MATH 411: INTRODUCTION TO ABSTRACT ALGEBRA
HOMEWORK #5

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Problem 1

Problem Statement: Transpositions $(12), (23), \cdots, (n-1, n)$ (in other words, transpositions of the form $(i, i+1)$) are called adjacent transpositions.

(1) Prove that every transposition (ab) can be written as a product of adjacent transpositions

(2) Prove that S_n has a generating set $\{(12), (23), \cdots, (n-1, n)\}$

Solution. (1) Without loss of generality, assume that $a < b$. Then we can write

$$(ab) = (a+1, a)(a+2, a+1)\cdots(b-1, b-2)(b-1, b)(b-2, b-1)\cdots(a+1, a+2)(a, a+1)$$

Indeed, this will map:

$$\begin{aligned} &a \to a+1 \to a+2 \to \cdots \to b-1 \to b \\ &b \to b-1 \to b-2 \to \cdots \to a+1 \to a \\ &a+i \to a+i+1 \to a+i \forall 1 \le i \le b-a-1 \end{aligned}$$

(2) Denote $G = \{(12), (23), \cdots, (n-1, n)\}$.
First, notice that $G \in S_n$, and the compositions of an arbitrary number of elements in G is closed in S_n (1)
Second, we will show that any element in S can be represented as a product of elements in G.
Indeed, we have shown in class that every permutation $\sigma \in S_n$ can be written as a product of transpositions $\sigma = \tau_1 \cdot \tau_2 \cdot \cdots \cdot \tau_k$
But according to part (1), any transposition τ_i can be written as a product of adjacent transpositions $\tau_i = \tau_{i_1}\tau_{i_2}\cdots$.
Therefore,

$$\sigma = (\tau_{1_1}\tau_{1_2}\cdots)(\tau_{2_1}\tau_{2_2}\cdots)(\tau_{k_1}\tau_{k_2}\cdots)$$

, which is a product of adjacent transpositions. (2)
From (1) and (2), we can conclude that G is a generating set of S_n

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Problem 2

Problem Statement:

(1) Prove that a product of 2 transpositions $(xy)(zt)$ in S_n can be written as a product of certain $3-$cycles (ijk) for every n.

(2) Prove that A_n has a generating set that consists of all $3-cycles$

Solution. (1) We consider the following 3 cases:
Case 1: $x = z, y = t$
Then $(xy)(zt) = Id$ or a product of zero 3 cycles
Case 2: $x = z, y \neq t$
Then $(xy)(zt) = (xy)(xt) = (xty)$
Case 3: $x \neq z, y \neq t$
Then $(xy)(zt) = (tzy)(xzy)$
Thus, all products of 2 transpositions $(xy)(zt)$ in S_n can be written as a product of certain 3-cycles (ijk) for every n

(2) Since A_n is the set of all even permutations, for $\sigma \in A_n$, we can write $\sigma = \tau_1\tau_2\cdots\tau_{2k} = (\tau_1\tau_2)(\tau_3\tau_4)\cdots(\tau_{2k-1}\tau_{2k})$, where τ_i are transpositions in S_n
And since each pair $(\tau_{2i-1}2i$ can be written as a product of certain 3 cycles in S_n, σ can be written as a product of 3 cycles. (1)
Second, every 3-cycle (ijk) is even, as $(ijk) = (ik)(ij)$. And the product of even permutations is also an even permutation.
Therefore, the set of all 3-cycles is closed under the binary operation in A_n. (2)
From (1) and (2), we conclude that A_n has a generating set that consists of all 3-cycles.
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Problem 3

Problem Statement: Find (with proof) a permutation in S_n with

(1) the smallest

(2) the largest

number of disorders (see lecture notes)

Solution. (1) The number of disorders must be non-negative. Therefore, since $D(Id) = 0$, and $\forall \sigma \in S_n, D(\sigma) \geq 0 = D(Id)$, the identity is a permutation with the smallest possible number of disorders.

(2) The number of disorders must not exceed the number of pairs, which is: $\binom{n}{2} = \frac{n(n-1)}{2}$ (1)
Consider $\rho = \begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ n & (n-1) & (n-2) & \cdots & 1 \end{bmatrix}$. Then $\forall i < j, \rho_i = n - i > n - j = \rho_j$. Therefore, every pair in ρ is a disorder.
In other words, $D(\rho) = \binom{n}{2} = \frac{n(n-1)}{2}$ (2).
From (1), and (2), we conclude that ρ is a permutation with the largest number of disorders in S_n

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Problem 4

Problem Statement: Let $\rho \in S_n$ and let τ be the transposition (ρ_i, ρ_j) for $i < j$. Let $\sigma = \tau\rho$. Consider the set X of all pairs $\{i,k\}$ and $\{k,j\}$ such that $i < k < j$. Prove that the subsets of disorders in X for ρ and for σ have the same parity.

Solution. Let ρ_i, ρ_j, ρ_k be the element at index i, j, k in ρ. We consider the following 4 cases of the relative value of ρ_k versus ρ_i, ρ_j

	{i,k}	{k,j}	Disorder Change
$\rho_k > \rho_i$ $\rho_k > \rho_j$	Order for ρ Order for σ	Disorder for ρ Disorder for σ	0
$\rho_k < \rho_i$ $\rho_k > \rho_j$	Disorder for ρ Order for σ	Disorder for ρ Order for σ	-2
$\rho_k > \rho_i$ $\rho_k < \rho_j$	Order for ρ Disorder for σ	Order for ρ Disorder for σ	+2
$\rho_k < \rho_i$ $\rho_k < \rho_j$	Disorder for ρ Disorder for σ	Order for ρ Order for σ	0

Because the disorder always change by an even amount, the subsets of disorders in X for ρ and σ have the same parity. ■

Problem 5

Problem Statement: Let $\rho \in S_n$ and let τ be the transposition (ρ_i, ρ_j) for $i < j$. Let $\sigma = \tau\rho$. Consider the set X of all pairs $\{i, k\}$ and $\{j, k\}$ such that $i < j < k$. Prove that the subsets of disorders in X for ρ and for σ have the same parity.

Solution. Let ρ_i, ρ_j, ρ_k be the element at index i, j, k in ρ. We consider the following 4 cases of the relative value of ρ_k versus ρ_i, ρ_j

	{i,k}	{k,j}	Disorder Change
$\rho_k > \rho_i$ $\rho_k > \rho_j$	Order for ρ Order for σ	Order for ρ Order for σ	0
$\rho_k < \rho_i$ $\rho_k > \rho_j$	Disorder for ρ Order for σ	Order for ρ Disorder for σ	0
$\rho_k > \rho_i$ $\rho_k < \rho_j$	Order for ρ Disorder for σ	Disorder for ρ Order for σ	0
$\rho_k < \rho_i$ $\rho_k < \rho_j$	Disorder for ρ Disorder for σ	Disorder for ρ Disorder for σ	0

Because the disorder always change by an even amount, the subsets of disorders in X for ρ and σ have the same parity. ■

Problem 6

Problem Statement: Use Mathematica code for the group activity and Problem 1 to give a computer-assisted proof that every permutation of 7 corner cubes of the $2 \times 2 \times 2$ Rubik's cube (other than the "fixed" corner cube) can be achieved by a sequence of face rotations. Note that this problem requires a combination of a theoretical argument and a computer code. Merge the PDF file of your Wolfram Cloud code and output with your homework file. In addition, provide a link to your Wolfram Cloud code.

Solution. Let the number "fixed" corner be one of the bottom corners. Label it as 0. Then we number the other corners on the bottom $0, 1, 2, 3$ counter-clockwise. Similarly, starting with the corner right above 0 as 4, we label the corners on the upper level $4, 5, 6, 7$ counter-clockwise.
So if we rotate the faces, we get the following cycles:

$$1 \to 5 \to 6 \to 2$$
$$4 \to 7 \to 6 \to 5$$
$$2 \to 6 \to 7 \to 3$$

Running the Mathematica on the order of the subgroup of S_7 generated by $(1562), (4765), (2673)$, we get 5040.
Since this is equal to the order of S_7, the subgroup is equal to S_7 itself.
In other words, all permutation of 7 corner cubes of the $2 \times 2 \times 2$ Rubik's cube can be achieved by a sequence of face rotations.

The link to the Wolfram Mathematica code: https://www.wolframcloud.com/env/tmdang/TrungDangHW5.6.nb

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In[20]:=

```
rotA = Cycles[{{1, 5, 6, 2}}]
rotB = Cycles[{{4, 7, 6, 5}}]
rotC = Cycles[{{2, 6, 7, 3}}]
RubikGroup222 = PermutationGroup[{rotA, rotB, rotC}]
GroupOrder[RubikGroup222]
GroupOrder[SymmetricGroup[7]]
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Out[20]=

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Cycles[{{1, 5, 6, 2}}]
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Out[21]=

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Cycles[{{4, 7, 6, 5}}]
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Out[22]=

```
Cycles[{{2, 6, 7, 3}}]
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Out[23]=

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PermutationGroup[{Cycles[{{1, 5, 6, 2}}], Cycles[{{4, 7, 6, 5}}], Cycles[{{2, 6, 7, 3}}]}]
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Out[24]=

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5040
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Out[25]=

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5040
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Problem 7

Problem Statement: Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \sigma_1 & \sigma_2 & \sigma_3 & \cdots & \sigma_n \end{pmatrix} \in S_n$. Prove the following formula:

$$\sigma(i_1 i_2 i_2 \cdots i_k)\sigma^{-1} = (\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k})$$

Solution. We will prove the formula by investigating where the elements are mapped to on the RHS and the LHS.
For $i_j \in \{i_1, i_2, \cdots, i_k\}$,

$$\begin{cases} RHS(\sigma_{i_j}) = \sigma_{i_{j+1}}, \forall 1 \leq j \leq k-1 \\ RHS(\sigma_{i_k}) = \sigma_{i_1} \end{cases} \quad (1)$$

Meanwhile, $\sigma^{-1}(\sigma_{i_j}) = i_j$; $(i_1 i_2 \cdots i_k)$ sends $i_j \to i_{j+1 \mod k}$; and $\sigma(i_{j+1 \mod k}) = \sigma_{i_{j+1 \mod k}}$ To sum up,

$$\begin{cases} LHS(\sigma_{i_j}) = \sigma_{i_{j+1}}, \forall 1 \leq j \leq k-1 \\ LHS(\sigma_{i_k}) = \sigma_{i_1} \end{cases} \quad (2)$$

For any $l \notin \{i_1, i_2, \cdots i_k\}$,

$$RHS(\sigma_l) = \sigma_l = LHS(\sigma_l), \forall l \notin \{i_1, \cdots i_k\} (3)$$

From (1), (2), (3), we have the RHS and the LHS maps every element to the same destination, therefore they are equivalent ■

Problem 8

Problem Statement: Permutations τ and ρ in S_n are called conjugate if $\rho = \sigma\tau\sigma^{-1}$ for some $\sigma \in S_n$. Prove that ρ and τ are conjugate if and only if they have the same cycle structure (in other words, lengths of cycles in the independent cycle decomposition should give the same partition of n)

Solution. First, we show that ρ and τ have the same cycle structure. Indeed, assume that $\tau(i) = j$ for some $i, j \in \{1, 2, 3, \cdots, n\}$. Then,

$$\rho(\sigma(i)) = \sigma\tau\sigma^{-1}(\sigma(i)) = \sigma\tau(i) = \sigma(j)$$

Therefore, since σ is a permutation, it is also a bijection. Thus, ρ and τ are isomorphisms, where each element $i \in \tau$ becomes $\sigma(i) \in \rho$.
Thus, they have the same cycle structures. (1)
Second, we show that if ρ and τ have the same cycle structure then they are conjugates.
List the cycles of τ above the cycles of ρ, aligning cycles of the same length with one another. Now interpret this as the two-line presentation of a permutation, and call it σ; then $\sigma\tau\sigma^{-1} = \rho$ by the claim.

For example, if $\tau = (1324)(56)$ and $\rho = (5231)(64)$, then by performing the algorithm above, we get

$$\begin{matrix} 1 & 3 & 2 & 4 & 5 & 6 \\ 5 & 2 & 3 & 1 & 6 & 4 \end{matrix}$$

And thus the desired permutation σ is $(1564)(23)$ (2)
From (1) and (2), we have shown that ρ and τ are conjugate if and only if they have the same cycle structure. ■

Problem 9

Problem Statement: Prove that S_n has a generating set $\{(12), (123\cdots n)\}$

Solution. For brevity, we denote $(123\cdots n) = \sigma$.
Then, one can verify that:

$$\sigma(12)\sigma^{-1} = (23)$$

Indeed,

$$\begin{cases} \sigma(12)\sigma^{-1}(1) = \sigma(12)(n) = \sigma(n) = 1 \\ \sigma(12)\sigma^{-1}(2) = \sigma(12)(1) = \sigma(2) = 3 \\ \sigma(12)\sigma^{-1}(3) = \sigma(12)(2) = \sigma(1) = 2 \\ \sigma(12)\sigma^{-1}(i) = \sigma(12)(i-1) = \sigma(i-1n) = n, \forall i \neq 1,2,3 \end{cases}$$

Similarly, we can observe that:

$$\sigma(23)\sigma^{-1} = (34)$$
$$\sigma(34)\sigma^{-1} = (45)$$
$$\cdots$$
$$\sigma(n-2, n-1)\sigma^{-1} = (n-1, n)$$

Thus, from $\{(12), (123\cdots n)\}$, we can generate $\{(12), (23), \cdots (n-1, n)\}$.
And since we have proven in problem 1 that this latter set generates S_n, $\{(12), (123\cdots n)\}$ also generates S_n. ■