Computational Fluid Dynamics

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Contents

Preface and recommended literature

These lecture notes have been prepared for a new graduate course on compuational fluid dynamics in the Guelph–Waterloo Institute for Physics (GWIP), the joint graduate school of the Universities of Guelph and Waterloo, as well as Perimeter Institute for Theoretical Physics (Fall 2020). Many textbooks helped me compose these lectures and lecture notes. Some suggested textbooks include:

Numerical Methods

- Toro (2009): *Riemann Solvers and Numerical Methods for Fluid Dynamics* (Springer, 3rd edition, 2009)
- Leveque (2002): *Finite Volume Methods for Hyperbolic Problems* (Cambridge Univ. Press, Cambridge Texts in Applied Mathematics, 2002)

Mathematically inclined literature:

- Kröner (1997): *Numerical Schemes for Conservation Laws* (Wiley, 1997)
- Evans (2010): *Partial Differential Equations* (Graduate Studies in Mathematics, American Mathematical Society, 2nd edition, 2010)

Other recommended literature

- Bodenheimer et al. (2006): *Numerical Methods in Astrophysics* (Taylor & Francis, 2007)
- Anile (1989): *Relativistic fluids and magneto-fluids* (Cambridge Univ. Press, 1990)
- Rieutord (2015): *Fluid Dynamics* (Springer, 2015)
- LeVeque (2007): *Finite Difference Methods for Ordinary and Partial Differential Equations* (SIAM, 2007)
- Morton & Mayers (2005): *Numerical Solution of Partial Differential Equations* (Cambridge Univ. Press, 2nd edition, 2005)

Chapter 1

Basic Notions of Partial Differential Equations

1.1 PDEs of 2nd order

Recommended reading: Evans (2010) Chap. 1, Chap. 6.1, 7.1.1, 7.2.1, Strauss (2007) Chap. 1.6, most introductory books on PDEs.

1.2 PDEs of 1st order

Recommended reading: Evans (2010) Chap. 7.3.1, 11.1, Toro (2009) Chap. 2.

1.3 Some properties of 1st order hyperbolic systems

Recommended reading: Toro (2009) Chap. 2, Leveque (2002) Chap. 2.9–2.11, 3.1–3.6, 11.2, 18.5.

1.3.1 Characteristics

1.3.2 Domain of dependence and range of influence

Chapter 2

Basic Equations of Computational Fluid Dynamics

Recommended reading: Bodenheimer et al. (2006) Chap. 1, Rieutord (2015) Chap. 11, Lifshitz & Pitaevskii (1981) Chap. 1, Anile (1989) Chap. 2.

2.1 Continuous media and the Boltzmann equation

More on the N-body approach: Bodenheimer et al. (2006) Chap. 3
A compilation of the largest cosmological N-body simulations accomplished to date can be found in Tab. II of Cheng et al. (2020).

2.2 From the Boltzmann equation to the Euler equations

2.3 Navier-Stokes equations

More on viscosity: Landau & Lifshitz (2004) §15, Shakura & Sunyaev (1973), Balbus & Hawley (1991).

2.4 Magnetohydrodynamics

2.5 Radiation transfer

2.6 Relativistic Hydrodynamics

Recommended reading: Gourgoulhon (2012) Chaps. 4 & 6, Gourgoulhon (2006), Baumgarte & Shapiro (2010) Chap. 5, Alcubierre (2008) Chap. 2.2 & 7.

2.7 Relativistic radiation transfer

Recommended reading: Thorne (1981), Shibata et al. (2011), Straumann (2013) Sec. 3.11.

Chapter 3

Finite Difference methods for PDEs

Recommended reading: LeVeque (2007) Chap. 1, 9, 10, Bodenheimer et al. (2006) Chap. 2, Choptuik (2006) Sec. 1., Kröner (1997) Chap. 2.4, Toro (2009) Chap. 5.1.

3.1 Basic notions of discretization

3.2 Finite difference approximations

3.2.1 Partial derivatives & differential operators

3.2.2 Sample discretizations

3.3 Consistency, stability, convergence

Recommended reading: Morton & Mayers (2005) Chap. 5.

3.4 Stability analysis and the CFL condition

3.5 Diffusion and dispersion

3.6 Error analysis and convergence

Chapter 4

Properties of Conservation Laws

Recommended reading: Evans (2010) Chap. 11.1, 11.2, 11.4, Kröner (1997) Chap. 4.1., Leveque (2002) Chap. 13, Toro (2009) Chap. 2.4.

Recommended original literature: Lax (1957)

4.1 Local existence of classical solutions

4.2 Weak solutions

4.2.1 Definition

4.2.2 Behavior near discontinuities (Jump conditions)

4.3 Entropy condition

4.4 Riemann's Problem

4.4.1 First examples

4.4.2 Riemann invariants & characteristic fields

4.4.3 Special solutions: rarefaction waves

4.4.4 Special solutions: shocks and the Lax entropy condition

4.4.5 Special solutions: contact discontinuities

4.4.6 General solution to Riemann's problem

Chapter 5

Riemann Problem for the Euler Equations

Recommended reading: Toro (2009) Chap. 4, Leveque (2002) Chap. 14, Kröner (1997) Chap. 4.1.

5.1 General setup

5.2 Solution strategy

Chapter 6

Numerical Schemes for Conservation Laws

Chapter 7

Approximate Riemann Solvers

Recommended reading: Toro (2009) Chaps. 10 & 11, Kröner (1997) Chap. 4.4, Leveque (2002) Chap. 15.3

7.1 The Riemann Solver of Roe

Recommended reading: Roe (1981), Roe & Pike (1985)

7.2 The HLL family of Riemann solvers

Recommended reading: Harten (1983), Davis (1988), Einfeldt (1988), Einfeldt et al. (1991)

7.2.1 The HLL solver

7.2.2 The HLLC solver

7.2.3 Wave speed estimates

Chapter 8

Extensions to Balance Laws and multi-D, higher-order methods

Recommended reading: Toro (2009) Chap. 15–17, Leveque (2002) Chap. 17–19

8.1 Source terms

8.2 Multidimensional systems of conservation laws

8.2.1 Dimensional splitting

8.2.2 Unsplit finite volume schemes

8.2.3 Higher-order multi-D schemes: MUSCL-Hancock methodsa

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