MATH 3110 - Spring 2023

Learning Activity 13

1. For an $n \times n$ matrix A, find a formula to compute the determinants of dA in terms of $\det(A)$, where d is a scalar. Show how to get to the formula.

   Answer: We have

   $$\det(dA) = d^n \det(A)$$

   Justification:: From A to dA, we multiply each row by d. That is there are n row scalings, by d, as A is an $n \times n$ matrix.

   $$A \xrightarrow{dR_1 \to R_1} \xrightarrow{dR_2 \to R_2} \cdots \xrightarrow{dR_n \to R_n} dA$$

2. Let C and D be 3×3 matrices. Write the determinant of D in terms of the determinant of C in the following cases (e.g. $\det(D) = 2\det(C)$). Do not compute the determinant of C. Justify your answers.

   (a) $D = C^2$

   Answer: $\det(D) = \det(C)^2$

   Justification

   $$\det(D) = \det(C^2) = \det(CC) = \det(C)\det(C) = \det(C)^2$$

   (b) $C = \begin{bmatrix} 1 & -1 & 1 \\ -3 & 5 & 2 \\ 2 & -1 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} -k & k & -k \\ -3 & 5 & 2 \\ 2 & -1 & 3 \end{bmatrix}$.

   Answer: $\det(D) = -k\det(C)$

   Justification: We have a single row operation $C \xrightarrow{-kR_1 \to R_1} D$. Hence $\det(D) = -k\det(C)$.

   (c) $D = 3C$

   Answer: note that we assume (from the question) that C and D are 3×3 matrices. We have

   $$\det(D) = 3^3 \det(C) = 27\det(C)$$

   Justification: $D = 3C$ means that every row of D is 3 times the row of C, so we have

   $$C \xrightarrow{3R_1 \to R_1} \xrightarrow{3R_2 \to R_2} \xrightarrow{3R_3 \to R_3} D$$

   (d) $C = \begin{bmatrix} 2 & 0 & -3 \\ -1 & -1 & 0 \\ 2 & -3 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2-k & -k & -3 \\ -1 & -1 & 0 \\ 2 & -3 & 1 \end{bmatrix}$.

   Answer: $\det(D) = \det(C)$.

   Justification: We have a single row replacement $C \xrightarrow{R_1 + kR_2 \to R_1} D$, so $\det(C) = \det(D)$.

   (e) $C = \begin{bmatrix} 0 & 4 & -3 \\ 1 & 2 & -3 \\ 0 & -1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 4 & -3 \\ 0 & -1+4k & 1-3k \end{bmatrix}$

   Answer: $\det(D) = -\det(C)$.

   Justification: We have a row interchanging and a row replacement $C \xrightarrow{R_1 \leftrightarrow R_2} \xrightarrow{R_3 + kR_2 \to R_3} D$, hence $\det(C) = -\det(D)$. They might compute the determinant of C and D directly by using cofactor expansion and then they write the relation between the two determinants. That is still fine

3. Compute the determinant of the matrix M **using elementary row operations (all row operation must be shown)**.

$$\begin{bmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 5 & 4 & -3 \\ -3 & -7 & -5 & 2 \end{bmatrix} \quad R_3 - 2R_1 \to R_3 \quad \begin{bmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & -1 & -2 & 5 \\ -3 & -7 & -5 & 2 \end{bmatrix}$$

$$R_4 + 3R_1 \to R_4. \quad \begin{bmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & -1 & -2 & 5 \\ 0 & 2 & 4 & -10 \end{bmatrix} \quad R_3 + R_2 \to R_3$$

$$\begin{bmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & -10 \end{bmatrix} \quad R_4 - 2R_2 \to R_4 \quad \begin{bmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We can stop here because of the row of zeros and conclude that $\det(M) = 0$

We used only row replacements, so we have

$$\det(M) = (1)(1)(0)(0) = 0$$