Sec.4.2: Null Spaces, Column Spaces, and Linear Transformations

Objectives:

- Definition of Null Spaces and Column Spaces
- Kernel and Image of a linear transformation

The Null Space of a Matrix

Definition

Let A be an $m \times n$ matrix. The null space of A, denoted by $\mathrm{Nul}(A)$, is the set of all solutions of the homogeneous equation $A\vec{x} = \vec{0}$. That is

$$\mathrm{Nul}(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}.$$

Clearly, $\mathrm{Nul}(A)$ is a subset of $\mathbb{R}^n$.

$A = [\vec{a}_1, \vec{a}_2, \ldots \vec{a}_n]$

Proposition

If A is an $m \times n$ matrix, then the null space $\mathrm{Nul}(A)$ of A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to a system $A\vec{x} = \vec{0}$ is a subspace of $\mathbb{R}^n$.

Explain why $\mathrm{Nul}(A)$ is a subspace of $\mathbb{R}^n$?

Example

Let $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$. Are $\vec{u} = \begin{bmatrix} -10 \\ -6 \\ 4 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ in $\mathrm{Nul}(A)$?

Example

Let H be a subset of $\mathbb{R}^4$ defined by

$$H = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \mid a - 2b + 5c = d \text{ and } c - a = b \right\}$$

Show that H is a subspace of $\mathbb{R}^4$ by expressing it as the null space of a matrix A (find the matrix A).

An explicit description of $\mathrm{Nul}(A)$

We want to write $\mathrm{Nul}(A)$ as $\mathrm{Nul}(A) = \mathrm{Span}(\vec{v}_1, \ldots, \vec{v}_p)$. That is, we are looking for a spanning set $\{\vec{v}_1, \ldots, \vec{v}_p\}$ of $\mathrm{Nul}(A)$.

Finding a Spanning set for $\mathrm{Nul}(A)$

1. Solve $A\vec{x} = \vec{0}$ and write the solutions in parametric vector form.
2. Note that parameters are only the free variables.
3. If the solutions of $A\vec{x} = \vec{0}$ are of the form

   (**) $\quad \vec{x} = t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_p\vec{v}_p,$ where $t_1, t_2, \ldots, t_p \in \mathbb{R}$

   then $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ spans $\mathrm{Nul}(A)$.
   That is, $\mathrm{Nul}(A) = \mathrm{Span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p)$.

Example

Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

1. If $\mathrm{Nul}(A)$ is a subspace of $\mathbb{R}^k$, what is k?
2. Find a spanning set of $\mathrm{Nul}(A)$.
3. Give a nonzero element of $\mathrm{Nul}(A)$.

Suppose $Nul(A) = Span\left(\begin{bmatrix}-1\\2\\0\end{bmatrix}, \begin{bmatrix}0\\-3\\3\end{bmatrix}\right)$

Is $\vec{v} = \begin{bmatrix}-3\\18\\-12\end{bmatrix}$ in $Nul(A)$?

Yes if $\vec{v}$ lin. comb. of spanned vectors

If A available, check $A(\vec{v}) = \vec{0}$

· $\vec{v}$ is in $Nul(A)$ if $\vec{v} = c_1\begin{bmatrix}-1\\2\\0\end{bmatrix} + c_2\begin{bmatrix}0\\-3\\3\end{bmatrix}$

$$\left[\begin{array}{cc|c}-1 & 0 & -3\\2 & -3 & 18\\0 & 3 & -12\end{array}\right] \xrightarrow{REF} \left[\begin{array}{cc|c}-1 & 0 & -13\\0 & -3 & 12\\0 & 0 & 0\end{array}\right] \therefore \begin{array}{l}x_1 = 3\\x_2 = -4\end{array}$$

$\vec{v} = 3\vec{v}_1$

The Column Space of a Matrix

Definition

Let $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}$ be an $m \times n$ matrix. The column space of A, denoted by $\mathrm{Col}(A)$, is the set of all linear combinations of the columns of A. That is

$$\begin{aligned} \mathrm{Col}(A) &= \mathrm{Span}(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n) \\ &= \{c_1\vec{a}_1 + c_2\vec{a}_2 + \cdots + c_n\vec{a}_n \mid c_1, c_2, \ldots c_n \in \mathbb{R}\}. \end{aligned}$$

The column space of A is a subset of $\mathbb{R}^m$.

Theorem

The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

Note

Let $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}$ be an $m \times n$ matrix.

1. By definition, $\mathrm{Col}(A)$ is a subspace of $\mathbb{R}^m$ spanned by $\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\}$.
2. The following statements are equivalent (all true or all false):
   - $\mathrm{Col}(A) = \mathbb{R}^m$.
   - For every $\vec{b}$ in $\mathbb{R}^m$, the equation $A\vec{x} = \vec{b}$ has a solution.
   - The linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A is onto $\mathbb{R}^m$.

Example

Find a matrix A such that $W = \mathrm{Col}(A)$ where

$$W = \left\{ \begin{bmatrix} a+b \\ 2a-b \\ -3a \end{bmatrix} \;\middle|\; a, b \text{ in } \mathbb{R} \right\}$$

Example

Let $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 1 & 2 \end{bmatrix}$

1. If the column space of A is a subspace of $\mathbb{R}^k$, what is k?
2. Give a nonzero element $\mathrm{Col}(A)$.
3. Is $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ in $\mathrm{Col}(A)$?

Let A be an $m \times n$ matrix.

Null Space

- $\mathrm{Nul}(A)$ is a subspace of $\mathbb{R}^n$.
- Null space of A is implicitly defined: the vectors in $\mathrm{Nul}(A)$ solve $A\vec{x} = \vec{0}$.
- It takes time to describe $\mathrm{Nul}(A)$ as row reduction of $\begin{bmatrix} A & \vec{0} \end{bmatrix}$ is required.
- There is no obvious relation between $\mathrm{Nul}(A)$ and the entries in A.

Column Space

- $\mathrm{Col}(A)$ is a subspace of $\mathbb{R}^m$.
- $\mathrm{Col}(A)$ is explicitly defined since it is spanned by described vectors (the columns of A).
- It is easy to find vectors in $\mathrm{Col}(A)$ since they are linear combinations of the columns of A.
- There is an obvious relation between $\mathrm{Col}(A)$ and the entries of A since the columns of A generate $\mathrm{Col}(A)$.

- A typical vector $\vec{v}$ in $\mathrm{Nul}(A)$ has the property $A\vec{v} = \vec{0}$.
- It is easy to check if a given vector $\vec{v}$ is in $\mathrm{Nul}(A)$ by computing $A\vec{v}$.
- $\mathrm{Nul}(A) = \{\vec{0}\}$ if and only if the equation $A\vec{x} = \vec{0}$ has only the trivial solution.
- $\mathrm{Nul}(A) = \{\vec{0}\}$ if and only the linear transformation $\vec{x} \mapsto A\vec{x}$ is one-to-one.

- A typical vector $\vec{v}$ in $\mathrm{Col}(A)$ has the property that $A\vec{x} = \vec{v}$ is consistent.
- It may take time to check if a given vector $\vec{v}$ is in $\mathrm{Col}(A)$ as row reduction of $\begin{bmatrix} A & \vec{v} \end{bmatrix}$ is required.
- $\mathrm{Col}(A) = \mathbb{R}^m$ if and only if the equation $A\vec{x} = \vec{b}$ has a solution for every $\vec{b}$ in $\mathbb{R}^m$.
- $\mathrm{Col}(A) = \mathbb{R}^m$ if and only if the linear transformation $\vec{x} \mapsto A\vec{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$.

Linear Transformations of Vector Spaces

Definition

Let V and W be vector spaces. A linear transformation T from V to W, denoted by $T : V \to W$, is a rule that assigns to each vector $\vec{x}$ in V a unique vector $T(\vec{x})$ in W, such that

1. $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v}$ in V,
2. $T(c\vec{u}) = cT(\vec{u})$ for all $\vec{u}$ in V and all scalars c.

T: V → W is lin. tran. if ...

Example

Show that the following map is a linear transformation:

$$\begin{aligned} T : \mathbb{P}_2 \quad &\longrightarrow \quad \mathbb{R}^2 \\ p(t) \quad &\mapsto \quad T(p(t)) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix} \end{aligned}$$

Definition

Let $T : V \to W$ be a linear transformation.

1. V is called the domain of T, and W is its codomain.
2. The **kernel** of T, denoted by $\ker(T)$, is the set of all vectors $\vec{u}$ in V such that $T(\vec{u}) = \vec{0}$. That is
$$\ker(T) = \{\vec{u} \text{ in } V \mid T(\vec{u}) = \vec{0}\}$$
3. The **range** or **image** of T, denoted by $\mathrm{range}(T)$ or $\mathrm{im}(T)$, is the set of all vectors in W which are of the form $T(\vec{v})$ for some $\vec{v}$ in V. That is
$$\mathrm{range}(T) = \mathrm{im}(T) = \{T(\vec{v}) \mid \vec{v} \text{ in } V\}$$

Fact

Let $T : V \to W$ be a linear transformation of vector spaces. Then,

1. $\ker(T)$ is a subspace of V,
2. $\mathrm{im}(T)$ is a subspace W.
3. if $V = \mathbb{R}^n, W = \mathbb{R}^m$, and A is the standard matrix of T, then $\ker(T)$ the null space $\mathrm{Nul}(A)$ of A, and $\mathrm{im}(T)$ is the column space $\mathrm{Col}(A)$ of A.

Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation with standard matrix $A = \begin{bmatrix} 1 & -2 \\ -2 & 4 \\ -1 & 2 \end{bmatrix}$.

1. Is $\vec{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ in $\ker(T)$? (If $T(\vec{u}) = \vec{0}$)
2. Determine a spanning set of $\ker(T)$.
3. Give a nonzero vector of $\mathrm{Im}(T)$.
4. Is $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ in $\mathrm{Im}(T)$? (If there exists $\vec{v}$ in $\mathbb{R}^2$ with $T(\vec{u}) = \vec{v}$, $\therefore$ $A\vec{x} = \vec{u}$)