1. Let $H = \left\{ \begin{bmatrix} a+b \\ 2a \\ 3a-b \\ -b \end{bmatrix} \mid a, b \text{ in } \mathbb{R} \right\}$ be a subspace of $\mathbb{R}^3$.

   (a) Find a basis for H.

   Computation: You must explain why they form a basis.

   Note that for any vector in H, we have

$$\begin{bmatrix} a+b \\ 2a \\ 3a-b \\ -b \end{bmatrix} = a \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix}$$

   So $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix} \right\}$ is a spanning set for H.

   In addition, $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix} \right\}$ is linearly independent since the vectors are not scalar multiple of each other.

   So, the set is a basis for H.

   $\boxed{\text{A basis for } H\text{: } \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix} \right\}}$

   (b) What is the dimension of H?

   $\boxed{\dim(H) = 2}$

2. Let $W = \{a + bt^3 \mid a, b \text{ in } \mathbb{R}\}$ be a subspace of $\mathbb{P}_4$.

   (a) Find a basis for W.

   Computation: You must explain why they form a basis.

   For any polynomial in W, we have

$$a + bt^3 = a(1) + bt^3$$

   so it is a linear combination of the polynomials 1 and t^3. It follows that $\{1, t^3\}$ is a spanning set for W. Furthermore, the set is linearly independent as the polynomials are not scalar multiple of each other. Thus, $\{1, t^3\}$ is a basis for W.

   $\boxed{\text{A basis for } W\text{: } \{1, t^3\}}$

   (b) What is the dimension of W?

   $\boxed{\dim(W) = 2}$

3. Determine $\dim(\text{Nul}(A))$ and $\dim(\text{Col}(A))$ where $A = \begin{bmatrix} 1 & 3 & -2 & 5 \\ 0 & 1 & -1 & 2 \\ 2 & 1 & 1 & 1 \end{bmatrix}$.

   Computation: We have

$$\begin{bmatrix} 1 & 3 & -2 & 5 \\ 0 & 1 & -1 & 2 \\ 2 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{REF} \begin{bmatrix} 1 & 3 & -2 & 5 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

   So $A\vec{x} = \vec{0}$ has one free variable so $\dim(\text{Nul}(A)) = 1$, and A has 3 pivot columns so $\dim(\text{Col}(A)) = 3$.

$\dim(\text{Nul}(A)) = 1$
$\dim(\text{Col}(A)) = 3$