MATH 3110 - Spring 2023

Learning Activity 12

1. Let $A = \begin{bmatrix} 2 & 3 & -4 \\ 4 & 0 & 5 \\ 5 & 1 & 6 \end{bmatrix}$ be a 3×3 matrix.

(a) Write the formula for $\det(A)$ using cofactor expansion across the second row (use the terms a_{ij} and C_{ij}, do not use the symbol $\sum$).

Formula: $\det A = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$

(b) Write the formulas for the cofactors C_{21}, C_{22} and C_{23} of A.

$C_{21} = (-1)^{2+1}\det A_{21}$ $\quad$ $C_{22} = (-1)^{2+2}\det A_{22}$ $\quad$ $C_{23} = (-1)^{2+3}\det A_{23}$

(c) Compute C_{21}, C_{22} and C_{23}.

Computation:

$$C_{21} = (-1)^{2+1}\det A_{21} = -1\begin{vmatrix} 3 & -4 \\ 1 & 6 \end{vmatrix} = -22 \qquad C_{22} = (-1)^{2+2}\det A_{22} = \begin{vmatrix} 2 & -4 \\ 5 & 6 \end{vmatrix} = 32$$

$$C_{23} = (-1)^{2+3}\det A_{23} = -1\begin{vmatrix} 2 & 3 \\ 5 & 1 \end{vmatrix} = 13$$

$C_{21} = -22$ $\quad$ $C_{22} = 32$ $\quad$ $C_{23} = 13$

(d) Determine $\det(A)$ using cofactor expansion across the second row.

Computation: Using the formula from (a) and the cofactors from (b), we have

$$\begin{aligned} \det A &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} \\ &= (4)(-22) + (0)(32) + (5)(13) = -23 \end{aligned}$$

$$\boxed{\det(A) = -23}$$

2. Compute the determinant of the matrix M using cofactor expansion.

$$M = \begin{bmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & 0 \\ 2 & 5 & 4 & -3 \\ -3 & -7 & -5 & 2 \end{bmatrix}$$

Computation (mention which row/column are you using and show all works): Here, I am choosing the 2nd row. Recall that m_{ij} is the entry of M at row number i and column number j, and

M_{ij} is the matrix obtained by deleting row number i and column number j of M. One may use as well notation for cofactor $C_{ij} = (-1)^{i+j}\det(M_{ij})$.

If row number i is chosen

$$\det(M) = \sum_{j=1}^{4}(-1)^{i+j}m_{ij}\det(M_{ij}) = \sum_{j=1}^{4} m_{ij}C_{ij}$$

If column number j is chosen

$$\det(M) = \sum_{i=1}^{4}(-1)^{i+j}m_{ij}\det(M_{ij}) = \sum_{i=1}^{4} m_{ij}C_{ij}$$

For row number 2:

$$\begin{aligned} \det(M) &= -0\det(M_{21}) + 1\det(M_{22}) - 2\det(M_{23}) + 0\det(M_{24}) \\ &= \det(M_{22}) - 2\det(M_{23}) \\ &= \det\begin{bmatrix} 1 & 3 & -4 \\ 2 & 4 & -3 \\ -3 & -5 & 2 \end{bmatrix} - 2\det\begin{bmatrix} 1 & 3 & -4 \\ 2 & 5 & -3 \\ -3 & -7 & 2 \end{bmatrix} \\ &= \underbrace{1\begin{vmatrix} 4 & -3 \\ -5 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & -3 \\ -3 & 2 \end{vmatrix} + (-4)\begin{vmatrix} 2 & 4 \\ -3 & -5 \end{vmatrix}}_{\text{I am using row 1}} - 2\left(\underbrace{1\begin{vmatrix} 5 & -3 \\ -7 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & -3 \\ -3 & 2 \end{vmatrix} + (-4)\begin{vmatrix} 2 & 5 \\ -3 & -7 \end{vmatrix}}_{\text{I am using row 1}}\right) \\ &= (8-15) - 3(4-9) - 4(-10+12) - 2\Big((10-21) - 3(4-9) - 4(-14+15)\Big) = 0 - 2(0) = 0 \end{aligned}$$

$$\boxed{\det(M) = 0}$$