1 Sobolev Embeddings

1.1 On $\mathbb{T}$

Let $k > \frac{1}{2}$. Then

$$\|f\|_{C(\mathbb{T})} \leq C\|f\|_{H^k(\mathbb{T})}$$

That is, if f has at least half of a derivative on a 1-dimensional compact subset of $\mathbb{R}$, then it is continuous.

1.2 On compact subsets of $\mathbb{R}^2$

Let $\Omega \Subset \mathbb{R}^2$ and $kp > 2$. Then

$$\|f\|_{L^\infty(\Omega)} \leq C\|f\|_{W^{k,p}(\Omega)}$$

That is, functions on compact intervals with enough derivatives, and/or are p-integrable enough, are bounded.

1.3 On $\mathbb{R}^n$, with $k = 1$

Let $p > n$. Then

$$\|f\|_{C^0(\mathbb{R}^n)} \leq \underbrace{\|f\|_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)}}_{\text{Hölder-norm}} \leq C\|f\|_{W^{1,p}(\mathbb{R}^n)}$$

That is, in higher dimensions, it takes higher regularity to ensure continuity.

1.4 On $\mathbb{R}^n$, in general

Let $kp > n$. Then

$$\|f\|_{C^{k-\lfloor\frac{n}{p}\rfloor-1}(\mathbb{R}^n)} \leq \underbrace{\|f\|_{C^{k-\lfloor\frac{n}{p}\rfloor-1,\gamma}(\mathbb{R}^n)}}_{\text{Hölder-norm}} \leq C\|f\|_{W^{k,p}(\mathbb{R}^n)}$$

2 Gagliardo-Nirenburg-Sobolev Inequalities

2.1 On $\mathbb{R}^n$, with $k = 1$

Let $1 \leq p < n$, and define $p^* := \dfrac{np}{n-p}$. Then $\forall f \in W^{1,p}(\mathbb{R}^n)$,

$$\|f\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Df\|_{L^p(\mathbb{R}^n)}$$

2.2 On $\mathbb{R}^n$, in general

Let $1 \leq kp < n$, and define $p^* := \dfrac{np}{n-kp}$. Then $\forall f \in W^{k,p}(\mathbb{R}^n)$,

$$\|f\|_{L^{p^*}(\mathbb{R}^n)} \leq C\left\|D^k f\right\|_{L^p(\mathbb{R}^n)}$$

Furthermore, if Ω is open, bounded, and has C^1 boundary, then

$$\|f\|_{L^{p^*}(\Omega)} \leq C\|f\|_{W^{k,p}(\Omega)}$$

3 Young's Inequality

Let $1 + \dfrac{1}{r} = \dfrac{1}{p} + \dfrac{1}{q}$. Then

$$\|f * g\|_r \leq \|f\|_p \|g\|_q$$

4 Hölder's Inequality

Let $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

5 Interpolation in Lebesgue Space

Let $\frac{1}{p} = \frac{a}{q} + \frac{1-a}{r}$. Then

$$\|f\|_p \leq \|f\|_q^a + \|f\|_r^{1-a}$$

That is, functions in L^q and L^r are in L^p for any $p \in (q, r)$.

6 Tips and Tricks to Remember

- Bounding a convolution? Use Young's.
- Proving continuity or countinuous differentiability? Use Sobolev Embeddings.
- Proving regularity? Use Gagliardo-Nirenburg-Sobolev Inequalities.
- Can you prove a function in L^1 is bounded? Use Interpolation.
- Bounding an integral? Use Hölder's.

7 Spectrum of Bounded Linear Operators on Hilbert Spaces

7.1 In General

Let $A \in \mathcal{B}(\mathcal{H})$. Then

- $\sigma(A) \subset B_{\|A\|}(0)$.
- $\sigma(A)$ is closed.
- $\sigma(A) \neq \emptyset$
- $r(A) = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}$.
- $\lambda \in \mathrm{resi}(A) \implies \overline{\lambda}$ is an eigenvalue of A^*.

7.2 For Self-Adjoint Operators

Let $A \in \mathcal{B}(\mathcal{H})$ such that $A = A^*$. Then

- $\sigma(A) \subset \mathbb{R}$.
- $\mathrm{resi}(A) = \emptyset$.
- $r(A) = \|A\|$.
- Eigenvectors correspoding to distinct eigenvalues are orthogonal.

7.3 For Compact, Self-Adjoint Operators

Let $A \in \mathcal{B}(\mathcal{H})$ such that $A = A^*$ and A is compact. Then

- $\sigma(A)$ consists entirely of eigenvalues, except possibly 0, which may be in the continuous spectrum.
- Every nonzero eigenvalue has finite multiplicity, that is, the dimension of the eigenspace is finite, i.e. $\dim \ker[A-\lambda I] < \infty$.
- If $\sigma(A)$ has an accumulation point, it must be 0. There are no other accumulation points.
- A can be represented as a convergent (in operator norm) series of prejections onto eigenspaces. That is,

$$A = \sum_{\lambda\in\sigma(A)} \lambda P_{\ker[A-\lambda I]}$$