

Consider an $n \times n$ matrix A with complex entries. A number λ is an eigenvalue of A if there is a nonzero vector u such that

$$Au = \lambda u$$
.

If λ is an eigenvector such that $Au = \lambda u$, then u is called an eigenvector of A corresponding to λ .

A matrix A is diagonalizable if there is a basis $\{u_1, \ldots, u_n\}$ of \mathbb{C}^n such that there are eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$, which may not be distinct, such that

$$Au_k = \lambda_k u_k$$
, for $k = 1, \dots, n$.

The spectrum of a finite-dimensional linear operator A, denoted $\sigma(A)$, consists of the eigenvalues of the matrix representing A.

Let $\{u_1, \ldots, u_n\}$ be an orthonormal basis of \mathbb{C}^n . Then construct the matrix $U = (u_1 \ u_2 \ \ldots \ u_n)$ where the columns of U are the basis vectors. Then denote $\{e_1, \ldots, e_n\}$ as the standard basis of \mathbb{C}^n . It follows that

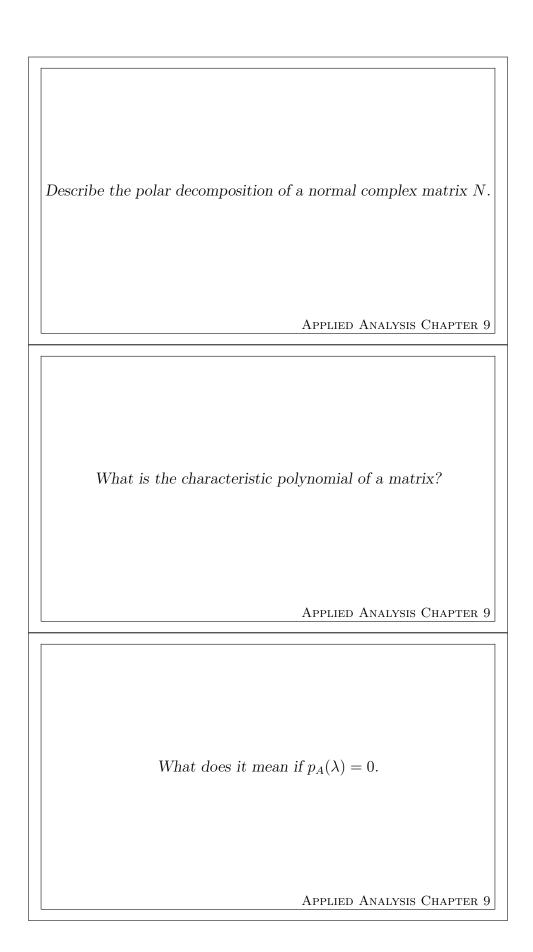
$$Ue_k = u_k, U^*u_k = e_k, i.e. U^* = U^{-1}$$

Next, suppose the basis vectors $\{u_1, \ldots, u_n\}$ are eigenvectors of A, i.e. $\exists \lambda_k$ such that $Au_k = \lambda_k u_k$ for $k = 1, \ldots, n$. It then follows that

$$U^*AUe_k = \lambda_k e_k,$$

which shows $D = U^*AU$ is a diagonal matrix with the eigenvalues on the diagonal, so $A = UDU^*$ where $D = (d_{ij})$ and $d_{ij} = \delta_{ij}\lambda_i$. Indeed, if $A = UDU^*$ with U unitary and D diagonal, then the columns of U form an orthonormal basis of \mathbb{C}^n consisting of eigenvectors of A. A is a normal operator.

An $n \times n$ complex matrix A is normal if and only if \mathbb{C}^n has an orthonormal basis consisting of eigenvectors of A.



We want to decompose a normal matrix N in to the product of a unitary matrix V and a nonnegative matrix A.

Since N is normal, it can decomposition $N = UDU^*$ where U is unitary and D is diagonal $(D = (d_{ij})$ where $d_{ij} = \delta_{ij}\lambda_i$ and λ_i are the eigenvalues of N). We can rewrite D as $D = \Phi|D|$ where Φ is a diagonal matrix consisting of $\arg \lambda_i$ and |D| is a diagonal matrix consisting of $|\lambda_i|$. Then

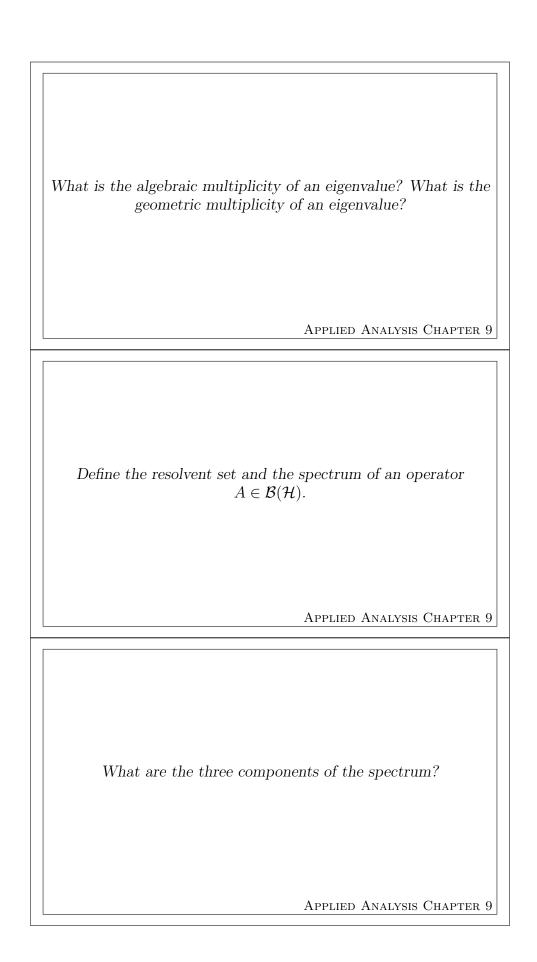
$$N = VA$$
, where $V = U\Phi U^*$ and $A = U|D|U^*$.

Also, A is non-negative, meaning $u^*Au \geq 0$ for all $u \in \mathbb{C}^n$.

This is the matrix analog of the polar decomposition of a complex number $z = re^{i\theta}$ in to the non-negative part r and the complex number $e^{i\theta}$.

The eigenvalues of a matrix A are the roots of the characteristic polynomial p_A of A, given by $p_A(\lambda) = \det(A - \lambda I)$.

If $p_A(\lambda) = 0$, then $A - \lambda I$ is singular, and in particular, $\ker(A - \lambda I) \neq \{0\}$. This means λ is an eigenvector.



The algebraic multiplicity of an eigenvalue λ is the power on the factor $(x - \lambda)$ in $p_A(\lambda)$.

The geometric multiplicity is the dimension of the eigenspace associated with λ , that is, the dimension of $\ker(A - \lambda I)$.

The geometric multiplicity of an eigenvalue is never greater than the algebraic multiplicity.

The resolvent set of an operator $A \in \mathcal{B}(\mathcal{H})$, denoted $\rho(A)$, is the set of complex numbers such that $(A - \lambda I) : \mathcal{H} \to \mathcal{H}$ is one-to-one and onto.

The spectrum of A, denoted $\sigma(A)$, is the complement of the resolvent set:

$$\sigma(A) = \mathbb{C} \setminus \rho(A).$$

- (a) The point spectrum of A consists of all $\lambda \in \sigma(A)$ such that $A \lambda I$ is not one-to-one. In this case, $\ker(A \lambda I) \neq \{0\}$, and λ is called an eigenvalue of A.
- (b) The continuous spectrum of A consists of all $\lambda \in \sigma(A)$ such that $A \lambda I$ is one-to-one but not onto, and ran $(A \lambda I)$ is dense in \mathcal{H} .
- (c) The residual spectrum of A consists of all $\lambda \in \sigma(A)$ such that $A \lambda I$ is one-to-one but not onto, and ran $(A \lambda I)$ is not dense in \mathcal{H} .

Define $M \in \mathcal{B}(L^2([0,1]))$ by Mf(x) = xf(x). Find and classiy the spectrum of M.

Applied Analysis Chapter 9

Define the resolvent of A at λ .

APPLIED ANALYSIS CHAPTER 9

What are some basic properties of the resolvent set of A? What does this say about the spectrum of A? Define the spectral radius of A.

APPLIED ANALYSIS CHAPTER 9

If $Mf = \lambda f$, then $(x - \lambda)f = 0$, and so f = 0. Since eigenvectors cannot be 0, then there are no eigenvalues of M.

If $\lambda \notin [0,1]$, then $(x-\lambda)f(x) \in L^2([0,1])$ since $x-\lambda$ is bounded away from 0 on [0,1]. Thus $\mathbb{C} \setminus [0,1] \in \rho(M)$.

If $\lambda \in [0,1]$, then $M-\lambda I$ is not onto since $f(x) \equiv c \in L^2$ but if $(M-\lambda I)g = f$ then $(x-\lambda)g(x) = c$, and so $g(x) = \frac{c}{x-\lambda}$, which is not an L^2 function. Let $f \in L^2$. Then define $f_n \in L^2([0,1])$ by

$$f_n(x) = \mathcal{X}_{[B_{1/n}(\lambda)]^C} f(x).$$

Then $f_n \in \text{ran}(M - \lambda I)$ since $f_n(x) = (M - \lambda I) \frac{f_n(x)}{x - \lambda}$ and $\frac{f_n(x)}{x - \lambda} \in L^2([0, 1])$. Also, $f_n \to f$ in $L^2([0, 1])$. Thus ran $(M - \lambda I)$ is dense for all $\lambda \in [0, 1]$. Thus the spectrum of M is completely continuous, and is $\sigma(M) = [0, 1]$.

Suppose $\lambda \in \rho(A)$. Then $(A - \lambda I)$ is invertible. Define the resolvent of A at λ , denoted $R(\lambda)$, by $R(\lambda) = (A - \lambda I)^{-1}$. The resolvent is an operator valued function defined on $\rho(A)$, that is,

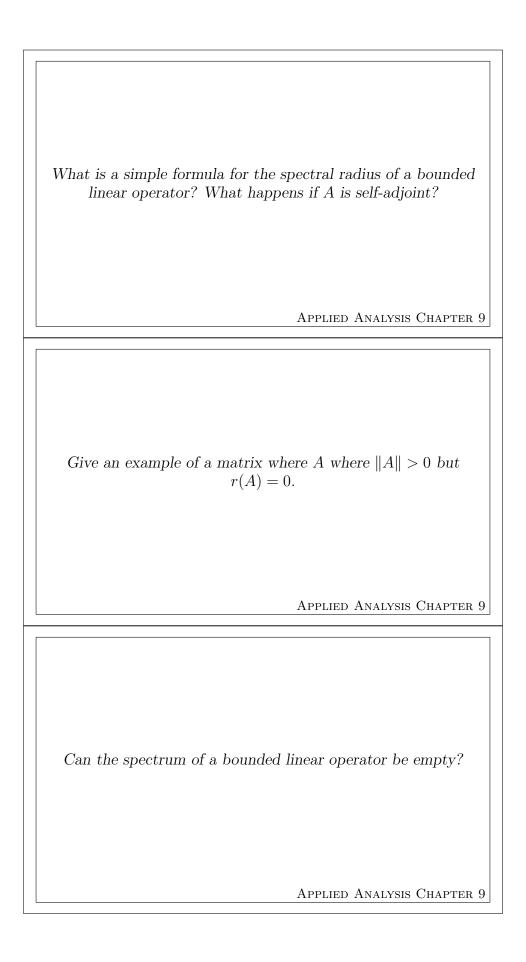
$$R: \rho(A) \to \mathcal{B}(\mathcal{H}).$$

The resolvent set $\rho(A)$ of an operator A on a Hilbert space \mathcal{H} is open and contains the exterior disc $\lambda \in \mathbb{C} \mid |\lambda| > ||A||$.

Since the $\sigma(A) = [\rho(A)]^C$, then $\sigma(A) \subset B_{\|A\|}(0)$.

The spectral radius of A, denoted r(A), is the radius of the smallest disk which contains $\sigma(A)$, that is,

$$r(A) = \sup\{|\lambda| \mid \lambda \in \sigma(A)\} = \inf\{r \mid \sigma(A) \subset B_r(0)\}\$$



For any bounded linear operator A,

$$r(A) = \lim_{n \to \infty} \|A^n\|^{\frac{1}{n}}.$$

When A is self adjoint, r(A) = ||A||.

Let A be the $n \times n$ matrix defined by

$$N = \left(\begin{array}{cccc} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{array}\right)$$

Then $N^n=0$, and thus $r(N)=\lim_{n\to\infty}\|N^n\|^{\frac{1}{n}}=0$, but $\|N\|=1$.

N is called nilpotent since r(N) = 0.

No. $\sigma(A)$ must consist of at least a single point.

Describe the spectrum of compact, self-adjoint operators.
Applied Analysis Chapter 9
Show that the eigenvalues of a bounded, self-adjoint linear operator are real, and eigenvalues associated with different eigenvectors are orthogonal.
Applied Analysis Chapter 9
What is an invariant subspace, and why are they important?
Applied Analysis Chapter 9

Let K be a compact, self-adjoint operator with $K = K^*$.

Then $\sigma(K)$ consists entirely of eigenvectors, except possibly 0, which may belong to the continuous spectrum.

Let λ be an eigenvalue of A with eigenvector u. Then

$$\lambda(u,u) = (\lambda u,u) = (Au,u) = (u,Au) = (u,\lambda u) = \overline{\lambda}(u,u),$$

which shows $\lambda = \overline{\lambda}$, i.e. $\lambda \in \mathbb{R}$.

Let λ be an eigenvalue with eigenvector u and μ be an eigenvalue with eigenvector v. Then

$$\lambda(u, v) = (\lambda u, v) = (Au, v) = (u, Av) = (u, \mu v) = \overline{\mu}(u, v) = \mu(u, v).$$

Thus, $(\lambda - \mu)(u, v) = 0$. So if $\lambda \neq \mu$, then (u, v) = 0, which shows eigenvectors corresponding to different eigenvalues are orthogonal.

Let M be a linear subspace of a Hilbert space \mathcal{H} . Then M is called an invariant subspace of a linear operator A if

$$Ax \in M, \quad \forall x \in M$$

This means $A|_M$ is a linear operator on M.

Suppose M and N are invariant subspaces of an operator A with $\mathcal{H} = M \oplus N$. Then each $x \in \mathcal{H}$ can be written uniquely as x = m + n where $m \in M$ and $n \in N$. Then

$$Ax = A|_{M}m + A|_{N}n.$$

Invariant subspaces of an operator A are important because the action of A on \mathcal{H} is completely determined by its actions on the invariant subspaces of A.

Show that for bounded, self-adjoint operators, the orthogonal complement of an invariant subspace is invariant. Applied Analysis Chapter 9	
APPLIED ANALYSIS CHAPTER 9	╛

a Hilbert sp Also, since adjoint, 0 =	an invariant subspace \mathcal{H} . Choose ar $Ax \in M$, then $(A = (Ax, y) = (x, Ay)$. Thus M^{\perp} is an invariant subspace.	a arbitrary $y \in (x, y) = 0$ for ϵ x, y , and thus Ay	M^{\perp} . Then $\forall x \in M$. Then $\forall x \in M$. Then $\perp x$ for all $x \in M$.	$x \in M, (x, y) = 0$ ten since A is sel	0. lf-