Chap. 4: properties of conservation laws (theoretical background)

consider system of conservation laws (1D):

$$(\text{IVP}) \qquad \begin{aligned} u_t + f(u)_x &= 0 \quad \text{in } \mathbb{R} \times (0,\infty) \\ u(x,0) &= u_0(x) \quad \text{on } \mathbb{R} \times \{t=0\} \end{aligned}$$

$$u = (u^1, \dots, u^m) : \mathbb{R} \times [0,\infty) \to \mathbb{R}^m$$

- show that in general classical solutions (smooth $\rightsquigarrow C^1$) do not exist (globally)
- define weak solutions & entropy condition to guarantee uniqueness of weak solution
- Riemann problem: discontinuous initial data & non-linear waves
  $\to$ centrepiece of numerical algorithms

4.1 Local existence of classical solutions

Even in the case of smooth data f and u_0 there are in general no continuous solutions that exist <u>globally</u> in time.

<u>Example:</u> <u>Inviscid Burger's equ</u>

Consider scalar conservation law

$$u_t + f(u)_x = 0 \quad \text{with } f(u) = \frac{1}{2}u^2$$

and $u_0 \in C^\infty(\mathbb{R})$, $u_0(x) = \begin{cases} 1, & x \in (-\infty, -1] \\ 0, & x \in [1, \infty) \\ u_0' \leq 0 \end{cases}$

Remember (Sec. 1.3.1): characteristic $\Gamma_{x_0}^t = (\gamma(t), t)$ defined by

$$\gamma'(t) = \frac{d\gamma}{dt} = \lambda(u) = f'(u).$$
$$\gamma(0) = x_0$$

$$\leadsto \begin{cases} \frac{d}{dt} u(\gamma(t), t) = \partial_t u + \gamma' \partial_x u = u_t + f(u)_x = 0 \\ \gamma'(t) = f'(u(\gamma(t), t)) = f'(u(\gamma(0), 0)) = f'(u_0(x_0)) \\ \qquad = \text{const.} \end{cases}$$

$\Rightarrow$ u const. along $\Gamma_{x_0}^t$

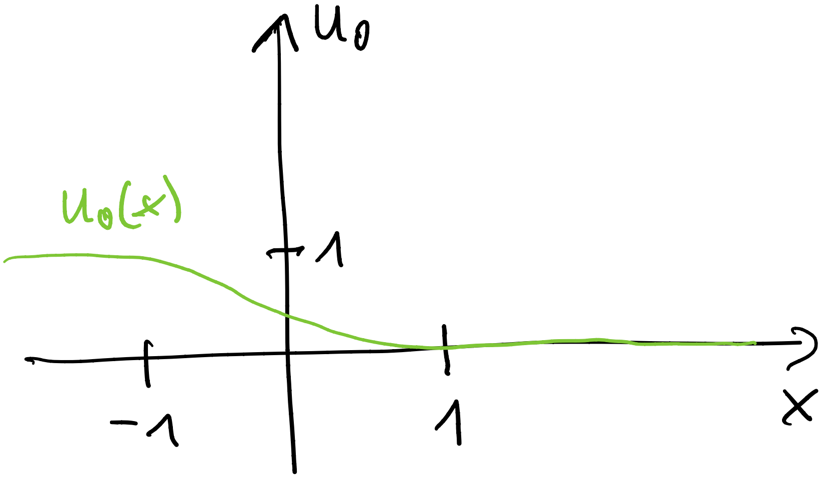
$\Gamma_{x_0}^t$ is a straight line

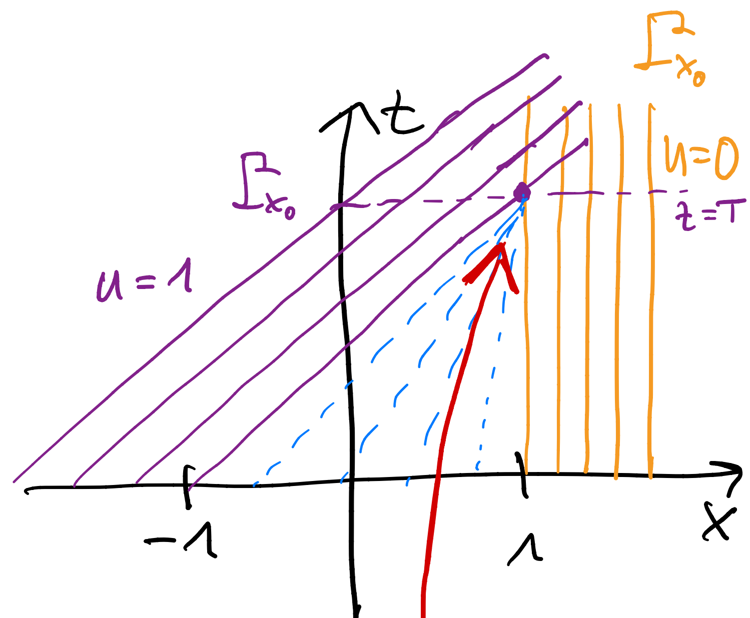
Here: $\gamma'(t) = f'(u_0(\gamma(0))) = u_0(\gamma(0))$

$$u(\gamma(t),t) = u(\gamma(0),0) = u_0(\gamma(0))$$

$$\Gamma^2_{x_0} \text{ starting at } \left\{\begin{matrix} x_0 \in (-\infty,-1] \\ x_0 \in [1,\infty) \end{matrix}\right\} \text{ have slope } \left\{\begin{matrix} 1 \\ 0 \end{matrix}\right\},$$

and $u = \left\{\begin{matrix} 1 \\ 0 \end{matrix}\right\}$ along these lines.



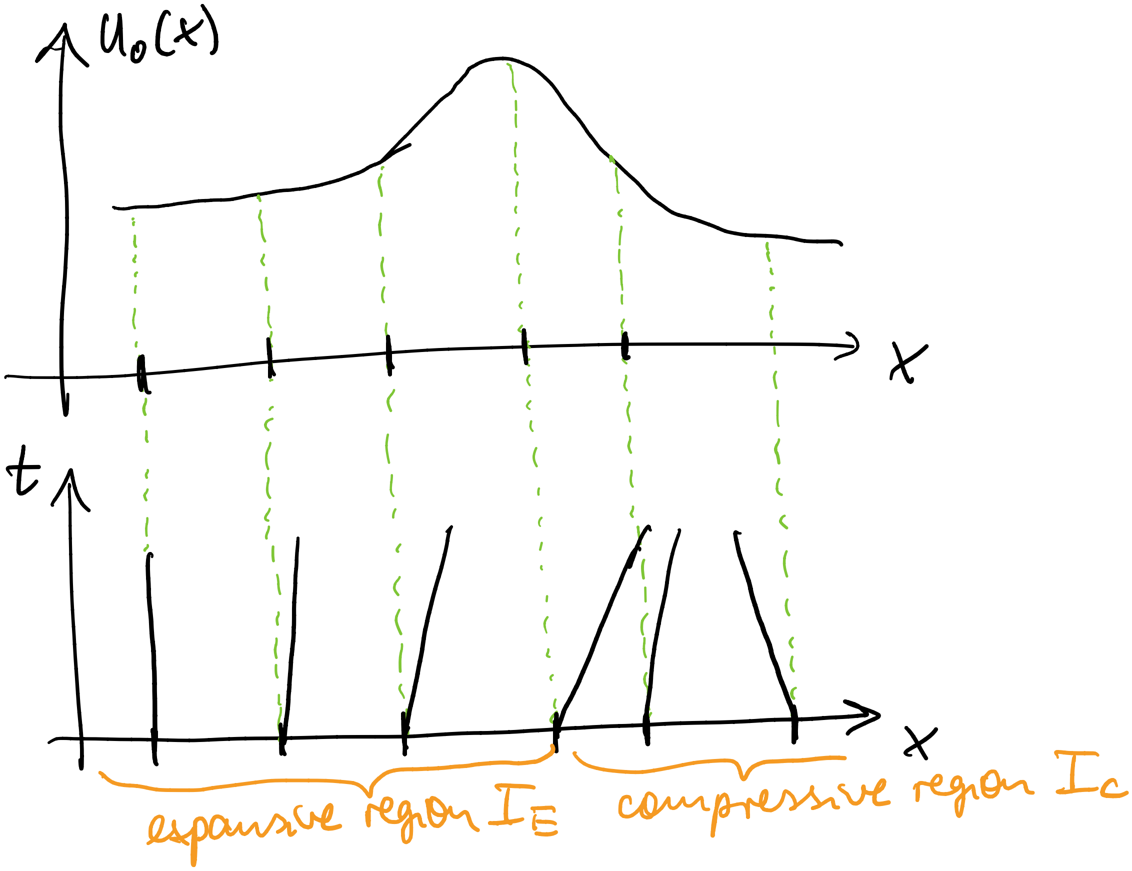


characteristics meet at finite time T

$\Rightarrow$ u <u>cannot be continuous</u>

i.e. $u \notin C^0(\mathbb{R}\times[0,\infty))$!

Example: wave steepening Consider scalar CL with smooth initial data $u_0(x)$ and convex flux function $\lambda'(u) = f''(u) > 0$



- "larger values of $u_0(x)$ will travel faster than smaller values of $u_0(x)$"
- I_E & I_C reversed for concave flux
- existence of I_E & I_C leads to crossing of characteristics and multi-valued solutions

$\rightsquigarrow$ "the wave breaks" $(u_x \to \infty)$ at first crossing

write $u(x,t)$ in terms of characteristics $\gamma(t)$
(see existence result below)

$$u(x,t) = u(\gamma(t),t) = u_0(\underbrace{x - \lambda(u_0(x_0))t}_{=x_0})$$

$$\rightsquigarrow u_x = u_0'(x_0)\frac{\partial x_0}{\partial x} \quad , \quad x = x_0 + \lambda(u_0(x_0))t$$

$$\frac{\partial x}{\partial x_0} = 1 + \lambda'(u_0(x_0))\, u_0'(x_0)t = 1 + \lambda_x t$$

$$\frac{1}{u_x} = 0 \iff \frac{\partial x}{\partial x_0} = 0$$

$$\Updownarrow$$

$$\boxed{t_{break} = -\frac{1}{\lambda_x(u_0(x_0))}}$$

breaking <u>first</u> occurs for characteristic $\Gamma_{x_0}^{+}$ for which $\lambda_x(u_0(x_0)) < 0$ and $|\lambda_x(u_0(x_0))|$ is maximal.

<u>Theorem (local existence of classical solution):</u>

Assume $f \in C^2(\mathbb{R})$, $u_0 \in C^1(\mathbb{R})$, $|f''|, |u_0'| \leq$ const.
Then there exists $T > 0$ such that (IVP) for <u>scalar conservation law</u> has a classical solution $u \in C^1(\mathbb{R} \times [0,T))$.

<u>Proof:</u> For $(x,t) \in \mathbb{R} \times [0,\infty)$ consider characteristic $\Gamma_{x_0} = (\gamma(t),t)$ passing through (x,t):

(i) $$x_0 = x - \lambda(u_0(x_0))t = x - f'(u_0(x_0))t$$
$$= x - f'(u(x,t))t$$

(Note: u const. along Γ_{x_0})

(ii) $$u(x,t) = u(x_0,0) = u_0(x_0) = u_0(x - f'(u(x,t))t)$$

Define $F(v,x,t) \equiv v - u_0(x - f'(v)t)$, then

$F(u_0(x),x,0) = 0$ and $\partial_v F = 1 + u_0' f''(v) t \neq 0$

Implicit function theorem $\Rightarrow$ for $t < T$ suff. small $\exists$ function u satisfying (ii)

differentiate (i):

$$\frac{\partial x}{\partial t} = 0 = \frac{\partial x_0}{\partial t} + f'(u_0(x_0)) + f''(u_0(x_0))\, u_0'(x_0) \frac{\partial x_0}{\partial t} t$$
$$= f'(u_0(x_0)) + \left[1 + f''(u_0(x_0))\, u_0'(x_0)\, t\right] \frac{\partial x_0}{\partial t}$$
$$\Leftrightarrow \frac{\partial x_0}{\partial t} = - \frac{f'(u_0(x_0))}{1 + f''(u_0(x_0))\, u_0'(x_0)\, t}$$

$$\frac{\partial x}{\partial x} = 1 = \frac{\partial x_0}{\partial x} + f''(u_0(x_0))\, u_0'(x_0)\, t \frac{\partial x_0}{\partial x}$$

$$\Leftrightarrow \frac{\partial x_0}{\partial x} = \frac{1}{1+f''(u_0(x_0))\,u_0'(x_0)\,t}$$

differentiate (ii): (u const. along Γ)

$$u_t = u_0'(x)\,\frac{\partial x_0}{\partial t} = -\frac{u_0'\,f'(u)}{1+f''(u)\,u_0'\,t}$$

$$u_x = u_0'(x)\,\frac{\partial x_0}{\partial x} = \frac{u_0'}{1+f''(u)\,u_0'\,t}$$

$\Rightarrow$ $u_t + f'(u)\,u_x = 0$ and $u \in C^1(\mathbb{R})$ for $(x,t) \in \mathbb{R}\times[0,T)$ and T sufficiently small.

$\square$

<u>Remark:</u> <u>Systems of CLs in 1D:</u>

<u>Theorem:</u> Let $u_0 \in C_0^1(\mathbb{R};\mathbb{R}^m)$ and $u_t + f(u)_x = 0$ strictly hyperbolic. Then $\exists\, T>0$ such that the IVP admits a classical solution $u : [0,T)\times\mathbb{R}\to\mathbb{R}^m$.

<u>Proof:</u> Bressan pp. 67–70.

4.2 Weak solutions

4.2.1 Definition

Previous examples suggest to allow discontinuous (classically non-differentiable) solutions.

<u>Idea:</u> consider integral form of conservation law, then functions must only be integrable, e.g. $u \in L^\infty(\mathbb{R}\times[0,\infty);\mathbb{R}^m)$ or L^p

$$\|u\|_p = \left[\int_0^\infty \int_{\mathbb{R}} |u|^p \, dx\, dt\right]^{1/p} < \infty \,, \quad \|u\|_\infty = \inf\left\{ C \geq 0 \,\middle|\, \begin{array}{c} |u(x)| \leq C \\ \text{almost everywhere} \end{array} \right\} < \infty$$

$\rightsquigarrow$ consider for now u smooth solution of (IVP) and $v \in \underbrace{C_0^\infty}_{\text{compact support}}(\mathbb{R}\times[0,\infty);\mathbb{R}^m)$:

$$\Rightarrow \quad v\cdot u_t + v\cdot f(u)_x = 0$$

$\downarrow$ integrate

$$\int_0^\infty\int_{\mathbb{R}} v\cdot u_t \, dx\,dt + \int_0^\infty\int_{\mathbb{R}} v\cdot f(u)_x = 0$$

$\downarrow$ integrate by parts

$$\int_0^\infty\int_{\mathbb{R}} v_t\cdot u\,dx\,dt - \int_{\mathbb{R}} \Big[v\cdot u\Big]_{t=0}^{t=\infty} dx$$

$$+ \int_0^\infty\int_{\mathbb{R}} v_x\cdot f(u)\,dx\,dt - \int_0^\infty \underbrace{\Big[v\cdot f(u)\Big]_{x=-\infty}^{x=\infty}}_{=0} dt = 0$$

$$\Leftrightarrow \int_0^\infty\int_{\mathbb{R}} \big(v_t\cdot u + v_x\cdot f(u)\big)\,dx\,dt + \int_{\mathbb{R}} v(x,0)\,u_0(t)\,dt = 0$$

<u>Def</u>: Consider (IVP) with $u_0 \in L^\infty(\mathbb{R};\mathbb{R}^m)$.
Then $u \in L^\infty(\mathbb{R}\times[0,\infty);\mathbb{R}^m)$ is called a <u>weak solution</u> (integral solution, solution in the distributional sense) if and only if

$$\int_0^\infty \int_{\mathbb{R}} \big(u\cdot v_t + f(u)\cdot v_x\big)\,dx\,dt + \int_{\mathbb{R}} u_0\cdot v(x,0)\,dx = 0$$

for all $v \in C_0^\infty(\mathbb{R}\times[0,\infty);\mathbb{R}^m)$.

compact support

Remark: if u is a weak solution that happens to be $C^1(\mathbb{R}\times[0,\infty);\mathbb{R}^m)$, then u is a classical solution of the (IVP).

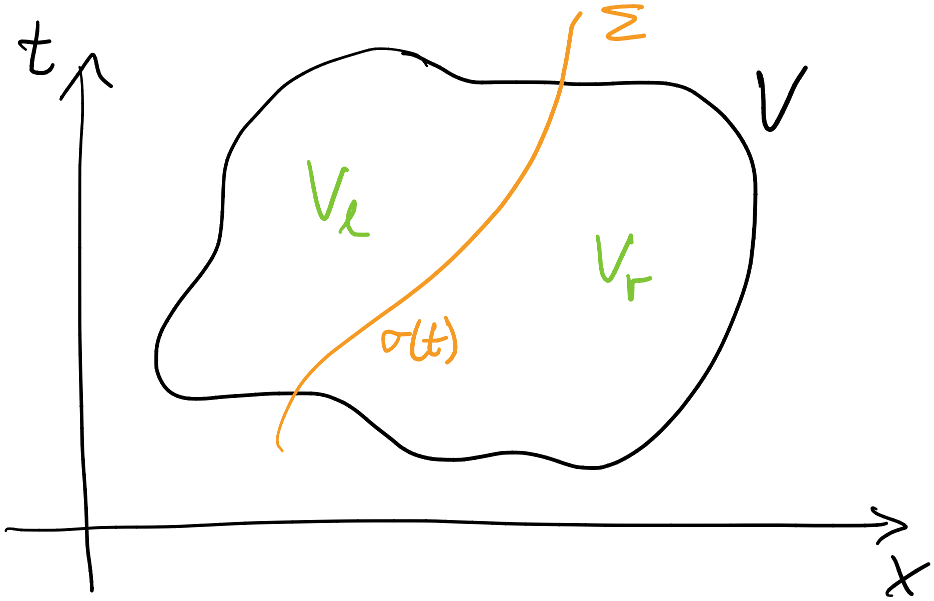
4.2.2 Behaviour near discontinuities (Jump conditions)

Theorem (Rankine – Hugoniot):

Consider $V\subset\mathbb{R}\times(0,\infty)$ separated by a smooth curve $\Sigma: t\mapsto(\sigma(t),t)$ in two parts V_l and V_r. Let $u\in L^1(\mathbb{R}\times(0,\infty))$ such that $u_l\equiv u|_{V_l}\in C^1(\overline{V_l})$ and $u_r\equiv u|_{V_r}\in C^1(\overline{V_r})$ and u_l, u_r locally satisfy the IVP on V_l, V_r in the classical sense. Then u is a weak solution of the IVP if and only if

$$\text{(RH)}\quad f(u_l(\sigma(t),t))-f(u_r(\sigma(t),t))=\left[u_l(\sigma(t),t)-u_r(\sigma(t),t)\right]\sigma'(t)$$

for all $t>0$.



<u>Notation:</u> (RH) is often written

$$f_l - f_r = \sigma'(u_l - u_r) \quad \text{along } \Sigma$$

or

$$[\![f]\!] = \sigma' [\![u]\!]$$

where $[\![\ ]\!]$ means "jump across the curve Σ"

<u>Proof:</u> Take $v \in C_0^\infty(V)$ (otherwise can always enlarge V)

compact support in V

does not necessarily vanish along Σ

Let $\nu = (\nu^1, \nu^2)$ denote the outer unit normal of V_l $\Rightarrow$ $\nu(t) = \frac{1}{\sqrt{1+\sigma'(t)^2}}\left(1, -\sigma'(t)\right)$. Then:

$$0 \overset{\text{Def}}{=} \int_0^\infty \int_{\mathbb{R}} [u \cdot v_t + f(u) \cdot v_x] \, dx\,dt + \overbrace{\int_{\mathbb{R}} u_0 \cdot v(x,0) \, dx}^{=0,\ v \in C_0^\infty(V)}$$

$$= \iint_{V_\ell} [u \cdot v_t + f(u) \cdot v_x] \, dx\,dt + \iint_{V_r} [u \cdot v_t + f(u) \cdot v_x] \, dx\,dt$$

$u \in C^1$ on V_ℓ, V_r

$$= -\int_{V_\ell} \underbrace{[u_t + f(x)_x]}_{=0 \text{ on } V_\ell} \cdot v \, dx\,dt + \int_{\partial V_\ell} (u \cdot v\, \nu^2 + f(u) \cdot v\, \nu^1) \, d\ell$$

$\nu_r = -\nu_\ell = -\nu$

$$- \int_{V_r} \underbrace{[u_t + f(u)_x]}_{=0 \text{ on } V_r} \cdot v \, dx\,dt - \int_{\partial V_r} (u \cdot v\, \nu^2 + f(u) \cdot v\, \nu^1) \, d\ell$$

v vanishes on $\partial V_\ell \setminus \Sigma$, $\partial V_r \setminus \Sigma$

$$= \int_\Sigma \left[(u_\ell - u_r) \nu^2 + (f(u_\ell) - f(u_r)) \nu^1 \right] \cdot v \, d\ell$$

Since v arbitrary

$$\Rightarrow \quad f(u_\ell) - f(u_r) = \sigma (u_\ell - u_r)$$

$\square$

4.3 Entropy condition

In general, the (IVP) has <u>no unique weak solution</u>, i.e. the (IVP) for weak solutions is <u>not well-posed</u>.

→ we will define a "<u>selection criterion</u>" (<u>entropy condition</u>) to pick the correct physical solution and restore well-posedness.

<u>Example (non-uniqueness):</u> Consider 1D CL:

inviscid Burger's eqn.

$$u_t + \partial_x\left(\frac{u^2}{2}\right) = 0, \quad x \in \mathbb{R}, \; t > 0$$

$$u_0(x) = \begin{cases} 0, & x \le 0 \\ 1, & x > 0 \end{cases}$$

Define

$$u_1(x,t) \equiv \begin{cases} 0, & x < t/2 \\ 1, & x > t/2 \end{cases}$$

$$u_2(x,t) \equiv \begin{cases} 0, & x < 0 \\ x/t, & 0 \le x \le t \\ 1, & t < x \end{cases}$$

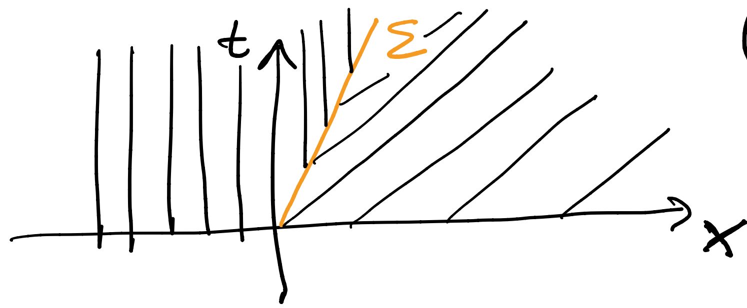
Both u_1, u_2 are piecewise C^1 and satisfy initial condition.

- u_1 satisfies RH conditions along $\Sigma = (\sigma(t), t)$ with $\sigma(t) = \frac{1}{2}t$:

$$\sigma'(t) = \frac{1}{2} \quad \text{and} \quad \frac{f(u_l) - f(u_r)}{u_l - u_r} = \frac{\frac{1}{2}u_l^2 - \frac{1}{2}u_r^2}{u_l - u_r}$$

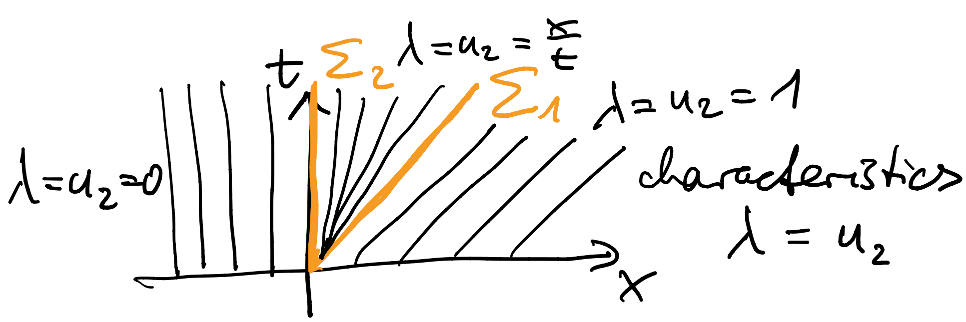
$$= \frac{1}{2}(u_l + u_r) = \frac{1}{2} \quad \checkmark$$

$\Rightarrow$ u_1 weak solution



(rarefaction shock, entropy violating)

- can check: u_2 satisfies RH conditions along $\Sigma_1 = (\sigma_1(t), t) = (t, t)$ and $\Sigma_2 = (\sigma_2(t), t) = (0, t)$



$\Rightarrow$ u_2 also weak solution

In reality, discontinuities are never arbitrarily sharp, but are rather "smeared out" by some intrinsic viscosity of the fluid. Physically correct solutions should arise as the limit of solutions to the "regularized system"

$$u_t^\varepsilon + f(u^\varepsilon)_x - \underbrace{\varepsilon u_{xx}^\varepsilon}_{\text{"small viscosity effect"}} = 0 \quad \text{in } \mathbb{R}\times(0,\infty)$$

as $\varepsilon \to 0$, i.e. as the problem approaches the inviscid problem.

Following Theorem provides the "viscosity method" of how to select the correct weak solution:

Theorem: Let $u_0 \in L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and $f_i, B \in C^2(\mathbb{R}^n \times (0,\infty)\times\mathbb{R})$ with bounded derivatives. Then for any $\varepsilon > 0$ there exists a uniquely defined classical solution u^ε of

$$u_t + \sum_{i=1}^n \partial_i f_i(x,t,u) + B(x,t,u) = \varepsilon \Delta u \text{ in } \mathbb{R}^n \times (0,\infty)$$

$$u(x,0) = u_0(x) \text{ in } \mathbb{R}^n$$

such that $u^\varepsilon \xrightarrow{\varepsilon \to 0} u$ almost everywhere in $\mathbb{R}^n \times (0,\infty)$ for some u that is a weak solution of ("viscosity limit")

$$\partial_t u + \sum_{i=1}^n \partial_i f_i(x,t,u) + B(x,t,u) = 0 \text{ in } \mathbb{R}^n \times (0,\infty)$$

$$u(x,0) = u_0(x) \text{ in } \mathbb{R}^n$$

Proof: omitted, see references in Kroener (p. 22).

Remark: For general systems the previous theorem is still an open problem.

Now: want necessary criterion for a weak soln. u to be the viscosity limit defined in the theorem.

Definition: Two smooth functions $\Phi, \Psi \in C^2(\mathbb{R}^m; \mathbb{R})$ are called an entropy/entropy-flux pair for the system of CLs $u_t + f(u)_x = 0$ if

(i) Φ is convex $(D^2\Phi(z) y \cdot y > 0 \ \forall z, y \in \mathbb{R}^m)$

(ii) $D\Phi(z) Df(z) = D\Psi(z), \quad z \in \mathbb{R}^m$

Definition: A weak solution u of the (IVP) is called an entropy solution if for any entropy pair Φ, Ψ it satisfies the inequality

$$\boxed{\Phi(u)_t + \Psi(u)_x \leq 0 \quad \text{in } \mathbb{R}\times(0,\infty)}$$

in the distributional sense, i.e.

$$\int_0^\infty\int_{\mathbb{R}} \Phi(u)v_t + \Psi(u)v_x \,dx\,dt \geq 0 \quad \forall \underset{v\geq 0}{v\in C_0^\infty(\mathbb{R}\times(0,\infty))}.$$

(Note: the latter expression is obtained by assuming u smooth, multiplying by v and integrating by parts)

Theorem: The viscosity limit $u = \lim_{\varepsilon\to 0} u^\varepsilon$ of

$$u_t^\varepsilon + f(u^\varepsilon)_x - \varepsilon u_{xx}^\varepsilon = 0 \quad \text{in } \mathbb{R}\times(0,\infty)$$
$$u^\varepsilon = u_0 \quad \text{on } \mathbb{R}\times\{t=0\}$$

is an entropy solution of

$$u_t + f(u)_x = 0 \quad \text{on } \mathbb{R}\times(0,\infty)$$
$$u = u_0 \quad \text{on } \mathbb{R}\times\{t=0\}$$

provided that u^ε is uniformly bounded in L^∞ and $u^\varepsilon \to 0$ as $|x|\to\infty$ sufficiently rapidly.

<u>Proof</u>: 1) choose entropy pair Φ, Ψ.

$$D\Phi(u^\varepsilon) \cdot \Big(u^\varepsilon_t + \underbrace{f(u^\varepsilon)_x}_{= Df(u^\varepsilon)\, u^\varepsilon_x} - \varepsilon\, u^\varepsilon_{xx}\Big) = 0$$

$$\Longleftrightarrow \quad \Phi(u^\varepsilon)_t + \Psi(u^\varepsilon)_x = \varepsilon\, D\Phi(u^\varepsilon)\, u^\varepsilon_{xx}$$

$$= \varepsilon\, \Phi(u^\varepsilon)_{xx} - \varepsilon\, \underbrace{\big(D^2\Phi(u^\varepsilon)\, u^\varepsilon_x\big) \cdot u^\varepsilon_x}_{\substack{\geq 0 \\ (\Phi \text{ convex})}}$$

$$\Big\downarrow \begin{array}{l} \text{multiply by } \begin{cases} v \in C_0^\infty(\mathbb{R}\times(0,\infty)) \\ v \geq 0 \end{cases} \\ \text{integrate} \end{array}$$

- $$\int_0^\infty \int_{\mathbb{R}} \Phi(u^\varepsilon)_t\, v + \Psi(u^\varepsilon)_x\, v \; dx\, dt$$

$$\overset{\substack{\text{integrate} \\ \text{by parts}}}{=} \underbrace{\int_{\mathbb{R}} \big[\Phi(u^\varepsilon) v\big]_0^\infty dx + \int_0^\infty \big[\Psi(u^\varepsilon) v\big]_{-\infty}^\infty dx}_{= 0,\ v \in C_0^\infty(\mathbb{R}\times(0,\infty))} - \int_0^\infty \int_{\mathbb{R}} \big[\Phi(u^\varepsilon) v_t + \Psi(u^\varepsilon) v_x\big] \, dx\, dt$$

$$\Rightarrow \int_0^\infty \int_{\mathbb{R}} \big[\Phi(u^\varepsilon) v_t + \Psi(u^\varepsilon) v_x\big] dx\, dt \overset{\substack{\Phi \text{ convex} \\ v \geq 0}}{\geq} - \int_0^\infty \int_{\mathbb{R}} \varepsilon\, \Phi(u^\varepsilon)_{xx}\, v \, dx\, dt$$

$$\overset{\substack{2\times \text{ integrate} \\ \text{by parts}}}{=} - \int_0^\infty \int_{\mathbb{R}} \varepsilon\, \Phi(u^\varepsilon)\, v_{xx}\, dx\, dt$$

$$\Big\downarrow \begin{array}{l} \varepsilon \to 0 \\ u^\varepsilon \to u \end{array}$$

$$\int_0^\infty \int_{\mathbb{R}} \left[\Phi(u)\, v_t + \Psi(u)\, v_x\right] dx\, dt \geq 0$$

$\Rightarrow \quad u = \lim_{\varepsilon \to 0} u^\varepsilon$ <u>satisfies the entropy condition</u>

2) choose $v \in C_0^\infty(\mathbb{R} \times [0,\infty); \mathbb{R}^m)$:

$$\int_0^\infty \int_{\mathbb{R}} \left[u^\varepsilon_t + f(u^\varepsilon)_x - \varepsilon u^\varepsilon_{xx}\right] \cdot v \, dx\, dt = 0$$

$\downarrow$ integrate by parts

$$-\int_0^\infty \int_{\mathbb{R}} u^\varepsilon \cdot v_t \, dx\, dt + \int_{\mathbb{R}} \underbrace{\left[u^\varepsilon \cdot v\right]_0^\infty dx}_{= -\int_{\mathbb{R}} u^\varepsilon(x,0) \cdot v \, dx \;\; (u^\varepsilon \text{ falls off } |x| \to \infty)}$$

$$-\int_0^\infty \int_{\mathbb{R}} f(u^\varepsilon)\, v_x \, dx\, dt - \int_0^\infty \int_{\mathbb{R}} \varepsilon u^\varepsilon v_{xx} \, dx\, dt = 0$$

$$\Leftrightarrow \int_0^\infty \int_{\mathbb{R}} \left[u^\varepsilon \cdot v_t + f(u^\varepsilon) \cdot v_x + \varepsilon u^\varepsilon \cdot v_{xx}\right] dx\, dt + \int_{\mathbb{R}} u^\varepsilon_0 \cdot v \, dx = 0$$

$$\Big\downarrow \begin{array}{l} \varepsilon \to 0 \\ u^\varepsilon \to u \end{array}$$

$$\int_0^\infty \int_{\mathbb{R}} \left[u \cdot v_t + f(u) \cdot v_x \right] dx\, dt + \int_{\mathbb{R}} u_0 \cdot v \, dx = 0$$

$\Rightarrow$ u is weak solution.

$\square$

<u>Theorem (Uniqueness of entropy solution for scalar conservation laws):</u>

There exists – up to a measure zero – at most one entropy solution for the scalar conservation law

$$\begin{aligned} u_t + f(u)_x &= 0 \qquad \text{on } \mathbb{R} \times (0, \infty) \\ u &= u_0(x) \quad \text{on } \mathbb{R} \times \{t = 0\} \end{aligned}$$

Proof: Evans Sec. 11.4.3, p. 652

<u>Remarks:</u> 1) <u>Well-posedness for weak soln.</u>

Note that the previous theorem restores well-posedness, given the entropy selection/admissibility criterion

2) <u>Uniqueness:</u>

- Note entropy solution is a weak solution that satisfies the entropy inequalities for <u>any</u> entropy pair Φ, Ψ of the CL.

- 1D scalar equations:
  There exists at most one weak solution that satisfies the entropy inequalities for ALL entropy pairs of the CL.
  Note that ONE entropy pair may be sufficient to rule out a given weak solution as entropy solution.

- 1D systems of CLs: similar uniqueness theorem (but more involved to proof)

- multi-D system: uniqueness still an open problem

3) Existence:

- 1D scalar CL ($m=1$): any Φ convex

  $\rightsquigarrow$ obtain corresponding flux: $\Psi(z) \equiv \int_{z_0}^{z} \Phi'(y) f'(y) dy$

  $z \in \mathbb{R}$

- $m=2$: find Φ, Ψ, with Φ convex, and

$$(*) \quad \underbrace{(\Phi_{z_1}, \Phi_{z_2})}_{D\Phi} Df(z) = \underbrace{\begin{pmatrix} \Psi_{z_1} \\ \Psi_{z_2} \end{pmatrix}}_{D\Psi}, \quad z \in \mathbb{R}^2$$

  $m > 2$: $(*)$ over-determined (2 unknowns Φ, Ψ but $m > 2$ equations)

  $\longrightarrow$ no solution in general

- hydrodynamics: thanks to 2nd law of thermodynamics, one can write down an entropy pair using the thermodynamic entropy (modulo a minus sign)

  $\Phi(u) \equiv -S$

  $\Psi(u)$: -entropy flux

  Note: mathematical entropy can only decrease:

$$\Phi(u)_t + \Psi(u)_x \leq 0$$

$\rightsquigarrow$ physical entropy evolves according to its flux and can never decrease, but increase across discontinuities ($\to$ shocks)

$\rightsquigarrow$ entropy inequalities serve as additional constraint the conservation law "does not know about" and that select the thermodynamically / physically admissible solution to the CL

4) Physical interpretation

- entropy pair $\to$ see 3)
- viscosity limit & entropy solution

Note that the entropy solution for the 1D Euler equs is the unique solution to the 1D Navier-Stokes equs in the limit viscosity $\varepsilon \to 0$

Navier-Stokes equs (incompressible fluid)

$$u_t + f(u)_x = -\varepsilon u_{xx}$$

Euler equs

$$u_t + f(u)_x = 0$$

$$\underset{\substack{\text{classical solution}\\ \text{(unique)}}}{u^\varepsilon} \xrightarrow{\varepsilon \to 0} \underset{\substack{\text{entropy solution}\\ \text{(unique)}}}{u}$$

4.4 The Riemann problem & Lax entropy condition

In this section we shall discuss the <u>Riemann problem</u> for hyperbolic systems of conservation laws, which is the IVP:

(Riemann 1860)

$$u_t + f(u)_x = 0$$

$$u(x,0) = u_0(x) = \begin{cases} u_l \,, & x<0 \\ u_r \,, & x>0 \end{cases}$$

with u_l, u_r constants. The solution of this IVP plays a central role in constructing numerical schemes to find weak solutions of systems of CLs.

4.4.1 First examples

Consider simple ~~example~~ example (inviscid Burgers equ.)

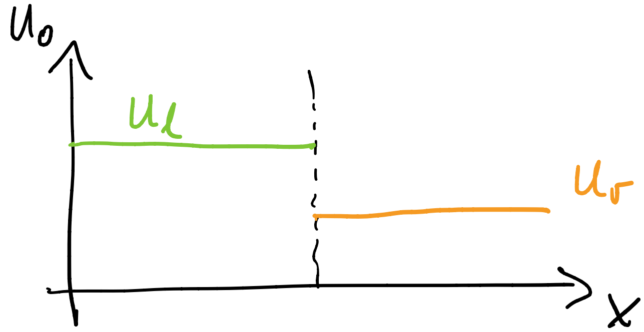
$$u_t + f(u)_x = 0 \,, \quad f(u) = \frac{u^2}{2} \quad \text{convex}$$

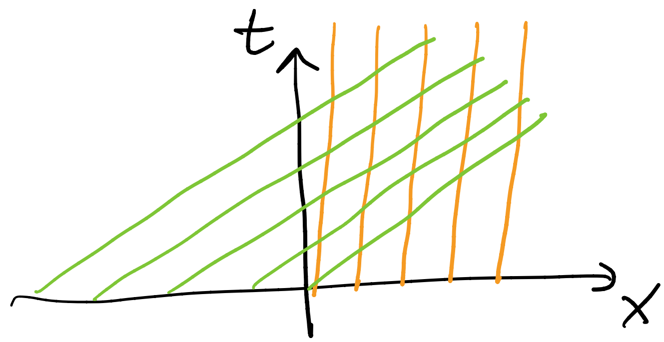
$$u(x,0) = u_0(x) = \begin{cases} u_l \,, & x<0 \\ u_r \,, & x>0 \end{cases}$$

$\rightsquigarrow$ const. characteristic speeds: $\lambda'(u) = f''(u) = 1 > 0$

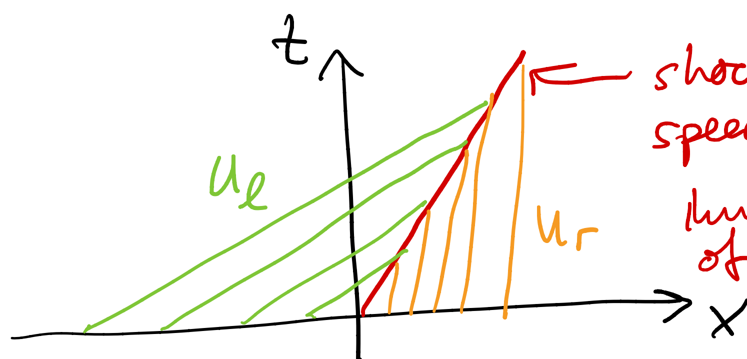
① <u>assume $u_\ell > u_r$:</u> "compressive data"

$$\lambda_\ell = \lambda(u_\ell) = u_\ell > u_r = \lambda(u_r) = \lambda_r$$





characteristics



shock Σ with speed $\sigma'(t) = s$

immediate crossing of characteristics!

RH jump conditions:

$$s = \sigma'(t) = \frac{f(u_\ell) - f(u_r)}{u_\ell - u_r} = \frac{1}{2}\,\frac{u_\ell^2 - u_r^2}{u_\ell - u_r}$$

$$= \frac{1}{2}(u_\ell + u_r)$$

weak solution:

$$u(x,t) = \begin{cases} u_\ell, & x - st < 0 \\ u_r, & x - st > 0 \end{cases}$$

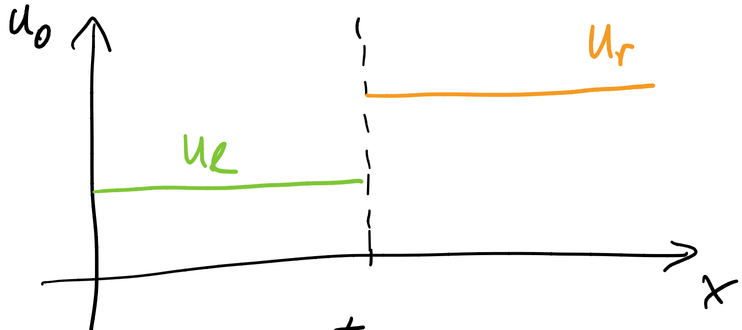
$$\boxed{\lambda(u_\ell) > s > \lambda(u_r)}$$

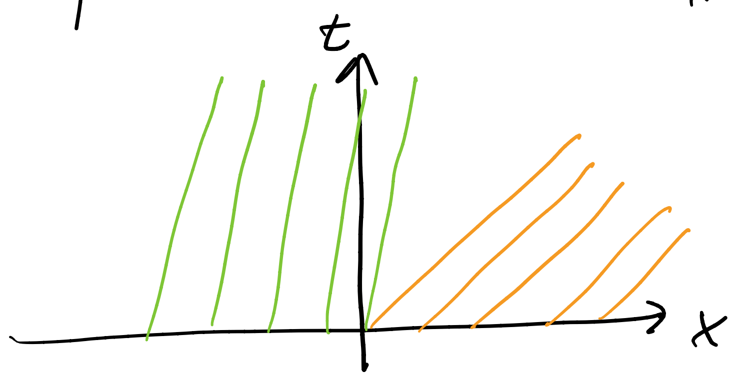
Lax entropy condition

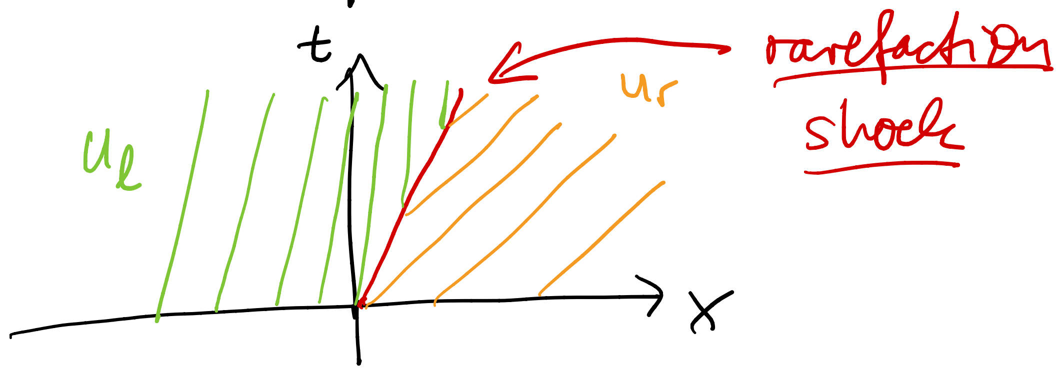
(→ will see satisfies entropy condition)

② assume $u_\ell < u_r$: "expansive data"

$$\lambda_\ell = \lambda(u_\ell) \qquad \lambda_r = \lambda(u_r)$$







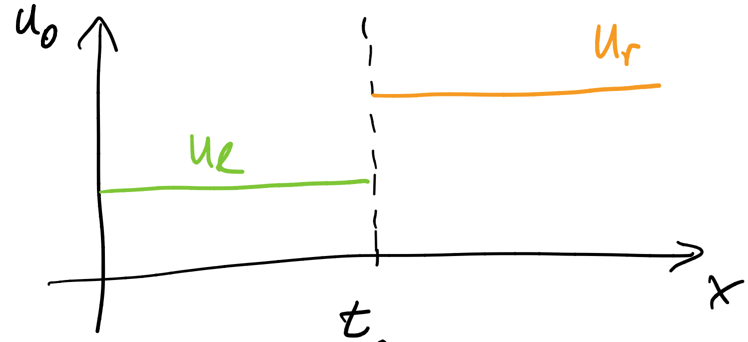
Similar to above, one weak solution is:

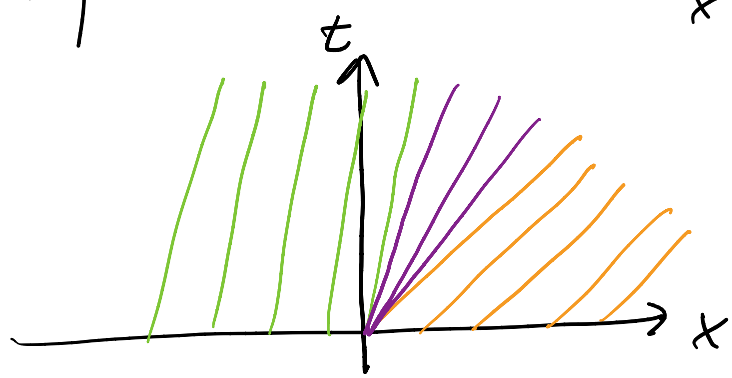
$$s = \frac{1}{2}(u_\ell + u_r), \quad u(x,t) = \begin{cases} u_\ell, & x - st < 0 \\ u_r, & x - st > 0 \end{cases}$$

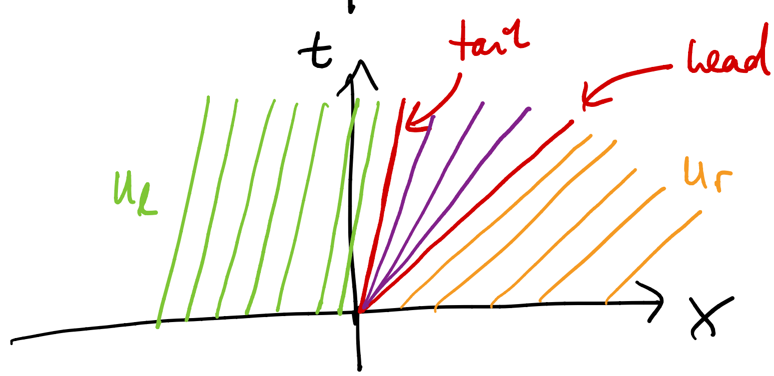
But: $\boxed{\lambda_l < s < \lambda_r}$ entropy-violating shock

discontinuity has not arisen from compression, characteristics diverge from the discontinuity

Another possibility:







all $\gamma(t)$ of wave emanate from same point

$$\begin{cases} u(x,t) = u_l, & \frac{x}{t} \le \lambda_l = u_l \\ u(x,t) = \frac{x}{t}, & \lambda_l < \frac{x}{t} < \lambda_r \\ u(x,t) = u_r, & \frac{x}{t} \ge \lambda_r = u_r \end{cases}$$

centered rarefaction wave

larger values of $u_0(x)$ propagate faster than smaller values $\to$ wave spreads and flattens

$\leadsto$ "rarefaction" (non-linear phenomenon such as shocks)

③ <u>Complete solution:</u>

$$u_\ell > u_r : \; u(x,t) = \begin{cases} u_\ell, & x < st \\ u_r, & x > st \end{cases}, \quad s = \frac{1}{2}(u_\ell + u_r)$$

$$u_r \leq u_\ell : \; u(x,t) = \begin{cases} u_\ell, & \frac{x}{t} \leq u_\ell \\ \frac{x}{t}, & u_\ell < \frac{x}{t} < u_r \\ u_r, & \frac{x}{t} \geq u_r \end{cases}$$

Now: general Riemann problem for systems of CLs

(Riemann 1860)

$$u_t + f(u)_x = 0$$

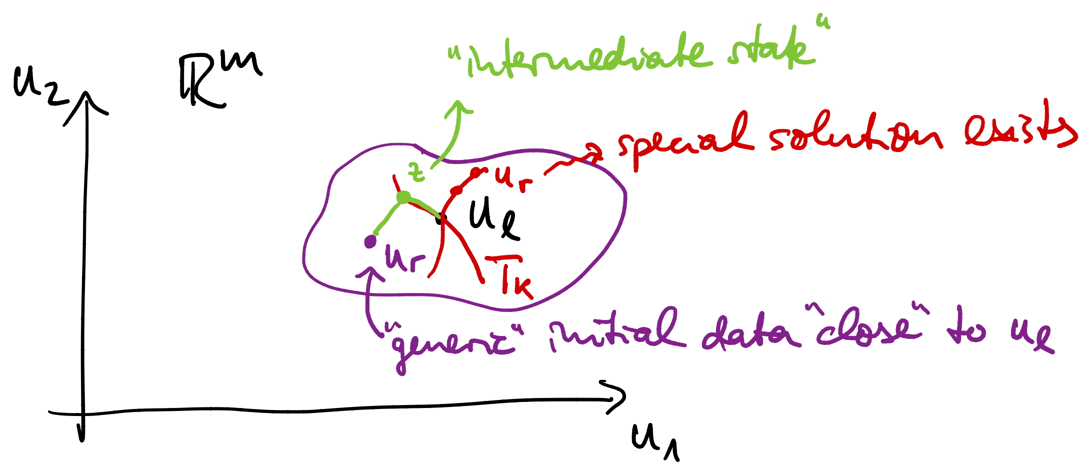
$$u(x,0) = u_0(x) = \begin{cases} u_l, & x<0 \\ u_r, & x>0 \end{cases}$$

with u_l, u_r constants.

Idea: multi-step approach to the problem

→ find special solutions first (rarefaction waves, shocks, contact discontinuities)

→ represent general solution as a combination of special solutions



4.4.2 Riemann invariants & characteristic fields

First we'll need to define characteristics for systems of CLs:

Def (k-characteristic): Let $u = u(x,t) \in \mathbb{R}^m$ be a smooth solution of the hyperbolic system

$$u_t + f(u)_x = 0 \, , \quad x \in \mathbb{R}, \; t \in [0,T]$$

and let $\lambda_1(u) \leq \lambda_2(u) \leq \dots \leq \lambda_m(u)$ be the eigenvalues of $Df(u)$. A curve $\Gamma_k = (\gamma_k(t), t) \in C^1([0,\tau])$ with

$$\gamma'(t) = \lambda_k(u(\gamma(t),t)), \quad 0 \leq t \leq \tau$$

is called a k-characteristic.

In general $u(x,t)$ is not constant along a k-characteristic ($\rightarrow$ k-characteristics are not straight lines) but there are other suitable functions $w(u)$:

Lemma & Def (Riemann invariant):

Let $u = u(x,t)$ be a smooth solution of the system $u_t + f(u)_x = 0$ and let $\Gamma_k = (\gamma_k(t), t)$ be a k-characteristic w.r.t. λ_k.

Let $w \in C^1(\mathbb{R}^m, \mathbb{R})$ with

$$(*) \qquad Df(u)^T \cdot \nabla w = \lambda_k \nabla w,$$

then $w(u)$ is <u>constant along Γ_k</u>

$$\frac{d}{dt} w\big(u(\gamma_k(t), t)\big) = 0.$$

A function w with the property $Df(v)^T \cdot \nabla w(v) = \lambda_k \nabla w(v)$ for all $v \in \mathbb{R}^m$ is called a <u>Riemann invariant.</u>

<u>Proof:</u>

$$\frac{d}{dt} w\big(u(\gamma(t), t)\big) = \sum_{l=1}^{m} \Big[\partial_x u_l \, \gamma'(t) + \underbrace{\partial_t u_l}_{= -[f(u)_x]_l}\Big] \partial_l w$$

γ k-charact.

$$= \sum_{l=1}^{m} \Big[\lambda_k \partial_x u_l - \big(Df(u) \partial_x u\big)_l\Big] \partial_l w$$

$$= \sum_{j=1}^{m} \lambda_k \partial_x u_j \, \partial_j w - \sum_{l=1}^{m} \Big[\sum_{j=1}^{m} \partial_j f_l(u) \partial_x u_j\Big] \partial_l w$$

$$= \sum_{j=1}^{m} \left[\lambda_k \partial_j w - \sum_{\ell=1}^{m} \partial_j f_\ell(u) \partial_\ell w \right] \partial_x u_j$$

$$= \underbrace{\left[\lambda_k \nabla w - Df(u)^T \cdot \nabla w \right]}_{=0} \cdot \partial_x u$$

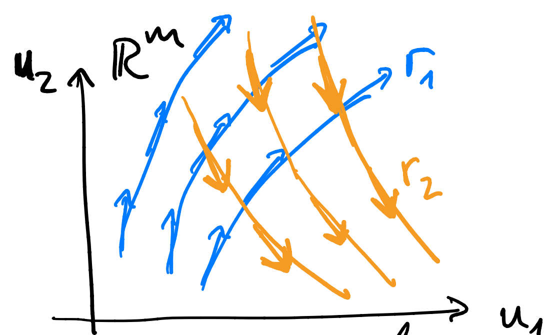
$$= 0$$

$\square$

Def & Lemma (k-Riemann invariant):

Consider a system $u_t + f(u)_x = 0$, $u = u(x,t) \in \mathbb{R}^m$ and let $r_k(u)$ denote the eigenvectors of $Df(u)$ corresponding to the eigenvalues $\lambda_k(u)$.

Consider an integral curve $(v \in C^1(\mathbb{R}; \mathbb{R}^m) : \xi \mapsto v(\xi))$ of r_k in state space:



$$v'(\xi) = \mu(\xi)\, r_k(v(\xi))$$

for some $\mu \in C^0(\mathbb{R}; \mathbb{R})$ that depends on the parametrization ξ of the curve v.

A function $w \in C^1(\mathbb{R}^m; \mathbb{R})$ with

$$\nabla w(v(\xi)) \cdot r_k(v(\xi)) = 0$$

for an integral curve v of r_k is <u>constant along the integral curve</u>,

$$\frac{d}{dt} w(v(\xi)) = 0.$$

Such a w is called a <u>k-Riemann invariant</u>.

<u>Proof:</u>
$$\frac{d}{dt} w(v(\xi)) = \nabla w(v(\xi)) \cdot v'(\xi)$$
$$= \nabla w(v(\xi)) \cdot \mu(\xi)\, r_k(v(\xi))$$
$$= 0$$

□

<u>Remarks:</u> 1) Existence

One can show (e.g. Kroener Theorem 4.1.12): There are $m-1$ k-Riemann invariants $w_1, \dots, w_{m-1} \in C^1(\mathbb{R}^m, \mathbb{R})$ such that their gradients $\nabla w_1, \dots, \nabla w_{m-1}$ are linearly independent.

2) Riemann invariant $\Rightarrow$ k-Riemann invariant

Let w be a Riemann invariant with

respect to $\lambda_j(u)$. Then w is a k-Riemann invariant for all $k \neq j$:

Let $v \in \mathbb{R}^m$, then: $\lambda_j(v) \cdot r_k(v)^T \nabla w(v)$

$$= r_k(v)^T \cdot Df(v)^T \cdot \nabla w(v)$$

dropping v

$$= \left[Df \cdot r_k\right]^T \cdot \nabla w$$

$$= \lambda_k \, r_k^T \cdot \nabla w$$

$$\Rightarrow \quad (\lambda_j - \lambda_k) \, \nabla w \cdot r_k = 0$$

$$j \neq k: \quad \nabla w \cdot r_k = 0$$

Now: more on behaviour of characteristics

$\rightsquigarrow$ note that the change of λ_k along an integral curve $v(\xi)$ of r_k can be computed as:

$$\frac{d}{d\xi} \lambda_k(v(\xi)) = \nabla\lambda_k(v(\xi)) \cdot v'(\xi)$$
$$= \mu(\xi)\, \nabla\lambda_k(v(\xi)) \cdot r_k(v(\xi))$$

Def: Let $D \subseteq \mathbb{R}^m$. A k-characteristic is called

- linearly degenerate in D if and only if $\nabla\lambda_k \cdot r_k = 0$ in D
- genuinely non-linear in D if and only if $\nabla\lambda_k \cdot r_k \neq 0$ in D

  (λ_k varies monotonically along $v(\xi)$)

A system $u_t + f(u)_x = 0$, $u(x,t) \in \mathbb{R}^m$ is called linearly degenerate or genuinely non-linear in D if the property holds for all k-characteristics in D.

<u>Remarks:</u> 1) for genuinely non-linear k-characteristics we can always use some scaled r_k, such that $\nabla \lambda_k \cdot r_k = 1$.

2) For scalar equns, $\lambda^1(v) = f'(v)$ and $r^1(v) = 1$

$\rightsquigarrow$ gen. non-linear $\Leftrightarrow$ $f''(v) \neq 0$

(convexity requirement cf. Sec. 1.3.1)

3) For constant coeff. linear hyperbolic systems $u_t + A u_x = 0$

$\nabla \lambda_k(v) \equiv 0$ everywhere by construction

$\Rightarrow$ linearly degenerate

For non-linear linearly degenerate systems λ_k constant along a given integral curve, but takes different values along different integral curves

4.4.3 Special solutions: rarefaction waves

Motivation: are there <u>continuous</u> solutions to the Riemann problem?

<u>Def (k-rarefaction waves)</u>: Let $D \subset \mathbb{R} \times (0,\infty)$ and $u \in C^1(D;\mathbb{R}^m)$ be a solution of the strictly hyperbolic system $\partial_t u + f(u)_x = 0$ with $\lambda_k(u)$ being the eigenvalues of $Df(u)$.

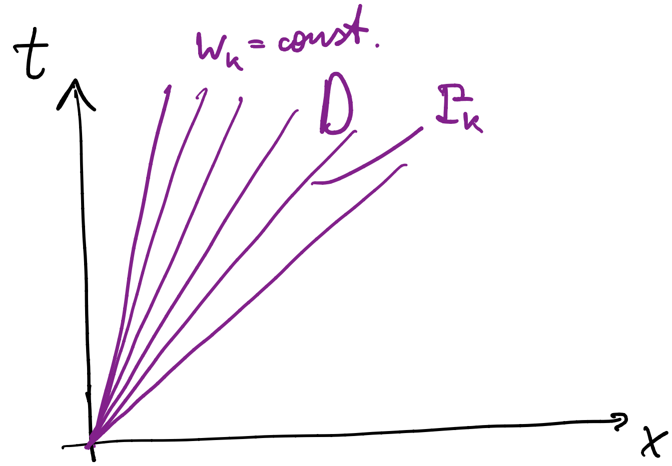
If all k-Riemann invariants w_j are constant in D,

$$\frac{d}{dt} w_j(u(\gamma(t),t)) = 0 \quad \forall\, j = 1,\dots,m-1$$

with $\gamma'(t) = \lambda_k$ for $0 \le t \le \tau$ and $(\gamma(t),t) \in D$, the solution u is called a <u>k-rarefaction wave</u>.

<u>Remark</u>: k-rarefaction waves have the same property as solutions for scalar conservation laws (u const. along characteristics)

Theorem: Let the system $u_t + f(u)_x = 0$ be strictly hyperbolic and let u be a k-rarefaction wave in $D \subseteq \mathbb{R} \times (0,\infty)$. Then the k-characteristics are straight lines in D along which u is constant.



Proof: Consider k-characteristic γ of corresp eigenvalue λ_k with left eigenvector l_k, $l_k Df(u) = \lambda_k l_k$. Set $n \equiv \begin{pmatrix} \lambda_k \\ 1 \end{pmatrix}$, then

$$l_k \frac{d}{dt} u(\gamma(t), t) = l_k \underbrace{(\underbrace{\gamma'(t)}_{=\lambda_k} u_x + u_t)}_{n \cdot \nabla u = \partial_n u}$$

$$= l_k (\underbrace{u_t + Df(u) u_x}_{=0}) = 0$$

$$\Rightarrow \partial_n u = 0$$

Also: $0 = \frac{d}{dt} w_j(u(\gamma(t),t)) = \nabla w_j (\underbrace{u_x \gamma' + u_t}_{n \cdot \nabla u = \partial_n u})$

$= \nabla w_j \cdot \partial_n u$

$$\Rightarrow \underbrace{\begin{pmatrix} l_k \\ \nabla w_1 \\ \vdots \\ \nabla w_{m-1} \end{pmatrix}}_{\text{linearly independent (Sec. 4.4.2)}} \partial_n u = 0 \qquad (*)$$

$$\alpha_0 l_k + \sum_{j=1}^{m-1} \alpha_j \nabla w_j \overset{!}{=} 0 \qquad | \cdot r_k$$

$\nabla w_j \cdot r_k \overset{\text{Def}}{=} 0 \downarrow$

$$\alpha_0 l_k \cdot r_k = 0$$

Observe: $\lambda_k l_k r_\ell = l_k Df r_\ell = l_k \lambda_\ell r_\ell$

$$\Rightarrow (\lambda_k - \lambda_\ell) l_k \cdot r_\ell = 0$$

$$\Rightarrow l_k \cdot r_\ell = 0 \quad k \neq \ell$$

$$\Rightarrow l_k r_k \neq 0$$

$\Rightarrow \{l_k, \nabla w_j\}$ linearly independent

$$\overset{(*)}{\Rightarrow} \partial_n u = \frac{d}{dt} u(\gamma(t),t) = 0$$

$$\Rightarrow \gamma'(t) = \lambda_k(\underbrace{u(\gamma(t),t)}_{= \text{const.}}) = \text{const.}$$

$\square$

Theorem (existence and structure of rarefaction waves):

Consider the strictly hyperbolic system

$$u_t + f(u)_x = 0 \quad \text{in } \mathbb{R}\times(0,\infty)$$

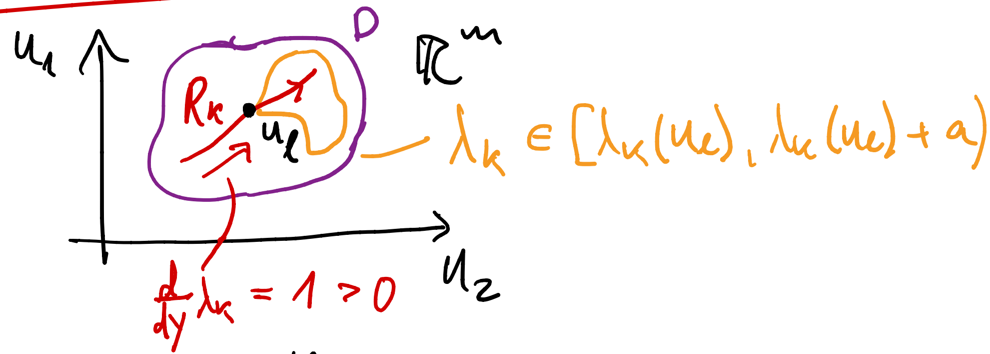
with $Df(u) \in C^1(\mathbb{R}^m, \mathbb{R}^{m\times m})$ and let λ_k be a genuinely non-linear characteristic field on $D \subset \mathbb{R}^m$ with $\nabla\lambda_k(v)\cdot r_k(v) > 0 \ (\equiv 1) \ \forall v \in D$ and let $u_\ell \in D$. Then there exists $a>0$ and a function $v \in C^1([\lambda_k(u_\ell), \lambda(u_\ell)+a]; \mathbb{R}^m)$ such that $v(\lambda_k(u_\ell)) = u_\ell$ and such that

the Riemann problem with initial values

$$u(x,0) = u_0(x) = \begin{cases} u_\ell, & x<0 \\ \underbrace{v(y)}_{\text{"}u_r\text{"}}, & x>0 \end{cases}$$

for all $y \in [\lambda_k(u_\ell), \lambda_k(u_\ell)+a)$ can be solved by a continuous weak solution, which we call a k-rarefaction wave. One has $\frac{d}{dy}\lambda_k(v(y)) > 0$, i.e. $\lambda_k(u_\ell) < \lambda_k(u_r)$, and

$$(*) \quad \boxed{u(x,t) = \begin{cases} u_\ell, & \frac{x}{t} < \lambda_k(u_\ell) \\ v(x/t), & \lambda_k(u_\ell) < \frac{x}{t} < y \\ v(y), & y < \frac{x}{t}. \end{cases}} \quad \begin{matrix} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III} \end{matrix}$$



Proof: $\exists\, a > 0$ sufficiently small such that the IVP

$$\begin{cases} v'(y) = r_k(v(y)), & \lambda_k(u_\ell) < y < \lambda_k(u_\ell) + a \\ v(\lambda_k(u_\ell)) = u_\ell \end{cases}$$

has a unique solution (ODE!).

Also: $$\frac{d}{dy}\lambda_k(v(y)) = \nabla\lambda_k(v(y))\, v'(y) = \nabla\lambda_k \cdot r_k \overset{\text{assumption}}{=} 1$$

$$\Rightarrow \text{ since } \lambda_k(\underbrace{v(\lambda_k(u_\ell))}_{u_\ell}) = \lambda_k(u_\ell)$$

slope: +1

$$(**) \quad \lambda_k(v(y)) = y \quad \text{all along } \lambda_k(u_\ell) < y < \lambda_k(u_\ell) + a$$

For ansatz $(*)$ we have $\partial_t u + Df(u)\, u_x = 0$ in the distributional sense:

In I and III $u = \text{const.} \Rightarrow u_t + Df(u)u_x = 0$ ✓

In II: $v'(x/t)\left(-\frac{x}{t^2}\right) + Df(v(x/t))\, v'(x/t)\frac{1}{t}$

$$= r_k(v(x/t))\left(-\frac{x}{t^2}\right) + \frac{1}{t}\underbrace{\lambda_k(v(x/t))}_{\overset{(**)}{=}\frac{x}{t}} r_k(v(x/t))$$

$= 0$ ✓

Let w be a k-Riemann invariant in D.

Then: $\frac{d}{dt} w(\underbrace{u(\gamma(t),t)}_{\substack{=\text{const}\\ \text{in I, III}}}) \overset{\text{I, III}}{=} 0$

$$\frac{d}{dt} w(u(\gamma(t),t)) = \nabla w(\underbrace{u_x\gamma' + u_t}_{\substack{=0 \text{ in II}\\ \text{see above}}}) = 0$$

where γ is the corresponding k-characteristic with $\gamma'(t) = \lambda_k(u(\gamma(t),t))$.

$\Rightarrow$ any w is constant in D

$\Rightarrow$ u is rarefaction wave. □

<u>Remarks:</u> 1) This rarefaction wave solution is the uniquely determined weak solution satisfying the entropy condition for $u_\ell < u_r$.

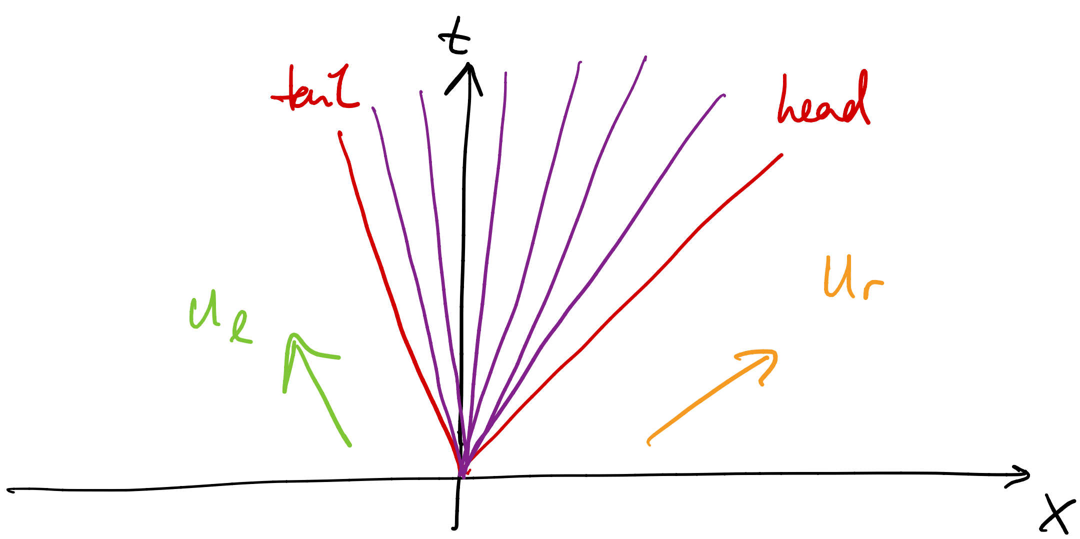
(satisfies entropy-pair jump conditions, cf. (4.1.57) in Kroener)

2) This rarefaction wave solution is called a centered rarefaction wave as all variation in the "solution flow" arises from $x = t = 0$. Note that it is a similarity solution (only depends on $\frac{x}{t}$ $\leadsto$ looks similar on all scales) and is constant along every $\frac{x}{t} =$ const. ray.

3) Note that from $\frac{d}{dy} \lambda_k(v(y)) > 0$ it follows that

$$\lambda_k(u_\ell) < \lambda_k(u_r)$$

$\leadsto$ divergence of characteristics

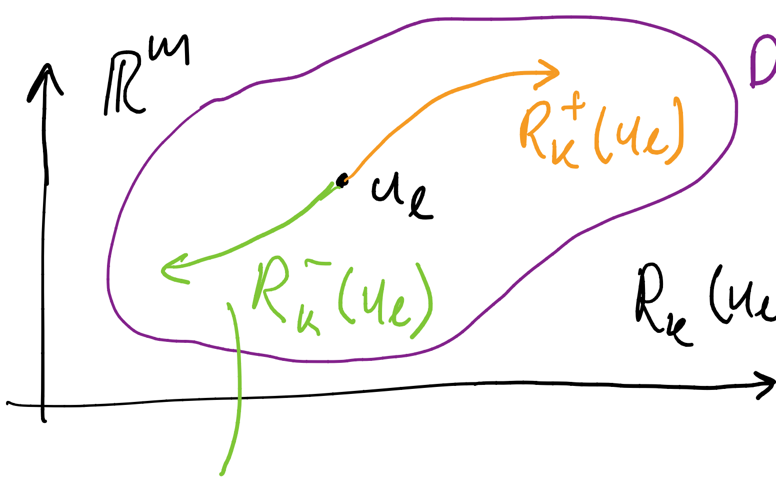


A rarefaction wave is a continuous solution to the Riemann problem that "smooths out" the initial discontinuity!

Def: Given a fixed state $u_\ell \in \mathbb{R}^m$ we define the integral curve $v'(\xi) = r_k(v(\xi))$, $v(0) = u_\ell$, $\xi \in (-a,a)$ and $a>0$ sufficiently small, considered in the previous theorem as the <u>k-rarefaction curve</u> $R_k(u_\ell)$. If $(\lambda_k(u_\ell), r_k(u_\ell))$ are genuinely non-linear, we write:

$$R_k^+(u_\ell) \equiv \{ z \in R_k(u_\ell) \mid \lambda_k(z) > \lambda_k(u_\ell) \}$$

$$R_k^-(u_\ell) \equiv \{ z \in R_k(u_\ell) \mid \lambda_k(z) < \lambda_k(u_\ell) \}$$



$R_k(u_\ell) = R_k^+(u_\ell) \cup \{u_\ell\} \cup R_k^-(u_\ell)$

compressive data $\Rightarrow$ no single rarefaction solution possible

4.4.4 Special solutions: shocks and the Lax entropy condition

Now: look for solution that "keeps the discontinuity"!

Reconsider behaviour of weak solutions across a discontinuity: Consider weak solution of strictly hyperbolic system

$$u_t + f(u)_x = 0 \quad \text{in } \mathbb{R}\times(0,\infty)$$

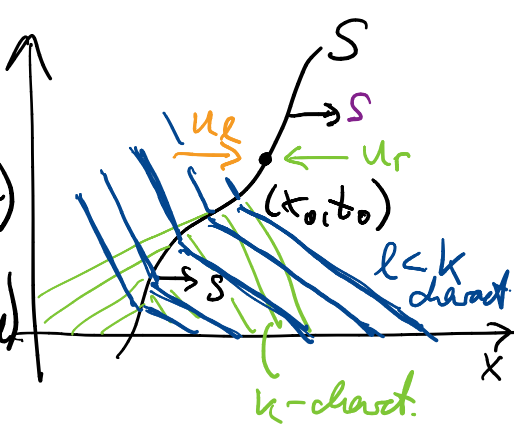
that is piecewise smooth with a jump along $S = (\sigma(t), t)$ for a curve σ with $\sigma' \equiv s$ the shock velocity. Let $(x_0, t_0) \in S$ and u_l, u_r the limit values of u from left and right at (x_0, t_0):

$$u_l = \lim_{x\nearrow x_0} u(x, t_0), \quad u_r = \lim_{x\searrow x_0} u(x, t_0).$$

Assume

$$\lambda_1(u_r) < \ldots < \lambda_k(u_r) < s < \lambda_{k+1}(u_r) < \ldots < \lambda_n(u_r)$$

$$\lambda_1(u_l) < \ldots < \lambda_j(u_l) < s < \lambda_{j+1}(u_l) < \ldots < \lambda_n(u_l)$$



where $\lambda_1(u), \dots, \lambda_m(u)$ are the eigenvalues of $Df(u)$.

$\rightsquigarrow$ k-conditions on u_r at (x_0, t_0) from initial data at $t=0$ on right side due to <u>impinging</u> ℓ-characteristics $\lambda_1, \dots, \lambda_k < s$

$m-j$ conditions on u_ℓ from initial data on left side due to impinging ℓ-characteristics $\lambda_{j+1}, \dots, \lambda_m > s$

m conditions relating u_r & u_ℓ (jump cond.)

$$s(u_r - u_\ell) = f(u_r) - f(u_\ell)$$

$\rightsquigarrow$ $2m+1$ unknowns:

$$u_\ell = \begin{pmatrix} u_{\ell 1} \\ \vdots \\ u_{\ell m} \end{pmatrix}, \quad u_r = \begin{pmatrix} u_{r1} \\ \vdots \\ u_{rm} \end{pmatrix}, \quad s$$

Necessary condition for computing the unknowns for arbitrary given initial data at $t=0$ is that

$$k + m - j + m = 2m + k - j \overset{!}{=} 2m+1$$

$$\Leftrightarrow j = k-1$$

$$\Leftrightarrow \quad \lambda_k(u_r) < s < \lambda_{k+1}(u_r)$$
$$\lambda_{k-1}(u_\ell) < s < \lambda_k(u_\ell)$$

(Lax Comm. Pure and Applied Math. 10 (1957) 537, Sec. 7)

or $(*)$ $$\boxed{\lambda_k(u_r) < s < \lambda_k(u_\ell)}$$ Lax entropy condition

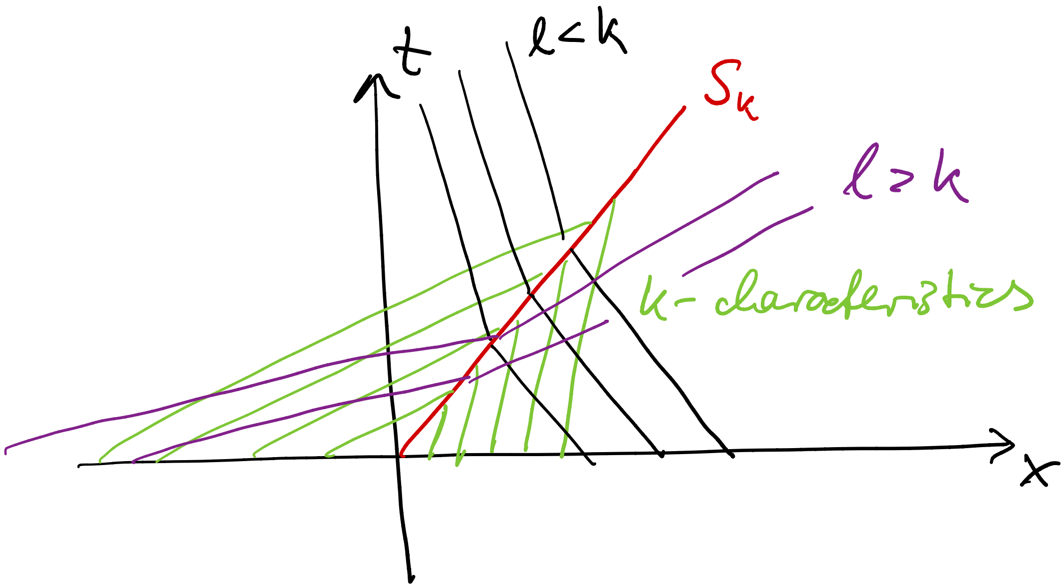
Def (Lax entropy condition, k-shock):

A discontinuous weak solution $u(x,t)$ satisfying the RH jump conditions and the Lax entropy condition (*) is called a k-shock.

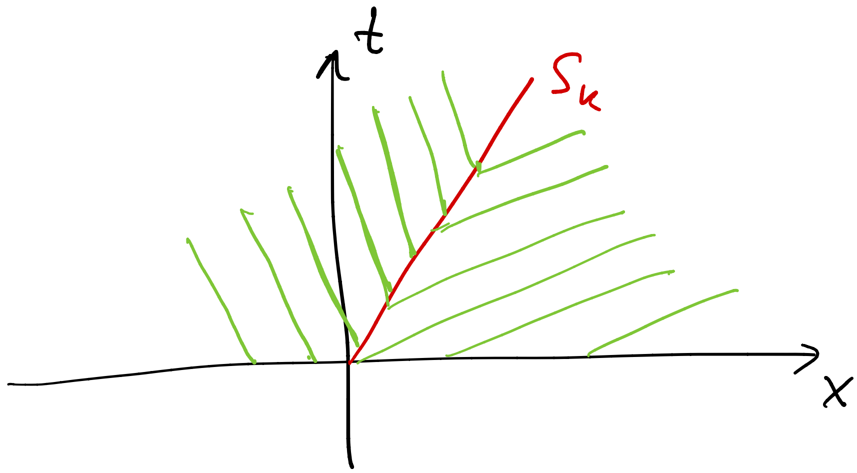
Remarks: 1) The Lax entropy condition states that the k-characteristics impinge on the discontinuity while the other characteristics cross it:

$$\lambda_\ell(u_\ell) < s \quad \& \quad \lambda_\ell(u_r) < s \quad \text{for } \ell < k$$

$$\lambda_\ell(u_\ell) > s \quad \& \quad \lambda_\ell(u_r) > s \quad \text{for } \ell > k$$



2) A <u>rarefaction shock</u> $\lambda_k(u_r) > s > \lambda_k(u_\ell)$ cannot be uniquely determined by above conditions due to <u>expansive</u> nature, need additional conditions on the discontinuity



$\leadsto$ Lax entropy condition excludes rarefaction shocks

<u>Existence of k-shock solutions:</u>

<u>**Def:**</u> Given state $u_\ell \in \mathbb{R}^m$, define <u>shock set</u>

$$S(u_\ell) \equiv \{ z \in \mathbb{R}^m \mid f(z) - f(u_\ell) = s(z - u_\ell) \text{ for a constant } s = s(z, u_\ell) \}$$

where f is the flux of a system of conservation laws.

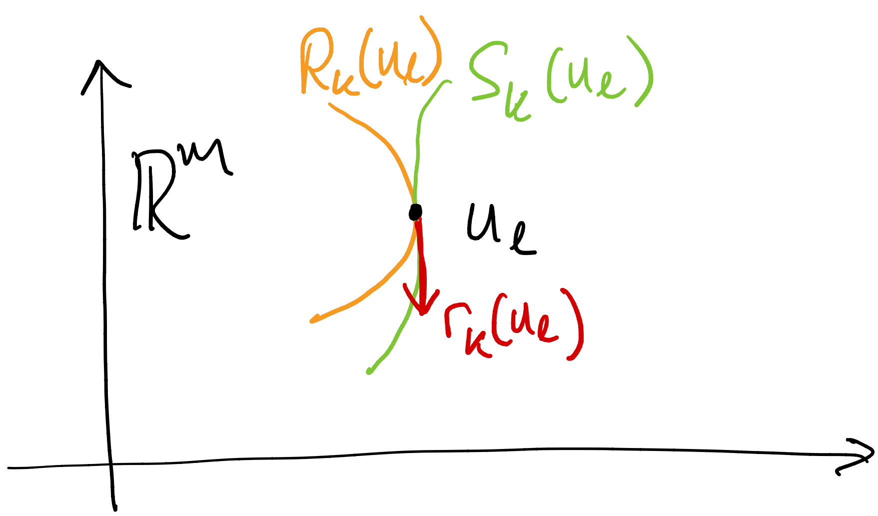
Theorem (properties of shock set): Consider the RP for a strictly hyperbolic system of CLs $u_t + f(u)_x = 0$, and let $u_\ell \in \mathbb{R}^m$ be a given left state of the RP. Then there exists a neighborhood of u_ℓ in which $S(u_\ell)$ is the union of m smooth curves $S_k(u_\ell)$ $(k = 1, \dots, m)$ with

(i) $S_k(u_\ell)$ passes through u_ℓ with tangent $r_k(u_\ell)$, i.e. $s(0) = u_\ell$, $s'(0) = r_k(u_\ell)$

(ii) $\lim\limits_{\substack{z \to u_\ell \\ z \in S_k(u_\ell)}} s(z, u_\ell) = \lambda_k(u_\ell)$

(iii) $s(z, u_\ell) = \dfrac{\lambda_k(z) + \lambda_k(u_\ell)}{2} + \mathcal{O}(|z - u_\ell|^2)$

as $z \in S_k(u_\ell) \to u_\ell$



Proof: Evans Theorem 2, Sec. 11.2.3 p627.

Idea: use strict hyperbolicity & implicit function theorem to show existence of m curves $\{s_k(t)\}_{k=1,\dots,m}$ with $s_k(0)=u_e$, $s_k'(0) = r_k(u_e)$ ($\Rightarrow$ (i)) and of a function $s: \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$, such that

$$(*) \quad f(s_k(t)) - f(u_e) = s(s_k(t), u_e)(s_k(t) - u_e)$$

$$\Big\downarrow \left.\frac{d}{dt}\right|_{t=0}$$

$$\lambda_k(u_e)\, r_k(u_e) = Df(u_e)\, s_k'(0) = \sigma(u_e, u_e)\, s_k'(0) = s(u_e, u_e)\, r_k(u_e)$$

$$\Rightarrow \text{(ii)}$$

$$\left.\frac{d^2}{dt^2}\right|_{t=0} (*) \rightsquigarrow \quad 2s(s_k(t), u_e) - \lambda_k(u_e) - \lambda_k(s_k(t)) = \mathcal{O}(t^2) \text{ as } t \to 0$$

$$\Rightarrow \text{(iii)}$$

$\square$

Remarks: 1) $R_k(u_\ell)$ vs. $S_k(u_\ell)$:

Note the curves $R_k(u_\ell)$ and $S_k(u_\ell)$ are different, but share the same tangent vector at u_ℓ and agree to first order in $|z - u_\ell|$ at u_ℓ.

2) Structure of k-shocks:

Let (λ_k, r_k) be genuinly non-linear pair in $D \subseteq \mathbb{R}^m$ with $u_r \in S_k(u_\ell)$, then

$$\boxed{u(x,t) = \begin{cases} u_\ell & x < st \\ u_r & x > st \end{cases}}$$

is a weak solution to the RP with $s = s(u_r, u_\ell)$.

Note: u satisfies $u_t + f(u)_x = 0$ since piecewise smooth and $u_t = u_x = 0$; u satisfies RH jump condition by construction through choice of u_r (and s).

$$\frac{d}{dt}\Big|_{t=0} \lambda_k(S_k(t)) = D\lambda_k(S_k(t))\, S_k'(t)\Big|_{t=0}$$

$$= \nabla\lambda_k(u_\ell)\, r_k(u_\ell) \neq 0$$

(genuinly non-linear)

$\Rightarrow \lambda_k(S_k(t))$ cannot be constant

$\overset{\text{(iii) theorem}}{\Rightarrow}$ $\lambda_k(u_r) < s(u_r, u_\ell) < \lambda_k(u_\ell)$

or $\lambda_k(u_\ell) < s(u_r, u_\ell) < \lambda_k(u_r)$ Lax entropy violating

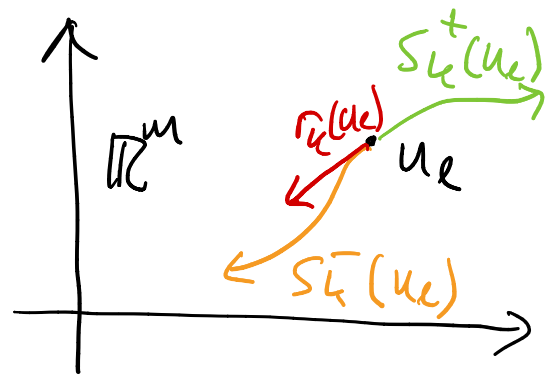
$\leadsto$ can only admit u_r satisfying first relation!

<u>Def:</u> Let $u_\ell \in D \subseteq \mathbb{R}^m$ and (λ_k, r_k) be a genuinely nonlinear pair (in D) of a strictly hyperbolic system of CLs with shock set $S(u_\ell)$. Then define:

$$S_k^+(u_\ell) \equiv \{ z \in S_k(u_\ell) \mid \lambda_k(u_\ell) < \sigma(z, u_\ell) < \lambda_k(z) \}$$

$$S_k^-(u_\ell) \equiv \{ z \in S_k(u_\ell) \mid \lambda_k(z) < \sigma(z, u_\ell) < \lambda_k(u_\ell) \}$$

<u>Remarks:</u>



- $S_k(u_\ell) = S_k^+(u_\ell) \cup \{u_\ell\} \cup S^-(u_\ell)$
- (u_r, u_ℓ) admissible if and only if $u_r \in S_k^-(u_\ell)$

Theorem (equivalence of entropy conditions):

Consider the hyperbolic and genuinely non-linear system $u_t + f(u)_x = 0$ in $\mathbb{R} \times (0,\infty)$ and assume $u = u(x,t)$ is a solution with a weak shock, i.e. with sufficiently small jump $u_l - u_r$. Then the entropy condition along the discontinuity is equivalent to the Lax entropy condition.

Proof: see Kroener Theorem 4.1.25 p. 298.