

If X is a linear space and M is a linear subspace of X, then N is a complementary subspace of M if every $x \in X$ can be uniquely represented by x = n + m where $n \in N$ and $m \in M$. This implies $M \cap N = \{0\}$.

If M is a linear subspace of X, then M may have infinitely many complementary subspaces. All of the complementary subspaces of M have the same dimension, which is called the codimension of M.

A projection on a linear space X is a linear map $P: X \to X$ such that $P^2 = P$.

First note that x = Px if and only if $x \in \operatorname{ran} P$, since if x = Py then $Px = P^2x = Py = x$.

Let $x \in \ker P \cap \operatorname{ran} P$. Then x = Px = 0.

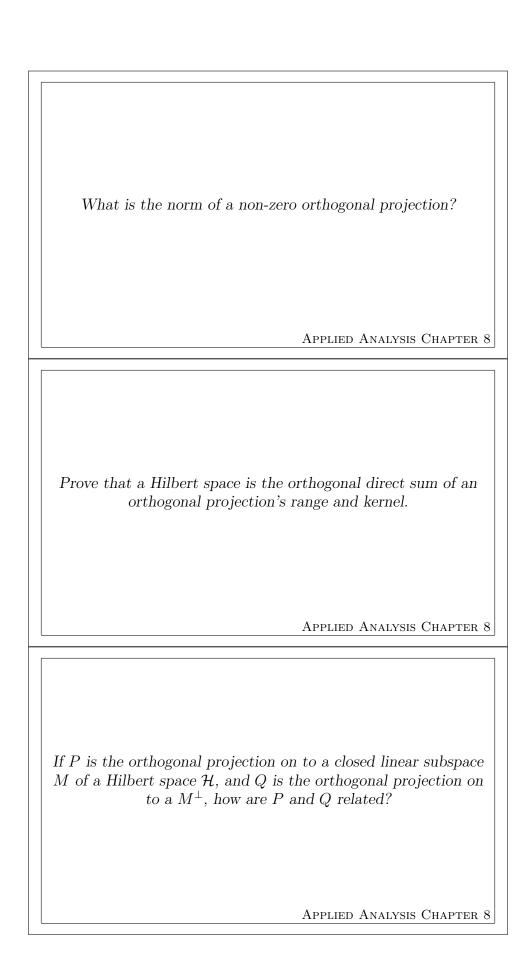
Let $x \in X$. Then $x - Px \in \ker P$ since

$$P(x - Px) = Px - P^2x = Px - Px = 0.$$

Then x = Px + (x - Px).

An orthogonal projection P, on a Hilbert space \mathcal{H} , is a function $P:\mathcal{H}\to\mathcal{H}$ such that

$$P^2 = P$$
 and $(Px, y) = (x, Py) \ \forall x, y \in \mathcal{H}$



Using the Cauchy-Schwarz Inequality,

$$\|Px\| = \frac{\|Px\|^2}{\|Px\|} = \frac{(Px, Px)}{\|Px\|} = \frac{(x, P^2x)}{\|Px\|} = \frac{(x, Px)}{\|Px\|} \le \frac{\|x\| \|Px\|}{\|Px\|} = \|x\|,$$

so $\|P\| \le 1$. However, there is an $x \in \operatorname{ran} P$ with $\|x\| \ne 0$ (since P is non-zero. Then

$$||Px|| = ||x||,$$

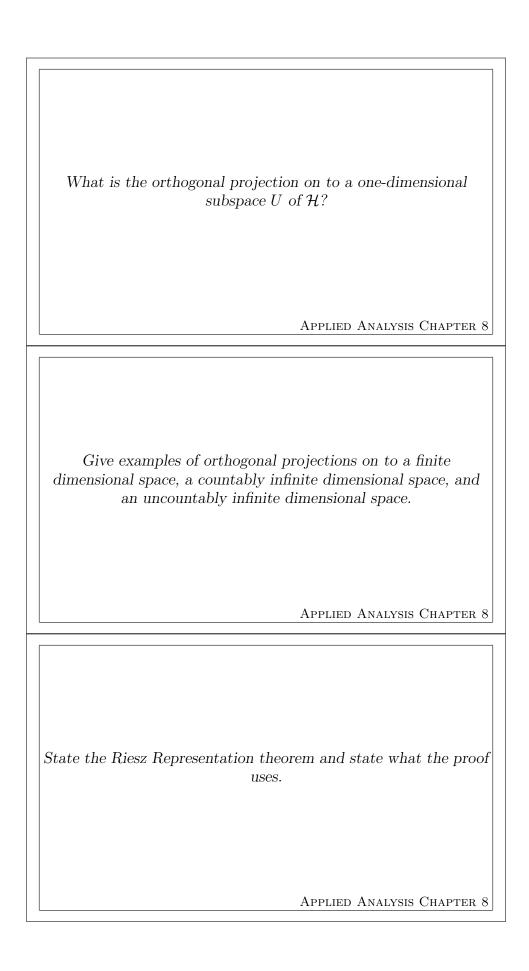
which shows $||P|| \ge 1$, and thus ||P|| = 1.

Let P be an orthogonal projection on \mathcal{H} . We know $\mathcal{H} = \ker P \oplus \operatorname{ran} P$ where \oplus is just a (not necessarily orthogonal) direct sum. However, if $x = Py \in \operatorname{ran} P$ and $z \in \ker P$, then

$$(x,z) = (Py,z) = (y,Pz) = (y,0) = 0$$

and thus $\ker P \perp \operatorname{ran} P$.

$$I - P = Q$$
.



Let $\{u\}$ be a basis of U. Then define P_U by

$$P_{u}x = \frac{(u,x)}{\|u\|^2}u.$$

Let $\mathcal{H} = \mathbb{R}^n$ and \mathbf{u} be any unit vector. The orthogonal projection in the direction of \mathbf{u} is the rank one matrix $\mathbf{u}\mathbf{u}^T$. The component of a vector \mathbf{x} in the direction of \mathbf{u} , i.e. the projection of \mathbf{x} on to $[\{\mathbf{u}\}]$ is

$$P_{\mathbf{u}}\mathbf{x} = \frac{(\mathbf{u}, \mathbf{x})}{\|\mathbf{u}\|^2}\mathbf{u} = (\mathbf{u}^T\mathbf{x})\mathbf{u}.$$

Let $\mathcal{H} = \ell^2(\mathbb{Z})$ and $u = e_n = (\delta_{k,n})_{k=-\infty}^{\infty}$ and $x = (x_k)$. Then $P_{e_n}x = x_n e_n$ gives a vector of all 0s except for the n^{th} component of x in the n^{th} position. Let $\mathcal{H} = L^2(\mathbb{T})$ and $u(x) \equiv \frac{1}{\sqrt{2\pi}}$, which is the constant function with ||u|| = 1. Then P_u maps a function f to its mean $\langle f \rangle$, i.e.

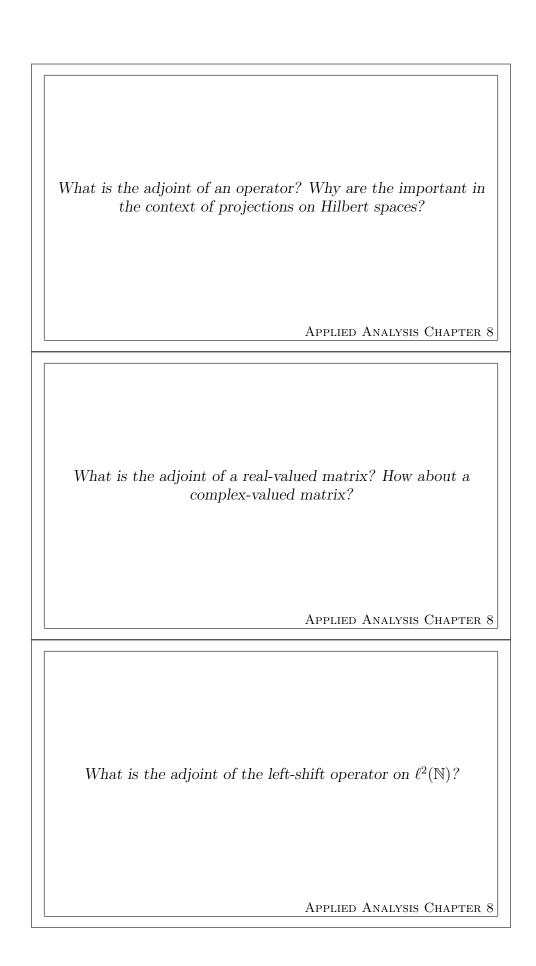
$$P_u f = \frac{1}{2\pi} \int_{\mathbb{T}} f(x) \mathrm{d}x.$$

Then $f = \langle f \rangle + \tilde{f}$ is the decomposition of a function into a constant mean part and a fluctuating, 0 mean part.

If ϕ is a bounded linear functional on a Hilbert space \mathcal{H} , then there is a unique $y \in \mathcal{H}$ such that $\phi(x) = (y, x)$ for all $x \in \mathcal{H}$.

The proof uses

- The kernel of a bounded linear operator is a closed subspace.
- Defining a clevel orthogonal projection P whose kernel is equal to ker ϕ .
- Arbitrary vectors can be decomposed by $\mathcal{H} = \ker P \oplus \operatorname{ran} P$.



The adjoint of an bounded linear operator A on a Hilbert space \mathcal{H} is denoted A^* and is the unique operator in $\mathcal{B}(\mathcal{H})$ such that

$$(x, Ay) = (A^*x, y) \quad \forall x, y \in \mathcal{H}.$$

An operator A is called self-adjoint if $A = A^*$. All orthogonal projects are self-adjoint, that is, for any projection P on \mathcal{H} , we have

$$(Px, y) = (x, Py) \quad \forall x, y \in \mathcal{H}.$$

This is not true for all projections - just orthogonal projections.

If A is a real-valued matrix in $\mathbb{R}^{n \times n}$, then $A^* = A^T$. That is, the adjoint is the transpose.

If A is a complex-valued matrix in $\mathbb{C}^{n\times n}$, then $A^* = \overline{A^T}$. That is, the adjoint is the Hermitian conjugate matrix.

Let $T \in \mathcal{B}(\ell^2(\mathbb{N}))$ by

$$T(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots).$$

Then $T^* = S$, the right-shift operator, given by

$$S(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots)$$

because

$$(x,Ty) = \sum_{i=1}^{\infty} \overline{x_i} y_{i+1} = (Sx, y).$$

This is analogous to the transpose of a matrix since the transpose of the infinite matrix representing T is the infinite matrix representing S.

What is the adjoint of a Fredholm integral operator $K \in \mathcal{B}(L^2([0,1]))$?

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If $A: \mathcal{H} \to \mathcal{H}$ is a bounded linear operator, show $\overline{\operatorname{ran} A} = (\ker A^*)^{\perp}$, and $\ker A = (\operatorname{ran} A^*)^{\perp}$.

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Suppose that $A: \mathcal{H} \to \mathcal{H}$ is a bounded linear operator on a Hilbert space \mathcal{H} with closed range. When does the equation Ax = y have a solution?

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Let $K \in \mathcal{B}(L^2([0,1]))$ by

$$Kf(x) = \int_0^1 k(x, y) f(y) dy$$

for some continuous function $k : [0,1] \times [0,1] \to \mathbb{C}$. Then K^* is given explicitly by integration against the complex conjugate, transpose kernel:

$$K^*f(x) = \int_0^1 \overline{k(y,x)} f(y) \mathrm{d}y.$$

This is analogous to the Hermitian conjugate of a matrix.

Let $x \in \operatorname{ran} A$. Then x = Ay for some $y \in \mathcal{H}$. Then $(x, z) = (Ay, z) = (y, A^*z) = 0$ for any $z \in \ker A^*$. Thus $x \in (\ker A^*)^{\perp}$, which is closed, and so $\operatorname{ran} A \subset (\ker A^*)^{\perp}$.

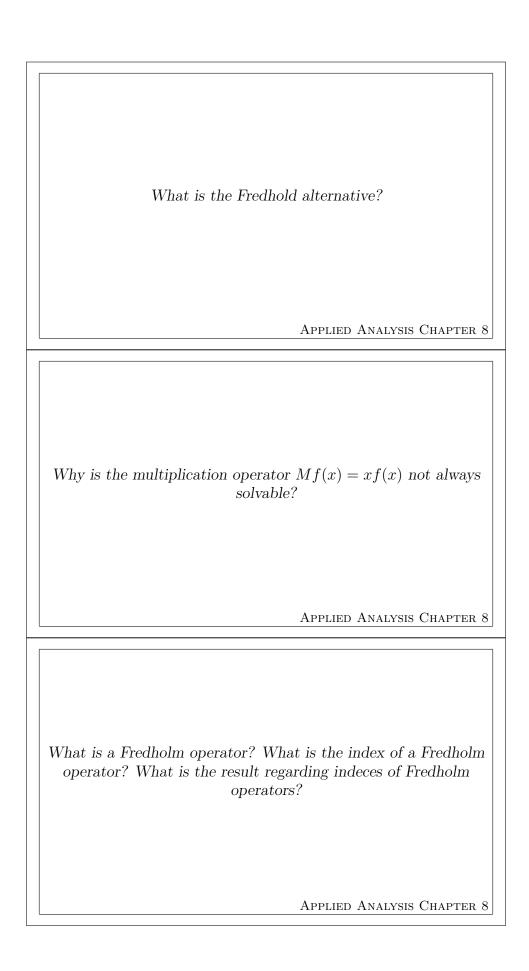
If $x \in (\operatorname{ran} A)^{\perp}$, then $0 = (Ay, x) = (y, A^*x)$ for every $y \in \mathcal{H}$, which shows $A^*x = 0$ for every $x \in (\operatorname{ran} A)^{\perp}$, i.e. $x \in \ker A^*$, so $(\operatorname{ran} A)^{\perp} \subset \ker A^*$. Then $(\ker A^*)^{\perp} \subset (\operatorname{ran} A)^{\perp \perp} = \overline{\operatorname{ran} A}$, which shows $(\ker A^*)^{\perp} = \overline{\operatorname{ran} A}$. The clever move here was $X \subset Y \Longrightarrow Y^{\perp} \subset X^{\perp}$.

Then taking $A = A^*$ in the above equality gives $(\ker A)^{\perp} = \overline{\operatorname{ran} A^*}$. Then taking orthogonal complements gives $\ker A = \overline{\operatorname{ran} A^*}^{\perp} = (\operatorname{ran} A^*)^{\perp}$.

A succint way of stating this theorem is

$$\mathcal{H} = \overline{\operatorname{ran} A} \oplus (\ker A^*)^{\perp}.$$

The equation Ax = y has a solution for x if and only if $y \perp \ker A^*$.



A bounded linear operator A on a Hilbert space \mathcal{H} satisfies the Fredholm alternative if either

- (a) either Ax = 0, $A^*x = 0$ have only the zero solution, and the equations Ax = y, $A^*x = y$ have a unique solution for every $y \in \mathcal{H}$;
- (b) or Ax = 0, $A^*x = 0$ have nontrivial, finite-dimensional solution spaces of the same dimension, Ax = y has a (nonunique) solution if and only if $y \perp z$ for every solution z of $A^*z = 0$, and $A^*x = y$ has a (nonunique) solution if and only if $y \perp z$ for every solution z of Az = 0.

In English, either

- (a) A and A^* are bijective;
- (b) or A and A^* are not injective, but have the same nullity.

Even though $\ker M^* = \ker M = \{0\}$, and hence every $g \in L^2([0,1])$ is orthogonal to $\ker M^*$, Mf = g is not always solvable since ran M is properly dense in $L^2([0,1])$.

A bounded linear operator A on a Hilbert space \mathcal{H} is a Fredhold operator if

- (a) $\operatorname{ran} A$ is closed;
- (b) $\ker A$ and $\ker A^*$ are finite-dimensional.

The index of a Fredholm operator A, denoted ind A, is the integer

 $\operatorname{ind} A = \dim \ker A - \dim \ker A^*$.

If A is Fredholm and K is compact, then A + K is Fredholm and

$$\operatorname{ind} A = \operatorname{ind} (A + K).$$

That is, the index of a Fredholm is unchanged by compact perturbations. Since I is Fredholm, and ind I = 0, then ind (I + K) = 0 for compact K.

What is the most important sesquilinear form derived from a bounded linear operator A on a Hilbert space \mathcal{H} ? What is the associated quadratic form? What is it mean to be a non-negative operator? How about a positive (positive definite) operator? APPLIED ANALYSIS CHAPTER 8 How can we define an inner product on a Hilber space \mathcal{H} given a positive definite bounded linear operator A? APPLIED ANALYSIS CHAPTER 8 If A is a bounded, self-adjoint operator on a Hilbert space \mathcal{H} ,

what is an easy formula for ||A||?

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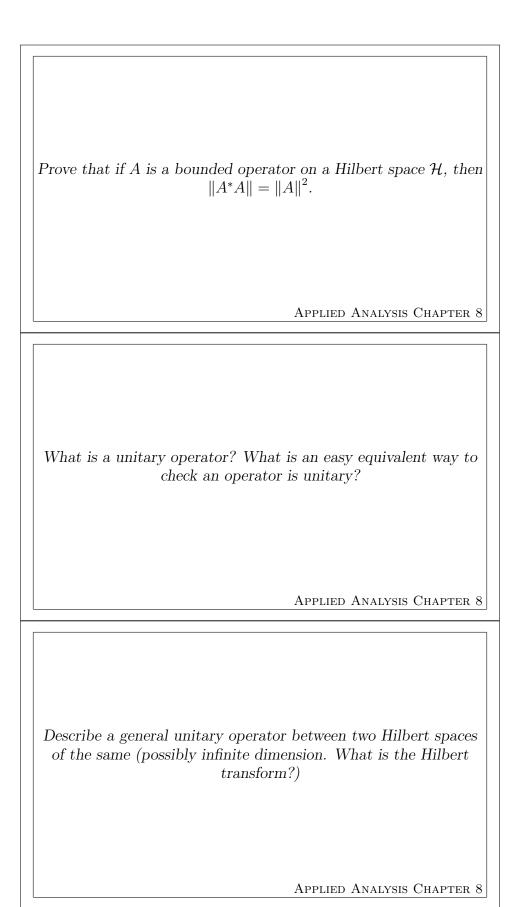
Given a linear operator A, define the sesquilinear form $a: \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ by a(x,y)=(x,Ay). The associated quadrative form is $q_A(x)=a(x,x)$, or $q_A(x)=(x,Ax)$.

A is called nonnegative if its quadratic form is nonnegative for all $x \in \mathcal{H}$, i.e. $q_A(x) \geq 0, \forall x \in \mathcal{H}$. A is positive definite if $q_A(x) > 0, \forall x \in \mathcal{H}$.

$$(x,y)_A := (x,Ay)$$

defines an inner product on \mathcal{H} . In addition, $(\cdot, \cdot)_A$ is equivalent to (\cdot, \cdot) .

 $||A|| = \sup_{||x||=1} |q_A(x)|,$ where q is the quadratic form $q_A(x) = (x, Ax).$



$$||A||^2 = \sup_{||x||=1} ||Ax||^2 = \sup_{||x||=1} |(Ax, Ax)| = \sup_{||x||=1} |q_{A^*A}(x)| = ||A^*A||$$

If A is self adjoint, the $A^* = A$ and $||A||^2 = ||A^2||$.

A linear map $U: \mathcal{H}_1 \to \mathcal{H}_2$ between real or complex Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 is said to be orthogonal (real) or unitary (complex), respectively, if it is invertible and if

$$(Ux, Uy) = (x, y), \quad \forall x, y \in \mathcal{H}_1$$

Unitary operators preserve inner products.

Two Hilbert spaces are isomorphic if there exists a unitary operator between them.

An operator U is unitary if and only if its inverse is it's adjoint. That is, U is unitary if and only if $U^* = U^{-1}$.

Let \mathcal{H}_1 and \mathcal{H}_2 have the same (possibly infinite) dimension. Then their bases can be indexed by the same index set. Suppose \mathcal{H}_1 has basis $\{u_{\alpha}\}$ and \mathcal{H}_2 has basis $\{v_{\alpha}\}$. Then any $x \in \mathcal{H}_1$ can be written as

$$x = \sum_{\alpha} c_{\alpha} u_{\alpha}, \quad \text{where } c_{\alpha} = (u_{\alpha}, x).$$

Let λ_{α} be complex numbers with $|\lambda_{\alpha}| = 1$ and define $U : \mathcal{H}_1 \to \mathcal{H}_2$ by

$$Ux = \sum_{\alpha} \lambda_{\alpha}(u_{\alpha}, x)v_{\alpha}.$$

Then U is unitary, and thus $\mathcal{H}_1 \cong \mathcal{H}_2$. The Hilbert transform \mathbb{H} is of this form. Define \mathbb{H} : $\mathcal{H}_0 \subset L^2(\mathbb{T}) \to \mathcal{H}_0$ ($\mathcal{H}_0 = \{f \in L^2(\mathbb{T}) \mid \langle f \rangle = 0\}$ is the space of L^2 functions with 0 mean) by

$$\mathbb{H}f = \mathbb{H}\left(\sum_{n \in \mathbb{N}} \hat{f}_n e^{inx}\right) = \sum_{n \in \mathbb{N}} \left(i(\operatorname{sgn}[n])\hat{f}_n e^{inx}\right).$$

1	What is the most famous unitary operator (besides maybe the identity operator)?
	Applied Analysis Chapter 8
	What is a normal operator?
	Applied Analysis Chapter 8

The most famous (and most useful) unitary operator is the Fourier transform, $\mathcal{F}: L^2(\mathbb{T}) \to \ell^2(\mathbb{N})$, which is given by

$$\mathcal{F}f = \mathcal{F}\left(\sum_{n\in\mathbb{N}} c_n e^{inx}\right) = (c_n)_{n\in\mathbb{N}}, \quad \text{where } c_n = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} f(x) e^{-inx} dx.$$

A operator A is said to be normal if it commutes with its adjoint, that is,

$$AA^* = A^*A.$$