State the Divergence Theorem

State Green's First and Second Identities.

What is a "test function"?

Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain and $w = (w_1, \ldots, w_n) \in \mathcal{C}^1(\overline{\Omega})$ with outward pointing normal N. Then

$$\int_\Omega \boldsymbol{\nabla} \cdot w \, \mathrm{d}x = \int_{\partial\Omega} w \cdot N \mathrm{d}S$$

Let Ω be a smooth domain and let $u \in \mathcal{C}^2(\overline{\Omega})$ and $v \in \mathcal{C}^1(\overline{\Omega})$-funtions. Then we have Green's First Identity:

$$\int_\Omega \boldsymbol{\nabla} v \cdot \boldsymbol{\nabla} u + v \nabla^2 u \mathrm{d}x = \int_\Omega \boldsymbol{\nabla} \cdot (v \boldsymbol{\nabla} u) \mathrm{d}x = \int_{\partial\Omega} v \frac{\partial u}{\partial N} \mathrm{d}S$$

Exchanging u and v in Green's First Identity and finding the difference gives Green's Second Identity:

$$\int_\Omega \left(v \nabla^2 u - u \nabla^2 v\right) \mathrm{d}x = \int_{\partial\Omega} \left[v \frac{\partial u}{\partial N} - u \frac{\partial v}{\partial N} \right] \mathrm{d}S$$

Test functions are smooth functions with compact support, i.e.

$$\mathcal{C}_c^\infty = \{ u \in \mathcal{C}^\infty \ : \ \operatorname{supp} u \subset \mathcal{V} \Subset \Omega \} \, .$$

What is a weak derivative of an L^1_{loc} function?

Why does f have a weak-derivative, but g does not?

$$f(x) = \begin{cases} x & \text{if } x \in (0,1) \\ 1 & \text{if } x \in (1,2) \end{cases}$$

$$g(x) = \begin{cases} x & \text{if } x \in (0,1) \\ 2 & \text{if } x \in (1,2) \end{cases}$$

Define $W^{1,p}(\Omega)$ for $1 \le p \le \infty$. Then define $W^{k,p}(\Omega)$ for $1 \le p \le \infty$ and $k \in \mathbb{N}$. What is the norm in $W^{k,p}$?

Let $u \in L^1_{\text{loc}}(\Omega)$. Then $v^\alpha \in L^1_{\text{loc}}(\Omega)$ is called the α^{th} weak derivative of u, written $v^\alpha = D^\alpha u$, if

$$\int_\Omega u(x) D^\alpha \phi(x) \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega v^\alpha(x) \phi(x) \mathrm{d}x \qquad \forall \phi \in \mathcal{C}_c^\infty(\Omega),$$

where $\alpha \in \mathbb{N}^n$ is a multi-index with $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

We can explicitly calculate the weak derivative (using integration by parts) of f. For g, however, assuming a weak derivative exists results in a contradiction by exploiting the boundary terms in the integration by parts that don't cancel each other out.

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) \mid \text{the weak derivative } u' \text{ of } u \text{ exists, and } u' \in L^p(\Omega)\}$$

$$W^{k,p}(\Omega) = \left\{u \in L^1_{\text{loc}}(\Omega) \mid D^\alpha u \text{ exists and is in } L^p(\Omega) \text{ for } |\alpha| \leq k\right\}$$

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}$$

$$\|u\|_{W^{k,\infty}(\Omega)} = \sum_{\alpha \leq k} \|D^\alpha u\|_{L^\infty(\Omega)}$$

What is the “simple version” of the Sobolev Embedding Theorem?

Let $kp > 2$ and $u \in \mathcal{C}_c^\infty(\mathbb{R}^2)$. Then

$$\|u\|_{L^\infty(\mathbb{R}^2)} \leq C\|u\|_{W^{k,p}(\mathbb{R}^2)}$$

In English, $W^{k,p}$ functions are bounded in $\mathbb{R}^2$.