Determinant

Let A be an $n\times n$ matrix $\quad (A = (a_{ij})_{ij}$

Definition: $\det(A) = a_{11}\det A_{11} - a_{12}\det A_{12} + \cdots + (-1)^{1+n}\det A_{1n}$

A_{ij} : obtained by deleting row i and column j of the matrix A.

Cofactors of A : $\quad C_{ij} = (-1)^{i+j}\det A_{ij}$

Cofactor expansion across row i

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

Cofactor expansion down column j

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

$$\Longrightarrow \quad \det(A) = \det(A^T)$$

Properties

- If $A \xrightarrow{R_i \to R_i + cR_j} B$, $\quad \det(A) = \det(B)$
- If $A \xrightarrow{R_i \leftrightarrow R_j} B$, $\quad -\det(A) = \det(B)$
- If $A \xrightarrow{R_i \to cR_i} B$, $\quad c\det(A) = \det(B)$
- If $A \xrightarrow{REF} U$: no use of row scaling

$$\det(A) = (-1)^r \det(U)$$

r: # of row interchanging

U: in REF (triangular matrix) $\quad \det(U)$ = product of the main diagonal entries of U

- $\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA)$ $\quad (AB \neq BA)$
- A^{-1} exists $(\det(A) \neq 0)$

$$\det(AA^{-1}) = \det(I_n) = 1$$

$$\det(A)\det(A^{-1}) = 1 \Longrightarrow \det(A^{-1}) = \frac{1}{\det(A)}$$

- **Inverse formula:**

If $\det(A) \neq 0$

$$A^{-1} = \frac{1}{\det A} \operatorname{Adj} A, \quad \operatorname{Adj} A = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & & & \\ C_{12} & C_{2n} & & C_{nn} \end{bmatrix}$$

Vector spaces.

$(V, +, \cdot)$ a vector space

- A subset H of V is a subspace of V if
  - $\vec{0}$ is in H
  - If $\vec{u}$ and $\vec{v}$ are in H, then $\vec{u} + \vec{v}$ is in H
  - If $\vec{u}$ is in H and c is in $\mathbb{R}$, then $c\vec{u}$ is in H

- Any subset $W = \text{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p)$ where $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ are in $\vec{V}$, is a subspace of V.

- Let A be an $m \times n$ matrix. $(A = [\vec{a}_1 \ \vec{a}_2 \cdots \vec{a}_n])$

  $\text{Nul}(A) = \{ \vec{x} \text{ in } \mathbb{R}^n \mid A\vec{x} = \vec{0} \}$ is a subspace of $\mathbb{R}^n$.

  $\text{Col}(A) = \text{Span}(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n)$ is a subspace of $\mathbb{R}^m$.

  $\text{Row}(A) = \text{Span}(\text{rows of } A)$ is a subspace of $\mathbb{R}^n$.

- $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ a linear transformation with standard matrix A
  $(T(\vec{x}) = A\vec{x})$

  $$\text{Ker}(T) = \{ \vec{x} \text{ in } \mathbb{R}^n \mid T(\vec{x}) = \vec{0} \}$$

  $$T(\vec{x}) = A\vec{x}$$

  $$= \{ \vec{x} \text{ in } \mathbb{R}^n \mid A\vec{x} = \vec{0} \} = \text{Nul}(A)$$

  $$\text{Im}(T) = \{ T(\vec{x}) \mid \vec{x} \text{ in } \mathbb{R}^n \} = \text{Col}(A)$$

- **Bases.**

  V a vector space

  - A basis for V is a spanning set of V which is linearly independent.

- If $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p$ is the parametric vector form of the solution of $A\vec{x} = \vec{0}$

  $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is a basis for $Nul(A)$.

- The pivot columns of A form a basis for $Col(A)$
- The non-zero rows of an REF of A form a basis for $Row(A)$

<u>Spanning Set Theorem</u>

V is a vector space and H a subspace of V

$$H = Span(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p)$$

If $\vec{v}_i = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_{i-1}\vec{v}_{i-1} + c_{i+1}\vec{v}_{i+1} + \cdots + c_p\vec{v}_p$, then

$$H = Span(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_p)$$

<u>Coordinate systems</u>

Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ be a basis for V. For any $\vec{u}$ in V, there exists a unique set of scalars $c_1, c_2, \ldots, c_n$ such that

$$\vec{u} = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_n\vec{b}_n$$

$$[\vec{u}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \leftarrow \mathcal{B}\text{-coordinate vector of } \vec{u}.$$

- <u>Change of basis</u>

Let $\mathcal{C} = \{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\}$ be another basis for V.

The change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$ is

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \left[[\vec{b}_1]_{\mathcal{C}} \quad [\vec{b}_2]_{\mathcal{C}} \quad \cdots \quad [\vec{b}_n]_{\mathcal{C}} \right]$$

$$[\vec{u}]_{\mathcal{C}} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} [\vec{u}]_{\mathcal{B}}$$

$$\underset{\mathcal{B} \leftarrow \mathcal{C}}{P} = \left[[\vec{c}_1]_{\mathcal{B}} \quad [\vec{c}_2]_{\mathcal{B}} \quad \cdots \quad [\vec{c}_n]_{\mathcal{B}} \right]$$

$$[\vec{u}]_{\mathcal{B}} = \underset{\mathcal{B} \leftarrow \mathcal{C}}{P} [\vec{u}]_{\mathcal{C}}$$

$$\left(P_{\mathcal{E} \leftarrow \mathcal{B}}\right)^{-1} = P_{\mathcal{B} \leftarrow \mathcal{E}}$$

- $\dim(V) = \#$ vectors in a basis for V.

  $\dim(\mathrm{Nul}(A)) = \#$ free variables in $A\vec{x} = \vec{0}$

  $\dim(\mathrm{Col}(A)) = \#$ pivot columns of A.

  $\dim(\mathrm{Row}(A)) = \#$ pivots in A

- Let A be an $m \times n$ matrix

  $$\mathrm{rank}(A) = \dim(\mathrm{Col}(A)) = \dim(\mathrm{Row}(A))$$

  $$\mathrm{Row}(A) = \mathrm{Col}(A^T)$$

  $$\dim(\mathrm{Col}(A^T)) = \dim(\mathrm{Row}(A)) = \mathrm{rank}(A)$$

  $$\| \quad \mathrm{rank}(A^T)$$

  $(\mathrm{rank}(A^T) = \mathrm{rank}(A))$

- $\mathrm{rank}(A) + \dim(\mathrm{Nul}(A)) = n = \#$ columns of A.

- $\mathrm{rank}(A) \leq \min(m, n)$.