

# Computational Fluid Dynamics

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# Preface and recommended literature

These lecture notes have been prepared for a new graduate course on computational fluid dynamics in the Guelph–Waterloo Institute for Physics (GWIP), the joint graduate school of the Universities of Guelph and Waterloo, as well as Perimeter Institute for Theoretical Physics (Fall 2020). Many textbooks helped me compose these lectures and lecture notes. Some suggested textbooks include:

## Numerical Methods

- [Toro \(2009\)](#): *Riemann Solvers and Numerical Methods for Fluid Dynamics* (Springer, 3rd edition, 2009)
- [Leveque \(2002\)](#): *Finite Volume Methods for Hyperbolic Problems* (Cambridge Univ. Press, Cambridge Texts in Applied Mathematics, 2002)

## Mathematically inclined literature:

- [Kröner \(1997\)](#): *Numerical Schemes for Conservation Laws* (Wiley, 1997)
- [Evans \(2010\)](#): *Partial Differential Equations* (Graduate Studies in Mathematics, American Mathematical Society, 2nd edition, 2010)

## Other recommended literature

- [Bodenheimer et al. \(2006\)](#): *Numerical Methods in Astrophysics* (Taylor & Francis, 2007)
- [Anile \(1989\)](#): *Relativistic fluids and magneto-fluids* (Cambridge Univ. Press, 1990)
- [Rieutord \(2015\)](#): *Fluid Dynamics* (Springer, 2015)
- [LeVeque \(2007\)](#): *Finite Difference Methods for Ordinary and Partial Differential Equations* (SIAM, 2007)
- [Morton & Mayers \(2005\)](#): *Numerical Solution of Partial Differential Equations* (Cambridge Univ. Press, 2nd edition, 2005)



# Chapter 1

## Basic Notions of Partial Differential Equations

### 1.1 PDEs of 2nd order

**Recommended reading:** [Evans \(2010\)](#) Chap. 1, Chap. 6.1, 7.1.1, 7.2.1, [Strauss \(2007\)](#) Chap. 1.6, most introductory books on PDEs

### 1.2 PDEs of 1st order

**Recommended reading:** [Evans \(2010\)](#) Chap. 7.3.1, 11.1, [Toro \(2009\)](#) Chap. 2

### 1.3 Some properties of 1st order hyperbolic systems

**Recommended reading:** [Toro \(2009\)](#) Chap. 2, [Leveque \(2002\)](#) Chap. 2.9–2.11, 3.1–3.6, 11.2, 18.5

#### 1.3.1 Characteristics

#### 1.3.2 Domain of dependence and range of influence





## Chapter 2

# Basic Equations of Computational Fluid Dynamics

**Recommended reading:** [Bodenheimer et al. \(2006\)](#) Chap. 1, [Rieutord \(2015\)](#) Chap. 11, [Lifshitz & Pitaevskii \(1981\)](#) Chap. 1, [Anile \(1989\)](#) Chap. 2

### 2.1 Continuous media and the Boltzmann equation

More on the  $N$ -body approach: [Bodenheimer et al. \(2006\)](#) Chap. 3

A compilation of the largest cosmological  $N$ -body simulations accomplished to date can be found in Tab. II of [Cheng et al. \(2020\)](#).

### 2.2 From the Boltzmann equation to the Euler equations

### 2.3 Navier-Stokes equations

More on viscosity: [Landau & Lifshitz \(2004\)](#) §15, [Shakura & Sunyaev \(1973\)](#), [Balbus & Hawley \(1991\)](#)

### 2.4 Magnetohydrodynamics

### 2.5 Radiation transfer

### 2.6 Relativistic Hydrodynamics

**Recommended reading:** [Gourgoulhon \(2012\)](#) Chaps. 4 & 6, [Gourgoulhon \(2006\)](#), [Baumgarte & Shapiro \(2010\)](#) Chap. 5, [Alcubierre \(2008\)](#) Chap. 2.2 & 7

### 2.7 Relativistic radiation transfer

**Recommended reading:** [Thorne \(1981\)](#), [Shibata et al. \(2011\)](#), [Straumann \(2013\)](#) Sec. 3.11.



## Chapter 3

# Finite Difference methods for PDEs

**Recommended reading:** [LeVeque \(2007\)](#) Chap. 1, 9, 10, [Bodenheimer et al. \(2006\)](#) Chap. 2, [Choptuik \(2006\)](#) Sec. 1., [Kröner \(1997\)](#) Chap. 2.4, [Toro \(2009\)](#) Chap. 5.1

### 3.1 Basic notions of discretization

### 3.2 Finite difference approximations

#### 3.2.1 Partial derivatives & differential operators

#### 3.2.2 Sample discretizations

### 3.3 Consistency, stability, convergence

**Recommended reading:** [Morton & Mayers \(2005\)](#) Chap. 5.

### 3.4 Stability analysis and the CFL condition

### 3.5 Diffusion and dispersion

### 3.6 Error analysis and convergence



## Chapter 4

# Properties of Conservation Laws

**Recommended reading:** [Evans \(2010\)](#) Chap. 11.1, 11.2, 11.4, [Kröner \(1997\)](#) Chap. 4.1., [Leveque \(2002\)](#) Chap. 13, [Toro \(2009\)](#) Chap. 2.4

**Recommended original literature:** [Lax \(1957\)](#)

### 4.1 Local existence of classical solutions

### 4.2 Weak solutions

#### 4.2.1 Definition

#### 4.2.2 Behavior near discontinuities (Jump conditions)

### 4.3 Entropy condition

### 4.4 Riemann's Problem

#### 4.4.1 First examples

#### 4.4.2 Riemann invariants & characteristic fields

#### 4.4.3 Special solutions: rarefaction waves

#### 4.4.4 Special solutions: shocks and the Lax entropy condition

#### 4.4.5 Special solutions: contact discontinuities

#### 4.4.6 General solution to Riemann's problem



## Chapter 5

# Riemann Problem for the Euler Equations

**Recommended reading:** [Toro \(2009\)](#) Chap. 4, [Leveque \(2002\)](#) Chap. 14, [Kröner \(1997\)](#) Chap. 4.1

### 5.1 Overview

### 5.2 Solution strategy





## Chapter 6

# Numerical Schemes for Conservation Laws

**Recommended reading:** [Kröner \(1997\)](#) Chap. 4.4, [Toro \(2009\)](#) Chaps. 5.3 & 6, [Leveque \(2002\)](#) Chaps. 12 & 15

### 6.1 Conservative Schemes

### 6.2 Convergence: Lax-Wendroff theorem & entropy condition

### 6.3 Godunov's method for nonlinear systems of conservation laws



## Chapter 7

# Approximate Riemann Solvers

**Recommended reading:** [Toro \(2009\)](#) Chaps. 10 & 11, [Kröner \(1997\)](#) Chap. 4.4, [Leveque \(2002\)](#) Chap. 15.3

### 7.1 The Riemann Solver of Roe

**Recommended reading:** [Roe \(1981\)](#), [Roe & Pike \(1985\)](#)

### 7.2 The HLL family of Riemann solvers

**Recommended reading:** [Harten \(1983\)](#), [Davis \(1988\)](#), [Einfeldt \(1988\)](#), [Einfeldt et al. \(1991\)](#)

#### 7.2.1 The HLL solver

#### 7.2.2 The HLLC solver

#### 7.2.3 Wave speed estimates



## Chapter 8

# Extensions to Balance Laws and multi-D, higher-order methods

Recommended reading: [Toro \(2009\)](#) Chap. 15–17, [Leveque \(2002\)](#) Chap. 17–19

### 8.1 Source terms

### 8.2 Multidimensional systems of conservation laws

#### 8.2.1 Dimensional splitting

#### 8.2.2 Unsplit finite volume schemes

#### 8.2.3 Higher-order multi-D schemes: MUSCL-Hancock methods



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