State the Divergence Theorem

State Green's First and Second Identities.

What is a "test function"?

Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain and $w = (w_1, \dots, w_n) \in \mathcal{C}^1(\overline{\Omega})$ with outward pointing normal N. Then

$$\int_\Omega \boldsymbol{\nabla} \cdot w \, \mathrm{d}x = \int_{\partial\Omega} w \cdot N \mathrm{d}S$$

Let Ω be a smooth domain and let $u \in \mathcal{C}^2(\overline{\Omega})$ and $v \in \mathcal{C}^1(\overline{\Omega})$-funtions. Then we have Green's First Identity:

$$\int_\Omega \boldsymbol{\nabla} v \cdot \boldsymbol{\nabla} u + v\nabla^2 u \mathrm{d}x = \int_\Omega \boldsymbol{\nabla} \cdot (v \boldsymbol{\nabla} u) \mathrm{d}x = \int_{\partial\Omega} v \frac{\partial u}{\partial N} \mathrm{d}S$$

Exchanging u and v in Green's First Identity and finding the difference gives Green's Second Identity:

$$\int_\Omega \left(v\nabla^2 u - u\nabla^2 v\right) \mathrm{d}x = \int_{\partial\Omega} \left[v \frac{\partial u}{\partial N} - u \frac{\partial v}{\partial N} \right] \mathrm{d}S$$

Test functions are smooth functions with compact support, i.e.

$$\mathcal{C}_c^\infty = \{ u \in \mathcal{C}^\infty \ : \ \operatorname{supp} u \subset \mathcal{V} \Subset \Omega \} .$$

What is a weak derivative of an L^1_{loc} function?

Why does f have a weak-derivative, but g does not?

$$f(x) = \begin{cases} x & \text{if } x \in (0,1) \\ 1 & \text{if } x \in (1,2) \end{cases}$$

$$g(x) = \begin{cases} x & \text{if } x \in (0,1) \\ 2 & \text{if } x \in (1,2) \end{cases}$$

Define $W^{1,p}(\Omega)$ for $1 \leq p \leq \infty$. Then define $W^{k,p}(\Omega)$ for $1 \leq p \leq \infty$ and $k \in \mathbb{N}$. What is the norm in $W^{k,p}$?

Let $u \in L^1_{\text{loc}}(\Omega)$. Then $v^\alpha \in L^1_{\text{loc}}(\Omega)$ is called the α^{th} weak derivative of u, written $v^\alpha = D^\alpha u$, if

$$\int_\Omega u(x) D^\alpha \phi(x) \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega v^\alpha(x) \phi(x) \mathrm{d}x \qquad \forall \phi \in \mathcal{C}_c^\infty(\Omega),$$

where $\alpha \in \mathbb{N}^n$ is a multi-index with $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

We can explicitly calculate the weak derivative (using integration by parts) of f. For g, however, assuming a weak derivative exists results in a contradiction by exploiting the boundary terms in the integration by parts that don't cancel each other out.

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) \mid \text{the weak derivative } u' \text{ of } u \text{ exists, and } u' \in L^p(\Omega)\}$$

$$W^{k,p}(\Omega) = \left\{u \in L^1_{\text{loc}}(\Omega) \mid D^\alpha u \text{ exists and is in } L^p(\Omega) \text{ for } |\alpha| \leq k\right\}$$

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}$$

$$\|u\|_{W^{k,\infty}(\Omega)} = \sum_{\alpha \leq k} \|D^\alpha u\|_{L^\infty(\Omega)}$$

What is the Sobolev Embedding Theorem for $n = 2$?

Shkoller Analysis Chapter 2.1

How can one explicitly approximate $W^{k,p}$ functions by smooth functions?

Shkoller Analysis Chapter 2.1

Is it true that $\mathcal{C}^\infty(\Omega)$ is dense in $W^{k,p}(\Omega)$?

Is it true that $\mathcal{C}^\infty(\overline{\Omega})$ is dense in $W^{k,p}(\overline{\Omega})$?

Shkoller Analysis Chapter 2.1

Only consider $\mathbb{R}^2$.

Let $kp > 2$ and $u \in \mathcal{C}_c^\infty(\mathbb{R}^2)$. Then

$$\|u\|_{L^\infty(\mathbb{R}^2)} \leq C\|u\|_{W^{k,p}(\mathbb{R}^2)}$$

In English, $W^{k,p}$ functions are bounded in $\mathbb{R}^2$.

Choose $u \in W^{k,p}(\Omega)$, and let $(\eta_\varepsilon n)$ be the standard mollifiers (increasingly concentrated Gaussian curves). For each ε, define

$$u^\varepsilon = \eta_\varepsilon * u$$

and

$$\Omega_\varepsilon = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \varepsilon\}.$$

Then

(A) $u^\varepsilon \in C^\infty(\Omega_\varepsilon)$ for each $\varepsilon > 0$, and

(B) $u^\varepsilon \to u$ in $W^{k,p}_{\mathrm{loc}}(\Omega)$ as $\varepsilon \to 0$.

Yes and yes.

What is the norm in $\mathcal{C}^0$? What is the norm in $\mathcal{C}^1$? What is the is the space $\mathcal{C}^{0,\gamma}$? What is its norm? Is it Banach?

What is Morrey's Inequality?

What is the Sobolev Embedding Theorem for $k = 1$? How is it an extension of the Sobolev Embedding Theorem for $n = 2$?

$\mathcal{C}^0$ is the space of continuous functions and $\mathcal{C}^1$ is the space of continuously differentiable functions. They both are Banach spaces and their norms are

$$\|u\|_{\mathcal{C}^0(\overline{\Omega})} = \sup_{x\in\overline{\Omega}} \|u(x)\|$$
$$\|u\|_{\mathcal{C}^1(\overline{\Omega})} = \|u\|_{\mathcal{C}^0(\overline{\Omega})} + \|Du\|_{\mathcal{C}^0(\overline{\Omega})}$$

The space $\mathcal{C}^{0,\gamma}(\overline{\Omega})$ is called a Hölder space. The Hölder norm is

$$\|u\|_{\mathcal{C}^{0,\gamma}(\overline{\Omega})} = \|u\|_{\mathcal{C}^0(\overline{\Omega})} + \left[u\right]_{\mathcal{C}^{0,\gamma}(\overline{\overline{\Omega}})}$$

where

$$\left[u\right]_{\mathcal{C}^{0,\gamma}(\overline{\Omega})} = \max_{\substack{x,y\in\Omega \\ x\neq y}} \left(\frac{|u(x)-u(y)|}{|x-y|^{\gamma}} \right).$$

Hölder spaces are Banach.

Let $B_r \subset \mathbb{R}^n$ be a ball of radius r, and let $n < p \leq \infty$. For $x, y \in B_r$, we have Morrey's Inequality:

$$|u(x) - u(y)| \leq C|x-y|^{1-\frac{n}{p}} \|Du\|_{L^p(B_r)} \qquad \forall u \in W^{1,p}(B_r).$$

The Sobolev Embedding Theorem is: there is a constant $C = C(p, n)$ such that

$$\|u\|_{\mathcal{C}^{0,1-\frac{n}{p}}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)} \qquad \forall u \in W^{1,p}(\mathbb{R}^n).$$

The Sobolev Embedding Theorem for $n = 2$ only applies to C_C^∞ functions in $\mathbb{R}^2$. To get back to that result from this theorem, choose $n = 2$, $p = \infty$, $k = 1$, and $u \in C^\infty(\Omega)$ where $\Omega \Subset \mathbb{R}^2$. Then we see that

$$\|u\|_{\mathcal{C}^1(\Omega)} \leq C\|u\|_{W^{1,\infty}(\Omega)}.$$

Then since the continuous functions on compact sets are bounded, the L^∞ norm is bounded by the $\mathcal{C}^1$ norm, and we get the result.

What is the Sobolev Embedding Theorem for $kp > n$?

Shkoller Analysis Chapter 2.1

What can we say about the differentiability of functions in $W^{1,p}_{loc}(\Omega)$ where $\Omega \subset \mathbb{R}^n$ and $n < p \leq \infty$?

Shkoller Analysis Chapter 2.1

What is the Gagliardo-Nirenberg-Sobolev Inequality for $k = 1$? Why do we need it?

Shkoller Analysis Chapter 2.1

There is a constant $C = C(k, p, n)$ such that

$$\|u\|_{\mathcal{C}^{k-\lfloor \frac{n}{p} \rfloor -1,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{W^{k,p}(\mathbb{R}^n)}$$

where

$$\gamma = \begin{cases} \left\lfloor \frac{n}{p} \right\rfloor + 1 - \frac{n}{p} & \text{if } \frac{n}{p} \notin \mathbb{N} \\ \text{any } \alpha \in \mathbb{R} \cap (0, 1) & \text{if } \frac{n}{p} \in \mathbb{N} \end{cases}$$

This inequality relates a function's differentiability with its smoothness. That is, if a function is in $W^{k,p}$, it is "differentiable" a certain number of times, i.e. $W^{k,p}$ functions necessarily have an amount of "smoothness".

If $u \in W^{1,p}_{\text{loc}}(\Omega)$, then u is differentiable (a.e.) in Ω and its derivative is equal to the weak derivative (a.e.).

For $1 \leq p < n$, let $p^* = \dfrac{np}{n-p}$, i.e. $\dfrac{1}{p} - \dfrac{1}{p^*} = \dfrac{1}{n}$. Then

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C(p, n)\|Du\|_{L^p(\mathbb{R}^n)} \qquad \forall u \in W^{1,p}(\mathbb{R}^n)$$

This inequality relates the decay of a function's derivative to its own decay. If a function's derivative decays L^p fast, then the function decays even faster, specifically, L^{p^*}.

What is the Gagliardo-Nirenberg-Sobolev Inequality for $1 \leq kp < n$?

What is Morrey's Inequality for $n < kp$?

What is the general Interpolation Inequality for $W^{k,p}$?

Suppose $D^k u \in L^p(\mathbb{R}^n)$. Then $u \in L^{\frac{np}{n-kp}}(\mathbb{R}^n)$ and

$$\|u\|_{L^{\frac{np}{n-kp}}(\mathbb{R}^n)} \leq C\left\|D^k u\right\|_{L^p(\mathbb{R}^n)}$$

Also, if $\Omega \subset \mathbb{R}^n$ is open, bounded, has a $\mathcal{C}^1$ boundary, then

$$\|u\|_{L^{\frac{np}{n-kp}}(\Omega)} \leq C\|u\|_{W^{k,p}(\Omega)}$$

Let $u \in W^{k,p}(\mathbb{R}^n)$. Then

$$\|u\|_{\mathcal{C}^{k-1-\lfloor \frac{n}{p} \rfloor, 1+\lfloor \frac{n}{p} \rfloor - \frac{n}{p}}(\mathbb{R}^n)} \leq C\|u\|_{W^{k,p}(\mathbb{R}^n)}$$

Let $n \in \mathbb{N}$ and p, q, r, j, k, ℓ satisfy the relations

$$j \leq k < \ell \qquad p, q, r \geq 1 \qquad 0 < \alpha \leq 1$$

$$\frac{1}{p} - \frac{k}{n} = \alpha\left(\frac{1}{q} - \frac{\ell}{n}\right) + (1-\alpha)\left(\frac{1}{r} - \frac{j}{n}\right)$$

$$\frac{1}{p} - \frac{k}{n} > \frac{1}{q} - \frac{\ell}{n} \geq \frac{1}{p} - \frac{k+1}{n}$$

Then $\exists C = C(p, q, r, j, k, \ell)$ such that

$$\|u\|_{W^{k,p}(\mathbb{R}^n)} \leq C\|u\|^{\alpha}_{W^{\ell,q}(\mathbb{R}^n)}\|u\|^{1-\alpha}_{W^{j,r}(\mathbb{R}^n)}.$$

That is, if a function has a q-summable ℓ^{th} derivative and an r-summable j^{th} derivative, then it has a p-summable k^{th} derivative.