MATH 3110 - Spring 2023

Learning Activity 17 - Solution

1. Let $p_1(t) = 1 + t^2, p_2(t) = t + t^2$ and $p_3(t) = 1 + 2t + t^2$ be polynomials in $\mathbb{P}_2$. Is the set $\mathcal{B} = \{p_1(t), p_2(t), p_3(t)\}$ linearly independent? Justify your answer. (Mark the right answer by X and then explain your answer)

$\mathcal{B}$ is linearly independent	$\mathcal{B}$ is linearly dependent.
X	

Justify your answer: By definition, $\mathcal{B} = \{p_1(t), p_2(t), p_3(t)\}$ is linearly independent if the equation $x_1p_1(t) + x_2p_2(t) + x_3p_3(t) = 0$ has only the trivial solution. Solve the equation

$$\begin{aligned} (x_1 + x_1t^2) + (x_2t + x_2t^2) + (x_3 + 2x_3t + x_3t^2) &= 0 \\ (x_1 + x_3) + (x_2 + 2x_3)t + (x_1 + x_2 + x_3)t^2 &= 0 \end{aligned}$$

This is an equality of two polynomials so the corresponding coefficients are equal. So

$$\begin{aligned} x_1 + x_3 &= 0 \\ x_2 + 2x_3 &= 0 \\ x_1 + x_2 + x_3 &= 0 \end{aligned}$$

Reducing the augmented matrix to REF

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow{REF} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right]$$

So, every variable is a pivot variable, thus the only solution is $(x_1, x_2, x_3) = (0, 0, 0)$. It follows that $\mathcal{B}$ is linearly independent.

2. Let $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 3 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 5 \\ 0 \\ 2 \\ 1 \end{bmatrix}$ be vectors in $\mathbb{R}^4$.

(a) Show that the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent.

Your answer: By definition, the set is linearly dependent if the equation $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$ has nonzero (nontrivial solution). If it has a free variable, then it has nonzero solutions. Reduce the augmented matrix to REF

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 0 \\ -1 & 2 & 0 & 0 \\ 2 & -2 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{REF} \left[\begin{array}{ccc|c} 1 & 3 & 5 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So x_3 is a free variable and the set is linearly dependent.

They can also directly give a dependence relation among $\vec{v}_1, \vec{v}_2, \vec{v}_3$, for example $2\vec{v}_1 + \vec{v}_2 - \vec{v}_3 = \vec{0}$, so they are linearly dependent.

(b) Let $H = \text{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ be the subspace of $\mathbb{R}^4$ spanned by the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. Find a basis of H.

Computation: Recall that a basis for H is a spanning set of H that is linearly independent. Here $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a spanning set of H but, from the previous question, it is still linearly dependent. So we can eliminate a vector from the spanning set that is a linear combination of the others (that depends on the others).

From the previous question, $2\vec{v}_1 + \vec{v}_2 - \vec{v}_3 = \vec{0}$ is dependence relation. So we have $\vec{v}_3 = 2\vec{v} + \vec{v}_2$. So $\vec{v}_3$ can be eliminated from the spanning set and $H = \text{Span}(\vec{v}_1, \vec{v}_2)$ (by the Spanning set theorem). In addition, $\vec{v}_1$ and $\vec{v}_2$ are not scalar multiple of each other, so they are linearly independent. It follows that $\{\vec{v}_1, \vec{v}_2\}$ is a basis for H.

With similar reasons, $\{\vec{v}_1, \vec{v}_3\}$ and $\{\vec{v}_2, \vec{v}_3\}$ are also bases for H **(the linear independence must be shown)**.

A basis for H: $\{\vec{v}_1, \vec{v}_2\}$
or $\{\vec{v}_1, \vec{v}_3\}$
or $\{\vec{v}_2, \vec{v}_3\}$

3. Consider the matrix $A = \begin{bmatrix} 0 & 3 & -6 & 6 \\ 3 & -7 & 8 & -5 \\ 3 & -9 & 12 & -9 \end{bmatrix}$.

(a) Suppose $\text{Nul}(A)$ is a subspace of $\mathbb{R}^n$ and $\text{Col}(A)$ a subspace of $\mathbb{R}^k$. What is the value of n? What is the value of k?

$n = 4$	$k = 3$.

(b) Find a basis for $\text{Nul}(A)$.

Computation: To find a basis for $\text{Nul}(A)$, write the solution of $A\vec{x} = \vec{0}$ in parametric vector form and collect the vectors.

$$\left[\begin{array}{cccc|c} 0 & 3 & -6 & 6 & 0 \\ 3 & -7 & 8 & -5 & 0 \\ 3 & -9 & 12 & -9 & 0 \end{array}\right] \xrightarrow{REF} \left[\begin{array}{cccc|c} 3 & -7 & 8 & -5 & 0 \\ 0 & 3 & -6 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ so } \begin{aligned} x_1 &= 2x_3 - 3x_4 \\ x_2 &= 2x_3 - 2x_4 \\ x_3 &= x_3 \\ x_4 &= x_4 \end{aligned}$$

So

$$\vec{x} = x_3 \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

A basis for $\text{Nul}(A)$: $\left\{ \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

(c) Use your REF from the previous question to find a basis for $\text{Col}(A)$.

The pivot columns of A form a basis for $\text{Col}(A)$ (note that the columns of A are considered, not the columns of the REF). According to REF above, column 1 and 2 are pivot columns

A basis for $\text{Col}(A)$: $\left\{ \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -7 \\ -9 \end{bmatrix} \right\}$