Define Banach Space

Applied Analysis Chapter 5

$\mathbb{R}^n$ *and* $\mathbb{C}^n$ *are Banach spaces with respect to which norms?*

Applied Analysis Chapter 5

$C([a,b])$ *is a Banach spaces with respect to which norm?*

Applied Analysis Chapter 5

A normed linear space which is complete with respect to its norm.

n-tuples are Banach with respect to the max norm (∞ norm), sum norm, and any p-norm in between.

Continuous functions are Banach with respect to the sup norm (∞ norm, uniform norm).

$C^k([a,b])$ is a Banach spaces with respect to which norm?

Define $\ell^p(\mathbb{N})$ for $1 \leq p \leq \infty$

Define $L^p([a,b])$ for $1 \leq p \leq \infty$

k-continuously-differentiable functions are Banach with respect to the C^k norm, which is the sum of the sup norms of all derivatives, from 0 to k.

$$\|f\|_{C^k} = \sum_{i=0}^{k} \left\| f^{(i)} \right\|_\infty$$

For $1 \leq p < \infty$, $\ell^p(\mathbb{N})$ is the space of all p-summable sequences, that is,

$$\ell^p(\mathbb{N}) = \left\{ (x_n)_{n=1}^\infty \mid \sum_{i=1}^\infty |x_i|^p < \infty \right\} \quad \text{with} \quad \|(x_n)\|_{\ell^p} = \left(\sum_{i=1}^\infty |x_i|^p \right)^{\frac{1}{p}}.$$

$\ell^\infty(\mathbb{N})$ is the space of all bounded sequences, that is,

$$\ell^\infty = \left\{ (x_n)_{n=1}^\infty \mid \sup_{i=1}^\infty |x_i| < \infty \right\} \quad \text{with} \quad \|(x_n)\|_{\ell^\infty} = \sup_{i=1}^\infty |x_i|.$$

$\ell^p(\mathbb{N})$ is Banach for $1 \leq p \leq \infty$.

For $1 \leq p < \infty$, $L^p([a,b])$ is the space of all Lebesgue-measurable functions which are p-integrable, that is,

$$L^p([a,b]) = \left\{ f \mid \int_a^b |f(x)|^p \mathrm{d}x < \infty \right\} \quad \text{with} \quad \|(x_n)\|_{L^p} = \left(\int_a^b |f(x)|^p \mathrm{d}x \right)^{\frac{1}{p}}.$$

$L^\infty([a,b])$ is the space of all Lebesgue-measurable functions which are essentially bounded (bounded on a subset of $[a,b]$ whose complement has measure 0), that is,

$$\begin{aligned} &L^\infty([a,b]) = \{ f \mid \exists M < \infty \ : \ |f(x)| \leq M \text{ a.e. in } [a,b] \} \quad \text{with} \\ &\quad \|f\|_{L^\infty} = \inf \{ M \mid |f(x)| \leq M \text{ a.e. in } [a,b] \} \end{aligned}$$

$L^p([a,b])$ is Banach for $1 \leq p \leq \infty$.

Define Sobolev Spaces $W^{k,p}((a,b))$

Is the space of polynomial functions Banach?

Is the space of continuous functions with $f(0) = 0$ *Banach?*

The Sobolev spaces consist of functions whose derivatives satisfy an integrability condition. Namely all derivatives up to the k^{th} are in L^p. The $W^{k,p}$ norm is defined as follows:

$$\|f\|_{W^{k,p}} = \left(\sum_{j=0}^{k} \int_a^b \left| f^{(j)}(x) \right|^p \mathrm{d}x \right)^{\frac{1}{p}} = \left(\sum_{j=0}^{k} \left\| f^{(j)} \right\|_{L^p}^p \right)^{\frac{1}{p}}$$

All Sobolev Spaces are Banach.

It is a linear subspace, but no, it is not Banach. We use the Bernstein Polynomials to show it is dense in $C([0,1])$. However, it is not closed since limits of polynomials may not be polynomials. Since it is not closed, it is not complete, and thus not Banach.

Yes, it is a closed linear subspace of a Banach space (namely, $C([0,1])$) and hence is Banach.

Define Linear Operator

Define Bounded *Linear Operator*

Define the Operator Norm for bounded linear operators

A linear operator T between linear spaces X and Y is a function $T : X \to Y$ such that

$$T(\lambda x + \mu y) = \lambda T(x) + \mu T(y) \qquad \forall \lambda, \mu \in \mathbb{R} \text{ (or } \mathbb{C}\text{) and } x, y \in X.$$

A linear operator T is bounded if $\exists M \geq 0$ such that

$$\|Tx\| \leq M\|x\| \qquad \forall x \in X.$$

The norm of an operator T is given by the following (all four are equivalent):

$$\begin{aligned}
\|T\| & \\
&= \inf\{M \mid \|Tx\| \leq M\|x\| \ \forall x \in X\}, \\
&= \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}, \\
&= \sup_{\|x\| \leq 1} \|Tx\|, \\
&= \sup_{\|x\| = 1} \|Tx\|.
\end{aligned}$$

Describe the four common matrix norms

How do the matrix norms extend to ℓ^∞ and L^∞?

Linear maps are bounded if and only if they are continuous.

If $T : \mathbb{R}^n \to \mathbb{R}^m$ where $\mathbb{R}^n$ and $\mathbb{R}^m$ are equipped with the 2-norm (Euclidean norm), then $\|T\|_2 = \sqrt{r(A^TA)}$, where r is the spectral radius and A is the matrix of the operator T. If $\mathbb{R}^n$ and $\mathbb{R}^m$ are equipped with the 1-norm (sum norm), then $\|T\|_1$ is the max column sum, i.e. $\|T\|_1 = \max_{1\leq j\leq n}\{\sum_{i=1}^m |a_{ij}|\}$. If $\mathbb{R}^n$ and $\mathbb{R}^m$ are equipped with the ∞-norm (max norm), then $\|T\|_\infty$ is the max row sum, i.e. $\|T\|_\infty = \max_{1\leq i\leq m}\left\{\sum_{j=1}^n |a_{ij}|\right\}$. There is also the Hilbert-Schmidt norm of matrices, which is not derived from norms of $\mathbb{R}^n$ and $\mathbb{R}^m$. It is basically the 2-norm of the $m \times n$ tuple: $\|T\|_{\text{HS}} = \left(\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2\right)^{\frac{1}{2}}$.

Suppose $T : \ell^\infty(\mathbb{N}) \to \ell^\infty(\mathbb{N})$, each equipped with the sup-norm. Then T can be represented by an infinite matrix, and its norm is the max row sum: $\|T\| = \sup_{i\in\mathbb{N}}\left\{\sum_{j=1}^\infty |a_{ij}|\right\}$. T is only bounded if this norm is bounded.
Now suppose $K : C([0,1]) \to C([0,1])$, each equipped with the uniform norm, and suppose $k : [0,1]^2 \to \mathbb{R}$. Define K explicitly as

$$Kf(x) = \int_0^1 k(x,y)f(y)\mathrm{d}y.$$

(Note that this is called a Fredholm integral operator.) Then the norm is the "max row sum" of the function k:

$$\|K\| = \max_{0\leq x\leq 1}\left\{\int_0^1 |k(x,y)|\mathrm{d}y\right\}$$

This is finite since k is continuous on a compact set.

Bounded $\implies$ continuous uses

- Linearlity.

Continuous $\implies$ bounded uses:

- Continuous $\implies$ continuous specifically at 0.
- Choose $\varepsilon = 1$, obtain δ from definition of continuity, and scale any point to be of magnitude δ.
- Linearity.

Bounded Linear Transformation (BLT) Theorem: Suppose the domain of the bounded linear map T is a dense subset M of X. Then there is a unique extension $\bar{T}$ with domain X, $\bar{T}x = Tx$ for all $x \in M$, and $\|\bar{T}\| = \|T\|$.

Describe the differences between linearly, topologically, and isometrically isomorphic linear spaces

Define equivalent norms

- Define $\bar{T}x$ as the limit of images from M, which exists since bounded linear maps send Cauchy sequences to Cauchy sequences.
- Show that $\bar{T}x$ is well-defined by considering two sequences in M.
- Show that $\bar{T}$ is in fact an extension of T.
- Show $\|\bar{T}\| = \|T\|$ by simple inequalities and extension.
- Show uniqueness by considering two extensions and showing they are equal on all of X.

- T is a linear isomorphism if it is bijective.
- T is a topological isomorphism if both T and T^{-1} are bounded.
- T is a isometric isomorphism if T also preserves norms, i.e. $\|Tx\| = \|x\|$ for all x.

Two norms are equivalent if each can bound the other, i.e. $\exists c, C \in \mathbb{R}$ such that

$$c\|x\|_1 \leq \|x\|_2 \leq C\|x\|_1$$

State the Open Mapping Theorem:

Define the kernel and range of a linear map

The kernel and range of a linear map T are linear subspaces of the domain and codomain, rspectively. If T is bounded, the kernel is closed.

If $T : X \to Y$ is a bijective, bounded linear map between Banach spaces, then $T^{-1} : Y \to X$ is bounded.

The kernel of a map T is any point in the domain which is mapped to 0. The range of a map T is any point in the codomain which is mapped to from at least one point in the domain.

$$\ker T = \{x \in X \mid Tx = 0\}$$

$$\operatorname{ran} T = \{y \in Y \mid \exists x \in X : Tx = y\}$$

To show linear subspaces, use linearity of T. The show closure, use continuity of T.

Is it possible for finite dimensional linear maps to be surjective and not injective or vice-versa? How about infinite dimensional linear maps?

What is the operator norm of the Volterra Integral Operator K acting on $C([a,b])$ with the maximum norm?

$$Kf(x) = \int_a^x f(y)\mathrm{d}y$$

State the Leibniz Integral Rule

For finite-dimensional maps, surjectivity is equivalent to injectivity. However, a counter-example in finite dimensions are the left and right shift operators on $\ell^\infty(\mathbb{N})$.

$\|K\| = b - a$ since

$$\|Kf\| \leq \sup_{a \leq x \leq b} \int_a^x |f(y)| \mathrm{d}y \leq \int_a^b |f(y)| \mathrm{d}y \leq \int_a^b \|f\| \mathrm{d}y = (b-a)\|f\|$$

so $\|K\| \leq (b-a)$. However, let $g \equiv 1$. Then $\|g\| = 1$ and

$$\|Kg\| = \left\| \int_a^x \mathrm{d}y \right\| = \|x - a\| = b - a$$

Thus $\|K\| = b - a$.

Let $f(x,t)$ be a function such that the partial derivative of f with respect to t exists and is continuous. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{a(t)}^{b(t)} f(x,t) \mathrm{d}x \right) = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t} \mathrm{d}x + f(b(t),t) \cdot b'(t) - f(a(t),t) \cdot a'(t).$$

Let T be a bounded linear map between two Banach spaces X and Y. Then the following are equivalent:

(a) there is a constant $c > 0$ such that

$$c\|x\| \leq \|Tx\| \quad \forall x \in X;$$

(b) T has closed range, and the only solution of the equation $Tx = 0$ is $x = 0$.

Given a finite-dimensional Banach space, the components of a vector with respect to any basis of a finite-dimensional space can be bounded by the norm of the vector. Also, the norm of a vector can be bounded by the 1-norm of that vector. In particular, let $\{e_1, \ldots, e_n\}$ be a basis of a finite-dimensional Banach space X with norm $\|\cdot\|$. Then $\exists m, M > 0$ such that if $x = \sum_{i=1}^n x_i e_i$, then

$$m \sum_{i=1}^{n} |x_i| \leq \|x\| \leq M \sum_{i=1}^{n} |x_i|.$$

Every finite-dimensional normed linear space is a Banach space.

(a) $\implies$ (b) uses:

- Bounded linear maps send Cauchy sequences to Cauchy sequences
- Completeness of Banach spaces
- Continuity of T

(b) $\implies$ (a) uses:

- Closed subspaces of Banach spaces are Banach
- Open Mapping Theorem
- Definition of inverse map

The proof uses:

- Homogeneity of norm
- Heine Borel Theorem
- Compositions of continuous functions are continuous
- Continuous functions on compact domains acheive their supremum and infimum

The proof uses:

- Components of vectors can be bounded by the vectors' norms.
- Completeness gives limits to Cauchy sequences.

Every linear operator on a finite-dimensional Banach space is bounded.

Any two norms on a finite-dimensional space are equivalent.

The space of bounded linear maps $\mathcal{B}(X, Y)$ is a normed linear space

The proof uses:

- Linearity of T
- Components of vectors can be bounded by the vectors' norms.

The proof uses:

- Components of vectors can be bounded by the vectors' norms . . . twice.

Addition and scalar multiplication are pointwise

$$(S+T)x = Sx + Tx, \qquad (\lambda T)x = \lambda(Tx).$$

The operator norm defines a norm on $\mathcal{B}(X,Y)$.

$$\|T\| = \sup_{x\neq 0} \frac{\|Tx\|}{\|x\|}$$

Compositions of bounded linear maps are bounded linear maps and their norms are bounded by the product of the norms of the components.

$$\|ST\| \leq \|S\|\|T\|.$$

Define uniform convergence of operators.

Let $X = C([0,1])$ equipped with the uniform norm. Give an example of a sequence of Fredholm integral operators on X that converges to 0.

For all x,

$$\|STx\| \leq \|S\|\|Tx\| \leq \|S\|\|T\|\|x\|.$$

If (T_n) is a sequence of operators in $\mathcal{B}(X,Y)$ and

$$\lim_{n\to\infty} \|T_n - T\| = 0$$

for some $T \in \mathcal{B}(X,Y)$, then we say that T_n converges uniformly to T, or that T_n converges to T in the uniform, or operator norm, topology on $\mathcal{B}(X,Y)$.

Let K_n be given by

$$K_n f(x) = \int_0^1 xy^n f(y)\mathrm{d}y.$$

Then $K_n \to 0$ uniformly since

$$\|K_n - 0\| = \|K_n\| = \max_{0\leq x\leq 1} \left\{ \int_0^1 |xy^n|\mathrm{d}y \right\} \to 0 \quad \text{as} \quad n \to 0.$$

If X is a normed linear space and Y is a Banach space, then $\mathcal{B}(X, Y)$ is a Banach space.

Define a compact operator

(a) If S and T are compact operators, any linear combination is compact.
(b) If (T_n) is a sequence of compact operators that comverges uniformly to T, then T is compact.
(c) If T is an operator with finite-dimensional range, then T is compact.
(d) If S is compact and T is bounded, or if S is bounded and T is compact, TS is compact.

Let (T_n) be a Cauchy sequence in $\mathcal{B}(X,Y)$. Then for any $x \in X$, $\|T_nx - T_mx\| \leq \|T_n - T_m\|\|x\|$ by linearity and the definition of the operator norm. Since (T_n) is Cauchy, we have (T_nx) is Cauchy for any $x \in X$. Since Y is complete, then for each x, $\exists y_x \in Y$ such that $T_nx \to y_x$. Next we define our candidate limit operator T by $Tx = y_x$. T is clearly linear, but we still need to show it is bounded and the uniform limit of the sequence (T_n). Choose an $\varepsilon > 0$ and note by the triangle inequality and definition of operator norm,

$$\|T_nx - Tx\| \leq \|T_n - T_m\|\|x\| + \|T_mx - Tx\|.$$

By the definition of Cauchy, $\exists N_\varepsilon$ such that $n, m \geq N_\varepsilon \implies \|T_n - T_m\| < \frac{\varepsilon}{2}$. Then since $T_nx \to y_x = Tx$, $\exists M_x$ such that $m > M_x \implies \|T_mx - Tx\| < \frac{\varepsilon}{2}$. This gives $\|T_nx - Tx\| < \varepsilon$. Finally, $\|Tx\| \leq \|Tx - T_nx\| + \|T_nx\| < \|T_nx\| + \varepsilon$. Since ε was arbitrary, T is bounded. Thus $T_n - T$ is bounded, and $\|T_n - T\| < \varepsilon$. Again, since ε was arbitrary $T_n \to T$ uniformly.

A linear operator $T : X \to Y$ is compact if $T(B)$ is a precompact subset of Y for every bounded subset B of X.

OR

A linear operator T is compact if every bounded sequence in X has a subsequence whose image converges in Y.

(a) Uses a subsequence of a subsequence.

(b) Uses subsequences of subsequences of ... and a diagonal subsubsequence argument.

(c) Uses Bolzano Weierstrass

(d) One uses compactness and continuity, and the other uses boundedness and compactness.

Define strong convergence of operators

Uniform convergence of operators implies strong convergence.

Give examples of strongly convergent sequences of operators that do not converge uniformly

A sequence (T_n) of operators converges strongly to T if $T_n x \to Tx$ for every $x \in X$.

Suppose $T_n \to T$ uniformly. Then $\|T_n - T\| \to 0$. Then for any $x \in X$,

$$\|T_n x - Tx\| = \|(T_n - T)x\| \leq \|T_n - T\|\|x\| \to 0,$$

which shows $T_n x \to Tx$.

Let $X = \ell^p(\mathbb{N})$ for $1 \leq p < \infty$. Then for $n \in \mathbb{N}$, define P_n as the projection

$$P_n(x_1, x_2, \dots, x_n, x_{n+1}, x_{n+2}, \dots) = (x_1, x_2, \dots, x_n, 0, 0, \dots)$$

Note $\|P_n - P_m\| = 1$ for $n > m$ since $x = (0, 0, \dots, 0, 1, 0, \dots)$, where the $(n - m + 1)^{\text{st}}$ component of x is 1, gets mapped to itself. So (P_n) is not a Cauchy sequence and thus cannot converge uniformly. However, for any given sequence x, $\|P_n x - Ix\|_{\ell^p} = \|(0, 0, \dots, 0, x_{n+1}, x_{n+2}, \dots)\|_{\ell^p} \to 0$ since sequences in $\ell^p(\mathbb{N})$ decay to 0 for $1 \leq p < \infty$. Thus $P_n \to I$ strongly.

Let $X = C([0,1])$ and let K_n be a sequence of functionals given by $K_n f = \int_0^1 \sin(n\pi x) f(x)\mathrm{d}x$. Use integration by parts to show $K_n p \to 0$ for all polynomials p and use Weierstrass Approximation Theorem to say the same for any continuous function. Thus $K_n \to 0$ strongly. However, let $g_n(x) = \sin(n\pi x)$. Then $P_n g_n = \frac{1}{2}$, which shows $\|K_n\| \geq \frac{1}{2}$ for each $n \in \mathbb{N}$. Thus K_n does not converge uniformly to 0.

Define linear functional, algebraic dual space, and topological dual space

The dual space of a finite dimensional space X is linear isomorphic to X.

State the Hahn-Banach Theorem

A linear functional is a scalar-valued linear map from a linear space to $\mathbb{R}$ (or $\mathbb{C}$).

An algebraic dual space consists of all linear functionals on a linear space.

A topological dual space consists of all continuous linear functionals on a linear space. Given a space X, the topological dual space is denoted X^*.

Suppose $\{e_1, \ldots, e_n\}$ is a basis for X. We can form functionals $\{\omega_1, \ldots, \omega_n\}$ such that $\omega_i(e_j) = \delta_{i,j}$ (Kronecker delta function). This set of functionals forms a basis for the dual space X^*.

If Y is a linear subspace of a normed linear space X and $\psi : Y \to \mathbb{R}$ is a bounded linear functional on Y with $\|\psi\| = M$, then $\exists$ a bounded linear functional $\phi : X \to \mathbb{R}$ such that ϕ restricted to Y is equal to ψ and $\|\phi\| = M$.

Define the bidual of a Banach space, and reflexivity of Banach spaces

Define weak convergence of a sequence in X and weak-$*$ convergence of a sequence in X^*. Then define weak convergence of a sequence in X^* and weak-$*$ convergence of a sequence in X^{**}.

State the Banach-Alaoglu Theorem

The bidual of a Banach space X is the dual of its dual, i.e. X^{**}. For each $x \in X$, we can define a linear functional $F_x \in X^{**}$ by $F_x(\phi) = \phi(x)$. This means there is an embedding of X inside X^{**}. If X and X^{**} are isomorphic, we say X is reflexive.

A sequence $(x_n) \in X$ converges weakly to x, denoted $x_n \rightharpoonup x$, if $\phi(x_n) \to \phi(x)$ for every bounded linear functional $\phi \in X^*$. A sequence $(\phi_n) \in X^*$ converges weak-$*$ly to ϕ, denoted $\phi_n \rightharpoonup^* \phi$, if $\phi_n(x) \to \phi(x)$ for every $x \in X$.

A sequence $(\phi_n) \in X^*$ converges weakly to ϕ, denoted $\phi_n \rightharpoonup \phi$, if $F(\phi_n) \to F(\phi)$ for every bounded linear functional $F \in X^{**}$. A sequence $(F_n) \in X^{**}$ converges weak-$*$ly to F, denoted $F_n \rightharpoonup^* F$, if $F_n(\phi) \to F(\phi)$ for every $\phi \in X^*$.

Let X^* be the dual space of a Banach space X. The closed unit ball in X^* is weak-$*$ compact. In other words, let (ϕ_n) be a sequence in the unit ball of X^*. Then there is a subsequence (ϕ_{n_k}) and a linear functional ϕ in the unit ball of X^* such that $\phi_{n_k} \rightharpoonup^* \phi$.