Discussion 4: October 23rd.

Topics.

We are now working through sections 4.1-4.5. 4.1 Random variables. 4.2 Discrete random variables. 4.3 Expected value. 4.4 Expectation of a function of random variables. 4.5 Variance.

- A *random variable* is a function from a probability space to another set: $X(w) : \Omega \to \mathbb{R}$.
- The *probability mass (density) function (pmf or pdf or simply density)* for a discrete random variable X is a function $p_X(k)$ such that $p_X(k) = \Pr(X = k)$.
- The *cumulative distribution function (cdf or simply distribution)* for a random variable X is a function $F_X(k) = \Pr(X \le k)$.
- The *expected value (or expectation or mean)* of a discrete random variable X is defined $\mathbb{E}[X] = \sum_{k \in \text{image(X)}} k \cdot \Pr(X = k)$.
- The *expected value (or expectation or mean)* of a function of discrete random variable X is defined $\mathbb{E}[f(X)] = \sum_{k \in \text{image(X)}} f(k) \cdot \Pr(X = k)$.
- The *variance* of a discrete random variable X is defined as $\mathbb{E}[(X - E[X])^2]$.

Problems.

Exercise 1. *Say that we experiment with tossing 3 fair coins. Let Y denote the number of heads in the outcome. What is the pdf of Y? What if we change the number of coins to n? To verify your answer, make sure the values sum to 1.*

Solution 1. We have that

$$\Pr(Y = 0) = \Pr(\{TTT\}) = \frac{1}{8} \tag{1}$$

$$\Pr(Y = 1) = \Pr(\{TTH, THT, HTT\}) = \frac{3}{8} \tag{2}$$

$$\Pr(Y = 2) = \Pr(\{THH, HTH, HHT\}) = \frac{3}{8} \tag{3}$$

$$\Pr(Y = 3) = \Pr(\{HHH\}) = \frac{1}{8}, \tag{4}$$

$$\tag{5}$$

which clearly sums to 1.

If we have n coins, then we have a sample space of size 2^n, where each event has probability 2^{-n}. The number of ways to get k heads in n tosses is given by $\binom{n}{k}$, hence we have

$$\Pr(Y = k) = \binom{n}{k} \frac{1}{2^n}.$$

Summing these together we get the expression

$$\Pr(Y \le n) = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{2^n},$$

which sums to 1 by the binomial theorem.

Exercise 2. *Say that we repeatedly flip a biased coin with probability p of getting heads. Let Y denote the number of flips before we observe a heads. What is the pdf of Y? To verify your answer, make sure the values sum to* 1*.*

Solution 2. Let's start with a few examples and look for a pattern.

$$\Pr(Y = 0) = 0$$

$$\Pr(Y = 1) = \Pr(\{H\}) = p$$
$$\Pr(Y = 2) = \Pr(\{TH\}) = (1-p)p$$
$$\Pr(Y = 3) = \Pr(\{TTH\}) = (1-p)^2 p,$$

which leads us to

$$\Pr(Y = k) = \Pr(\{TT \ldots TH\}) = (1-p)^{k-1} p.$$

Let's sum these to verify

$$\Pr(Y \ge 1) = \sum_{k=1}^{\infty} (1-p)^{k-1} p = p \sum_{k=0}^{\infty} (1-p)^k = \frac{p}{1-(1-p)} = \frac{p}{p} = 1.$$

Exercise 3. *Suppose that for some random variable we have a density function $p(k) = 2^{-k}$ for $k \ge 1$. Verify that this is in fact a density that sums to 1. Now suppose we have the density $p(k) = A \cdot k^{-\alpha}$ with $\alpha > 1$ for $k \ge 1$. Note that here we have a constant $A \in \mathbb{R}$ that is unspecified. Find an expression for A that guarantees that this is an actual density function, by summing the density and setting it equal to* 1*.*

Solution 3. Let's sum the first density

$$\sum_{k \ge 1} \left(\frac{1}{2}\right)^k = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1.$$

Now let's sum the second

$$\sum_{k \geq 1} A \left(\frac{1}{k}\right)^{\alpha} = AC < \infty,$$

because the p-series converge for $p > 1$. Hence, $A = \frac{1}{C}$, works. (This sum can be calculated for some values of α, for example for $\alpha = 2$, $C = \frac{\pi^2}{6}$ [see the Basel problem], but for many others all we know is that it's finite and can only get numerical approximations).