Define complementary subspace, codimension, and prejection.

Show $X = \ker P \oplus \operatorname{ran} P$ *for any projection* P *on a linear space* X*.*

Define orthogonal projection.

If X is a linear space and M is a linear subspace of X, then N is a complementary subspace of M if every $x \in X$ can be uniquely represented by $x = n + m$ where $n \in N$ and $m \in M$. This implies $M \cap N = \{0\}$.

If M is a linear subspace of X, then M may have infinitely many complementary subspaces. All of the complementary subspaces of M have the same dimension, which is called the codimension of M.

A projection on a linear space X is a linear map $P : X \to X$ such that $P^2 = P$.

First note that $x = Px$ if and only if $x \in \operatorname{ran} P$, since if $x = Py$ then $Px = P^2x = Py = x$.

Let $x \in \ker P \cap \operatorname{ran} P$. Then $x = Px = 0$.

Let $x \in X$. Then $x - Px \in \ker P$ since

$$P(x - Px) = Px - P^2x = Px - Px = 0.$$

Then $x = Px + (x - Px)$.

An orthogonal projection P, on a Hilbert space $\mathcal{H}$, is a function $P : \mathcal{H} \to \mathcal{H}$ such that

$$P^2 = P \qquad \text{and} \qquad (Px, y) = (x, Py) \quad \forall x, y \in \mathcal{H}$$

What is the norm of a non-zero orthogonal projection?

Prove that a Hilbert space is the orthogonal direct sum of an orthogonal projection's range and kernel.

If P is the orthogonal projection on to a closed linear subspace M of a Hilbert space $\mathcal{H}$, and Q is the orthogonal projection on to a $M^{\perp}$, how are P and Q related?

Using the Cauchy-Schwarz Inequality,

$$\|Px\| = \frac{\|Px\|^2}{\|Px\|} = \frac{(Px, Px)}{\|Px\|} = \frac{(x, P^2 x)}{\|Px\|} = \frac{(x, Px)}{\|Px\|} \leq \frac{\|x\|\|Px\|}{\|Px\|} = \|x\|,$$

so $\|P\| \leq 1$. However, there is an $x \in \operatorname{ran} P$ with $\|x\| \neq 0$ (since P is non-zero. Then

$$\|Px\| = \|x\|,$$

which shows $\|P\| \geq 1$, and thus $\|P\| = 1$.

Let P be an orthogonal projection on $\mathcal{H}$. We know $\mathcal{H} = \ker P \oplus \operatorname{ran} P$ where $\oplus$ is just a (not necessarily orthogonal) direct sum. However, if $x = Py \in \operatorname{ran} P$ and $z \in \ker P$, then

$$(x, z) = (Py, z) = (y, Pz) = (y, 0) = 0$$

and thus $\ker P \perp \operatorname{ran} P$.

$$I - P = Q.$$

What is the orthogonal projection on to a one-dimensional subspace U of $\mathcal{H}$?

Give examples of orthogonal projections on to a finite dimensional space, a countably infinite dimensional space, and an uncountably infinite dimensional space.

State the Riesz Representation theorem and state what the proof uses.

Let $\{u\}$ be a basis of U. Then define P_U by

$$P_u x = \frac{(u,x)}{\|u\|^2} u.$$

Let $\mathcal{H} = \mathbb{R}^n$ and $\mathbf{u}$ be any unit vector. The orthogonal projection in the direction of $\mathbf{u}$ is the rank one matrix $\mathbf{u}\mathbf{u}^T$. The component of a vector $\mathbf{x}$ in the direction of $\mathbf{u}$, i.e. the projection of $\mathbf{x}$ on to $[\{\mathbf{u}\}]$ is

$$P_{\mathbf{u}}\mathbf{x} = \frac{(\mathbf{u},\mathbf{x})}{\|\mathbf{u}\|^2}\mathbf{u} = \left(\mathbf{u}^T\mathbf{x}\right)\mathbf{u}.$$

Let $\mathcal{H} = \ell^2(\mathbb{Z})$ and $u = e_n = (\delta_{k.n})_{k=-\infty}^{\infty}$ and $x = (x_k)$. Then $P_{e_n}x = x_n e_n$ gives a vector of all 0s except for the n^{th} component of x in the n^{th} position. Let $\mathcal{H} = L^2(\mathbb{T})$ and $u(x) \equiv \frac{1}{\sqrt{2\pi}}$, which is the constant function with $\|u\| = 1$. Then P_u maps a function f to its mean $\langle f \rangle$, i.e.

$$P_u f = \frac{1}{2\pi}\int_{\mathbb{T}} f(x)\mathrm{d}x.$$

Then $f = \langle f \rangle + \tilde{f}$ is the decomposition of a function into a constant mean part and a fluctuating, 0 mean part.

If ϕ is a bounded linear functional on a Hilbert space $\mathcal{H}$, then there is a unique $y \in \mathcal{H}$ such that $\phi(x) = (y,x)$ for all $x \in \mathcal{H}$.
The proof uses

- The kernel of a bounded linear operator is a closed subspace.
- Defining a clevel orthogonal projection P whose kernel is equal to $\ker\phi$.
- Arbitrary vectors can be decomposed by $\mathcal{H} = \ker P \oplus \operatorname{ran} P$.

What is the adjoint of an operator? Why are the important in the context of projections on Hilbert spaces?

What is the adjoint of a real-valued matrix? How about a complex-valued matrix?

What is the adjoint of the left-shift operator on $\ell^2(\mathbb{N})$?

The adjoint of an bounded linear operator A on a Hilbert space $\mathcal{H}$ is denoted A^* and is the unique operator in $\mathcal{B}(\mathcal{H})$ such that

$$(x, Ay) = (A^* x, y) \qquad \forall x, y \in \mathcal{H}.$$

An operator A is called self-adjoint if $A = A^*$. All orthogonal projects are self-adjoint, that is, for any projection P on $\mathcal{H}$, we have

$$(Px, y) = (x, Py) \qquad \forall x, y \in \mathcal{H}.$$

This is not true for all projections - just orthogonal projections.

If A is a real-valued matrix in $\mathbb{R}^{n\times n}$, then $A^* = A^T$. That is, the adjoint is the transpose.

If A is a complex-valued matrix in $\mathbb{C}^{n\times n}$, then $A^* = \overline{A^T}$. That is, the adjoint is the Hermitian conjugate matrix..

Let $T \in \mathcal{B}(\ell^2(\mathbb{N}))$ by

$$T(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots).$$

Then $T^* = S$, the right-shift operator, given by

$$S(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots)$$

because

$$(x, Ty) = \sum_{i=1}^{\infty} \overline{x_i} y_{i+1} = (Sx, y).$$

This is analagous to the transpose of a matrix since the transpose of the infinite matrix representing T is the infinite matrix representing S.

What is the adjoint of a Fredholm integral operator $K \in \mathcal{B}(L^2([0,1]))$?

If $A : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator, show

$$\overline{\operatorname{ran} A} = (\ker A^*)^\perp, \qquad \text{and} \qquad \ker A = (\operatorname{ran} A^*)^\perp.$$

Suppose that $A : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator on a Hilbert space $\mathcal{H}$ with closed range. When does the equation $Ax = y$ have a solution?

Let $K \in \mathcal{B}(L^2([0,1]))$ by

$$Kf(x) = \int_0^1 k(x,y)f(y)\mathrm{d}y$$

for some continuous function $k : [0,1] \times [0,1] \to \mathbb{C}$. Then K^* is given explicitly by integration against the complex conjugate, transpose kernel:

$$K^*f(x) = \int_0^1 \overline{k(y,x)}f(y)\mathrm{d}y.$$

This is analagous to the Hermitian conjugate of a matrix.

Let $x \in \operatorname{ran} A$. Then $x = Ay$ for some $y \in \mathcal{H}$. Then $(x,z) = (Ay,z) = (y, A^*z) = 0$ for any $z \in \ker A^*$. Thus $x \in (\ker A^*)^\perp$, which is closed, and so $\overline{\operatorname{ran} A} \subset (\ker A^*)^\perp$.

If $x \in (\operatorname{ran} A)^\perp$, then $0 = (Ay, x) = (y, A^*x)$ for every $y \in \mathcal{H}$, which shows $A^*x = 0$ for every $x \in (\operatorname{ran} A)^\perp$, i.e. $x \in \ker A^*$, so $(\operatorname{ran} A)^\perp \subset \ker A^*$. Then $(\ker A^*)^\perp \subset (\operatorname{ran} A)^{\perp\perp} = \overline{\operatorname{ran} A}$, which shows $(\ker A^*)^\perp = \overline{\operatorname{ran} A}$. The clever move here was $X \subset Y \implies Y^\perp \subset X^\perp$.

Then taking $A = A^*$ in the above equality gives $(\ker A)^\perp = \overline{\operatorname{ran} A^*}$. Then taking orthogonal complements gives $\ker A = \overline{\operatorname{ran} A^*}^\perp = (\operatorname{ran} A^*)^\perp$.

A succint way of stating this theorem is

$$\mathcal{H} = \overline{\operatorname{ran} A} \oplus (\ker A^*)^\perp.$$

The equation $Ax = y$ has a solution for x if and only if $y \perp \ker A^*$.

What is the Fredhold alternative?

Why is the multiplication operator $Mf(x) = xf(x)$ not always solvable?

What is a Fredholm operator? What is the index of a Fredholm operator? What is the result regarding indeces of Fredholm operators?

A bounded linear operator A on a Hilbert space $\mathcal{H}$ satisfies the Fredholm alternative if either

(a) either $Ax = 0$, $A^*x = 0$ have only the zero solution, and the equations $Ax = y$, $A^*x = y$ have a unique solution for every $y \in \mathcal{H}$;

(b) or $Ax = 0$, $A^*x = 0$ have nontrivial, finite-dimensional solution spaces of the same dimension, $Ax = y$ has a (nonunique) solution if and only if $y \perp z$ for every solution z of $A^*z = 0$, and $A^*x = y$ has a (nonunique) solution if and only if $y \perp z$ for every solution z of $Az = 0$.

In English, either

(a) A and A^* are bijective;

(b) or A and A^* are not injective, but have the same nullity.

Even though $\ker M^* = \ker M = \{0\}$, and hence every $g \in L^2([0,1])$ is orthogonal to $\ker M^*$, $Mf = g$ is not always solvable since $\operatorname{ran} M$ is properly dense in $L^2([0,1])$.

A bounded linear operator A on a Hilbert space $\mathcal{H}$ is a Fredhold operator if

(a) $\operatorname{ran} A$ is closed;

(b) $\ker A$ and $\ker A^*$ are finite-dimensional.

The index of a Fredholm operator A, denoted $\operatorname{ind} A$, is the integer

$$\operatorname{ind} A = \dim \ker A - \dim \ker A^*.$$

If A is Fredholm and K is compact, then $A + K$ is Fredholm and

$$\operatorname{ind} A = \operatorname{ind} (A + K).$$

That is, the index of a Fredholm is unchanged by compact perturbations. Since I is Fredholm, and $\operatorname{ind} I = 0$, then $\operatorname{ind} (I + K) = 0$ for compact K.

What is the most important sesquilinear form derived from a bounded linear operator A on a Hilbert space $\mathcal{H}$? What is the associated quadratic form? What is it mean to be a non-negative operator? How about a positive (positive definite) operator?

How can we define an inner product on a Hilber space $\mathcal{H}$ given a positive definite bounded linear operator A?

If A is a bounded, self-adjoint operator on a Hilbert space $\mathcal{H}$, what is an easy formula for $\|A\|$?

Given a linear operator A, define the sesquilinear form $a : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ by $a(x, y) = (x, Ay)$. The associated quadrative form is $q_A(x) = a(x, x)$, or $q_A(x) = (x, Ax)$.

A is called nonnegative if its quadratic form is nonnegative for all $x \in \mathcal{H}$, i.e. $q_A(x) \geq 0$, $\forall x \in \mathcal{H}$. A is positive definite if $q_A(x) > 0$, $\forall x \in \mathcal{H}$.

$$(x, y)_A \coloneqq (x, Ay)$$

defines an inner product on $\mathcal{H}$. In addition, $(\cdot, \cdot)_A$ is equivalent to $(\cdot, \cdot)$.

$$\|A\| = \sup_{\|x\|=1} |q_A(x)|, \qquad \text{where } q \text{ is the quadratic form } q_A(x) = (x, Ax).$$

Prove that if A is a bounded operator on a Hilbert space $\mathcal{H}$, then $\|A^*A\| = \|A\|^2$.

What is a unitary operator? What is an easy equivalent way to check an operator is unitary?

Describe a general unitary operator between two Hilbert spaces of the same (possibly infinite dimension. What is the Hilbert transform?)

$$\|A\|^2 = \sup_{\|x\|=1} \|Ax\|^2 = \sup_{\|x\|=1} |(Ax, Ax)| = \sup_{\|x\|=1} |q_{A^*A}(x)| = \|A^*A\|$$

If A is self adjoint, the $A^* = A$ and $\|A\|^2 = \left\|A^2\right\|$.

A linear map $U : \mathcal{H}_1 \to \mathcal{H}_2$ between real or complex Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ is said to be orthogonal (real) or unitary (complex), respectively, if it is invertible and if

$$(Ux, Uy) = (x, y), \qquad \forall x, y \in \mathcal{H}_1$$

Unitary operators preserve inner products.

Two Hilbert spaces are isomorphic if there exists a unitary operator between them.

An operator U is unitary if and only if its inverse is it's adjoint. That is, U is unitary if and only if $U^* = U^{-1}$.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ have the same (possibly infinite) dimension. Then their bases can be indexed by the same index set. Suppose $\mathcal{H}_1$ has basis $\{u_\alpha\}$ and $\mathcal{H}_2$ has basis $\{v_\alpha\}$. Then any $x \in \mathcal{H}_1$ can be written as

$$x = \sum_\alpha c_\alpha u_\alpha, \qquad \text{where } c_\alpha = (u_\alpha, x).$$

Let λ_α be complex numbers with $|\lambda_\alpha| = 1$ and define $U : \mathcal{H}_1 \to \mathcal{H}_2$ by

$$Ux = \sum_\alpha \lambda_\alpha (u_\alpha, x) v_\alpha.$$

Then U is unitary, and thus $\mathcal{H}_1 \cong \mathcal{H}_2$. The Hilbert transform $\mathbb{H}$ is of this form. Define $\mathbb{H} : \mathcal{H}_0 \subset L^2(\mathbb{T}) \to \mathcal{H}_0$ ($\mathcal{H}_0 = \left\{f \in L^2(\mathbb{T}) \mid \langle f \rangle = 0\right\}$ is the space of L^2 functions with 0 mean) by

$$\mathbb{H}f = \mathbb{H}\left(\sum_{n\in\mathbb{N}} \hat{f}_n e^{inx}\right) = \sum_{n\in\mathbb{N}} \left(i(\operatorname{sgn}[n])\hat{f}_n e^{inx}\right).$$

What is the most famous unitary operator (besides maybe the identity operator)?

What is a normal operator?

The most famous (and most useful) unitary operator is the Fourier transform, $\mathcal{F} : L^2(\mathbb{T}) \to \ell^2(\mathbb{N})$, which is given by

$$\mathcal{F}f = \mathcal{F}\left(\sum_{n\in\mathbb{N}} c_n e^{inx}\right) = (c_n)_{n\in\mathbb{N}}, \qquad \text{where } c_n = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{T}} f(x)e^{-inx}\mathrm{d}x.$$

A operator A is said to be normal if it commutes with its adjoint, that is,

$$AA^* = A^*A.$$