Discussion 2: October 9th.

Multinomial coefficients.

Recall that if we want to split n objects into r many distinct groups of $n_1, \ldots, n_r$ each (with $n_1 + \ldots + n_r = n$), then the number of such groupings is

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1! n_2! \ldots n_r!}$$

Note: this counts the different orderings of the groups; if we don't care about group ordering either, we have to divide by $r!$.

Exercise 1. *In a game of basketball, 12 children divide themselves into teams of 3 each. How many divisions are possible?*

Solution 1. Following the multinomial reasoning, the number of team divisions is $\binom{12}{3,3,3,3}$. Now if we consider the teams to be indistinguishable, we should also divide by 4! to remove their permutations, so the final answer is $\binom{12}{3,3,3,3}\frac{1}{4!}$.

Conditional probability is important, useful, and important.

The probability of E conditioned on F is:

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

The chain/multiplication rule:

$$P(A \text{ and } B) = P(A \cap B) = P(A)P(B|A)$$

$$P(A_1 \cap \cdots \cap A_n) = P(A_1)P(A_2|A_1)P(A_3|A_2, A_1) \ldots P(A_n|A_{n-1}, \ldots, A_1).$$

Exercise 2. *An urn contains 6 white and 9 black balls. If 4 balls are to be randomly selected without replacement, what is the probability that the first 2 selected are white and the last 2 black?*

Solution 2. Without the chain rule:

- choose 2 white balls: $\binom{6}{2}$,
- choose 2 black balls: $\binom{9}{2}$,
- choose one of the two permutations for the white balls (i.e. W_1W_2 and W_2W_1): $\binom{2}{1} = 2$,
- choose one of the two permutations for the black balls (i.e. W_1W_2 and W_2W_1): $\binom{2}{1} = 2$.

Hence, the probability is

$$\frac{\binom{6}{2}\binom{9}{2}\cdot 2\cdot 2}{\binom{15}{4}}.$$

With the chain rule: (omiting specific definition of the events) the probability we seek is

$$\Pr(WWBB) = \Pr(W)\Pr(W|W)\Pr(B|WW)\Pr(BB|WW) = \frac{6}{15}\cdot\frac{5}{14}\cdot\frac{9}{13}\cdot\frac{8}{12}.$$

Exercise 3. *A total of n balls are sequentially and randomly chosen, without replacement, from an urn containing r red and b blue balls ($n \le r+b$). Given that k of the n balls are blue, what is the conditional probability that the first ball chosen is blue?*

Solution 3. Call B the event that the first ball is blue and call B_k the event that we select k blue balls in our selection of n balls. We are looking for

$$P(B|B_k) = \frac{P(B_k|B)P(B)}{P(B_k)}.$$

Let's figure out the terms on the right hand side. $P(B) = \frac{b}{r+b}$ for obvious reasons. $P(B_k) = \frac{\binom{b}{k}\binom{r}{n-k}}{\binom{b+r}{n}}$, where we choose k blue balls out of the b and then we choose $n-k$ red balls out of r. Similarly, $P(B_k|B)$ is similar to $P(B_k)$, but with one blue ball already taken out, hence $P(B_k|B) = \frac{\binom{b-1}{k-1}\binom{r}{n-k}}{\binom{b+r-1}{n-1}}$. Therefore

$$P(B|B_k) = \frac{\frac{\binom{b-1}{k-1}\binom{r}{n-k}}{\binom{b+r-1}{n-1}}\frac{b}{r+b}}{\frac{\binom{b}{k}\binom{r}{n-k}}{\binom{b+r}{n}}} = \frac{k}{n}.$$

Exercise 4. *An ordinary deck of cards of 52 playing cards is randomly divided into 4 piles of 13 each. What is the probability that there is exactly one ace in each pile?*

Solution 4. Let's set up the following events: E_1 - ace of spades is in a pile, E_2 - ace of diamonds is in a different pile from ace of spades, E_3 - ace of hearts is in a different pile from ace of spades and ace of diamonds, E_4 - all aces are in different piles.

Note that the probability we seek is

$$P(E_4) = P(E_1\cap E_2\cap E_3\cap E_4) = P(E_1)P(E_2|E_1)P(E_3|E_2,E_1)P(E_4|E_3,E_2,E_1).$$

$P(E_1)$ is clearly just 1. $P(E_2|E_1) = \frac{39}{51}$ because there are 39 (i.e. $51-12$) slots left for the ace of diamonds to be placed, once the ace of spades has been placed in a pile. $P(E_3|E_2,E_1) = \frac{26}{50}$, since there are $2\cdot 13$ slots left for the ace of hearts out of 50 to be in a different pile. $P(E_4|E_3,E_2,E_1) = \frac{13}{49}$ for similar reasons. Hence

$$P(E_4) = \frac{39}{51}\frac{26}{50}\frac{13}{49} \approx \frac{3}{4}\frac{1}{2}\frac{1}{4} = \frac{3}{32} \approx 0.10.$$

The law of total probability for more than two hypotheses.

If Ω is the disjoint union of $B_1, B_2, B_3, \ldots$ and $A \subset \Omega$, then

$$P(A) = P(B1)P(A|B1) + P(B2)P(A|B2) + P(B3)P(A|B3) + \ldots$$

Exercise 5. *Suppose that we have 3 cards that are identical except that both sides of the first card are red, both sides of the second are black, and one side of the third is red and the other side is black. The 3 cards are mixed up in a hat, one is chosen at random, and is placed face up on a table. If the upper side of the chosen card is red, then what is the probability that the other side is red?*

Solution 5. Let us create the following events: BB we draw the black-black card from the hat, RB we draw the black-red card, and RR we draw the red-red card from the hat. We are interested in finding

$$P(RR|R) = \frac{P(R|RR)P(RR)}{P(R)},$$

where we have used Bayes theorem to get the right-hand side. We clearly have $P(R|RR) = 1$. Also $P(RR) = \frac{1}{3}$. Now we have

$$P(R) = P(R|RR)P(RR) + P(R|BR)P(BR) + P(R|BB)P(BB) = 1 \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} = \frac{1}{2}$$

and hence

$$P(RR|R) = \frac{1 \cdot \frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}.$$

More exercises.

Exercise 6. *What is the number of ways to get: a) two aces and three kings in a hand of six? b) two aces and three kings in a hand of six and have an ace and a king share a suit?*

Solution 6. a) Choose two aces: $\binom{4}{2}$. Choose three kings: $\binom{4}{3}$. Choose the wild card: $\binom{44}{1}$. Hence, the number of ways is: $\binom{4}{2}\binom{4}{3}\binom{44}{1}$.

b) Choose a suit for the same suit cards: $\binom{4}{1}$. Choose one more ace: $\binom{3}{1}$. Choose a king that is not the same suit as the extra ace just chosen: $\binom{2}{1}$. Choose a wild card: $\binom{44}{1}$. Hence, the number of ways is: $\binom{4}{1}\binom{3}{1}\binom{2}{1}\binom{44}{1}$.

Exercise 7. *Consider a group of 20 people. If every person shakes hands with another person, how many handshakes are there?*

Solution 7. This question is equivalent to asking: how many distinct pairs can we make out of 20 people? This is given by $\binom{20}{2} = \frac{20 \cdot 19}{2}$.

Exercise 8. *A person has 8 friends, but will only invite 5 to a party. (a) How many choices are there if 2 of the friends just broke up and will not show up together? (b) How many choices are there if 2 friends only attend events together?*

Solution 8. a) Split into two disjoint cases. First, the case that neither of the broken up friends get invited is $\binom{6}{5}$. The second is that one gets invited, so choose one $\binom{2}{1}$ and then choose from the rest of the guests for the 4 remaining spots $\binom{6}{4}$. Hence, there are $\binom{6}{5} + \binom{2}{1}\binom{6}{4}$ ways.

b) Similar disjoint cases. If neither get invited, then $\binom{6}{5}$. If both get invited, then we have 3 spots left for the six guests, so $\binom{6}{3}$. Hence, $\binom{6}{5} + \binom{6}{3}$.

Exercise 9. *What is the probability of getting a "straight" (a sequence of consecutive card values not all of the same suit) in a poker hand (5 cards)?*

Solution 9. First choose a straight location $\binom{10}{1}$ (a straight can start anywhere from ace (which can count as either a one or a 14) to 10, so there are 10 options). Then choose a suit for each of the cards $\binom{4}{1}^5$. Then cut out the possibilities with all card values to having the same suit: $\binom{4}{1}$. Hence, the number of ways is $\binom{10}{1}\left(\binom{4}{1}^5 - \binom{4}{1}\right)$.