1. Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ and $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$ where $\vec{b}_1 = \begin{bmatrix} 7 \\ 5 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} -3 \\ -1 \end{bmatrix}, \vec{c}_1 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$, and $\vec{c}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

   (a) Find the change of coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

   Computation: There are two methods:

   - Fist method: using definition, that is $P_{\mathcal{C}\leftarrow\mathcal{B}} = [[\vec{b}_1]_{\mathcal{C}} \; [\vec{b}_2]_{\mathcal{C}}]$ and compute each column by solving $x_1\vec{c}_1 + x_2\vec{c}_2 = \vec{b}_1$ and $\alpha_1\vec{c}_1 + \alpha_2\vec{c}_2 = \vec{b}_2$ so that $[\vec{b}_1]_{\mathcal{C}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $[\vec{b}_2]_{\mathcal{C}} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$
   - Second method: by using $\left[\begin{array}{cc|cc} \vec{c}_1 & \vec{c}_2 & \vec{b}_1 & \vec{b}_2 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{c|c} I_2 & P_{\mathcal{C}\leftarrow\mathcal{B}} \end{array}\right]$.

   For example, using the second method

$$\left[\begin{array}{cc|cc} 1 & -2 & 7 & -3 \\ -5 & 2 & 5 & -1 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|cc} 1 & 0 & -3 & 1 \\ 0 & 1 & -5 & 2 \end{array}\right]$$

$$\boxed{P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix}}$$

   (b) Find the change of coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$.

   Computation: There are three methods:

   - Fist method: using definition, that is $P_{\mathcal{B}\leftarrow\mathcal{C}} = [[\vec{c}_1]_{\mathcal{B}} \; [\vec{c}_2]_{\mathcal{B}}]$ and compute each column by solving $x_1\vec{b}_1 + x_2\vec{b}_2 = \vec{c}_1$ and $\alpha_1\vec{b}_1 + \alpha_2\vec{b}_2 = \vec{c}_2$ so that $[\vec{c}_1]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $[\vec{c}_2]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$
   - Second method: by using $\left[\begin{array}{cc|cc} \vec{b}_1 & \vec{b}_2 & \vec{c}_1 & \vec{c}_2 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{c|c} I_2 & P_{\mathcal{B}\leftarrow\mathcal{C}} \end{array}\right]$.
   - Third method: $P_{\mathcal{B}\leftarrow\mathcal{C}} = P^{-1}_{\mathcal{C}\leftarrow\mathcal{B}}$

   For example for the third method, we have computed $P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix}$ from the previous question. So

$$P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 1 \\ -5 & 3 \end{bmatrix}$$

$$\boxed{P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} -2 & 1 \\ -5 & 3 \end{bmatrix}}$$

2. Note that $\mathcal{B} = \{1 - 2t + t^2, 3 - 5t + 4t^2, 2t + 3t^2\}$ is basis for $\mathbb{P}_2$.

   (a) Find the change-of-coordinate matrix from $\mathcal{B}$ to the standard basis $\mathcal{C} = \{1, t, t^2\}$.

   Computation: By definition

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = [[p_1(t)]_{\mathcal{C}} \; [p_2(t)]_{\mathcal{C}} \; [p_3(t)]_{\mathcal{C}}] = \begin{bmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{bmatrix}$$

$$\boxed{P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{bmatrix}}$$

   (b) Let $p(t)$ be in $\mathbb{P}_2$ such that the $\mathcal{B}$-coordinate vector of $p(t)$ is $[p(t)]_{\mathcal{B}} = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$. Determine the $\mathcal{C}$-coordinate vector of $p(t)$.

   Computation: We have

$$\begin{aligned} [p(t)]_{\mathcal{C}} &= P_{\mathcal{C}\leftarrow\mathcal{B}} \; [p(t)]_{\mathcal{B}} \\ &= \begin{bmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \end{aligned}$$

$$\boxed{[p(t)]_{\mathcal{C}} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}}$$