

Define a field and a σ – field

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Define a probability measure and give some of its properties

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List some properties of probability measures

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A field on a set Ω is a collection of subsets \mathcal{F} such that:

1. (at least contains two sets) $\emptyset, \Omega \in \mathcal{F}$,
2. (closure under complement) if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$,
3. (closure under finite union) if $A, B \in \mathcal{F}$, then $A \cup B \in \mathcal{F}$

A σ -field \mathcal{F} on the set Ω also has:

4. (closure under countable union) if $A_1, \dots \in \mathcal{F}$, then $\cup_i A_i \in \mathcal{F}$.

A probability measure on a set Ω with field \mathcal{F} is a function $P : \mathcal{F} \rightarrow [0, \infty)$ with:

1. $0 \leq P(A) \leq 1, \forall A \in \mathcal{F}$,
2. $P(\emptyset) = 0$ and $P(\Omega) = 1$,
3. if A_1, \dots are disjoint and $\cup_i A_i \in \mathcal{F}$, then

$$P(\cup_i A_i) = \sum_i P(A_i)$$

A probability measure P on Ω with field \mathcal{F} has

1. (monotonicity), if $A \subset B$, then $P(A) \leq P(B)$,
2. (inclusion-exclusion) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ and more generally

$$P(\cup A_n) = \sum_i P(A_i) - \sum_{i < j} P(A_i \cap A_j) + \sum_{i < j < k} P(A_i \cap A_j \cap A_k) \\ + \dots + (-1)^{n+1} P(A_1 \cap \dots \cap A_n),$$

3. (countably subadditive) if $A_1, \dots \in \mathcal{F}$ and $\cup_i A_i \in \mathcal{F}$, then $P(\cup_i A_i) \leq \sum_i P(A_i)$,
4. (continuous from below) if $A_1 \subset A_2 \subset \dots \subset A$, then $P(A_n) \uparrow P(A)$
5. (continuous from above) if $A_1 \supset A_2 \supset \dots \supset A$, then $P(A_n) \downarrow P(A)$

Briefly describe the process of showing the Lebesgue measure exists

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Define $\limsup_n A_n$ and $\liminf_n A_n$ of sets and give an interpretation and an inequality

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Define measure-theoretic independence (events, collections of events, σ – field)

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Theorem 3.1 (Cartheodory extension theorem): a probability measure on a field can be uniquely extended to the generated σ – field if the measure is σ -finite.

Hence, to construct the Lebesgue measure, first we define the Lebesgue measure that assigns to half-open intervals the interval length, second we verify that this is a well-defined measure on the Borel field, and then we apply theorem 3.1.

Proving theorem 3.1 is involved. Also, studying $\sigma(\mathcal{B}_0)$ is necessary, where \mathcal{B}_0 is the field of finite unions and intersections of intervals.

$\limsup_n A_n = \bigcup_n \bigcap_{k \geq n} A_k$. If w is in LHS, then for every n , there exists some $k \geq n$ so that $w \in A_k$, hence w is in infinitely many of the A_n . “Infinitely often”.

$\liminf_n A_n = \bigcap_n \bigcup_{k \geq n} A_k$. If w is in LHS, then there exists n such that for all $k \geq n$, $w \in A_k$ for all k . Hence, w is in all but finitely many A_n . “Eventually”.

$$\begin{aligned} P(\liminf_n A_n) &\leq \liminf_n P(A_n) \\ &\leq \limsup_n P(A_n) \leq P(\limsup_n A_n) \end{aligned}$$

- Two events A and B are independent if $P(A \cap B) = P(A)P(B)$.

- A collection of events $\{A_1, \dots, A_n\}$ are independent if

$$P(A_{k_1} \cap \dots A_{k_j}) = P(A_{k_1}) \dots P(A_{k_j})$$

for all $2 \leq j \leq n$ and $1 \leq k_1 < \dots < k_n \leq n$.

- A collection of classes $\mathcal{A}_1, \dots, \mathcal{A}_n$ in a σ – field \mathcal{F} are independent if for each choice of $A_i \in \mathcal{A}_i$, the collection $\{A_n\}$ is independent.
- Two σ – fields \mathcal{A} and \mathcal{B} are independent if for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$, we have $\mu(A \cap B) = \mu(A)\mu(B)$.

State the Borel-Cantelli Lemma 1 and proof

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State the Borel-Cantelli Lemma 2 and proof

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What's a use case of the Borel-Cantelli lemmas (both)?

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If $\sum P(A_n) < \infty$, then $P(\limsup_n A_n) = 0$.

Proof: Observe that $\limsup_n A_n \subset \cup_{k \geq m} A_k$ for all m . This implies that

$$P(\limsup_n A_n) \leq P(\cup_{k \geq m} A_k) \leq \sum_{k \geq m} P(A_k).$$

Since this holds for arbitrary m and the right hand side sum goes to 0 if the infinite sum converges, the lemma follows.

If $\{A_n\}$ are independent and $\sum P(A_n) = \infty$ then $P(\limsup_n A_n) = 1$.

Proof: It is enough to show that $P(\cup_n \cap_{k \geq n} A_k^c) = 0$ for which it is enough to show that $P(\cap_{k \geq n} A_k^c) = 0$ for all n . Note that $1 - x \leq e^{-x}$, then (by independence)

$$P(\cap_{k=n}^{n+j} A_k^c) = \prod_{k=n}^{n+j} (1 - P(A_k)) \leq \exp\{-\sum_{k=n}^{n+j} P(A_k)\}.$$

But since the sum diverges, as $j \rightarrow \infty$, the RHS goes to 0, hence

$$P(\cap_{k=n}^{\infty} A_k^c) = \lim_j P(\cap_{k=n}^{n+j} A_k^c) = 0$$

Define tail σ – field and state the Kolmogorov 0-1 law

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What's a use case of the 0-1 law?

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Simple random variables

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Given a sequence of events A_1, A_2, \dots in a probability space (Ω, \mathcal{F}, P) , the tail σ – field associated with the sequence $\{A_n\}$ is

$$\mathcal{T} = \cap_{n=1}^{\infty} \sigma(A_n, A_{n+1}, \dots).$$

The $\limsup_n A_n$ and $\liminf_n A_n$ are events in the tail σ – field.

The Kolmogorov zero-one law: if A_1, A_2, \dots are independent, then for each event A in the tail σ – field, $P(A)$ is either 0 or 1.



A random variables X on (Ω, \mathcal{F}) is simple iff it can be written as

$$X(w) = \sum_i x_i I_{A_i}$$

for some finite set of x_i and $A_i \in \mathcal{F}$.

Simple random variables X_n converge to X with probability 1 ($\lim_n X_n = X$) iff $\forall \epsilon > 0$,

$$P(|X_n - X| > \epsilon \text{ i.o.}) = 0$$

which, if the above holds, implies that

$$\lim_n P(|X_n - X| > \epsilon) = 0.$$

Note that

$$\{\lim_n X_n = X\}^c = \cup_{\epsilon} \{|X_n - X| \geq \epsilon \text{ i.o.}\} = \cup_{\epsilon} \cup_n \cap_{k \geq n} \{|X_k - X| \geq \epsilon\}.$$

State and prove the Markov's inequality

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What's a use case of Markov's inequality?

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State and prove Chebyshev's inequality

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Markov's inequality: For a random variable X , nonnegative, then for positive α , we have

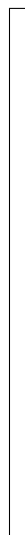
$$P(X \geq \alpha) \leq \frac{1}{\alpha} \mathbb{E}[X].$$

Proof: Note that for any convex f and any set A , we have that

$$\min_{x \in A} f(x) \mathbf{1}_A \leq E[X \mathbf{1}_A] \leq E[X]$$

Hence, with $f(x) = x$ and $A = [\alpha, \infty)$, the result follows. If we use $f(x) = |x|^k$, then we have for positive α :

$$\Pr(|X| \geq \alpha) \leq \frac{1}{\alpha^k} \mathbb{E}[|X|^k]$$



Chebyshev's inequality: for a random variable X , we have

$$\Pr(|X - m| \geq \alpha) \leq \frac{1}{\alpha^2} \text{Var}(X)$$

Proof: Applying Markov's inequality with $k = 2$ and subtracting $m = \mathbb{E}[X]$, we obtain the desired result.

What's a use case of Chebyshev's inequality?

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State and prove Jensen's inequality (finite case).

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What's a use case of Jensen's inequality?

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Jensen's inequality says that for a convex function $\phi(x)$ and a random variable X , we have

$$\mathbb{E}[\phi(X)] \geq \phi(\mathbb{E}[X])$$

Proof: the proof follows by induction (base case follows from convexity; induction step follows from grouping n of the sum terms together).
More details [here](#).

Markov's inequality. Non-negativity of the Kullback-Leibler divergence and hence mutual information. The fact that $H(X) \leq \log |\mathcal{X}|$. The fact that $H(X|Y) \leq H(X)$. Independence bound on entropy (entropy of a collection of random variables is bounded above by the sum of their individual entropies).

State and prove Holder's inequality.

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What's a use case of Holder's inequality?

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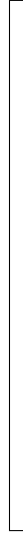
State and prove the strong law of large numbers.

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Suppose that $\frac{1}{p} + \frac{1}{q} = 1$ for $p, q > 1$. Then:

$$\mathbb{E}[|XY|] \leq \mathbb{E}[|X|^p]^{\frac{1}{p}} \mathbb{E}[|Y|^q]^{\frac{1}{q}}$$

Proof: Young's. Here.



If X_n are iid and $\mathbb{E}[X_n] = m$, then

$$\Pr\left(\lim_n n^{-1} S_n = m\right) = 1$$

Proof: WLOG $m = 0$. It is enough to show that $\Pr(|n^{-1} S_n| \geq \epsilon \text{ i.o.}) = 0$ for each ϵ .

Let $\mathbb{E}[X_i^2] = \sigma^2$ and $\mathbb{E}[X_i^4] = \xi^4$. By independence, we have

$$\mathbb{E}[S_n^4] = n\xi^4 + 3n(n-1)\sigma^4 \leq Kn^2$$

where K does not depend on n . By Markov's inequality for $k = 4$,

$$\Pr(|S_n| \geq n\epsilon) \leq Kn^{-2}\epsilon^{-4},$$

so the result follows by the first Borel-Cantelli lemma (the event probs are summable, hence the lim sup is 0).

State and prove the weak law of large numbers.

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Demonstrate a use of the strong law.

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Demonstrate a use of the weak law.

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If X_n are iid and $\mathbb{E}[X_n] = m$, then for all ϵ

$$\lim_n \Pr(|n^{-1}S_n - m| \geq \epsilon) = 0.$$

Proof: By appealing to the strong law, we have

$$\Pr(|n^{-1}S_n - m| \geq \epsilon) \leq \frac{\text{Var}(S_n)}{n^2\epsilon^2} = \frac{n\text{Var}(X_1)}{n^2\epsilon^2} \rightarrow 0.$$



What is a measurable mapping? What are some properties?

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What is the pushforward measure?

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What is the change-of-variables formula?

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For two measure spaces (Ω, \mathcal{F}) and (Ω', \mathcal{F}') , a transformation $T : \Omega \rightarrow \Omega'$ is measurable \mathcal{F}/\mathcal{F}' iff for all $A \in \mathcal{F}'$, $T^{-1}(A) \in \mathcal{F}$.

If $T^{-1}(A) \in \mathcal{F}$ for each $A \in \mathcal{A}$, where \mathcal{A} generates \mathcal{F}' , then T is \mathcal{F}/\mathcal{F}' measurable.

A random vector is measurable iff each component function is measurable.

Continuous functions are measurable. If $f_k : \Omega \rightarrow \mathbb{R}$ are measurable \mathcal{F} , then $g(f_1(w), \dots, f_k(w))$ is measurable \mathcal{F} if $g : \mathbb{R}^k \rightarrow \mathbb{R}$ is measurable.

Composition of measurable functions is measurable. Sum, sup, lim sup, product are measure-preserving. A limit of measurable functions is measurable if the limit exists everywhere. We can construct a sequence of simple measurable functions that increase to any given measurable function.

Given (Ω, \mathcal{F}) and (Ω', \mathcal{F}') and a measurable transformation $T : \Omega \rightarrow \Omega'$ and a measure μ on \mathcal{F} , then $\mu T^{-1}(A') = \mu(T^{-1}(A'))$ is a pushforward measure on \mathcal{F}' .

A measurable function g on Ω' is integrable with respect to the pushforward measure $\mu T^{-1} = T(\mu)$ iff the composition $g \circ T$ is integrable with respect to the measure μ . In that case,

$$\int_{\Omega'} g d(\mu T^{-1}) = \int_{\Omega} g \circ T d\mu$$

Say a few things about distribution functions.

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How is the integral defined for a non-negative measurable function? Properties?

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State some properties of the general integral (addition, limits)

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A distribution function for a random variable X on \mathbb{R} is $F(x) = \Pr(X \leq x)$. It is non-decreasing, right-continuous (by continuity from above). By continuity from below, $\lim_{y \uparrow x} F(y) = F(x^-) = \Pr(X < x)$.

For every non-decreasing, right-continuous function with $\lim_{x \rightarrow -\infty} F(x) = 0$ and $\lim_{x \rightarrow \infty} F(x) = 1$, there exists on some probability space a random variable X for F is the distribution function.

If $\lim_n F_n(x) = F(x)$ for all x , then we write $F_n \implies F$ and say that the distributions converge weakly and their corresponding random variables converge weakly.

$\int_{\Omega} f d\mu = \sup \sum_i (\inf_{w \in A_i} f(w)) \mu(A_i)$ where the sup is taken over all partitions of Ω .

If $f \leq g$ then $\int f \leq \int g$.

If $f_n \uparrow f$ then $\int f_n \uparrow \int f$.

The integral is linear.

If $f = 0$ a.e., then $\int f = 0$. If the measure of the set where f is non-zero is positive, then the integral is positive. If the integral exists, then $f < \infty$ a.e.

- Monotonicity: (i) if $f \leq g$ and are integrable, then $\int f d\mu \leq \int g d\mu$,
- Linearity: if f, g are integrable, then $\int (\alpha f + \beta g) d\mu \leq \alpha \int f d\mu + \beta \int g d\mu$.
- Monotone convergence: if $0 \leq f_n \uparrow f$ a.e., then $\int f_n d\mu \uparrow \int f d\mu$,
- Fatou's lemma: for non-negative f_n , $\int \liminf f_n d\mu \leq \liminf \int f_n d\mu$,
- Dominated convergence: if $|f_n| \leq g$ a.e., where g is integrable, and if $f_n \rightarrow f$ a.e., then f and f_n are integrable and $\int f_n d\mu \rightarrow \int f d\mu$.

Give an application of MCT.

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Give an application of DOM.

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Give an application of Fatou's lemma.

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Consider the space $\{1, 2, \dots\}$ with the counting measure. If for all m , we have $0 \leq x_{n,m} \uparrow x_m$ as $n \rightarrow \infty$, then $\lim_n \sum_m x_{n,m} = \sum_m x_m$. For an infinite sequence of measures μ_n on \mathcal{F} , $\mu(A) = \sum_n \mu_n(A)$ defines another measure (countably additive because sums can be reversed in a non-negative double series). You can show that $\int f d\mu = \sum_n \int f d\mu_n$ holds for all nonnegative f .

Consider the sequence $f_n = X1_{A_n}$ where $A_n \downarrow A$ with $\mu(A) = 0$ and $A_1 = \Omega$. Assuming that f_1 is absolutely integrable, note that $|f_1| \geq |f_n|$ hence the sequence is dominated and therefore $\lim_n \int f_n d\mu = \int X1_A d\mu = 0$.

Consider on $(\mathbb{R}, \mathcal{R}, \lambda)$ the functions $f_n = n^2 I_{(0, n^{-1})}$ and $f = 0$ satisfy $f_n \rightarrow 0$ for each x , but $\int f d\lambda = 0$ and $\int f_n d\lambda = n \rightarrow \infty$. Hence Fatou's lemma inequality can be strict. Note that DOM and MCT do not apply here, as f_n are unbounded.

What is the continuous mapping theorem?

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When do the integral values determine the integrands?

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What is a density?

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A sequence of random variables X_n converging in distribution/probability/as to X implies that $g(X_n)$ converges to $g(X)$ if g is continuous.

If g is just a function on $\mathbb{R} \rightarrow \mathbb{R}$ and it's a composition, this is fine. However, if you want this to state something about the expectation, then $g(X) := \mathbb{E}[X^k]$ (so g is now an operator), this is only continuous if the moments are bounded. However even in a space of bounded moment random variables (aka. $L^p(X)$), one can find a sequence of random variables whose integrals will blow up, so this g is not a bounded operator, hence not continuous.

If random variables converge in distribution, then do their moments converge?

No: $Pr(X_n = n^2) = 1/n, Pr(X_n = 0) = 1 - 1/n$.

If f and g are nonnegative and $\int_A f d\mu = \int_A g d\mu$ for all $A \in \mathcal{F}$ and if μ is σ -field then $f = g$ a.e.

If f and g are integrable and $\int_A f d\mu = \int_A g d\mu$ for all $A \in \mathcal{F}$ then $f = g$ a.e.

δ is a density if it is a nonnegative function so that for two measures ν and μ we have

$$\nu(A) = \int_A \delta d\mu, \quad A \in \mathcal{F}$$

What is Scheffe's theorem?

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What is uniform integrability?

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What is an application of uniform integrability?

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Suppose $\nu_n(A) = \int_A \delta_n d\mu$ and $\nu(A) = \int_A \delta d\mu$ for densities δ_n and δ . If

$$\nu_n(\Omega) = \nu(\Omega) < \infty, \quad n = 1, 2, \dots,$$

and if $\delta_n \rightarrow \delta$ except on a set of μ -measure 0, then

$$\sup_{A \in \mathcal{F}} |\nu(A) - \nu_n(A)| \leq \int_{\Omega} |\delta - \delta_n| d\mu \rightarrow 0$$

In other words, the total variation of two measures is bounded above by the L^1 difference of the densities, hence if the densities converge, then the measures converge.

A sequence f_n is uniformly integrable if

$$\lim_{\alpha \rightarrow \infty} \sup_n \int_{|f_n| \geq \alpha} |f_n| d\mu = 0$$

Suppose $\mu(\Omega) < \infty$ and $f_n \rightarrow f$ a.e.

(i) if f_n are uniformly integrable, then f is integrable and

$$\int f_n d\mu \rightarrow \int f d\mu$$

(ii) if f and f_n are nonnegative and integrable, then the conclusion of (i) implies that f_n are uniformly integrable.

If $\mu(\Omega) < \infty$, f and f_n are integrable, and $f_n \rightarrow f$ a.e., then the following are equivalent:

- f_n are uniformly integrable,
- $\int |f - f_n| d\mu \rightarrow 0$,
- $\int |f_n| d\mu \rightarrow \int |f| d\mu$.

What is Riemann integrability?

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What is a product space? What is the product measure?

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What is Fubini's theorem?

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A real function f on $(a, b]$ is Riemann integrable with integral r if: for all ϵ there exists a δ with

$$|r - \sum_i f(x_i)\lambda(I_i)| < \epsilon$$

if $\{I_i\}$ is any finite partition of $(a, b]$ into subintervals satisfying $\lambda(I_i) < \delta$ and if $x_i \in I_i$ for each i .

Given two measure spaces (X, \mathcal{X}) and (Y, \mathcal{Y}) , the product space is $(X \times Y, \mathcal{X} \times \mathcal{Y})$ where $\mathcal{X} \times \mathcal{Y}$ is the set of sets $A \times B$ where $A \in \mathcal{X}$ and $B \in \mathcal{Y}$.

Given two measures μ, ν , the product measure is a measure π with $\pi(A \times B) = \mu(A)\nu(B)$. This measure is unique.

If (X, \mathcal{X}, μ) and (Y, \mathcal{Y}, ν) are σ -field measure spaces, then

$$\int_{X \times Y} f(x, y) \pi(d(x, y)) = \int_X \left[\int_Y f(x, y) \nu(dy) \right] \mu(dx)$$

$$\int_{X \times Y} f(x, y) \pi(d(x, y)) = \int_Y \left[\int_X f(x, y) \mu(dx) \right] \nu(dy)$$

What is convergence in probability?

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What is the Glivenko-Cantelli theorem?

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*What is the definition of expected value? Same some things
about it and limits.*

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Random variables X_n converge in probability to X , written $X_n \rightarrow_P X$ if for each positive ϵ

$$\lim_n P(|X_n - X| \geq \epsilon) = 0$$

A necessary and sufficient condition for convergence in probability is that each subsequence X_{n_i} contain a further subsequence $X_{n_{k(i)}}$ such that $X_{n_{k(i)}} \rightarrow X$ with probability 1 as $i \rightarrow \infty$.

In nonprobabilistic contexts, convergence in probability becomes convergence in measure.

Suppose that X_1, X_2, \dots are independent and have a common distribution function F ; put $D_n(w) = \sup_x |F_n(x, w) - F(x)|$, where

$$F_n(x, w) = \frac{1}{n} \sum_{i=1}^n I_{(-\infty, x]}(X_i(w))$$

Then $D_n \rightarrow 0$ with probability 1.

$$\mathbb{E}[X] = \int X dP = \int_{\Omega} X(w) P(dw).$$

$$\mathbb{E}[X 1_A] = \int_A X dP.$$

$$\mathbb{E}[g(X)] = \int_{\Omega} g(X(w)) P(dw).$$

$$\text{Absolute moments } \mathbb{E}[|X|^k] = \int_{-\infty}^{\infty} |x|^k P(dx).$$

By the way $P(dx)$ intuitively refers to the P measure of a small change in x .

If X_n are dominated by an integrable random variable (or uniformly integrable), then $\mathbb{E}[X_n] \rightarrow \mathbb{E}[X]$ follows if $X_n \rightarrow_P X$.

*What is the MGF? What is it for normal, exponential, Poisson,
and Binomial dists?*

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The MGF is

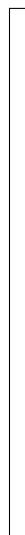
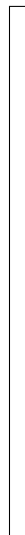
$$M_X(t) = \mathbb{E}[e^{tX}] = \int_{-\infty}^{\infty} e^{sx} \mu(dx)$$

The standard normal distribution: $M_X(t) = e^{t^2/2}$.

The exponential: $M_X(t) = \frac{\alpha}{\alpha - s}$ defined for $s < \alpha$.

The Poisson: $M_X(t) = e^{\lambda(e^t - 1)}$.

The Binomial: $M_X(t) = pe^t + (1 - p)$.



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