1. Note that the set $\mathcal{B} = \{1 - 2t + t^2, 3 - 5t + 4t^2, 2t + 3t^2\}$ is a basis for $\mathbb{P}_2$. Compute the $\mathcal{B}$-coordinates vector for $p(t) = -1 + 2t$.

   Computation: If $p(t) = x_1\,(1-2t+t^2) + x_2\,(3-5t+4t^2) + x_3\,(2t+3t^2)$ then the $\mathcal{B}$-coordinates of $p(t)$ is $[p(t)]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

   Solve the equation

$$\begin{aligned}(x_1 - 2x_1t + x_1t^2) + (3x_2 - 5x_2t + 4x_2t^2) + (2x_3t + 3x_3t^2) &= -1 + 2t \\ (x_1 + 3x_2) + (-2x_1 - 5x_2 + 2x_3)t + (x_1 + 4x_2 + 3x_3)t^2 &= -1 + 2t\end{aligned}$$

   So

$$\begin{aligned} x_1 + 3x_2 \quad &= -1 \\ -2x_1 - 5x_2 + 2x_3 &= 2 \\ x_1 + 4x_2 + 3x_3 &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ -2 & -5 & 2 & 2 \\ 1 & 4 & 3 & 0 \end{array}\right] \xrightarrow{REF} \left[\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

   So $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$

$$\boxed{[p(t)]_{\mathcal{B}} = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}}$$

2. The set $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$, where $\vec{b}_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix}, \vec{b}_3 = \begin{bmatrix} 2 \\ -2 \\ 8 \end{bmatrix}$, is a basis for $\mathbb{R}^3$.

   (a) Find the $\mathcal{B}$-coordinates vector of $\vec{x} = \begin{bmatrix} -7 \\ 8 \\ -10 \end{bmatrix}$.

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   Computation: If $\vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2 + c_3\vec{b}_3$ then $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$. Solve the vector equation.

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & -7 \\ -1 & 3 & -2 & 8 \\ -2 & 4 & 8 & -10 \end{array}\right] \xrightarrow{REF} \left[\begin{array}{ccc|c} 1 & -2 & 2 & -7 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 12 & -24 \end{array}\right]$$

   So

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$$

$$\boxed{[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}}$$

   (b) Find the vector $\vec{u}$ such that $[\vec{u}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$.

   Computation: We have

$$\vec{u} = -1\vec{b}_1 + 0\vec{b}_2 + 3\vec{b}_3 = -\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + \vec{0} + 3\begin{bmatrix} 2 \\ -2 \\ 8 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \\ 26 \end{bmatrix}$$

$$\boxed{\vec{u} = \begin{bmatrix} 5 \\ -5 \\ 26 \end{bmatrix}}$$