Sec.4.5: Dimension of a Vector Space

Objective: To determine the dimension of a given vector space V or any subspace H.

Theorem

Let V be a vector space that has a basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$. Then any set S of vectors of V containing more than n vectors is linearly dependent. That is if $\#S > n$, then S is linearly dependent.

$\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ a basis for V

$S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ $\vec{v}_i$'s in V. $(p > n)$

$S' = \{[\vec{v}_1]_{\mathcal{B}}, [\vec{v}_2]_{\mathcal{B}}, \ldots, [\vec{v}_p]_{\mathcal{B}}\}$ a set of p vectors in $\mathbb{R}^n$.

$p > n \Rightarrow S'$ is lin. dependent in $\mathbb{R}^n$.

$\Rightarrow S$ is lin. dependent in V.

Example

Consider the set $S = \{1 + t^2, -4 + t - 3t^3, t - 7t^2, 5 - t, 11 + t + 4t^3\}$ of polynomials in $\mathbb{P}_3$. Explain why the set S is linearly dependent.

Note that $\mathcal{B} = \{1, t, t^2, t^3\}$ is a basis for $\mathbb{P}_3$ which contains four polynomials.
By the previous theorem, any set in $\mathbb{P}_3$ that contains more than four polynomials is linearly dependent.
Since S has $5 > 4$ polynomials, then S is linearly dependent .

Theorem

If V is a vector space with a basis of size n, then every basis for V has exactly n vectors.

$\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots \vec{b}_n\}$ a basis for V
$\mathcal{C} = \{\vec{c}_1, \vec{c}_2, \dots \vec{c}_p\}$ another basis for V
then $p = n$

- If $p > n$: $\mathcal{C}$ is lin. dep $\Rightarrow$ $\mathcal{C}$ is not a basis
- If $p < n$: $\mathcal{C}$ is not a spanning set $\Rightarrow$ $\mathcal{C}$ is not a basis

Definition (Dimension of a Vector space)

If V is vector space spanned by a finite set, then V is said to be finite-dimensional.

- The **dimension** of V, denoted by $\dim(V)$ is the number of the vectors in a basis for V.
- The dimension of the vector space $\{\vec{0}\}$ is defined to be 0.
- If V is not spanned by a finite set, the V is said to be infinite-dimensional.

Examples

Determine the dimensions of $\mathbb{R}^n, \mathbb{P}_n$, and $M_{2\times 2}(\mathbb{R})$.

1. $\mathcal{E} = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ is a basis for $\mathbb{R}^n$, so $\dim(\mathbb{R}^n) = n$.
2. $\mathcal{B} = \{1, t, t^2, \ldots, t^n\}$ is a basis for $\mathbb{P}_n$, so $\dim(\mathbb{P}_n) = n + 1$.
3. $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for $M_{2\times 2}(\mathbb{R})$ so $\dim(M_{2\times 2}(\mathbb{R})) = 4$.

$M_{m\times n}(\mathbb{R}) = m \times n$

Theorem (Basis Theorem)

Let V be a vector space such that $\dim(V) = p$, with $p \geq 1$ (we also can say that V is a p-dimensional space).

- Any linearly independent set of exactly p elements in V is automatically a basis for V.
- Any set of exactly p elements that spans V is automatically a basis for V.

$\dim(V) = p$

- $S = \{\vec{b}_1, \ldots, \vec{b}_p\}$ is lin. indep $\Rightarrow$ a basis.

$\dim(V) = p$

- $S = \{\vec{b}_1, \ldots, \vec{b}_p\}$ and $V = \text{Span}(\vec{b}_1, \ldots, \vec{b}_p) \Rightarrow S$ is a basis.

Subspaces of a finite-dimensional space

Theorem

Let V be a finite-dimensional vector space and let H be a subspace of V.

- Then any linearly independent subset of H can be expanded to a basis for H.
- Furthermore H is also finite-dimensional and $\dim(H) \leq \dim(V)$.

Example

Find the dimension of the subspace H of $\mathbb{R}^4$ defined by

$$H = \{ \begin{bmatrix} a - 3b + 6c \\ 4a + 4d \\ b - 2c - d \\ 5d \end{bmatrix} \mid a, b, c, d \in \mathbb{R} \}$$

First, find a basis for H (spanning set which is linearly independent).
For any $\vec{u}$ in H

$$\vec{u} = \begin{bmatrix} a - 3b + 6c \\ 4a + 4d \\ b - 2c - d \\ 5d \end{bmatrix} = \begin{bmatrix} a \\ 4a \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3b \\ 0 \\ b \\ 0 \end{bmatrix} + \begin{bmatrix} 6c \\ 0 \\ -2c \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 4d \\ -d \\ 5d \end{bmatrix}$$

$$\begin{aligned}\vec{u} &= a\begin{bmatrix}1\\4\\0\\0\end{bmatrix}+b\begin{bmatrix}-3\\0\\1\\0\end{bmatrix}+c\begin{bmatrix}6\\0\\-2\\0\end{bmatrix}+d\begin{bmatrix}0\\4\\-1\\5\end{bmatrix}\\ &= a\vec{a}_1+b\vec{a}_2+c\vec{a}_3+d\vec{a}_4\end{aligned}$$

So, $H = \mathrm{Span}(\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4) = \mathrm{Col}(A)$ where $A = \begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4\end{bmatrix}$.
Therefore, a basis for H is formed by the pivot columns of A.

$$A = \begin{bmatrix}1 & -3 & 6 & 0\\ 4 & 0 & 0 & 4\\ 0 & 1 & -2 & -1\\ 0 & 0 & 0 & 5\end{bmatrix} \xrightarrow{REF} \begin{bmatrix}1 & -3 & 6 & 0\\ 0 & 12 & -24 & 4\\ 0 & 0 & 0 & -4/3\\ 0 & 0 & 0 & 0\end{bmatrix}$$

So $\{\vec{a}_1, \vec{a}_2, \vec{a}_4\}$ is a basis for H and $\dim(H) = 3$.

The Dimension of $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

Proposition

- The dimension of $\mathrm{Nul}(A)$ is the number of free variables in the equation $A\vec{x} = \vec{0}$.
- The dimension of $\mathrm{Col}(A)$ is the number of pivots columns in A.

Example

Let $A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & 2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$. Determine $\dim(\mathrm{Nul}(A))$ and $\dim(\mathrm{Col}(A))$.

$$A \xrightarrow{\text{REF}} \begin{bmatrix} 1 & -2 & 0 & 1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

x_2, x_4, x_5 : free variables $\to \dim(\mathrm{Nul}(A)) = 3$

Col 1 and 3 : pivot columns $\to \dim(\mathrm{Col}(A)) = 2$.