Define L^p space for $1 \le p \le \infty$.

Lieb and Loss Chapter 2

For functions in $L^p \cap L^\infty$, how is the L^p norm related to the L^∞ norm?

Lieb and Loss Chapter 2

What is a convex set? Convex function? What does it mean to be strictly convex? Concave? How are these related to continuity?

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L^p is the space of all p^{th} power summable functions.

Let Ω be a measure space with a positive measure μ and let $1 \leq p < \infty$. Then

$$L^p(\Omega, \mu) \coloneqq \{ f \mid f : \Omega \to \mathbb{C},\ f \text{ is } \mu\text{-summable and } |f|^p \text{ is } \mu\text{-summable} \}.$$

The norm of L^p is given by $\|f\|_{L^p} = \left(\int_\Omega |f|^p \mathrm{d}\mu \right)^{\frac{1}{p}}$.
For $p = \infty$,

$$L^\infty(\Omega, \mu) \coloneqq \{ f \mid f : \Omega \to \mathbb{C},\ f \text{is } \mu\text{-measurable and } \exists \text{ constant } K \text{ such that } |f(x)| < K \text{ for } \mu \text{ almost every } x \in \Omega \}$$

with norm $\|f\|_{L^\infty} = \inf \{ K \mid |f(x)| < K \text{ for } \mu \text{ almost every } x \in \Omega \}$.

If $f \in L^p \cap L^\infty$, then $f \in L^q$ for all $q > p$ and $\|f\|_\infty = \lim_{p \to \infty} \|f\|_p$.

- A convex set $K \subset \mathbb{R}^n$ is one for which $\lambda x + (1-\lambda)y \in K$ for all $x, y \in K$ and $0 \leq \lambda \leq 1$.
- A convex function f on a convex set K is a real-valued function satisfying $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$ for all $x, y \in K$ and $0 \leq \lambda \leq 1$.
- A function is strictly convex if equality never holds whenever $x \neq y$ and $0 < \lambda < 1$.
- A function is concave if the inequality is reversed.
- If K is open then convex functions are continuous.

What is Jensen's Inequality?

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What is Hölder's Inequality? What is the Schwarz Inequality? What is the generalization for m functions?

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What is Hanner's Inequality?

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Let $J : \mathbb{R} \to \mathbb{R}$ be a convex function and f a real-valued function on some finite measurable set Ω. Define $\langle \cdot \rangle$ to be the average of a function, i.e.

$$\langle f \rangle \coloneqq \frac{1}{\mu(\Omega)} \int_\Omega f.$$

Then

(i) $[J \circ f]_- \in L^1(\Omega)$;

(ii) $\langle F \circ f \rangle \geq J(\langle f \rangle)$.

In English,

(i) The negative part of the composition is absolutely summable;

(ii) The average of the composition is at least the composition of the average.

Let $1 \leq p \leq \infty$ and let q be the dual index of p. Then if $f \in L^p$ and $g \in L^q$, then $fg \in L^1$ and

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

The Schwarz Inequality is the special case when $p = q = 2$. We have

$$\|fg\|_1 \leq \|f\|_2 \|g\|_2$$

To generalize, for $i = 1, 2, \ldots, n$, let $f_i \in L^{p_i}$ and $\frac{1}{p_1} + \cdots + \frac{1}{p_n} = 1$. Then

$$\prod_{i=1}^{n} f_i \in L^1 \qquad \text{and} \qquad \left\| \prod_{i=1}^{n} f_i \right\|_1 \leq \prod_{i=1}^{n} \|f_i\|_{p_i}$$

Let $f, g \in L^p$. If $1 \leq p \leq 2$, then (Parallelogram Identity)

$$\|f+g\|_p^p + \|f-g\|_p^p \leq \Big(\|f\|_p + \|g\|_p\Big)^p + \Big|\|f\|_p - \|g\|_p\Big|^p$$

and

$$\Big(\|f+g\|_p + \|f-g\|_p\Big)^p + \Big|\|f+g\|_p - \|f-g\|_p\Big|^p \leq 2^p\Big(\|f\|_p^p + \|g\|_p^p\Big).$$

If $2 \leq p < \infty$, the inequalities are reversed.

Is L^p complete (and thus Banach)?

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State the projection theorem for convex subsets of L^p.

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Define weak convergence in L^p.

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Yes. Let $1 \leq p \leq \infty$ and let (f_i) be a Cauchy sequence in L^p, i.e. $\|f_i - f_j\|_p \to 0$ as $i, j \to \infty$. Then there is a unique function $f \in L^p$ such that $\|f_i - f\|_p \to 0$ as $i \to \infty$, i.e.

$$f_i \to f \qquad \text{say "} f_i \text{ converges strongly to } f\text{".}$$

Let $1 < p < \infty$ and let K be a convex subset of L^p. Let $f \in L^p$ such that $f \notin K$ and define

$$D := \operatorname{dist}(f, K) \inf_{g \in K} \|f - g\|_p.$$

Then $\exists h \in K$ such that

$$\|f - h\|_p = D.$$

Let (f_i) be a sequence in L^p. If $L(f_i) \to L(f)$ for every bounded linear functional L on L^p, then we say $f_i \rightharpoonup f$, or f_i weakly converges to f.
It can be shown that for $1 \leq p < \infty$, $(L^p)^* \cong L^q$, where q is the dual index of p, and that every bounded linear functional $L \in (L^p)^*$ can be represented as integration against a unique L^q function, i.e. $\forall L \in (L^p)^*$, $\exists! g \in L^q$ such that

$$L(f) = \int fg$$

for every $f \in L^p$. Thus, (f_i) converges weakly in L^p if

$$\int f_i g \to \int fg$$

for every $g \in L^q$.

State the "Linear functionals separate" theorem.

State the Uniform Boundedness Principle.

Lieb and Loss Chapter 2

Define convolution for functions on $\mathbb{R}^n$.

Lieb and Loss Chapter 2

Suppose $f \in L^p$ with $L(f) = 0$ for all $L \in (L^p)^*$. Then $f = 0$.

Consequently, if $f_i \rightharpoonup g$ and $f_i \rightharpoonup h$, then $g = h$.

Let (f_i) be a sequence in L^p such that $\forall L \in (L^p)^*$ the sequence $(L(f_i))$ is bounded in $\mathbb{C}$. Then $\left(\|f_i\|_p\right)$ is a bounded sequence in $\mathbb{R}$.

For $f, g : \mathbb{R}^n \to \mathbb{C}$, we define the convolution of f and g, denoted $f * g$, as

$$(f * g)(x) := \int_{\mathbb{R}^n} f(x - y) g(y) \mathrm{d}y.$$

Define a mollification. Why are they useful?

Is $L^p(\mathbb{R}^n)$ separable? What does that mean?

State the Banach-Alaoglu Theorem.

Let $j \in L^1(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} j = 1$. For $\varepsilon > 0$, define j_ε as

$$j_\varepsilon(x) := \frac{1}{\varepsilon^n} j\Big(\frac{x}{\varepsilon}\Big).$$

so that $\|j_E\|_1 = \|j\|_1$ and $\int_{\mathbb{R}^n} j_\varepsilon = 1$.
Define the mollification of a function $f \in L^p(\mathbb{R}^n)$ for some $1 \leq p < \infty$, denoted f_ε, as the convolution of f and j_ε for some ε, that is,

$$f_\varepsilon = f * j_\varepsilon.$$

Then $f_\varepsilon \in L^p(\mathbb{R}^n)$ and $\|f_\varepsilon\|_p \leq \|f\|_p \|j\|_1$. Also, $f_\varepsilon \to f$ strongly in L^p, that is, $\|f_\varepsilon - f\|_p \to 0$.

In addition, if $j \in C_C^\infty$, then $f_\varepsilon \in C^\infty$. This is a concrete construction which shows that C^∞ functions are dense in L^p.

Yes, $L^p(\mathbb{R}^n)$ is separable. This means there is a countable dense subset of $L^p(\mathbb{R}^n)$, that is, $\exists \Phi = \{\phi_1, \phi_2, \dots\} \subset L^p(\mathbb{R}^n)$ such that $\forall f \in L^p$ and $\varepsilon > 0$, $\exists \phi_j \in \Phi$ such that $\|f - \phi_j\|_p < \varepsilon$.

Let $\Omega \subset \mathbb{R}^n$ be a measurable set and consider $L^p(\Omega)$ with $1 < p < \infty$. Let (f_i) be a bounded sequence in L^p. Then there exists a subsequence (f_{i_j}) and $f \in L^p$ such that $f_{i_j} \rightharpoonup f$ in L^p. That is, bounded sets in L^p are weakly compact.

How does Urysohn's Lemma give that C_C^∞ is dense in L^p?

What is special about convolutions of functions in dual L^p spaces?

Let $\Omega \subset \mathbb{R}^n$ be an open set and let $K \subset \Omega$ be compact. Then there is a functin $J_K \in C_C^\infty(\Omega)$ such that $0 \leq J_K(x) \leq 1$ for all $x \in \Omega$ and $J_K(x) = 1$ for all $x \in K$.

As a consequence, there is a sequence of functions $(g_i) \in C_C^\infty$ that take values in $[0,1]$ and such that $\lim_{j\to\infty} g_j(x) = 1$ for every $x \in \Omega$.

As a second consequence, given a sequence of functions $(f_i) \in C^\infty$ such that f_i converges strongly to a function $f \in L^p$, the sequence $(h_i) = (g_i f_i) \in C_C^\infty$ and $h_i \to f$ strongly.

This shows, since C^∞ is dense in L^p, that C_C^∞ is dense in L^p.

If $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, where q is the dual index of p, then $f * g$ is continuous and $(f * g)$ tends to 0 at infinity.