Chap.III: Determinants
Sec.3.1: Introduction to Determinants

Objectives:

- Define determinants.
- Compute determinants using cofactor expansions.

Where does the determinant of a matrix come from?

Recall

An $n \times n$ matrix A is invertible if and only if it has n pivots.

1. **The case of a** 2×2 **matrix:** Assume $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is invertible, i.e. A has two pivots.
Reduce A to REF and determine the condition imposed on the entries of A so that it has two pivots.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \xrightarrow{a_{11}R_2 \to R_2} \begin{bmatrix} a_{11} & a_{12} \\ a_{21}a_{11} & a_{22}a_{11} \end{bmatrix} \xrightarrow{R_2 - a_{21}R_1 \to R_2}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22}a_{11} - a_{21}a_{12} \end{bmatrix}$$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\det(A) = ad - bc$

Thus A has two pivots if and only if $a_{22}a_{11} - a_{21}a_{12} \neq 0$.
A is invertible if and only if $\det(A) = a_{22}a_{11} - a_{21}a_{12} \neq 0$.

2 The case of a 3×3 matrix

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a 3×3 matrix. Assume that A is invertible.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow[a_{11}R_3 \to R_3]{a_{11}R_2 \to R_2} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11}a_{21} & a_{11}a_{22} & a_{11}a_{23} \\ a_{11}a_{31} & a_{11}a_{32} & a_{11}a_{33} \end{bmatrix}$$

$$\xrightarrow[R_3 - a_{31}R_1 \to R_3]{R_2 - a_{21}R_1 \to R_2} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{33} - a_{13}a_{31} \end{bmatrix}$$

Since A is invertible, either the $(2,2)$-entry or the $(3,2)$-entry of the last matrix is nonzero. We assume that the $(2,2)$-entry is nonzero, otherwise, we make a row interchange. We continue the row echelon form.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow{REF} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & 0 & \alpha \end{bmatrix}$$

where, after cancellation and factoring,

$$\begin{aligned} \alpha &= a_{11}\big(a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21} - a_{13}a_{31}a_{22} \\ &\quad - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}\big) \\ &= a_{11}\Delta. \end{aligned}$$

Since A is invertible and $a_{11} \neq 0$, the number

$$\Delta = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21} - a_{13}a_{31}a_{22} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

must be nonzero (A must have a pivot at the 3rd column).
The quantity Δ is the determinant of the 3×3 matrix A.

Write the determinant of the previous 3×3 matrix in terms of the determinant of 2×2 matrices.

$$\begin{aligned}\Delta &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21} \\ &\quad - a_{13}a_{31}a_{22} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} \\ &= \left(a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32}\right) - \left(a_{12}a_{21}a_{33} - a_{12}a_{23}a_{31}\right) \\ &\quad + \left(a_{13}a_{32}a_{21} - a_{13}a_{31}a_{22}\right) \\ &= a_{11} \cdot \det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \cdot \det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} \\ &\quad + a_{13} \cdot \det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}\end{aligned}$$

$= a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}$

Definition

Let A be an $n \times n$ square matrix. We define A_{ij} to be the submatrix of A formed by deleting row i and column j of A (A_{ij} is also called ijth minor of A).

Be careful with the notation a_{ij} (the (i,j)-entry of A) and A_{ij}.

Example

Let $A = \begin{bmatrix} -1 & 3 & 4 & -3 \\ 0 & 0 & -2 & 1 \\ 3 & 5 & 6 & -4 \\ -4 & 0 & -3 & -3 \end{bmatrix}$. Write the matrix A_{34}. $\begin{bmatrix} -1 & 3 & 4 \\ 0 & 0 & -2 \\ -4 & 0 & -3 \end{bmatrix}$

Note

The determinant of a 3×3 matrix is given by

$$\Delta = a_{11} \cdot \det(A_{11}) - a_{12} \cdot \det(A_{12}) + a_{13} \cdot \det(A_{13}).$$

Definition (Determinant of an $n \times n$ matrix)

For $n \geq 2$, the determinant of an $n \times n$ matrix A is the sum of n terms of the form $\pm a_{1j} \det(A_{1j})$, with plus and minus signs alternating, where $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A. In particular

$$\begin{aligned}\det(A) &= a_{11} \cdot \det(A_{11}) - a_{12} \cdot \det(A_{12}) + \cdots + (-1)^{n+1} \cdot \det(A_{1n}) \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \cdot \det(A_{1j})\end{aligned}$$

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, we also denote $\det(A)$ by

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Examples

1. Compute the determinant of $A = \begin{bmatrix} 2 & 3 & -4 \\ 4 & 0 & 5 \\ 5 & 1 & 6 \end{bmatrix}$.

$$\begin{aligned}\det(A) &= 2 \cdot \det(A_{11}) - 3\det(A_{12}) + (-4) \cdot \det(A_{13}) \\ &= 2\det\begin{bmatrix} 0 & 5 \\ 1 & 6 \end{bmatrix} - 3\det\begin{bmatrix} 4 & 5 \\ 5 & 6 \end{bmatrix} - 4\det\begin{bmatrix} 4 & 0 \\ 5 & 1 \end{bmatrix} \\ &= 2(-5) - 3(-1) - 4(4) = -23\end{aligned}$$

Compute determinant of $A = \begin{bmatrix} 1 & 5 & -1 \\ 2 & 4 & -1 \\ 1 & -2 & 0 \end{bmatrix}$

- Expand along 3rd row

$$\det(A) = (+1)(1)\begin{vmatrix} 5 & -1 \\ 4 & -1 \end{vmatrix} + (-1)(-2)\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} + (1)(0)\begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix}$$

$$= 1(-5-(-4)) + 2(-1-(-2))$$

$$= -1 + 2(1)$$

$$= 1$$

❷ Compute the determinant of the matrix $B = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 2 & 0 & 3 & 0 \\ 1 & 2 & 0 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

By definition

$$\begin{aligned} \det(B) &= 1\det(B_{11}) - 0\det(B_{12}) + 2\det(B_{13}) - 0\det(B_{14}) \\ &= \begin{vmatrix} 0 & 3 & 0 \\ 2 & 0 & 3 \\ 1 & 1 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & 0 & 0 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix} \\ &= -3\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} + 2\cdot 2\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} \\ &= (-3)(2-3) + 4(2-3) = -1 \end{aligned}$$

Cofactor expansion

Definition (Cofactors of a matrix)

The (i,j)-cofactor of an $n \times n$ matrix A is the number C_{ij} given by

$$C_{ij} = (-1)^{i+j} \cdot \det(A_{ij})$$

Rewriting the definition $\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \cdot \det(A_{1j})$ using cofactors gives

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}.$$

This formula is the cofactor expansion across the first row of A.

Theorem

Let A be an $n \times n$ square matrix. Then, the determinant $\det(A)$ of A can be computed by a cofactor expansion across any row or down any column.

1. The expansion across the ith row is (i is fixed)

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = \sum_{j=1}^{n} a_{ij}C_{ij}$$

2. The expansion down the jth column is (j is fixed)

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} = \sum_{i=1}^{n} a_{ij}C_{ij}$$

Note

The plus or minus sign in the (i, j)-cofactor depends on the position of a_{ij} in the matrix. The sign of $(-1)^{i+j}$ is determined in the following pattern.

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Example

Use a cofactor expansion down the third column to compute the determinant of $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & 0 \\ 3 & 3 & 1 \end{bmatrix}$.

We have

$$\begin{aligned} \det(A) &= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} \\ &= 0 \cdot (-1)^{1+3}\det(A_{13}) + 0 \cdot (-1)^{2+3}\det(A_{23}) \\ &\quad +1 \cdot (-1)^{3+3}\det(A_{33}). \\ &= 0 - 0 + \det \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix} \\ &= 0 - 0 + (2 - (-2)) = 4 \end{aligned}$$

Definition (triangular matrix)

An $n \times n$ matrix A is triangular if all of its entries above (or below) the main diagonal are all zero.

The matrix $B = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ is an (upper) triangular matrix.

Theorem

If A is a triangular matrix, then $\det(A)$ is the product of the entries on the main diagonal of A, i.e. $\det(A) = a_{11}a_{22}\cdots a_{nn}$.

$$\det(B) = (1)(2)(-3)(-1) = 6$$

If A triangular, then det(A) product of entries on main diagonal line

$$\begin{vmatrix} -5 & 2 & -5 & -2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & -4 \end{vmatrix} = (1)(-5)\begin{vmatrix} 1 & -2 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & -4 \end{vmatrix}$$

$$= -5\left(1\begin{vmatrix} 3 & 2 \\ 0 & -4 \end{vmatrix} + (-1)(0)\begin{vmatrix} -2 & 1 \\ 0 & -4 \end{vmatrix} + (1)(0)\begin{vmatrix} -2 & 1 \\ 3 & 2 \end{vmatrix}\right)$$

$$= (-5)(1)\left(3(-4) - 0(2)\right)$$

$$= (-5)(1)(3)(-4)$$

Practice

Compute the determinant of $D = \begin{bmatrix} 2 & 5 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 8 \\ 4 & 10 & -4 & -1 \end{bmatrix}$