What is an eigenvalue? What is an eigenvector? What does it mean for a matrix to be diagonalizable? What does the spectrum of a finite-dimensional linear operator consist of?

Describe the diagonalization procedure for operators on finite-dimensional Hilbert spaces.

When does a finite dimensional Hilbert space $\mathbb{C}^n$ have an orthonormal basis consisting of eigenvectors of an operator A?

Consider an $n \times n$ matrix A with complex entries. A number λ is an eigenvalue of A if there is a nonzero vector u such that

$$Au = \lambda u.$$

If λ is an eigenvector such that $Au = \lambda u$, then u is called an eigenvector of A corresponding to λ.

A matrix A is diagonalizable if there is a basis $\{u_1, \ldots, u_n\}$ of $\mathbb{C}^n$ such that there are eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$, which may not be distinct, such that

$$Au_k = \lambda_k u_k, \qquad \text{for } k = 1, \ldots, n.$$

The spectrum of a finite-dimensional linear operator A, denoted $\sigma(A)$, consists of the eigenvalues of the matrix representing A.

Let $\{u_1, \ldots, u_n\}$ be an orthonormal basis of $\mathbb{C}^n$. Then construct the matrix $U = (u_1\ u_2\ \ldots\ u_n)$ where the columns of U are the basis vectors. Then denote $\{e_1, \ldots, e_n\}$ as the standard basis of $\mathbb{C}^n$. It follows that

$$Ue_k = u_k, \qquad U^* u_k = e_k, \qquad \text{i.e. } U^* = U^{-1}$$

Next, suppose the basis vectors $\{u_1, \ldots, u_n\}$ are eigenvectors of A, i.e. $\exists \lambda_k$ such that $Au_k = \lambda_k u_k$ for $k = 1, \ldots, n$. It then follows that

$$U^* AUe_k = \lambda_k e_k,$$

which shows $D = U^* AU$ is a diagonal matrix with the eigenvalues on the diagonal, so $A = UDU^*$ where $D = (d_{ij})$ and $d_{ij} = \delta_{ij}\lambda_i$.
Indeed, if $A = UDU^*$ with U unitary and D diagonal, then the columns of U form an orthonormal basis of $\mathbb{C}^n$ consisting of eigenvectors of A.
A is a normal operator.

An $n \times n$ complex matrix A is normal if and only if $\mathbb{C}^n$ has an orthonormal basis consisting of eigenvectors of A.

Describe the polar decomposition of a normal complex matrix N.

What is the characteristic polynomial of a matrix?

What does it mean if $p_A(\lambda) = 0$.

We want to decompose a normal matrix N in to the product of a unitary matrix V and a nonnegative matrix A.

Since N is normal, it can decomposition $N = UDU^*$ where U is unitary and D is diagonal ($D = (d_{ij})$ where $d_{ij} = \delta_{ij}\lambda_i$ and λ_i are the eigenvalues of N). We can rewrite D as $D = \Phi|D|$ where Φ is a diagonal matrix consisting of $\arg\lambda_i$ and $|D|$ is a diagonal matrix consisting of $|\lambda_i|$. Then

$$N = VA, \qquad \text{where } V = U\Phi U^* \text{ and } A = U|D|U^*.$$

Also, A is non-negative, meaning $u^*Au \geq 0$ for all $u \in \mathbb{C}^n$.

This is the matrix analog of the polar decomposition of a complex number $z = re^{i\theta}$ in to the non-negative part r and the complex number $e^{i\theta}$.

The eigenvalues of a matrix A are the roots of the characteristic polynomial p_A of A, given by $p_A(\lambda) = \det(A - \lambda I)$.

If $p_A(\lambda) = 0$, then $A - \lambda I$ is singular, and in particular, $\ker(A - \lambda I) \neq \{0\}$. This means λ is an eigenvector.

What is the algebraic multiplicity of an eigenvalue? What is the geometric multiplicity of an eigenvalue?

Define the resolvent set and the spectrum of an operator $A \in \mathcal{B}(\mathcal{H})$.

What are the three components of the spectrum?

The algebraic multiplicity of an eigenvalue λ is the power on the factor $(x - \lambda)$ in $p_A(\lambda)$.

The geometric multiplicity is the dimension of the eigenspace associated with λ, that is, the dimension of $\ker(A - \lambda I)$.

The geometric multiplicity of an eigenvalue is never greater than the algebraic multiplicity.

The resolvent set of an operator $A \in \mathcal{B}(\mathcal{H})$, denoted $\rho(A)$, is the set of complex numbers such that $(A - \lambda I) : \mathcal{H} \to \mathcal{H}$ is one-to-one and onto.

The spectrum of A, denoted $\sigma(A)$, is the complement of the resolvent set:

$$\sigma(A) = \mathbb{C} \setminus \rho(A).$$

(a) The point spectrum of A consists of all $\lambda \in \sigma(A)$ such that $A - \lambda I$ is not one-to-one. In this case, $\ker(A - \lambda I) \neq \{0\}$, and λ is called an eigenvalue of A.

(b) The continuous spectrum of A consists of all $\lambda \in \sigma(A)$ such that $A - \lambda I$ is one-to-one but not onto, and $\operatorname{ran}(A - \lambda I)$ is dense in $\mathcal{H}$.

(c) The residual spectrum of A consists of all $\lambda \in \sigma(A)$ such that $A - \lambda I$ is one-to-one but not onto, and $\operatorname{ran}(A - \lambda I)$ is not dense in $\mathcal{H}$.

Define $M \in \mathcal{B}(L^2([0,1]))$ by $Mf(x) = xf(x)$. Find and classiy the spectrum of M.

Define the resolvent of A at λ.

What are some basic properties of the resolvent set of A? What does this say about the spectrum of A? Define the spectral radius of A.

If $Mf = \lambda f$, then $(x-\lambda)f = 0$, and so $f = 0$. Since eigenvectors cannot be 0, then there are no eigenvalues of M.
If $\lambda \notin [0,1]$, then $(x-\lambda)f(x) \in L^2([0,1])$ since $x-\lambda$ is bounded away from 0 on $[0,1]$. Thus $\mathbb{C} \setminus [0,1] \in \rho(M)$.
If $\lambda \in [0,1]$, then $M - \lambda I$ is not onto since $f(x) \equiv c \in L^2$ but if $(M-\lambda I)g = f$ then $(x-\lambda)g(x) = c$, and so $g(x) = \frac{c}{x-\lambda}$, which is not an L^2 function.
Let $f \in L^2$. Then define $f_n \in L^2([0,1])$ by

$$f_n(x) = \mathcal{X}_{[B_{1/n}(\lambda)]^C} f(x).$$

Then $f_n \in \operatorname{ran}(M-\lambda I)$ since $f_n(x) = (M-\lambda I)\frac{f_n(x)}{x-\lambda}$ and $\frac{f_n(x)}{x-\lambda} \in L^2([0,1])$. Also, $f_n \to f$ in $L^2([0,1])$. Thus $\operatorname{ran}(M-\lambda I)$ is dense for all $\lambda \in [0,1]$. Thus the spectrum of M is completely continuous, and is $\sigma(M) = [0,1]$.

Suppose $\lambda \in \rho(A)$. Then $(A-\lambda I)$ is invertible. Define the resolvent of A at λ, denoted $R(\lambda)$, by $R(\lambda) = (A-\lambda I)^{-1}$. The resolvent is an operator valued function defined on $\rho(A)$, that is,

$$R \ : \ \rho(A) \to \mathcal{B}(\mathcal{H}).$$

The resolvent set $\rho(A)$ of an operator A on a Hilbert space $\mathcal{H}$ is open and contains the exterior disc $\lambda \in \mathbb{C} \mid |\lambda| > \|A\|$.

Since the $\sigma(A) = [\rho(A)]^C$, then $\sigma(A) \subset B_{\|A\|}(0)$.

The spectral radius of A, denoted $r(A)$, is the radius of the smallest disk which contains $\sigma(A)$, that is,

$$r(A) = \sup\{|\lambda| \mid \lambda \in \sigma(A)\} = \inf\{r \mid \sigma(A) \subset B_r(0)\}$$

What is a simple formula for the spectral radius of a bounded linear operator? What happens if A is self-adjoint?

Give an example of a matrix where A where $\|A\| > 0$ but $r(A) = 0$.

Can the spectrum of a bounded linear operator be empty?

For any bounded linear operator A,

$$r(A) = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}.$$

When A is self adjoint, $r(A) = \|A\|$.

Let A be the $n \times n$ matrix defined by

$$N = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

Then $N^n = 0$, and thus $r(N) = \lim_{n\to\infty} \|N^n\|^{\frac{1}{n}} = 0$, but $\|N\| = 1$.

N is called nilpotent since $r(N) = 0$.

No. $\sigma(A)$ must consist of at least a single point.

Describe the spectrum of compact, self-adjoint operators.

Show that the eigenvalues of a bounded, self-adjoint linear operator are real, and eigenvalues associated with different eigenvectors are orthogonal.

What is an invariant subspace, and why are they important?

Let K be a compact, self-adjoint operator with $K = K^*$.
Then $\sigma(K)$ consists entirely of eigenvectors, except possibly 0, which may belong to the continuous spectrum.

Let λ be an eigenvalue of A with eigenvector u. Then

$$\lambda(u, u) = (\lambda u, u) = (Au, u) = (u, Au) = (u, \lambda u) = \overline{\lambda}(u, u),$$

which shows $\lambda = \overline{\lambda}$, i.e. $\lambda \in \mathbb{R}$.

Let λ be an eigenvalue with eigenvector u and μ be an eigenvalue with eigenvector v. Then

$$\lambda(u, v) = (\lambda u, v) = (Au, v) = (u, Av) = (u, \mu v) = \overline{\mu}(u, v) = \mu(u, v).$$

Thus, $(\lambda - \mu)(u, v) = 0$. So if $\lambda \neq \mu$, then $(u, v) = 0$, which shows eigenvectors corresponding to different eigenvalues are orthogonal.

Let M be a linear subspace of a Hilbert space $\mathcal{H}$. Then M is called an invariant subspace of a linear operator A if

$$Ax \in M, \qquad \forall x \in M$$

This means $A|_M$ is a linear operator on M.

Suppose M and N are invariant subspaces of an operator A with $\mathcal{H} = M \oplus N$. Then each $x \in \mathcal{H}$ can be written uniquely as $x = m + n$ where $m \in M$ and $n \in N$. Then

$$Ax = A|_M m + A|_N n.$$

Invariant subspaces of an operator A are important because the action of A on $\mathcal{H}$ is completely determined by its actions on the invariant subspaces of A.

Show that for bounded, self-adjoint operators, the orthogonal complement of an invariant subspace is invariant.

Prove that if λ belongs to the residual spectrum of a bounded linear operator A on a Hilbert Space, then $\overline{\lambda}$ is an eigenvalue of A^*.

Describe the spectrum of a bounded self-adjoint operator A on a Hilbert space.

Let M be an invariant subspace of the bounded, self-adjoint operator A on a Hilbert space $\mathcal{H}$. Choose an arbitrary $y \in M^{\perp}$. Then $\forall x \in M$, $(x, y) = 0$. Also, since $Ax \in M$, then $(Ax, y) = 0$ for all $x \in M$. Then since A is self-adjoint, $0 = (Ax, y) = (x, Ay)$, and thus $Ay \perp x$ for all $x \in M$, which shows $Ay \in M^{\perp}$. Thus $M^{\perp}$ is an invariant subspace of A.

Let λ be in the residual spectrum of A. Then $\operatorname{ran}(A - \lambda I)$ is not dense in $\mathcal{H}$. By the projection theorem, $\exists x \in \mathcal{H}$ with $x \neq 0$ such that $x \perp \operatorname{ran}(A - \lambda I)$. This implies $x \in \ker((A - \lambda I)^*) = \ker(A^* - \overline{\lambda} I)$. Thus x is an eigenvector of A^* corresponding to $\overline{\lambda}$.

The spectrum is real and contained in $[-\|A\|, \|A\|]$.

Nonzero eigenvalues of compact operators have something that nonzero eigenvalues of non-compact, infinite-dimensional operators have. What is it?

State the spectral theorem for compact, self-adjoint operators.

Nonzero eigenvalues of compact operators have finite multiplicity.

Let $A : \mathcal{H} \to \mathcal{H}$ be a compact, self-adjoint operator on a Hilbert space $\mathcal{H}$. Then there is an orthonormal basis of $\mathcal{H}$ consisting of eigenvectors of A. The nonzero eigenvalues of A form a finite or countably infinite set $\{\lambda_k\}$ of real numbers, and

$$A = \sum_k \lambda_k P_k$$

where P_k is the orthogonal projection onto the finite-dimensional eigenspace of eigenvectors with eigenvalue λ_k. If the number of nonzero eigenvalues is countably infinite, then the series above converges to A in the operator norm.

In other words, every compact, self-adjoint operator on a Hilbert space can be decomposed into the sum of orthogonal projections onto its eigenspaces.