# Chapter 6: Numerical Schemes for hyperbolic systems of CLs

Consider hyperbolic system of CLs:

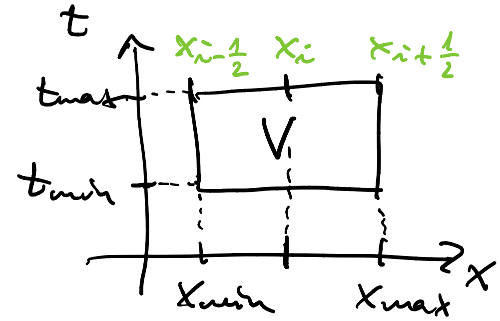
$$(\mathrm{I}) \qquad u_t + f(u)_x = 0 \quad \text{in } \mathbb{R}\times(0,\infty)$$

$$(\mathrm{II}) \qquad u(x,0) = u_0(x) \quad \text{in } \mathbb{R}\times\{0\}$$

## 6.1 Conservative schemes

Motivation:

Integrate (I) over a rectangular control volume
$V = (x_{min}, x_{max}) \times (t_{min}, t_{max}) \subseteq \mathbb{R}\times(0,\infty)$:




$$\iint \left[u_t + f(u)_x\right] dt\, dx$$

$$= \int \left[u(x, t_{max}) - u(x, t_{min})\right] dx$$

$$+ \int \left[f(u(x_{max}, t)) - f(u(x_{min}, t))\right] dt = 0$$

$\downarrow$ set $V$ to volume of one grid cell $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [t^n, t^{n+1}]$

$$(\ast)\quad \underbrace{\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left[u(x,t^{n+1}) - u(x,t^n)\right] dx}_{\text{change in } u \text{ in the volume } [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \text{ during } [t^n, t^{n+1}]} + \underbrace{\int_{t^n}^{t^{n+1}} \left[f(u(x_{i+\frac{1}{2}},t)) - f(u(x_{i-\frac{1}{2}},t))\right] dt}_{-\text{ flow difference through cell interfaces during } [t^n, t^{n+1}]} = 0$$

Assume there is a function $g: \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ ("numerical flux") such that ($\simeq$: "approximates"):

average conserved variable in cell $i$

$$u_i^n \simeq \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x,t^n)\, dx$$

average flux through interface in $\Delta t$

$$g_{i+\frac{1}{2}}^n = g(u_i^n, u_{i+1}^n) \simeq \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} f(u(x_{i+\frac{1}{2}},t))\, dt$$

$$g_{i-\frac{1}{2}}^n = g(u_{i-1}^n, u_i^n) \simeq \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} f(u(x_{i-\frac{1}{2}},t))\, dt$$

$$\rightsquigarrow \quad (\ast) \Leftrightarrow u_i^{n+1} - u_i^n = -\frac{\Delta t}{\Delta x}\left(g_{i+\frac{1}{2}}^n - g_{i-\frac{1}{2}}^n\right)$$

## Consistency:

- We need to ensure that $g$ is consistent with physical flux:

$$g(v,v) = f(v) \quad \forall\, v \in \mathbb{R}^m$$

This is because of the following example:

Consider 1D $u_0(x) = \text{const.}$

$\Rightarrow$ $u = u_0$ is solution of CL and

$$g^0_{i+\frac{1}{2}} = g(u^0_i, u^0_{i+1}) = g(u_0, u_0)$$

$$g^0_{i+\frac{1}{2}} = \frac{1}{\Delta t}\int_0^{\Delta t} f(u(x_{i+\frac{1}{2}}, t))\, dt = \frac{1}{\Delta t}\int_0^{\Delta t} f(u_0)\, dt = f(u_0)$$

- Also need to expect continuity as $u_{i+1}, u_i$ vary

i.e. $g(u_i, u_{i+1}) \longrightarrow f(v)$ as $u_{i+1}, u_i \to v$

$\leadsto$ require Lipschitz continuity: there exist constants $L_1, L_2$ such that

$$\| g(u_{i+1}, u_i) - f(v) \| \leq L_1 \|u_{i+1} - v\|^\alpha + L_2 \|u_i - v\|^\alpha$$

<u>Def (Conservative scheme)</u>: Let $f \in C^1(\mathbb{R}^m; \mathbb{R}^m)$ and $g \in C^{0,1}(\mathbb{R}^m \times \mathbb{R}^m; \mathbb{R}^m)$ be consistent with the system of CLs (I), i.e., $g(v, v) = f(v)$ $\forall v \in \mathbb{R}^m$.
Assume we have a numerical grid $\mathcal{G} = \{(i\Delta x, n\Delta t) \mid i \in \mathbb{Z}, n \in \mathbb{N}\}$ and discretized initial data $u^0_i \in \mathbb{R}^m$.
A scheme of the form

$$u^{n+1}_i = u^n_i - \frac{\Delta t}{\Delta x}\left[ g(u^n_i, u^n_{i+1}) - g(u^n_{i-1}, u^n_i) \right]$$

is said to be in conservation form with numerical flux $g$.

**Notation:** $C^{0,1}$: Lipschitz continuous $\left( \begin{array}{l} h \in C^{k,\alpha}: k \text{ continuous derivatives} \\ \text{and } |h(x)-h(y)| \leq C \|x-y\|^{\alpha} \end{array} \right)$

**Remarks:**

1) The above definition is an "abstract" definition in the sense that the exact way $u_i^n$ and $g^n_{i+\frac{1}{2}}$ approximate volume / time averages of $u$ and $f(u)$ differ from scheme to scheme and, in particular, do not exactly equal the integral expressions used above to motivate them.

2) The definition can be generalized to

$$u_i^{n+1} - u_i^n = -\frac{\Delta t}{\Delta x}\Big\{ \theta \big[ g(u_i^{n+1}, u_{i+1}^{n+1}) + g(u_{i-1}^{n+1}, u_i^{n+1}) \big] - (1-\theta)\big[ g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, g_i^n) \big] \Big\}$$

for $0 \leq \theta \leq 1$. For $\theta = 0$: explicit scheme ($\Leftrightarrow$ Def)
$\theta = 1$: implicit scheme

3) The important property of a scheme in conservative form is that it guarantees the conservation property of the solution on the discretized

level:

$$\sum_i \left(g^n_{i+\frac{1}{2}} - g^n_{i-\frac{1}{2}}\right) = \sum_i \left[g(u^n_i, u^n_{i+1}) - g(u^n_{i-1}, u^n_i)\right] = 0$$

all counterparts cancel

$$\Rightarrow \boxed{\sum_i u_i^{n+1} = \sum_i u_i^n}$$

conservation on discretized level

4) Consistency of $g$ $\Rightarrow$ local truncation error is of order one (1st-order scheme)
Lemma 2.2.4 Kroener p. 45

# 6.2 Convergence: Lax-Wendroff theorem & entropy condition

Theorem (Lax-Wendroff 1960): (Comm. Pure Appl. Math. 13 (1960), 217–237)

Let $(u_\ell)_{\ell \in \mathbb{N}}$ be a sequence of discrete solutions of a scheme in conservation form with respect to $(h_\ell = \Delta x)_\ell$ and $(k_\ell = \Delta t)_\ell$, where $h_\ell, k_\ell \xrightarrow{\ell \to \infty} 0$ with $\frac{k_\ell}{h_\ell} = \text{const.}$

Assume that there exists $C \in \mathbb{R}$ such that

$$\sup_\ell \sup_{\mathbb{R} \times (0,\infty)} |u_\ell(x,t)| \leq C \qquad (u_\ell \text{ uniformly bounded})$$

and $u_\ell \xrightarrow{\ell \to \infty} u$ almost everywhere in $\mathbb{R} \times (0,\infty)$.

Then $u$ is a weak solution of the system of CLs.

Proof: Consider explicit case $(\theta = 0)$, see above.
Implicit case is analogous.

Choose $\varphi \in C_0^\infty(\mathbb{R}^m \times (0,\infty))$ and multiply scheme by $\varphi$:
$\uparrow$ compact support

$$(*) \quad \Delta x \left(u_i^{n+1} - u_i^n\right) \varphi(x_i, t^n) = -\Delta t \left(g^n_{i+\frac{1}{2}} - g^n_{i-\frac{1}{2}}\right) \varphi(x_i, t)$$

① Analyze convergence of LHS:

$$(**) \quad \sum_{n=0}^{\infty} \left(u_i^{n+1} - u_i^n\right) \varphi(x_i, t^n) = \sum_{n=1}^{\infty} u_i^n \left[\varphi(x_i, t^{n-1}) - \varphi(x_i, t^n)\right] - u_i^0 \varphi(x_i, 0)$$

(i) partial summation:

$$\sum_{k=m}^{n} f_k (g_{k+1} - g_k) = \big(\underbrace{f_n g_{n+1} - f_m g_m}_{=0}\big) - \sum_{k=m+1}^{n} g_k (f_k - f_{k-1})$$

$= 0$ as $\varphi$ has compact support

(ii) Abel's formula

$$A(t) = \sum_{0 \le n \le t} a_n \quad \text{partial sum function}$$

$$\sum_{x < n \le y} a_n \phi(n) = A(y)\phi(y) - A(x)\phi(x) - \int_x^y A(z)\phi'(z)\,dz$$

for $\phi$ differentiable function on $[x, y]$

set $x = -1$

$$\leadsto \sum_{n=0}^{x} a_n \phi(n) = A(x)\phi(x) - \int_0^x A(z)\phi'(z)\,dz$$

$$\leadsto \cancel{(\ast\ast)} = -\int_0^\infty u_\ell(x_i,t)\,\partial_t \varphi(x_i,t)\,dt - u_i^0 \varphi(x_i,0)$$

for any $\ell \in \mathbb{N}$

Add sum over $x_i$:

$$\Delta x \sum_i \sum_{n=0}^{\infty} (u_i^{n+1} - u_i^n)\varphi(x_i,t^n)$$

$$= -\sum_i \Delta x \left\{ \int_0^\infty u_\ell(x_i,t)\,\partial_t\varphi(x_i,t)\,dt - u_i^0\varphi(x_i,0)\right\}$$

$$= -\int_{\mathbb{R}}\int_0^\infty u_\ell(x,t)\,\partial_t\varphi(x,t)\,dt\,dx - \int_{\mathbb{R}} u^0(x)\varphi(x,0)\,dx$$

$$+\mathcal{O}(\Delta x)$$

(note: both sums are finite due to $\varphi \in C_0^\infty$)

$$\Big\downarrow \ \ell \to \infty$$

$$= -\int_{\mathbb{R}}\int_0^\infty u(x,t)\,\partial_t\varphi(x,t)\,dt\,dx - \int_{\mathbb{R}} u^0(x)\varphi(x,0)\,dx$$

② Analyze convergence of RHS:

$$-\Delta t \sum_i \sum_{n=0}^{\infty} \left(g^n_{i+\frac{1}{2}} - g^n_{i-\frac{1}{2}}\right) \varphi(x_i, t^n)$$

partial summation

$$\overset{\nearrow}{=} -\Delta t \sum_i \sum_{n=0}^{\infty} g^n_{i+\frac{1}{2}} \left[\varphi(x_i, t^n) - \varphi(x_{i+1}, t^n)\right]$$

$$= \Delta x \Delta t \sum_i \sum_{n=0}^{\infty} g^n_{i+\frac{1}{2}} \partial_x \varphi(x_i, t^n) + \mathcal{O}(\Delta x)$$

Set: $g_\ell(x,t) \equiv g^n_{i+\frac{1}{2}}$ $\quad x_i \le x < x_{i+1}$, $\quad t^n < t \le t^{n+1}$

$$= g(u^n_i, u^n_{i+1}), \quad u^n_i = (u_\ell)^n_i$$

$$= \sum_i \sum_{n=0}^{\infty} \int_{t^n}^{t^{n+1}} \int_{x_i}^{x_{i+1}} g_\ell(x,t)\, \partial_x \varphi(x,t)\, dx\, dt + \mathcal{O}(\Delta x)$$

$$= \int_{\mathbb{R}} \int_0^{\infty} g_\ell(x,t) \partial_x \varphi(x,t)\, dt\, dx + \mathcal{O}(\Delta x)$$

$\Big\downarrow \; \ell \to \infty$ $\qquad$ g consistent & Lipschitz continuous

$$\int_{\mathbb{R}} \int_0^{\infty} \underbrace{g(u(x,t), u(x,t))}_{= f(u)} \partial_x \varphi(x,t)\, dt\, dx$$

To see how this convergence arises:

extend grid function $u_i^n$ to $\mathbb{R}\times(0,\infty)$:

$$u_\ell(x,t) = u_i^n \quad \text{for} \quad t^n < t \leq t^{n+1}$$
$$x_i - \frac{\Delta x}{2} < x \leq x_i + \frac{\Delta x}{2}$$

$$\rightsquigarrow g_\ell(x,t) = g\left(u_\ell\left(x-\frac{\Delta x}{2}\right), u_\ell\left(x+\frac{\Delta x}{2}\right)\right)$$

and $\left|g_\ell(x,t) - g(u(x,t),u(x,t))\right|$

g Lipschitz continuous

$$= \left|g\left(u_\ell\left(x-\frac{\Delta x}{2}\right), u_\ell\left(x+\frac{\Delta x}{2}\right)\right) - g(u(x,t),u(x,t))\right|$$
$$\leq L_1\left|u_\ell\left(x-\frac{\Delta x}{2}\right) - u(x,t)\right| + L_2\left|u_\ell\left(x+\frac{\Delta x}{2}\right) - u(x,t)\right|$$

$(\cancel{**})\downarrow \ell\to\infty$

$$0 \quad \Rightarrow \quad g_\ell(x,t) \to g(u(x,t),u(x,t)) = f(u)$$

This is because (dropping $t$):

$$\int_0^\infty\int_{\mathbb{R}} u_\ell\left(x+\frac{\Delta x}{2}\right)\varphi(x)\,dx\,dt \underset{x\to x-\frac{\Delta x}{2}}{=} \int_0^\infty\int_{\mathbb{R}} u_\ell(x)\,\varphi\left(x-\frac{\Delta x}{2}\right)dx\,dt$$

$$\xrightarrow{\ell\to\infty} \int_0^\infty\int_{\mathbb{R}} u(x)\,\varphi(x,t)\,dx\,dt$$

$\Rightarrow u_\ell\left(x+\frac{\Delta x}{2}\right) \to u(x)$ weak

$\Rightarrow u_\ell^2\left(x+\frac{\Delta x}{2}\right) \to u^2(x)$ weak

$L^2_{loc}$ convergence:

$$\int_0^\infty \int_{\mathbb{R}} |u_\ell(x+\tfrac{\Delta t}{2}) - u|^2 \varphi(x,t)\,dx\,dt = \int_0^\infty \int_{\mathbb{R}} u_\ell(x-\tfrac{\Delta t}{2})^2 \varphi$$

$$-2\iint u_\ell(x+\tfrac{\Delta t}{2})\,u(x)\,\varphi + \iint u^2\varphi \xrightarrow{\ell\to\infty} 0$$

$\Rightarrow$ (∗∗∗)

<u>1 + 2</u> $\Rightarrow$ (∗)

$$\Big\downarrow \ell \to \infty$$

$$\int_{\mathbb{R}} \int_0^\infty \left(u \cdot \varphi_t + f(u)\cdot \varphi_x\right) dt\,dx + \int_{\mathbb{R}} u_0 \cdot \varphi(x,0)\,dx = 0$$

$\Rightarrow$ $u$ weak solution

$\square$

<u>Remark</u>: The Lax-Wendroff theorem only guarantees that a conservative scheme converges to a weak solution (if it converges)

$\rightsquigarrow$ how do we ensure to obtain the entropy solution?