What is the Fourier basis for $L^2(\mathbb{T})$?

What are trigonometric polynomials? Why are they important?

Define a convolution of two continuous functions

The Fourier basis is $\{e_n \mid n \in \mathbb{Z}\}$ where

$$e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}.$$

A "trigonometric polynomial" is a special name for finite linear combinations of the Fourier basis. They are important since they are dense in $C(\mathbb{T})$. Proving they are dense in $C(\mathbb{T})$, along with showing orthonormality, proves the Fourier basis is indeed a basis of $L^2(\mathbb{T})$.

The convolution of two continuous functions $f, g : \mathbb{T} \to \mathbb{C}$, denoted $f * g$, is a continuous function defined by the following integral:

$$(f * g)(x) = \int_{\mathbb{T}} f(x-y)g(y)\mathrm{d}y = \int_{\mathbb{T}} f(y)g(x-y)\mathrm{d}y$$

What is an approximate identity? Why are they important?

If $\{\phi_n \in C(\mathbb{T}) \mid n \in \mathbb{N}\}$ *is an approximate identity and* $f \in C(\mathbb{T})$*, how does* $\phi_n * f$ *converge to* f*? Uniformly, or just pointwise?*

The trigonometric polynomials are dense in $C(\mathbb{T})$ *with respect to the uniform norm.*

A family of functions $\{\phi_n \in C(\mathbb{T}) \mid n \in \mathbb{N}\}$ is an approximate identity if

(a) $\phi_n(x) \geq 0$;

(b) $\displaystyle\int_{\mathbb{T}} \phi_n(x)\mathrm{d}x = 1$ for every $n \in \mathbb{N}$;

(c) $\displaystyle\lim_n \to \infty \int_{\delta \leq |x| \leq \pi} \phi_n(x)\mathrm{d}x = 0$ for every $0 < \delta \leq \pi$.

In English:

(a) All functions are non-negative.

(b) The area under each curve is 1.

(c) As $n \to \infty$, most of the area accumulates near 0.

Approximate identities are important since for large n, the convolution of f with ϕ_n gives a local average of f.

Uniformly. The proof uses:

- $f(x) = \int_{\mathbb{T}} \phi_n(y) f(x)\mathrm{d}y$ (clever since $f(x)$ is independent of y)
- Splitting the integral in to y in a δ-ball around 0 and y outside of that δ-ball
- Different components of the product are small on different subsets of $\mathbb{T}$.

The proof uses:

- A specific approximate identity: $\phi_n(x) = c_n(1 + \cos x)^n$, where c_n is chosen so $\int_{\mathbb{T}} \phi_n = 1$.
- Since $\phi_n * f \to f$ uniformly, just show it is in fact a trigonometric polynomial.
- Noting ϕ_n is a trigonometric polynomial: $\phi_n(x) = \sum_{k=-n}^{n} a_{n,k} e^{ikx}$ where $2^{-n} c_n \binom{2n}{n+k}$.

Why can any $f \in L^2(\mathbb{T})$ be expanded as a Fourier series? What are its coefficients?

What is Parseval's Identity and why is it important in Fourier theory?

If $f, g \in L^2(\mathbb{T})$, then $f * g$ is continuous ($f * g \in C(\mathbb{T})$) and

$$\|f * g\|_\infty \leq \|f\|_2 \|g\|_2$$

Since trigonometric polynomials are dense in $C(\mathbb{T})$, which is dense in $L^2(\mathbb{T})$, any $f \in L^2(\mathbb{T})$ can be written as

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}_n e_n(x)$$

where $e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$, and so

$$f(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \hat{f}_n e^{inx}.$$

The orthonormality of $\{e_n\}$ provides a concrete calculation of $\hat{f}_n$:

$$\hat{f}_n = (e_n, f) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} f(x) e^{-inx} \mathrm{d}x.$$

Parseval's Identity is $(f, g) = \sum_{n \in \mathbb{N}} \overline{f_n} g_n$ where f_n and g_n are the coefficients of the expansions of f and g with respect to an orthonormal basis. In Fourier theory, we have

$$\int_{\mathbb{T}} \overline{f(x)} g(x) \mathrm{d}x = (f, g) = \sum_{n=-\infty}^{\infty} \overline{\hat{f}_n} \hat{g}_n$$

Taking $g = f$ gives

$$\|f\|_{L^2(\mathbb{T})}^2 = \int_{\mathbb{T}} |f(x)|^2 \mathrm{d}x = \sum_{n=-\infty}^{\infty} \left|\hat{f}_n\right|^2 = \left\|\left(\hat{f}_n\right)\right\|_{\ell^2(\mathbb{Z})}^2$$

and thus we can define a Hilbert space isomorphism $\mathcal{F}$ between $L^2(\mathbb{T})$ and $\ell^2(\mathbb{T})$ by

$$\mathcal{F} f = \left(\hat{f}_n\right)_{n=-\infty}^{\infty}.$$

The proof uses:

- The Cauchy-Schwarz inequality applied to the definition of convolution
- The convolution of continuous functions in continuous
- Forming continuous approximations of f and g (sequences of continuous functions approaching f and g)
- Triangle Inequality
- Completeness of $C(\mathbb{T})$

This is a special case of Young's inequality.

State and prove the Convolution Theorem

What are other useful bases of $L^2(\mathbb{T})$?

Generalize the Fourier series to a d-dimensional 2π-periodic function.

For $f, g \in L^2(\mathbb{T})$, $\widehat{(f*g)}_n = \sqrt{2\pi}\hat{f}_n\hat{g}_n$.
First let $f, g \in C(\mathbb{T})$. Then

$$\widehat{(f*g)}_n = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{T}}(f*g)(x)e^{-inx}\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{T}}\left(\int_{\mathbb{T}} f(x-y)g(y)\mathrm{d}y\right)e^{-inx}\mathrm{d}x$$

We can use Fubini's Theorem since the integrand is continuous:

$$\widehat{(f*g)}_n = \int_{\mathbb{T}}\frac{1}{\sqrt{2\pi}}\left(\int_{\mathbb{T}} f(x-y)e^{-in(x-y)}\mathrm{d}x\right)g(y)e^{iny}\mathrm{d}y$$

by multiplying and dividing by e^{iny}. Then

$$\widehat{(f*g)}_n = \hat{f}_n\int_{\mathbb{T}} g(y)e^{-iny}\mathrm{d}y = \sqrt{2\pi}\hat{f}_n\hat{g}_n$$

By density of $C(\mathbb{T})$ in $L^2(\mathbb{T})$, and continuity of the Fourier transform and the convolution with respect to L^2 convergence, this result holds for $L^2(\mathbb{T})$ functions too.

$\{e_n(x) = e^{inx} \mid n \in \mathbb{Z}\}$ is an orthogonal (not orthonormal) basis. Then

$$f(x) = \sum_{n=-\infty}^{\infty}\hat{f}_n e^{inx}, \qquad \text{where} \qquad \hat{f}_n = \frac{1}{2\pi}\int_{\mathbb{T}} f(x)e^{-inx}\mathrm{d}x$$

$\{1, \cos(nx), \sin(nx) \mid n = 1, 2, \ldots\}$ is an orthogonal basis. Then

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(nx) + b_n\sin(nx)\right]$$

where $a_n = \frac{1}{\pi}\int_{\mathbb{T}} f(x)\cos(nx)\mathrm{d}x$ and $b_n(x) = \frac{1}{\pi}\int_{\mathbb{T}}\sin(nx)\mathrm{d}x$. These are useful since odd(even) functions have sin(cos) expansions. Finally, any function defined on $[0, \pi]$ can be extended to an odd(even) function on $[-\pi, \pi]$ and can thus be represented by a sin(cos) expansion.

A function $f : \mathbb{R}^d \to \mathbb{C}$ is 2π-periodic in each variable if

$$f(x_1, x_2, \ldots, x_i + 2\pi, \ldots, x_d) = f(x_1, x_2, \ldots, x_i, \ldots, x_d) \qquad \text{for } i = 1, \ldots, d.$$

An orthonormal basis of $L^2(\mathbb{T}^d)$ is

$$e_{\mathbf{n}}(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}}e^{i\mathbf{n}\cdot\mathbf{x}}, \qquad \text{where } \mathbf{x} \in \mathbb{T}^d,\ \mathbf{n} \in \mathbb{Z}^d,\ \text{and } \mathbf{n}\cdot\mathbf{x} = \sum_{i=1}^{d} n_i x_i.$$

The Fourier series expansion of a function $f \in L^2(\mathbb{T}^d)$ is

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}}\sum_{\mathbf{n}\in\mathbb{Z}^d}\hat{f}_{\mathbf{n}}e^{i\mathbf{n}\cdot\mathbf{x}}, \qquad \text{where } \hat{f}_{\mathbf{n}} = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{T}^d} f(\mathbf{x})e^{-i\mathbf{n}\cdot\mathbf{x}}\mathrm{d}\mathbf{x}$$

What is the connection between smoothness of a function and its Fourier coefficients?

How is differentiation made easier by fourier series? How does this expand the notion of derivative?

Define the Sobolev space $H^1(\mathbb{T})$ in terms of Fourier coefficients.

The smoother the function, the faster its Fourier coefficients decay. Smooth functions contain small amounts of high-frequency components.

The Fourier coefficients of a derivative is a scalar multiple (in particular, in) of the Fourier coefficients of the original function.

$$\widehat{f'}_n = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{inx} f'(x) \mathrm{d}x.$$

Integration by parts gives

$$\widehat{f'}_n = \frac{1}{\sqrt{2\pi}}[f(2\pi) - f(0)] - \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} (-in) e^{-inx} f(x) \mathrm{d}x = in \hat{f}_n.$$

Induction gives $\widehat{f^{(k)}}_n = (in)^k \hat{f}_n$. Heuristically, derivatives are a "roughing" operation, adding higher amounts of high-frequency components.
We can take a "derivative" of an arbitrary L^2 function by transforming it into an ℓ^2 sequence and multiplying each component by in. This leads to what is called a "weak derivative."

The Sobolev space $H^1(\mathbb{T}) = W^{1,2}(\mathbb{T})$ consists of all functions

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{N}} \hat{f}_n e^{inx} \in L^2(\mathbb{T}) \qquad \text{such that} \qquad \sum_{n\in\mathbb{N}} n^2 \left|\hat{f}_n\right|^2 < \infty.$$

That is, all functions $f \in L^2(\mathbb{T})$ such that the weak derivative of f, denoted f', is also in $L^2(\mathbb{T})$. f' is given by $f'(x) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{N}} in \hat{f}_n e^{inx}$. The inner product on $H^1(\mathbb{T})$ is given by

$$(f, g)_{H^1(\mathbb{T})} = \int_{\mathbb{T}} \left[\overline{f(x)} g(x) + \overline{f'(x)} g'(x)\right] \mathrm{d}x = (f, g)_{L^2(\mathbb{T})} + (f', g')_{L^2(\mathbb{T})},$$

and Parseval's Theorem gives $(f, g)_{H1(\mathbb{T})} = \sum_{n\in\mathbb{N}} (1 + n^2) \overline{\hat{f}_n} \hat{g}_n$.

Use Parseval's Theorem to show that weak derivatives satisfy integration by parts

Define a weak derivative of an L^2 function using test functions and the theory of bounded linear operators on Hilbert spaces.

Define a general Sobolev space $H^k(\mathbb{T})$ for $k \in \mathbb{R}^+$.

$$\int_{\mathbb{T}} f' g \mathrm{d}x = (\overline{f'}, g)_{L^2} = \sum_{n\in\mathbb{N}} -in\overline{\hat{f}_n}\hat{g}_n = -\sum_{n\in\mathbb{N}} \overline{\hat{f}_n} in\hat{g}_n = -(f, g')_{L^2} = -\int_{\mathbb{T}} f g' \mathrm{d}x$$

Let $f \in H^1(\mathbb{T})$ and define the bounded linear functional $F : C^1(\mathbb{T}) \subset L^2(\mathbb{T}) \to \mathbb{C}$ by

$$F(\phi) = -\int_{\mathbb{T}} f\phi' \mathrm{d}x.$$

Since F is a bounded linear functional defined on $C^1(\mathbb{T})$ and $C^1(\mathbb{T})$ is dense in $L^2(\mathbb{T})$, then we can uniquely extend F to a bounded linear functional on $L^2(\mathbb{T})$. Since $L^2(\mathbb{T})$ is a Hilbert space, the Riesz Representation Theorem states there is a unique $f' \in L^2$ such that $F(\phi) = (\overline{f'}, \phi)$ for all ϕ, which gives

$$\int_{\mathbb{T}} f'\phi \mathrm{d}x = -\int_{\mathbb{T}} f\phi' \mathrm{d}x.$$

The weak derivative of f, denoted f', is the unique element of $L^2(\mathbb{T})$ such that the above holds for all $\phi \in C^1(\mathbb{T})$. ϕ in this context is known as a smooth "test function."

For $k \geq 0$,

$$H^k(\mathbb{T}) = \left\{ f \in L^2 \mid f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad \sum_{n=-\infty}^{\infty} |n|^{2k} |c_n|^2 < \infty \right\}.$$

That is, an L^2 function is in H^k if its Fourier coefficients are $n^k c_n$ square summable.

When $k \in \mathbb{N}$, H^k corresponds to functions with k weak derivatives in L^2.

State the most simple Sobolev Embedding Theorem.

If $f \in H^k(\mathbb{T})$ and $k > \frac{1}{2}$, then $f \in C(\mathbb{T})$.

That is, if f has more than half of a weak derivative, then it is continuous.