Chap.III: Determinants
Sec.3.1: Introduction to Determinants

Objective: Compute the determinant of a matrix by using cofactor expansion.

Definitions

$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ $A_{23} = \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}$

Let $A = (a_{ij})$ be an $n \times n$ matrix.

- The submatrix A_{ij} of A is the matrix formed by deleting row i and column j of A (A_{ij} is also called ij-th minor of A).
- By definition, the determinant of A is the quantity

$$\det A = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + \cdots + (-1)^{n+1}\underset{a_{1n}}{}\det(A_{1n})$$

$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j})$$

- This definition is the extension of the definition of the determinant of a 2×2 matrix from Sec 2.2: the matrix A is invertible if and only if $\det A \neq 0$ (necessary and sufficient condition for A to have n pivots).
- Another common notation for the determinant of a matrix uses a pair of vertical lines in place of brackets. $\det(A_{23}) = \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = -6$

Cofactors

Let $A = (a_{ij})$ be an $n \times n$ matrix.

- The (i, j)-cofactor of A is the number C_{ij} given by

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

  Using cofactors, the definition of $\det(A)$ becomes

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

  This formula is called a cofactor expansion across the first row of A.
- The determinant of A can be computed by a cofactor across any row or down any column.
  The cofactor expansion across the ith row is given by

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

  The cofactor expansion down the jth column is given by

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Example

Compute the determinant of $A = \begin{bmatrix} 1 & 5 & -1 \\ 2 & 4 & -1 \\ 1 & -2 & 0 \end{bmatrix}$.

Expand along 3rd row

$$\det(A) = \underbrace{(+1)\, 1 \begin{vmatrix} 5 & 1 \\ 4 & -1 \end{vmatrix}}_{C_{11}} + \underbrace{(-1)(-2) \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix}}_{C_{12}} + \underbrace{(1)(0) \begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix}}_{C_{13}}$$

$$= 1(-5-4) + 2(-1-(-2)) + 0$$

$$= -1 + 2$$

$$= \boxed{1}$$

Theorem

If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A.

Down first column

$$\begin{vmatrix} -5 & 2 & -5 & -2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & -4 \end{vmatrix} = (1)(-5)\begin{vmatrix} 1 & -2 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & -4 \end{vmatrix} + (-1)(0)\det(A_{21}) + (1)(0)\det(A_{31}) + (-1)(0)\det(A_{41})$$

$$= (-5)\left(1 \begin{vmatrix} 3 & 2 \\ 0 & -4 \end{vmatrix} + (-1)(0)\begin{vmatrix} -2 & 1 \\ 0 & -4 \end{vmatrix} + (1)(0)\begin{vmatrix} -2 & 1 \\ 3 & 2 \end{vmatrix} \right)$$

$$= (-5)(1)\left(3(-4) - (0)(2)\right) + 0 + 0$$

$$= (-5)(1)(3)(-4)$$

Theorem

If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A.

$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A} \xrightarrow{aR_2 \to R_2} \overbrace{\begin{bmatrix} a & b \\ ac & ad \end{bmatrix}}^{\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}}$$

$$\xrightarrow{R_2 - cR_1 \to R_2} \underbrace{\overbrace{\begin{bmatrix} a & b \\ 0 & ad-bc \end{bmatrix}}^{\begin{bmatrix} 1 & 0 \\ -c & 1 \end{bmatrix}}}_{E_2E_1A}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\det(A) \neq \det(E_2E_1A)$$