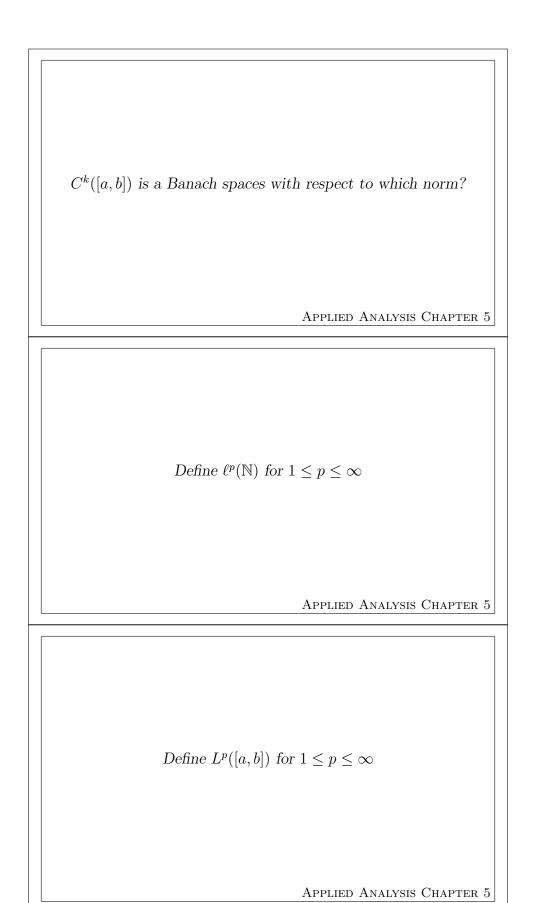
| Define Banach Space |
|---|
| Applied Analysis Chapter 5 |
| \mathbb{R}^n and \mathbb{C}^n are Banach spaces with respect to which norms? |
| Applied Analysis Chapter 5 |
| C([a,b]) is a Banach spaces with respect to which norm? Applied Analysis Chapter 5 |

| | _ |
|---|---|
| A normed linear space which is complete with respect to its norm. | |
| | |
| | |
| | |
| | |
| | |
| | |
| | |
| | _ |
| n -tuples are Banach with respect to the max norm (∞ norm), sum norm, and any p -norm in between. | |
| | |
| | |
| | |
| | |
| | |
| | |
| | |
| | |
| Continuous functions are Banach with respect to the sup norm (∞ norm, uniform norm). | |
| | |
| | |
| | |
| | |
| | |
| | |
| | |



k-continuously-differentiable functions are Banach with respect to the C^k norm, which is the sum of the sup norms of all derivatives, from 0 to k.

$$||f||_{C^k} = \sum_{i=0}^k ||f^{(i)}||_{\infty}$$

For $1 \leq p < \infty$, $\ell^p(\mathbb{N})$ is the space of all *p*-summable sequences, that is,

$$\ell^p(\mathbb{N}) = \left\{ (x_n)_{n=1}^{\infty} \mid \sum_{i=1}^{\infty} |x_i|^p < \infty \right\} \text{ with } \|(x_n)\|_{\ell^p} = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}}.$$

 $\ell^{\infty}(\mathbb{N})$ is the space of all bounded sequences, that is,

$$\ell^{\infty} = \left\{ (x_n)_{n=1}^{\infty} \mid \sup_{i=1}^{\infty} |x_i| < \infty \right\} \text{ with } \|(x_n)\|_{\ell^{\infty}} = \sup_{i=1}^{\infty} |x_i|.$$

 $\ell^p(\mathbb{N})$ is Banach for $1 \leq p \leq \infty$.

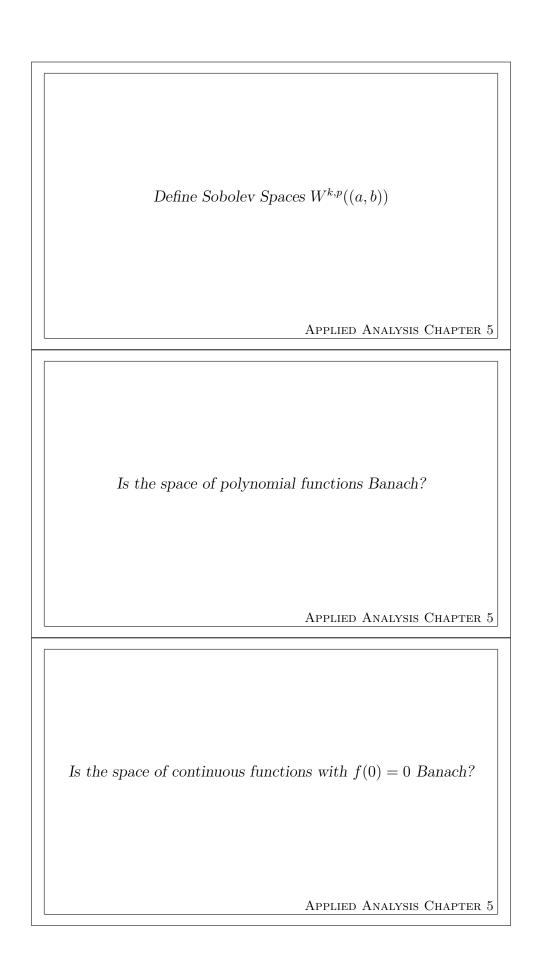
For $1 \leq p < \infty$, $L^p([a, b])$ is the space of all Lebesgue-measurable functions which are p-integrable, that is,

$$L^{p}([a,b]) = \left\{ f \mid \int_{a}^{b} |f(x)|^{p} dx < \infty \right\} \text{ with } \|(x_{n})\|_{L^{p}} = \left(\int_{a}^{b} |f(x)|^{p} dx \right)^{\frac{1}{p}}.$$

 $L^{\infty}([a,b])$ is the space of all Lebesgue-measurable functions which are essentially bounded (bounded on a subset of [a,b] whose complement has measure 0), that is,

$$L^{\infty}([a,b]) = \{ f \mid \exists M < \infty \ : \ |f(x)| \leq M \text{ a.e. in } [a,b] \} \text{ with } \\ \|f\|_{L^{\infty}} = \inf \{ M \mid |f(x)| \leq M \text{ a.e. in } [a,b] \}$$

 $L^p([a,b])$ is Banach for $1 \le p \le \infty$.



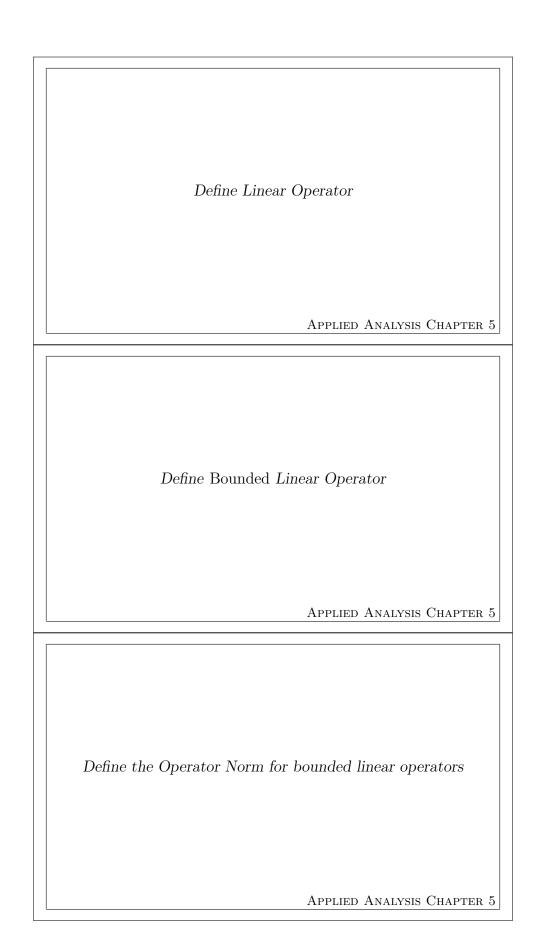
The Sobolev spaces consist of functions whose derivatives satisfy an integrability condition. Namely all derivatives up to the k^{th} are in L^p . The $W^{k,p}$ norm is defined as follows:

$$||f||_{W^{k,p}} = \left(\sum_{j=0}^{k} \int_{a}^{b} \left| f^{(j)}(x) \right|^{p} dx \right)^{\frac{1}{p}} = \left(\sum_{j=0}^{k} \left\| f^{(j)} \right\|_{L^{p}}^{p} \right)^{\frac{1}{p}}$$

All Sobolev Spaces are Banach.

It is a linear subspace, but no, it is not Banach. We use the Bernstein Polynomials to show it is dense in C([0,1]). However, it is not closed since limits of polynomials may not be polynomials. Since it is not closed, it is not complete, and thus not Banach.

Yes, it is a closed linear subspace of a Banach space (namely, C([0,1])) and hence is Banach.



A linear operator T between linear spaces X and Y is a function $T:X\to Y$ such that

$$T(\lambda x + \mu y) = \lambda T(x) + \mu T(y) \qquad \forall \lambda, \mu \in \mathbb{R} \text{ (or } \mathbb{C}) \text{ and } x, y \in X.$$

A linear operator T is bounded if $\exists M \geq 0$ such that

$$||Tx|| \le M||x|| \qquad \forall x \in X.$$

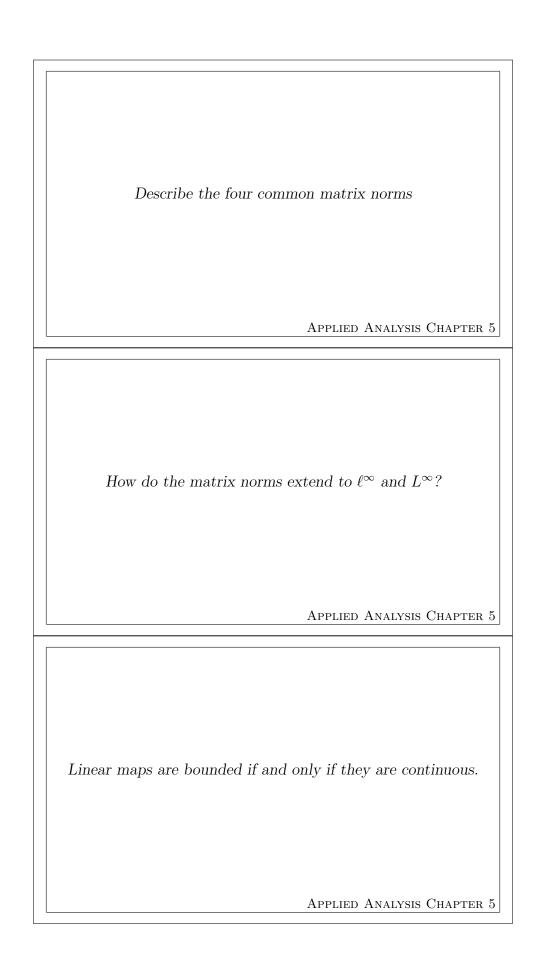
The norm of an operator T is given by the following (all four are equivalent):

$$= \inf \{ M \mid \|Tx\| \le M \|x\| \ \forall x \in X \},\$$

$$= \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|},$$

$$=\sup_{\|x\|\leq 1}\|Tx\|,$$

$$= \sup_{\|x\|=1} \|Tx\|.$$



If $T: \mathbb{R}^n \to \mathbb{R}^m$ where \mathbb{R}^n and \mathbb{R}^m are equipped with the 2-norm (Euclidean norm), then $\|T\|_2 = \sqrt{r(A^TA)}$, where r is the spectral radius and A is the matrix of the operator T. If \mathbb{R}^n and \mathbb{R}^m are equipped with the 1-norm (sum norm), then $\|T\|_1$ is the max column sum, i.e. $\|T\|_1 = \max_{1 \le j \le n} \{\sum_{i=1}^m |a_{ij}|\}$. If \mathbb{R}^n and \mathbb{R}^m are equipped with the ∞ -norm (max norm), then $\|T\|_{\infty}$ is the max row sum, i.e. $\|T\|_{\infty} = \max_{1 \le i \le m} \{\sum_{j=1}^n |a_{ij}|\}$. There is also the Hilbert-Schmidt norm of matrices, which is not derived from norms of \mathbb{R}^n and \mathbb{R}^m . It is basically the 2-norm of the $m \times n$ tuple: $\|T\|_{\mathrm{HS}} = \left(\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2\right)^{\frac{1}{2}}$.

Suppose $T: \ell^{\infty}(\mathbb{N}) \to \ell^{\infty}(\mathbb{N})$, each equipped with the sup-norm. Then T can be represented by an infinite matrix, and its norm is the max row sum: $||T|| = \sup_{i \in \mathbb{N}} \left\{ \sum_{j=1}^{\infty} |a_{ij}| \right\}$. T is only bounded if this norm is bounded. Now suppose $K: C([0,1]) \to C([0,1])$, each equipped with the uniform norm, and suppose $k: [0,1]^2 \to \mathbb{R}$. Define K explicitly as

$$Kf(x) = \int_0^1 k(x, y) f(y) dy.$$

(Note that this is called a Fredholm integral operator.) Then the norm is the "max row sum" of the function k:

$$||K|| = \max_{0 \le x \le 1} \left\{ \int_0^1 |k(x,y)| dy \right\}$$

This is finite since k is continuous on a compact set.

 $Bounded \implies continuous uses$

• Linearlity.

Continuous \implies bounded uses:

- Continuous \implies continuous specifically at 0.
- Choose $\varepsilon = 1$, obtain δ from definition of continuity, and scale any point to be of magnitude δ .
- Linearity.

Bounded Linear Transformation (BLT) Theorem: Suppose the domain of the bounded linear map T is a dense subset M of X. Then there is a unique extension \bar{T} with domain $X, \bar{T}x = Tx$ for all $x \in M$, and $\|\bar{T}\| = \|T\|$.

APPLIED ANALYSIS CHAPTER 5

Describe the differences between linearly, topologically, and isometrically isomorphic linear spaces

APPLIED ANALYSIS CHAPTER 5

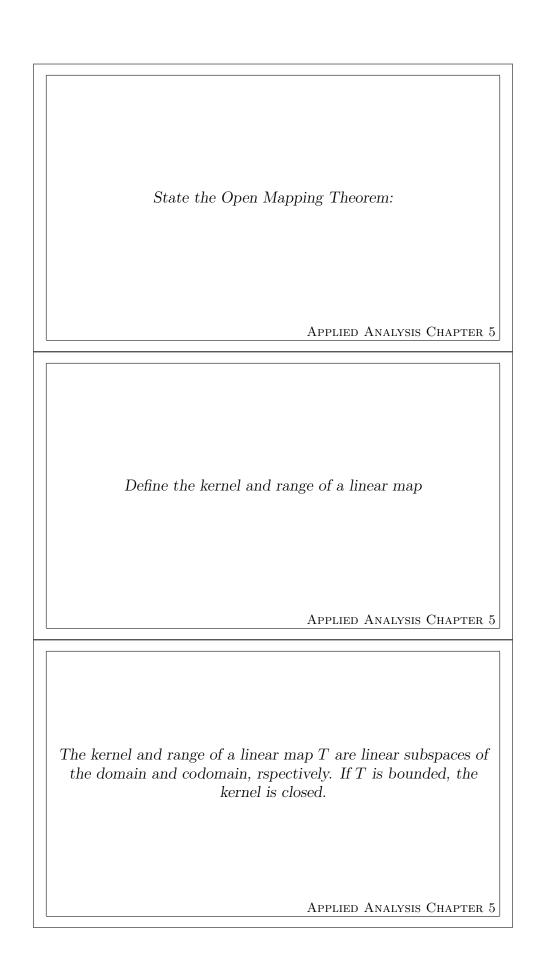
 $Define\ equivalent\ norms$

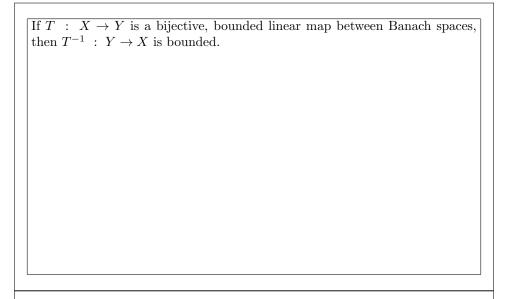
- Define Tx as the limit of images from M, which exists since bounded linear maps send Cauchy sequences to Cauchy sequences.
- Show that $\bar{T}x$ is well-defined by considering two sequences in M.
- Show that \bar{T} is in fact an extension of T.
- Show $||\bar{T}|| = ||T||$ by simple inequalities and extension.
- ullet Show uniqueness by considering two extensions and showing they are equal on all of X.

- \bullet T is a linear isomorphism if it is bijective.
- T is a topological isomorphism if both T and T^{-1} are bounded.
- T is a isometric isomorphism if T also preserves norms, i.e. ||Tx|| = ||x|| for all x.

Two norms are equivalent if each can bound the other, i.e. $\exists c, C \in \mathbb{R}$ such that

$$c\|x\|_1 \leq \|x\|_2 \leq C\|x\|_1$$





The kernel of a map T is any point in the domain which is mapped to 0. The range of a map T is any point in the codomain which is mapped to from at least one point in the domain.

$$\ker T = \{x \in X \mid Tx = 0\}$$

$$\operatorname{ran} T = \{y \in Y \mid \exists x \in X \ : \ Tx = y\}$$

To show linear subspaces, use linearity of T. The show closure, use continuity of T.

Is it possible for finite dimensional linear maps to be surjective and not injective or vice-versa? How about infinite dimensional linear maps?

APPLIED ANALYSIS CHAPTER 5

What is the operator norm of the Volterra Integral Operator K acting on C([a,b]) with the maximum norm?

$$Kf(x) = \int_{a}^{x} f(y) \mathrm{d}y$$

APPLIED ANALYSIS CHAPTER 5

State the Leibniz Integral Rule

For finite-dimensional maps, surjectivity is equivalent to injectivity. However, a counter-example in finite dimensions are the left and right shift operators on $\ell^{\infty}(\mathbb{N})$.

$$||K|| = b - a$$
 since

$$\|Kf\| \leq \sup_{a \leq x \leq b} \int_a^x |f(y)| \mathrm{d}y \leq \int_a^b |f(y)| \mathrm{d}y \leq \int_a^b \|f\| \mathrm{d}y = (b-a)\|f\|$$

so $||K|| \le (b-a)$. However, let $g \equiv 1$. Then ||g|| = 1 and

$$||Kg|| = \left\| \int_a^x dy \right\| = ||x - a|| = b - a$$

Thus ||K|| = b - a.

Let f(x,t) be a function such that the partial derivative of f with respect to t exists and is continuous. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\int_{a(t)}^{b(t)} f(x,t) \mathrm{d}x \right) = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t} \mathrm{d}x + f(b(t),t) \cdot b'(t) - f(a(t),t) \cdot a'(t).$$

Let T be a bounded linear map between two Banach spaces X and Y. Then the following are equivalent:

(a) there is a constant c > 0 such that

$$c||x|| \le ||Tx|| \ \forall x \in X;$$

(b) T has closed range, and the only solution of the equation Tx = 0 is x = 0.

APPLIED ANALYSIS CHAPTER 5

Given a finite-dimensional Banach space, the components of a vector with respect to any basis of a finite-dimensional space can be bounded by the norm of the vector. Also, the norm of a vector can be bounded by the 1-norm of that vector. In particular, let $\{e_1,\ldots,e_n\}$ be a basis of a finite-dimensional Banach space X with norm $\|\cdot\|$. Then $\exists m, M > 0$ such that if $x = \sum_{i=1}^n x_i e_i$, then

$$m\sum_{i=1}^{n}|x_i| \le ||x|| \le M\sum_{i=1}^{n}|x_i|.$$

APPLIED ANALYSIS CHAPTER 5

Every finite-dimensional normed linear space is a Banach space.

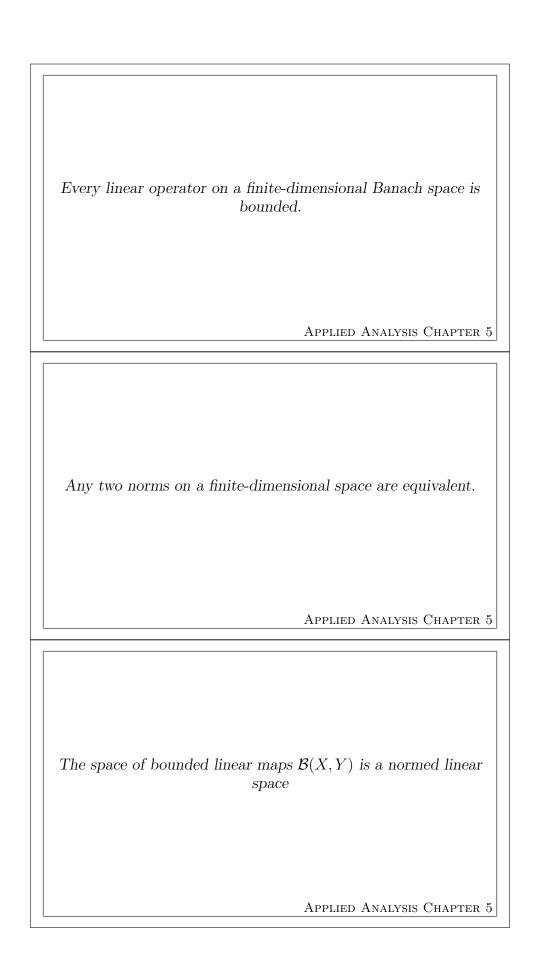
- $(a) \implies (b)$ uses:
 - Bounded linear maps send Cauchy sequences to Cauchy sequences
 - Completeness of Banach spaces
 - ullet Continuity of T
- (b) \implies (a) uses:
 - Closed subspaces of Banach spaces are Banach
 - Open Mapping Theorem
 - Definition of inverse map

The proof uses:

- Homogeneity of norm
- Heine Borel Theorem
- Compositions of continuous functions are continuous
- Continuous functions on compact domains acheive their supremum and infimum

The proof uses:

- Components of vectors can be bounded by the vectors' norms.
- \bullet Completeness gives limits to Cauchy sequences.



The proof uses:

- ullet Linearity of T
- Components of vectors can be bounded by the vectors' norms.

The proof uses:

• Components of vectors can be bounded by the vectors' norms . . . twice.

Addition and scalar multiplication are pointwise

$$(S+T)x = Sx + Tx,$$
 $(\lambda T)x = \lambda (Tx).$

The operator norm defines a norm on $\mathcal{B}(X,Y)$.

$$||T|| = \sup_{x \neq 0} \frac{||Tx||}{||x||}$$

Compositions of bounded linear maps are bounded linear maps and their norms are bounded by the product of the norms of the components.

 $||ST|| \le ||S|| ||T||.$

APPLIED ANALYSIS CHAPTER 5

Define uniform convergence of operators.

APPLIED ANALYSIS CHAPTER 5

Let X = C([0,1]) equipped with the uniform norm. Give an example of a sequence of Fredholm integral operators on X that converges to 0.

For all x,

$$||STx|| \le ||S|| ||Tx|| \le ||S|| ||T|| ||x||.$$

If (T_n) is a sequence of operators in $\mathcal{B}(X,Y)$ and

$$\lim_{n \to \infty} ||T_n - T|| = 0$$

for some $T \in \mathcal{B}(X,Y)$, then we say that T_n converges uniformly to T, or that T_n converges to T in the uniform, or operator norm, topology on $\mathcal{B}(X,Y)$.

Let K_n be given by

$$K_n f(x) = \int_0^1 x y^n f(y) \mathrm{d}y.$$

Then $K_n \to 0$ uniformly since

$$||K_n - 0|| = ||K_n|| = \max_{0 \le x \le 1} \left\{ \int_0^1 |xy^n| dy \right\} \to 0 \text{ as } n \to 0.$$

If X is a normed linear space and Y is a Banach space, then $\mathcal{B}(X,Y)$ is a Banach space.

APPLIED ANALYSIS CHAPTER 5

Define a compact operator

APPLIED ANALYSIS CHAPTER 5

- (a) If S and T are compact operators, any linear combination is compact.
- (b) If (T_n) is a sequence of compact operators that converges uniformly to T, then T is compact.
- (c) If T is an operator with finite-dimensional range, then T is compact.
- (d) If S is compact and T is bounded, or if S is bounded and T is compact, TS is compact.

Let (T_n) be a Cauchy sequence in $\mathcal{B}(X,Y)$. Then for any $x \in X$, $||T_nx - T_mx|| \le ||T_n - T_m|| ||x||$ by linearity and the definition of the operator norm. Since (T_n) is Cauchy, we have (T_nx) is Cauchy for any $x \in X$. Since Y is complete, then for each x, $\exists y_x \in Y$ such that $T_nx \to y_x$. Next we define our candidate limit operator T by $Tx = y_x$. T is clearly linear, but we still need to show it is bounded and the uniform limit of the sequence (T_n) . Choose an $\varepsilon > 0$ and note by the triangle inequality and definition of operator norm,

$$||T_n x - Tx|| \le ||T_n - T_m|| ||x|| + ||T_m x - Tx||.$$

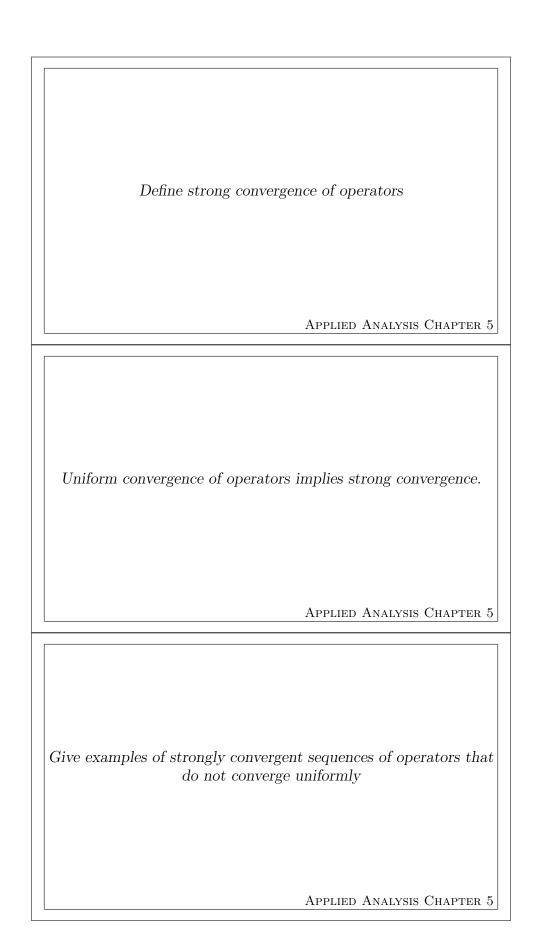
By the definition of Cauchy, $\exists N_{\varepsilon}$ such that $n, m \geq N_{\varepsilon} \Longrightarrow \|T_n - T_m\| < \frac{\varepsilon}{2}$. Then since $T_n x \to y_x = Tx$, $\exists M_x$ such that $m > M_x \Longrightarrow \|T_m x - Tx\| < \frac{\varepsilon}{2}$. This gives $\|T_n x - Tx\| < \varepsilon$. Finally, $\|Tx\| \leq \|Tx - T_n x\| + \|T_n x\| < \|T_n x\| + \varepsilon$. Since ε was arbitrary, T is bounded. Thus $T_n - T$ is bounded, and $\|T_n - T\| < \varepsilon$. Again, since ε was arbitrary $T_n \to T$ uniformly.

A linear operator $T: X \to Y$ is compact if T(B) is a precompact subset of Y for every bounded subset B of X.

OR

A linear operator T is compact if every bounded sequence in X has a subsequence whose image converges in Y.

- (a) Uses a subsequence of a subsequence.
- (b) Uses subsequences of subsequences of ... and a diagonal subsubsequence argument.
- (c) Uses Bolzano Weierstrass
- (d) One uses compactness and continuity, and the other uses boundedness and compactness.



A sequence (T_n) of operators converges strongly to T if $T_n x \to T x$ for every $x \in X$.

Suppose $T_n \to T$ uniformly. Then $||T_n - T|| \to 0$. Then for any $x \in X$,

$$||T_n x - Tx|| = ||(T_n - T)x|| \le ||T_n - T|| ||x|| \to 0,$$

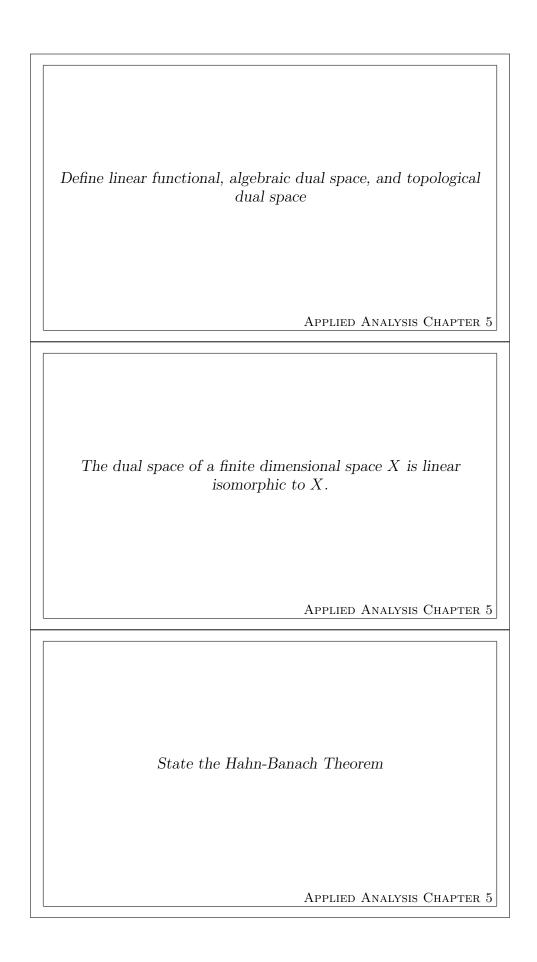
which shows $T_n x \to T x$.

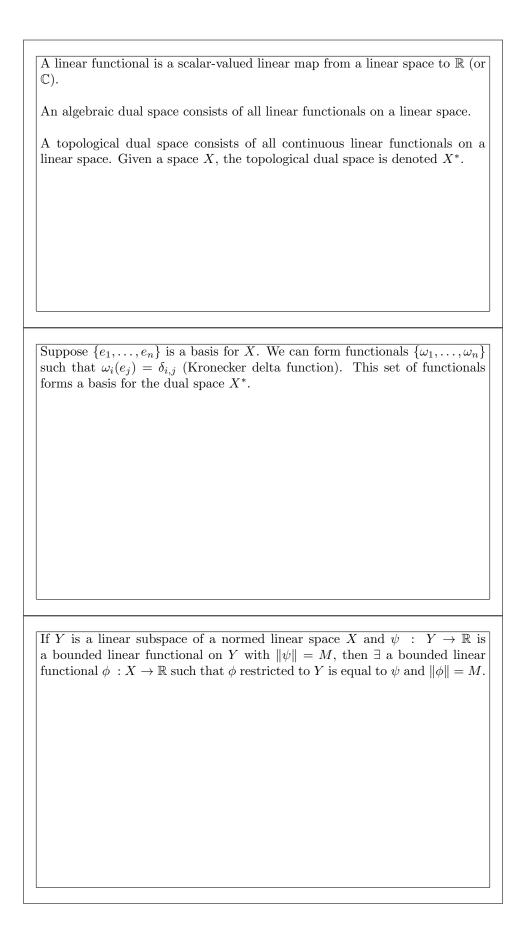
Let $X = \ell^p(\mathbb{N})$ for $1 \leq p < \infty$. Then for $n \in \mathbb{N}$, define P_n as the projection

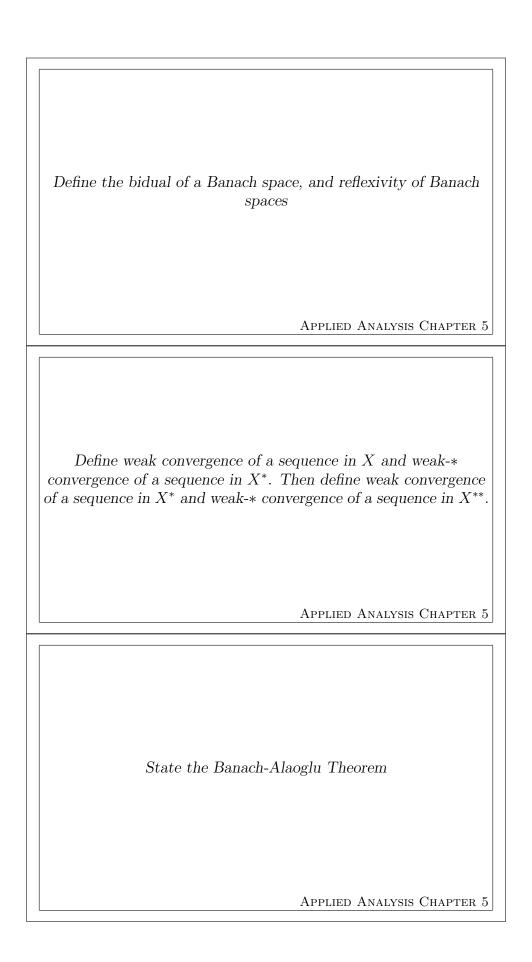
$$P_n(x_1, x_2, \dots, x_n, x_{n+1}, x_{n+2}, \dots) = (x_1, x_2, \dots, x_n, 0, 0, \dots)$$

Note $||P_n - P_m|| = 1$ for n > m since $x = (0, 0, \dots, 0, 1, 0, \dots)$, where the $(n - m + 1)^{\text{st}}$ component of x is 1, gets mapped to itself. So (P_n) is not a Cauchy sequence and thus cannot converge uniformly. However, for any given sequence x, $||P_n x - Ix||_{\ell^p} = ||(0, 0, \dots, 0, x_{n+1}, x_{n+2}, \dots)||_{\ell^p} \to 0$ since sequences in $\ell^p(\mathbb{N})$ decay to 0 for $1 \le p < \infty$. Thus $P_n \to I$ strongly.

Let X = C([0,1]) and let K_n be a sequence of functionals given by $K_n f = \int_0^1 \sin(n\pi x) f(x) dx$. Use integration by parts to show $K_n p \to 0$ for all polynomials p and use Weierstrass Approximation Theorem to say the same for any continuous function. Thus $K_n \to 0$ strongly. However, let $g_n(x) = \sin(n\pi x)$. Then $P_n g_n = \frac{1}{2}$, which shows $||K_n|| \ge \frac{1}{2}$ for each $n \in \mathbb{N}$. Thus K_n does not converge uniformly to 0.







The bidual of a Banach space X is the dual of its dual, i.e. X^{**} . For each $x \in X$, we can define a linear functional $F_x \in X^{**}$ by $F_x(\phi) = \phi(x)$. This means there is an embedding of X inside X^{**} . If X and X^{**} are isomorphic, we say X is reflexive.

A sequence $(x_n) \in X$ converges weakly to x, denoted $x_n \rightharpoonup x$, if $\phi(x_n) \rightarrow \phi(x)$ for every bounded linear functional $\phi \in X^*$. A sequence $(\phi_n) \in X^*$ converges weak-*ly to ϕ , denoted $\phi_n \rightharpoonup^* \phi$, if $\phi_n(x) \rightarrow \phi(x)$ for every $x \in X$.

A sequence $(\phi_n) \in X^*$ converges weakly to ϕ , denoted $\phi_n \rightharpoonup \phi$, if $F(\phi_n) \rightarrow F(\phi)$ for every bounded linear functional $F \in X^{**}$. A sequence $(F_n) \in X^{**}$ converges weak-*ly to F, denoted $F_n \rightharpoonup^* F$, if $F_n(\phi) \rightarrow F(\phi)$ for every $\phi \in X^*$.

Let X^* be the dual space of a Banach space X. The closed unit ball in X^* is weak-* compact. In other words, let (ϕ_n) be a sequence in the unit ball of X^* . Then there is a subsequence (ϕ_{n_k}) and a linear functional ϕ in the unit ball of X^* such that $\phi_{n_k} \rightharpoonup^* \phi$.