

Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain and $w = (w_1, \dots, w_n) \in \mathcal{C}^1(\overline{\Omega})$ with outward pointing normal N. Then

$$\int_{\Omega} \mathbf{\nabla \cdot } w \, \, \mathrm{d}x = \int_{\partial \Omega} w \cdot N \mathrm{d}S$$

Let Ω be a smooth domain and let $u \in \mathcal{C}^2(\overline{\Omega})$ and $v \in \mathcal{C}^1(\overline{\Omega})$ -funtions. Then we have Green's First Identity:

$$\int_{\Omega} \nabla v \cdot \nabla u + v \nabla^2 u dx = \int_{\Omega} \nabla \cdot (v \nabla u) dx = \int_{\partial \Omega} v \frac{\partial u}{\partial N} dS$$

Exchanging u and v in Green's First Identity and finding the difference gives Green's Second Identity:

$$\int_{\Omega} \left(v \nabla^2 u - u \nabla^2 v \right) dx = \int_{\partial \Omega} \left[v \frac{\partial u}{\partial N} - u \frac{\partial v}{\partial N} \right] dS$$

Test functions are smooth functions with compact support, i.e.

$$C_c^{\infty} = \{ u \in C^{\infty} : \operatorname{supp} u \subset V \subseteq \Omega \}.$$

What is a weak derivative of an L^1_{loc} function?

SHKOLLER ANALYSIS CHAPTER 2.1

Why does f have a weak-derivative, but g does not?

$$f(x) = \begin{cases} x & \text{if } x \in (0,1) \\ 1 & \text{if } x \in (1,2) \end{cases}$$

$$g(x) = \begin{cases} x & \text{if } x \in (0,1) \\ 2 & \text{if } x \in (1,2) \end{cases}$$

SHKOLLER ANALYSIS CHAPTER 2.1

Define $W^{1,p}(\Omega)$ for $1 \leq p \leq \infty$. Then define $W^{k,p}(\Omega)$ for $1 \leq p \leq \infty$ and $k \in \mathbb{N}$. What is the norm in $W^{k,p}$?

SHKOLLER ANALYSIS CHAPTER 2.1

Let $u \in L^1_{loc}(\Omega)$. Then $v^{\alpha} \in L^1_{loc}(\Omega)$ is called the α^{th} weak derivative of u, written $v^{\alpha} = D^{\alpha}u$, if

$$\int_{\Omega} u(x) D^{\alpha} \phi(x) dx = (-1)^{|\alpha|} \int_{\Omega} v^{\alpha}(x) \phi(x) dx \qquad \forall \phi \in \mathcal{C}_{c}^{\infty}(\Omega),$$

where $\alpha \in \mathbb{N}^n$ is a multi-index with $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

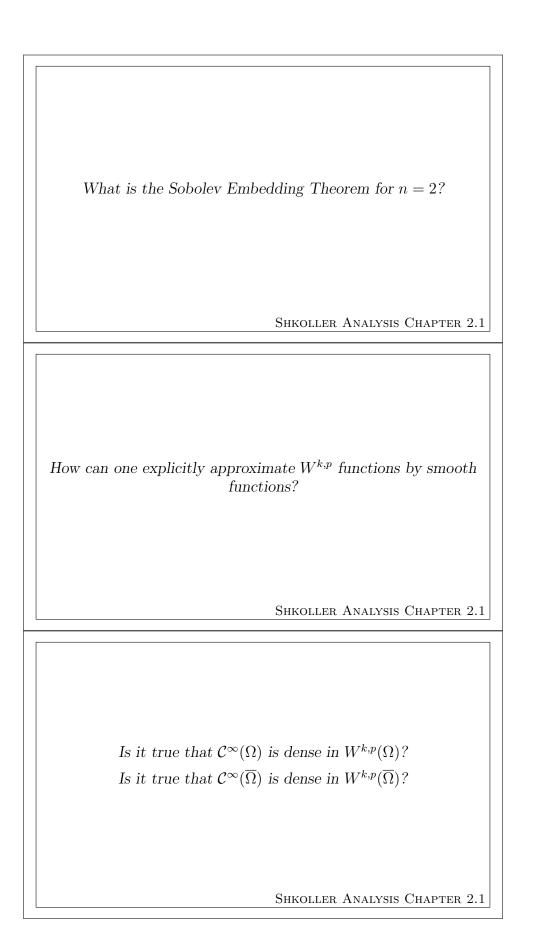
We can explicitly calculate the weak derivative (using integration by parts) of f. For g, however, assuming a weak derivative exists results in a contradiction by exploiting the boundary terms in the integration by parts that don't cancel each other out.

 $W^{1,p}(\Omega) = \{u \in L^p(\Omega) \mid \text{the weak derivative } u' \text{ of } u \text{ exists, and } u' \in L^p(\Omega)\}$

 $W^{k,p}(\Omega) = \left\{ u \in L^1_{\mathrm{loc}}(\Omega) \mid D^{\alpha}u \text{ exists and is in } L^p(\Omega) \text{ for } |\alpha| \leq k \right\}$

$$||u||_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \le k} ||D^{\alpha}u||_{L^p(\Omega)}^p\right)^{\frac{1}{p}}$$

$$||u||_{W^{k,\infty}(\Omega)} = \sum_{\alpha \le k} ||D^{\alpha}u||_{L^{\infty}(\Omega)}$$



Only consider \mathbb{R}^2 .

Let kp > 2 and $u \in \mathcal{C}_c^{\infty}(\mathbb{R}^2)$. Then

$$||u||_{L^{\infty}(\mathbb{R}^2)} \le C||u||_{W^{k,p}(\mathbb{R}^2)}$$

In English, $W^{k,p}$ functions are bounded in \mathbb{R}^2 .

Choose $u \in W^{k,p}(\Omega)$, and let $(\eta_{\varepsilon}n)$ be the standard mollifiers (increasingly concentrated Gaussian curves). For each ε , define

$$u^{\varepsilon} = \eta_{\varepsilon} * u$$

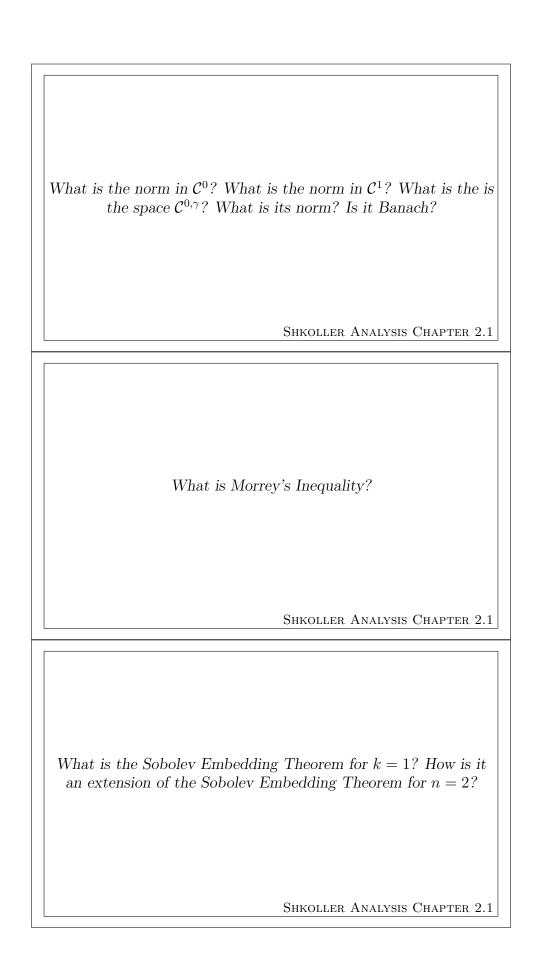
and

$$\Omega_{\varepsilon} = \{ x \in \Omega \mid \operatorname{dist}(x, \partial \Omega) > \varepsilon \}.$$

Then

- (A) $u^{\varepsilon} \in C^{\infty}(\Omega_{\varepsilon})$ for each $\varepsilon > 0$, and
- (B) $u^{\varepsilon} \to u$ in $W_{\text{loc}}^{k,p}(\Omega)$ as $\varepsilon \to 0$.

Yes and yes.



 \mathcal{C}^0 is the space of continuous functions and \mathcal{C}^1 is the space of continuously differentiable functions. They both are Banach spaces and their norms are

$$\begin{split} \|u\|_{\mathcal{C}^0(\overline{\Omega})} &= \sup_{x \in \overline{\Omega}} \|u(x)\| \\ \|u\|_{\mathcal{C}^1(\overline{\Omega})} &= \|u\|_{\mathcal{C}^0(\overline{\Omega})} + \|Du\|_{\mathcal{C}^0(\overline{\Omega})} \end{split}$$

The space $C^{0,\gamma}(\overline{\Omega})$ is called a Hölder space. The Hölder norm is

$$||u||_{\mathcal{C}^{0,\gamma}(\overline{\Omega})} = ||u||_{\mathcal{C}^{0}(\overline{\Omega})} + [u]_{\mathcal{C}^{0,\gamma}(\overline{\overline{\Omega}})}$$

where

$$\left[u\right]_{\mathcal{C}^{0,\gamma}(\overline{\Omega})} = \max_{\substack{x,y \in \Omega \\ x \neq y}} \left(\frac{|u(x) - u(y)|}{|x - y|^{\gamma}}\right).$$

Hölder spaces are Banach.

Let $B_r \subset \mathbb{R}^n$ be a ball of radius r, and let $n . For <math>x, y \in B_r$, we have Morrey's Inequality:

$$|u(x) - u(y)| \le C|x - y|^{1 - \frac{n}{p}} ||Du||_{L^p(B_r)}$$
 $\forall u \in W^{1,p}(B_r).$

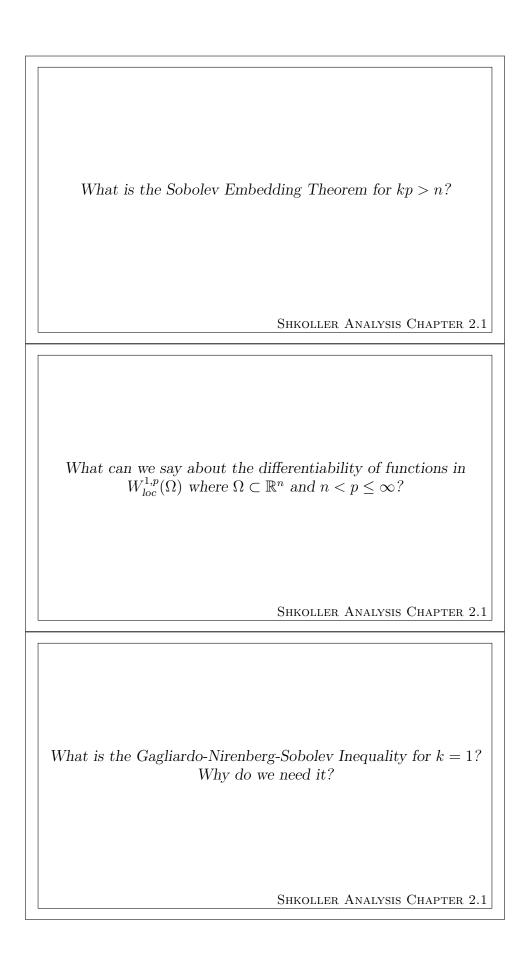
The Sobolev Embedding Theorem is: there is a constant C = C(p, n) such that

$$||u||_{\mathcal{C}^{0,1-\frac{n}{p}}(\mathbb{R}^n)} \le C||u||_{W^{1,p}(\mathbb{R}^n)} \qquad \forall u \in W^{1,p}(\mathbb{R}^n).$$

The Sobolev Embedding Theorem for n=2 only applies to C_C^{∞} functions in \mathbb{R}^2 . To get back to that result from this theorem, choose $n=2, p=\infty$, k=1, and $u\in C^{\infty}(\Omega)$ where $\Omega\subseteq \mathbb{R}^2$. Then we see that

$$||u||_{\mathcal{C}^1(\Omega)} \le C||u||_{W^{1,\infty}(\Omega)}.$$

Then since the continuous functions on compact sets are bounded, the L^{∞} norm is bounded by the C^1 norm, and we get the result.



There is a constant C = C(k, p, n) such that

$$\|u\|_{\mathcal{C}^{k-\lfloor \frac{n}{p} \rfloor - 1, \gamma}(\mathbb{R}^n)} \le C\|u\|_{W^{k, p}(\mathbb{R}^n)}$$

where

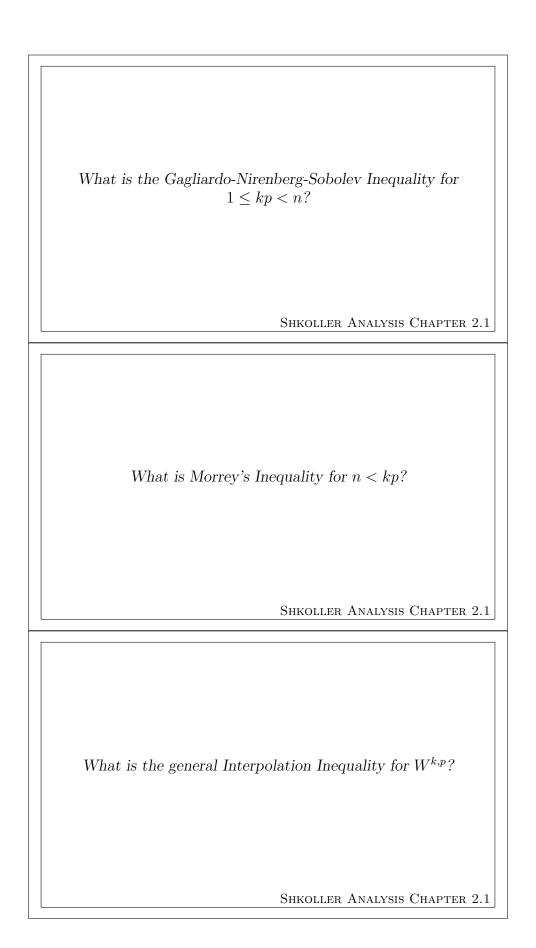
$$\gamma = \begin{cases} \left\lfloor \frac{n}{p} \right\rfloor + 1 - \frac{n}{p} & \text{if } \frac{n}{p} \notin \mathbb{N} \\ \text{any } \alpha \in \mathbb{R} \cap (0, 1) & \text{if } \frac{n}{p} \in \mathbb{N} \end{cases}$$

This inequality relates a function's differentiability with its smoothness. That is, if a function is in $W^{k,p}$, it is "differentiable" a certain number of times, i.e. $W^{k,p}$ functions necessarily have an amount of "smoothness".

If $u \in W^{1,p}_{loc}(\Omega)$, then u is differentiable (a.e.) in Ω and its derivative is equal to the weak derivative (a.e.).

For
$$1 \le p < n$$
, let $p^* = \frac{np}{n-p}$, i.e. $\frac{1}{p} - \frac{1}{p^*} = \frac{1}{n}$. Then
$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \le C(p,n) \|Du\|_{L^p(\mathbb{R}^n)} \qquad \forall u \in W^{1,p}(\mathbb{R}^n)$$

This inequality relates the decay of a function's derivative to its own decay. If a function's derivative decays L^p fast, then the function decays even faster, specifically, L^{p^*} .



Suppose $D^k u \in L^p(\mathbb{R}^n)$. Then $u \in L^{\frac{np}{n-kp}}(\mathbb{R}^n)$ and

$$||u||_{L^{\frac{np}{n-kp}}(\mathbb{R}^n)} \le C||D^k u||_{L^p(\mathbb{R}^n)}$$

Also, if $\Omega \subset \mathbb{R}^n$ is open, bounded, has a \mathcal{C}^1 boundary, then

$$||u||_{L^{\frac{np}{n-kp}}(\Omega)} \le C||u||_{W^{k,p}(\Omega)}$$

Let $u \in W^{k,p}(\mathbb{R}^n)$. Then

$$\|u\|_{\mathcal{C}^{k-1-\left\lfloor\frac{n}{p}\right\rfloor,1+\left\lfloor\frac{n}{p}\right\rfloor-\frac{n}{p}}(\mathbb{R}^n)} \leq C\|u\|_{W^{k,p}(\mathbb{R}^n)}$$

Let $n \in \mathbb{N}$ and p, q, r, j, k, ℓ satisfy the relations

$$\begin{split} j \leq k < \ell & p, q, r \geq 1 & 0 < \alpha \leq 1 \\ & \frac{1}{p} - \frac{k}{n} = \alpha \left(\frac{1}{q} - \frac{\ell}{n}\right) + (1 - \alpha) \left(\frac{1}{r} - \frac{j}{n}\right) \\ & \frac{1}{p} - \frac{k}{n} > \frac{1}{q} - \frac{\ell}{n} \geq \frac{1}{p} - \frac{k+1}{n} \end{split}$$

Then $\exists C = C(p, q, r, j, k, \ell)$ such that

$$||u||_{W^{k,p}(\mathbb{R}^n)} \le C||u||_{W^{\ell,q}(\mathbb{R}^n)}^{\alpha}||u||_{W^{j,r}(\mathbb{R}^n)}^{1-\alpha}.$$

That is, if a function has a q-summable $\ell^{\rm th}$ derivative and an r-summable $j^{\rm th}$ derivative, then it has a p-summable $k^{\rm th}$ derivative.