What is a topology? What is an open set? What is a closed set? What is a topological space?

What is the trivial topology? What is the discrete topology?

What is a metric topology?

A topology $\mathcal{T}$ on a nonempty set X is a collection of subsets of X, such that:

(a) $\emptyset, X \in \mathcal{T}$.

(b) The union of an arbitrary collection of open sets (may be uncountable) is open.

(c) The intersection of a finite number of open sets is open.

Sets in $\mathcal{T}$ are called open sets. A set is called closed if its complement is open. Sets can be both open and closed, or neither open nor closed.

A topological space is the pair $(X, \mathcal{T})$, or just X if $\mathcal{T}$ is clear from the context.

The trivial topology (also known as the indiscrete topology) is $\{\emptyset, X\}$. It is called indiscrete because it cannot distinguish points.

The discrete topology is the power set of X, $\mathcal{P}(X)$.

Let (X, d) be a metric space. Then let $\mathcal{T}$ be the set of all open sets as defined in the context of metric spaces. Then $\mathcal{T}$ is called the metric topology on X.

What is an induced topology?

What is a topological neighborhood? What is a Hausdorff topology?

Define convergence, continuity, and homeomorphisms using topological neighborhoods.

Let $(X, \mathcal{T})$ be a topological space and $Y \subset X$. The induced (or relative) topology of Y in X, denoted $\mathcal{T}_Y$, are all subsets of Y which are the intersection of Y with some open set in $\mathcal{T}$, that is,

$$\mathcal{T}_Y = \{H \subset Y \mid H = G \cap Y \text{ for some } G \in \mathcal{T}\}.$$

A subset V of X is a topological neighborhood of a point x if there is an open set G such that $x \in G \subset V$.

A topology $\mathcal{T}$ on X is called Hausdorff if for every $x \neq y$ there are neighborhoods V_x or x and V_y of y such that $V_x \cap V_y = \emptyset$. That is, every pair of distinct points has a pair of nonintersecting neighborhoods.

A sequence x_n converges to a point $x \in X$ if for every neighborhood V of x there is a number N such that $x_n \in V$ for every $n \geq N$.

A function $f : X \to Y$ is continuous at $x \in X$ if for every neighborhood W of $f(x)$ there is a neighborhod V of x such that $f(V) \subset W$.

A function $f : X \to Y$ is a homeomorphism if it is bijective and both f and f^{-1} are continuous. If there is a homeomorphism between X and Y, we say X and Y are homeomorphic. Homeomorphic topological spaces are indistinguishable in the sense that $G \in \mathcal{T}_X \iff f(G) \in \mathcal{T}_Y$, and $(x_n) \to x \in X \iff (f(x_n)) \to f(x) \in Y$.

What is the topological definition of compactness? What can we say when the topological space is a metric space?

What is a base for a topology? What is a neighborhood base? What does it mean for a topological space to be first-countable? Second-countable? How are bases and neighborhood bases related?

What does it mean for one topology to be stronger or weaker than another? What are the implications for convergence? Can we always compare any two topologies?

A subset K of a topological space X is compact if every open cover of K contains a finite open subcover.

In metric spaces, compactness is equivalent to sequential compactness, which is that every sequence has a convergent subsequence.

A subset $\mathcal{B}$ of a topology $\mathcal{T}$ is a base for $\mathcal{T}$ if for every $G \in \mathcal{T}$ there is a collection of sets $B_\alpha \in \mathcal{B}$ such that $G = \bigcup_\alpha B_\alpha$. That is, every open set is the union of basis elements.

A colection $\mathcal{N}$ of neighborhoods of a point $x \in X$ is a neighborhood base for x if for each neighborhood V of x, there is a neighborhood $W \in \mathcal{N}$ with $W \subset V$. That is, every neighborhood contains a neighborhood base element.

A topological space is first-countable if every $x \in X$ has a countable neighborhood base. A topological space is second-countable if it has a countable base.

A collection of sets $\mathcal{B}$ is a base if and only if it contains a neighborhood base for every $x \in X$.

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on the same space X. Then $\mathcal{T}_1$ is stronger (sometimes referred to as finer) than $\mathcal{T}_2$ if $\mathcal{T}_2 \subset \mathcal{T}_1$, that is, $\mathcal{T}_1$ has more open sets. We can also say $\mathcal{T}_2$ is weaker (or coarser) than $\mathcal{T}_1$.

If a sequence converges with respect to a topology, than it converges with respect to any weaker topology.

It is not always possible to compare topologies because it is possible that $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on X but $\mathcal{T}_1 \not\subset \mathcal{T}_2$ and $\mathcal{T}_2 \not\subset \mathcal{T}_1$.

What is the relationship between topological comparisons and continuity?

Suppose $f : (X, \mathcal{T}_1) \to (Y, \mathcal{S}_1)$ is continuous. Then

(a) If $\mathcal{T}_2$ is finer than $\mathcal{T}_1$, then $f : (X, \mathcal{T}_2) \to (Y, \mathcal{S}_1)$ is continuous.

(b) If $\mathcal{S}_2$ is coarser than $\mathcal{S}_1$, then $f : (X, \mathcal{T}_1) \to (Y, \mathcal{S}_2)$ is continuous.

The identity map $I : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is continuous if and only if $\mathcal{T}_1$ is finer than $\mathcal{T}_2$. That is, I is continuous if and only if it loses information.