

If X is a linear space and M is a linear subspace of X, then N is a complementary subspace of M if every  $x \in X$  can be uniquely represented by x = n + m where  $n \in N$  and  $m \in M$ . This implies  $M \cap N = \{0\}$ .

If M is a linear subspace of X, then M may have infinitely many complementary subspaces. All of the complementary subspaces of M have the same dimension, which is called the codimension of M.

A projection on a linear space X is a linear map  $P: X \to X$  such that  $P^2 = P$ .

First note that x = Px if and only if  $x \in \operatorname{ran} P$ , since if x = Py then  $Px = P^2x = Py = x$ .

Let  $x \in \ker P \cap \operatorname{ran} P$ . Then x = Px = 0.

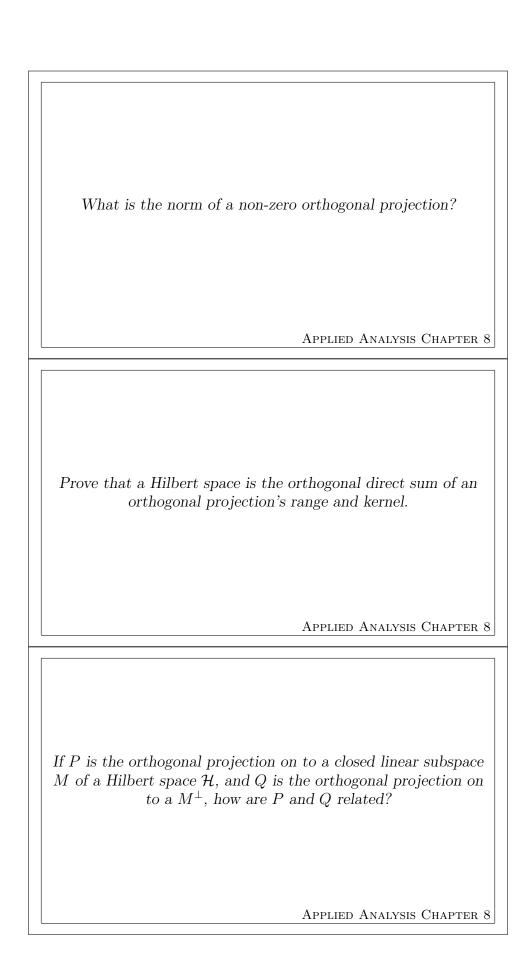
Let  $x \in X$ . Then  $x - Px \in \ker P$  since

$$P(x - Px) = Px - P^2x = Px - Px = 0.$$

Then x = Px + (x - Px).

An orthogonal projection P, on a Hilbert space  $\mathcal{H}$ , is a function  $P:\mathcal{H}\to\mathcal{H}$  such that

$$P^2 = P$$
 and  $(Px, y) = (x, Py) \ \forall x, y \in \mathcal{H}$ 



Using the Cauchy-Schwarz Inequality,

$$\|Px\| = \frac{\|Px\|^2}{\|Px\|} = \frac{(Px, Px)}{\|Px\|} = \frac{(x, P^2x)}{\|Px\|} = \frac{(x, Px)}{\|Px\|} \le \frac{\|x\| \|Px\|}{\|Px\|} = \|x\|,$$

so  $\|P\| \le 1$ . However, there is an  $x \in \operatorname{ran} P$  with  $\|x\| \ne 0$  (since P is non-zero. Then

$$||Px|| = ||x||,$$

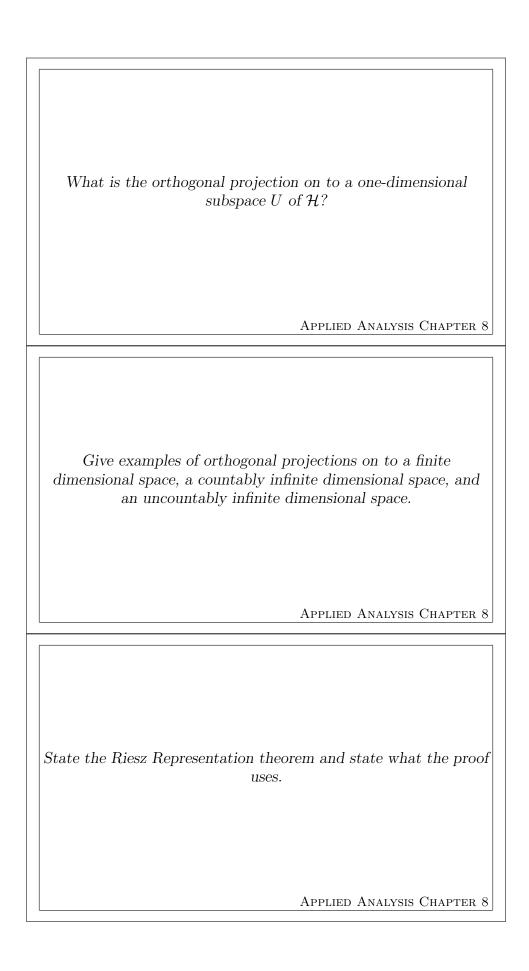
which shows  $||P|| \ge 1$ , and thus ||P|| = 1.

Let P be an orthogonal projection on  $\mathcal{H}$ . We know  $\mathcal{H} = \ker P \oplus \operatorname{ran} P$  where  $\oplus$  is just a (not necessarily orthogonal) direct sum. However, if  $x = Py \in \operatorname{ran} P$  and  $z \in \ker P$ , then

$$(x,z) = (Py,z) = (y,Pz) = (y,0) = 0$$

and thus  $\ker P \perp \operatorname{ran} P$ .

$$I - P = Q$$
.



Let  $\{u\}$  be a basis of U. Then define  $P_U$  by

$$P_{u}x = \frac{(u,x)}{\|u\|^2}u.$$

Let  $\mathcal{H} = \mathbb{R}^n$  and  $\mathbf{u}$  be any unit vector. The orthogonal projection in the direction of  $\mathbf{u}$  is the rank one matrix  $\mathbf{u}\mathbf{u}^T$ . The component of a vector  $\mathbf{x}$  in the direction of  $\mathbf{u}$ , i.e. the projection of  $\mathbf{x}$  on to  $[\{\mathbf{u}\}]$  is

$$P_{\mathbf{u}}\mathbf{x} = \frac{(\mathbf{u}, \mathbf{x})}{\|\mathbf{u}\|^2}\mathbf{u} = (\mathbf{u}^T\mathbf{x})\mathbf{u}.$$

Let  $\mathcal{H} = \ell^2(\mathbb{Z})$  and  $u = e_n = (\delta_{k,n})_{k=-\infty}^{\infty}$  and  $x = (x_k)$ . Then  $P_{e_n}x = x_n e_n$  gives a vector of all 0s except for the  $n^{\text{th}}$  component of x in the  $n^{\text{th}}$  position. Let  $\mathcal{H} = L^2(\mathbb{T})$  and  $u(x) \equiv \frac{1}{\sqrt{2\pi}}$ , which is the constant function with ||u|| = 1. Then  $P_u$  maps a function f to its mean  $\langle f \rangle$ , i.e.

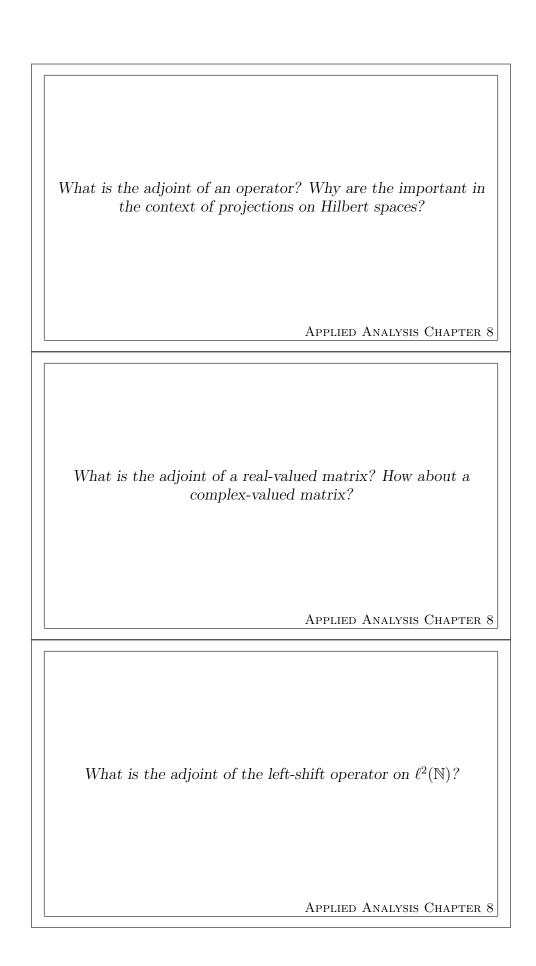
$$P_u f = \frac{1}{2\pi} \int_{\mathbb{T}} f(x) \mathrm{d}x.$$

Then  $f = \langle f \rangle + \tilde{f}$  is the decomposition of a function into a constant mean part and a fluctuating, 0 mean part.

If  $\phi$  is a bounded linear functional on a Hilbert space  $\mathcal{H}$ , then there is a unique  $y \in \mathcal{H}$  such that  $\phi(x) = (y, x)$  for all  $x \in \mathcal{H}$ .

The proof uses

- The kernel of a bounded linear operator is a closed subspace.
- Defining a clevel orthogonal projection P whose kernel is equal to ker  $\phi$ .
- Arbitrary vectors can be decomposed by  $\mathcal{H} = \ker P \oplus \operatorname{ran} P$ .



The adjoint of an bounded linear operator A on a Hilbert space  $\mathcal{H}$  is denoted  $A^*$  and is the unique operator in  $\mathcal{B}(\mathcal{H})$  such that

$$(x, Ay) = (A^*x, y) \quad \forall x, y \in \mathcal{H}.$$

An operator A is called self-adjoint if  $A = A^*$ . All orthogonal projects are self-adjoint, that is, for any projection P on  $\mathcal{H}$ , we have

$$(Px, y) = (x, Py) \quad \forall x, y \in \mathcal{H}.$$

This is not true for all projections - just orthogonal projections.

If A is a real-valued matrix in  $\mathbb{R}^{n \times n}$ , then  $A^* = A^T$ . That is, the adjoint is the transpose.

If A is a complex-valued matrix in  $\mathbb{C}^{n\times n}$ , then  $A^* = \overline{A^T}$ . That is, the adjoint is the Hermitian conjugate matrix.

Let  $T \in \mathcal{B}(\ell^2(\mathbb{N}))$  by

$$T(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots).$$

Then  $T^* = S$ , the right-shift operator, given by

$$S(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots)$$

because

$$(x,Ty) = \sum_{i=1}^{\infty} \overline{x_i} y_{i+1} = (Sx, y).$$

This is analogous to the transpose of a matrix since the transpose of the infinite matrix representing T is the infinite matrix representing S.

What is the adjoint of a Fredholm integral operator  $K \in \mathcal{B}(L^2([0,1]))$ ?

APPLIED ANALYSIS CHAPTER 8

If  $A: \mathcal{H} \to \mathcal{H}$  is a bounded linear operator, show  $\overline{\operatorname{ran} A} = (\ker A^*)^{\perp}$ , and  $\ker A = (\operatorname{ran} A^*)^{\perp}$ .

APPLIED ANALYSIS CHAPTER 8

Suppose that  $A: \mathcal{H} \to \mathcal{H}$  is a bounded linear operator on a Hilbert space  $\mathcal{H}$  with closed range. When does the equation Ax = y have a solution?

APPLIED ANALYSIS CHAPTER 8

Let  $K \in \mathcal{B}(L^2([0,1]))$  by

$$Kf(x) = \int_0^1 k(x, y) f(y) dy$$

for some continuous function  $k : [0,1] \times [0,1] \to \mathbb{C}$ . Then  $K^*$  is given explicitly by integration against the complex conjugate, transpose kernel:

$$K^*f(x) = \int_0^1 \overline{k(y,x)} f(y) \mathrm{d}y.$$

This is analogous to the Hermitian conjugate of a matrix.

Let  $x \in \operatorname{ran} A$ . Then x = Ay for some  $y \in \mathcal{H}$ . Then  $(x, z) = (Ay, z) = (y, A^*z) = 0$  for any  $z \in \ker A^*$ . Thus  $x \in (\ker A^*)^{\perp}$ , which is closed, and so  $\operatorname{ran} A \subset (\ker A^*)^{\perp}$ .

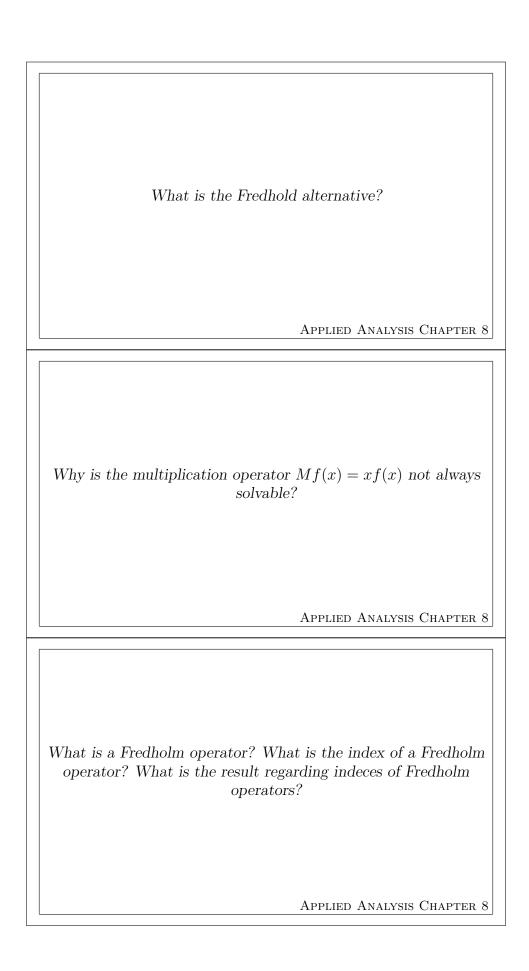
If  $x \in (\operatorname{ran} A)^{\perp}$ , then  $0 = (Ay, x) = (y, A^*x)$  for every  $y \in \mathcal{H}$ , which shows  $A^*x = 0$  for every  $x \in (\operatorname{ran} A)^{\perp}$ , i.e.  $x \in \ker A^*$ , so  $(\operatorname{ran} A)^{\perp} \subset \ker A^*$ . Then  $(\ker A^*)^{\perp} \subset (\operatorname{ran} A)^{\perp \perp} = \overline{\operatorname{ran} A}$ , which shows  $(\ker A^*)^{\perp} = \overline{\operatorname{ran} A}$ . The clever move here was  $X \subset Y \Longrightarrow Y^{\perp} \subset X^{\perp}$ .

Then taking  $A = A^*$  in the above equality gives  $(\ker A)^{\perp} = \overline{\operatorname{ran} A^*}$ . Then taking orthogonal complements gives  $\ker A = \overline{\operatorname{ran} A^*}^{\perp} = (\operatorname{ran} A^*)^{\perp}$ .

A succint way of stating this theorem is

$$\mathcal{H} = \overline{\operatorname{ran} A} \oplus (\ker A^*)^{\perp}.$$

The equation Ax = y has a solution for x if and only if  $y \perp \ker A^*$ .



A bounded linear operator A on a Hilbert space  $\mathcal{H}$  satisfies the Fredholm alternative if either

- (a) either Ax = 0,  $A^*x = 0$  have only the zero solution, and the equations Ax = y,  $A^*x = y$  have a unique solution for every  $y \in \mathcal{H}$ ;
- (b) or Ax = 0,  $A^*x = 0$  have nontrivial, finite-dimensional solution spaces of the same dimension, Ax = y has a (nonunique) solution if and only if  $y \perp z$  for every solution z of  $A^*z = 0$ , and  $A^*x = y$  has a (nonunique) solution if and only if  $y \perp z$  for every solution z of Az = 0.

In English, either

- (a) A and  $A^*$  are bijective;
- (b) or A and  $A^*$  are not injective, but have the same nullity.

Even though  $\ker M^* = \ker M = \{0\}$ , and hence every  $g \in L^2([0,1])$  is orthogonal to  $\ker M^*$ , Mf = g is not always solvable since ran M is properly dense in  $L^2([0,1])$ .

A bounded linear operator A on a Hilbert space  $\mathcal{H}$  is a Fredhold operator if

- (a)  $\operatorname{ran} A$  is closed;
- (b)  $\ker A$  and  $\ker A^*$  are finite-dimensional.

The index of a Fredholm operator A, denoted ind A, is the integer

 $\operatorname{ind} A = \dim \ker A - \dim \ker A^*$ .

If A is Fredholm and K is compact, then A + K is Fredholm and

$$\operatorname{ind} A = \operatorname{ind} (A + K).$$

That is, the index of a Fredholm is unchanged by compact perturbations. Since I is Fredholm, and ind I = 0, then ind (I + K) = 0 for compact K.

What is the most important sesquilinear form derived from a bounded linear operator A on a Hilbert space  $\mathcal{H}$ ? What is the associated quadratic form? What is it mean to be a non-negative operator? How about a positive (positive definite) operator? APPLIED ANALYSIS CHAPTER 8 How can we define an inner product on a Hilber space  $\mathcal{H}$  given a positive definite bounded linear operator A? APPLIED ANALYSIS CHAPTER 8 If A is a bounded, self-adjoint operator on a Hilbert space  $\mathcal{H}$ ,

what is an easy formula for ||A||?

APPLIED ANALYSIS CHAPTER 8

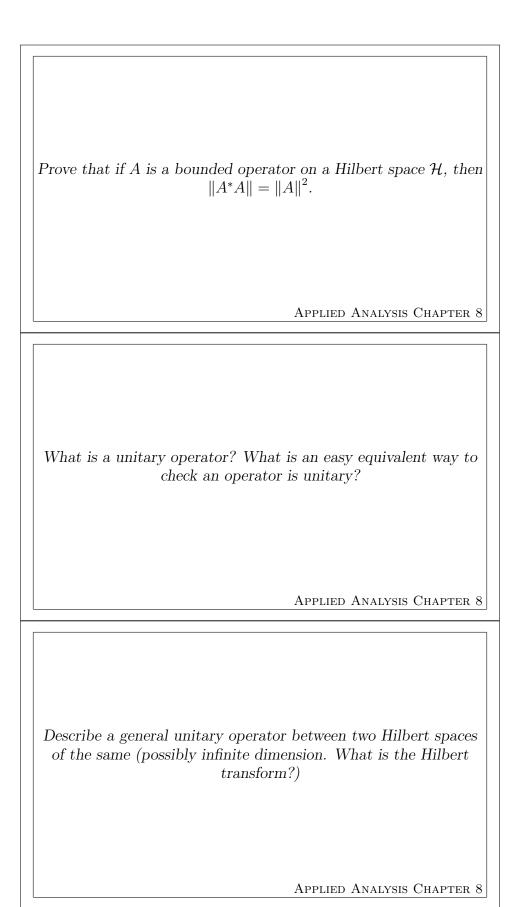
Given a linear operator A, define the sesquilinear form  $a: \mathcal{H} \times \mathcal{H} \to \mathbb{C}$  by a(x,y)=(x,Ay). The associated quadrative form is  $q_A(x)=a(x,x)$ , or  $q_A(x)=(x,Ax)$ .

A is called nonnegative if its quadratic form is nonnegative for all  $x \in \mathcal{H}$ , i.e.  $q_A(x) \geq 0, \forall x \in \mathcal{H}$ . A is positive definite if  $q_A(x) > 0, \forall x \in \mathcal{H}$ .

$$(x,y)_A := (x,Ay)$$

defines an inner product on  $\mathcal{H}$ . In addition,  $(\cdot, \cdot)_A$  is equivalent to  $(\cdot, \cdot)$ .

 $||A|| = \sup_{||x||=1} |q_A(x)|,$  where q is the quadratic form  $q_A(x) = (x, Ax).$ 



$$||A||^2 = \sup_{||x||=1} ||Ax||^2 = \sup_{||x||=1} |(Ax, Ax)| = \sup_{||x||=1} |q_{A^*A}(x)| = ||A^*A||$$

If A is self adjoint, the  $A^* = A$  and  $||A||^2 = ||A^2||$ .

A linear map  $U: \mathcal{H}_1 \to \mathcal{H}_2$  between real or complex Hilbert spaces  $\mathcal{H}_1$  and  $\mathcal{H}_2$  is said to be orthogonal (real) or unitary (complex), respectively, if it is invertible and if

$$(Ux, Uy) = (x, y), \quad \forall x, y \in \mathcal{H}_1$$

Unitary operators preserve inner products.

Two Hilbert spaces are isomorphic if there exists a unitary operator between them.

An operator U is unitary if and only if its inverse is it's adjoint. That is, U is unitary if and only if  $U^* = U^{-1}$ .

Let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  have the same (possibly infinite) dimension. Then their bases can be indexed by the same index set. Suppose  $\mathcal{H}_1$  has basis  $\{u_{\alpha}\}$  and  $\mathcal{H}_2$  has basis  $\{v_{\alpha}\}$ . Then any  $x \in \mathcal{H}_1$  can be written as

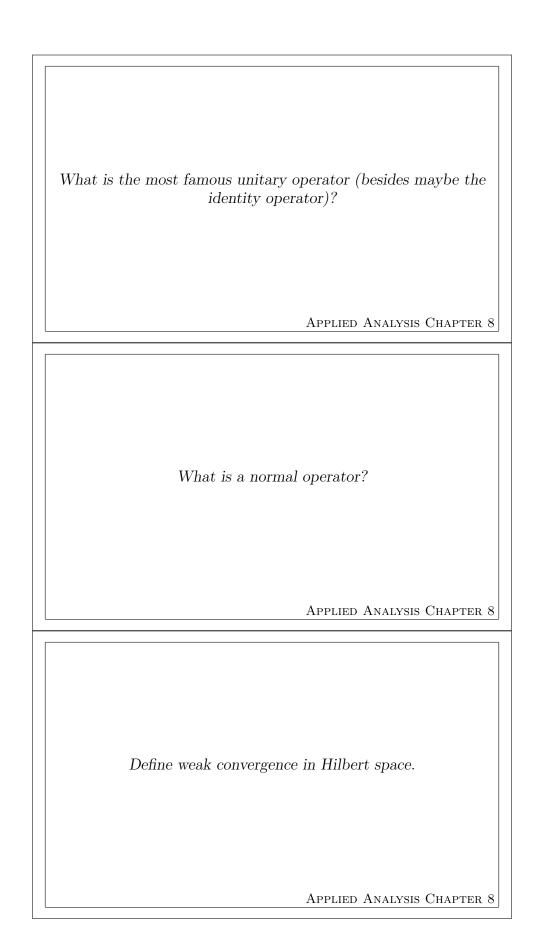
$$x = \sum_{\alpha} c_{\alpha} u_{\alpha}, \quad \text{where } c_{\alpha} = (u_{\alpha}, x).$$

Let  $\lambda_{\alpha}$  be complex numbers with  $|\lambda_{\alpha}| = 1$  and define  $U : \mathcal{H}_1 \to \mathcal{H}_2$  by

$$Ux = \sum_{\alpha} \lambda_{\alpha}(u_{\alpha}, x)v_{\alpha}.$$

Then U is unitary, and thus  $\mathcal{H}_1 \cong \mathcal{H}_2$ . The Hilbert transform  $\mathbb{H}$  is of this form. Define  $\mathbb{H}$ :  $\mathcal{H}_0 \subset L^2(\mathbb{T}) \to \mathcal{H}_0$  ( $\mathcal{H}_0 = \{f \in L^2(\mathbb{T}) \mid \langle f \rangle = 0\}$  is the space of  $L^2$  functions with 0 mean) by

$$\mathbb{H}f = \mathbb{H}\left(\sum_{n \in \mathbb{N}} \hat{f}_n e^{inx}\right) = \sum_{n \in \mathbb{N}} \left(i(\operatorname{sgn}[n])\hat{f}_n e^{inx}\right).$$



The most famous (and most useful) unitary operator is the Fourier transform,  $\mathcal{F}: L^2(\mathbb{T}) \to \ell^2(\mathbb{N})$ , which is given by

$$\mathcal{F}f = \mathcal{F}\left(\sum_{n \in \mathbb{N}} c_n e^{inx}\right) = (c_n)_{n \in \mathbb{N}}, \quad \text{where } c_n = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} f(x) e^{-inx} dx.$$

A operator A is said to be normal if it commutes with its adjoint, that is,

$$AA^* = A^*A.$$

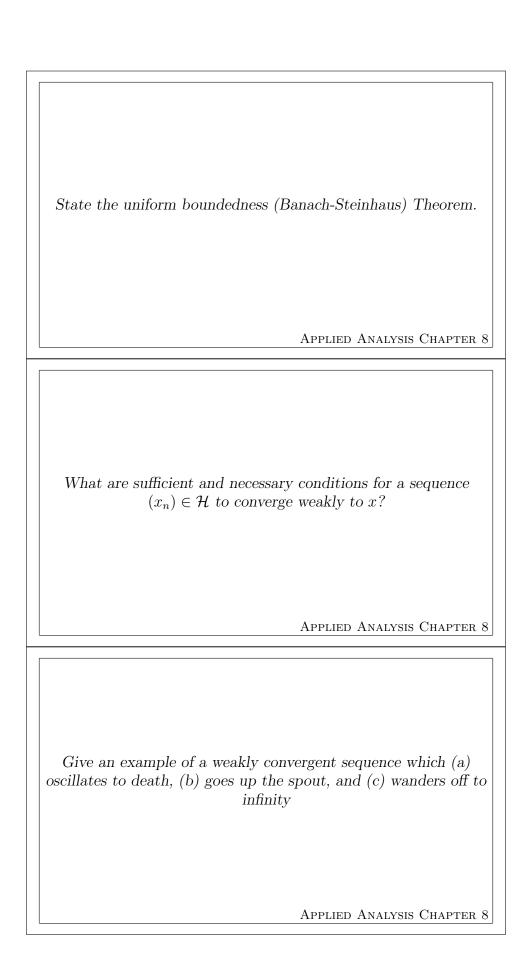
Let  $\mathcal{H}$  be a Hilbert space. A sequence  $(x_n) \in \mathcal{H}$  converges weakly to x, denoted  $x_n \rightharpoonup x$  if

$$\lim_{n \to \infty} (x_n, y) = (x, y), \qquad \forall y \in \mathcal{H}.$$

This is a special case of weak convergence in a Banach space X, which states  $x_n \rightharpoonup x$  if

$$\phi(x_n) \to x, \quad \forall \phi \in X^*.$$

In Hilbert space,  $\mathcal{H}^* \cong \mathcal{H}$ , and by the Reisz Representation Theorem, every linear functional can be uniquely represented as an inner product against an element in  $\mathcal{H}$ .



Suppose that  $\{\phi_n \in X^* \mid n \in \mathbb{N}\}$  is a set of linear functionals on a Banach space X such that  $\{\phi_n(x) \mid n \in \mathbb{N}\}$  is bounded for every  $x \in X$ . Then  $\{\|\phi_n\| \mid n \in \mathbb{N}\}$  is bounded.

That is, if the values of the countable set of functionals evaluated at each point is bounded, then the functionals themselves are bounded in operator norm.

Let D be a dense subset of a Hilbert space  $\mathcal{H}$ . Then  $x_n \rightharpoonup x$  if and only if

- (a)  $\exists M \in \mathbb{R}$  such that  $||x_n|| \leq M$  for  $n \in \mathbb{N}$ ;
- (b)  $(x_n, y) \to (x, y)$  for each  $y \in D$ .

- (a) Oscillates to death: Let  $f_n = \sin(n\pi x) \in L_2([0,1])$ . Then  $\int_0^1 f(x)\sin(n\pi x)dx \to 0$  as  $n \to \infty$ , but  $||f_n|| = \frac{1}{\sqrt{2}}$  for al  $n \in \mathbb{N}$ . So  $f_n \to 0$  but  $f_n \not\to 0$ .
- (b) Goes up the spout: Let  $f_n = \mathcal{X}_{[0,\frac{1}{n}]}\sqrt{n} \in L^2([0,1])$ . Then for any polynomial p,  $\int_0^1 p(x)f_n(x)\mathrm{d}x = \sqrt{n}\int_0^{\frac{1}{n}} p(x)\mathrm{d}x \to 0$  as  $n \to \infty$ . Also,  $||f_n|| = 1$  for each  $n \in \mathbb{N}$ . Then since polynomials are dense in  $L^2([0,1])$ ,  $f_n \to 0$  but  $f_n \to 0$ .
- (c) Wanders off to infinity: Let  $f_n = \mathcal{X}_{[n,n+1]} \in L^2(\mathbb{R})$ . Then  $||f_n|| = 1$  for  $n \in \mathbb{N}$ . Also, for any function c(x) with compact support,  $\int_{\mathbb{R}} f_n(x)c(x)\mathrm{d}x \to 0$  as  $n \to \infty$ . Then since functions with compact support are dense in  $L^2(\mathbb{R})$ ,  $f_n \rightharpoonup 0$  but  $f_n \not\to 0$ .

In the context of Hilbert spaces, what in addition to weak convergence does one need in order to show strong convergence?
Applied Analysis Chapter 8
What is the Banach-Alaoglu Theorem? What is it's analog in finite-dimensional Hilbert space?
Applied Analysis Chapter 8

Suppose  $x_n \rightharpoonup x$ . In general,

$$||x|| \le \liminf_{n \to \infty} ||x_n||.$$

If  $\lim_{n\to\infty} ||x_n|| = ||x||$ , then  $x_n \to x$ . The proof uses:

- continuity of the inner product
- Cauchy-Schwarz Inequality
- Definition of convergence in norm (strong convergence)

The closed unit ball of a Hilbert space is weakly compact.

A set is weakly precompact if and only if it is bounded.

This is the infinite-dimensional analog of the Heine-Borel Theorem (for finite-dimensional Hilbert spaces, closed and bounded  $\iff$  compact).

The proof uses

- a diagonal argument on sequences that converge using Heine-Borel on
- The Reisz Representation Theorem
- The Bounded Linear Transformation Theorem (Bounded linear transformations on dense subsets of a Banach space X can be uniquely extended to bounded linear transformations on X.)