Define an inner product
Applied Analysis Chapter 6
What is a pre-Hilbert space?
Applied Analysis Chapter 6
Can we always define a norm given an inner product? Applied Analysis Chapter 6

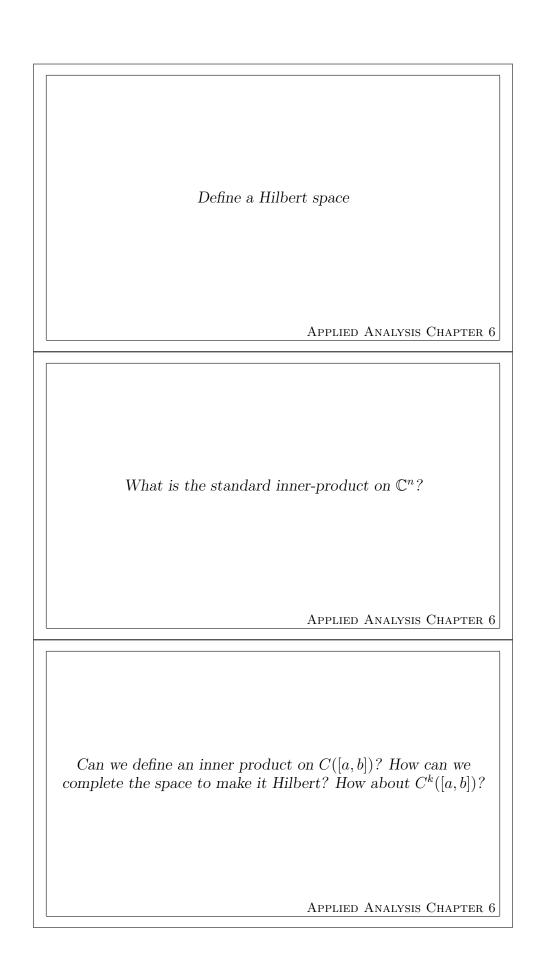
An inner product on a complex linear space X is a map $(\cdot,\cdot):X\times X\to\mathbb{C}$ such that for all $x,y,z\in X$ and $\lambda,\mu\in\mathbb{C}$,

- (a) $(x, \lambda y + \mu z) = \lambda(x, y) + \mu(x, z)$ (linear in the second argument);
- (b) $(y,x) = \overline{(x,y)}$ (Hermitian symmetric);
- (c) $(x, x) \ge 0$ (nonnegative);
- (d) $(x, x) = 0 \iff x = 0$ (positive definite);

A pre-Hilbert space (or inner-product space) is a linear space with an inner product defined.

Yes. In fact the most common norm we will see is the following:

$$||x|| = \sqrt{(x,x)}$$



A Hilbert space is a complete inner-product space. That is, a Banach space with a norm derived from a defined inner-product.

Given $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in \mathbb{C}^n , then

$$(x,y) = \sum_{i=1}^{n} \overline{x_i} y_i$$

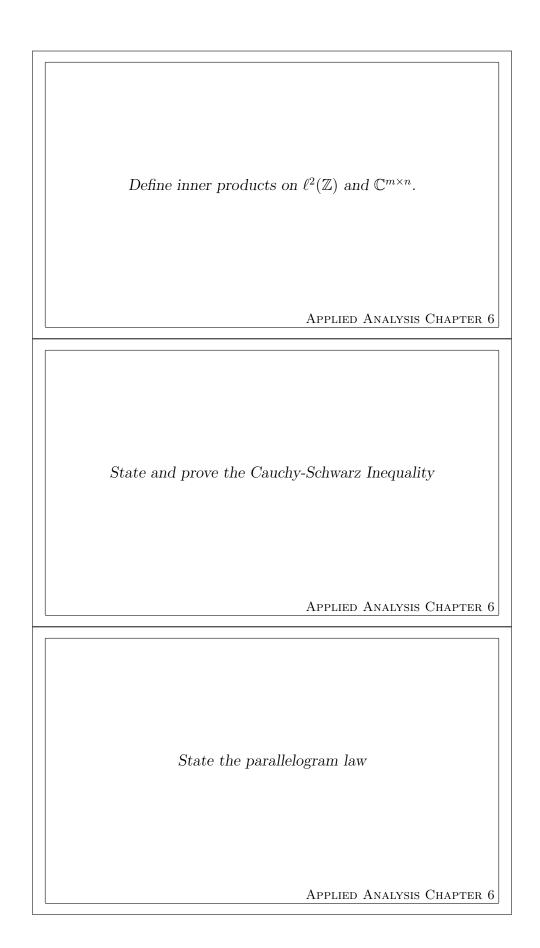
Let $f,g \in C([a,b])$. Then, analogous to the inner-product on \mathbb{C}^n , we can define (f,g) as follows:

$$(f,g) = \int_a^b \overline{f(x)}g(x)dx$$

This makes C([a,b]) a pre-Hilbert space. The completion of C([a,b]) is $L^2([a,b])$, which is the only Hilbert L^p space. For $f,g\in C^k([a,b])$,

$$(f,g) = \sum_{i=0}^{k} \int_{a}^{b} \overline{f^{(i)}(x)} g^{(i)}(x) dx = \sum_{i=0}^{k} \left(f^{(i)}, g^{(i)} \right)_{C([a,b])}$$

This makes $C^k([a,b])$ a pre-Hilbert space. The completion of $C^k([a,b])$ is the Sobolev space $W^{k,2}((a,b))$, which is also denoted $H^k((a,b))$.



For $x = (x_n)_{n=-\infty}^{\infty}$ and $y = (y_n)_{n=-\infty}^{\infty}$ in $\ell^2(\mathbb{Z})$,

$$(x,y) = \sum_{n=-\infty}^{\infty} \overline{x_n} y_n$$

For $A = (a_{ij})$ and $B = (b_{ij})$ in $\mathbb{C}^{m \times n}$,

$$(A,B) = \operatorname{tr}(A^*B) = \sum_{i=1}^m \sum_{j=1}^n \overline{a_{ij}} b_{ij}$$

The corresponding norm is the Hilbert-Schmidt norm.

If $x, y \in X$, where X is an inner product space, then

$$|(x,y)| \le ||x|| ||y||.$$

For any $\lambda \in \mathbb{C}$, by nonnegativity, $0 \leq (x - \lambda y, x - \lambda y)$, which implies, by linearity in the second argument, anti-linearity in the first argument, and definition of norm,

$$\lambda(x,y) + \overline{\lambda}(y,x) \le ||x||^2 + |\lambda|^2 ||y||^2$$

Now choose $\lambda = \frac{(y,x)}{\|y\|^2}$. Then substitution gives

$$2\frac{|(x,y)|^2}{\|y\|^2} \le \|x\|^2 + \frac{|(x,y)|^2}{\|y\|^2}.$$

Simple algebra gives the result.

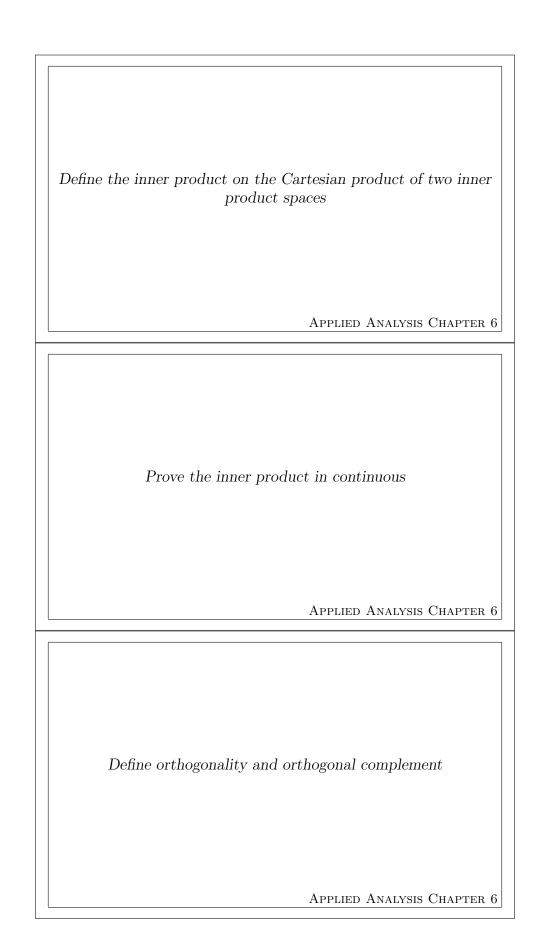
A normed linear space X is an inner product space with a norm derived from the inner product by $||x|| = \sqrt{(x,x)}$ if and only if the following holds:

$$||x + y||^2 + ||x - y||^2 = 2||x||^2 + 2||y||^2$$
 $\forall x, y \in X$

Geometrically, the sum of the squares of the diagonals of a parallelogram equal the sum of the squares of the sides. If the above equation holds, then

$$(x,y) = \frac{1}{4} \Big(\|x+y\|^2 - \|x-y\|^2 - i\|x+iy\|^2 + i\|x-iy\|^2 \Big)$$

defines an inner product on X. This is called the polarization formula.



Let $(X, (\cdot, \cdot)_X)$ and $(Y, (\cdot, \cdot)_Y)$ be two inner product spaces. Then the Cartesian product space is the space containing all tuples,

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}$$

and the natural inner product is simply the sum of the inner products of the components:

$$((x_1, y_1), (x_2, y_2))_{X \times Y} = (x_1, x_2)_X + (y_1, y_2)_Y.$$

This gives rise to the natural norm on $X \times Y$:

$$\|(x,y)\| = \sqrt{\|x\|^2 + \|y\|^2}$$

Let X be an inner product space. Choose $(x_1, y_1), (x_2, y_2) \in X \times X$ such that

$$\|(x_1, y_1) - (x_2, y_2)\| < \delta = \frac{1}{2} \max \left\{ \frac{\varepsilon}{\|y_1\|}, \frac{\varepsilon}{\|x_2\|} \right\}.$$

Assume for now $\delta \neq 0$. Then in particular, $||x_1 - x_2|| < \delta$ and $||y_1 - y_2|| < \delta$. Then

$$|(x_1, y_1) - (x_2, y_2)| = |(x_1, y_1) - (x_2, y_1) + (x_2, y_1) - (x_2, y_2)|$$

Then by linearity, triangle inequality, and the Cauchy-Schwarz inequality,

$$|(x_1, y_1) + (x_2, y_2)| \le ||x_1 - x_2|| ||y_1|| + ||x_2|| ||y_1 - y_2|| < \delta(||y_1|| + ||x_2||) = \varepsilon$$

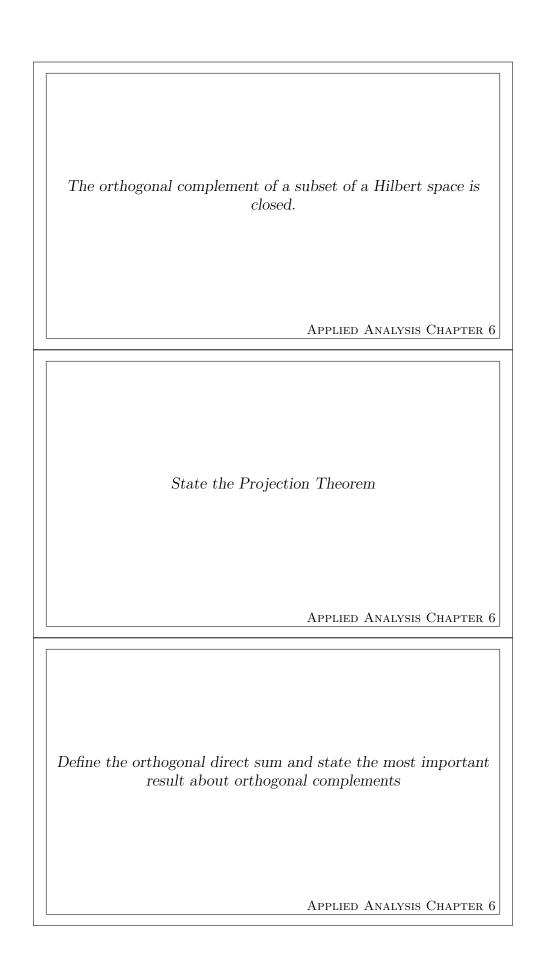
Two vectors x and y are called orthogonal, denoted $x \perp y$, if their inner product is equal to 0:

$$(x,y) = 0$$

Subsets A and B are orthogonal if every element in A is orthogonal to every element in B. That is, $A \perp B$ if $a \perp b$, $\forall a \in A$ and $b \in B$.

The orthogonal complement of a subset A, denoted A^{\perp} , in a Hilbert space \mathcal{H} is the set of all elements in \mathcal{H} orthogonal to every element in A. That is,

$$A^{\perp} = \{ x \in \mathcal{H} : x \perp a, \ \forall a \in A \}$$



The proof uses:

- linearity of the inner product
- continuity of the inner product

Let M be a closed linear subspace of a Hilbert space \mathcal{H} . Then

- (a) For each $x \in \mathcal{H}$, $\exists !$ closest point $y \in M$ such that $||x-y|| = \min_{z \in M} ||x-z||$;
- (b) The point $y \in M$ closest to x is the unique element of M with the property that $(x y) \perp M$.

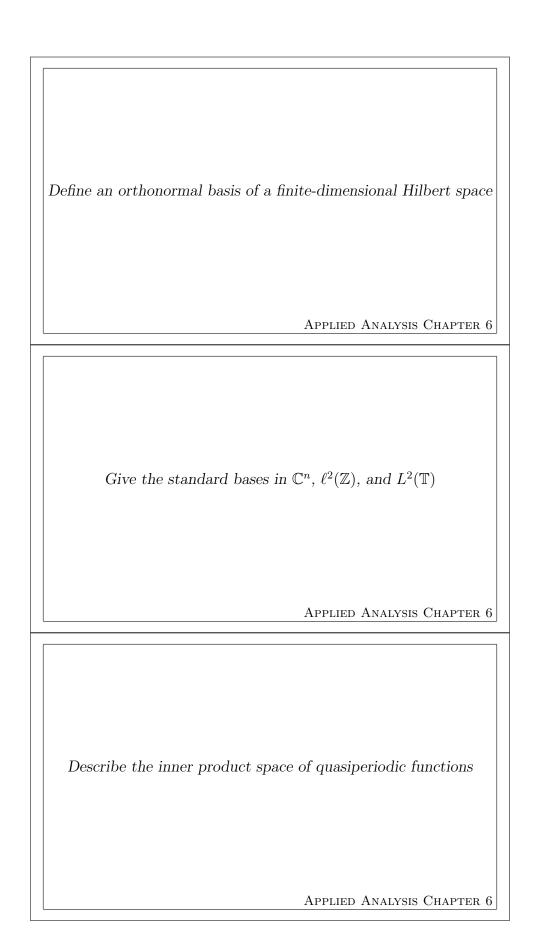
The proof uses:

- Definition of Infimum
- Parallelogram Law
- Norm and Inner product are continuous
- Cauchy \implies convergent in complete spaces
- Convexity of normed linear spaces

Given two orthogonal closed linear subspaces M and N, the orthogonal direct sum of M and N, denoted $M \oplus N$, is the smallest linear subspace containing M and N, i.e.

$$M \oplus N = \{m+n \mid m \in M \text{ and } n \in N\}$$

If M is a closed linear subspace of \mathcal{H} , then $M \oplus M^{\perp} = \mathcal{H}$. If M is not closed, we still have $\overline{M} \oplus M^{\perp} = \mathcal{H}$.



Let \mathcal{H} be a finite-dimensional Hilbert space. A set of vectors $\{e_1, \ldots, e_n\}$ is an orthonormal basis of \mathcal{H} if $||e_i|| = 1$ for each $i = 1, \ldots, n$, $e_i \perp e_j$ for $i \neq j$, and for all $x \in \mathcal{H}$, $\exists ! x_k \in \mathbb{C}$ such that

$$x = \sum_{i=1}^{n} x_i e_i$$

The standard basis in \mathbb{C}^n is $\{e_1, \ldots, e_n\}$ where $e_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0, 0)$ where all components are 0 except for a 1 in the i component.

The standard basis for $\ell^2(\mathbb{Z})$ is $\{\ldots, e_{-1}, e_0, e_1, \ldots\}$, where $e_i = (\delta_{ij})_{j=-\infty}^{\infty}$ and δ_{ij} is the Kronecker delta function.

The standard basis for $L^2(\mathbb{T})$ is $\{\ldots, e_{-1}, e_0, e_1, \ldots\}$ where $e_n(x) = \frac{1}{\sqrt{2\pi}}e^{inx}$

Define X to be the space of all functions of the form $a(t) = \sum_{k=1}^{n} a_k e^{i\omega_k t}$. We can define an inner product on X by

$$(a,b) = \lim_{T \to \infty} \int_{-T}^{T} \overline{a(t)} b(t) dt$$

which simplifies to

$$(a,b) = \sum_{k=1}^{n} \overline{a_k} b_k.$$

Note the set

$$\Omega = \{e^{i\omega t} \mid \omega \in \mathbb{R}\}$$

is an uncountable orthonormal set, which means X is not separable. Ω is, in fact, an orthonormal basis of X.

Define unconditional convergence
Applied Analysis Chapter 6
Define absolute convergence
Applied Analysis Chapter 6
What does it mean to be a Cauchy unordered series?
APPLIED ANALYSIS CHAPTER 6

Let $\{x_{\alpha} \in X \mid \alpha \in I\}$ be an indexed set in a Banach space X and I may be uncountable. For each finite subset J of I, define the partial sum S_J by

$$S_J = \sum_{\alpha \in J} x_\alpha$$

The unordered sum of the indexed set converges unconditionally to $x \in X$, denoted

$$x = \sum_{\alpha \in I} x_{\alpha},$$

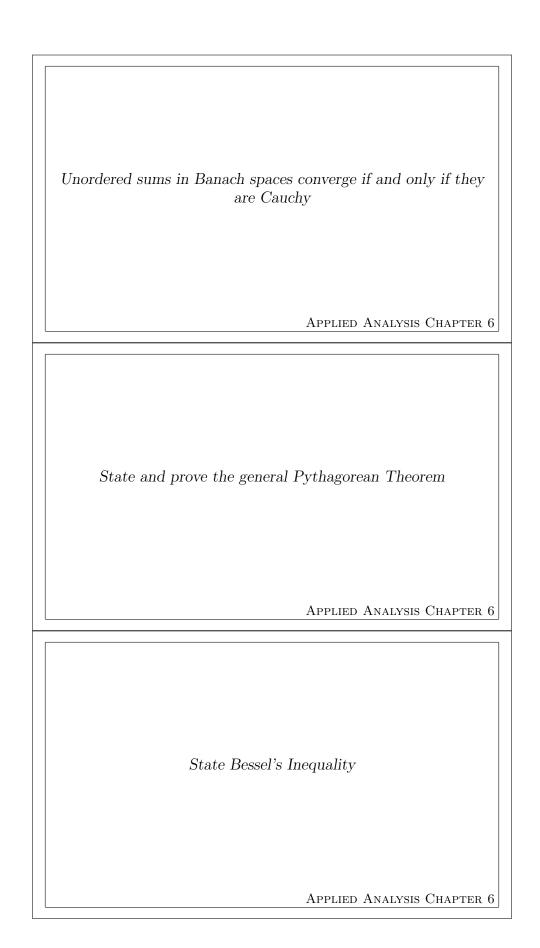
if for every $\varepsilon > 0$, there is a finite subset J^{ε} of I such that $||S_J - x|| < \varepsilon$ for every finite subset J of I which contains J^{ε} .

All unconditionally convergent series have only countably many nonzero terms.

A sum $\sum_{\alpha \in I} x_i$ converges absolutely if $\sum_{\alpha \in I} ||x_i||$ converges unconditionally.

Absolute convergence implies unconditional convergence.

An unordered sum $\sum_{\alpha \in I}$ is Cauchy if for every ε , there is a finite subset J^{ε} of I such that $||S_J|| < \varepsilon$ for every finite $J \subset I \setminus J^{\varepsilon}$



Cauchy \implies convergent uses the following outline:

- Define an increasing class (J_n) of finite subsets of I.
- Show that (S_{J_n}) is Cauchy.
- Since X is banach, (S_{J_n}) converges to some limit $x \in X$. Use x as the candidate limit for the original Cauchy series.
- \bullet Use an $\frac{\varepsilon}{2}$ trick with the Cauchy criterion and the definition of x to complete the proof

Convergent \implies Cauchy uses the following:

- Set operations (\setminus, \cup)
- Triangle inequality

Let $U = \{u_{\alpha} \mid \alpha \in I\}$ be an indexed, orthogonal subset of a Hilbert space \mathcal{H} . The sum $\sum_{\alpha \in I} u_{\alpha}$ converges unconditionally if an only if $\sum_{\alpha \in I} \|u_{\alpha}\|^2$ converges unconditionally, and in that case,

$$\left\| \sum_{\alpha \in I} u_{\alpha} \right\|^2 = \sum_{\alpha \in I} \|u_{\alpha}\|^2.$$

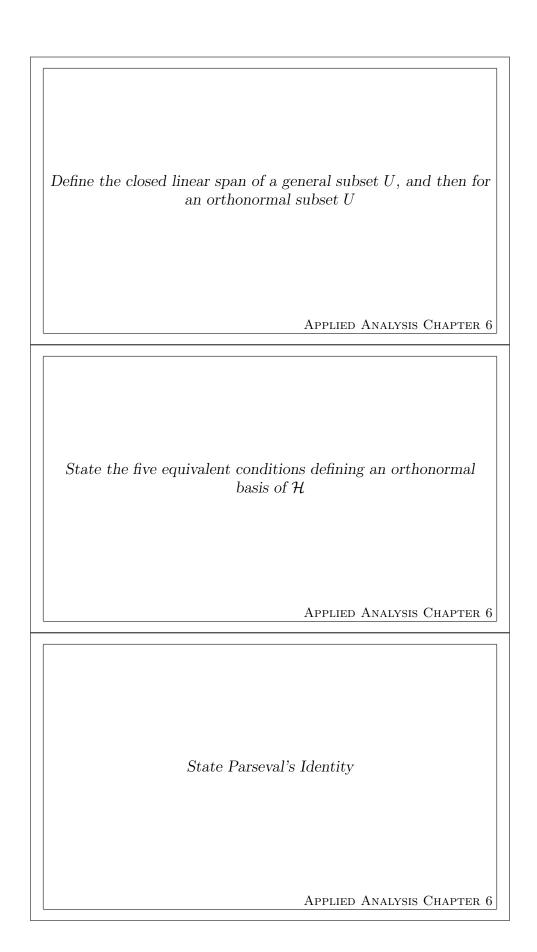
For any finite subset J of I,

$$\left\| \sum_{\alpha \in J} u_{\alpha} \right\|^{2} = \sum_{\alpha, \beta \in J} (u_{\alpha}, u_{\beta}) = \sum_{\alpha \in J} (u_{\alpha}, u_{\alpha}) = \sum_{\alpha \in J} \left\| u_{\alpha} \right\|^{2}.$$

It then follows that $\sum_{\alpha \in I} u_{\alpha}$ converges if and only if $\sum_{\alpha \in I} \|u_{\alpha}\|^2$ converges. Then since the norm is continuous, the result holds.

Let $U = \{u_{\alpha} \mid \alpha \in I\}$ be an orthnormal set in a Hilbert space \mathcal{H} , and choose any $x \in \mathcal{H}$. Then

- (a) $\sum_{\alpha \in I} |(u_{\alpha}, x)|^2 \le ||x||^2$.
- (b) $x_U := \sum_{\alpha \in I} (u_\alpha, x) u_\alpha$ converges. $(x_U \text{ is the projection of } x \text{ on to the subspace spanned by } U.)$
- (c) $x x_U \in U^{\perp}$



The closed linear space of a subset U of a Hilbert space \mathcal{H} , denoted [U], is given by

$$[U] = \left\{ \sum_{u \in U} c_u u \mid c_u \in \mathbb{C} \text{ and } \sum_{u \in U} c_u u \text{ converges unconditionally} \right\}$$

If $U = \{u_{\alpha} \mid \alpha \in I\}$ is an orthonormal set, then

$$[U] = \left\{ \sum_{\alpha \in I} c_{\alpha} u_{\alpha} \mid c_{\alpha} \in \mathbb{C} \text{ and } \sum_{\alpha \in I} |c_{\alpha}|^{2} < \infty \right\}$$

This simplification follows from the Pythagorean Theorem.

Uf $U = \{u_{\alpha} \mid \alpha \in I\}$ is an orthonomal subset of a Hilbert space \mathcal{H} , then the following conditions are equivalent:

- (a) $(u_{\alpha}, x) = 0$ for all $\alpha \in I$ implies x = 0;
- (b) $x = x_U = \sum_{\alpha \in I} (u_\alpha, x) u_\alpha$ for all $x \in \mathcal{H}$; (c) $||x||^2 = \sum_{\alpha \in I} |(u_\alpha, x)|^2$ for all $x \in \mathcal{H}$; (d) $[U] = \mathcal{H}$;
- (e) U is a maximal orthonormal set.

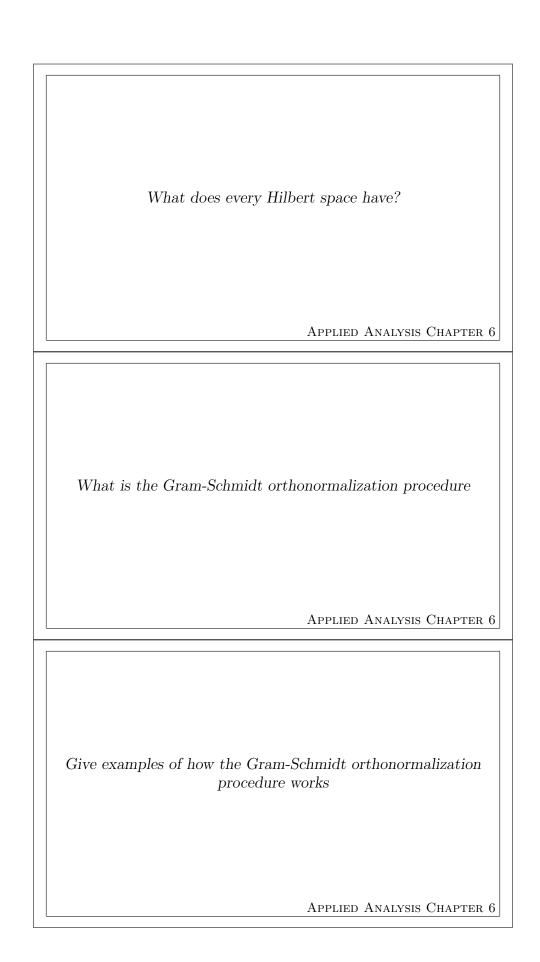
In English,

- (a) The only element orthogonal to every element in U is 0.
- (b) Every element is equal to its own projection onto [U].
- (c) The Pythagorean Theorem, simplified, since $||u_{\alpha}|| = 1$ for all α
- (d) U spans all of \mathcal{H}
- (e) No non-zero orthogonal vector can be added to the set U.

Suppose $U = \{u_{\alpha} \mid \alpha \in I\}$ is an orthonormal basis of \mathcal{H} . Define $x = \sum_{\alpha \in I} x_{\alpha} u_{\alpha}$ and $y = \sum_{\alpha \in I} y_{\alpha} u_{\alpha}$. Then

$$(x,y) = \sum_{\alpha \in I} \overline{x_{\alpha}} y_{\alpha}$$

sicne $x_{\alpha} = (u_{\alpha}, x)$ and $y_{\alpha} = (u_{\alpha}, y)$ for $\alpha \in I$.



Every Hilbert space has an orthonormal basis. Also, given any orthonormal subset U of a Hilbert space \mathcal{H} , there is an orthonormal basis of \mathcal{H} containing U. In other words, one can always extend an orthonormal set to an orthonormal basis.

The Gram-Schmidt orthonormalization procedure is a way of constructing a countable orthonormal set U given a countable set of linearly independent vectors V such that [U] = [V]. Define u_n as follows:

$$u_1 = \frac{v_1}{\|v_1\|}$$
 and $u_{n+1} = c_{n+1} \left(v_{n+1} - \sum_{k=1}^n (u_k, v_{k+1}) u_k \right)$

where c_{n+1} is chosen to $||u_{n_1}|| = 1$.

Define a weighted inner-product on the continuous functions C([a,b]) by

$$(f,g) = \int_{a}^{b} w(x)\overline{f(x)}g(x)dx$$

Denote $C_w([a,b])$ as the set of functions whos norm is finite, i.e.

$$C_w([a,b]) = \{f : [a,b] \to \mathbb{C} \mid f \text{ is continuous and } (f,f) < \infty \}$$

and complete this space to obtain the Hilbert space $L_w([a,b])$.

Set $M = \{x^n \mid n \in \mathbb{N}\}$. This set is linearly independent, but may not orthonormal. Given $L_w([-1,1])$ with $w(x) \equiv 1$, the G-S procedure on M produces the Legendre polynomials. Given $L_w([-1,1])$ with $w(x) = \sqrt{1-x^2}$, the G-S procedure on M produces the Tchebyschev polynomials. Given $L_w(\mathbb{R})$ with $w(x) = \exp\left[-\frac{x^2}{2}\right]$, the G-S procedure on M produces the Hermite polynomials.