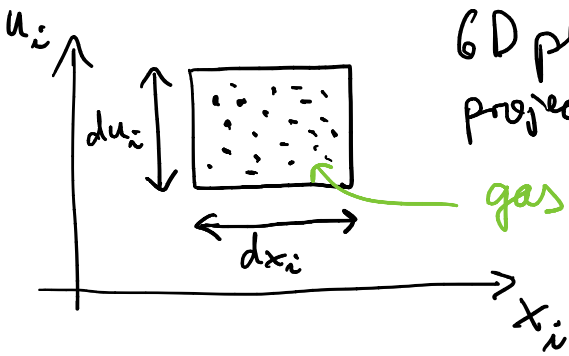
# Chapter 2: Basic Equations

## 2.1 Continuous media: Boltzmann Eq




6D phase space projected onto $x_i - u_i$ plane

In principle evolution of system fully determined by Newtonian mechanics (N-body equations):

$$\frac{d\vec{x}_i}{dt} = u_i \,, \quad \frac{d\vec{u}_i}{dt} = \underbrace{\vec{F}_i}_{\text{force on } i}(x_j, u_j, t) \quad \forall \text{ particles } i, j$$

But: 1 mole $\hat{=}$ $N = 6 \times 10^{23}$ particles $\rightarrow$ computationally prohibitive

$\rightarrow$ statistical approach (continuous media $\rightarrow$ hydrodynamics)

$$dN = f(\vec{x}, \vec{u}, t)\, d\vec{x}\, d\vec{u}$$

$dN$: number of particles in phase-space control volume $d\vec{x}\, d\vec{u}$

$f$: distribution function

- ignore internal degrees of freedom $\rightarrow$ vibration, rotation
- identical particles of mass $m$
- ignore QM effects

Assume particles subject to external force per unit mass $\vec{\vec{F}}$ ($\approx$ constant over typical inter-particle separation)

$\rightsquigarrow$ particle number $f(\vec{x},\vec{u},t)\,d\vec{x}\,d\vec{u}$ conserved along their trajectories in phase space (Liouville's theorem) in absence of interactions:

$$(*)\quad \Delta f = f(\vec{x}+\underbrace{\vec{u}dt}_{\text{change in position}},\ \vec{u}+\underbrace{\vec{\vec{F}}dt}_{\text{change in velocity due to }\vec{\vec{F}}},\ t+dt) - f(\vec{x},\vec{u},t) = \underbrace{[\Delta f]_{coll}}_{\text{change in } f \text{ in } dt \text{ due to collisions}}$$

$dt \to 0$

$\Delta f \to \frac{df}{dt}$

$$\boxed{\frac{df}{dt} = \frac{\partial f}{\partial t} + u_i \frac{\partial f}{\partial x_i} + F_i \frac{\partial f}{\partial u_i} = \left[\frac{\partial f}{\partial t}\right]_{coll}}$$

Boltzmann's transport equation

→ evolution of distribution function in 6-D phase space $(\vec{x},\vec{u})$

→ change $dN$ in number of particles within a fixed volume $d\vec{x}\,d\vec{u}$ equals net number of particles entering or leaving this volume

(→ Liouville's theorem)

Some definitions:

- $$n(\vec{x},t) \equiv \int \underbrace{f(\vec{x},\vec{u},t)\,d\vec{u}}_{\#\text{particles per unit volume in } \vec{u},\,\vec{u}+d\vec{u}}$$ total number of particles per unit volume

- $\ell \equiv n^{1/3}$ mean particle separation

  Note: any physical length scale $dx$ we are interested in must be $\gg \ell$ for statistical approach to be valid

- $\rho(\vec{x},t) \equiv \int m f(\vec{x},\vec{u},t)\, d\vec{u}$ mass density

  mass of particle

- $\vec{v}(\vec{x},t) \equiv \frac{1}{\rho} \int \vec{u}\, m f(\vec{x},\vec{u},t)\, d\vec{u}$ bulk velocity

- $\rho\varepsilon(\vec{x},t) \equiv \frac{1}{2} \int m \tilde{u}^2 f(\vec{x},\vec{u},t)\, d\vec{u}$ specific internal energy $\varepsilon$

  $\tilde{\vec{u}} \equiv \vec{u} - \vec{v}$ peculiar velocity

<u>Special case</u>:
- elastic collisions (energy & momentum conserved)
- low density (collisions of 3 & more particles can be neglected)
- absence of external forces $\vec{F} \equiv 0$

unique solution $\downarrow$ statistical mechanics

$$f(\vec{x},\vec{u},t)\, d\vec{u} = n(\vec{x},t) \left[\frac{m}{2\pi k_B T(\vec{x},t)}\right]^{3/2} \exp\left[-\frac{m(\underbrace{\vec{u}-\vec{v}}_{\equiv \tilde{u}})^2}{2k_B T(\vec{x},t)}\right] d\vec{u}$$

peculiar velocity

Maxwellian velocity distribution

(equilibrium state $t \to \infty$, $\frac{\partial f}{\partial t} = 0$)

# 2.2 From Boltzmann to Euler

**Def:** k-th moment of the Boltzmann equation

$$\int U_k \left[ \frac{\partial f}{\partial t} + u_i \frac{\partial f}{\partial x_i} + F_i \frac{\partial f}{\partial u_i} \right] du = \int U_k \left[ \frac{\partial f}{\partial t} \right]_{coll} d\vec{u},$$

where $U_k = \vec{u}^k$, i.e. $U_0 = 1$, $U_1 = \vec{u}$, $U_2 = u^2$ etc.

## General properties of collision term:

If collisions are elastic and neither create nor destroy particles, then

$$\int \left[ \frac{\partial f}{\partial t} \right]_{coll} d\vec{u} = 0$$ number of particles conserved

$$\int \left[ \frac{\partial f}{\partial t} \right]_{coll} u_i \, d\vec{u} = 0$$ total momentum conserved

$$\int \left[ \frac{\partial f}{\partial t} \right]_{coll} u^2 \, d\vec{u} = 0$$ total energy conserved

$$\lim_{u \to \infty} u^k f = 0$$ total number, momentum energy must be finite

**Also:** $\left( \int_{\Omega} v \frac{\partial u}{\partial x_i} = \int_{\partial\Omega} u v n_i \, d\Gamma - \int_{\Omega} u \frac{\partial v}{\partial x_i} \right)$

$$\int \frac{\partial f}{\partial u_i} \, d\vec{u} \overset{p.i.}{=} \int_{|u| = \pm\infty} f \, n_i \, d\Gamma - \int \underbrace{\frac{\partial(1)}{\partial u_i}}_{=0} f \, d\vec{u} = 0$$

$$\int u_j \frac{\partial f}{\partial u_i} \, d\vec{u} \overset{p.i.}{=} \underbrace{\int_{|u| = \pm\infty} u_j f \, n_i \, d\Gamma}_{=0} - \int \underbrace{\frac{\partial u_j}{\partial u_i}}_{=\delta_{ij}} f \, d\vec{u}$$

$$= -\delta_{ij} \frac{\rho}{m}$$

$$\frac{1}{2} \int u^2 \frac{\partial f}{\partial u_i} \, d\vec{u} = \underbrace{\frac{1}{2} \int_{|u| = \pm\infty} u^2 f \, n_i \, d\Gamma}_{=0} - \frac{1}{2} \int \underbrace{\frac{\partial u^2}{\partial u_i}}_{\frac{\partial (\sum_j u_j^2)}{\partial u_i} = 2 u_i} f \, d\vec{u}$$

$$= - \int u_i f \, d\vec{u} = - \frac{\rho}{m} v_i$$

(1) From <u>0-moment</u> of Boltzmann equ:

$$m \int \frac{\partial f}{\partial t} \, d\vec{u} + m \int u_i \frac{\partial f}{\partial x_i} \, d\vec{u} + m F_i \underbrace{\int \frac{\partial f}{\partial u_i} \, d\vec{u}}_{=0} = \underbrace{\int \left[\frac{\partial f}{\partial t}\right]_{coll} d\vec{u}}_{=0}$$

$u_i, x_i$ independent

$\downarrow$

$$\Longleftrightarrow \frac{\partial}{\partial t} \int m f \, d\vec{u} + \frac{\partial}{\partial x_i} \int u_i m f \, d\vec{u} = 0$$

$$\Leftrightarrow \quad \boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0}$$ continuity equation

<u>Note:</u> $V$ large enough

$$\rightsquigarrow \int \frac{\partial \rho}{\partial t}\, dV + \underbrace{\int \nabla\cdot(\rho\vec{v})\, dV}_{\overset{\text{Gauss}}{=} \int_{\partial V} \rho v^i n_i \, d\Omega \; = 0}$$

$n_i$: unit normal; $\vec{v}=0$ on $\partial V$

$$= \frac{\partial}{\partial t}\int_V \rho\, dV = \frac{\partial M_V}{\partial t} = 0 \quad \text{mass conservation}$$

② From <u>1st-moment</u> of Boltzmann equ:

$$m\int u_i \frac{\partial f}{\partial t}\, d\vec{u} + m\int u_i u_j \frac{\partial f}{\partial x_i}\, d\vec{u} + m F_j \underbrace{\int u_i \frac{\partial f}{\partial u_j}\, d\vec{u}}_{= -\delta_{ij}\frac{\rho}{m}}$$

$$= \int u_i \left[\frac{\partial f}{\partial t}\right]_{coll} d\vec{u} = 0$$

$$\Leftrightarrow \frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_i}\int m u_i u_j f\, d\vec{u} - \rho F_i = 0$$

$$\int m u_i u_j f \, d\vec{u} = \int m (\tilde{u}_i + v_i)(\tilde{u}_j + v_j) f \, d\vec{u}$$

cross terms vanish

$\int \tilde{u}_i f \, d\vec{u} = 0$

peculiar velocities

$$= \int m v_i v_j f \, d\vec{u} + \int m \tilde{u}_i \tilde{u}_j f \, d\vec{u}$$

$$\equiv \rho v_i v_j + P_{ij}$$

$$P_{ij} \equiv \int m \tilde{u}_i \tilde{u}_j f \, d\vec{u}$$ pressure tensor

Most astrophysical cases:

isotropic pressure $\Rightarrow$ $P_{ij} = P \delta_{ij}$

$$P \equiv \frac{1}{3} \int m \tilde{u}^2 f \, d\vec{u}$$

$$\left(P = \frac{2}{3} \rho \varepsilon\right)$$

$$\Leftrightarrow \quad \frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j) = -\frac{\partial P}{\partial x_i} + \rho F_i$$

$$\Leftrightarrow \quad \boxed{\begin{array}{l} \dfrac{\partial}{\partial t}(\rho \vec{v}) + \nabla \cdot \Pi = \rho \vec{F} \\[2ex] \Pi \equiv \rho v_i v_j + P \delta_{ij} \quad \text{momentum flux tensor} \end{array}}$$ momentum equation

Note: • pressure gradient and external forces act as <u>source (or sink) terms</u>

- force due to pressure gradient results from exchange of energy between bulk flow and peculiar motions

③ From 2nd-moment of Boltzmann equ: (Exercise):

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{v^2}{2}+\varepsilon\right)\right]+\frac{\partial}{\partial x_i}\left[\rho v_i\left(\frac{v^2}{2}+\varepsilon\right)\right]=-\frac{\partial h_i}{\partial x_i}-\frac{\partial}{\partial x_i}(P v_i)$$

$$+\rho v_i F_i$$

where $h_i \equiv \int \frac{m}{2} \tilde{u}_i \tilde{u}^2 f \, d\vec{u}$ conduction heat flux

If $\vec{h}$ can be neglected (typical for astrophys. systems):

$$\leadsto \boxed{\frac{\partial E}{\partial t}+\frac{\partial}{\partial x_i}\left[(E+P)v_i\right]=\rho v_i F_i}$$ total energy equation

$$E \equiv \frac{1}{2}\rho v^2+\rho\varepsilon$$ total energy density

Remarks: 1) ①, ②, ③ are known as the Euler equations. They can be rewritten as a system of conservation laws:

$$\boxed{\begin{aligned} u_t + f^i(u)_{x_i} &= S \\ \partial_t u + \frac{\partial}{\partial x_i} f^i(u) &= S \end{aligned}}$$

where

$$u = \begin{pmatrix} \rho \\ \rho \vec{v} \\ E \end{pmatrix}$$ "conserved variables"

$$f^i = \begin{pmatrix} \rho v^i \\ \rho v^i v^j + \delta_{ij} p \\ v^i (E+p) \end{pmatrix}$$ $j = 1,2,3$ (momentum components), $i = 1,2,3$ (flux directions) "fluxes"

$$S = \begin{pmatrix} 0 \\ \rho \vec{F} \\ \rho \vec{v} \cdot \vec{F} \end{pmatrix}$$ "sources"

2) Equation of state: The Euler equs. are five equations for six unknowns $\rho, \{v^i\}, P, e \equiv \rho \varepsilon$

$\rightsquigarrow$ require relation $P = P(\rho, e)$ to close the system $\to$ "equation of state"

Ideal gas: $p = (\gamma - 1) e = (\gamma - 1) \rho \varepsilon$

← internal energy density

with $\gamma = \frac{c_p}{c_v}$ the ratio of specific heats (adiabatic constant)

In general: $p = p(\rho, e, X_i, \dots)$ can be very

↑ composition

complicated (→ different chemical elements, ionization states, compl. reactions etc.)

<u>Exercise:</u> Show strict hyperbolicity of the Euler equations.

3) <u>Validity of continuous medium approximation</u>

require (see above):

(i) $\lambda \ll dx \ll l_{sys}$ ← characteristic size of physical system

$\lambda$: mean free path of particle

$dx$: finite size element of medium

$f$ should not vary over $dx$ & $dN$ should be large enough for averaging to be meaningful

(ii) interparticle forces must be short-range i.e. on distances $l_{force} \ll dx$, otherwise energy & momentum exchanged with fluid elements far away

example for long-range: <u>external grav. field</u>
→ external force

Note: <u>self-gravity</u> cannot be obtained from Boltzmann equ due to long-range nature
→ require additional Poisson equ.

(iii) peculiar motion close to boundary of element $dx$ carries particles into adjacent elements with different $\rho(\vec{x})$, $v(\vec{x})$, $T(\vec{x})$ ⇝ "<u>diffusion</u>"

⇝ <u>friction forces</u> appear ⇒ microscopic exchange of momentum & energy neglected in Euler equation

→ need to take conduction heat flux $\vec{h}$ and other terms into account (see below)