① $\mathcal{M} = \langle S, A, R, T, \gamma \rangle$. $|S \times A| < \infty$, $0 \le \gamma < 1$.

$\{X \to Y\} \triangleq \{f \mid f : X \to Y\}$.

$Q, Q' \in \{S \times A \to \mathbb{R}\}$. $Q(s,a) \ge Q'(s,a) \; \forall (s,a)$.

$$\left|\max_{a \in A} Q(s,a) - \max_{a' \in A} Q'(s,a')\right| < \max_{a \in A} |Q(s,a) - Q'(s,a)|$$

$\max Q(s,a) > Q(s,a') \Rightarrow LHS = \max Q(s,a) - \max Q'(s,a')$

$\le Q(s,a) - Q'(s,a)$

$\le \max_{a \in A} |Q(s,a) - Q'(s,a')|$.

② $|\min Q(s,a) - \min Q'(s,a')|$

$\min Q(s,a) = Q(s,a) \ge Q'(s,a) \ge \min Q'(s,a') = Q'(s,a')$

$\Rightarrow LHS = Q(s,a) - Q'(s,a') < Q(s,\dot{a}) - Q'(s,\dot{a})$.

③ $$\left|\frac{1}{|A|}\sum_{a \in A} Q(s,a) - \frac{1}{|A|}\sum_{a' \in A} Q'(s,a')\right| \le \max_{a \in A} |Q(s,a) - Q'(s,a')|$$

$$= \left|\frac{1}{|A|}\sum_{a \in A}\left[Q(s,a) - Q(s,a')\right]\right| \le \max |Q(s,a) - Q'(s,a')|$$

④ $$\frac{1}{\omega}\log\left(\frac{1}{|A|}\sum e^{\omega Q(s,a)}\right) - \frac{1}{\omega}\log\left(\frac{1}{|A|}\sum e^{\omega Q'(s,a')}\right)$$

$$= \frac{1}{\omega}\log\left(\sum e^{\omega Q(s,a)}\right) - \frac{1}{\omega}\log\left(\sum e^{\omega Q'(s,a')}\right)$$

$$= \frac{1}{\omega}\log\left(\frac{\sum e^{\omega Q(s,a)}}{\sum e^{\omega Q'(s,a')}}\right) = \frac{1}{\omega}\log\left(\frac{\sum e^{\omega Q'(s,a')\frac{\omega}{\Delta}(a)}}{\sum e^{\omega Q'(s,a')}}\right).$$

$$\le \frac{1}{\omega}\log\left(e^{\omega \max(\Delta a)}\right) = \log\left(e^{\max(\Delta a)}\right) = \max(\Delta a).$$

$\otimes : \{S \times A \to \mathbb{R}\} \to \{S \to \mathbb{R}\}$

$$\|\otimes Q - \otimes Q'\|_\infty \leq \|Q - Q'\|_\infty .$$

$$V_0(s) = 0 \ , \ k = 1 \ , \ s \in S .$$

$$V_k(s) = \otimes_{a \in A} \left(R(s,a) + \gamma \sum_{s' \in S} T(s'|s,a) V_{k-1}(s') \right)$$

$$= \otimes_{a \in A} Q(s,a) \quad \gamma\text{-contraction:}$$

(5) $$\|BV_1 - BV_2\| = \max_{s \in S} |BV_1(s) - BV_2(s)|$$

$$= \max \left| \otimes (R(s,a) + \gamma \sum T(s'|s,a) V_1(s') - \otimes \left[R(s,a') + \gamma T(s'|s,a) V_2(s') \right] \right|$$

$$= \max \left| \gamma \sum T(s'|s,a) [V_1(S') - V_2(s')] \right|$$

$$= \gamma \max_{s,a \in S,A} \left| \max_{s \in S} [V_1(s') - V_2(s')] \right| \leq \gamma [V_1(s) - V_2(s)]$$

$$= \gamma \|V_1 - V_2\|_\infty .$$

(6) $$\otimes_\lambda = \lambda \otimes_1 + (1-\lambda) \otimes_2 .$$

$$\|\otimes_\lambda Q - \otimes_\lambda Q'\| = \| \lambda \otimes_1 Q + (1-\lambda) \otimes_2 Q - \lambda \otimes_1 Q' - (1-\lambda) \otimes_2 Q' \|$$

$$= \| \lambda [\otimes_1 Q - \otimes_1 Q'] + (1-\lambda) [\otimes_2 Q - \otimes_2 Q'] \|$$

$$\leq \| \lambda [\otimes_1 Q - \otimes_1 Q' \| + \| (1-\lambda) [\otimes_2 Q - \otimes_2 Q' \| \leq (1-\lambda+\lambda) \|Q-Q'\|$$

$$= \|Q - Q'\|$$

(7) $$\otimes_1 Q(s) = \frac{1}{|A|} \sum_{a \in A} Q(s,a) \qquad \otimes_2 Q = \max_{a \in A} Q(s,a) .$$

$$\otimes_\varepsilon Q = \varepsilon \otimes_1 + (1-\varepsilon) \otimes_2$$