Sec.3.2: Properties for Determinants

Objective: Computing determinants using elementary row operations.

If A and B are $n \times n$ matrices that are row equivalent, then their determinant are related (one can be expressed in terms of the other).

Theorem

Let A and B be $n \times n$ matrices. Then

1. If $A \xrightarrow{R_i \leftrightarrow R_j} B$, then $\det(A) = -\det(B)$.
2. If $A \xrightarrow{R_i + cR_j \to R_i} B$, then $\det(A) = \det(B)$.
3. if $A \xrightarrow{cR_i \to R_i} B$, then $c\det(A) = \det(B)$.

2nd Method to computing the determinant of A:

- The strategy is to reduce A to echelon form
- then use the fact that the determinant of a triangular matrix is the product of the diagonal entries
- Avoid row scaling when row-reducing

Example

Compute the determinant of $A = \begin{pmatrix} 0 & 3 & -1 \\ 1 & -2 & 1 \\ -1 & -4 & 3 \end{pmatrix}$.

$$A = \begin{pmatrix} 0 & 3 & -1 \\ 1 & -2 & 1 \\ -1 & -4 & 3 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \text{(this will reverses the sign)}$$

$$B = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 3 & -1 \\ -1 & -4 & 3 \end{pmatrix} \xrightarrow{R_3 + R_1 \to R_3} \text{(does not change)}$$

$\det(B) = \det(C)$

$$C = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 3 & -1 \\ 0 & -6 & 4 \end{pmatrix} \xrightarrow{R_3 + 2R_2 \to R_3} \text{(does not change)}$$

$\det(C) = \det(U)$

$$\begin{pmatrix} 1 & -2 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & 2 \end{pmatrix} = U$$

$\det(A) = -\det(B) = -\det(C) = -\det(U) = -(1)(3)(2) = -6$

Case of Elementary Matrices

Recall the definition of an elementary matrix and determine its determinant.

- Elementary matrix: $I_n \xrightarrow{R_i + cR_j \to R_i} E$, $\det(E) = \det(I_n) = 1$.
- Elementary matrix: $I_n \xrightarrow{R_i \leftrightarrow R_j} E$, $\det(E) = -\det(I_n) = -1$
- Elementary matrix: $I_n \xrightarrow{rR_i \to R_i} E$, $\det(E) = r\det(I_n) = r.1 = r$

Note

Let E be an elementary matrix. Then

$$\det(E) = \begin{cases} 1 & \text{if } E \text{ is from a row replacement} \\ -1 & \text{if } E \text{ is from an interchange} \\ r & \text{if } E \text{ is from a row scaling by } r \end{cases}$$

The general case

Let U be a REF of a matrix A obtained through the use or row replacements and row interchanges only. Express $\det(A)$ in terms of $\det(U)$.

Reduce A to a REF U without row scaling

$$A \xrightarrow{R_1} \xrightarrow{R_2} \cdots \xrightarrow{R_k} U \leftarrow \text{REF}$$

assume: There were r number of interchanging.

$$\det(A) = \underbrace{(-1)(-1)\cdots(-1)}_{r \text{ times}} \det(U)$$

Proposition

Suppose A is an $n \times n$ matrix and U is an echelon form of A obtained through the use of row replacements and row interchanges **only**. Let r denote the number of row interchanges used to obtain U. Then

$$\begin{aligned}\det(A) &= (-1)^r u_{11} u_{22} \cdots u_{nn} \\ &= \begin{cases} (-1)^r \cdot (\text{product of pivots in } U) & \text{when } A \text{ is invertible} \\ 0 & \text{when } A \text{ is not invertible} \end{cases}\end{aligned}$$

Theorem

A square matrix A is invertible if and only if $\det(A) \neq 0$.

Corollary

Let A be a square matrix.

1. Then $\det(A) = 0$ when the columns of A are linearly dependent.
2. If A has two rows or two columns the same, then $\det(A) = 0$ and A is nonivertible (singular).

Proposition

Let A and B be $n \times n$ matrices.

1. $\det(A^T) = \det(A)$.
2. $\det(AB) = \det(A)\det(B)$

Note

In general $\det(A + B) \neq \det(A) + \det(B)$.

Examples: Consequences of $\det(AB) = \det(A)\det(B)$

1. Let A be an invertible matrix. Determine $\det(A^{-1})$.
2. Determine $\det(B^n)$ for any positive integer n.

① Let A an invertible matrix (A^{-1} exists)

$AA^{-1} = I_n \Rightarrow \det(AA^{-1}) = \det(I_n)$

$\Rightarrow \det(A)\det(A^{-1}) = \det(I_n)$

$\Rightarrow \det(A)\det(A^{-1}) = 1.$

$\Rightarrow \det(A^{-1}) = \dfrac{1}{\det(A)}$

$(\det(A) \neq 0)$

② B is an $n\times n$ matrix. , $k \geq 0$

$\det(B^k) = \det(\underbrace{B\times B\times\cdots\times B}_{k \text{ times}}) = \underbrace{\det(B)\det(B)\cdots\det(B)}_{k \text{ times}}$

$= (\det(B))^k$

Practice

1. Assume $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$, find $\begin{vmatrix} a & b & c \\ 7d+g & 7e+h & 7f+i \\ g & h & i \end{vmatrix}$
2. .
   Find the determinant by row reduction to echelon form.

$$\begin{vmatrix} 1 & 3 & 1 & 2 \\ 0 & 1 & 2 & -5 \\ -1 & 2 & 9 & -4 \\ 4 & -2 & -2 & 4 \end{vmatrix}$$

1. Assume $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$, find $\begin{vmatrix} a & b & c \\ 7d+g & 7e+h & 7f+i \\ g & h & i \end{vmatrix}$

$$\underbrace{\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}}_{A} \xrightarrow{7R_2 \to R_2} \underbrace{\begin{pmatrix} a & b & c \\ 7d & 7e & 7f \\ g & h & i \end{pmatrix}}_{B} \xrightarrow{R_2+R_3 \to R_2} \underbrace{\begin{pmatrix} a & b & c \\ 7d+g & 7e+h & 7f+i \\ g & h & i \end{pmatrix}}_{C}$$

$7\det(A) = \det(B) = \det(C)$

$\det(C) = 7\det(A) = 7 \cdot 7 = 49.$