1 A few more combinatorics problems.

Exercise 1. In a game of basketball, 10 children divide themselves into teams of 5 each. How many divisions are possible?

Solution 1. The number of labeled teams (A,B) corresponds to the number of ways that we can make team A (since team B is determined once A is chosen). Hence, the different team A's is $\binom{10}{5}$. However, if we are interested in divisions, or unlabeled teams, then we should divide out by 2 to account for the permutation of the team labels, so the number of divisions is $\frac{1}{2} \cdot \binom{10}{5}$.

Exercise 2 (Poker Hands.)**.** Suppose we play a game of poker in which players are dealt a hand of 5 cards out of a standard deck with the values $\{A, K, Q, J, 10, 9, 8, 7, 6, 5, 4, 3, 2\}$ and 4 suits of each (clubs, spades, diamonds, and hearts). Calculate the probability of obtaining the following hands: (a) two pairs (defined as two pairs of the same value and another card of a different value, e.g. $\{J, J, 9, 9, 3\}$), (b) flush (defined as five cards all of the same suit, e.g. $\{K, 2, 4, Q, 10\}$ with all cards spades).

Solution 2. (a) Let's focus on the numerators:

- Choose two card values to be pairs: $\binom{13}{2}$
- Choose a card value for the fifth card: $\binom{11}{1}$
- Choose two suits for the first pair of cards: $\binom{4}{2}$
- Choose two suits for the second pair of cards: $\binom{4}{2}$
- Choose a suit for the fifth card: $\binom{4}{1}$

Hence, the probability of two pairs is

$$\frac{\binom{13}{2}\binom{11}{1}\binom{4}{2}\binom{4}{2}\binom{4}{1}}{\binom{52}{5}}.$$

(b) Similarly:

- Choose a suit for the cards: $\binom{4}{1}$
- Choose five card values for the cards: $\binom{13}{5}$

Hence, the probability of a flush is

$$\frac{\binom{4}{1}\binom{13}{5}}{\binom{52}{5}}.$$

Exercise 3. A new employee checks in the hats of n people at a restaurant, forgetting to put the claim check number in their hats. When customers return for their hats, the employee gives them back hats chosen at random from the remaining hats. What is the probability that no one receives the correct hat? (Hint: use inclusion-exclusion on the bad properties, $E_i^c = \{\text{customer } i \text{ gets their hat back}\}$.)

Solution 3. To use the inclusion-exclusion formula, we calculate the following probabilities:

$$\begin{aligned} P(E_i^c) &= \frac{(n-1)!}{n!} \\ P(E_i^c \cap E_j^c) &= \frac{(n-2)!}{n!} \\ &\vdots \\ P(E_{i_1}^c \cap \ldots \cap E_{i_m}^c) &= \frac{(n-m)!}{n!}. \end{aligned}$$

Applying the inclusion-exclusion formula, we obtain:

$$\begin{aligned} P(\cap_{i=1}^n E_i) &= 1 - P(\cup_{i=1}^n E_i^c) \\ &= 1 - \left(\sum_i P(E_i) - \sum_{i<j} P(E_i \cap E_j) + \cdots + (-1)^n P(E_1 \cap \ldots \cap E_n) \right) \\ &= 1 - \left(\binom{n}{1} \frac{(n-1)!}{n!} - \binom{n}{2} \frac{(n-2)!}{n!} + \cdots + (-1)^n \binom{n}{n} \frac{1}{n!} \right) \\ &= 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!} \end{aligned}$$

As $n \to \infty$, this approaches $e^{-1} \approx 0.36$ (using that the Taylor series for $e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$).

Exercise 4 (Tricky.)**.** How many solutions does $x_1 + x_2 + x_3 = 11$ have, where x_1, x_2, x_3 are nonnegative integers with $x_1 \leq 3, x_2 \leq 4$, and $x_3 \leq 6$? (Hint: first, think of 11 as a sum of 11 units $1 + 1 + \cdots + 1$ and the variables x_1, x_2, x_3 as the result of partitioning these units into three sections by inserting two bars between the units. For example, $1 + 1 + |1 + 1 + 1 + |1 + \cdots + 1$ corresponds to $(x_1, x_2, x_3) = (2, 3, 6)$. Now, using the fact that the number of solutions is identical to the number of ways places we can place the bars, find a formula for the number of solutions. Then use inclusion-exclusion on the bad properties: $E_1^c = \{x_1 > 3\}, E_2^c = \{x_2 > 4\}, E_3^c = \{x_3 > 6\}$.)

Solution 4. First note that the number of ways to create a string out of 11 ones and 2 bars is $\binom{11+2}{11}$ (since the string has $11 + 2$ characters total and we must choose where to place 11 units; see the wiki article on "stars and bars" for a more in-depth explanation). Following this, note these counts:

$$\begin{aligned}
|N| &= \{\vec{x} \mid \langle \vec{x}, \vec{1} \rangle = 11,\ \vec{x} \in \mathbb{Z}^3\} = \binom{11+2}{11} = 78 \\
|E_1^c| &= \{\vec{x} \mid \langle \vec{x}, \vec{1} \rangle = 11,\ \vec{x} \in \mathbb{Z}^3,\ x_1 > 3\}| = \binom{7+2}{7} = 36 \\
|E_2^c| &= \{\vec{x} \mid \langle \vec{x}, \vec{1} \rangle = 11,\ \vec{x} \in \mathbb{Z}^3,\ x_2 > 4\}| = \binom{6+2}{6} = 28 \\
|E_3^c| &= \{\vec{x} \mid \langle \vec{x}, \vec{1} \rangle = 11,\ \vec{x} \in \mathbb{Z}^3,\ x_3 > 6\}| = \binom{4+2}{4} = 15 \\
|E_1^c \cap E_2^c| &= \{\vec{x} \mid \langle \vec{x}, \vec{1} \rangle = 11,\ \vec{x} \in \mathbb{Z}^3,\ x_1 > 3,\ x_2 > 4\}| = \binom{2+2}{2} = 6 \\
|E_1^c \cap E_3^c| &= \{\vec{x} \mid \langle \vec{x}, \vec{1} \rangle = 11,\ \vec{x} \in \mathbb{Z}^3,\ x_1 > 3,\ x_3 > 6\}| = \binom{0+2}{0} = 1 \\
|E_2^c \cap E_3^c| &= \{\vec{x} \mid \langle \vec{x}, \vec{1} \rangle = 11,\ \vec{x} \in \mathbb{Z}^3,\ x_2 > 4,\ x_3 > 6\}| = 0 \\
|E_1^c \cap E_2^c \cap E_3^c| &= \{\vec{x} \mid \langle \vec{x}, \vec{1} \rangle = 11,\ \vec{x} \in \mathbb{Z}^3,\ x_1 > 3,\ x_2 > 4,\ x_3 > 6\}| = 0
\end{aligned}$$

Combining these together with the inclusion-exclusion formula, we get

$$P(E_1 \cap E_2 \cap E_3) = \frac{78 - 36 - 28 - 15 + 6 + 1 + 0 - 0}{78} = \frac{1}{13}.$$