Sec.4.6: The Rank of a Matrix

A a matrix

$\dim(Nul(A)) = \#$ free variable of $A\vec{x} = \vec{0}$

$\dim(Col(A)) = \#$ pivot columns

$= \#$ pivot variable of $A\vec{x} = \vec{0}$

Main Objective: to understand the "Rank Theorem" which is a relationship between the $\dim(Col(A)) = \dim(Row(A))$ and $\dim(Nul(A))$.

The Row Space

Let A be an $m \times n$ matrix. Then we can view A as a collection of rows instead of a collection of columns.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \begin{matrix} \leftarrow r_1 \\ \leftarrow r_2 \\ \leftarrow r_3 \end{matrix} \qquad A = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

Note that if A is an $m \times n$ matrix then each row of A has n entries, so we can identify each row as a vector in $\mathbb{R}^n$.

Definition

Let A be an $m \times n$ matrix. The row space of A, denoted by $\text{Row}(A)$, is the set of all linear combinations of the rows of A.

That is if $A = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{bmatrix}$, then $\text{Row}(A) = \text{Span}(r_1, r_2, \ldots, r_m)$.

So $\text{Row}(A)$ is a subspace of $\mathbb{R}^n$, since each $r_i \in \mathbb{R}^n$.

Note

If $A = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{bmatrix}$, then $A^T = \begin{bmatrix} r_1 & r_2 & \cdots & r_m \end{bmatrix}$ so $\text{Row}(A) = \text{Col}(A^T)$

$\text{Row}(A) = \text{Span}(r_1, r_2, \ldots, r_m)$
$\shortparallel$
$\text{Col}(A^T) = \text{Span}(r_1, r_2, \ldots, r_m)$

Theorem

- If two matrices A and B are row equivalent then $\mathrm{Row}(A) = \mathrm{Row}(B)$.
- In the previous case, if B is in row echelon form then the nonzero rows of B form a basis for $\mathrm{Row}(A)$ and $\mathrm{Row}(B)$.

Example

Find a basis for $\mathrm{Row}(A)$ where $A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & -1 & 5 \\ 0 & 2 & 4 \end{bmatrix}$.

Example

The matrix $A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & -1 & 5 \\ 0 & 2 & 4 \end{bmatrix}$ is row equivalent to $B = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

- $\mathrm{Row}(A) = \mathrm{Row}(B)$.
- the set $\{(1, -2, 3), (0, 1, 2)\}$ is a basis for $\mathrm{Row}(A)$ (and for $\mathrm{Row}(B)$).

dim (Row(A)) = 2 = # nonzero rows of B

= # pivots

= # pivot columns

= dim(Col(A))

Note

The previous Theorem implies that the dimension of $\mathrm{Row}(A)$ equals the numbers of pivots in A, which is the same as the dimension of $\mathrm{Col}(A)$, i.e.

$$\dim(\mathrm{Row}(A)) = \dim(\mathrm{Col}(A)) = \#\text{pivots}$$

Row (A) = Col (A^T)

dim (Row (A)) = dim (Col A^T)

= dim (Col(A))

Example

Find bases for $\mathrm{Row}(A), \mathrm{Col}(A)$ and $\mathrm{Nul}(A)$ where

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}.$$

We have
$$A \xrightarrow{REF} B = \begin{array}{c} \begin{array}{ccccc} & & x_3 & & x_5 \end{array} \\ \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

dim (Row(A)) = 3

A basis for $\mathrm{Row}(A)$ is $\{(1,3,-5,1,5),(0,1,-2,2,-7),(0,0,0,-4,20)\}$

A basis for $\mathrm{Col}(A)$ is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}$

dim (Col(A)) = 3

The parametric vector form of the solutions for $A\vec{x} = \vec{0}$ is given by

$$\vec{x} = x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix}, \quad x_3, x_5 \in \mathbb{R}$$

So a basis for $\mathrm{Nul}(A)$ is given by $\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$

dim (Nul(A)) = 2.

The rank of a matrix

Definition

Let A be a matrix. The **rank** of A is the dimension of the column space of A (which is the same as the dimension of $\mathrm{Row}(A)$).

$$\mathrm{rank}(A) = \dim(\mathrm{Row}(A)) = \dim(\mathrm{Col(A)})$$

If A is an $m \times n$ matrix, then $\mathrm{rank}(A) \leq n$ and $\mathrm{rank}(A) \leq m$.
$\Rightarrow$ The maximum number of linearly independent columns of A and the maximum number of linearly independent rows of A are the same.

Example

What is the rank of the matrix $A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$?

$\dim(\mathrm{Row}(A)) = \dim(\mathrm{Col}(A)) = 3$ (the REF of A has 3 pivots). Hence $\mathrm{rank}(A) = 3$.

Theorem (Rank-Nullity Theorem)

Let A be an $m \times n$ matrix, then

$$\mathrm{rank}(A) + \dim(\mathrm{Nul}(A)) = n$$

If A has n columns then

- $n = \#$ variables in $A\vec{x} = \vec{0}$
- $\mathrm{rank}(A) = \dim(\mathrm{Col}(A)) = \#$ pivot variables in $A\vec{x} = \vec{0}$
- $\dim(\mathrm{Nul}(A)) = \#$ free variables in $A\vec{x} = \vec{0}$.

$$\underbrace{\#\text{ pivot variables}}_{\mathrm{rank}(A)} + \underbrace{\#\text{ free variables}}_{\dim(\mathrm{Null}(A))} = \underbrace{\#\text{ variables}}_{n}$$

$$\mathrm{rank}(A) + \dim(\mathrm{Null}(A)) = n.$$

Examples

1. If A is a 7×9 matrix with a two-dimensional null space, what is the rank of A?
   By the rank Theorem

$$\begin{aligned} \mathrm{rank}(A) + \dim(\mathrm{Nul}(A)) &= 9 \\ \mathrm{rank}(A) + 2 &= 9 \end{aligned}$$

   Then $\mathrm{Rank}(A) = 9 - 2 = 7$

2. Could a 6×9 matrix B have a two-dimensional null space?
   If $\dim(\mathrm{Nul}(B)) = 2$, then

$$\mathrm{rank}(B) + 2 = 9$$

   So $\mathrm{rank}(B) = \dim(\mathrm{Col}(B)) = 9 - 2 = 7$. But the columns of B are vectors in $\mathbb{R}^6$ so $\dim(\mathrm{Col}(B))$ cannot exceed 6. So B cannot have a two-dimensional null space.

Theorem (Invertible Matrix Theorem (continued))

Let A be an $n \times n$ matrix. Then, the following statements are equivalent:

1. A is invertible
2. $\mathrm{Col}(A) = \mathbb{R}^n$
3. $\dim(\mathrm{Col}(A)) = n$
4. $\mathrm{rank}(A) = n$
5. $\dim(\mathrm{Nul}(A)) = 0$.
6. $\mathrm{Nul}(A) = \{\vec{0}\}$