Computational Fluid Dynamics
Problem Set 5

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Problem 1 (Project: Riemann solver for the Euler equations)

Implement (in a language of your choice, but excluding Fortran) an 'exact' Riemann solver to solve the Riemann problem for the one-dimensional time-dependent Euler equations for ideal gases. To verify your implementation and explore solutions, use an adiabatic constant of $\gamma = 1.4$ and consider the following initial data for the primitives $\mathbf{w} = (\rho, v, p)$:

Problem	ρ_l	v_l	p_l	ρ_r	u_r	p_r
1	1.0	0.0	1.0	0.125	0.0	0.1
2	1.0	-2.0	0.4	1.0	2.0	0.4

Problem 1 results in a left rarefaction wave and a right shock, while Problem 2 results in a two-rarefaction wave solution. Plot density, pressure, velocity and internal energy at time $t = 0.25$ for Problem 1 and $t = 0.15$ for Problem 2 on a domain $x \in [-1, 1]$ and verify that the behavior of these quantities across the three non-linear wave structures looks as expected.

Hints:

1. Follow the solution strategy as discussed in the lectures. You will find additional detail in Chap. 4 of Toro's book. You can check your solutions against Tabs. 4.2 and 4.3 in Toro's book.

2. First, find the pressure in the star region p^*, using a Newton-Raphson scheme on the pressure function $f(p)$ discussed in the lectures. You may use $p^*_{\text{guess}} = \frac{1}{2}(p_l + p_r)$ as initial guess for p^* and a relative tolerance of 10^{-6} for the root-finding procedure.

3. Second, compute the solution for the primitives $\mathbf{w}(x,t) = (\rho(x,t), v(x,t), p(x,t))$. Use a spatial grid $x \in [-1, 1]$ with the initial discontinuity at $x = 0$. Due to the self-similar nature of the solution, sample the solution for the primitives $\mathbf{w}(x,t)$ as a function of the 'speed' $S = x/t$. The corresponding solutions were discussed in the lectures. Once you specify the time t, the solution profiles become a function of x only.

Problem 2 (Bonus question—discrete entropy condition)

Consider the situation of the Lax-Wendroff theorem and assume that the conservative scheme is consistent with the discrete entropy condition. Show that the scheme then converges to a weak solution of the conservation law that satisfies the entropy condition.

Hint: The proof is similar to the proof of the Lax-Wendroff theorem.