MATH 3110 - Spring 2023

Learning Activity 15 - Solutions

1. Determine if W is a subspace of $\mathbb{R}^3$ in the following cases. **Justify your answer.** (Mark the right answer by X and then explain your answer).

To determine if a given subset is a subspace, we can consider two methods:

- By using the definition: showing that $\vec{0} \in W$, W is closed under addition (if $\vec{u}, \vec{v} \in W$, then $\vec{u}+\vec{v} \in W$), and W is closed under scalar multiplication (if $c \in \mathbb{R}$ and $\vec{u} \in W$, the $c\vec{v} \in W$)). So if one of these is not satisfied, then W is not a vector space.
- By writing W as spanned of some vectors, i.e. $W = \text{Span}(\vec{v}_1, \dots, \vec{v}_r)$.

(a) $W = \left\{ \begin{bmatrix} a+c \\ a-b \\ b+c \end{bmatrix} \mid a, b, c \text{ in } \mathbb{R} \right\}$.

It's a subspace of $\mathbb{R}^3$	It's not a subspace of $\mathbb{R}^3$.
X	

Justification:

First Method: We use the definition (checking the three conditions).

- Choosing $a = b = c = 0$, we have $\vec{0} = \begin{bmatrix} 0+0 \\ 0-0 \\ 0+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. So $\vec{0}$ is in W.
- For $\vec{u} = \begin{bmatrix} a+c \\ a-b \\ b+c \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} a'+c' \\ a'-b' \\ b'+c' \end{bmatrix}$ in W, we have $\vec{u}+\vec{v} = \begin{bmatrix} (a+a')+(c+c') \\ (a+a')-(b+b') \\ (b+b')+(c+c') \end{bmatrix}$. So, $\vec{u}+\vec{v}$ still has the form of the elements of W so it's in W. Thus W is closed under addition.
- For $\vec{u} = \begin{bmatrix} a+c \\ a-b \\ b+c \end{bmatrix}$ and a scalar α, we have $\alpha\vec{u} = \begin{bmatrix} \alpha a+\alpha c \\ \alpha a-\alpha b \\ \alpha b+\alpha c \end{bmatrix}$. So $\alpha\vec{v}$ still have the form of the elements of W so it is in W. Thus, W is closed under scalar multiplication.

Second method: We write W as span of some vectors in $\mathbb{R}^3$. For every $\vec{u}$ in W, we have

$$\vec{u} = \begin{bmatrix} a+c \\ a-b \\ b+c \end{bmatrix} = \begin{bmatrix} a \\ a \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -b \\ b \end{bmatrix} + \begin{bmatrix} c \\ 0 \\ c \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + c\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_2, \quad a, b, c \in \mathbb{R}$$

Therefore, the element of W are linear combinations of $\vec{v}_1, \vec{v}_2, \vec{v}_3$, so $W = \text{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$. Therefore, W is a subspace of $\mathbb{R}^3$.

(b) $W = \left\{ \begin{bmatrix} 1 \\ a+3b \\ 3a-2b \end{bmatrix} \mid a, b \text{ in } \mathbb{R} \right\}$.

It's a subspace of $\mathbb{R}^3$	It's not a subspace of $\mathbb{R}^3$.
	X

Justification: We check the three conditions (first method).

Note that the first entry of the elements of W must be 1. Since the first entry of the zero vector is not 1, the zero vector is not in W. It follows that W is not a subspace of $\mathbb{R}^3$.

(c) $W = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \mid a, b, c \in \mathbb{R} \text{ and } a+b+c = 1 \right\}$.

It's a subspace of $\mathbb{R}^3$	It's not a subspace of $\mathbb{R}^3$.
	X

Justification: We check the three conditions (first method).

Note that the sum of the entries of an element of W must be 1. Since the sum of the entries of the zero vector is $0 \neq 1$, the zero vector is not in W. Therefore W is not a subspace of $\mathbb{R}^3$.

2. Let $M_{2\times 2}(\mathbb{R})$ be the set of all 2×2 matrices (with real number entries). That is

$$M_{2\times 2}(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \text{ in } \mathbb{R} \right\}$$

The set $M_{2\times 2}(\mathbb{R})$ with the zero matrix $\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, with the usual addition and scalar multiplication is a vector space. Let $H = \left\{ \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \mid a, c, d \text{ in } \mathbb{R} \right\}$ be a subset of $M_{2\times 2}(\mathbb{R})$.

(a) Give a nonzero element of H.

Answer:

Note that the (1,2)-entry of an element of H must be 0. So the matrix must be a 2×2 matrix such that $a_{12} = 0$.

The matrix $A = \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix}$ is a nonzero element of H since its (1,2)-entry is 0.

(b) Give an element of $M_{2\times 2}(\mathbb{R})$ which is not in H.

Answer:

The matrix must be a 2×2 matrix such that $a_{12} \neq 0$.

The matrix $B = \begin{bmatrix} 0 & 1 \\ -1 & 5 \end{bmatrix}$ is an element of $M_{2\times 2}(\mathbb{R})$ but not in H, since its (1,2)-entry is not zero.

(c) Is H a subspace of $M_{2\times 2}(\mathbb{R})$? Justify your answer.

It's a subspace of $M_{2\times 2}(\mathbb{R})$	It's not a subspace of $M_{2\times 2}(\mathbb{R})$.
X	

Justification:

First method (using the definition). We check the three conditions.

- Since the $(1,2)$-entry of the zero matrix $\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is 0, the zero matrix is in H.
- If $A = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} a' & 0 \\ c' & d' \end{bmatrix}$ are in H, then $A+B = \begin{bmatrix} a+a' & 0 \\ c+c' & d+d' \end{bmatrix}$ is also in H, since its $(1,2)$-entry is 0. So H is closed under addition.
- If α is a scalar and $A = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$ is in H, then $\alpha A = \begin{bmatrix} \alpha a & 0 \\ \alpha c & \alpha d \end{bmatrix}$ is also in H, since its $(1,2)$-entry is in 0. So H is closed under scalar multiplication.

Since the three conditions are satisfied, H is a subspace of $M_{2\times 2}(\mathbb{R})$.

Second method (write H as spanned by some matrices). For any element of H, we have

$$\begin{bmatrix} a & 0 \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

So every element of H is a linear combination of the matrices $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

It follows that $H = \text{Span}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$. Thus, H is a subspace of $M_{2\times 2}(\mathbb{R})$.