Sec.4.3: Linearly Independent Sets, Bases

Objective: To determine if a given subset S of a vector space V is a basis for V.

Definition (Linearly independent set)

A subset $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ of a vector space V is linearly independent if the equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_p\vec{v}_p = \vec{0}$$

has only the trivial solution $x_1 = x_2 = \cdots = x_p = 0$.
Otherwise, S is said to be linearly dependent.

As in the case of $\mathbb{R}^n$, if V is a vector space, then

1. A single vector $\vec{v}$ in V is linearly independent if and only if $\vec{v} \neq \vec{0}$.
2. A set of two vectors is linearly dependent if and only if one of the vectors is a scalar multiple of the other. $\begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -4 \\ 2 & -2 \end{bmatrix}$, $A_2 = 2A_1$ Dependent
3. Any set of vectors containing the zero vector is linearly dependent.

$S = \{1 + t^2 + 3t^3, 0, t^2 - t^3\} \Rightarrow$ Dependent as contains zero vector

Theorem

Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ be a subset of a vector space V, with $p \geq 2$ and $\vec{v}_1 \neq \vec{0}$. Then S is linearly dependent if and only if there is some $\vec{v}_j$, $j > 1$, such that $\vec{v}_j$ is a linear combination of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{j-1}$.

Show that $S = \left\{ A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$ is lin. independent

$$x_1 A_1 + x_2 A_2 + x_3 A_3 + x_4 A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & x_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ x_3 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & x_4 \end{bmatrix} = \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}$$

$$\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$x_1, x_2, x_3, x_4 = 0$ is only solution, so

$\{A_1, A_2, A_3, A_4\}$ is lin. independent

Examples

1. Consider the vector space $\mathbb{P}_2$ (the set of all polynomials of degree at most 2). Let

$$p_1(t) = t - 1, \quad p_2(t) = t^2 + 2t - 3, \quad p_3(t) = t^2 - 1.$$

Determine if $\{p_1(t), p_2(t), p_3(t)\}$ is linearly independent or not.

Consider the equation $x_1p_1(t) + x_2p_2(t) + x_3p_3(t) = 0$.

$$\begin{aligned} (x_1t - x_1) + (x_2t^2 + 2x_2t - 3x_2) + (x_3t^2 - x_3) &= 0 \\ (-x_1 - 3x_2 - x_3) + (x_1 + 2x_2)t + (x_2 + x_3)t^2 &= 0 \end{aligned}$$

So

$$\begin{aligned} -x_1 - 3x_2 - x_3 &= 0 \\ x_1 + 2x_2 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$$

The corresponding augmented matrix is reduced to the following REF

$$\left[\begin{array}{ccc|c} -1 & -3 & -1 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{\text{REF}} \left[\begin{array}{ccc|c} -1 & -3 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since there is a free variable, the system has nonzero (nontrivial) solutions. That is $x_1p_1(t) + x_2p_2(t) + x_3p_3(t) = 0$ has nontrivial solutions, so the set $\{p_1(t), p_2(t), p_3(t)\}$ is linearly dependent.

Check if the following sets are linearly independent.

2. In $\mathbb{P}_2$, $S = \{p_1(t) = 2 + t, p_2(t) = 2t + t^2\}$.

Since $p_1(t)$ and $p_2(t)$ are not scalar multiple of each other, they are linearly independent.

3. In $M_{2\times2}(\mathbb{R})$, $S = \{A_1 = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}, A_2 = \begin{bmatrix} 2 & -4 \\ 0 & -6 \end{bmatrix}\}$.

Since $A_2 = -2A_1$, A_1 and A_2 are (scalar) multiple of each other, hence there are linearly dependent.

4. In $\mathcal{C}((0,1))$, $S = \{\sin t, \cos t\}$.

4. $\{\cos t, \sin t\}$ is linearly independent in $\mathcal{C}([0,1])$ since $\cos t$ and $\sin t$ are not multiple of one another.

Definition (Basis of a Vector space)

An indexed subset $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_p\}$ of a vector space V is a **basis** for V if $\mathcal{B}$ is a **linearly independent** set and V **is spanned by** $\mathcal{B}$, that is

$$V = \mathrm{Span}(\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_p)$$

Similarly if H is a subspace of V, then a basis for H is collection of vectors $\mathcal{B}_H = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r\}$ in H that is linearly independent and $H = \mathrm{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r)$.

if $\vec{u}$ in V, $\vec{u} = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_p\vec{b}_p$

$E = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is basis for $\mathbb{R}^2$
$\quad\ \ \vec{e}_1 \quad\ \vec{e}_2$

- Lin. Ind. as e_1 not multiple of e_2
- For any $\begin{bmatrix} a \\ b \end{bmatrix}$ in $\mathbb{R}^2$, $\begin{bmatrix} a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix} = a\vec{e}_1 + b\vec{e}_2$

$\Rightarrow \mathbb{R}^2 = \mathrm{Span}(\vec{e}_1, \vec{e}_2)$

$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$ is basis for $\mathbb{R}^2$?

- b_1 and b_2 ind.
- $\begin{bmatrix} a \\ b \end{bmatrix}$ in $\mathbb{R}^2$, $\begin{bmatrix} a \\ b \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2\begin{bmatrix} \ \\ 1 \end{bmatrix}$

$\therefore$ Basis for $\mathbb{R}^2$

Examples

1. Let $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \end{bmatrix}$ be an invertible $n \times n$ matrix. Explain why $\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\}$ is basis for $\mathbb{R}^n$.

   (In particular the set $\left\{ \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, \vec{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^n$ (called standard basis))

If $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \end{bmatrix}$ is invertible, then, by the Invertible matrix theorem

- $\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\}$ is linearly independent.
- $\mathbb{R}^n = \mathrm{Span}(\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n)$

So $\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\}$ is a basis for $\mathbb{R}^n$.

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2. Find a basis for $\mathbb{P}_n$.

- If $x_0 \cdot 1 + x_1 t + x_2 t^2 + \cdots + x_n t^n = 0$ then $x_0 = x_1 = \cdots = x_n = 0$. So $\{1, t, t^2, \dots, t^n\}$ is linearly independent.
- For any polynomial $p(t)$ in $\mathbb{P}_n$, we have

$$p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$$

So $\mathbb{P}_n = \mathrm{Span}(1, t, t^2, \dots, t^n)$

Facts

Let $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$ be a set of vectors in $\mathbb{R}^n$.

1. If $p > n$, then there are more vectors than entries, so S is linearly dependent. Thus S is not a basis for $\mathbb{R}^n$.
2. If $p < n$, then $\mathbb{R}^n \neq \mathrm{Span}(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p)$, so S is not a basis for $\mathbb{R}^n$.

$\mathbb{R}^2$

$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \end{bmatrix} \right\}$ $\quad p = 3 > n = 2$

$\Longrightarrow$ $\mathcal{B}$ is lin. dependent

$\Longrightarrow$ $\mathcal{B}$ is not a basis for $\mathbb{R}^2$.

$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \right\}$ $\quad p = 2 < n = 3 \quad \mathbb{R}^3 \neq \mathrm{Span}(\vec{v}_1, \vec{v}_2)$

The Spannning Set Theorem

Example

Let $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$ be vectors in $\mathbb{R}^3$ and let $H = \mathrm{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ be a subspace of $\mathbb{R}^3$. Is $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ a basis for H ? If not find a basis for H.

- $H = \mathrm{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ so $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a spanning set for H.
- Augmented matrix of $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$.

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ -1 & 0 & 2 & 0 \\ 3 & 3 & -3 & 0 \end{array}\right] \xrightarrow{REF} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

There is a free variable so $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent and it is not a basis for H.

Theorem (The Spanning Set Theorem)

Let V be a vector space, let $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ be a set of vectors in V, and let $H = \mathrm{Span}(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n)$.

1. If there exists i such that $\vec{v}_i$ is a linear combination of the other vectors in S, then the set $S' = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots \vec{v}_n\}$ (we remove $\vec{v}_i$ from S) still spans H. That is

$$H = \mathrm{Span}(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots \vec{v}_n).$$

2. If $H \neq \{\vec{0}\}$, then some subset of S is a basis for H.

Example (continued)

Find a basis for the subspace in the previous example by using spanning set theorem.

For $H = \mathrm{Span}(\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix})$, we have seen

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ 3 & 3 & -3 & 0 \end{array}\right] \xrightarrow{REF} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \text{ so } \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

For $x_3 = 1$, $\vec{x} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ is a solution so $2\vec{v}_1 - \vec{v}_2 + \vec{v}_3 = \vec{0}$. So $\vec{v}_2 = 2\vec{v}_1 + \vec{v}_3$, and by the spanning set theorem, $\vec{v}_2$ can be removed from the spanning set of H.Thus

$$H = \mathrm{Span}(\vec{v}_1, \vec{v}_3)$$

In addition, $\vec{v}_1$ and $\vec{v}_3$ are not scalar multiple of each other so they are linearly independent.
Therefore, $\{\vec{v}_1, \vec{v}_3\}$ is a basis for H.

$$H = \text{Span}\left(\begin{array}{ccc} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \\ \begin{bmatrix}1\\0\\1\end{bmatrix}, & \begin{bmatrix}2\\0\\0\end{bmatrix}, & \begin{bmatrix}3\\0\\1\end{bmatrix}\end{array}\right) \quad \text{subspace of } \mathbb{R}^3$$

Find basis for H:

- $\{b_1, b_2, b_3\}$ is spanning set for H
- We have $\vec{b}_3 = \vec{b}_1 + \vec{b}_2 \Rightarrow \{b_1, b_2, b_3\}$ linear dep.
  - Not basis
- By spanning set theorem...
  - Remove $\vec{b}_3$ from spanning set
  - $H = \text{Span}(\vec{b}_1, \vec{b}_2)$
- Since $\vec{b}_1$ and $\vec{b}_2$ not scalar multiples...
  - $\{\vec{b}_1, \vec{b}_2\}$ lin ind

$\{\vec{b}_1, \vec{b}_2\}$ is a linear independent spanning set for H so it is a spanning set for H

Bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

Note: Basis for $\mathrm{Nul}(A)$

To find a basis for $\mathrm{Nul}(A)$:

- Write the solutions of $A\vec{x} = \vec{0}$ in parametric vector form, i.e.

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p, \quad c_1, \ldots, c_p \in \mathbb{R}$$

- The set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ forms a basis for $\mathrm{Nul}(A)$.

Example

Let $A = \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Find a basis for $\mathrm{Nul}(A)$.

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}$$
$$\left[\begin{array}{cccc|c} \boxed{1} & -2 & 0 & -1 & 0 \\ 0 & 0 & \boxed{-1} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \leftarrow \text{REF}$$

x_2 and x_4: free

$-x_3 + x_4 = 0 \Rightarrow x_3 = x_4$

$x_1 - 2x_2 - x_4 = 0 \Rightarrow x_1 = 2x_2 + x_4$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 \\ x_2 \\ x_4 \\ x_4 \end{bmatrix} = \begin{pmatrix} 2x_2 \\ x_2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} x_4 \\ 0 \\ x_4 \\ x_4 \end{pmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ is a basis for $\mathrm{Nul}(A)$

Theorem (A basis for $\mathrm{Col}(A)$)

The pivot columns of a matrix A form a basis for $\mathrm{Col}(A)$.

Example

Let $A = \begin{bmatrix} 0 & -3 & -6 & 4 \\ -1 & -2 & -1 & 3 \\ -2 & -3 & 0 & 3 \\ 1 & 4 & 5 & -9 \end{bmatrix}$. Find a basis for $\mathrm{Col}(A)$.

$$A \xrightarrow{\text{REF}} \begin{bmatrix} 1 & 4 & 5 & -9 \\ 0 & 2 & 4 & -6 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So $\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ -1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 3 \\ -9 \end{bmatrix} \right\}$ is a basis for $\mathrm{Col}(A)$.

Note

A basis is

- a spanning set which is as small as possible,
- a linearly independent set which is as big as possible.