Define the support of a continuous function.

Define $C^k(\Omega)$ *and* $C^\infty(\Omega)$. *Then define* $C_C^\infty(\Omega)$.

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Define Urysohn's Lemma in the context of $\mathbb{R}^n$.

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Let $f : \mathbb{R}^n \to \mathbb{C}$ be a continuous function. Then the support of f, denoted $\operatorname{supp} f$, is the closure of the set on which $f(x) \neq 0$. That is,

$$\operatorname{supp} f = \overline{\{x \in \mathbb{R}^n \mid f(x) \neq 0\}}.$$

$C^k(\Omega)$ is the set of k-times differentiable functions on Ω. Functions in $C^k(\Omega)$ for every $k > 0$ are said to be in $C^\infty(\Omega)$, that is, infinitely differentiable functions. $C_C^\infty(\Omega)$ is the set of infinitely differentiable functions on Ω which have support bounded and contained in Ω (compact when $\Omega = \mathbb{R}^n$). That is,

$$C^k(\Omega) = \left\{ f : \Omega \to B \mid \frac{\partial^i f}{\partial x^i} \text{ for } i = 0, \ldots, k \in C(\Omega) \right\}$$

$$C^\infty(\Omega) = \left\{ f : \Omega \to B \mid f \in C^k(\Omega) \text{ for } k \in \mathbb{N} \right\}$$

$$C_C^\infty(\Omega) = \{ f \in C^\infty(\Omega) \mid \operatorname{supp}(f) \text{ is compact} \}$$

Let $\Omega \subset \mathbb{R}^n$ be an open set and let $K \subset \Omega$ be compact. Then there exists a nonnegative function $\psi \in C_C^\infty$ with $\psi(x) = 1$ for $x \in K$.

Define σ-algebra

What are the Borel sets?

What is a measure on a σ-algebra?

Let Σ be a collection of subsets of Ω. Then Σ is called a σ-algebra if

(i) If $A \in \Sigma$, then $A^C \in \Sigma$;

(ii) If $A_1, A_2, \ldots$ is a countable family of sets in Σ, then $\bigcup_{n=1}^{\infty} A_i \in \Sigma$;

(iii) and $\Omega \in \Sigma$.

In English,

(i) Σ is closed under complemets;

(ii) Σ is closed under countable unions;

(iii) and Σ contains the entire set Ω.

The Borel sets is the smallest σ-algebra containing the open sets of $\mathbb{R}^n$, i.e the smallest σ-algebra generated by the open balls of $\mathbb{R}^n$ (sets of the form $B_{x,R} = \{y \in \mathbb{R}^n \mid |x-y| < R\}$).

A measure $\mu : \Sigma \to \mathbb{R}_0^+ \cup \infty$ is a function from Σ into the nonnegative real numbers (including infinity) such that

(i) $\mu(\emptyset) = 0$,

(ii) and $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A)i)$ for any sequence of disjoint sets (A_i) in Σ.

In English, a measure is a function which sends the empty set to 0 and has "countable additivity".

Define measure space

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A measure space consists of a set Ω, a σ-algebra Σ of Ω, and a measure μ on Σ. A measure space is denoted (Ω, Σ, μ).