Sec.4.7: Change of Basis

Objective: To determine the change-of-coordinates matrix from a basis $\mathcal{B}_1$ to a different basis $\mathcal{B}_2$.

Recall

Let V be a vector space with a basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$. Then if

$$\vec{u} = x_1\vec{b}_1 + x_2\vec{b}_2 + \cdots + x_n\vec{b}_n$$

is a vector in V, then the coordinate vector of $\vec{u}$ relative to $\mathcal{B}$ (or the $\mathcal{B}$-coordinate vector of $\vec{u}$) is given by

$$[\vec{u}]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Let $\mathcal{C} = \{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\}$ be another basis for V.
Given the $\mathcal{B}$-coordinate vector of $\vec{u}$, how to compute $\mathcal{C}$-coordinate vector of $\vec{u}$?

Example

Let V be a 2-dimensional vector space and let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ and $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$ be bases for V. Suppose

$$\vec{b}_1 = 4\vec{c}_1 + \vec{c}_2 \quad \text{and} \quad \vec{b}_2 = -6\vec{c}_1 + \vec{c}_2$$

Let $\vec{u}$ be in V such that

$$\vec{u} = 3\vec{b}_1 + \vec{b}_2, \quad \text{i.e.} \quad [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Find $[\vec{u}]_{\mathcal{C}}$.

Theorem

Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ and $\mathcal{C} = \{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\}$ be bases of an n-dimensional vector space V. Then there is a unique matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ such that

$$[\vec{u}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}\ [\vec{u}]_{\mathcal{B}}$$

The columns of $P_{\mathcal{C}\leftarrow\mathcal{B}}$ are the $\mathcal{C}$-coordinates of the vectors in the basis $\mathcal{B}$. That is

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} [\vec{b}_1]_{\mathcal{C}} & [\vec{b}_2]_{\mathcal{C}} & \cdots & [\vec{b}_n]_{\mathcal{C}} \end{bmatrix}$$

Definition (Change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$)

The matrix $P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} [\vec{b}_1]_{\mathcal{C}} & [\vec{b}_2]_{\mathcal{C}} & \cdots & [\vec{b}_n]_{\mathcal{C}} \end{bmatrix}$ is called the **change-of-coordinates** matrix from $\mathcal{B}$ to $\mathcal{C}$. So for every $\vec{u} \in V$:

$$[\vec{u}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}\ [\vec{u}]_{\mathcal{B}}$$

Note

The columns of $P_{\mathcal{C}\leftarrow\mathcal{B}}$ are linearly independent, hence $P_{\mathcal{C}\leftarrow\mathcal{B}}$ is invertible. Multiplying the equality

$$[\vec{u}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}\ [\vec{u}]_{\mathcal{B}}$$

by $\left(P_{\mathcal{C}\leftarrow\mathcal{B}}\right)^{-1}$ we have

$$\left(P_{\mathcal{C}\leftarrow\mathcal{B}}\right)^{-1}\ [\vec{u}]_{\mathcal{C}} = \left(P_{\mathcal{C}\leftarrow\mathcal{B}}\right)^{-1}\ P_{\mathcal{C}\leftarrow\mathcal{B}}\ [\vec{u}]_{\mathcal{B}}$$

So

$$\left(P_{\mathcal{C}\leftarrow\mathcal{B}}\right)^{-1}\ [\vec{u}]_{\mathcal{C}} = [\vec{u}]_{\mathcal{B}}$$

Thus the matrix $\left(P_{\mathcal{C}\leftarrow\mathcal{B}}\right)^{-1}$ is the matrix that convert $\mathcal{C}$-coordinates to $\mathcal{B}$-coordinates, that is

$$P_{\mathcal{B}\leftarrow\mathcal{C}} = \left(P_{\mathcal{C}\leftarrow\mathcal{B}}\right)^{-1}$$

Change of Basis in $\mathbb{R}^n$

Let $\mathcal{E} = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ be the standard basis for $\mathbb{R}^n$, and let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ be a nonstandard basis for $\mathbb{R}^n$.
We have seen from Sec.4.4 that the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{E}$ is given by

$$P_{\mathcal{B}} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \end{bmatrix}$$

That is

$$P_{\mathcal{B}} = P_{\mathcal{E}\leftarrow\mathcal{B}}$$

Example

Let $\vec{b}_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$, $\vec{c}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, and $\vec{c}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$. Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ and $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$ be bases for $\mathbb{R}^2$. Find the change of coordinates matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$.

Observe from the previous example that

$$\left[\begin{array}{cc|cc} \vec{c}_1 & \vec{c}_2 & \vec{b}_1 & \vec{b}_2 \end{array}\right] \sim \left[\begin{array}{c|c} I_2 & P_{\mathcal{C}\leftarrow\mathcal{B}} \end{array}\right]$$

Note

In general, if $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ and $\mathcal{C} = \{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\}$ are two different bases for $\mathbb{R}^n$, then

$$\left[\begin{array}{ccc|ccc} \vec{c}_1 & \cdots & \vec{c}_n & \vec{b}_1 & \cdots & \vec{b}_n \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{c|c} I_n & P_{\mathcal{C}\leftarrow\mathcal{B}} \end{array}\right]$$

and

$$\left[\begin{array}{ccc|ccc} \vec{b}_1 & \cdots & \vec{b}_n & \vec{c}_1 & \cdots & \vec{c}_n \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{c|c} I_n & P_{\mathcal{B}\leftarrow\mathcal{C}} \end{array}\right]$$

Fact (Relation between $P_{\mathcal{B}} = P_{\mathcal{E}\leftarrow\mathcal{B}}$, $P_{\mathcal{C}} = P_{\mathcal{E}\leftarrow\mathcal{C}}$, and $P_{\mathcal{C}\leftarrow\mathcal{B}}$)

Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ and $\mathcal{C} = \{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\}$ be nonstandard bases for $\mathbb{R}^n$. For each $\vec{x}$ in $\mathbb{R}^n$, we have

$$\begin{aligned} \vec{x} &= P_{\mathcal{B}}\ [\vec{x}]_{\mathcal{B}} \\ \vec{x} &= P_{\mathcal{C}}\ [\vec{x}]_{\mathcal{C}} \end{aligned}$$

with $P_{\mathcal{B}} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \end{bmatrix}$ and $P_{\mathcal{C}} = \begin{bmatrix} \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \end{bmatrix}$. Therefore

$$[\vec{x}]_{\mathcal{C}} = P_{\mathcal{C}}^{-1}\ \vec{x} = P_{\mathcal{C}}^{-1}\ P_{\mathcal{B}}\ [\vec{x}]_{\mathcal{B}}$$

So

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = P_{\mathcal{C}}^{-1}\ P_{\mathcal{B}}$$