

LIEB AND LOSS CHAPTER 2

 $L^p$  is the space of all  $p^{\text{th}}$  power summable functions.

Let  $\Omega$  be a measure space with a positive measure  $\mu$  and let  $1 \leq p < \infty$ . Then

$$L^p(\Omega,\mu) \coloneqq \left\{f \mid f \ : \ \Omega \to \mathbb{C}, \ f \text{ is $\mu$-summable and } \ \left|f\right|^p \text{ is $\mu$-summable}\right\}.$$

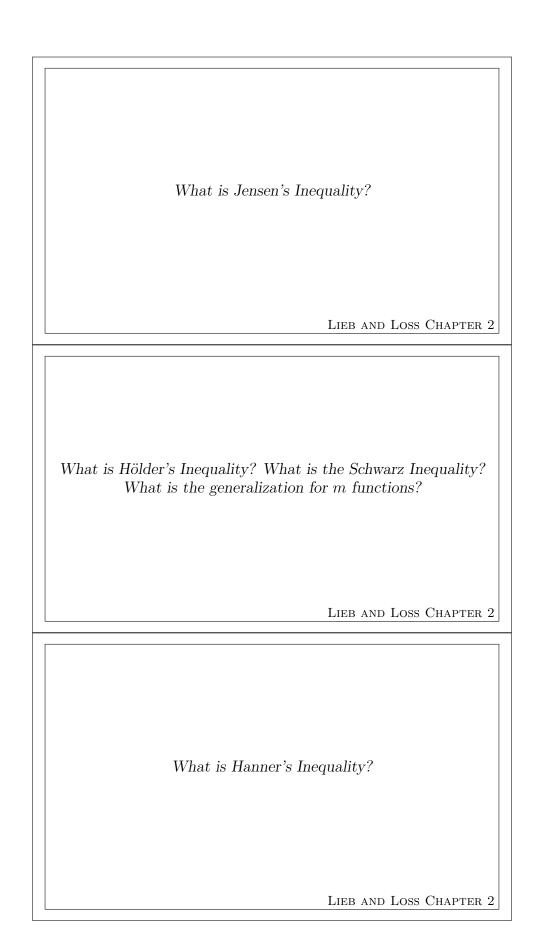
The norm of 
$$L^p$$
 is given by  $||f||_{L^p} = \left(\int_{\Omega} |f|^p d\mu\right)^{\frac{1}{p}}$ .  
For  $p = \infty$ ,

$$L^{\infty}(\Omega, \mu) := \{ f \mid f : \Omega \to \mathbb{C}, \text{ fis } \mu\text{-measurable and } \exists \text{ constant } K \text{ such that } |f(x)| < K \text{ for } \mu \text{ almost every } x \in \Omega \}$$

with norm  $\|f\|_{L^{\infty}} = \inf \{ K \mid |f(x)| < K \text{ for } \mu \text{ almost every } x \in \Omega \}.$ 

If 
$$f \in L^p \cap L^\infty$$
, then  $f \in L^q$  for all  $q > p$  and  $||f||_{\infty} = \lim_{p \to \infty} ||f||_p$ .

- A convex set  $K \subset \mathbb{R}^n$  is one for which  $\lambda x + (1 \lambda)y \in K$  for all  $x, y \in K$  and  $0 \le \lambda \le 1$ .
- A convex function f on a convex set K is a real-valued function satisfying  $f(\lambda x + (1 \lambda)y) \leq \lambda f(x) + (1 \lambda)f(y)$  for all  $x, y \in K$  and  $0 \leq \lambda \leq 1$ .
- A function is strictly convex if equality never holds whenever  $x \neq y$  and  $0 < \lambda < 1$ .
- A function is concave if the inequality is reversed.
- If K is open then convex functions are continuous.



Let  $J: \mathbb{R} \to \mathbb{R}$  be a convex function and f a real-valued function on some finite measurable set  $\Omega$ . Define  $\langle \cdot \rangle$  to be the average of a function, i.e.

$$\langle f \rangle \coloneqq \frac{1}{\mu(\Omega)} \int_{\Omega} f.$$

Then

- (i)  $[J \circ f]_{-} \in L^{1}(\Omega);$
- (ii)  $\langle F \circ f \rangle \geq J(\langle f \rangle)$ .

In English,

- (i) The negative part of the composition is absolutely summable;
- (ii) The average of the composition is at least the composition of the average.

Let  $1 \le p \le \infty$  and let q be the dual index of p. Then if  $f \in L^p$  and  $g \in L^q$ , then  $fg \in L^1$  and

$$||fg||_1 \le ||f||_p ||g||_q.$$

The Schwarz Inequality is the special case when p = q = 2. We have

$$||fg||_1 \le ||f||_2 ||g||_2$$

To generalize, for  $i=1,2,\ldots,n,$  let  $f_i\in L^{p_i}$  and  $\frac{1}{p_1}+\cdots+\frac{1}{p_n}=1.$  Then

$$\prod_{i=1}^{n} f_i \in L^1 \quad \text{and} \quad \left\| \prod_{i=1}^{n} f_i \right\|_1 \le \prod_{i=1}^{n} \|f_i\|_{p_i}$$

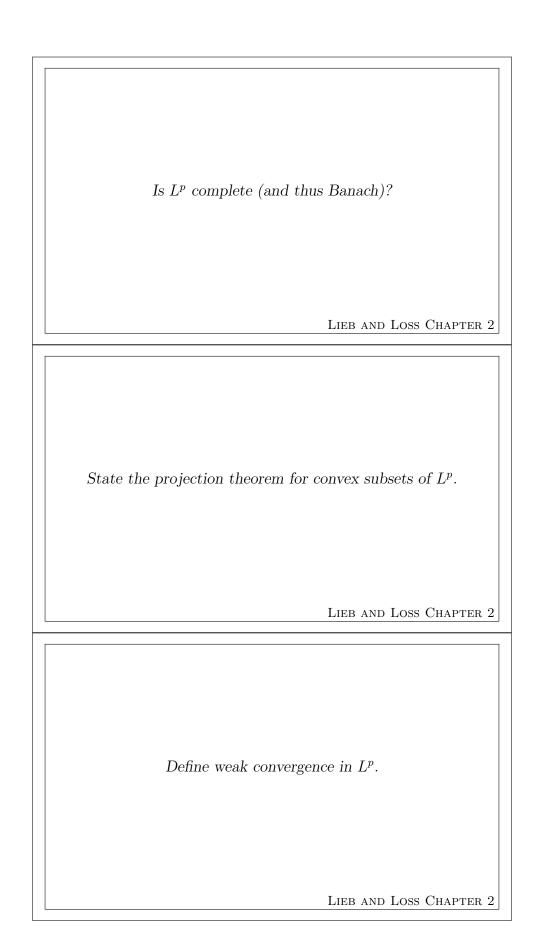
Let  $f, g \in L^p$ . If  $1 \le p \le 2$ , then (Parallelogram Identity)

$$||f+g||_p^p + ||f-g||_p^p \le (||f||_p + ||g||_p)^p + ||f||_p - ||g||_p|^p$$

and

$$\Big(\|f+g\|_p + \|f-g\|_p\Big)^p + \Big|\|f+g\|_p - \|f-g\|_p\Big|^p \leq 2^p \Big(\|f\|_p^p + \|g\|_p^p\Big).$$

If  $2 \le p < \infty$ , the inequalities are reversed.



Yes. Let  $1 \leq p \leq \infty$  and let  $(f_i)$  be a Cauchy sequence in  $L^p$ , i.e.  $||f_i - f_j||_p \to 0$  as  $i, j \to \infty$ . Then there is a unique function  $f \in L^p$  such that  $||f_i - f||_p \to 0$  as  $i \to \infty$ , i.e.

 $f_i \to f$  say " $f_i$  converges strongly to f".

Let  $1 and let K be a convex subset of <math>L^p$ . Let  $f \in L^p$  such that  $f \notin K$  and define

$$D\coloneqq \mathrm{dist}\,(f,K)\inf_{g\in K}\|f-g\|_p.$$

Then  $\exists h \in K$  such that

$$\|f - h\|_p = D.$$

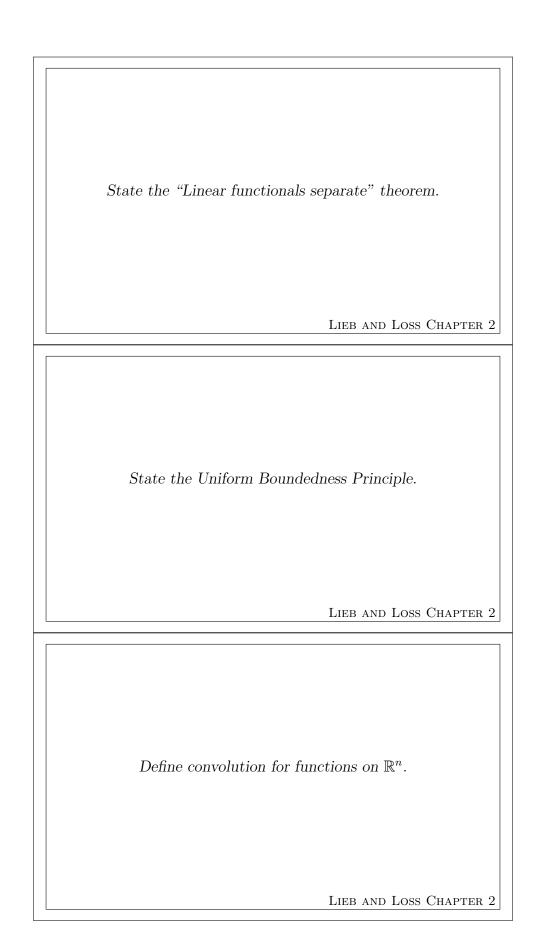
Let  $(f_i)$  be a sequence in  $L^p$ . If  $L(f_i) \to L(f)$  for every bounded linear functional L on  $L^p$ , then we say  $f_i \to f$ , or  $f_i$  weakly converges to f. It can be shown that for  $1 \le p < \infty$ ,  $(L^p)^* \cong L^q$ , where q is the dual index of p, and that every bounded linear functional  $L \in (L^p)^*$  can be represented as integration against a unique  $L^q$  function, i.e.  $\forall L \in (L^p)^*$ ,  $\exists ! g \in L^q$  such that

$$L(f) = \int fg$$

for every  $f \in L^p$ . Thus,  $(f_i)$  converges weakly in  $L^p$  if

$$\int f_i g \to \int f g$$

for every  $g \in L^q$ .



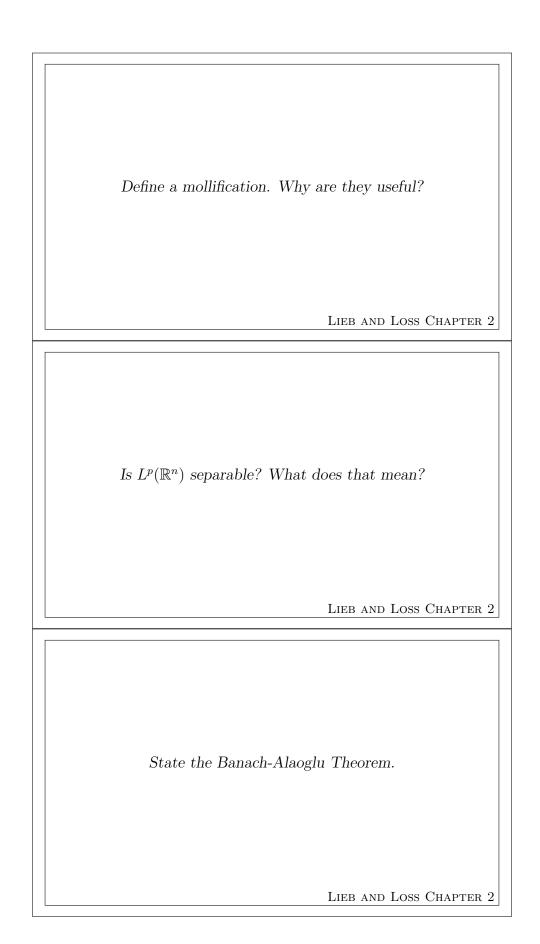
Suppose  $f \in L^p$  with L(f) = 0 for all  $L \in (L^p)^*$ . Then f = 0.

Consequently, if  $f_i \rightharpoonup g$  and  $f_i \rightharpoonup h$ , then g = h.

Let  $(f_i)$  be a sequence in  $L^p$  such that  $\forall L \in (L^p)^*$  the sequence  $(L(f_i))$  is bounded in  $\mathbb{C}$ . Then  $(\|f_i\|_p)$  is a bounded sequence in  $\mathbb{R}$ .

For  $f, g : \mathbb{R}^n \to \mathbb{C}$ , we define the convolution of f and g, denoted f \* g, as

$$(f * g)(x) := \int_{\mathbb{R}^n} f(x - y)g(y)dy.$$



Let  $j \in L^1(\mathbb{R}^n)$  with  $\int_{\mathbb{R}^n} j = 1$ . For  $\varepsilon > 0$ , define  $j_{\varepsilon}$  as

$$j_{\varepsilon}(x) \coloneqq \frac{1}{\varepsilon^n} j\left(\frac{x}{\varepsilon}\right).$$

so that  $||j_E||_1 = ||j||_1$  and  $\int_{\mathbb{R}^n} j_{\varepsilon} = 1$ .

Define the mollification of a function  $f \in L^p(\mathbb{R}^n)$  for some  $1 \leq p < \infty$ , denoted  $f_{\varepsilon}$ , as the convolution of f and  $j_{\varepsilon}$  for some  $\varepsilon$ , that is,

$$f_{\varepsilon} = f * j_{\varepsilon}.$$

Then  $f_{\varepsilon} \in L^p(\mathbb{R}^n)$  and  $||f_{\varepsilon}||_p \le ||f||_p ||j||_1$ . Also,  $f_{\varepsilon} \to f$  strongly in  $L^p$ , that is,  $||f_{\varepsilon} - f||_p \to 0$ .

In addition, if  $j \in C_C^{\infty}$ , then  $f_{\varepsilon} \in C^{\infty}$ . This is a concrete construction which shows that  $C^{\infty}$  functions are dense in  $L^p$ .

Yes,  $L^p(\mathbb{R}^n)$  is separable. This means there is a countable dense subset of  $L^p(\mathbb{R}^n)$ , that is,  $\exists \Phi = \{\phi_1, \phi_2, \dots\} \subset L^p(\mathbb{R}^n)$  such that  $\forall f \in L^p$  and  $\varepsilon > 0$ ,  $\exists \phi_j \in \Phi$  such that  $\|f - \phi_j\|_p < \varepsilon$ .

Let  $\Omega \subset \mathbb{R}^n$  be a measurable set and consider  $L^p(\Omega)$  with  $1 . Let <math>(f_i)$  be a bounded sequence in  $L^p$ . Then there exists a subsequence  $(f_{i_j})$  and  $f \in L^p$  such that  $f_{i_j} \rightharpoonup f$  in  $L^p$ . That is, bounded sets in  $L^p$  are weakly compact.

How does Urysohn's Lemma give that $C_C^{\infty}$ is dense in $L^p$ ?
Lieb and Loss Chapter 2
What is special about convolutions of functions in dual $L^p$ spaces?  Lieb and Loss Chapter 2

Let  $\Omega \subset \mathbb{R}^n$  be an open set and let  $K \subset \Omega$  be compact. Then there is a function  $J_K \in C_C^{\infty}(\Omega)$  such that  $0 \leq J_K(x) \leq 1$  for all  $x \in \Omega$  and  $J_K(x) = 1$  for all  $x \in K$ .

As a consequence, there is a sequence of functions  $(g_i) \in C_C^{\infty}$  that take values in [0,1] and such that  $\lim_{j \to \infty} g_j(x) = 1$  for every  $x \in \Omega$ .

As a second consequence, given a sequence of functions  $(f_i) \in C^{\infty}$  such that  $f_i$  converges strongly to a function  $f \in L^p$ , the sequence  $(h_i) = (g_i f_i) \in C_C^{\infty}$  and  $h_i \to f$  strongly.

This shows, since  $C^{\infty}$  is dense in  $L^p$ , that  $C_C^{\infty}$  is dense in  $L^p$ .

If  $f \in L^p(\mathbb{R}^n)$  and  $g \in L^q(\mathbb{R}^n)$ , where q is the dual index of p, then f \* g is continuous and (f \* g) tends to 0 at infinity.