Chapter 7: Approximate Riemann Solvers

<u>Idea</u>: • Godunov-type methods are computationally expensive, as one must solve a Riemann problem at every cell interface at each time step.

→ Riemann problem is most demanding task in the method

However, all that is needed about the local solution to the Riemann problem is the solution along the ray $\frac{x}{t} = 0$ ($v^n(x_{i+\frac{1}{2}}, t^n) =$ const. along the ray, i.e. on $[t^n, t^{n+1}]$), in order to compute the flux $f(v^n(x_{i+\frac{1}{2}}, t^n))$.

→ complicated iterative procedure for little information!

• replace "<u>exact</u>" Riemann solver by an "approximate" (cheaper) Riemann solver.

→ gives rise to class of <u>Godunov-type methods</u>

6.1 Roe's Riemann solver

<u>Idea:</u> replace actual Riemann problem by an (approximate) linearized problem defined locally at each cell interface

Def (Riemann solver of Roe):

Replace the RP
$$u_t + f(u)_x = 0 \quad \text{in } \mathbb{R}\times(0,\infty)$$
$$u(x,0) = \begin{cases} u_l & x<0 \\ u_r & x\geq 0\end{cases}$$

by the linear problem

$$(*) \quad w_t + A_{lr} w_x = 0 \quad \text{in } \mathbb{R}\times(0,\infty)$$
$$w(x,0) = \begin{cases} w_l & x<0 \\ w_r & x\geq 0\end{cases}$$

where $A_{lr} = A(u_l, u_r) \in \mathrm{Mat}(\mathbb{R}^{m\times m})$ satisfies the conditions

(i) $f(v) - f(w) = A(v,w)(v-w)$
(conservation across discontinuities)

(ii) $\|A(v,w) - Df(v)\| \to 0$ as $\|w-v\|\to 0$
(consistency with exact Jacobian)

(iii) $A(v,w)$ has only real eigenvalues
and has a complete set of eigenvectors
(system is hyperbolic and RP thus solvable)

The Roe scheme is defined by replacing the exact solution of the local RP in Godunov's method by the exact solution to (*).

Motivation:

- linearized problem justified at most cell interfaces

$$\| u_{i+1}^n - u_i^n \| \sim \mathcal{O}(\Delta x)$$

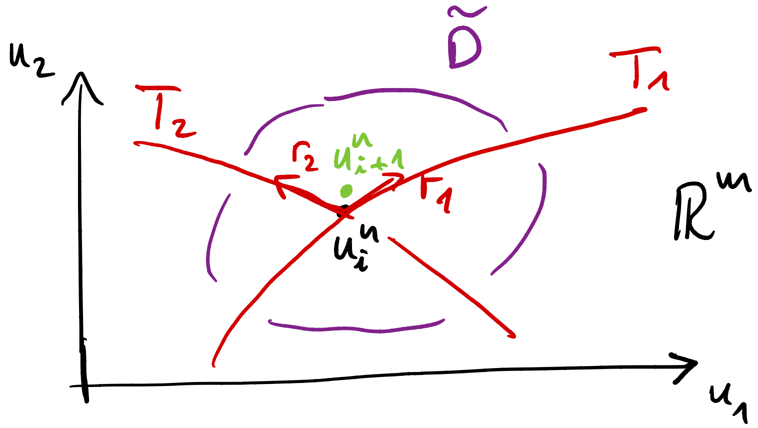
$$Df(u_{i+1}) \approx Df(u_i)$$

constant matrix

appropriate average state $\bar{u} = \bar{u}(u_{i+1}, u_i)$

$$u_t + f(u)_x = 0 \quad \rightsquigarrow \quad u_t + Df(\bar{u}) u_x = 0$$

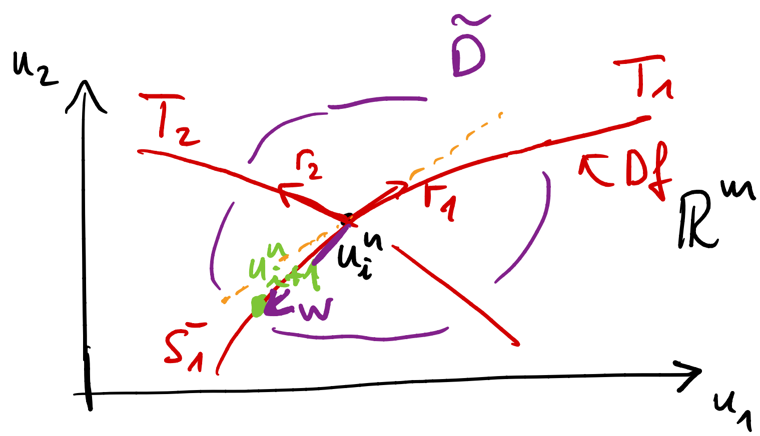
State space: integral curves & shock sets needed to connect u_{i+1}^n, u_i^n are nearly straight lines



$$u^n_{i+1} - u^n_i = \sum_{k=1}^{m} \alpha_k r_k$$

infinitesimal (linearized) Riemann problem

- near shocks: u^n_{i+1}, u^n_i may be separated far in state space at least in one direction $k = k_s$ along a shock set $S^-_{k_s}$.



$\rightsquigarrow$ make $w \equiv u^n_{i+1} - u^n_i$ an eigenvalue of $A(u_i, u_{i+1})$ instead of r_1

Note: u^n_{i+1} cannot be on more than one shock set at a time. For all other families $k \neq k_s$ u^n_{i+1} is "close" to u^n_i $\longrightarrow$ "there is at most one strong wave"

$\longrightarrow$ need to make sure that the corresponding single shock wave

$w \equiv u_{i+1}^n - u_i^n$ is an eigenvector of $A(u_{i+1}^n, u_i^n)$, so that the linearized problem "captures" the shock "outside of the linear domain"

RH jump conditions: $f(v) - f(w) = s\,(v-w)$

Require: $A(v,w)\,(v-w) \overset{!}{=} s\,(v-w) = f(v) - f(w)$

$\Rightarrow$ requirement (i)

Construction of $A(v,w)$:

- Consider the line integral along the path

$$u(\xi) = u_i^n + (u_{i+1}^n - u_i^n)\,\xi\,, \quad 0 \le \xi \le 1$$

$$f(u_{i+1}) - f(u_i) = \int_0^1 \frac{df(u(\xi))}{d\xi}\, d\xi = \int_0^1 Df(u(\xi))\, u'(\xi)\, d\xi$$

$$= (u_{i+1}^n - u_i^n) \underbrace{\int_0^1 Df(u(\xi))\, d\xi}_{A(u_i,\, u_{i+1})}$$

$\rightarrow$ $A(u_i, u_{i+1})$ satisfies (i) & (ii), but (iii) is not guaranteed.

Also: integral difficult to evaluate in general (closed form may not exist)

- Roe (1981): introduce coordinate transformation $z(u)$

path: $z(\xi) = z_i + (z_{i+1} - z_i)\,\xi \,, \quad z_i = z(u_i)$

$$\rightsquigarrow \; f(u_{i+1}) - f(u_i) = \int_0^1 \frac{d\, f(z(\xi))}{d\xi}\, d\xi$$

$$= (z_{i+1} - z_i) \underbrace{\int_0^1 Df_z(z(\xi))\, d\xi}_{\equiv\, C(u_i,\, u_{i+1})}$$

Jacobian w.r.t. $z = z(u)$

$$u_{i+1} - u_i = \int_0^1 \frac{d\, u(z(\xi))}{d\xi}\, d\xi$$

$$= \int_0^1 \frac{d\, u(z(\xi))}{dz}\, z'(\xi)\, d\xi$$

$$= (z_{i+1} - z_i) \underbrace{\int_0^1 \frac{d\, u(z(\xi))}{dz}\, d\xi}_{\equiv\, B(u_i,\, u_{i+1})}$$

$$\rightsquigarrow \; A(u_i, u_{i+1}) = C(u_i, u_{i+1})\, B(u_i, u_{i+1})$$

Harten, Lax, van Leer (1983): this procedure guarantees (iii) if the system has a (convex) entropy function Φ $\leadsto$ choose $z(u) = \nabla\Phi(u)$

wrt. to u

$\leadsto$ $A = A_{HLL}$ is similar to a symmetric matrix and hence has real eigenvalues $\rightarrow$ symmetric hyperbolic

<u>Proposition:</u> Assume that there exists a matrix $A(v,w)$ as in the previous definition. Then the Roe scheme as defined above can be written in conservative form

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left[g(w_i^n, w_{i+1}^n) - g(w_{i-1}^n, w_i^n)\right]$$

where the numerical flux g is given by

$$g(v,w) = \frac{1}{2}\left[f(v) - f(w)\right] - \frac{1}{2}\sum_{k=1}^{m} |\lambda_k|\, \alpha_k r_k$$

with (λ_k, r_k) being the eigenvalues and eigenvectors of $A(v,w)$. The coefficients ("wave strengths") α_k are determined by

$$w - v = \sum_{k=1}^{m} \alpha_k r_k$$

Proof: see Kroener Lemma 4.4.8 p. 342
Toro Sec. 11.1.3

Remarks: 1) Note that the exact solution to the linearized RP can be easily written down:

$$w(x,t) = \begin{cases} u_l & \frac{x}{t} < \lambda_1 \\ u_k & \lambda_k \leq \frac{x}{t} < \lambda_{k+1}, \; k \in \{1,\dots,m-1\} \\ u_r & \lambda_m \leq \frac{x}{t} \end{cases}$$

where $u_k \equiv u_l + \sum_{j=1}^{k} \alpha_j r_j$ and α_j are given by $u_r - u_l = \sum_{j=1}^{m} \alpha_j r_j$.

2) In general, the Roe scheme does not approximate the entropy solution (see counter example Remark 4.4.9 Kroener p. 345)
→ <u>additional entropy "fixes" required</u>
see Toro Sec 11.4, Leveque Sec. 15.3

3) Construction of $A(v,w)$ is complicated and expensive in general
→ Roe-Pike method (Toro Sec. 11.3)

offers approach to compute all quantities needed for flux computation etc. <u>without</u> actually constructing $A(v,w)$

<u>Roe's scheme for the Euler equations</u>:

Consider Euler's equs in conservative form

$$u_t + f(u)_x = 0$$

Roe (1981): $z = \frac{u}{\sqrt{\rho}}$ $\rightsquigarrow$ find that

$$A(u_l, u_r) = Df(\bar{u}), \text{ where}$$

wrt. to u

$$\bar{u} = \begin{pmatrix} \bar{\rho} \\ \bar{\rho}\bar{u} \\ \bar{e} \end{pmatrix} \quad \text{and}$$

$$\bar{\rho} = \sqrt{\rho_r \rho_l}$$

$$\bar{u} = \frac{\sqrt{\rho_l}\, u_l + \sqrt{\rho_r}\, u_r}{\sqrt{\rho_l} + \sqrt{\rho_r}}$$

$$\bar{H} = \frac{\sqrt{\rho_l}\, H_l + \sqrt{\rho_r}\, H_r}{\sqrt{\rho_l} + \sqrt{\rho_r}},$$

"<u>Roe mean values</u>"

$H \equiv \frac{e+p}{\rho}$ specific enthalpy

then $A(u_l, u_r)$ satisfies conditions (i)–(iii).

Remark: For the Roe-scheme with mean values as defined above the density and pressure may become negative $\rho < 0$, $p < 0$ (see Kroener lemma 4.4.14 p. 350)
disadvantageous for problems where low densities are expected!

$\rightarrow$ additional fixes are required

see Einfeldt et al. J. Comp. Phys. 92, 273 (1991)

$\rightarrow$ Einfeldt et al. show that for certain Riemann problems there is no linearization that preserves positivity of ρ, e!

6.2 The HLL family of Riemann solvers

<u>Idea:</u> The HLL and HLLC Riemann solvers represent two- and three-wave approximations to the Riemann problem, respectively.
The HLL solver plus wave speed estimates by Davis and Einfeldt is known as the <u>HLLE</u> solver.

6.2.1 The HLL Riemann solver

Consider again the system

$$u_t + f(u)_x = 0 \quad , \quad x \in [0, L]$$

ICs $\quad u(x,0) = u^0(x)$

BCs $\quad u(0,t) = u_l^B(t) \, , \; u(L,t) = u_r^B(t)$

Conservative scheme:

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left[g_{i+\frac{1}{2}} - g_{i-\frac{1}{2}}\right]$$

Godunov original: $g_{i+\frac{1}{2}} \equiv g(u_i^n, u_{i+1}^n) = f(v^n(x_{i+\frac{1}{2}}, t^n))$

$$\equiv \text{"} f(v^n_{i+\frac{1}{2}}(0)) \text{"}$$

where $V^n_{i+\frac{1}{2}}(0)$ is the exact solution $u^n_{i+\frac{1}{2}}(\frac{x}{t})$ of the local Riemann problem

$$(*) \quad u_t + f(u)_x = 0$$

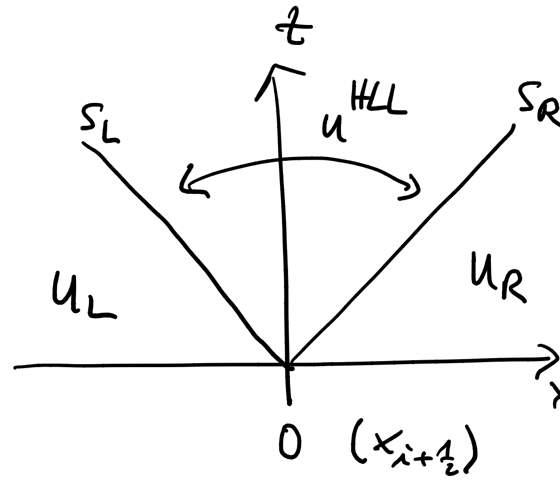
$$u(x,0) = \begin{cases} u_L (= u_i^n), & x < 0 \quad (x < x_{i+\frac{1}{2}}) \\ u_R (= u_{i+1}^n), & x \geq 0 \quad (x \geq x_{i+\frac{1}{2}}) \end{cases}$$

Harten, Lax, van Leer, SIAM Review 25(1):35 (1983):
replace exact solution $u_{i+\frac{1}{2}}(\frac{x}{t})$ by approximate solution

$$u(x,t) = \begin{cases} u_L, & \frac{x}{t} < S_L \\ u^{HLL}, & S_L \leq \frac{x}{t} \leq S_R \\ u_R, & \frac{x}{t} > S_R \end{cases}$$

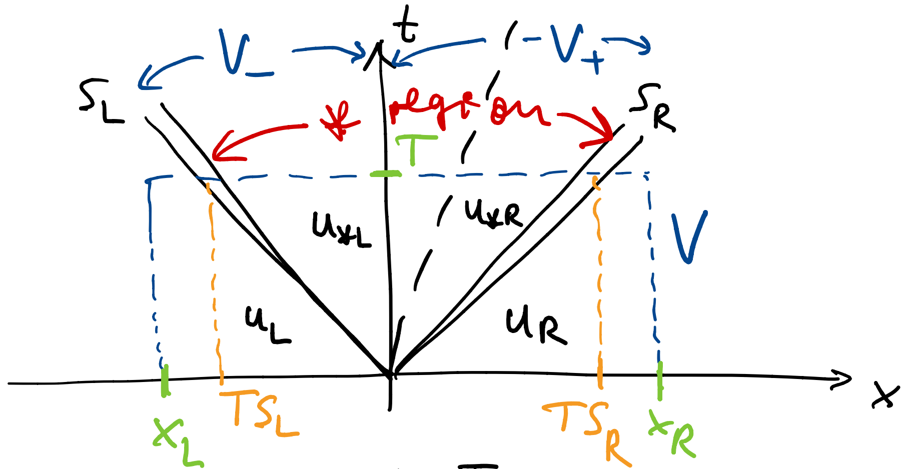
S_L, S_R: fastest wave speeds, assumed to be known

u^{HLL}: approximate state vector in * region



<u>Determine u^{HLL} using exact integral relations:</u>

Consider control volume $V = [x_L, x_R] \times [0, T]$ with $x_L \leq T S_L$ and $x_R \geq T S_R$



Integrate (*) over V: $\displaystyle\int_{x_L}^{x_R}\int_0^T u_t + f(u)_x \, dt\, dx$

$$\rightsquigarrow \int_{x_L}^{x_R} u(x,T)\,dx = \int_{x_L}^{x_R} \underbrace{u(x,0)}_{\substack{\text{const.}\\ \text{on } [x_L,0]\\ [0,x_R]}} dx + \int_0^T f(\underbrace{u(x_L,t)}_{=u_L})\,dt - \int_0^T f(\underbrace{u(x_R,t)}_{=u_R})\,dt$$

$$= x_R u_R - x_L u_L + T(f_L - f_R) \qquad \text{(I)}$$

Also:

$$\int_{x_L}^{x_R} u(x,T)\,dx = \underbrace{\int_{x_L}^{TS_L} \underbrace{u(x,T)}_{\text{const}}\,dx}_{(TS_L - x_L)u_L} + \int_{TS_L}^{TS_R} u(x,T)\,dx + \underbrace{\int_{TS_R}^{x_R} \underbrace{u(x,T)}_{\text{const.}}\,dx}_{(x_R - TS_R)u_R} \qquad \text{(II)}$$

(I),(II) $\Rightarrow$ average of u_* in $*$-region is constant!

$$\boxed{u^{HLL} \equiv \frac{1}{T(S_R - S_L)} \int_{TS_L}^{TS_R} u(x,T)\,dx = \frac{S_R u_R - S_L u_L + f_L - f_R}{S_R - S_L}}$$

Remarks: 1) this is true for any wave structure and number of waves in the Riemann problem

2) need to know the fastest wave speeds S_L, S_R

Useful exact flux expressions at $x = 0$:

Integrate (*) over $\begin{cases} V_- = [x_L, 0] \times [0, T] \\ V_+ = [0, x_R] \times [0, T] \end{cases}$

$$\overbrace{\underbrace{\int_{x_L}^{TS_L} \underbrace{u(x,T)}_{u_L}\,dx}_{u_L(TS_L - x_L)} + \int_{TS_L}^{0} u(x,T)\,dx}^{\int_{x_L}^{0} u(x,T)\,dx} - \underbrace{\int_{x_L}^{0} \underbrace{u(x,0)}_{u_L}\,dx}_{-x_L u_L} = -\int_0^T f(u(0,t)) - f(u(x_L,t))\,dt$$

$$\Leftrightarrow \int_{TS_L}^{0} u(x,T)\,dx = -TS_L u_L + T(f_L - f_{0L})$$

$$\Leftrightarrow \quad f_{0L} = f_L - S_L u_L - \underbrace{\frac{1}{T}\int_{TS_L}^{0} u(x,T)\,dx}_{S_L \times \text{ average in } * \text{-region where } x<0}$$

Similarly for V_+:

$$f_{0R} = f_R - S_R u_R + \frac{1}{T}\int_{0}^{TS_R} u(x,T)\,dx$$

Note: requirement of $f_{0L} = f_{0R} \Leftrightarrow$ (I).

HLL flux estimate at $x=0$:

2- wave approximation $\rightsquigarrow$ can replace averages in f_{0L}, f_{0R} by u^{HLL}:

$$f^{HLL} = f_{0L} = f_L + S_L\,(u^{HLL} - u_L)$$

$$f^{HLL} = f_{0R} = f_R + S_R\,(u^{HLL} - u_R)$$

($\Leftrightarrow$ Rankine-Hugoniot relations across R, L waves!)

$\downarrow$

$$f^{HLL} = \frac{S_R f_L - S_L f_R + S_L S_R\,(u_R - u_L)}{S_R - S_L}$$

$\rightsquigarrow$ set Godunov intercell flux to:

$$\boxed{g^{HLL}_{i+\frac{1}{2}} \equiv f^{HLL} = \begin{cases} f_L, & 0 \le S_L \\ \dfrac{S_R f_L - S_L f_R + S_L S_R (u_R - u_L)}{S_R - S_L}, & S_L \le 0 \le S_R \\ f_R, & S_R \le 0 \end{cases}}$$

where $f_L = f(u_i^n)$, $f_R = f(u_{i+1}^n)$.

Remarks: 1) Harten, Lax & van Leer (1983) showed that the Godunov scheme

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left[g_{i+\frac{1}{2}} - g_{i-\frac{1}{2}}\right]$$

using the HLL expression for $g_{i+\frac{1}{2}}$ derived above

- converges to a weak solution of the system of conservation laws (if convergent)
- converges to the entropy solution

2) Disadvantage: cannot resolve any intermediate waves (such as contact discontinuities or shear waves) by construction

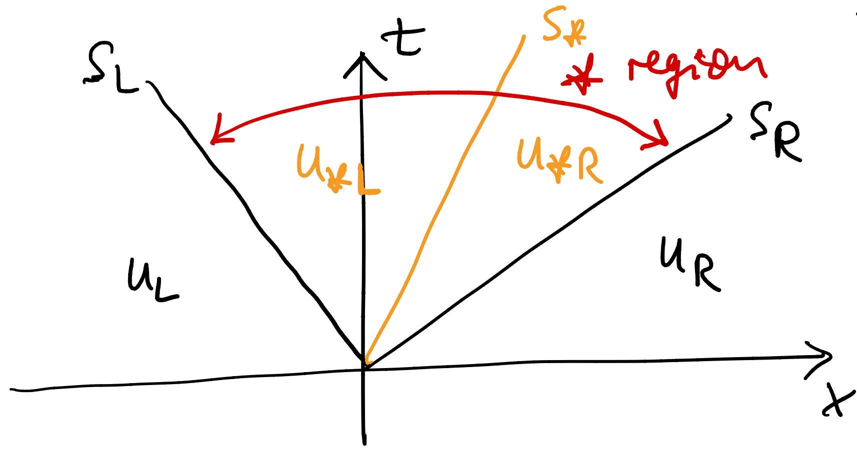
→ diffusive, "numerical smearing"

→ can be unacceptable for some problems (especially for stationary intermediate waves wrt. to grid)

6.2.2 The HLLC solver

<u>Idea:</u> add intermediate wave to restore deficiencies of HLL(E) solver, at least for Euler equations

Toro, Spruce & Speares, Shock Waves 4, 25 (1994)



C: "contact"

<u>Define</u> averages in analogy to U^{HLL} before:

$$U_{*L} \equiv \frac{1}{T(S_* - S_L)} \int_{TS_L}^{TS_*} U(x,T)\,dx$$

$$U_{*R} \equiv \frac{1}{T(S_R - S_*)} \int_{TS_*}^{TS_R} U(x,T)\,dx$$

and $U^{HLL} = \frac{1}{T(S_R - S_L)} \int_{TS_L}^{TS_R} U(x,T)\,dx = \frac{S_* - S_L}{S_R - S_L} U_{*L} + \frac{S_R - S_*}{S_R - S_L} U_{*R}$

(I)

HLLC approach: replace exact solution to local RP by

$$u(x,t) = \begin{cases} u_L & \frac{x}{t} \le S_L \\ u_{*L} & S_L \le \frac{x}{t} \le S_* \\ u_{*R} & S_* \le \frac{x}{t} \le S_R \\ u_R & S_R \le \frac{x}{t} \end{cases}$$

using the intercell Godunov flux

$$g^{HLLC}_{i+\frac{1}{2}} = \begin{cases} f_L & \frac{x}{t} \le S_L \\ f_{*L} & S_L \le \frac{x}{t} \le S_* \\ f_{*R} & S_* \le \frac{x}{t} \le S_R \\ f_R & S_R \le \frac{x}{t} \end{cases}$$

$f_{*L}, f_{*R}, u_{*L}, u_{*R}$ still to be determined from the Rankine-Hugoniot relations across the three waves:

(i) $f_{*L} = f_L + S_L(u_{*L} - u_L)$

(ii) $f_{*R} = f_{*L} + S_*(u_{*R} - u_{*L})$

(iii) $f_{*R} = f_R + S_R(u_{*R} - u_R)$

Note: these relations are consistent with (I)

Idea: 1) specify/estimate $S_* = S_*(S_L, S_R, u_L, u_R)$

$\rightarrow$ HLLC problem reduced to HLL problem of estimating S_L, S_R.

2) determine U_{*L}, U_{*R} and then use above expressions to find required fluxes F_{*L}, F_{*R}

Need additional conditions: (cf. Chap. 5.1)

(iv) $p_{*L} = p_{*R} = p_*$

(v) $S_* = \vec{v}_*^N$ (component of velocity normal to intermediate wave)

$$\left[1D: \quad S_* = u_* \; (v_*) \\ \qquad\qquad = \lambda_2 \right]$$

(i) & (iii) using the fact that $\vec{F}^i(\vec{U}) = v_i \vec{U} + p\vec{D}_i$ for the Euler eqns, where $\vec{D}_i = [0, e_i, v_i]$, $e_i = (1,0,0)$,
$\uparrow$
i

$\rightsquigarrow$ p_{*L}, p_{*R} as functions of L, R, S_*

$$p_{*L} = p_L + \rho_L (S_L - u_L)(S_* - u_L)$$

$$p_{*R} = p_R + \rho_R (S_R - u_R)(S_* - u_R)$$

(vi) $\overset{(iv)}{\rightsquigarrow}$ $S_* = S_*(S_L, S_R)$

$$= \frac{p_R - p_L + \rho_L v_L^N (S_L - v_L^N) - \rho_R v_R^N (S_R - v_R^N)}{\rho_L (S_L - v_L^N) - \rho_R (S_R - v_R^N)}$$

$\rightarrow$ reduced to HLL problem of specifying the wave speeds S_L, S_R.

Using (i)–(vi), several possible choices for $U_{*L}, U_{*R} \Rightarrow f_{*L}, f_{*R}$ (Toro Sec. 10.4.2)

For example: (i) & (iii) & p_{*L}, p_{*R} as above, find:

$$f_{*L,R} = f_{L,R} + S_{L,R}\,(U_{*L,R} - U_{L,R})$$

$$U_{*L,R} = \rho_{L,R}\left(\frac{S_{L,R} - u_{L,R}}{S_{L,R} - S_*}\right)\begin{pmatrix} 1 \\ S_* \\ \frac{E_{L,R}}{\rho_{L,R}} + (S_* - u_{L,R})\left[S_* + \frac{p_{L,R}}{\rho_{L,R}(S_{L,R} - u_{L,R})}\right] \end{pmatrix}$$

<u>Remark:</u> The HLL & HLLC Riemann solver discussed so far are independent of the equation of state. The EOS only enters through the wave speeds S_L, S_R, which we still need to specify.

6.2.3 Wave speed estimates

HLL & HLLC require estimates of wave speeds S_L, S_R.

Several choices: Davis SIAM J. Sci. Stat. Comput. 9, 445 (1988)
Einfeldt SIAM J. Numer. Anal. 25, 294 (1988)
Einfeldt et al. J. Comp. Phys. 92, 273 (1991)

1) Davis: $S_L = u_L - c_L, \quad S_R = u_R - c_R$

$$S_L = \min(u_L - c_L, u_R - c_R), \quad S_R = \max(u_L + c_L, u_R + c_R)$$

simple, but not recommended (see e.g. HLLE below)

2) Davis, Einfeldt: use Roe average eigenvalues

$$S_L = \bar{u} - \bar{c}, \quad S_R = \bar{u} + \bar{c}$$

$$\bar{u} = \frac{\sqrt{\rho_L}\, u_L + \sqrt{\rho_R}\, u_R}{\sqrt{\rho_L} + \sqrt{\rho_R}}, \quad \bar{c} = c(\bar{e}, \bar{\rho}) \overset{\text{ideal}}{\underset{\text{gas}}{=}} \left[(\gamma-1)\left(\bar{H} - \tfrac{1}{2}\bar{u}^2\right)\right]^{1/2}$$

$$\bar{H} = \frac{\sqrt{\rho_L}\, H_L + \sqrt{\rho_R}\, H_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \qquad H = \frac{e+p}{\rho} = \frac{E - \frac{1}{2}\rho u^2 + p}{\rho}$$

3) Einfeldt (HLLE): (inspired by Roe averages, holds for more general convex EOS if shocks sufficiently weak)

$$S_L = \tilde{u} - \tilde{d}, \quad S_R = \tilde{u} + \tilde{d}$$

$$\tilde{u} \equiv \bar{u} = \left(\sqrt{\rho_L}\, u_L + \sqrt{\rho_R}\, u_R\right) / \left(\sqrt{\rho_L} + \sqrt{\rho_R}\right)$$

$$\tilde{d}^2 \equiv \frac{\sqrt{\rho_L}\, c_L^2 + \sqrt{\rho_R}\, c_R^2}{\sqrt{\rho_L} + \sqrt{\rho_R}} + \eta_2 (u_R - u_L)^2$$

$$\eta_2 \equiv \frac{1}{2} \frac{\sqrt{\rho_L}\sqrt{\rho_R}}{\left(\sqrt{\rho_L} + \sqrt{\rho_R}\right)^2}$$

$\rightarrow$ "positively conservative Riemann solver"

(ρ, e remain > 0 during evolution)

"robust Godunov-type scheme"

4) $S_L = -S^+$, $S_R = S^+$, where

$$S^+ = S_{max}^{CFL} = C_{CFL} \frac{\Delta x}{\Delta t}$$

C_{CFL}: Courant number

maximum signal speed allowed by Courant stability condition

(for $C_{CFL} = 1$ would obtain Lax-Friedrichs numerical flux)