

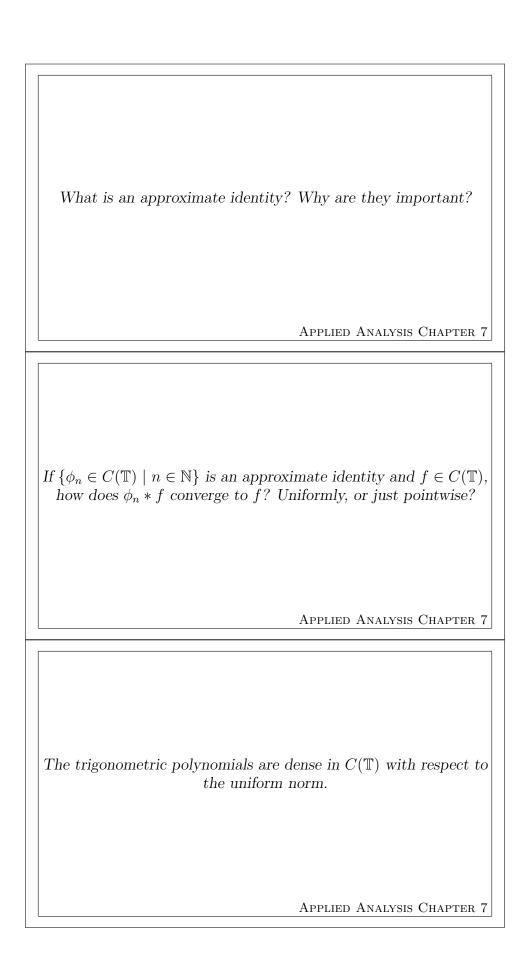
The Fourier basis is  $\{e_n \mid n \in \mathbb{Z}\}$  where

$$e_n(x) = \frac{1}{\sqrt{2\pi}}e^{inx}.$$

A "trigonometric polynomial" is a special name for finite linear combinations of the Fourier basis. They are important since they are dense in  $C(\mathbb{T})$ . Proving they are dense in  $C(\mathbb{T})$ , along with showing orthonormality, proves the Fourier basis is indeed a basis of  $L^2(\mathbb{T})$ .

The convolution of two continuous functions  $f,g:\mathbb{T}\to\mathbb{C}$ , denoted f\*g, is a continuous function defined by the following integral:

$$(f * g)(x) = \int_{\mathbb{T}} f(x - y)g(y)dy = \int_{\mathbb{T}} f(y)g(x - y)dy$$



A family of functions  $\{\phi_n \in C(\mathbb{T}) \mid n \in \mathbb{N}\}\$  is an approximate identity if

- (a)  $\phi_n(x) \ge 0$ ;
- (b)  $\int_{\mathbb{T}} \phi_n(x) dx = 1$  for every  $n \in \mathbb{N}$ ;
- (c)  $\lim_{n} \to \infty \int_{\delta \le |x| \le \pi} \phi_n(x) dx = 0$  for every  $0 < \delta \le \pi$ .

## In English:

- (a) All functions are non-negative.
- (b) The area under each curve is 1.
- (c) As  $n \to \infty$ , most of the area accumulates near 0.

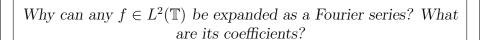
Approximate identities are important since for large n, the convolution of f with  $\phi_n$  gives a local average of f.

## Uniformly. The proof uses:

- $f(x) = \int_{\mathbb{T}} \phi_n(y) f(x) dy$  (clever since f(x) is independent of y)
- Splitting the integral in to y in a  $\delta$ -ball around 0 and y outside of that  $\delta$ -ball
- $\bullet$  Different components of the product are small on different subsets of  $\mathbb{T}.$

## The proof uses:

- A specific approximate identity:  $\phi_n(x) = c_n(1 + \cos x)^n$ , where  $c_n$  is chosen so  $\int_{\mathbb{T}} \phi_n = 1$ .
- Since  $\phi_n * f \to f$  uniformly, just show it is in fact a trigonometric polynomial.
- Noting  $\phi_n$  is a trigonometric polynomial:  $\phi_n(x) = \sum_{k=-n}^n a_{n,k} e^{ikx}$  where  $2^{-n} c_n \binom{2n}{n+k}$ .



## APPLIED ANALYSIS CHAPTER 7

What is Parseval's Identity and why is it important in Fourier theory?

APPLIED ANALYSIS CHAPTER 7

If 
$$f,g\in L^2(\mathbb{T})$$
, then  $f*g$  is continuous  $(f*g\in C(\mathbb{T}))$  and 
$$\|f*g\|_\infty \leq \|f\|_2 \|g\|_2$$

APPLIED ANALYSIS CHAPTER 7

Since trigonometric polynomials are dense in  $C(\mathbb{T})$ , which is dense in  $L^2(\mathbb{T})$ , any  $f \in L^2(\mathbb{T})$  can be written as

$$f(x) = \sum_{n = -\infty}^{\infty} \hat{f}_n e_n(x)$$

where  $e_n(x) = \frac{1}{\sqrt{2\pi}}e^{inx}$ , and so

$$f(x) = \frac{1}{2\pi} \sum_{n = -\infty}^{\infty} \hat{f}_n e^{inx}.$$

The orthonormality of  $\{e_n\}$  provides a concrete calculation of  $\hat{f}_n$ :

$$\hat{f}_n = (e_n, f) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} f(x)e^{-inx} dx.$$

Parseval's Identity is  $(f,g) = \sum_{n \in \mathbb{N}} \overline{f_n} g_n$  where  $f_n$  and  $g_n$  are the coefficients of the expansions of f and g with respect to an orthonormal basis. In Fourier theory, we have

$$\int_{\mathbb{T}} \overline{f(x)} g(x) dx = (f, g) = \sum_{n = -\infty}^{\infty} \overline{\hat{f}_n} \hat{g}_n$$

Taking g = f gives

$$||f||_{L^{2}(\mathbb{T})}^{2} = \int_{\mathbb{T}} |f(x)|^{2} dx = \sum_{n=-\infty}^{\infty} |\hat{f}_{n}|^{2} = ||(\hat{f}_{n})||_{\ell^{2}(\mathbb{Z})}^{2}$$

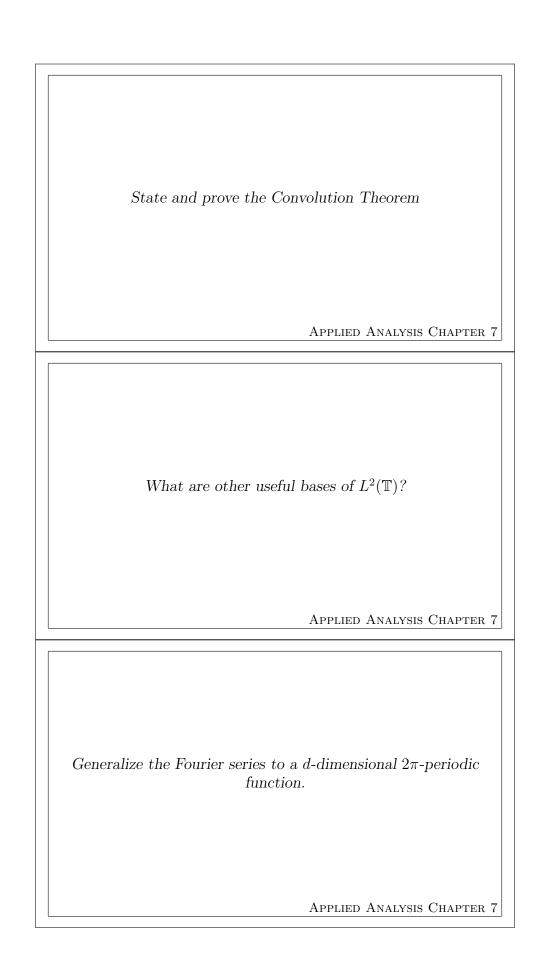
and thus we can define a Hilbert space isomorphism  $\mathcal{F}$  between  $L^2(\mathbb{T})$  and  $\ell^2(\mathbb{T})$  by

$$\mathcal{F}f = \left(\hat{f}_n\right)_{n=-\infty}^{\infty}.$$

The proof uses:

- The Cauchy-Schwarz inequality applied to the definition of convolution
- The convolution of continuous functions in continuous
- Forming continuous approximations of f and g (sequences of continuous functions approaching f and g)
- Triangle Inequality
- Completeness of  $C(\mathbb{T})$

This is a special case of Young's inequality.



For  $f, g \in L^2(\mathbb{T})$ ,  $\widehat{(f * g)}_n = \sqrt{2\pi} \hat{f}_n \hat{g}_n$ . First let  $f, g \in C(\mathbb{T})$ . Then

$$\widehat{(f*g)}_n = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} (f*g)(x) e^{-inx} dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} \left( \int_{\mathbb{T}} f(x-y) g(y) dy \right) e^{-inx} dx$$

We can use Fubini's Theorem since the integrand is continuous:

$$\widehat{(f * g)}_n = \int_{\mathbb{T}} \frac{1}{\sqrt{2\pi}} \left( \int_{\mathbb{T}} f(x - y) e^{-in(x - y)} dx \right) g(y) e^{iny} dy$$

by multiplying and dividing by  $e^{iny}$ . Then

$$\widehat{(f * g)}_n = \widehat{f}_n \int_{\mathbb{T}} g(y) e^{-iny} dy = \sqrt{2\pi} \widehat{f}_n \widehat{g}_n$$

By density of  $C(\mathbb{T})$  in  $L^2(\mathbb{T})$ , and continuity of the Fourier transform and the convolution with respect to  $L^2$  convergence, this result holds for  $L^2(\mathbb{T})$  functions too.

 $\{e_n(x) = e^{inx} \mid n \in \mathbb{Z}\}$  is an orthogonal (not orthonormal) basis. Then

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}_n e^{inx}, \quad \text{where} \quad \hat{f}_n = \frac{1}{2\pi} \int_{\mathbb{T}} f(x) e^{-inx} dx$$

 $\{1,\cos(nx),\sin(nx)\mid n=1,2,\dots\}$  is an orthogonal basis. Then

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

where  $a_n = \frac{1}{\pi} \int_{\mathbb{T}} f(x) \cos(nx) dx$  and  $b_n(x) = \frac{1}{\pi} \int_{\mathbb{T}} \sin(nx) dx$ . These are useful since odd(even) functions have  $\sin(\cos)$  expansions. Finally, any function defined on  $[0, \pi]$  can be extended to an odd(even) function on  $[-\pi, \pi]$  and can thus be represented by a  $\sin(\cos)$  expansion.

A function  $f: \mathbb{R}^d \to \mathbb{C}$  is  $2\pi$ -periodic in each variable if

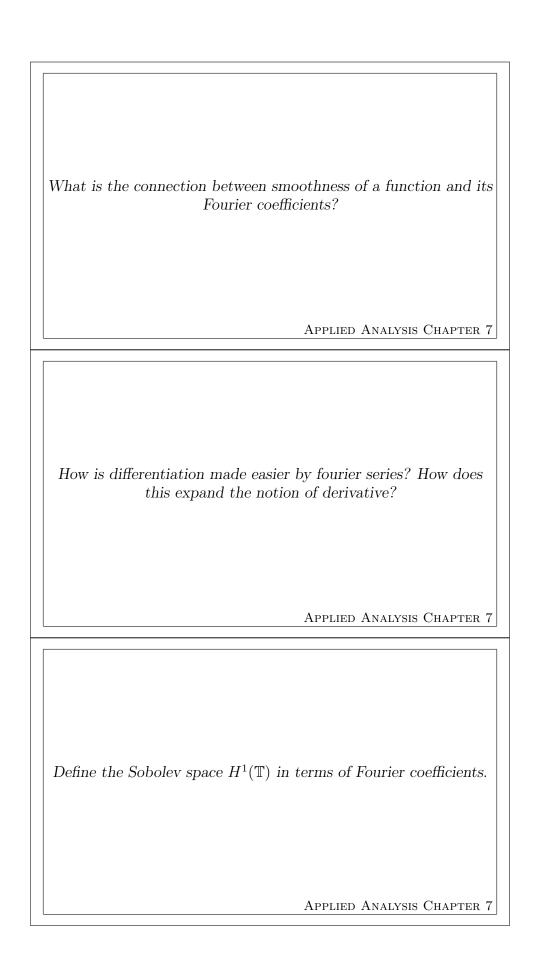
$$f(x_1, x_2, \dots, x_i + 2\pi, \dots, x_d) = f(x_1, x_2, \dots, x_i, \dots, x_d)$$
 for  $i = 1, \dots, d$ .

An orthonormal basis of  $L^2(\mathbb{T}^d)$  is

$$e_{\mathbf{n}}(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}} e^{i\mathbf{n}\cdot\mathbf{x}}, \quad \text{where } \mathbf{x} \in \mathbb{T}^d, \ \mathbf{n} \in \mathbb{Z}^d, \ \text{and } \mathbf{n} \cdot \mathbf{x} = \sum_{i=1}^d n_i x_i.$$

The Fourier series expansion of a function  $f \in L^2(\mathbb{T}^d)$  is

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}} \sum_{\mathbf{n} \in \mathbb{Z}^d} \hat{f}_{\mathbf{n}} e^{i\mathbf{n} \cdot \mathbf{x}}, \quad \text{where } \hat{f}_{\mathbf{n}} = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{T}^d} f(\mathbf{x}) e^{-i\mathbf{n} \cdot \mathbf{x}} d\mathbf{x}$$



The smoother the function, the faster its Fourier coefficients decay. Smooth functions contain small amounts of high-frequency components.

The Fourier coefficients of a derivative is a scalar multiple (in particular, in) of the Fourier coefficients of the original function.

$$\widehat{f'}_n = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{inx} f'(x) \mathrm{d}x.$$

Integration by parts gives

$$\hat{f'}_n = \frac{1}{\sqrt{2\pi}} [f(2\pi) - f(0)] - \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} (-in)e^{-inx} f(x) dx = in\hat{f}_n.$$

Induction gives  $\widehat{f^{(k)}}_n = (in)^k \widehat{f}_n$ . Heuristically, derivatives are a "roughing" operation, adding higher amounts of high-frequency components.

We can take a "derivative" of an arbitrary  $L^2$  function by transforming it into an  $\ell^2$  sequence and multiplying each component by in. This leads to what is called a "weak derivative."

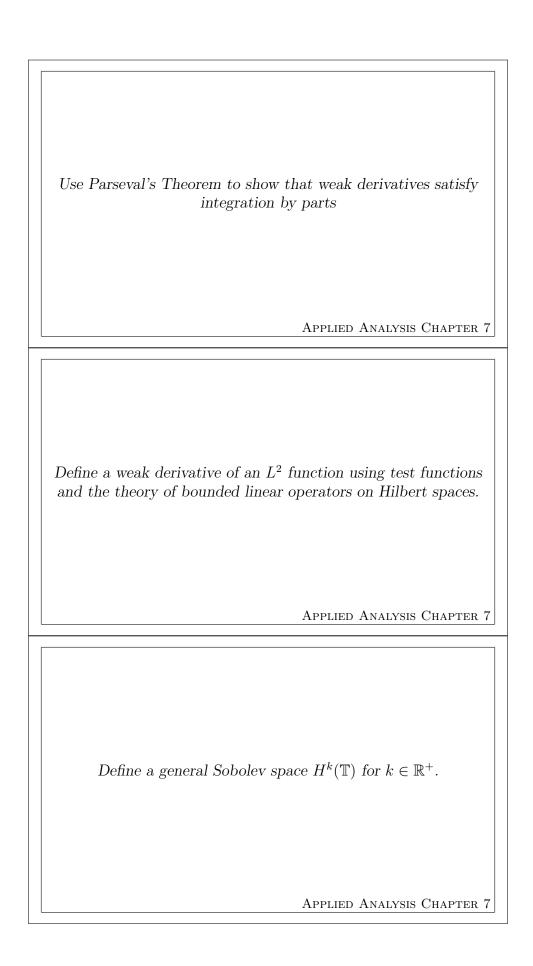
The Sobolev space  $H^1(\mathbb{T}) = W^{1,2}(\mathbb{T})$  consists of all functions

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{N}} \hat{f}_n e^{inx} \in L^2(\mathbb{T}) \quad \text{such that} \quad \sum_{n \in \mathbb{N}} n^2 \Big| \hat{f}_n \Big|^2 < \infty.$$

That is, all functions  $f \in L^2(\mathbb{T})$  such that the weak derivative of f, denoted f', is also in  $L^2(\mathbb{T})$ . f' is given by  $f'(x) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{N}} in \hat{f}_n e^{inx}$ . The inner product on  $H^1(\mathbb{T})$  is given by

$$(f,g)_{H^1(\mathbb{T})} = \int_{\mathbb{T}} \left[ \overline{f(x)} g(x) + \overline{f'(x)} g'(x) \right] dx = (f,g)_{L^2(\mathbb{T})} + (f',g')_{L^2(\mathbb{T})},$$

and Parseval's Theorem gives  $(f,g)_{H1(\mathbb{T})} = \sum_{n \in \mathbb{N}} (1+n^2) \overline{\hat{f}_n} \hat{g}_n$ .



$$\int_{\mathbb{T}} f'g dx = (\overline{f'}, g)_{L^2} = \sum_{n \in \mathbb{N}} -in\overline{\hat{f}_n} \hat{g}_n = -\sum_{n \in \mathbb{N}} \overline{\hat{f}_n} in\hat{g}_n = -(f, g')_{L^2} = -\int_{\mathbb{T}} fg' dx$$

Let  $f \in H^1(\mathbb{T})$  and define the bounded linear functional  $F: C^1(\mathbb{T}) \subset L^2(\mathbb{T}) \to \mathbb{C}$  by

$$F(\phi) = -\int_{\mathbb{T}} f \phi' dx.$$

Since F is a bounded linear functional defined on  $C^1(\mathbb{T})$  and  $C^1(\mathbb{T})$  is dense in  $L^2(\mathbb{T})$ , then we can uniquely extend F to a bounded linear functional on  $L^2(\mathbb{T})$ . Since  $L^2(\mathbb{T})$  is a Hilbert space, the Riesz Representation Theorem states there is a unique  $f' \in L^2$  such that  $F(\phi) = (\overline{f'}, \phi)$  for all  $\phi$ , which gives

$$\int_{\mathbb{T}} f' \phi dx = -\int_{\mathbb{T}} f \phi' dx.$$

The weak derivative of f, denoted f', is the unique element of  $L^2(\mathbb{T})$  such that the above holds for all  $\phi \in C^1(\mathbb{T})$ .  $\phi$  in this context is known as a smooth "test function."

For  $k \geq 0$ ,

$$H^{k}(\mathbb{T}) = \left\{ f \in L^{2} \mid f(x) = \sum_{n = -\infty}^{\infty} c_{n} e^{inx}, \quad \sum_{n = -\infty}^{\infty} |n|^{2k} |c_{n}|^{2} < \infty \right\}.$$

That is, an  $L^2$  function is in  $H^k$  if its Fourier coefficients are  $n^k c_n$  square summable.

When  $k \in \mathbb{N}$ ,  $H^k$  corresponds to functions with k weak derivatives in  $L^2$ .

If $f \in H^k(\mathbb{T})$ and $k > \frac{1}{2}$ , then $f \in C(\mathbb{T})$ .
That is, if $f$ has more than half of a weak derivative, then it is continuous.