Computational Fluid Dynamics

Problem Set 4

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Problem 1 (Jump conditions for the Euler equations)

Consider a 3-shock with shock velocity S_3 for the 1D Euler equations in conservative form, assuming an ideal gas with adiabatic constant γ. Let $(\rho_\star, v_\star, p_\star)$ and (ρ_r, v_r, p_r) denote the sets of primitive variables on the left and right side of the shock front, respectively. Derive the Rankine-Hugoniot jump conditions

$$\frac{\rho_\star}{\rho_r} = \frac{\frac{p_\star}{p_r} + \frac{\gamma-1}{\gamma+1}}{\frac{\gamma-1}{\gamma+1}\frac{p_\star}{p_r} + 1}, \qquad s_3 = v_r + c_r\sqrt{\frac{\gamma+1}{2\gamma}\frac{p_\star}{p_r} + \frac{\gamma-1}{2\gamma}}, \tag{1}$$

where $c = \sqrt{\gamma p/\rho}$ denotes the sound speed.

Problem 2 (Non-uniqueness of weak solutions)

A given initial value problem for conservation laws admits, in general, infinitely many weak solutions. Consider the Riemann problem for the inviscid Burgers' equation:

$$\begin{cases} \partial_t u + \partial_x \left(\frac{1}{2}u^2\right) = 0 \\ u(0,x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases} \end{cases} . \tag{2}$$

Show that for any $\alpha \in (0,1)$ the 'two-shock solution'

$$u_\alpha(t,x) \equiv \begin{cases} 0 & \text{if } x < \frac{\alpha}{2}t \\ \alpha & \text{if } \frac{\alpha}{2}t < x < \frac{1+\alpha}{2}t \\ 1 & \text{if } x > \frac{1+\alpha}{2}t \end{cases} . \tag{3}$$

is a weak solution to the Cauchy problem (2).

Problem 3 (Entropy pair & Riemann invariants)

Consider the 1D Euler equations for an ideal gas with entropy $S = S_0 + c_v \ln \frac{p}{\rho^\gamma}$, where S_0 and c_v are constants and γ is the adiabatic index.

1. *(Entropy pair)* Show that the entropy satisfies

$$S_t + vS_x = 0 \tag{4}$$

   where the solution is smooth. What is the entropy flux? This is to show that modulo a minus sign the physical entropy provides an entropy–entropy-flux pair for the Euler equations.

2. *(Riemann invariants)* Consider the Euler equations in primitive form using the primitives (ρ, v, S). Show that S is a Riemann invariant. Show that there exist the following k-Riemann invariants:

   - $k = 1$: S, $u + \frac{2}{\gamma-1}c$
   - $k = 2$: u, p
   - $k = 3$: S, $u - \frac{2}{\gamma-1}c$

3. *(Characteristic fields)* Show that the $k = 2$ characteristic field is linearly degenerate, while the $k = 1, 3$ characteristic fields are genuinely non-linear. *Hint:* It is easiest to work with the conservative formulation here.

Problem 4 (Solution to Burgers' equation)

Determine and sketch the exact entropy solution to Burgers' equation $u_t + (\frac{1}{2}u^2)_x = 0$ for all $t > 0$ for initial data given by

$$u^0(x) = \begin{cases} -1 & \text{if } x < -1 \\ 0 & \text{if } -1 < x < 1 \\ 1 & \text{if } x > 1 \end{cases} \tag{5}$$

Problem 5 (Bonus question, not required)

Consider a piecewise smooth weak solution u of the system $u_t + f(u)_x = 0$, and assume that an entropy pair to the equation is given by (Φ, Ψ). Show that the entropy condition along the discontinuity with velocity s leads to the inequality

$$s[\Phi(u_l) - \Phi(u_r)] \leq \Psi(u_l) - \Psi(u_r), \tag{6}$$

where u_l and u_r denote the values of u immediately left and right of the discontinuity. *Hint:* Consider the Rankine-Hugoniot theorem. *Remark:* This is the first step in showing the equivalence of the Lax entropy condition and the entropy inequalities.