Sec.4.4: Coordinate Systems

Objective: To compute the coordinates of a given vector in a vector space V relative to a given basis $\mathcal{B}$ of V.

Introduction

V: vector space

$\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_p\}$ is a basis for V

if
- $\mathcal{B}$ is lin. indep.
- $\mathcal{B}$ is a spanning set of V

$$V = \text{Span}(\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_p)$$

For any $\vec{u}$ in V

$$\longrightarrow \vec{u} = c_1 \vec{b}_1 + c_2 \vec{b}_2 + \cdots + c_p \vec{b}_p$$

Theorem (Unique Representation Theorem)

Let V be a vector space and let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ be a basis for V. Then for every $\vec{v}$ in V, there exists **a unique** set of scalars $c_1, c_2, \ldots, c_n$ such that

$$\vec{v} = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_n\vec{b}_n.$$

$$\vec{v} = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_n\vec{b}_n$$

$$- \quad \vec{v} = d_1\vec{b}_1 + d_2\vec{b}_2 + \cdots + d_n\vec{b}_n$$

$$\Longrightarrow \vec{0} = (c_1 - d_1)\vec{b}_1 + (c_2 - d_2)\vec{b}_2 + \cdots + (c_n - d_n)\vec{b}_n$$

Note that $\{\vec{b}_1, \ldots, \vec{b}_n\}$ is lin. indep

$$\Longrightarrow c_1 - d_1 = 0, \quad c_2 - d_2 = 0, \ldots, \quad c_n - d_n = 0$$

$$c_1 = d_1, \qquad c_2 = d_2, \qquad c_n = d_n.$$

Definition (Coordinates of a vector)

Let V be a vector space and let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ be a basis for V. Let $\vec{v}$ be a vector in V with

$$\vec{v} = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_n\vec{b}_n$$

The coordinates of $\vec{v}$ relative to $\mathcal{B}$ (or **$\mathcal{B}$-coordinates of $\vec{v}$**) are the weights $c_1, c_2, \ldots, c_n$. We write

$$\left[\ \vec{v}\ \right]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \quad \text{(note that this is vector in } \mathbb{R}^n\text{)}$$

We define the map

$$\begin{aligned} T : V &\to \mathbb{R}^n \\ \vec{v} &\mapsto \left[\ \vec{v}\ \right]_{\mathcal{B}} \end{aligned}$$

The map T is called **coordinate mapping** (determined by $\mathcal{B}$).

Examples

1. Let $\mathcal{E} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ be the standard basis for $\mathbb{R}^3$. What is the $\mathcal{E}$-coordinates vector of $\vec{v} = \begin{bmatrix} -2 \\ 7 \\ 6 \end{bmatrix}$? (Generalize it for the standard basis $\mathcal{E} = \{\vec{e}_1, \ldots, \vec{e}_n\}$ of $\mathbb{R}^n$).

We have

$$\vec{v} = \begin{bmatrix} -2 \\ 7 \\ 6 \end{bmatrix} = -2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 7\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 6\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = -2\vec{e}_1 + 7\vec{e}_2 + 6\vec{e}_3$$

So $[\vec{v}]_{\mathcal{E}} = \begin{bmatrix} -2 \\ 7 \\ 6 \end{bmatrix} = \vec{v}$.

Let $\mathcal{E} = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ be the standard basis for $\mathbb{R}^n$. Then, the $\mathcal{E}$-coordinates of $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is $[\vec{x}]_{\mathcal{E}} = \vec{x}$.

2. Find the coordinates of a polynomial $p(t) = a_0 + a_1 t + \cdots + a_n t^n \in \mathbb{P}_n$ relative to the standard basis $\mathcal{B} = \{1, t, \ldots, t^n\}$.

$$[p(t)]_{\mathcal{B}} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$ $\in$ a vector in $\mathbb{R}^{n+1}$

$\mathbb{P}_2$, $\mathcal{B} = \{1, t, t^2\}$

$p(t) = -7 + 3t^2$

$= -7 + 0t + 3t^2$

$[p(t)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$

3. Let $\vec{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$.
   - Show that $\mathcal{B}$ is a basis for $\mathbb{R}^2$.
   - Compute the $\mathcal{B}$-coordinates of $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. $\quad \vec{u} = [\vec{u}]_{\mathcal{E}}$
   - Find the vector $\vec{x}$ such that $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. $\leftarrow \quad \vec{x} = [\vec{x}]_{\mathcal{E}}$

- Since the the determinant $\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$, $\mathcal{B}$ is basis for $\mathbb{R}^2$.
- If $x_1\vec{b}_1 + x_2\vec{b}_2 = \vec{u}$ then $[\vec{u}]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

  Augmented matrix

$$\left[\begin{array}{cc|c} \vec{b}_1 & \vec{b}_2 & \vec{u} \end{array}\right] = \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & 2 \end{array}\right] \Rightarrow x_2 = 2, x_1 = 1 + x_2 = 3$$

  So $3\vec{b}_1 + 2\vec{b}_2 = \vec{u}$, and

$$[\vec{u}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

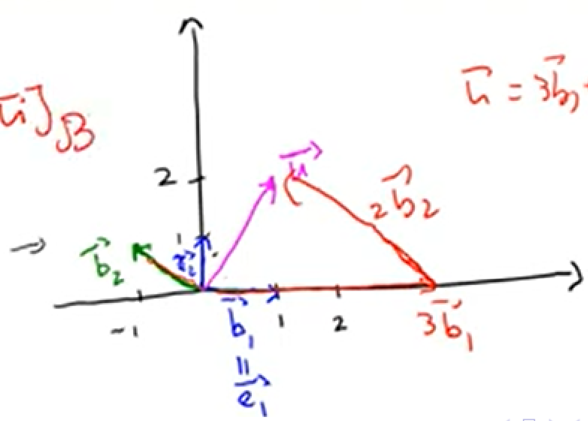
- $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, so

$$\vec{x} = -\vec{b}_1 + \vec{b}_2 = -\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$\vec{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad [\vec{u}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$[\vec{u}]_{\mathcal{E}} \longleftrightarrow [\vec{u}]_{\mathcal{B}}$

$\vec{u} = 3\vec{b}_1 + 2\vec{b}_2$



$\vec{b}_1 = \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = [\vec{u}]_{\mathcal{E}}$

Coordinates in $\mathbb{R}^n$

Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ be a basis for $\mathbb{R}^n$. Let $\vec{u}$ be in $\mathbb{R}^n$ with

$$\vec{u} = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_n\vec{b}_n \quad \text{that is} \quad [\ \vec{u}\ ]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

The above equality is the same as

$$\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \vec{u}$$

That is $\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \end{bmatrix} [\ \vec{u}\ ]_{\mathcal{B}} = \vec{u}$

Since $\mathcal{B}$ is basis, the matrix $\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \end{bmatrix}$ is invertible. So we also have

$$[\ \vec{u}\ ]_{\mathcal{B}} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \end{bmatrix}^{-1} \vec{u}$$

Definition

Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ be a basis for $\mathbb{R}^n$. The matrix $P_{\mathcal{B}} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \end{bmatrix}$ is called the change-of-coordinates matrix from $\mathcal{B}$ to the standard basis in $\mathbb{R}^n$.

If $\vec{u} \in \mathbb{R}^n$, then

- $\vec{u} = P_{\mathcal{B}} \ [\vec{u}]_{\mathcal{B}}$
- $[\vec{u}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1} \ \vec{u}$.

Coordinate Mapping

Proposition

Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ be a basis for a vector space V. Let $T : V \to \mathbb{R}^n$, with $T(\vec{v}) = [\ \vec{v}\ ]_{\mathcal{B}}$ be the coordinate mapping determined by $\mathcal{B}$. Then, T is a one-to-one linear transformation onto $\mathbb{R}^n$.

Facts

Since the coordinate mapping $T : V \to \mathbb{R}^n$ is a linear transformation, we have

1. $[\ \vec{u} + \vec{v}\ ]_{\mathcal{B}} = [\ \vec{u}\ ]_{\mathcal{B}} + [\ \vec{v}\ ]_{\mathcal{B}}$ for $\vec{u}, \vec{v}$ in V.
2. $[\ c\vec{v}\ ]_{\mathcal{B}} = c[\ \vec{v}\ ]_{\mathcal{B}}$ for $\vec{v}$ in V and scalar c.
3. $[\ \vec{v}\ ]_{\mathcal{B}} = \vec{0}$ if and only if $\vec{v} = \vec{0}$.

Definition (Isomorphic vector spaces)

Let V and W be vector spaces. We say that V and W are isomorphic if there exists a one-to-one linear transformation $T : V \to W$ onto W (we say that T is an isomorphism). In this case we write $V \cong W$.

Note

Let V be a vector space with a basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ with n vectors, then

1. V isomorphic to $\mathbb{R}^n$ as the coordinate mapping $T : V \to \mathbb{R}^n$, $T(\vec{v}) = [\ \vec{v}\ ]_{\mathcal{B}}$ is an isomorphism. This makes V act like $\mathbb{R}^n$.
2. A set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is linearly independent in V if and only if the set $\{[\ \vec{v}_1\ ]_{\mathcal{B}}, [\ \vec{v}_2\ ]_{\mathcal{B}}, \ldots, [\ \vec{v}_p\ ]_{\mathcal{B}}\}$ is linearly independent in $\mathbb{R}^n$.

Note

For the set $\mathbb{P}_n$ of polynomials of degree at most n, the standard basis is $\mathcal{B} = \{1, t, t^2, \ldots, t^n\}$. If $p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$, then

$$\left[\ p(t)\ \right]_{\mathcal{B}} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}.$$

The mapping coordinate $T : \mathbb{P}_n \to \mathbb{R}^{n+1}$, with $T(p(t)) = \left[\ p(t)\ \right]_{\mathcal{B}}$, is an isomorphism so we can study $\mathbb{P}_n$ as $\mathbb{R}^{n+1}$.

Example

Show that $p_1(t) = 1 + 2t^2$, $p_2(t) = 4 + t + 5t^2$ and $p_3(t) = 3 + 2t$ are linearly dependent in $\mathbb{P}_2$.

- Consider the standard basis $\mathcal{B} = \{1, t, t^2\}$ for $\mathbb{P}_2$.
- By the definition of linear independence, we should solve the equation $x_1 p_1(t) + x_2 p_2(t) + x_3 p_3(t) = 0$.
  However, since $\mathbb{P}_2$ acts like $\mathbb{R}^3$, $\{p_1(t), p_2(t), p_3(t)\}$ is linearly dependent if and only if $\{[p_1(t)]_{\mathcal{B}}, [p_2(t)]_{\mathcal{B}}, [p_3(t)]_{\mathcal{B}}\}$ is linearly dependent.
- We have
  $$[p_1(t)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \ [p_2(t)]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}, \ [p_3(t)]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

Let's check if the coordinate vectors are linearly dependent.

$$\left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & 5 & 0 & 0 \end{array}\right] \xrightarrow{\text{REF}} \left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

There is a free variable (so there re nonzero solutions), so $\{[p_1(t)]_{\mathcal{B}}, [p_2(t)]_{\mathcal{B}}, [p_3(t)]_{\mathcal{B}}\}$ is linearly dependent. So, $\{p_1(t), p_2(t), p_3(t)\}$ is linearly dependent.
Parametric vector form of the solutions of the corresponding equation

$$\vec{x} = x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$$

So $5p_1(t) - 2p_2(t) + p_3(t)$ is a dependence relation among $p_1(t), p_2(t)$ and $p_3(t)$