MATH 3110 - Spring 2023
PRACTICE II

This is a list of problems to help you prepare for Test 2. It is to serve as a study aid and it cannot be a substitute for study of your class notes. The problems in here are all about computations but students should understand the concepts. So, deeply learn first the definitions and properties before working on these problems. Questions in the test could be conceptual.

Topics: Chap 3 and Chap 4

Problem 1:

(1) Let A and B be $n \times n$ matrices. Show that $\det(AB) = \det(BA)$.

(2) Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a 3×3 matrix and suppose that $\det(A) = -2$. Let P be a 3×3 invertible matrix. Compute the determinant of the following matrices:

(a) $B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 2a_{31} & 2a_{32} & 2a_{33} \\ a_{21} + 3a_{11} & a_{22} + 3a_{12} & a_{23} + 3a_{13} \end{bmatrix}$

(b) $D = 2A$

(c) $E = A^n$, where n is a positive integer

(d) $F = PAP^{-1}$

Problem 2:

Let $\mathbb{P}_3$ be the set of polynomials of degree at most 3. The set $\mathbb{P}_3$ with the zero polynomial $p(t) = 0$, with the usual addition and scalar multiplication is a vector space.

(1) Let $H_1 = \{at^3 \mid a \in \mathbb{R}\}$ be a subset of $\mathbb{P}_3$.

(a) Give a nonzero element of H_1.

(b) Give an element of $\mathbb{P}_3$ which is not in H_1.

(c) Show that H_1 a subspace of $\mathbb{P}_3$.

(d) What is the dimension of H_1.

(2) Let $H_2 = \{a + t^3 \mid a \in \mathbb{R}\} \cup \{0\}$ be a subset of $\mathbb{P}_3$.

(a) Give a nonzero element of H_2.

(b) Give an element of $\mathbb{P}_3$ which is not in H_2.

(c) Is H_2 a subspace of $\mathbb{P}_3$. Justify your answer.

(3) Let $M_{2\times 2}(\mathbb{R})$ be the set of all 2×2 matrices (with real number entries), that is

$$M_{2\times 2}(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \text{ in } \mathbb{R} \right\}$$

The set $M_{2\times 2}(\mathbb{R})$ with the zero matrix $\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, with the usual addition and scalar multiplication is a vector space. Let B be a fixed matrix in $M_{2\times 2}(\mathbb{R})$ and let

$$W = \{A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid BA = 0\},$$

be a subset of $M_{2\times 2}(\mathbb{R})$. That is, a 2×2 matrix A is in W if the product BA is 0 (the zero matrix). Show that W is a subspace of $M_{2\times 2}(\mathbb{R})$.

Problem 3

Let $W = \left\{ \begin{bmatrix} 2a - b + c \\ -2a + 2b \\ 3a + b + 4c \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}$ be a subset of $\mathbb{R}^3$.

(1) Show that W is a subspace of $\mathbb{R}^3$.

(2) Show that the set $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$, with $\vec{b}_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ and $\vec{b}_2 = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$, is a basis for W.

(3) What is $\dim(W)$?

(4) Show that $\vec{x} = \begin{bmatrix} -5 \\ 4 \\ -10 \end{bmatrix}$ is in W.

(5) What is the $\mathcal{B}$-coordinate $[\vec{x}]_{\mathcal{B}}$ of $\vec{x}$?

(6) Let $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$ be another basis of W, such that $\vec{b}_1 = 2\vec{c}_1 - \vec{c}_2$ and $\vec{b}_2 = -3\vec{c}_1 + \vec{c}_2$. Determine change-of-coordinate matrix from $\mathcal{B}$ to $\mathcal{C}$.

(7) Compute the $\mathcal{C}$-coordinate $[\vec{x}]_{\mathcal{C}}$ of the vector $\vec{x}$ in (part (4)).

Problem 4:

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_2 \\ -x_1 + 3x_2 \\ x_1 + x_2 \end{bmatrix}$$

Let A be the standard matrix of T.

(1) Determine the matrix A.

(2) Is there a nonzero vector in $\operatorname{Ker}(T)$? Justify.

(3) Give a basis of $\ker(T)$.

(4) Give a nonzero element of $\operatorname{im}(T)$.

(5) Is $\vec{b} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$ an element of $\operatorname{im}(T)$?

Problem 6

(1) If A is a 3×4 matrix, what is the smallest possible dimension of $\operatorname{Nul}(A)$?

(2) Now, Let $A = \begin{bmatrix} 2 & 1 & -1 & 1 \\ 4 & 1 & -3 & 3 \\ -2 & 1 & 3 & -3 \end{bmatrix}$ be a 3×4 matrix.

(a) If $\operatorname{Nul}(A)$ is a subspace of $\mathbb{R}^m$ and $\operatorname{Col}(A)$ is a subspace of $\mathbb{R}^n$, what are the values of m and n.

(b) Is $\vec{z} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ in $\operatorname{Nul}(A)$?

(c) The matrix A is reduced to the matrix $B = \begin{bmatrix} 2 & 1 & -1 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

i. Determine the rank of A and the dimension of $\operatorname{Nul}(A)$.

ii. Find a basis $\mathcal{B}_C$ for $\operatorname{Col}(A)$.

iii. Find a basis $\mathcal{B}_N$ for $\operatorname{Nul}(A)$.

iv. Find a basis $\mathcal{B}_R$ of $\operatorname{Row}(A)$.