Define the support of a continuous function.

Define $C^k(\Omega)$ and $C^\infty(\Omega)$. Then define $C_C^\infty(\Omega)$.

State Urysohn's Lemma in the context of $\mathbb{R}^n$.

Let $f : \mathbb{R}^n \to \mathbb{C}$ be a continuous function. Then the support of f, denoted $\operatorname{supp} f$, is the closure of the set on which $f(x) \neq 0$. That is,

$$\operatorname{supp} f = \overline{\{x \in \mathbb{R}^n \mid f(x) \neq 0\}}.$$

$C^k(\Omega)$ is the set of k-times differentiable functions on Ω. Functions in $C^k(\Omega)$ for every $k > 0$ are said to be in $C^\infty(\Omega)$, that is, infinitely differentiable functions. $C^\infty_C(\Omega)$ is the set of infinitely differentiable functions on Ω which have support bounded and contained in Ω (compact when $\Omega = \mathbb{R}^n$). That is,

$$C^k(\Omega) = \left\{ f : \Omega \to B \mid \frac{\partial^i f}{\partial x^i} \text{ for } i = 0, \ldots, k \in C(\Omega) \right\}$$

$$C^\infty(\Omega) = \left\{ f : \Omega \to B \mid f \in C^k(\Omega) \text{ for } k \in \mathbb{N} \right\}$$

$$C^\infty_C(\Omega) = \{ f \in C^\infty(\Omega) \mid \operatorname{supp}(f) \text{ is compact} \}$$

Let $\Omega \subset \mathbb{R}^n$ be an open set and let $K \subset \Omega$ be compact. Then there exists a nonnegative function $\psi \in C^\infty_C$ with $\psi(x) = 1$ for $x \in K$.

Define σ-algebra

LIEB AND LOSS CHAPTER 1

What are the Borel sets?

LIEB AND LOSS CHAPTER 1

What is a measure on a σ-algebra?

LIEB AND LOSS CHAPTER 1

Let Σ be a collection of subsets of Ω. Then Σ is called a σ-algebra if

(i) If $A \in \Sigma$, then $A^C \in \Sigma$;

(ii) If $A_1, A_2, \ldots$ is a countable family of sets in Σ, then $\bigcup_{n=1}^{\infty} A_i \in \Sigma$;

(iii) and $\Omega \in \Sigma$.

In English,

(i) Σ is closed under complemets;

(ii) Σ is closed under countable unions;

(iii) and Σ contains the entire set Ω.

The Borel sets is the smallest σ-algebra containing the open sets of $\mathbb{R}^n$, i.e the smallest σ-algebra generated by the open balls of $\mathbb{R}^n$ (sets of the form $B_{x,R} = \{y \in \mathbb{R}^n \mid |x - y| < R\}$).

A measure $\mu : \Sigma \to \mathbb{R}_0^+ \cup \infty$ is a function from Σ into the nonnegative real numbers (including infinity) such that

(i) $\mu(\emptyset) = 0$,

(ii) and $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A)i)$ for any sequence of disjoint sets (A_i) in Σ.

In English, a measure is a function which sends the empty set to 0 and has "countable additivity".

Define measure space

Lieb and Loss Chapter 1

Describe the Dirac δ measure and the Lebesgue measure.

Lieb and Loss Chapter 1

What is the Lebesgue measure of a ball of radius r?

Lieb and Loss Chapter 1

A measure space consists of a set Ω, a σ-algebra Σ of Ω, and a measure μ on Σ. A measure space is denoted (Ω, Σ, μ).

Let $\Omega \subset \mathbb{R}^n$ and fix $y \in \Omega$ Let Σ be a σ-algebra on Ω. Define $\delta_y : \Sigma \to \mathbb{R} \cup \infty$ by

$$\delta_y(A) = \begin{cases} 1 & \text{if } y \in A \\ 0 & \text{if } y \notin A \end{cases}$$

In English, the Dirac measure simply measures whether or not a set contains a fixed point y.
The Lebesgue measure on subsets of $\mathbb{R}^n$ is the most common measure. It gives the Euclidean volume of "nice" sets (Borel sets). The Lebesgue measure is denoted $\mathcal{L}$, or simply $|\cdot|$.

Let $B_{x,r} \subset \mathbb{R}^n$ be an open n-dimensional ball centered around x with readius r. Then

$$\mathcal{L}(B_{x,r}) = |B_{0,1}| r^n = \frac{1}{n} \left| \mathbb{S}^{n-1} \right| r^n$$

where

$$\left| \mathbb{S}^{n-1} \right| = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}.$$

$\mathbb{S}^{n-1}$ denotes the sphere of radius 1 in $\mathbb{R}^n$.

What is inner- and outer-regularity for the Lebesgue measure?

What does it mean dor a measure space to be σ-finite? Tell why the measure space $(\mathbb{R}^n, \mathcal{B}, \mathcal{L}^n)$ is σ-finite.

Given two measure spaces $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$, define their product space. What is the section property?

The Lebesgue measure has inner-regularity:

$$\mathcal{L}^n(A) = \inf \{\mathcal{L}^n(O) \mid A \subset O \text{ and } O \text{ is open}\}$$

and outer-regularity:

$$\mathcal{L}^n(A) = \sup \{\mathcal{L}^n(C) \mid C \subset A \text{ and } C \text{ is compact}\}.$$

A measure space (Ω, Σ, μ) is called σ-finite if there is a sequence $(A_i) \in \Sigma$ with $\mu(A_i) < \infty$ for all $i \in \mathbb{N}$ such that $\Omega = \bigcup_{i=1}^{\infty} A_i$.
$(\mathbb{R}^n, \mathcal{B}, \mathcal{L}^n)$ is σ-finite since $\mathbb{R}^n$ is the union of the balls of radius 1 centered at every point $q = (q_1, \ldots, q_n) \in \mathbb{Q}^n$.

Let $\Omega = \Omega_1 \times \Omega_2 = \{(\omega_1, \omega_2) \mid \omega_1 \in \Omega_1 \text{ and } \omega_2 \in \Omega_2\}$. Then define a rectangle A in Ω of $A_1 \in \Sigma_1$ and $A_2 \in \Sigma_2$, by

$$A = A_1 \times A_2.$$

Define the measure Σ to be the smallest σ-algebra containing all rectangles from Σ_1 and Σ_2 (all sets in Ω of the above form). Finally, it can be shown there is a unique measure μ on Σ such that

$$\mu(A) = \mu(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2).$$

This is called the "product measure" μ on Σ. Then (Ω, Σ, μ) is a measure space.
This space has the "section property," which states that for any $A \in \Sigma$, the set $\{x \in \Omega_1 \mid (x, y) \in A\} \in \Sigma_1$ for any $y \in \Omega_2$, and the set $\{y \in \Omega_2 \mid (x, y) \in A\} \in \Sigma_2$ for any $x \in \Omega_1$.

Define monotone class and algebra of sets.

State the monotone class theorem. Describe the general method of proof used.

State the uniqueness of measures theorem.

A monotone class $\mathcal{M}$ is a collection of sets such that

(i) if $A_i \in \mathcal{M}$ for $i = 1, 2, \dots$, and if $A_1 \subset A_2 \subset \dots$, then $\bigcup_i A_i \in \mathcal{M}$,

(ii) if $B_i \in \mathcal{M}$ for $i = 1, 2, \dots$, and if $B_1 \supset B_2 \supset \dots$, then $\bigcap_i B_i \in \mathcal{M}$.

A collection of sets $\mathcal{A}$ is called an algbra of sets if for every A and B in $\mathcal{A}$, their differences $A \setminus B$ and $B \setminus A$, as well as their union $A \cup B$ are in $\mathcal{A}$. That is, algebras of sets are collections of sets closed under finite unions and complements.

Let Ω be a set an $\mathcal{A}$ be an algebra of sets on Ω with $\emptyset, \Omega \in \mathcal{A}$. Then there exists a smallest monotone class $\mathcal{S}$ containing $\mathcal{A}$, and $\mathcal{S}$ is also the smallest σ-algebra containing $\mathcal{A}$.

In general, define subsets of S which hold properties you want to hold for all of S. Prove those subsets are monotone classes, and thus, by the definition of S, these subsets must be equal to S.

Let Ω be a set and $\mathcal{A}$ an algebra of sets on Ω with $\emptyset, \Omega \in \mathcal{A}$. Let Σ be the smallest σ-algebra containing $\mathcal{A}$. Let μ_1 be a σ-finite measure in the sense that there exists a sequence of sets $(A_i) \in \mathcal{A}$ with $\mu_1(A_i) < \infty$ for all $i \in \mathbb{N}$ and $\Omega = \bigcup_{i=1}^{\infty} A_i$. Then suppose $\mu_1(A_i) = \mu_2(A_i)$ for all $i \in \mathbb{N}$. Then $\mu_1(A) = \mu_2(A)$ for all $A \in \Sigma$.

Define level sets and measureable functions.

Lieb and Loss Chapter 1

Define upper- and lower-semicontinuity. How do they relate to continuity?

Lieb and Loss Chapter 1

Define the essential support of a measurable function f.

Lieb and Loss Chapter 1

Let $f : \Omega \to \mathbb{R}$ be a real-valued function on a measure space (Ω, Σ, μ). A level set $L_f(t)$ is defined as all points $x \in \Omega$ with $f(x) > t$, that is

$$L_f(t) = \{x \in \Omega \mid f(x) > t\}.$$

We say f is a measurable function if every level set is measurable, i.e. f is measurable if $L_f(t) \in \Sigma$ for every $t \in \mathbb{R}$.
Complex valued functions are considered measurable if their real and imaginary parts are measurable.

A measurable function f is lower-semicontinuous if $L_f(t)$ is open for every $t \in \mathbb{R}$. f is upper-semicontinuous if $U_f(t)$ is open for every $t \in \mathbb{R}$, where

$$U_f(t) = \{x \in \Omega \mid f(x) < t\}.$$

A measurable function f is continuous if and only if it is both upper- and lower-semicontinuous.

Let f be a measurable function. Let $\tilde{\Omega}$ be the collection of all open sets ω with $f(x) = 0$ for μ-almost every $x \in \omega$. Define $\omega^* := \bigcup \tilde{\Omega}$. Then the essential support of f, denoted $\operatorname{ess\,supp} f$, is the complement of ω^*, that is

$$\operatorname{ess\,supp} f = (\omega^*)^C$$

The important difference between the support and essential support is the use of open sets. For example, $\mathcal{X}_{\mathbb{Q}}$ has support $\mathbb{R}$ but essential support $\emptyset$. This is because $\mathbb{Q}$ is not an open set.

Use Riemann integration to define the integral of a measurable function f over Ω. What is the qualitatively different intuition from classical Riemann integration?

What is a summable function? What is another name for summable functions?

State and sketch the proof of the Monotone Convergence Theorem.

Suppose f is a nonnegative, real-valued Σ-measurable function on Ω. Then

$$\int_\Omega f(x)\mu(\mathrm{d}x) \coloneqq \int_0^\infty \mu(L_f(t))\mathrm{d}t$$

where $L_f(t)$ is the lower level set

$$L_f(t) = \{x \in \Omega \mid f(x) > t\}.$$

While Riemann integration is a limit of Riemann approximations, i.e. splitting the function into vertically oriented rectangles, integration in general seeks to find the "volume" under the curve by splitting the function into horizonatally oriented rectangles.

A measurable function f is μ-summable, or μ-integrable, if $\int f \mathrm{d}\mu < \infty$.

Let $f_1, f_2, \ldots$ be an increasing sequence of summable functions on (Ω, Σ, μ). Define $f(x) \coloneqq \lim_{j\to\infty} f_j(x)$. Then f is measurable, and $\lim_{j\infty} \int f_j < \infty \iff \int f < \infty$, in which case

$$\int f = \lim_{j\to\infty} \int f_j.$$

Sketch: First note that $\bigcup_{j=1}^\infty L_{f_j}(t) = L_f(t)$, $(\mu(L_{f_j}(t)))$ is an increasing real sequence, and $\mu(L_{f_j}(t)) \to \mu(L_f(t))$. Then show, using convergence of Riemann sums, that

$$\int_0^\infty \mu(L_f(t)) = \lim_{j\to\infty} \int_0^\infty \mu(L_{f_j}(t))$$

State Fatou's Lemma.

State the Dominated Convergence Theorem.

State the theorem involving the "Missing Term in Fatou's Lemma".

Let $f_1, f_2, \ldots$ be a sequence of non-negative, summable functions on (Ω, Σ, μ), and define f to be the pointwise $\liminf$ of f_j, that is,

$$f(x) := \liminf_{j\to\infty} f_j(x).$$

Then

$$\liminf_{j\to\infty} \int f_j \geq \int f.$$

Intuitively, the least integral is at least as large as the integral of all the least values.

Let $f_1, f_2, \ldots$ be a sequence of complex-valued summable functions on (Ω, Σ, μ) and assume these functions converge to a function f pointwise almost everywhere. If there is a summable, nonnegative function G on (Ω, Σ, μ) such that $|f_j(x)| \leq G(x)$ for all $j = 1, 2, \ldots$, then $|f(x)| \leq G(x)$ and

$$\lim_{j\to\infty} \int f_j = \int f.$$

Let $f_1, f_2, \ldots$ be a sequence of complex-valued summable functions that converge pointwise almost everywhere to a function f. Also assume each f_j is uniformly p^{th} power summable, i.e.

$$\int_\Omega |f_j(x)|^p < C$$

for a constant C independent of j. Then

$$\lim_{j\to\infty} \int_\Omega ||f_j(x)|^p - |f_j(x) - f(x)|^p - |f(x)|^p| \mathrm{d}x = 0.$$

That is, if f_j is a bounded sequence in $L^p(\Omega)$ and $f_j \to f$ pointwise almost everywhere, then the above equality holds.

State Fubini's Theorem and give an example of when it cannot be used.

State Egorov's Theorem.

Define Simple functions and their integral.

Let $(\Omega_i, \Sigma_i, \mu_i)$, $i = 1, 2$, be two measure spaces and let f be a $\Sigma = \Sigma_1 \times \Sigma_2$-measurable function on $\Omega = \Omega_1 \times \Omega_2$ (also denote $\mu = \mu_1 \times \mu_2$). If $f \geq 0$, then the following three integrals are equal (in the sense that all three can be infinite):

1) $\displaystyle\int_{\Omega_1 \times \Omega_2} f(x, y)(\mu_1 \times \mu_2)(\mathrm{d}x\mathrm{d}y)$

2) $\displaystyle\int_{\Omega_1} \left(\int_{\Omega_2} f(x, y)\mu_2 \mathrm{d}y \right) \mu_1 \mathrm{d}x$

3) $\displaystyle\int_{\Omega_2} \left(\int_{\Omega_1} f(x, y)\mu_1 \mathrm{d}x \right) \mu_2 \mathrm{d}y$

Let (Ω, Σ, μ) be a finite measure space. Let $f_1, f_2, d \ldots$ a sequence of complex-valued measurable functions on Ω such that $f_n(x) \to f(x)$ pointwise almost everywhere on Ω. Then $\forall \varepsilon > 0$, $\exists A_\varepsilon \subset \Omega$ such that $\mu(A_\varepsilon) > \mu(\Omega) - \varepsilon$ and $f_n \to f$ in the uniform norm on A_ε. That is,

$$\sup_{x \in A_\varepsilon} |f_n(x) - f(x)| \to 0.$$

A simple function is a measurable function f that takes only finitely many values. That is,

$$f(x) = \sum_{i=1}^{N} C_i \mathcal{X}_{A_i}$$

where $C_i \in \mathbb{C}$ and A_i are measurable subsets of the measure space Ω. The integral of a simple function is simply the weighted sum of the measures of the sets A_i, that is,

$$\int f = \sum_{i=1}^{N} C_i \mu(A_i).$$

How are simple functions dense in a measure space?

Lieb and Loss Chapter 1

How are infinitely differentiable functions dense in a measurable subset of $\mathbb{R}^n$?

Lieb and Loss Chapter 1

Simple functions are dense in the L^1 norm. The theorem states:

Let (Ω, Σ, μ) be a measure space with Σ generated by an algebra $\mathcal{A}$. Assume that Ω is a σ-finite in the strong sense mentioned above. Let f be a simplex-valued summable function and let $\varepsilon > 0$. Then there is a simple function h_ε such that

$$\int_\Omega |f - h_\varepsilon| \mathrm{d}\mu < \varepsilon.$$

C^∞ functions are dense in the L^1 norm. The theorem states:

Let Ω be an open subset of $\mathbb{R}^n$ and let μ be a measure on the Borel σ-algebra of Ω. Let $\mathcal{A}$ be the algebra of half-open rectangles and assume Ω is σ-finite in the strong sense. Assume, also, that every finite, closed rectangle that is contained in Ω has finite μ-measure. If f is a μ-summable function, then, for each $\varepsilon > 0$, there is a $C^\infty(\mathbb{R}^n)$ function g_ε such that

$$\int_\Omega |f - g_\varepsilon| \mathrm{d}\mu < \varepsilon.$$