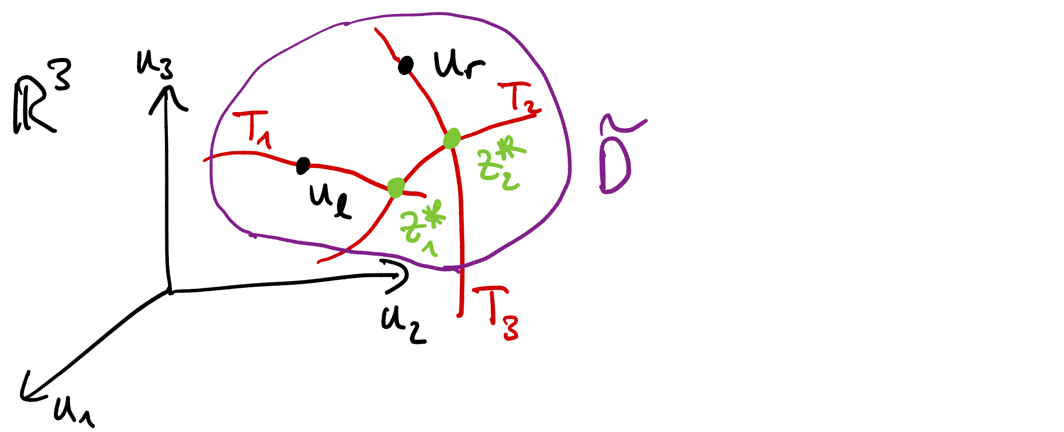
3-RI: $\quad s, \quad u - \dfrac{2c}{\gamma - 1}$

⇝ General structure of the RP for the Euler equations

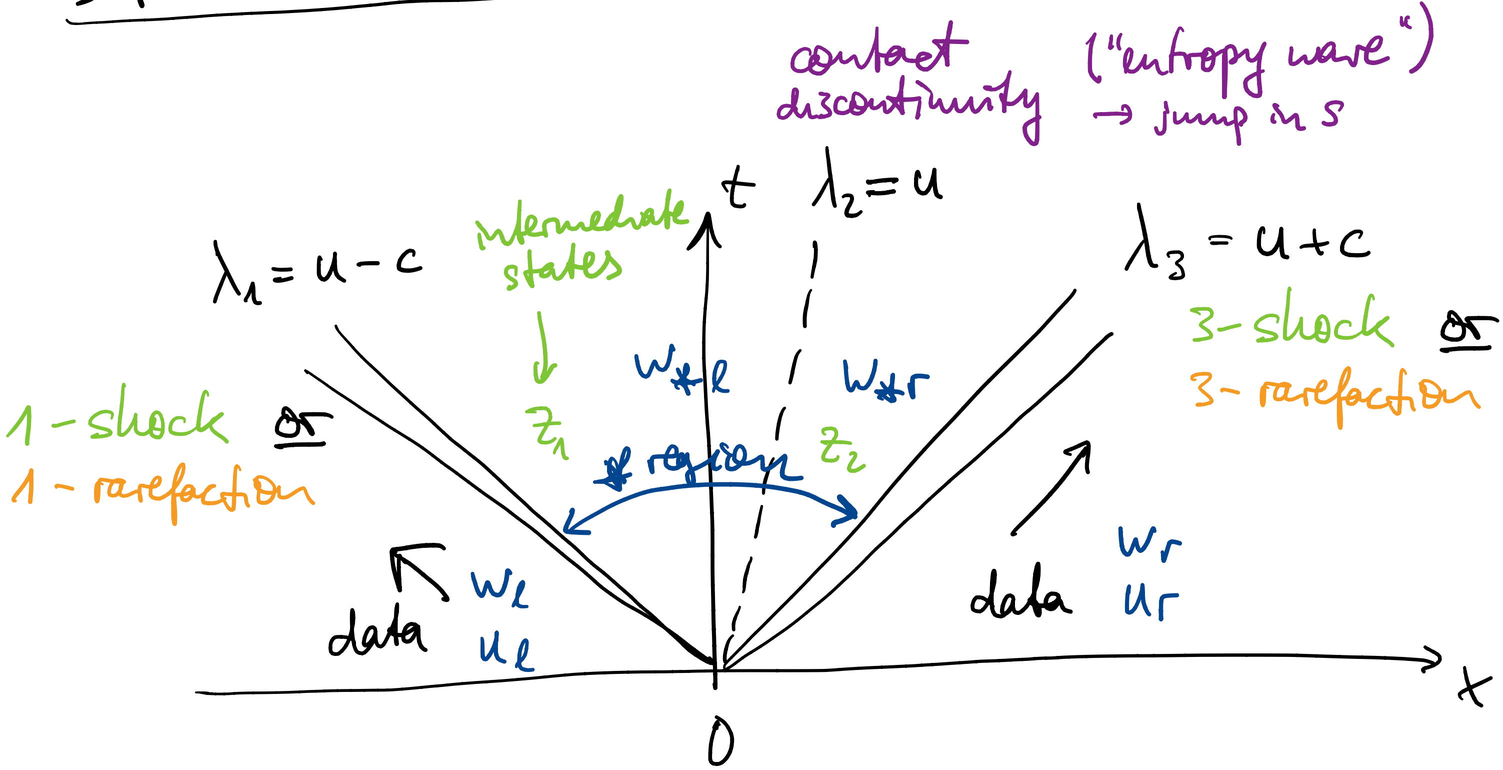
Situation in state space:

$m = 3$ equations

$u = (u_1, u_2, u_3) = (\rho, \rho u, E)$



Situation in real space:



Remarks: 1) Since both u & p are constant across contact discontinuities ($\to$ constant in $*$-region) it is easier to work with the

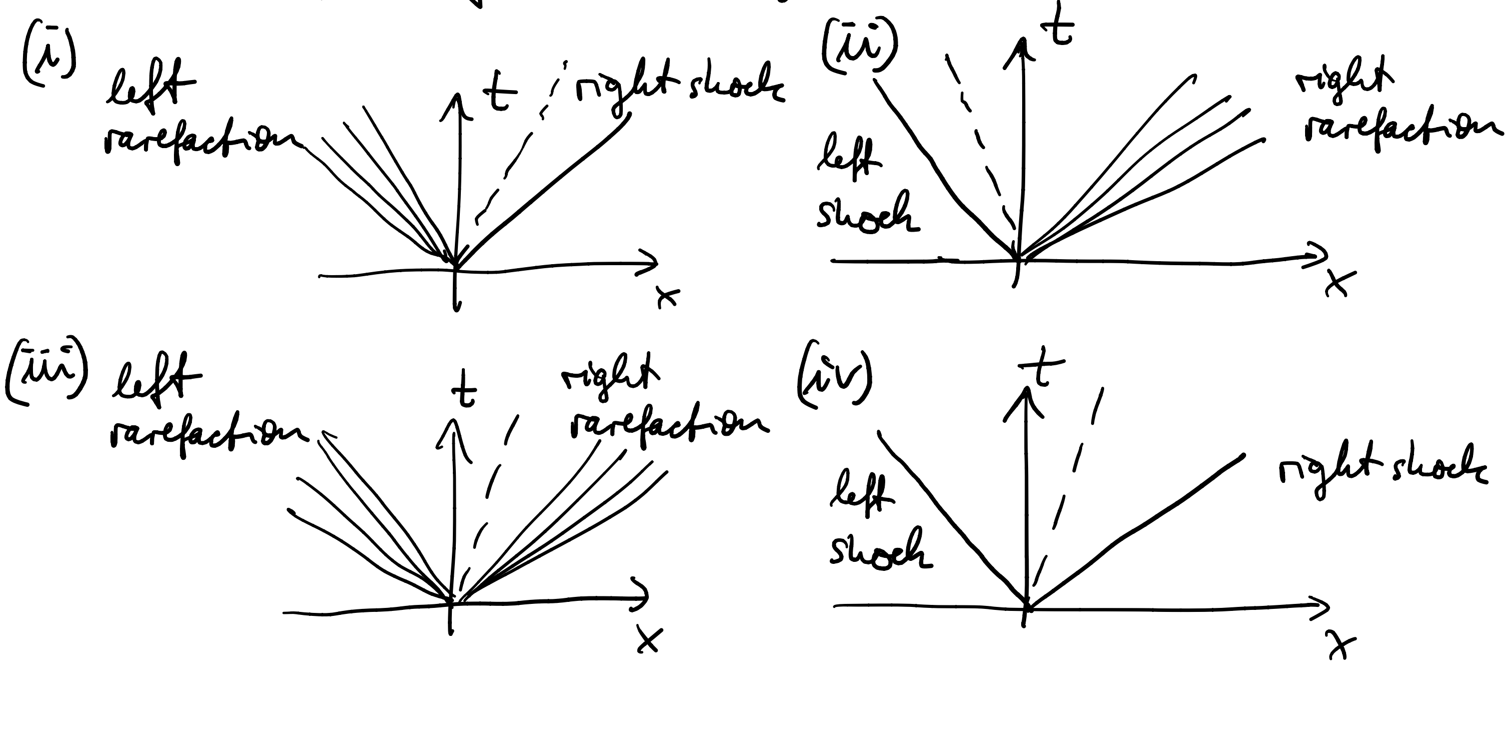
<u>primitive variables</u> $w = (\rho, u, p)$

rather than the

<u>conserved variables</u> $u \equiv (\rho, \rho u, E)$

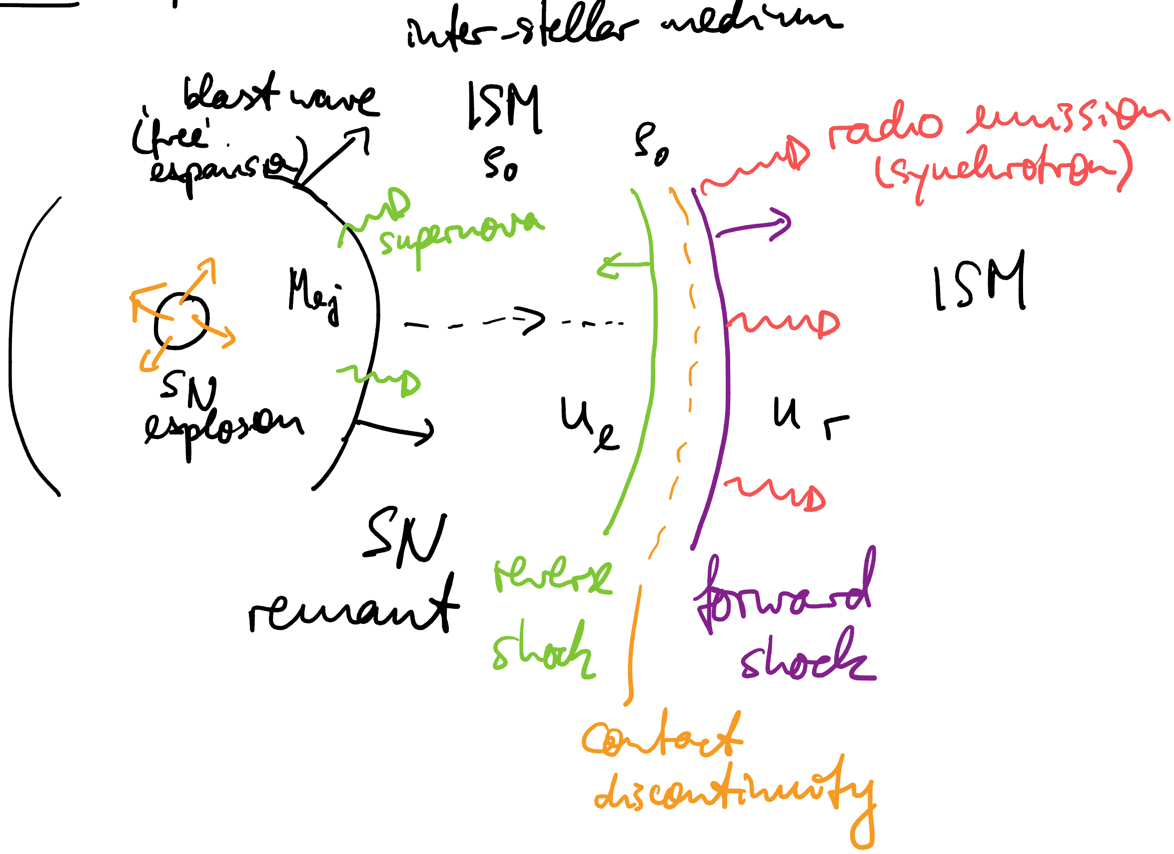
<u>Recall:</u> k-Riemann invariants are constant along integral curves of r_k, i.e. across rarefaction waves and contact discontinuities

2) There are 4 possible patterns for the solution of the RP:

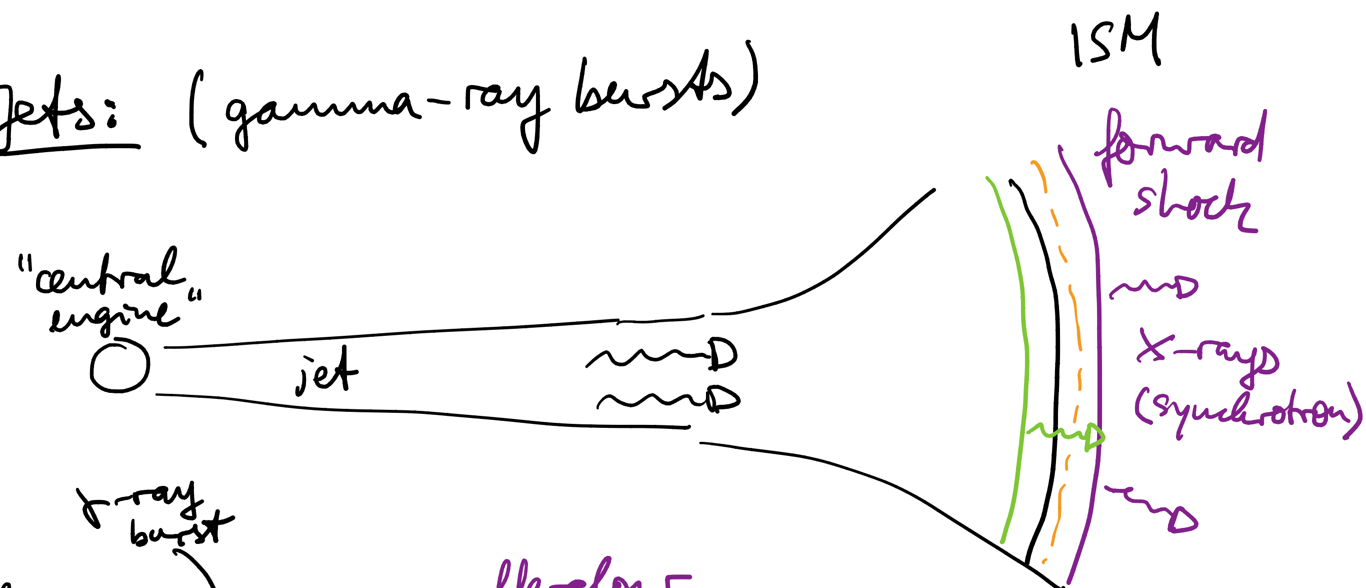


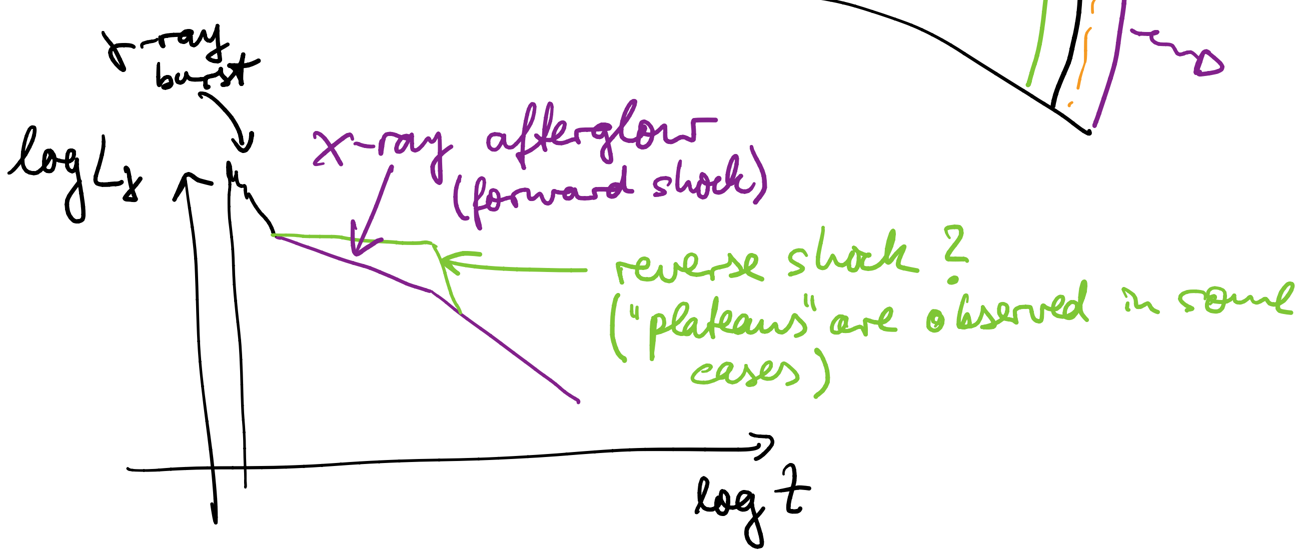
3) Some examples in astrophysics

Supernova explosion:



Jets: (gamma-ray bursts)





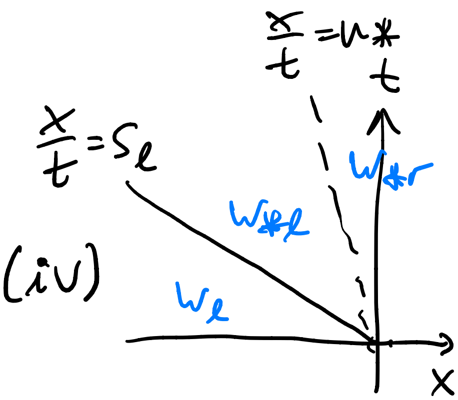
5.2 Solution strategy

① Assume we know u_*, p_* (= const. in $*$-region), evaluate the case (i) – (iv) that applies to this (u^*, p^*) configuration and write down the combined solution $w = (\rho, u, p)$, $w = w(x,t)$.

② Given a (reasonable) initial guess for (u_*^{in}, p_*^{in}), find an iterative scheme to determine (u_*, p_*) from which the final solution will follow with ①.

Let's do it:

① Treat regions left and right of the contact discontinuity separately:



(1) $S \equiv \frac{x}{t} \leq u_*$: <u>left side</u>

<u>$p_* > p_l$</u> $\Rightarrow$ left shock (ii) or (iv)

$$w(x,t)_{left} = \begin{cases} w^{sh}_{*l} & \text{if } S_l \leq \frac{x}{t} \leq u_* \\ w_l & \text{if } \frac{x}{t} \leq S_l \end{cases}$$

where $w^{sh}_{*l} = (\rho^{sh}_{*l}, \underbrace{u_*, p_*}_{\text{"known"}})$

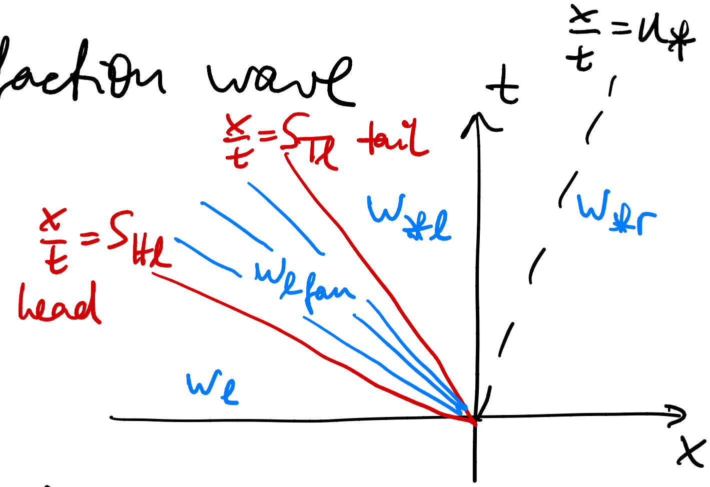
RH jump cond.

$$\rho_{*l} = \rho_l \frac{\frac{p_*}{p_l} + \frac{\gamma-1}{\gamma+1}}{\frac{\gamma-1}{\gamma+1}\frac{p_*}{p_l} + 1}$$

and $$S_l = u_l - c_l \sqrt{\frac{\gamma+1}{2\gamma}\frac{p_*}{p_l} + \frac{\gamma-1}{2\gamma}}$$

<u>Exercise:</u> Derive ρ_{*l}, S_l from RH jump conditions

$p_* \le p_\ell$: $\Rightarrow$ left rarefaction wave
(i) or (iii)



$$w(x,t)_{left} = \begin{cases} W_\ell \,, & \frac{x}{t} \le S_{H\ell} \\ W_{\ell fan} \,, & S_{H\ell} \le \frac{x}{t} \le S_{T\ell} \\ W_{*\ell} \,, & S_{T\ell} \le \frac{x}{t} \le u_* \end{cases}$$

where $\quad W_{*\ell} = (\rho_{*\ell}, \underbrace{u_*, p_*}_{\text{"known"}})$

$W_{\ell fan}$ = state "inside" rarefaction wave

Find: $\rho_{*\ell}$, $S_{H\ell}$, $S_{T\ell}$, $W_{\ell fan}$

$$\left.\begin{array}{ll} \mathrm{I} & s \\ \mathrm{II} & u + \dfrac{2c}{\gamma - 1} \end{array}\right\} = \text{1-Riemann invariants}$$

$\overset{\mathrm{I}}{\text{imp}}$ can use isentropic law $p = k\rho^\gamma$ ← adiab. const. given
across wave

$$p_\ell = k\rho_L^\gamma \Rightarrow k = \frac{p_\ell}{\rho_\ell^\gamma}$$

$$\boxed{\rho_{*\ell} = \rho_\ell \left(\frac{p_*}{p_\ell}\right)^{\frac{1}{\gamma}}}$$

then:

$$c = \sqrt{\left.\frac{\partial p}{\partial \rho}\right|_s} = \sqrt{\gamma K \rho^{\gamma-1}} = \sqrt{\gamma \frac{p}{\rho}}$$

$$c_{*l} = c_l \left[\frac{p_{*l}}{p_l} \left(\frac{p_{*l}}{p_l} \right)^{-\frac{1}{\gamma}} \right]^{1/2} = c_l \left(\frac{p_{*l}}{p_l} \right)^{\frac{\gamma-1}{2\gamma}}$$

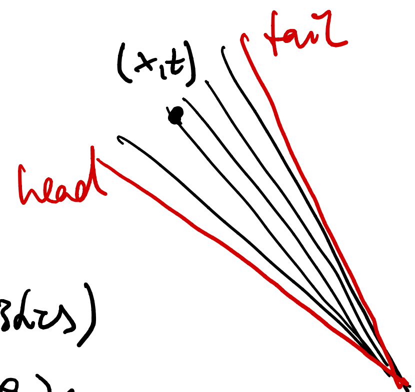
Head & tail speeds:

$$S_{Hl} = \lambda_1(u_l) = u_l - c_l$$

$$S_{Tl} = \lambda_1(u_*) = u_* - c_{*l}$$

$W_{l\,fan}$:

note solution constant on lines (1-characteristics) through the origin



$$\frac{dx}{dt} = \frac{x}{t} = \lambda_1(u) = u - c$$

relate to u_l, c_l via II:

$$u_l + \frac{2c_l}{\gamma-1} = u + \frac{2c}{\gamma-1}$$

combine to:

$$\boxed{W_{l\,fan} = \begin{cases} \rho = \rho_l \left[\frac{2}{\gamma+1} + \frac{\gamma-1}{(\gamma+1)c_l} \left(u_l - \frac{x}{t} \right) \right]^{\frac{2}{\gamma-1}} \\ u = \frac{2}{\gamma+1} \left[c_l + \frac{\gamma-1}{2} u_l + \frac{x}{t} \right] \\ p = p_l \left[\frac{2}{\gamma+1} + \frac{\gamma-1}{(\gamma+1)c_l} \left(u_l - \frac{x}{t} \right) \right]^{\frac{2\gamma}{\gamma-1}} \end{cases}}$$

(2) $S \equiv \frac{x}{t} \geq u_*$: <u>right side</u> analogous to above

<u>$p_* > p_r$</u> $\Rightarrow$ right shock (i) or (iv)

$$w(x,t)_{right} = \begin{cases} w_{*r}^{sh}, & u_* \leq \frac{x}{t} \leq S_r \\ w_r, & \frac{x}{t} \geq S_r \end{cases}$$

with $w_{*r}^{sh} = (\rho_{*r}^{sh}, u_*, p_*)$

$$\rho_{*r}^{sh} = \rho_r \frac{\frac{p_*}{p_r} + \frac{\gamma-1}{\gamma+1}}{\frac{\gamma-1}{\gamma+1}\frac{p_*}{p_r} + 1}$$

$$S_r = u_r + c_r \sqrt{\frac{\gamma+1}{2\gamma}\frac{p_*}{p_r} + \frac{\gamma-1}{2\gamma}}$$

<u>$p_* \leq p_r$</u> $\Rightarrow$ right rarefaction wave (ii) or (iii)

$$w(x,t)_{right} = \begin{cases} w_{*r\,fan}, & u_* \leq \frac{x}{t} \leq S_{Tr} \\ w_{r\,fan}, & S_{Tr} \leq \frac{x}{t} \leq S_{Hr} \\ w_r, & \frac{x}{t} \geq S_{Hr} \end{cases}$$

with

$$w_{*r\,fan} = (\rho_{*r}, u_*, p_*)$$

$$\rho_{*r} = \rho_r \left(\frac{p_*}{p_r}\right)^{1/\gamma}$$

$$S_{Hr} = \lambda_3(u_r) = u_r + c_r$$

$$S_{Tr} = \lambda_3(u_*) = u_* + c_*$$

$$W_{r\,fan} = \begin{cases} \rho = \rho_r \left[\frac{2}{\gamma+1} - \frac{\gamma-1}{(\gamma+1)c_r}\left(u_r - \frac{x}{t}\right)\right]^{\frac{2}{\gamma-1}} \\ u = \frac{2}{\gamma+1}\left[-c_r + \frac{\gamma-1}{2}u_r + \frac{x}{t}\right] \\ p = p_r\left[\frac{2}{\gamma+1} + \frac{\gamma-1}{(\gamma+1)c_r}\left(u_r - \frac{x}{t}\right)\right]^{\frac{2\gamma}{\gamma-1}} \end{cases}$$

2) Find (u_*, p_*) given initial guesses

<u>Proposition:</u> The solution p_* is given by the root of the function (assuming an ideal gas EOS):

$$f(p, w_\ell, w_r) \equiv f_\ell(p, w_\ell) + f_r(p, w_r) + u_r - u_\ell$$

where

$$f_\ell(p, w_\ell) = \begin{cases} (p - p_\ell)\sqrt{\frac{A_\ell}{p + B_\ell}}, & p > p_\ell \text{ (shock)} \\ \frac{2c_\ell}{\gamma-1}\left[\left(\frac{p}{p_\ell}\right)^{\frac{\gamma-1}{2\gamma}} - 1\right], & p \le p_\ell \text{ (rarefact.)} \end{cases}$$

$$f_r(p, w_r) = \begin{cases} (p - p_r)\sqrt{\dfrac{A_r}{p + B_r}}, & p > p_r \text{ (shock)} \\ \dfrac{2c_r}{\gamma - 1}\left[\left(\dfrac{p}{p_r}\right)^{\frac{\gamma-1}{2\gamma}} - 1\right], & p \leq p_r \text{ (rarefact.)} \end{cases}$$

with given constants

$$A_\ell = \frac{2}{(\gamma+1)\rho_\ell}, \quad B_\ell = \frac{\gamma-1}{\gamma+1} p_\ell$$

$$A_r = \frac{2}{(\gamma+1)\rho_r}, \quad B_r = \frac{\gamma-1}{\gamma+1} p_r$$

The solution for the velocity u_* is given by

$$u_* = \frac{1}{2}(u_\ell + u_r) + \frac{1}{2}\left[f_r(p_*) - f_\ell(p_*)\right].$$

<u>Proof:</u> ① f_ℓ for left shock ($p > p_\ell$):

transform RH jump conditions into frame comoving with the shock front "$\wedge$":

$\leftarrow S_\ell$		$\hat{S}_\ell = 0$	
ρ_ℓ	$\rho_{*\ell}$	ρ_ℓ	$\rho_{*\ell}$
u_ℓ	u_*	$\hat{u}_\ell$	$\hat{u}_*$
p_ℓ	p_*	p_ℓ	p_*

IV $\quad \hat{u}_\ell = u_\ell - S_\ell, \quad \hat{u}_* = u_* - \underbrace{S_\ell}_{\text{shock velocity}}$

mass flux through shock

$\rightsquigarrow$ I $\quad \rho_\ell \hat{u}_\ell = \rho_{*\ell} \hat{u}_* \equiv Q_\ell \quad$ ($\hat{S}_\ell = 0 \Leftrightarrow$ $f(\hat{u}_\ell) - f(\hat{u}_*) = 0$)

II $\quad \rho_\ell \hat{u}_\ell^2 + p_\ell = \rho_{*\ell} \hat{u}_*^2 + p_*$

III $\quad \hat{u}_\ell (\hat{E}_\ell + p_\ell) = \hat{u}_* (\hat{E}_{*\ell} + p_*)$

$$\mathrm{II}\ \&\ \mathrm{III} \Rightarrow Q_\ell = -\frac{p_* - p_\ell}{\hat{u}_* - \hat{u}_\ell} \qquad \mathrm{V}$$

$$\downarrow \ \mathrm{IV}\quad \hat{u}_\ell - \hat{u}_* = u_\ell - u_*$$

$$Q_\ell = -\frac{p_* - p_\ell}{u_* - u_\ell}$$

$$\mathrm{VI} \iff u_* = u_\ell - \frac{p_* - p_\ell}{Q_\ell}$$

alternatively: I in V: $Q_\ell^2 = -\dfrac{p_* - p_\ell}{\frac{1}{\rho_{*\ell}} - \frac{1}{\rho_\ell}}$

$$\rho_{*\ell} = \rho_\ell \frac{\frac{\gamma-1}{\gamma+1} + \frac{p_*}{p_\ell}}{\frac{\gamma-1}{\gamma+1}\frac{p_*}{p_\ell} + 1} \quad \text{(see above)}$$

$$Q_\ell = \sqrt{\frac{p_* + B_\ell}{A_\ell}}$$

in VI: $\boxed{u_* = u_\ell - f_\ell(p_*, w_\ell)}$

② f_ℓ for left rarefaction wave $(p \leq p_\ell)$

1- Riemann invariant: $u_\ell + \dfrac{2c_\ell}{\gamma - 1} = u_* + \dfrac{2c_{*\ell}}{\gamma - 1}$

using sound speed $c_{*l} = c_l \left(\frac{p_*}{p_l} \right)^{\frac{\gamma-1}{2\gamma}}$

(see above)

$$\rightsquigarrow \quad \boxed{u_* = u_l - f_l(p_*, w_l)}$$

③ Similarly for right shock & right rarefaction wave:

$$\boxed{u_* = u_r + f_r(p_*, w_r)}$$

④ eliminating u_* yields

$$u_l - f_l(p_*, w_l) = u_r + f_r(p_*, w_r)$$

$$\Leftrightarrow \quad f(p_*, w_l, w_r) = f_l(p_*, w_l) + f_r(p_*, w_r) + u_r - u_l \overset{!}{=} 0$$

⑤ with p_* solution of ④ u_* can be found by adding ② & ③:

$$u_* = \frac{1}{2}(u_l + u_r) + \frac{1}{2}\left[f_r(p_*) - f_l(p_*) \right]$$

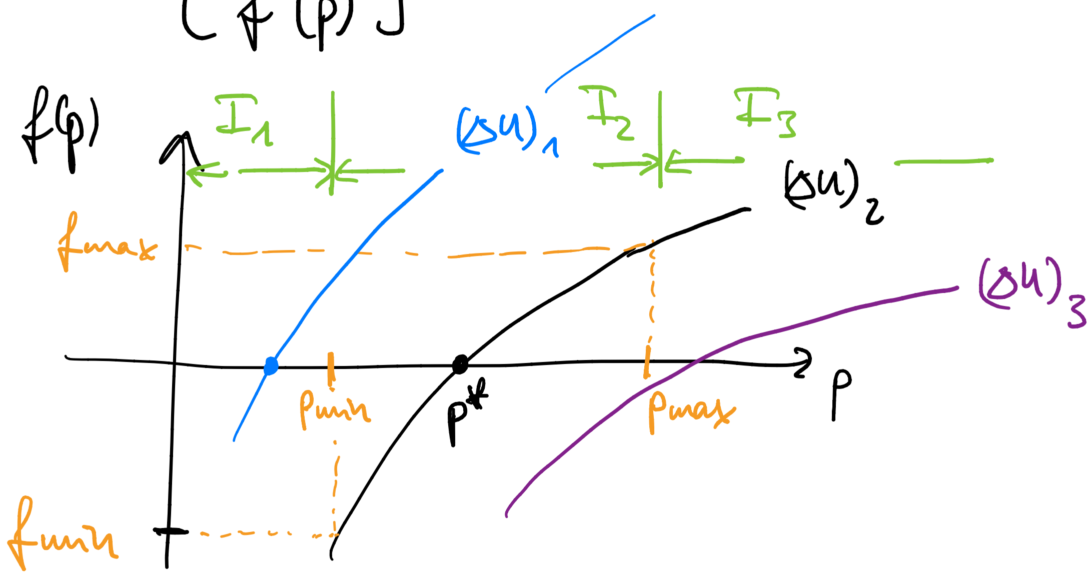
$\square$

Remarks: 1) Behavior of $f(p)$

Can show: (exercise)

- $f'(p) > 0 \rightarrow$ monotonically increasing
- $f''(p) < 0 \rightarrow$ concave
- $\left\{ \begin{matrix} f'(p) \\ f''(p) \end{matrix} \right\} \xrightarrow{p\to\infty} 0$

and



$$p_{min} = \min(p_l, p_r), \quad f_{min} = f(p_{min})$$
$$p_{max} = \max(p_l, p_r), \quad f_{max} = f(p_{max})$$

with ρ_l, p_l, ρ_r, p_r given the behavior of f depends on $\Delta u = u_r - u_l$

3 regimes:

- $f_{min}, f_{max} > 0 \Rightarrow p_* \in I_1 = (0, p_{min})$
- $f_{min} \leq 0, f_{max} \geq 0 \Rightarrow p_* \in I_2 = [p_{min}, p_{max}]$
- $f_{min}, f_{max} < 0 \Rightarrow p_* \in I_3 = (p_{max}, \infty)$

I_1: $p_* < p_l, p_r \Rightarrow$ 2 rarefaction waves

I_2: $\left.\begin{array}{l} p_l < p_* < p_r \\ \text{or} \\ p_r < p_* < p_l \end{array}\right\} \Rightarrow$ 1 rarefaction wave, 1 shock wave

I_3: $p_* > p_l, p_r \Rightarrow$ 2 shock waves

2) Vacuum

A physical solution to $f(p)$ requires $f(0) < 0$ (monotonicity property)

For $p_* = p_{crit} = 0$ we have $p_* \leq p_l, p_r$

$\Rightarrow$ left & right rarefaction

$$\rightsquigarrow \boxed{\begin{aligned}(u_r - u_l)_{crit} &\equiv -f_l(0) - f_r(0) \\ &= \frac{2c_l}{\gamma-1} + \frac{2c_r}{\gamma-1} > u_r - u_l\end{aligned}}$$

If $\Delta u \equiv u_r - u_l > \Delta u_{crit}$ vacuum ($p_* < 0$) is created $\rightarrow$ unphysical

$\rightsquigarrow$ explicit manifestation of "smallness constraint" of local Riemann problem (cf. Sec. 4.4.5)