Oct 2nd, Discussion 1
Counting

How many choices are possible for ...

1. one letter from A-Z and one digit from 0-9?
   Solution: by the basic principle of counting, we multiply together the sizes of the two sets to get the size of joint set of choices: $26 \times 10 = 260$.
2. a function from $\{A, \dots, Z\}$ to $\{0, \dots, 9\}$?
   Solution: a function is a choice of output for every input, hence, by the principle of counting, a choice of a function is the "product" of all the choices for every letter. This amounts to 26 multiplications of 10, hence the number of functions is 10^{26}.
3. a 7-place license plate of the form digit-letter-letter-letter-digit-digit-digit?
   Solution: again, by the basic principle of counting, we multiply together all the choices. We have 4 choices of digits and 3 choices of letters, hence there are $10^4 \times 26^3$ many possible license plates of this form (California).
4. a license plate as above, where you cannot use the same letter or digit twice?
   Solution: the new restriction in this problem makes the size of the sets decrease as we make our choices. Starting at the left most digit, we have 10 choices for slot 1 and 26 choices for slot 2, but once we get to the third slot, we must remove the letter chosen for slot 2 from our possibility set, leaving 25 choices. Similarly, the next slot only has 24 options. Following this reasoning, we obtain the following product as the total number of outcomes: $10 \cdot 26 \cdot 25 \cdot 24 \cdot 9 \cdot 8 \cdot 7 = \frac{10!}{6!} \cdot \frac{26!}{23!}$.
5. a 7-letter password, where you cannot use the same letter twice in a row?
   Solution: we have to be careful how we conceptualize the "experiments" in this case. It is important to settle on a choice of order for the experiments: let's say that we go from left to right in choosing the letters. For the first slot, we have 26 choices (since there are no nearby letters yet). The second slot must now choose from only 25, since the letter to its left can't be chosen again. For the third slot, the letter we put in the first slot comes back to our set of possibilities again, but the slot 2 letter leaves the set of possibilities; this leaves 25 letters as choices for the third slot. Similarly, the rest of the slots have 25 possibilities and therefore the number of outcomes is $26 \cdot 25^6$.
6. a password of 7 letters/digits, with at least one letter and at least one digit?
   Solution: a nice way to think about this problem is through complements. The idea is to count the total set and then subtract the number of outcomes we don't want, leaving us with the number of outcomes we do want. First, the total number of possibilities is 36^7. The outcomes we don't want are split into two classes: we have all letters or all digits. The number of outcomes with all letters is 26^7 and the number of outcomes with all digits is 10^7. Therefore, the answer is $36^7 - (26^7 + 10^7)$.
7. an arrangement of ten different math books on the shelf?
   Solution: this is a straightforward count of the permutations of ten objects, which we know is $10!$.

8. a team of five basketball players, from a group of twelve players?
   Solution: this is also a simple count of groups of 5 from a set of 12, which we know is $\binom{12}{5}$.
9. a partition of {Alice, Bob, Charlie, Dave} into two pairs?
   Solution: for this question, we have to be careful. It may appear to be a question about choosing two people from the set: making a pair constitutes choosing two people, which immediately determines the other pair. However, in this problem, it turns out that choosing two people and the complement of those two people is the exact same outcome. For example, the partition that places Alice and Bob on one team, is the same partition that places Charlie and Dave on one team. Hence, choosing "Alice and Bob" and choosing "Charlie and Dave" are the same. Therefore, we see that $\binom{4}{2}$ double counts and must be divided by two to give us the final answer of $\binom{4}{2}\frac{1}{2}$.
10. a partition of {Alice, Bob, Charlie, Dave} into two pairs, called team 1 and team 2?
    Solution: now in this question, because the teams are distinguished, the problem of choosing a partition is equivalent to the problem of choosing two people for, say, team 1 (i.e. the choices "Alice and Bob" and "Charlie and Dave" for team 1 are distinct). Hence, the number of partitions is $\binom{4}{2}$.

Uniform Probability Spaces

*Sometimes we assume that all outcomes in a sample space Ω are **equally likely**. In this case, for every event $A \subseteq \Omega$,*

$$P(A) = \frac{|A|}{|\Omega|} .$$

1. You roll two dice. What is the probability that they add up to 7?
   Solution: The number of two dice outcomes is 6^2. The number of ways to get two numbers to add up to 7 is 6 (we can manually count these: $(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)$). Hence, the probability is $\frac{1}{6}$.
2. You roll six dice. What is the probability that all numbers 1,2,3,4,5,6 appear?
   Solution: The total number of six dice outcomes is 6^6. The number of ways to get those numbers on six dice equals the number of permutations of those numbers, hence 6!. Hence, the probability is $\frac{6!}{6^6}$.
3. You toss 8 coins. What is the probability of 3 heads and 5 tails?
   Solution: The total number of outcomes is 2^8. The number of ways to get 3 heads out of 8 coin tosses is equivalent to $\binom{8}{3}$, since the counting problem is equivalent to choosing 3 of the 8 coins to be heads. Hence, the probability is $\frac{\binom{8}{3}}{2^8}$.
4. You roll ten dice. What is the probability that 6 appears exactly 5 times?
   Solution: The total number of outcomes is 6^{10}. Suppose we fix the location of all the sixes to be on the first 5 dice. Then there are 5^5 possibilities for the other dice (because they can't be sixes). Now we need to multiply this by the different arrangements of the sixes throughout the ten dice, which we know is $\binom{10}{5}$. Hence, the probability is $\frac{\binom{10}{5}5^5}{6^{10}}$.
5. What are the odds of winning the Powerball: guessing 5 numbers from $\{1,\dots,69\}$ and another one in $\{1,\dots,26\}$.
   Solution: the total number of possible powerball numbers is $69^5 \cdot 26$. The number of ways to guess them is 1. Hence, the probability is $\frac{1}{69^5 \cdot 26}$.
6. A deck of cards contains 52 different cards, 4 of which are aces. You deal the cards to 4 players, 13 cards to each one. What is the probability that each player gets one ace?
   Solution: The total number of ways to deal out the cards is 52!. If each player is forced to have an ace, then one of their cards is fixed. There are $\binom{13}{1}$ ways to choose which card is fixed for each player. There are also 4! ways to permute the aces among the 4 players. The rest of the cards are free to be anything from the remaining 48 cards, hence there 48! ways to choose the rest of the cards. Hence, the probability is $\frac{\binom{13}{1}^4 4!48!}{52!}$.