Discussion 3: October 16th.

Topics.

The midterm will cover sections 1.2-1.4, 2.2-2.5, 3.2-3.4.

1.2 Basic principle of counting. 1.3 Permutations. 1.4 Combinations. 2.2 Sample spaces and events. 2.3 Axioms of probability. 2.4 Simple propositions. 2.5 Sample spaces with equally likely outcomes. 3.2 Conditional probability. 3.3 Bayes formula. 3.4 Independent events.

2.3 Axioms of probability.

(1) For any $E \subset S$, $0 \leq P(E) \leq 1$. (2) $P(S) = 1$. (3) For any disjoint $E_1, E_2, \ldots$, we have $P(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$.

2.4 Simple propositions.

Complement: $P(E^c) = 1 - P(E)$. Monotonicity: if $A \subset B$ then $P(A) \leq P(B)$. Disjoint union additivity: for any A, B, we have $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Inclusion-exclusion identity: $P(A_1 \cup \cdots \cup A_n) = \sum_{i=1}^{n} P(E_i) - \sum_{i_1 < i_2} P(E_{i_1} \cap E_{i_2}) + \sum_{i_1 < i_2 < i_3} P(E_{i_1} \cap E_{i_2} \cap E_{i_3}) + \cdots + (-1)^{n+1} P(E_1 \cap \cdots \cap E_n)$.

3.2 Conditional probability. $P(A|B) = \frac{P(A \cap B)}{P(B)}$.

3.3 Bayes formula. $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$.

3.4 Independent events. A and B are independent if and only if $P(A \cap B) = P(A)P(B)$.

Problems.

Exercise 1. *Show that if A, B are independent and $A \subset B$ then $P(A) = 0$ or $P(B) = 1$.*

Solution 1. Since $A \subset B$, we have $P(A) = P(A \cap B) = P(A)P(B)$, from which the result follows by algebra.

Exercise 2 (4h independence, conditional probability, and regular counting)**.** *Say that we perform independent trials of rolling a pair of fair dice. An outcome is the sum of the dice. What is the probability that an outcome of 5 appears before an outcome of 7? (This can be solved via counting or via conditional probabilities.)*

Solution 2. The conditional probability approach goes as follows: set up the events E, F, G, H to denote the event that 1) we get a 5 before a 7, 2) we get a 5 on the first trial, 3) we get a 7 one the first trial, 4) we get neither on the first trial, respectively. Then, note that

$$\Pr(E) = \Pr(E|F)\Pr(F) + \Pr(E|G)\Pr(G) + \Pr(E|H)\Pr(H).$$

It is easy to check that $\Pr(F) = \frac{4}{36}, \Pr(G) = \frac{6}{36}, \Pr(H) = \frac{26}{36}$. Note that $\Pr(E|F) = 1, \Pr(E|G) = 0, \Pr(E|H) = \Pr(E)$, because F guarantees E, G cancels E, and H makes us run for E again and doesn't affect the probability (because of independence of trials). Substituting in we get

$$\Pr(E) = \frac{4}{36} + \Pr(E) \cdot \frac{26}{36},$$

from which it follows that $\Pr(E) = \frac{4}{10}$.

Exercise 3 (Book 3.57). *A simplified model for the movement of the price of a stock supposes that on each day the stock either moves up by 1 with probability p or goes down by 1 with probability $1-p$. The changes from day to day are assumed to be independent. (a) What is the probability that after 2 days the stock will be at its original price? (b) What is the probability that after 3 days the stock will have increased by 1 unit? (c) Given that after 3 days the stock price has gone up by 1, what is the probability that it went up on the first day?*

Solution 3. If S_n denotes the price value at day n, we assume $S_0 = 0$, and that $S_n = \sum_{i=1}^n X_i$, where X_i is the outcome of the stock movement on day i, we get:

a) $\Pr(S_2 = 0) = \Pr(X_1X_2 = (1,-1)) + \Pr(X_1X_2 = (-1,1)) = 2p(1-p)$.

b) $\Pr(S_3 = 1) = \Pr(X_1X_2X_3 = (1,1,-1)) + \Pr(X_1X_2X_3 = (1,-1,1)) + \Pr(X_1X_2X_3 = (-1,1,1)) = 3p^2(1-p)$.

c) $\Pr(S_1 = 1|S_3 = 1) = \frac{\Pr(S_1=1,S_3=1)}{\Pr(S_3=1)} = \frac{\Pr(X_1,X_2,X_3=(1,1,-1))+\Pr(X_1,X_2,X_3=(1,-1,1))}{\Pr(S_3=1)} = \frac{2p^2(1-p)}{3p^2(1-p)} = \frac{2}{3}$.

Exercise 4 (Book 3.58). *Suppose that we want to generate the outcome of a flip of a fair coin, but all we have access to is a biased coin that lands heads with probability p instead of $\frac{1}{2}$. Consider the following procedure: 1) Flip the coin, 2) Flip the coin again, 3) If both flips land on heads or both tails, return to step 1, 4) Let the result of the last flip be the result of the experiment. Show that the result is equally likely to be heads or tails.*

Solution 4. a) Let us calculate the probability of getting heads from this procedure P. This probability can be calculated by conditioning on three events: the event that the first trial gives us the same coin values E, the event that the first trial gives us different coin values F. So we get

$$\Pr(P = H) = \Pr(P = H|E)\Pr(E) + \Pr(P = H|F)\Pr(F),$$

where $\Pr(P = H|E) = \Pr(P = H)$ because the trials are independent and $\Pr(P = H|F) = \frac{1}{2}$ because exactly half the times we get the outcome TH guaranteeing a heads.

Exercise 5 (Theoretical question 3.18). *Let Q_n be the probability that no run of 3 consecutive heads appears in n flips of a fair coin. Show that $Q_n = \frac{1}{2}Q_{n-1} + \frac{1}{4}Q_{n-2} + \frac{1}{8}Q_{n-3}$ and $Q_0 = Q_1 = Q_2 = 1$. Then find Q_8. (Hint: condition on the first tail).*

Solution 5. Say that E_n is the event that there is no run of 3 consecutive heads in n coin flips and say that T_i is the event that the first tail occurs on the i-th trial. Then

$$\Pr(E_n) = \sum_{i=1}^{n} \Pr(E_n|T_i)\Pr(T_i) = \Pr(E_n|T_1)\Pr(T_1) + \Pr(E_n|T_2)\Pr(T_2) + \Pr(E_n|T_2)\Pr(T_2)$$

since for $i \geq 4$ we have $\Pr(E_n|T_i) = 0$ (because if the first tails occurs later than 4 spaces out, then we have a run of 3 consecutive heads). Finally, note that $\Pr(T_i) = 2^{-i}$ by the independent of fair coins, and $\Pr(E_n|T_1) = \Pr(E_{n-1})$, $\Pr(E_n|T_2) = \Pr(E_{n-2})$, and $\Pr(E_n|T_3) = \Pr(E_{n-2})$ because the problem reduces down to getting no runs of heads in a smaller set of coins we start with T, HT, or HHT. The result then follows by plugging these values in.

Further ideas for review: (1) Prof. Chaim's review questions (solutions will be up tomorrow). (2) Discussions 1 and 2 problems (solutions will be up tomorrow). (3) Review homeworks.