Define an inner product

What is a pre-Hilbert space?

Can we always define a norm given an inner product?

An inner product on a complex linear space X is a map $(\cdot, \cdot) : X \times X \to \mathbb{C}$ such that for all $x, y, z \in X$ and $\lambda, \mu \in \mathbb{C}$,

(a) $(x, \lambda y + \mu z) = \lambda(x, y) + \mu(x, z)$ (linear in the second argument);

(b) $(y, x) = \overline{(x, y)}$ (Hermitian symmetric);

(c) $(x, x) \geq 0$ (nonnegative);

(d) $(x, x) = 0 \iff x = 0$ (positive definite);

A pre-Hilbert space (or inner-product space) is a linear space with an inner product defined.

Yes. In fact the most common norm we will see is the following:

$$\|x\| = \sqrt{(x, x)}$$

Define a Hilbert space

What is the standard inner-product on $\mathbb{C}^n$?

Can we define an inner product on $C([a, b])$? How can we complete the space to make it Hilbert? How about $C^k([a, b])$?

A Hilbert space is a complete inner-product space. That is, a Banach space with a norm derived from a defined inner-product.

Given $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{C}^n$, then

$$(x, y) = \sum_{i=1}^{n} \overline{x_i} y_i$$

Let $f, g \in C([a, b])$. Then, analagous to the inner-product on $\mathbb{C}^n$, we can define (f, g) as follows:

$$(f, g) = \int_a^b \overline{f(x)} g(x) \mathrm{d}x$$

This makes $C([a, b])$ a pre-Hilbert space. The completion of $C([a, b])$ is $L^2([a, b])$, which is the only Hilbert L^p space. For $f, g \in C^k([a, b])$,

$$(f, g) = \sum_{i=0}^{k} \int_a^b \overline{f^{(i)}(x)} g^{(i)}(x) \mathrm{d}x = \sum_{i=0}^{k} \left(f^{(i)}, g^{(i)} \right)_{C([a,b])}$$

This makes $C^k([a, b])$ a pre-Hilbert space. The completion of $C^k([a, b])$ is the Sobolev space $W^{k,2}((a, b))$, which is also denoted $H^k((a, b))$.

Define inner products on $\ell^2(\mathbb{Z})$ and $\mathbb{C}^{m\times n}$.

State and prove the Cauchy-Schwarz Inequality

State the parallelogram law

For $x = (x_n)_{n=-\infty}^{\infty}$ and $y = (y_n)_{n=-\infty}^{\infty}$ in $\ell^2(\mathbb{Z})$,

$$(x, y) = \sum_{n=-\infty}^{\infty} \overline{x_n} y_n$$

For $A = (a_{ij})$ and $B = (b_{ij})$ in $\mathbb{C}^{m \times n}$,

$$(A, B) = \operatorname{tr}(A^* B) = \sum_{i=1}^{m} \sum_{j=1}^{n} \overline{a_{ij}} b_{ij}$$

The corresponding norm is the Hilbert-Schmidt norm.

If $x, y \in X$, where X is an inner product space, then

$$|(x, y)| \leq \|x\| \|y\|.$$

For any $\lambda \in \mathbb{C}$, by nonnegativity, $0 \leq (x - \lambda y, x - \lambda y)$, which implies, by linearity in the second argument, anti-linearity in the first argument, and definition of norm,

$$\lambda(x, y) + \overline{\lambda}(y, x) \leq \|x\|^2 + |\lambda|^2 \|y\|^2$$

Now choose $\lambda = \frac{(y,x)}{\|y\|^2}$. Then substitution gives

$$2 \frac{|(x, y)|^2}{\|y\|^2} \leq \|x\|^2 + \frac{|(x, y)|^2}{\|y\|^2}.$$

Simple algebra gives the result.

A normed linear space X is an inner product space with a norm derived from the inner product by $\|x\| = \sqrt{(x, x)}$ if and only if the following holds:

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 \qquad \forall x, y \in X$$

Geometrically, the sum of the squares of the diagonals of a parallelogram equal the sum of the squares of the sides. If the above equation holds, then

$$(x, y) = \frac{1}{4} \Big(\|x + y\|^2 - \|x - y\|^2 - i\|x + iy\|^2 + i\|x - iy\|^2 \Big)$$

defines an inner product on X. This is called the polarization formula.

Define the inner product on the Cartesian product of two inner product spaces

Prove the inner product in continuous

Define orthogonality and orthogonal complement

Let $(X, (\cdot,\cdot)_X)$ and $(Y, (\cdot,\cdot)_Y)$ be two inner product spaces. Then the Cartesian product space is the space containing all tuples,

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}$$

and the natural inner product is simply the sum of the inner products of the components:

$$((x_1, y_1), (x_2, y_2))_{X\times Y} = (x_1, x_2)_X + (y_1, y_2)_Y.$$

This gives rise to the natural norm on $X \times Y$:

$$\|(x, y)\| = \sqrt{\|x\|^2 + \|y\|^2}$$

Let X be an inner product space. Choose $(x_1, y_1), (x_2, y_2) \in X \times X$ such that

$$\|(x_1, y_1) - (x_2, y_2)\| < \delta = \frac{1}{2} \max\left\{\frac{\varepsilon}{\|y_1\|}, \frac{\varepsilon}{\|x_2\|}\right\}.$$

Assume for now $\delta \neq 0$. Then in particular, $\|x_1 - x_2\| < \delta$ and $\|y_1 - y_2\| < \delta$. Then

$$|(x_1, y_1) - (x_2, y_2)| = |(x_1, y_1) - (x_2, y_1) + (x_2, y_1) - (x_2, y_2)|$$

Then by linearity, triangle inequality, and the Cauchy-Schwarz inequality,

$$|(x_1, y_1) + (x_2, y_2)| \leq \|x_1 - x_2\|\|y_1\| + \|x_2\|\|y_1 - y_2\| < \delta(\|y_1\| + \|x_2\|) = \varepsilon$$

Two vectors x and y are called orthogonal, denoted $x \perp y$, if their inner product is equal to 0:

$$(x, y) = 0$$

Subsets A and B are orthogonal if every element in A is orthogonal to every element in B. That is, $A \perp B$ if $a \perp b$, $\forall a \in A$ and $b \in B$.

The orthogonal complement of a subset A, denoted $A^\perp$, in a Hilbert space $\mathcal{H}$ is the set of all elements in $\mathcal{H}$ orthogonal to every element in A. That is,

$$A^\perp = \{x \in \mathcal{H} \; : \; x \perp a, \; \forall a \in A\}$$

The orthogonal complement of a subset of a Hilbert space is closed.

State the Projection Theorem

Define the orthogonal direct sum and state the most important result about orthogonal complements

The proof uses:

- linearity of the inner product
- continuity of the inner product

Let M be a closed linear subspace of a Hilbert space $\mathcal{H}$. Then

(a) For each $x \in \mathcal{H}$, $\exists!$ closest point $y \in M$ such that $\|x - y\| = \min_{z \in M} \|x - z\|$;

(b) The point $y \in M$ closest to x is the unique element of M with the property that $(x - y) \perp M$.

The proof uses:

- Definition of Infimum
- Parallelogram Law
- Norm and Inner product are continuous
- Cauchy $\implies$ convergent in complete spaces
- Convexity of normed linear spaces

Given two orthogonal closed linear subspaces M and N, the orthogonal direct sum of M and N, denoted $M \oplus N$, is the smallest linear subspace containing M and N, i.e.

$$M \oplus N = \{m + n \mid m \in M \text{ and } n \in N\}$$

If M is a closed linear subspace of $\mathcal{H}$, then $M \oplus M^{\perp} = \mathcal{H}$. If M is not closed, we still have $\overline{M} \oplus M^{\perp} = \mathcal{H}$.

Define an orthonormal basis of a finite-dimensional Hilbert space

Give the standard bases in $\mathbb{C}^n$, $\ell^2(\mathbb{Z})$, and $L^2(\mathbb{T})$

Describe the inner product space of quasiperiodic functions

Let $\mathcal{H}$ be a finite-dimensional Hilbert space. A set of vectors $\{e_1, \dots, e_n\}$ is an orthonormal basis of $\mathcal{H}$ if $\|e_i\| = 1$ for each $i = 1, \dots, n$, $e_i \perp e_j$ for $i \neq j$, and for all $x \in \mathcal{H}$, $\exists! x_k \in \mathbb{C}$ such that

$$x = \sum_{i=1}^{n} x_i e_i$$

The standard basis in $\mathbb{C}^n$ is $\{e_1, \dots, e_n\}$ where $e_i = (0, 0, \dots, 0, 1, 0, \dots, 0, 0)$ where all components are 0 except for a 1 in the $i^{\cdot}$ component.

The standard basis for $\ell^2(\mathbb{Z})$ is $\{\dots, e_{-1}, e_0, e_1, \dots\}$, where $e_i = (\delta_{ij})_{j=-\infty}^{\infty}$ and δ_{ij} is the Kronecker delta function.

The standard basis for $L^2(\mathbb{T})$ is $\{\dots, e_{-1}, e_0, e_1, \dots\}$ where $e_n(x) = \dfrac{1}{\sqrt{2\pi}} e^{inx}$

Define X to be the space of all functions of the form $a(t) = \sum_{k=1}^{n} a_k e^{i\omega_k t}$. We can define an inner product on X by

$$(a, b) = \lim_{T \to \infty} \int_{-T}^{T} \overline{a(t)} b(t) \mathrm{d}t$$

which simplifies to

$$(a, b) = \sum_{k=1}^{n} \overline{a_k} b_k.$$

Note the set

$$\Omega = \{e^{i\omega t} \mid \omega \in \mathbb{R}\}$$

is an uncountable orthonormal set, which means X is not separable. Ω is, in fact, an orthonormal basis of X.

Define unconditional convergence

Define absolute convergence

What does it mean to be a Cauchy unordered series?

Let $\{x_\alpha \in X \mid \alpha \in I\}$ be an indexed set in a Banach space X and I may be uncountable. For each finite subset J of I, define the partial sum S_J by

$$S_J = \sum_{\alpha \in J} x_\alpha$$

The unordered sum of the indexed set converges unconditionally to $x \in X$, denoted

$$x = \sum_{\alpha \in I} x_\alpha,$$

if for every $\varepsilon > 0$, there is a finite subset J^ε of I such that $\|S_J - x\| < \varepsilon$ for every finite subset J of I which contains J^ε.

All unconditionally convergent series have only countably many nonzero terms.

A sum $\sum_{\alpha \in I} x_i$ converges absolutely if $\sum_{\alpha \in I} \|x_i\|$ converges unconditionally.

Absolute convergence implies unconditional convergence.

An unordered sum $\sum_{\alpha \in I}$ is Cauchy if for every ε, there is a finite subset J^ε of I such that $\|S_J\| < \varepsilon$ for every finite $J \subset I \setminus J^\varepsilon$

Unordered sums in Banach spaces converge if and only if they are Cauchy

State and prove the general Pythagorean Theorem

State Bessel's Inequality

Cauchy $\implies$ convergent uses the following outline:

- Define an increasing class (J_n) of finite subsets of I.
- Show that (S_{J_n}) is Cauchy.
- Since X is banach, (S_{J_n}) converges to some limit $x \in X$. Use x as the candidate limit for the original Cauchy series.
- Use an $\frac{\varepsilon}{2}$ trick with the Cauchy criterion and the definition of x to complete the proof

Convergent $\implies$ Cauchy uses the following:

- Set operations $(\setminus, \cup)$
- Triangle inequality

Let $U = \{u_\alpha \mid \alpha \in I\}$ be an indexed, orthogonal subset of a Hilbert space $\mathcal{H}$. The sum $\sum_{\alpha \in I} u_\alpha$ converges unconditionally if an only if $\sum_{\alpha \in I} \|u_\alpha\|^2$ converges unconditionally, and in that case,

$$\left\| \sum_{\alpha \in I} u_\alpha \right\|^2 = \sum_{\alpha \in I} \|u_\alpha\|^2.$$

For any finite subset J of I,

$$\left\| \sum_{\alpha \in J} u_\alpha \right\|^2 = \sum_{\alpha, \beta \in J} (u_\alpha, u_\beta) = \sum_{\alpha \in J} (u_\alpha, u_\alpha) = \sum_{\alpha \in J} \|u_\alpha\|^2.$$

It then follows that $\sum_{\alpha \in I} u_\alpha$ converges if and only if $\sum_{\alpha \in I} \|u_\alpha\|^2$ converges. Then since the norm is continuous, the result holds.

Let $U = \{u_\alpha \mid \alpha \in I\}$ be an orthnormal set in a Hilbert space $\mathcal{H}$, and choose any $x \in \mathcal{H}$. Then

(a) $\displaystyle\sum_{\alpha \in I} |(u_\alpha, x)|^2 \leq \|x\|^2$.

(b) $x_U := \sum_{\alpha \in I} (u_\alpha, x) u_\alpha$ converges. (x_U is the projection of x on to the subspace spanned by U.)

(c) $x - x_U \in U^\perp$

Define the closed linear span of a general subset U, and then for an orthonormal subset U

State the five equivalent conditions defining an orthonormal basis of $\mathcal{H}$

State Parseval's Identity

The closed linear space of a subset U of a Hilbert space $\mathcal{H}$, denoted $[U]$, is given by

$$[U] = \left\{ \sum_{u \in U} c_u u \mid c_u \in \mathbb{C} \text{ and } \sum_{u \in U} c_u u \text{ converges unconditionally} \right\}$$

If $U = \{u_\alpha \mid \alpha \in I\}$ is an orthonormal set, then

$$[U] = \left\{ \sum_{\alpha \in I} c_\alpha u_\alpha \mid c_\alpha \in \mathbb{C} \text{ and } \sum_{\alpha \in I} |c_\alpha|^2 < \infty \right\}$$

This simplification follows from the Pythagorean Theorem.

Uf $U = \{u_\alpha \mid \alpha \in I\}$ is an orthonomal subset of a Hilbert space $\mathcal{H}$, then the following conditions are equivalent:

(a) $(u_\alpha, x) = 0$ for all $\alpha \in I$ implies $x = 0$;
(b) $x = x_U = \sum_{\alpha \in I} (u_\alpha, x) u_\alpha$ for all $x \in \mathcal{H}$;
(c) $\|x\|^2 = \sum_{\alpha \in I} |(u_\alpha, x)|^2$ for all $x \in \mathcal{H}$;
(d) $[U] = \mathcal{H}$;
(e) U is a maximal orthonormal set.

In English,

(a) The only element orthogonal to every element in U is 0.
(b) Every element is equal to its own projection onto $[U]$.
(c) The Pythagorean Theorem, simplified, since $\|u_\alpha\| = 1$ for all α
(d) U spans all of $\mathcal{H}$
(e) No non-zero orthogonal vector can be added to the set U.

Suppose $U = \{u_\alpha \mid \alpha \in I\}$ is an orthonormal basis of $\mathcal{H}$. Define $x = \sum_{\alpha \in I} x_\alpha u_\alpha$ and $y = \sum_{\alpha \in I} y_\alpha u_\alpha$. Then

$$(x, y) = \sum_{\alpha \in I} \overline{x_\alpha} y_\alpha$$

sicne $x_\alpha = (u_\alpha, x)$ and $y_\alpha = (u_\alpha, y)$ for $\alpha \in I$.

What does every Hilbert space have?

What is the Gram-Schmidt orthonormalization procedure

Give examples of how the Gram-Schmidt orthonormalization procedure works

Every Hilbert space has an orthonormal basis. Also, given any orthonormal subset U of a Hilbert space $\mathcal{H}$, there is an orthonormal basis of $\mathcal{H}$ containing U. In other words, one can always extend an orthonormal set to an orthonormal basis.

The Gram-Schmidt orthonormalization procedure is a way of constructing a countable orthonormal set U given a countable set of linearly independent vectors V such that $[U] = [V]$. Define u_n as follows:

$$u_1 = \frac{v_1}{\|v_1\|} \qquad \text{and} \qquad u_{n+1} = c_{n+1}\left(v_{n+1} - \sum_{k=1}^{n}(u_k, v_{k+1})u_k\right)$$

where c_{n+1} is chosen to $\|u_{n_1}\| = 1$.

Define a weighted inner-product on the continuous functions $C([a, b])$ by

$$(f, g) = \int_a^b w(x)\overline{f(x)}g(x)\mathrm{d}x$$

Denote $C_w([a, b])$ as the set of functions whos norm is finite, i.e.

$$C_w([a, b]) = \{f \ : \ [a, b] \to \mathbb{C} \mid f \text{ is continuous and } (f, f) < \infty\}$$

and complete this space to obtain the Hilbert space $L_w([a, b])$.
Set $M = \{x^n \mid n \in \mathbb{N}\}$. This set is linearly independent, but may not orthonormal. Given $L_w([-1, 1])$ with $w(x) \equiv 1$, the G-S procedure on M produces the Legendre polynomials. Given $L_w([-1, 1])$ with $w(x) = \sqrt{1 - x^2}$, the G-S procedure on M produces the Tchebyschev polynomials. Given $L_w(\mathbb{R})$ with $w(x) = \exp\left[-\frac{x^2}{2}\right]$, the G-S procedure on M produces the Hermite polynomials.