**MATH 411: INTRODUCTION TO ABSTRACT ALGEBRA**
**HOMEWORK #6**

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# Problem 1

**Problem Statement:** Let $G$ be a group. Recall that elements $a, b \in G$ are called conjugate, notation $a \sim b$ if $b = gag^{-1}$ for some $g \in G$.

(1) Show that $\sim$ is an equivalence relation by checking 3 axioms (reflexivity, symmetry, transitivity). Equivalence classes for $\sim$ are called **conjugacy classes**.

(2) Describe all conjugacy classes in $S_5$. For each conjugacy class, compute the number of its elements.

(3) Repeat part (b) for the group $A_4$

(4) Repeat part (b) for the group $D_4$

*Solution.* (1) We show that $\sim$ is an equivalence class by checking the 3 axioms:

- *Reflexivity:* $b = ebe^{-1}$, therefore $b \sim b$
- *Transitivity:* $a \sim b, b \sim c$ then $a = gbg^{-1} = ghch^{-1}g^{-1}$. But then observe that $ghh^{-1}g^{-1} = e$. Thus, $h^{-1}g^{-1} = (gh)^{-1}$, and $a = (gh)c(gh)^{-1}$, so $a \sim c$
- *Symmetry:* $a \sim b$, then $a = gbg^{-1}$. Left multiply both sides by $g^{-1}$ and right multiply both sides by $g$, we get: $g^{-1}ag = b$, or $b \sim a$

Since all 3 axioms are satisfied, $\sim$ is an equivalence relation.

(2) As we have proven in class, 2 permutations are conjugate if and only if they have the same cycle structures. In other words, they must partition $S_5$ into disjoint cycles of equal length. Therefore, the conjugacy classes of $S_5$ can be described by the different ways to partition 5 elements into cycles, which are:

- $5 = 5$. Then there is only one cycle of length 5 in this partition, and there are 24 ways to arrange this cycle, for a total of 24 permutations in this conjugacy class
- $5 = 4+1$. Then there are 5 ways to choose the 4 elements in the cycle, and each one has 6 orientations. Thus this conjugacy class has 30 permutations.

- 5 = 3+2. Then there are $\binom{5}{3} = 10$ ways to choose 3 elements, each of the combination has 2 orientation (for instance (123) and (132)). The remaining 2 elements are automatically determined and also has only one orientation. Thus, this conjugacy class has 20 permutations
- 5 = 3+1+1. Similarly, there are 20 permutation in this conjugacy class
- 5 = 2+2+1. There are 15 ways to select 2 pairs of transpositions, each has only one orientation. Thus, this conjugacy class has 15 permutations
- 5 = 2+1+1+1. There are $\binom{5}{2} = 10$ ways to choose the transposition, thus this conjugacy class has 10 permutations
- 5 = 1+1+1+1+1. There is only 1 element in this conjugacy class, which is the identity permutation.

(3) Note that each cycle of length $n$ can be rewritten as a product of $n-1$ transpositions. Since the sum of the lengths of all cycles in $A_4$ is 4, then the number of cycles must be even (otherwise there will be an odd (4 − #number of cycles) number of transpositions). Thus, there are either 2, or 4 disjoint cycles.

- 4 = 3 + 1. Then there are 4 ways to choose the three elements of the cycles, and 2 ways to order them on the cycle, for a total of 8 permutations.
- 4 = 2 + 2. Then there are 3 ways to choose the 2 pairs, and only one way to order each pair's cycle, for a total of 3 permutations
- 4 = 1+ 1+ 1+ 1. Then this is the identity permutation, and there is only 1 permutation in this conjugacy class.

(4) We will find the conjugacy class for each element in $D_4$.
Apparently, the conjugacy class for $e$ is $\{e\}$ (1)
Let $R^n$ be any rotation.
When we conjugate $R^n$ by $R^m$, then we get $R^m R^n R^{-m} = R^n$, and when we conjugate $R^n$ by some reflection $S$ we get $SR^nS^{-1} = R^{-n}$ (which we will prove in Problem 5). Hence, for any rotation $R^n$, its conjugacy class is $\{R^n, R^{-n}\}$ (2)
Let $S$ be any reflection in $D_n$. Then if we take the conjugacy of $S$ by some $R^n$ we get $R^nSR^{-n} = R^nSR^{-n}(S^{-1}S) = R^n(SR^{-n-1})S = R^nR^nS = R^{2n}S$. Meanwhile, if we take the conjugacy of $S$ by some other reflection $SR^n$, then we get $SR^nS(SR^n)^{-1} = SR^nSSR^{-n} = S$. Because we have to take the rotations mod 4, for each reflection $S$, its conjugacy class is $\{S, SR^2\}$. (3)
From (1), (2), (3), the conjugacy classes of $D_4$ are: $\{\{e\}, \{R, R^3\}, \{R^2\}, \{S, SR^2\}, \{SR, SR^3\}\}$

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# Problem 2

**Problem Statement:** Find (with proof) all normal subgroups in (a) $S_3$; (b) $S_4$; (c) $A_4$; (d) $D_4$

*Solution.* (1) All conjugacy classes in $S_3$ are:

$$\{e\}, \{(12), (23), (13)\}, \{(123), (132)\}$$

Since $|S_3| = 6$, for $H$ to be a subgroup of $S_3$, $H$ must either have order 1,2,3 or 6. For $H$ to be a normal subgroup, $H$ must be the union of some conjugacy classes of $S_3$. Finally, $H$ must contain the indentity element (the identity conjugacy class must be included).

- If $|H| = 1$, then $H = \{e\}$, valid
- If $|H| = 2$, then there is no union of conjugacy classes with 2 elements, one of which is the identity.
- If $|H| = 3$, then $H = \{e, (123), (132)\}$, is a valid subgroup
- If $|H| = 6$, then $H = S_3$

Conclusion: all normal subgroups of $S_3$ are:

$$\{e\}, A_3, S_3$$

(2) All conjugacy classes in $S_4$ are:

$$\{e\}, \{(12), (13), (14), (23), (24), (34)\}, \{(123), (132), (124), (142), (134), (143), (234), (243)\},$$
$$\{(12)(34), (13)(24), (14)(23)\}, \{(1234), (1243), (1324), (1342), (1423), (1432)\}$$

Since $|S_4| = 24$, for $H$ to be a subgroup $H$ must either have order $1, 2, 3, 4, 6, 8, 12$, or 24. For $H$ to be a normal subgroup, $H$ must be the union of some conjugacy classes of $S_4$. H must also contain the identity element

- |H| = 1, then $H = \{e\}$
- |H| = 2, then there is no union of conjugacy classes with 2 elements.
- |H| =3, then there is no union of conjugacy classes with 3 elements, one of which is the identity
- |H| =4, then $H = \{e, (12)(34), (13)(24), (14)(23)\}$
- |H| = 6, then there is no union of conjugacy classes with 6 elements, one of which is the identity
- |H| = 8,then there is no union of conjugacy classes with 8 elements, one of which is the identity
- |H| = 12, then $H = \{e, (123), (132), (124), (142), (134), (143), (234), (243), (12)(34), (13)(24), (14)(23)\} = A_4$ is a valid subgroup of $S_4$
- |H| = 24, then $H = S_4$.

Conclusion, all normal subgroups of $S_4$ are:

$$\{e\}, \{e, (12)(34), (13)(24), (14)(23)\}, A_4, S_4$$

(3) All conjugacy classes in $A_4$ are:

$$\{e\},\{(12)(34),(13)(24),(14)(23)\},\{(123),(132),(124),(142),(134),(143),(234),(243)\}$$

Since $|A_4| = 12$, for $H$ to be a subgroup $H$ must either have order $1,2,3,4,6$, or 12. For $H$ to be a normal subgroup, $H$ must be the union of some conjugacy classes of $A_4$. H must also contain the identity element.

- |H| = 1 then $H = \{e\}$
- |H| = 2 then there is no union of conjugacy classes with 2 elements.
- |H| = 3 then there is no union of conjugacy classes with 3 elements, one of which is the identity
- |H| =4, then $H = \{e,(12)(34),(13)(24),(14)(23)\}$
- |H| = 6, then there is no union of conjugacy classes with 6 elements.
- |H| = 12 then $H = A_4$

Conclusion: all normal subgroups of $A_4$ are:

$$\{e\},\{e,(12)(34),(13)(24),(14)(23)\},A_4$$

(4) From Problem 1, we know that the conjugacy classes of $D_4$ are:

$$\{\{e\},\{R,R^3\},\{R^2\},\{S,SR^2\},\{SR,SR^3\}\}$$

Since $|D_4| = 8$, for $H$ to be a subgroup $H$ must either have order $1,2,4$, or 8. For $H$ to be a normal subgroup, $H$ must be the union of some conjugacy classes of $D_4$. H must also contain the identity element.

- If |H| = 1, then $H = \{e\}$
- If |H| = 2, then $H = \{e,R^2\}$
- If |H| = 4, then $H = \{e,R^2,R,R^3\}$ or $H = \{e,R^2,S,SR^2\}$ or $H = \{e,R^2,SR,SR^3\}$
- If |H| = 8, then $H = D_4$

Conclusion, all normal subgroups of $D_4$ are:

$$\{e\},\{e,R^2\},\{e,R^2,R,R^3\},\{e,R^2,S,SR^2\},\{e,R^2,SR,SR^3\},D_4$$

$\blacksquare$

# Problem 3

**Problem Statement:** Let $H \subset G$ be a subgroup of a finite group.

(1) Prove that the number $|G:H| := \frac{|G|}{|H|}$ is an integer. It is called the index of $H$

(2) Prove that every subgroup of index 2 is normal by proving that left cosets and right cosets are the same.

*Solution.* (1) By the Lagrange's Theorem, if $H \subset G$ is a subgroup of $G$, then $|G|$ is divisible by $|H|$. Therefore, $\frac{|G|}{|H|}$ is an integer.

(2) Since the left cosets of $H$ are either equal or disjoint, they partition $G$ into 2 sets, namely $H$ and $G \setminus H$.
Similarly, the right cosets of $H$ partition $G$ into 2 sets, namely $H$ and $G \setminus H$.
Therefore, the left cosets $G \setminus H$ and the right cosets $H \setminus G$ are the same.
Thus, every subgroup $H$ of index 2 is normal.

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# Problem 4

**Problem Statement:** For the following pairs $H \subset G$ of a group and its subgroup, list all left cosets of $H$ and right cosets of $H$. Decide if $H$ is normal or not.

(1) $H =< 7 >\subset \mathbb{Z}_{21} = G$

(2) $H =< 3 >\subset \mathbb{Z}_{13}^{*} = G$

(3) $H =< (12),(23) >\subset S_4 = G$

(4) $H = D_4 \subset S_4 = G$

*Solution.* (1) $H = < 7 >= \{0,7,14\}$.

Then, the left cosets and right cosets of $H$ are according to the following table:

| # mod 7 | Left cosets | Right cosets |
|---|---|---|
| 0 | $0+7Z \mod 21 = \{0,7,14\}$ | $7Z+0 \mod 21\{0,7,14\}$ |
| 1 | $1+7Z \mod 21 = \{1,8,15\}$ | $7Z+1 \mod 21\{1,8,15\}$ |
| 2 | $2+7Z \mod 21 = \{2,9,16\}$ | $7Z+2 \mod 21\{2,9,16\}$ |
| 3 | $3+7Z \mod 21 = \{3,10,17\}$ | $7Z+3 \mod 21\{3,10,17\}$ |
| 4 | $4+7Z \mod 21 = \{4,11,18\}$ | $7Z+4 \mod 21\{4,11,18\}$ |
| 5 | $5+7Z \mod 21 = \{5,12,19\}$ | $7Z+5 \mod 21\{5,12,19\}$ |
| 6 | $6+7Z \mod 21 = \{6,13,20\}$ | $7Z+6 \mod 21\{6,13,20\}$ |

Since the left cosets and right cosets of $H$ are the same, $H$ is a normal subgroup of $G$

(2) The subgroup generated by 3 in $\mathbb{Z}_{13}$ is: $H =< 3 >= \{3,9,1\}$

The left cosets and right cosets are according to the following table:

| $g$ mod 13 | Left cosets | Right cosets |
|---|---|---|
| 1 | 1H = H = {3,9,1} | H1 = {3,9,1} |
| 2 | 2H = {6,5, 2} | H2 = {6,5,2} |
| 4 | 4H = {12, 10, 4} | H4 = {12, 10 , 4} |
| 7 | 7H = {8, 11, 7} | H7 = {8, 11, 7} |

Note that $g$ is the smallest number that is not yet included in the left cosets, and right cosets of $H$. Since we can see that the left cosets and the right cosets of $H$ are the same, $H$ is a normal subgroup of $G$.

*Note:* We can also conclude quickly that $H$ is since $G$ is an abelian group, and every element in $H$ commutes with every elements in the group $G$.

(3) The subgroup generated by $\{(12),(23)\}$ is $H =<(12),(23)>= \{Id,(12),(23),(132),(123),(13)\}$
Then the left cosets and right cosets of $H$ are according to the following table:

| Left cosets | Right cosets |
|---|---|
| H = { Id, (23), (12), (123), (132), (13)} | H = { Id, (23), (12), (123), (132), (13)} |
| (34)H = { (34), (234), (12)(34), (1234), (1342), (134)} | H(34) = {(34), (243), (12)(34), (1243), (1432), (143)} |
| (243)H = { (243), (24), (1243), (124), (13)(24), (1324)}\| | H(234) = { (234), (24), (1342), (13)(24), (142), (1423)} |
| (1432)H = { (1432), (142), (143), (14), (1423), (14)(23)} | H(1234) = { (1234), (124), (134), (1324), (14), (14)(23)} |

It can be seen that in the left cosets, (1234) and (14) are in different sets, but (1234) and (14) are in the same right cosets. Therefore, the left cosets and right cosets of $H$ are not the same, thus $H$ is not normal.
*Note:* Another way to check this is to observe that $H$ does not contain the entire conjugacy class of (12) (for example, it does not contain (14)). Thus, $H$ is not normal.

(4) The left cosets and right cosets of $D_4$ in $S_4$ are:

| Left cosets | Right cosets |
|---|---|
| $D_4 : \{i,(1234),(13)(24),(1432),(12)(34),(14)(23),(13),(24)\}$ | $D_4 : \{i,(1234),(13)(24),(1432),(12)(34),(14)(23),(13),(24)\}$ |
| $(12)D_4 : \{(12),(234),(2413),(143),(34),(1423),(132),(124)\}$ | $D_4(12) : \{(12),(243),(1423),(134),(34),(1324),(123),(142)\}$ |
| $(14)D_4 : \{(14),(123),(1342),(243),(1243),(23),(134),(142)\}.$ | $D_4(14) : \{(14),(23),(132),(124),(143),(234),(1342),(1243)\}$ |

And since (14) and (123) are in the same left coset but not the right cosets, $D_4$ is not a normal subgroup of $S_4$

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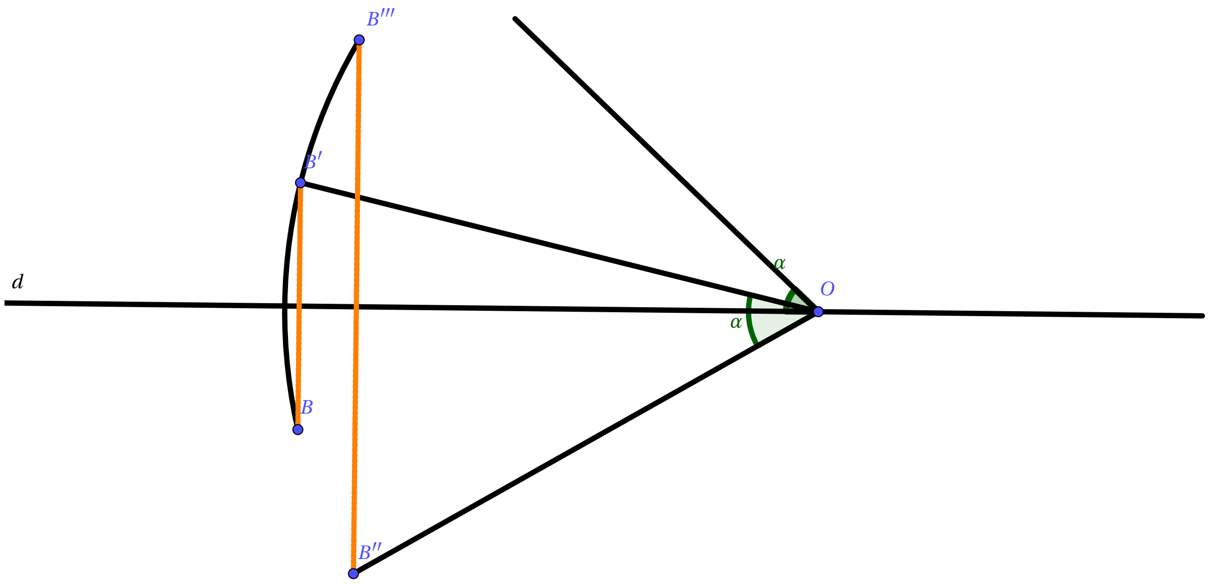
# Problem 5

**Problem Statement:**

(1) Let $S$ be a reflection in a line passing through a point $O \in \mathbb{R}^2$ and let $R$ be a rotation about $O$ by $\alpha^o$ counterclockwise. Compute (with proof) $SRS^{-1}$ explicitly: what kind of a transformation of $\mathbb{R}^2$ is it?

(2) Compute with proof the center of $D_n$. Notice that the cases of odd and even $n$ will be different

(3) Compute (with proof) the center of $GL_2(\mathbb{R})$




*Solution.*

(1) Let $B$ be an arbitrary point on the plane, and let $d$ be the line of reflection in $S$, $O$ be the center of rotation in $R$. Then, $S^{-1}(B) = B', R(B') = B'', S(B'') = B'''$. Since $B$ and $B'$ are symmetric through $d$ and $B''$ and $B'''$ are symmetric through $d$, we have:

$$\angle BOB''' = \angle BOd + \angle dOB''' = \angle dOB' + \angle B''Od = \angle B''OB' = -\angle B'OB'' = -\alpha$$

Since $\angle BOB''' = -\alpha$ constant, in the reverse direction of $R$, and $OB = OB' = OB'' = OB'''$, $SRS^{-1}$ equals the rotation about $O$ by $\alpha$ clockwise.

(2) Assume that $X$ is an element in $Z(D_n)$. Then for all other $Y$ in $D_n$, $XY = YX$, or $XYX^{-1} = X$. (1)
Apparently, if $X$ is the identity transformation, then $X \in Z(D_n)$ (2)

Now if $X$ is a reflection $S$, then let $Y$ be any rotation by $\alpha^o$ counterclockwise. Then from part (a), we know that $XYX = Y^{-1}$, or the rotation by $\alpha^o$ clockwise, is not a reflection. Therefore, (1) is not

satisfied, and $X$ cannot be a reflection. (3)
If $X$ is a rotation $R$, then we consider the other way to represent $X$: $XY = YX \Longrightarrow X = YXY^{-1}$.
If $Y$ is a rotation, then it is easy to see that $YXY^{-1}$ is also a rotation and it is equal to $X$.
Else, if $Y$ is a equal to a reflection $S$, then $X = SXS^{-1}$. From (a), we know that $SXS^{-1} = X^{-1}$, so $X = X^{-1}$.
Therefore, an element is in $Z(D_n)$ if and only if it is a rotation and it is the inverse of itself. This means $R$ can only be the rotation by $180^o$ and it only happens when $n$ is even. (4)
From (1), (2), (3), (4), we conclude that:

$$Z(D_n) = \begin{cases} \{Id\} \text{ if } n \text{ is odd} \\ \{Id, R_{n/2}\} \text{ if } n \text{ is even} \end{cases}$$

(3) Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an arbitrary $2 \times 2$ matrix in the center of $GL_2(\mathbb{R})$, and $\begin{bmatrix} e & f \\ g & h \end{bmatrix}$ be an arbitrary matrix in $GL_2(\mathbb{R})$ Then,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$\Longleftrightarrow \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix} = \begin{bmatrix} ea+fc & eb+fd \\ ga+hc & gb+hd \end{bmatrix}$$

This implies the following system of equations:

$$\begin{cases} bg = cf \\ af+bh = eb+fd \\ ce+dg = ga+hc \end{cases}$$
$$\Longleftrightarrow \begin{cases} bg = cf \\ f(a-d) = b(e-h) \\ c(e-h) = g(a-d) \end{cases}$$

Then if $b \neq 0$, from the first equation we have $g/f = c/b, \forall g, f \neq 0$, which is not possible since we choose $g, f$ arbitrarily. Therefore $b = 0$. Similarly, $c = 0$. Apply this to the second equation, we have $a - d = 0$, or $a = d$.
To sum up, all matrices in $Z(GL_2(\mathbb{R}))$ must be of the form $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
Reveresedly, every matrices of the form $aId$, where $a$ is a scalar and $Id$ is the identity matrix is apparently commutative with every other matrices in $GL_2(\mathbb{R})$.
Hence, $Z(GL_2(\mathbb{R})) = \{\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} | a \in \mathbb{R}\}$

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# Problem 6

**Problem Statement:** For every normal subgroup found in Problems 2 and 4,

(1) Write a multiplication table of the quotient group $G/H$

(2) Is $G/H$ cyclic in all these cases?

*Solution.* (1) (a) Problem (2a):

| H = {e} | eH | (12)H | (23)H | (13)H | (123)H | (132)H |
|---|---|---|---|---|---|---|
| eH | eH | (12)H | (23)H | (13)H | (123)H | (132)H |
| (12)H | (12)H | eH | (123)H | (132)H | (23)H | (13)H |
| (23)H | (23)H | (132)H | eH | (123)H | (13)H | (12)H |
| (13)H | (13)H | (123)H | (132)H | eH | (12)H | (23)H |
| (123)H | (123)H | (13)H | (12)H | (23)H | (132)H | eH |
| (132)H | (132)H | (23)H | (13)H | (12)H | eH | (123)H |

| H = $A_3$ | H | (12)H |
|---|---|---|
| H | H | (12)H |
| (12)H | (12)H | H |

| H = $S_3$ | H |
|---|---|
| H | H |

(b) Problem (2b): When $H = \{e\}$ the multiplication table of $S_4/H$ is the same as the multiplication table of $S_4$ and is:

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 |
| 2 | 2 | 1 | 4 | 3 | 6 | 5 | 8 | 7 | 10 | 9 | 12 | 11 | 14 | 13 | 16 | 15 | 18 | 17 | 20 | 19 | 22 | 21 | 24 | 23 |
| 3 | 3 | 5 | 1 | 6 | 2 | 4 | 9 | 11 | 7 | 12 | 8 | 10 | 15 | 17 | 13 | 18 | 14 | 16 | 21 | 23 | 19 | 24 | 20 | 22 |
| 4 | 4 | 6 | 2 | 5 | 1 | 3 | 10 | 12 | 8 | 11 | 7 | 9 | 16 | 18 | 14 | 17 | 13 | 15 | 22 | 24 | 20 | 23 | 19 | 21 |
| 5 | 5 | 3 | 6 | 1 | 4 | 2 | 11 | 9 | 12 | 7 | 10 | 8 | 17 | 15 | 18 | 13 | 16 | 14 | 23 | 21 | 24 | 19 | 22 | 20 |
| 6 | 6 | 4 | 5 | 2 | 3 | 1 | 12 | 10 | 11 | 8 | 9 | 7 | 18 | 16 | 17 | 14 | 15 | 13 | 24 | 22 | 23 | 20 | 21 | 19 |
| 7 | 7 | 8 | 13 | 14 | 19 | 20 | 1 | 2 | 15 | 16 | 21 | 22 | 3 | 4 | 9 | 10 | 23 | 24 | 5 | 6 | 11 | 12 | 17 | 18 |
| 8 | 8 | 7 | 14 | 13 | 20 | 19 | 2 | 1 | 16 | 15 | 22 | 21 | 4 | 3 | 10 | 9 | 24 | 23 | 6 | 5 | 12 | 11 | 18 | 17 |
| 9 | 9 | 11 | 15 | 17 | 21 | 23 | 3 | 5 | 13 | 18 | 19 | 24 | 1 | 6 | 7 | 12 | 20 | 22 | 2 | 4 | 8 | 10 | 14 | 16 |
| 10 | 10 | 12 | 16 | 18 | 22 | 24 | 4 | 6 | 14 | 17 | 20 | 23 | 2 | 5 | 8 | 11 | 19 | 21 | 1 | 3 | 7 | 9 | 13 | 15 |
| 11 | 11 | 9 | 17 | 15 | 23 | 21 | 5 | 3 | 18 | 13 | 24 | 19 | 6 | 1 | 12 | 7 | 22 | 20 | 4 | 2 | 10 | 8 | 16 | 14 |
| 12 | 12 | 10 | 18 | 16 | 24 | 22 | 6 | 4 | 17 | 14 | 23 | 20 | 5 | 2 | 11 | 8 | 21 | 19 | 3 | 1 | 9 | 7 | 15 | 13 |
| 13 | 13 | 19 | 7 | 20 | 8 | 14 | 15 | 21 | 1 | 22 | 2 | 16 | 9 | 23 | 3 | 24 | 4 | 10 | 11 | 17 | 5 | 18 | 6 | 12 |
| 14 | 14 | 20 | 8 | 19 | 7 | 13 | 16 | 22 | 2 | 21 | 1 | 15 | 10 | 24 | 4 | 23 | 3 | 9 | 12 | 18 | 6 | 17 | 5 | 11 |
| 15 | 15 | 21 | 9 | 23 | 11 | 17 | 13 | 19 | 3 | 24 | 5 | 18 | 7 | 20 | 1 | 22 | 6 | 12 | 8 | 14 | 2 | 16 | 4 | 10 |
| 16 | 16 | 22 | 10 | 24 | 12 | 18 | 14 | 20 | 4 | 23 | 6 | 17 | 8 | 19 | 2 | 21 | 5 | 11 | 7 | 13 | 1 | 15 | 3 | 9 |
| 17 | 17 | 23 | 11 | 21 | 9 | 15 | 18 | 24 | 5 | 19 | 3 | 13 | 12 | 22 | 6 | 20 | 1 | 7 | 10 | 16 | 4 | 14 | 2 | 8 |
| 18 | 18 | 24 | 12 | 22 | 10 | 16 | 17 | 23 | 6 | 20 | 4 | 14 | 11 | 21 | 5 | 19 | 2 | 8 | 9 | 15 | 3 | 13 | 1 | 7 |
| 19 | 19 | 13 | 20 | 7 | 14 | 8 | 21 | 15 | 22 | 1 | 16 | 2 | 23 | 9 | 24 | 3 | 10 | 4 | 17 | 11 | 18 | 5 | 12 | 6 |
| 20 | 20 | 14 | 19 | 8 | 13 | 7 | 22 | 16 | 21 | 2 | 15 | 1 | 24 | 10 | 23 | 4 | 9 | 3 | 18 | 12 | 17 | 6 | 11 | 5 |
| 21 | 21 | 15 | 23 | 9 | 17 | 11 | 19 | 13 | 24 | 3 | 18 | 5 | 20 | 7 | 22 | 1 | 12 | 6 | 14 | 8 | 16 | 2 | 10 | 4 |
| 22 | 22 | 16 | 24 | 10 | 18 | 12 | 20 | 14 | 23 | 4 | 17 | 6 | 19 | 8 | 21 | 2 | 11 | 5 | 13 | 7 | 15 | 1 | 9 | 3 |
| 23 | 23 | 17 | 21 | 11 | 15 | 9 | 24 | 18 | 19 | 5 | 13 | 3 | 22 | 12 | 20 | 6 | 7 | 1 | 16 | 10 | 14 | 4 | 8 | 2 |
| 24 | 24 | 18 | 22 | 12 | 16 | 10 | 23 | 17 | 20 | 6 | 14 | 4 | 21 | 11 | 19 | 5 | 8 | 2 | 15 | 9 | 13 | 3 | 7 | 1 |

When $H = \{e, (12)(34), (13)(24), (14)(23)\}$ then the multiplication table of $S_4/H$ is:

| $S_4/H$ | $H$ | $(12)H$ | $(13)H$ | $(14)H$ | $(23)H$ | $(24)H$ |
|---|---|---|---|---|---|---|
| $H$ | $H$ | $(12)H$ | $(13)H$ | $(14)H$ | $(23)H$ | $(24)H$ |
| $(12)H$ | $(12)H$ | $H$ | $(14)H$ | $(13)H$ | $(24)H$ | $(23)H$ |
| $(13)H$ | $(13)H$ | $(24)H$ | $H$ | $(14)H$ | $(12)H$ | $(34)H$ |
| $(14)H$ | $(14)H$ | $(23)H$ | $(34)H$ | $H$ | $(13)H$ | $(12)H$ |
| $(23)H$ | $(23)H$ | $(14)H$ | $(12)H$ | $(34)H$ | $H$ | $(13)H$ |
| $(24)H$ | $(24)H$ | $(13)H$ | $(34)H$ | $(12)H$ | $(14)H$ | $H$ |

When $H = A_4$ and $S_4$, the multiplication table of $S_4/H$ is:

| H = $A_4$ | H | (12)H |
|---|---|---|
| H | H | (12)H |
| (12)H | (12)H | H |

| H = $S_4$ | H |
|---|---|
| H | H |

(c) Problem (2c): When $H = \{e\}$, then the multiplication table of $A_4/H$ is the same as the multiplication table of $A_4$:

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| 2 | 2 | 3 | 1 | 6 | 4 | 5 | 8 | 9 | 7 | 12 | 10 | 11 |
| 3 | 3 | 1 | 2 | 5 | 6 | 4 | 9 | 7 | 8 | 11 | 12 | 10 |
| 4 | 4 | 7 | 10 | 1 | 8 | 11 | 2 | 5 | 12 | 3 | 6 | 9 |
| 5 | 5 | 9 | 11 | 3 | 7 | 12 | 1 | 6 | 10 | 2 | 4 | 8 |
| 6 | 6 | 8 | 12 | 2 | 9 | 10 | 3 | 4 | 11 | 1 | 5 | 7 |
| 7 | 7 | 10 | 4 | 11 | 1 | 8 | 5 | 12 | 2 | 9 | 3 | 6 |
| 8 | 8 | 12 | 6 | 10 | 2 | 9 | 4 | 11 | 3 | 7 | 1 | 5 |
| 9 | 9 | 11 | 5 | 12 | 3 | 7 | 6 | 10 | 1 | 8 | 2 | 4 |
| 10 | 10 | 4 | 7 | 8 | 11 | 1 | 12 | 2 | 5 | 6 | 9 | 3 |
| 11 | 11 | 5 | 9 | 7 | 12 | 3 | 10 | 1 | 6 | 4 | 8 | 2 |
| 12 | 12 | 6 | 8 | 9 | 10 | 2 | 11 | 3 | 4 | 5 | 7 | 1 |

When $H = \{Id, (12)(34), (13)(24), (14)(23)\}$

| | $H$ | $(123)H$ | $(132)H$ |
|---|---|---|---|
| $H$ | $H$ | $(123)H$ | $(132)H$ |
| $(123)H$ | $(123)H$ | $(132)H$ | $H$ |
| $(132)H$ | $(132)H$ | $H$ | $(123)H$ |

When $H = A_4$, then the multiplication table is:

| H = $A_4$ | H |
|---|---|
| H | H |

(d) Problem (2d)
When $H = \{e\}$, the multiplication table of $D_4/H$ is the same as the multiplication table of $D_4$:

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 2 | 2 | 1 | 4 | 3 | 6 | 5 | 8 | 7 |
| 3 | 3 | 7 | 1 | 5 | 4 | 8 | 2 | 6 |
| 4 | 4 | 8 | 2 | 6 | 3 | 7 | 1 | 5 |
| 5 | 5 | 6 | 7 | 8 | 1 | 2 | 3 | 4 |
| 6 | 6 | 5 | 8 | 7 | 2 | 1 | 4 | 3 |
| 7 | 7 | 3 | 5 | 1 | 8 | 4 | 6 | 2 |
| 8 | 8 | 4 | 6 | 2 | 7 | 3 | 5 | 1 |

When $H = \{e, R^2\}$, the multiplication table is:

| $D_4/H$ | $H$ | $R^1H$ | $S_1H$ | $S_2H$ |
|---|---|---|---|---|
| $H$ | $H$ | $R^2H$ | $S_1H$ | $S_2H$ |
| $R^1H$ | $R^2H$ | $H$ | $S_2H$ | $S_1H$ |
| $S_1H$ | $S_1H$ | $S_2H$ | $H$ | $R^2H$ |
| $S_2H$ | $S_2H$ | $S_1H$ | $R^2H$ | $H$ |

When $H = \{e, R^2, S, SR^2\}$ or $H = \{e, R^2, SR, SR^3\}$ or $H = D_4$:

| $H = \{e, R^2, S, SR^2\}$ | H | (SR)H |
|---|---|---|
| H | H | (SR)H |
| (SR)H | (SR)H | H |

| $H = \{e, R^2, SR, SR^3\}$ | H | SH |
|---|---|---|
| H | H | SH |
| SH | SH | H |

| $H = D_4$ | H |
|---|---|
| H | H |

(e) Problem (4a):
The multiplication table of $G/H$ is the same as the multiplication table of the group $\mathbb{Z}_7$:

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 2 | 2 | 3 | 4 | 5 | 6 | 7 | 1 |
| 3 | 3 | 4 | 5 | 6 | 7 | 1 | 2 |
| 4 | 4 | 5 | 6 | 7 | 1 | 2 | 3 |
| 5 | 5 | 6 | 7 | 1 | 2 | 3 | 4 |
| 6 | 6 | 7 | 1 | 2 | 3 | 4 | 5 |
| 7 | 7 | 1 | 2 | 3 | 4 | 5 | 6 |

This table generated in Wolfram Mathematica returned elements from 1 to 7 instead of from 0 to 6. To obtain the final value, we can deduct 1 from each element in the table.

(f) Problem (4b):
The multiplication table of $G/H$ is:

| | H | 2H | 4H | 7H |
|---|---|---|---|---|
| H | H | 2H | 4H | 7H |
| 2H | 2H | 4H | 7H | H |
| 4H | 4H | 7H | H | 2H |
| 7H | 7H | H | 2H | 4H |

(2) $G/H$ is not cyclic in some of the cases. For instance, when $H = \{e\}$, then $G/H$ is isomorphic to $G$. And since $S_4$ is not cyclic, $S_4/H$ is not cyclic.

■

# Problem 7

**Problem Statement:** For the following functions $f : G \to H$ between groups, decide (with proof) if $f$ is a homomorphism and if yes, describe its kernel $Ker(f)$:

(1) $f : \mathbb{Z} \to \mathbb{Z}, f(x) = 5x$

(2) $f : \mathbb{R}^* \to \mathbb{R}^*, f(x) = |x|$

(3) $f : \mathbb{R} \to \mathbb{R}^*, f(x) = e^x$

(4) $f : \mathbb{R} \to \mathbb{R}, f(x) = x^{43}$

(5) $f : \mathbb{Z}_{43} \to \mathbb{Z}_{43}, f(x) = x^{43}$

(6) $f : GL_n(\mathbb{R}) \to \mathbb{R}^*, f(x) = \det(x)$

*Solution.* (1) Let $a, b$ be arbitrary integers. Then $f(a+b) = 5(a+b) = 5a+5b = f(a)+f(b), \forall a, b \in \mathbb{Z}$. Thus, $f$ is a homomorphism. The identity of $\mathbb{Z}$ is 0, so an element is in $Ker(f)$ if and only if it is 0. Thus, $Ker(f) = \{0\}$

(2) Let $x, y$ be arbitrary real numbers. Then $f(xy) = |xy| = |x||y| = f(x)f(y)$. Thus, $f$ is a homomorphism. And $Ker(f) = \{1, -1\}$

(3) Let $a, b$ be arbitrary real numbers. Then $f(a+b) = e^{a+b} = e^a e^b = f(a)f(b)$. Then $f$ is a homomorphism. Since the idenity of $\mathbb{R}^*$ is 1, the kernel of $f$ contains all elements $y \in \mathbb{R}$ such that $e^y = 1$, or $y = 0$. Thus, $Ker(f) = \{0\}$

(4) Let $a, b$ be arbitrary real numbers. Then $f(a+b) = (a+b)^{43} \neq a^{43} + b^{43}$. Therefore,$f$ is not a homomorphism.

(5) Let $a, b$ be arbitrary integers in $\mathbb{Z}_{43}$. Since 43 is a prime number, using Fermat's little theorem, we have: $x^{43} \equiv x \mod 43$. Then,$f(a+b) = (a+b)^{43} \equiv a+b \equiv a^{43}+b^{43} = f(a)+f(b)$. Thus, $f$ is a homomorphism. The kernel of $f$ contains all elements such that $x^{43} \equiv 0 \mod 43$. Thus, $Ker(f) = \{0\}$

(6) Let $a, b$ be arbitrary matrices in $GL_n(\mathbb{R})$. Then since $\det(AB) = \det(A) \cdot \det(B)$, $f$ is a homomorphism. $Ker(f) = \{x : x \in GL_n(\mathbb{R}) \wedge \det(x) = 1\}$

■