MATH 411: INTRODUCTION TO ABSTRACT ALGEBRA
HOMEWORK #8

TRUNG DANG
33858723

Problem 1

Problem Statement: Let G be the group of rotations of the cube from the lectures

(1) Prove that G acts transitively on the set of four diagonals of the cube

(2) Prove that G is isomorphic to S_4

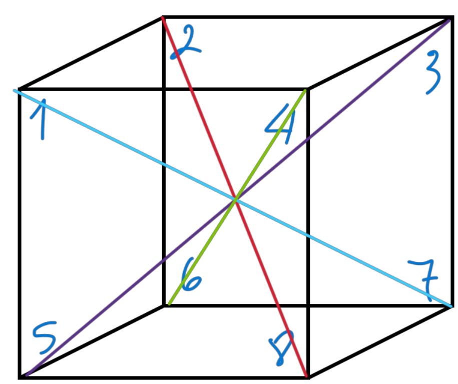


FIGURE 1

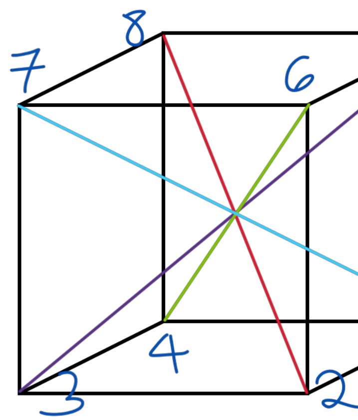
Solution. (1) Consider the cube with vertices labeled and 4 diagonals colored as above. Let X be the set of permutations of diagonals of the cube. We will show that $G \curvearrowright X$ satisfies the axioms of actions:

- $ex = e \star x = x \forall x \in X$
- Let g_1, g_2 be arbitrary rotations of the cube, then by definition of cube rotation, g_1 permutes the diagonals obtained by the permutation of g_2 on the original arrangement of permutations. Therefore, $(g_1 g_2) \star x = g_1 \star (g_2 \star x)$

Thus, G acts on the set of 4 diagonals of the cube.

(2) Assume that there is another element g of G that takes every diagonal to itself. Then consider the cube in the figure above.
Since g is not the identity, at least one of the diagonals must be affected. WLOG assumes

that g acts on the red diagonal resulting in 2 and 8 swapping places. Then, by Euler's Theorem, since rotations preserve the orientation of the cube, (1,7), (3,5), and (4,6) must also swap places. So we get:



But then the cube changed orientation, since viewing from the center, 1 initially goes to 2 counterclockwise, but now 1 goes to 2 clockwise.
Thus, the assumption is false, and the identity is the only element of G that takes every diagonal to itself.

(3) Let x be the original state of the cube as demonstrated in Figure 1. Then, we will show that G is isomorphic to the group of permutations of the diagonals X, which is equivalent to S_4. Define $f: G \to X$ as $f(g) = gx = g \star x, \forall g \in G$, and x defined above.
First, we observe that we can swap any two diagonals by applying a rotation on another diagonal: For instance, we can swap the purple and blue diagonals by rotating around the green diagonal 180 degrees. And since any permutations can be written as a product of transpositions, f is onto.
Second, assume that there are 2 rotations g, g' that result in the same diagonal permutations. Assume $g = hg'$ for some $h \in G$. Then, $gx = hg'x = h(g'x) = h(gx)$. Thus, h takes gx to gx, so from (2), h must be identity. Thus, $g = g'$, or f is one-to-one.
Third, since $f(g_1 g_2) = (g_1 g_2)x = g_1(g_2 x) = g_1 f(g_2) = f(g_1)f(g_2)$ by definition of action, so f is homomorphic.
Thus, f is an isomorphism between G and X, thus an isomorphism between G and S_4.

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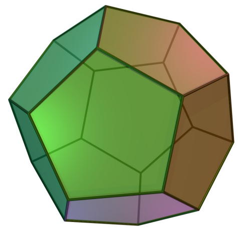
Problem 2

Problem Statement: Let G be the group of all orientation-preserving isometries of the dodecahedron. By analogy with the case of the cube done in class, give the list of all orientation-preserving isometries of the dodecahedron and prove that your list is complete using Orbit-Stabilizer Theorem.



Solution. First, note that a dodecahedron has 12 faces, 20 vertices, and 30 edges.
Let G be the group of all orientation-preserving isometries of the dodecahedron, X be the set of faces, and $x \in X$ be an arbitrary face.
Then similar to the cube, the only rotation that preserves a face must preserve the center of the face, thus must be a rotation about an axis that passes through the center of the dodecahedron and the face.
For each axis, we have 5 possible rotations.
Thus, $|G_x| = 5$.
And since each face can be permuted to any other face, we have $Orb(x) = 12$.
Hence, by Orbit-Stabilizer Theorem, we have:

$$|G| = |G_x||Orb(x)| = 5 \cdot 12 = \boxed{60}$$

There are 4 types of orientation preserving-isometries of the dodecahedron:

- The identity transformation - there is only 1 transformation
- The rotations about an axis through the center of 2 opposing faces:
  Since there are 12 faces, and for each axis there are 4 non-identity transformations, we have a total of: $6 \cdot 4 = 24$ transformations
- The rotations about an axis through 2 opposing vertices:
  Since there are 20 vertices, and for each axis there are 2 non-identity transformations, we have a total of: $10 \cdot 2 = 20$ transformations
- Finally, the rotations about an axis through 2 opposing edges:
  Since there are 30 edges, and for each axis there is only 1 non-identity transformation, we have: 15 transformation

This adds up to $1 + 24 + 20 + 15 = 60$ transformations, which is adequate by the Orbit-Stabilizer Theorem above. ■

Problem 3

Problem Statement: Let G be a finite group acting on a finite set X.

(1) Prove that G is transitive if and only if $|G| = \sum_{g\in G} |Fix(g)|$

(2) Suppose that G acts on X transitively and that $|X| \geq 2$. Prove Jordan's Theorem; there exists an element $g \in G$ such that $Fix(g) = \emptyset$. These elements are called derangements.

(3) Find the number of derangements for the action of S_5 on $\{1,2,3,4,5\}$

Solution. (1) The group G is transitive if there is only one orbit. By Burnside Lemma, we have:

$$1 = \#Orbit = \frac{1}{|G|}\sum_{g\in G}|Fix(g)|$$

which implies:

$$|G| = \sum_{g\in G}|Fix(g)|$$

(2) Assume that for all $g \in G$, $Fix(g) \neq \emptyset$. Then $\forall g \in G, |Fix(g)| \geq 1$. Also note that $Fix(e) = |X|$ with e being the identity of G. Then,

$$\sum_{g\in G}|Fix(g)| \geq (|G|-1) + |X| \geq |G| + 1 > |G|$$

, which is a contradiction.
Therefore, $\exists g \in G$ such that $Fix(g) = \emptyset$

(3) For $g \in S_5$ to have $Fix(g) = \emptyset$, then g must permute each element of $\{1,2,3,4,5\}$ to another position. Hence, g can only be either five-cycles, or a composition of a 3 cycle and a 2 cycle.
The number of 5 cycles is $4! = 24$, and the number of (3+2) cycles is $2 \cdot \binom{5}{3} = 20$, so the total number of derangements for the action of S_5 on $\{1,2,3,4,5\}$ is:

$$24 + 20 = 44 \text{ derangements}$$

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Problem 4

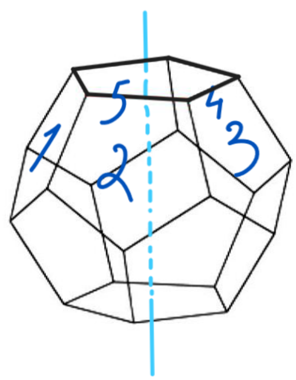
Problem Statement: Use the Burnside Lemma to find the number of ways to color the dodecahedron using at most two colors (colorings that differ by rotations are viewed as equivalent)

Solution. We will compute the number of possible colorings using the Burnside Lemma:

$$\#Colorings = \#Orbits = \frac{1}{|G|}\sum_{g\in G}|Fix(g)|$$

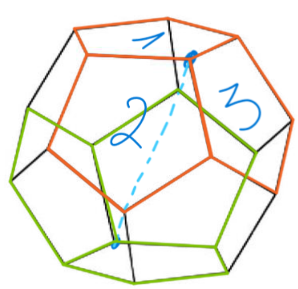
This means we have to calculate $|Fix(g)|$ for all rotations $g \in G$. For brevity, let $C(i)$ be the color of face i.

If g is a rotation about the center of 2 opposing faces:

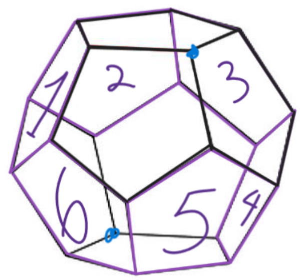


Then there are 2 ways to color the top face. There are also 2 ways to color face 1. If g is a rotation by $2\pi/5$ degrees then $C(1) = C(2) = C(3) = C(4) = C(5)$. Thus, there are only 4 ways to color the top half of the dodecahedron.
Similarly, there are only 4 ways to color the bottom half of the dodecahedron
Thus, there are a total of 16 ways to color the dodecahedron that is stabilized by a rotation about the center of 2 opposing faces. (1)

If g is a rotation about 2 opposing vertices:

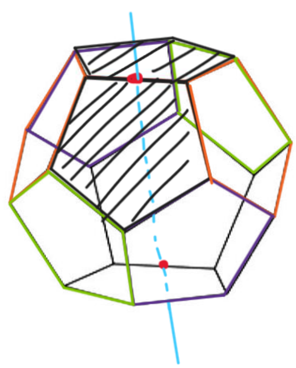


Then by a similar argument, there are 2 ways to choose the colors of 1, and $C(1) = C(2) = C(3)$. There are also 2 ways to choose the colors of the green faces. However, the remaining faces need not be the same color.



Indeed, a rotation by 120 degrees will send 1 to 3, 2 to 4, 3 to 5, 4 to 6, 5 to 1, and 6 to 2. Therefore, by the same argument, (1,3,5) must be of the same color, and (2,4,6) must be of the same color. To conclude this case, there are a total of $2 \cdot 2 \cdot 2 \cdot 2 = 16$ colorings that is stabilized by a rotation about 2 opposing vertices (2)

Lastly, if g is a rotation about the midpoint of 2 opposing edges.



. Then we can see that the set of faces are partitioned into pairs that permute into each other. So for each axis of rotation, there are 6 pairs, each with 2 ways to choose color from, for a total of 64 colorings. (3)

From problem 2, we have |G| = 60. Therefore, the number of orbits is:

$$\#Orbit = \frac{1}{60} \sum_{g \in G} |Fix(g)| = \frac{1}{60}(2^{1}2 \cdot 1 + 16 \cdot 24 + 16 \cdot 20 + 64 \cdot 15) = \boxed{96}$$

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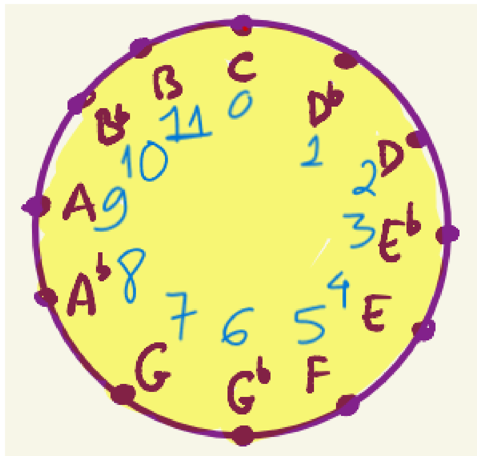
Problem 5

Problem Statement: In Western music, tones are equivalent if they differ by an octave. A triad is a subset made with three distinct tones, for example [C,E,G]. Two triples can be viewed as equivalent if one can be translated to another, for example [C,E,G] and [F,A,C] are equivalent because the chord [F,A,C] is a shift of 5 tones above the chord [C,E,G]. In addition, triples are equivalent if the tones are permuted with the same set of notes. For example, [C,E,G] = [G,C,E]. Use Burnside Lemma for $\mathbb{Z}_{12}$ to find the number of different triads.



Solution. We begin by relabeling the chords from 0 to 11 as shown in the figure. For each chord, we now represent it as (a,b,c) where $0 \le a < b < c \le 11$. Denote the set of all triads X. Also define the action $\mathbb{Z}_{12} \curvearrowright X$ be $d \star (a,b,c) = (a+d \mod 12, b+d \mod 12, c+d \mod 12)$, in other words shifting the chord up by d tones.
By Burnside Lemma, we have the number of different triads equal to the number of different orbits of this action, which is:

$$\#Orbit = \frac{1}{|\mathbb{Z}_{12}|} \sum_{d \in \mathbb{Z}_{12}} |Fix(d)|$$

Now assume that $(a,b,c) \in Fix(d)$ for some d. Then $(a+d, b+d, c+d) = (a,b,c)$.

- If $a+d \equiv a \mod 12$ then $d = 0$, thus $Fix(d) = \binom{12}{3} = 220$
- If $a+d \equiv b \mod 12$, then $b+d \equiv c$ (if $b+d = a \mod 12$ then $c+d = c \mod 12$ thus d $= 0$ as above), thus $c+d \equiv \mod 12$. Now since $d < 12$ and $a < b$, $a+d < b+12$ thus $a+d = b$, similarly, we have $b+d = c$, and $c+d = 12+a$. Solving the system of equations, we have $d = 4$, resulting in the following 4 triads: $(0,4,8), (1,5,9), (2,6,10), (3,7,11)$
- If $a+d \equiv c \mod 12$, then $b+d \equiv a \mod 12$, and $c+d \equiv b \mod 12$. With the same reasoning, we have: $a+d = c, b+d = a+12, c+d = b+12$, thus $d = 8$. And we have the same 4 triads.

Applying the Burnside Lemma, we have:

$$\#Orbit = \frac{1}{12}(220+4+4) = 19$$

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Problem 6

Problem Statement: For each of the five designs from Professor JT gift set, find one of the 17 wallpaper groups that describe it best and motivate your choice.

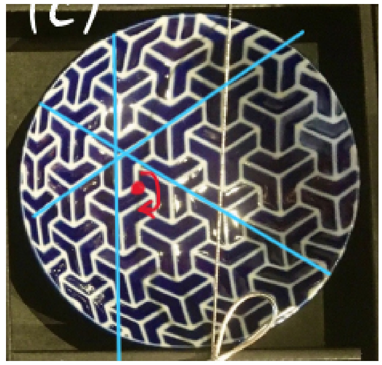
Solution. (1)

p4g: There are 90 degree rotations about the corner of the squares, and there vertical and horizontal reflection through the center of each square. However, all center of rotations do not lie on any axis of a reflection.

(2)

p6m: There are 60 degree rotations about the white dots, and reflections about the lines (highlighted in blue)

(3)

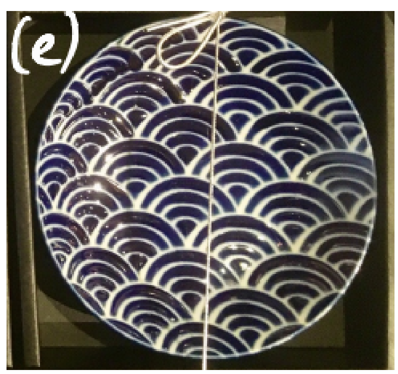


p31m: There is a 120 degree reflection without 60 degree rotations, and there is a center of 120 degrees of rotation not lying on any axis of a reflection (as shown on the figure)

(4)

p4m: There is a center of 90 degrees rotation lying on an axis of reflection

(5)



cm: There is a reflection, and there exists an axis of glide reflection that is not an axis of any reflection.

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Problem 7

Problem Statement: Let $R \in Isom(\mathbb{R}^2)$ be a rotation of order n. Prove that $< R > \subset Isom(\mathbb{R}^2)$ contains a rotation by angle $\frac{360^o}{n}$

Solution. Let R be a rotation by α degrees. Then by definition,

$$n\alpha = k \cdot 360 \text{ degrees}$$

Then $\alpha = \frac{k}{n}360$. And so the composition of multiple rotations R will always be a multiple of $\frac{1}{n}360$ degrees. Assume that for all angles $\beta \in \{\alpha, 2\alpha, \cdots, n\alpha\}, \beta \not\equiv \frac{360}{n} \mod 360$. Then, by Pigeonhole theorem, there exists $1 \le i < j \le n$ such that $i\alpha \equiv j\alpha \mod 360$. Therefore, $(i-j)\alpha \equiv 0 \mod 360$, thus the order of R is at most $i - j < n$, a contradiction.
Therefore, the assumption is false, and we must have a rotation by angle $\frac{360^o}{n}$ ∎

Problem 8

(1) Let $R \in Isom(\mathbb{R}^2)$ be a rotation and let $T \in Isom(\mathbb{R}^2)$ be an arbitrary isometry. Show that a conjugate TRT^{-1} is also a rotation (by the same angle as R) and find its center.

(2) Let $L \subset \mathbb{R}^2$ be a lattice. If T is a translation that preserves L and $P \in \mathbb{R}^2$ is a rotation point (see lectures) then $T(P)$ is also a rotation point.

Solution. (1) Let T be an arbitrary isometry in $Isom(\mathbb{R}^2)$, consider 4 cases:

- If T is a rotation, then since rotations are commutative, $TRT^{-1} = TT^{-1}R = R$
- If T is a translation, then by the analysis in class, we have TRT^{-1} is a rotation about the point $Q = T(O)$.
- If T is a reflection, then TR is a reflection, and TRT^{-1} is a composition of reflections along non-parallel lines with the same angle as R. Therefore, TRT^{-1} is a rotation with the same angle as R
- If T is a glide reflection, then $T = STr$, where S is a reflection and Tr is a translation. Then, $TRT^{-1} = (STr)R(STr)^{-1} = S(TrRTr^{-1})S^{-1} \approx SRS \approx R$ (As proven above)

(2) Let P be the rotation piont of a rotation R. Then $R(P) = P$, thus $T(P) = T(R(P)) = T \circ R(P) = TRT^{-1}T(P) = RT(P) = R(T(P))$ is also a rotation point.

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Problem 9

Problem Statement: Let G be a finite group.

(1) Show that the size of every conjugacy class of G is a divisor of $|G|$

(2) Prove that either G is Abelian or the number of conjugacy classes of G is at most equal to $\frac{3}{4}|G|$

Solution. (1) Define the action $G \curvearrowright G$ with the operation conjugate on G:

$$g \star x = gxg^{-1}$$

Then the orbit of each element $x \in G$ is the conjugacy class of x. Thus, for all x,

$$|Orb(x)| = \frac{|G|}{|G_x|}$$

thus is a divisor of $|G|$.

(2) By the Burnside lemma, we have the number of conjugacy classes equals:

$$\#Orbit = \frac{1}{|G|} \sum_{g \in G} |Fix(g)|$$

Notice that $x \in Fix(g) \iff g \in Fix(x)$. Thus, we compute the number of pairs of elements in G that commutes.

The total number of ordered pairs in $|G|$ is $|G|^2$. We will show that there are at least $|G|^2/4$ pairs that are not commute, or G is Abelian.

Indeed, there is at least one element x which does not commute with all other elements. Therefore, $|G_x| < |G|$, but $|G_x| \mid |G|$ so $|G_x| \leq |G|/2$. Thus, there are at least $|G|/2$ elements which also does not commute with x, thus the size of their stabilizers are also at most $|G|/2$.

By this, we have shown that there are at least: $|G|/2 \cdot |G|/2 = |G|^2/4$ ordered pairs that does not commute in $G \times G$. Thus,

$$\sum_{g \in G} |Fix(g)| \leq |G|^2 - |G|^2/4 = \frac{3}{4}|G|^2$$

or

$$\#Orbit \leq \frac{3}{4}|G|$$

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Problem 10

Problem Statement: Let G be a p-group acting on a finite set X. Prove that the number of fixed points in X (elements $x \in X$ such that $g \star x = x$ for every $g \in G$) is congruent to $|X|$ modulo p

Solution. Let the set of fixed points in X be Z. Then $\forall x \in Z, Orb(x) = \{x\}$. Denote the remaining orbits as $c_1, c_2, \cdots, c_n$. Then we have:

$$|X| = |Z| + \sum_{i=1}^{n} |c_i| \tag{1}$$

For every c_i, by the Orbit-Stabilizer theorem, we have:

$$\begin{aligned} |c_i| &= \frac{|G|}{|G_x|}, \text{ for some } x \in c_i \\ &= \frac{p^k}{|G_x|} \end{aligned}$$

Since $x \notin Z$, $|G_x| < |G|$, thus, $|c_i| = p^l$ for some $1 \leq l \leq k$. In other words, for all c_i, we have $p \mid |c_i|$. Then by equation (1), we have $|X| \equiv |Z| \mod p$, or the number of fixed points in X is congruent to |X| modulo p. ■