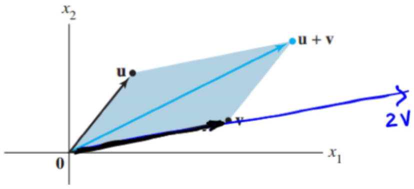
Chap 4:Vector Spaces
Sec.4.1: Vectors Spaces and Subspaces

Introduction

- 2-dimensional vectors: if **u** and **v** are 2-dimensional vectors:



then, $\mathbf{u}+\mathbf{v}$ and $c\mathbf{v}$ are still 2-dimensional vectors. All 2-dimensional vectors form the vector space $\mathbb{R}^2$.

- If **u** and **v** are 3-dimensional vectors and if c a scalar, then $\mathbf{u}+\mathbf{v}$ and $c\mathbf{u}$ are also 3-dimensional vectors. All 3-dimensional vectors form the vector space $\mathbb{R}^3$.
- We generalize this concept for the vector space $\mathbb{R}^n$, and even for some other vector spaces where the elements are not necessarily "vectors", such as matrices, functions, polynomials, $\ldots$.

Definition: Vetcor Spaces

A vector space is

- a nonempty set V of objects, called **vectors**,
- with two operations: **addition** and **scalar multiplication**, such that

1. for $\vec{u},\vec{v}\in V$, $\vec{u}+\vec{v}$ is in V (closure under addition).
2. for $\vec{u},\vec{v}\in V$, $\vec{u}+\vec{v}=\vec{v}+\vec{u}$ (commutative).
3. for $\vec{u},\vec{v},\vec{w}\in V$, $(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$ (associative).
4. There is a zero vector $\vec{0}$ in V such that for $\vec{u}\in V$, $\vec{0}+\vec{u}=\vec{u}$.
5. For each $\vec{u}\in V$, there is vector $-\vec{u}$ in V such that $\vec{u}+(-\vec{u})=\vec{0}$ (negatives).
6. The scalar multiple $c\vec{u}$ of $\vec{u}\in V$ by c is in V (closure under scalar multiplication).
7. for $\vec{u},\vec{v}\in V$, $c(\vec{u}+\vec{v})=c\vec{u}+c\vec{v}$ (distributive).
8. for $c,d\in\mathbb{R}$ and $\vec{u}\in V$, $(c+d)\vec{u}=c\vec{u}+d\vec{u}$ (distributive).
9. for $c,d\in\mathbb{R}$ and $\vec{u}\in V$, $c(d\vec{u})=(cd)\vec{u}$ (associative).
10. for $\vec{u}\in V$, $1\vec{u}=\vec{u}$ (identity).

Fact

For each vector $\vec{u}$ in a vector space V and for each scalar c:

1. $0\vec{u} = \vec{0}$.
2. $c\vec{0} = \vec{0}$.
3. $-\vec{u} = (-1)\vec{u}$ (negative $\vec{u}$ or the additive inverse of $\vec{u}$).

Examples

1. The set $\mathbb{R}^n$, where the addition and scalar multiplication are component-wise, **is a vector space**. Recall how to perform those operations.

- For $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$ and c in $\mathbb{R}$ (scalar), we have

$$\vec{u} + \vec{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix} \text{ and } c\vec{u} = \begin{bmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{bmatrix}$$

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- The negative of $\vec{u}$ is $-\vec{u} = \begin{bmatrix} -u_1 \\ -u_2 \\ \vdots \\ -u_n \end{bmatrix}$, and the zero vector is $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

2. For a positive integer n, let $\mathbb{P}_n$ denote the set of polynomials, with coefficients in $\mathbb{R}$, of degree at **most** n. How to define the sum and scalar multiplication in $\mathbb{P}_n$ to make it a vector space?

$$\mathbb{P}_n = \{a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n \mid a_i \text{ in } \mathbb{R}\}$$

In this case the polynomials are the vectors.

$n = 2$

$\mathbb{P}_2 = \{ a_0 + a_1 t + a_2 t^2 \mid a_0, a_0, a_1 \text{ in } \mathbb{R} \}$

$p(t) = -1 + 7t^2$ in $\mathbb{P}_2$

$g(t) = 3 + 5t$ in $\mathbb{P}_2$

$h(t) = 6 - t^3$ not in $\mathbb{P}_2$

$\mathbb{P}_3$

$p(t)$, $g(t)$, $h(t)$ are in $\mathbb{P}_3$

- Let $p(t)$ be a polynomial in $\mathbb{P}_n$

$$p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n, \quad a_i \in \mathbb{R}.$$

The degree of $p(t)$ is the highest power of t whose coefficient is not zero.

- If $p(t) = a_0$, where $a_0 \neq 0$ in $\mathbb{R}$, then the degree of $p(t)$ is zero.
- We call $p(t) = 0 + 0t + \cdots + 0t^{n-1} + 0t^n = 0$ the zero polynomial and we include it in $\mathbb{P}_n$ even if its degree is not defined.
- If $p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$ and $q(t) = b_0 + b_1 t + b_2 t^2 + \cdots + b_n t^n$ be polynomials in $\mathbb{P}_n$ and if c is a scalar, then we define
The sum $p + q$ of p and q

$$(p+q)(t) = p(t)+q(t) = (a_0+b_0)+(a_1+b_1)t+(a_2+b_2)t^2+\cdots+(a_n+b_n)t^n$$

the scalar multiplication cp

$$(cp)(t) = cp(t) = ca_0 + ca_1 t + ca_2 t^2 + \cdots + ca_n t^n$$

The polynomials $p + q$ and cp are polynomials of degree less than or equal to n, so they are elements of $\mathbb{P}_n$.
With these two operations, the set $\mathbb{P}_n$ is a vector space. The zero polynomial is the zero vector.

4. Let $\mathcal{C}((0,1))$ be the set of all continuous real-valued functions.

$$\mathcal{C}((0,1)) = \{f : (0,1) \to \mathbb{R} \mid f \text{ is continuous}\}.$$

Define the sum and scalar multiplication in $\mathcal{C}((0,1))$.

If f and g are in $\mathcal{C}((0,1))$ and if c is a scalar, we define the sum of $f+g$ of f and g

$$(f+g)(x) = f(x) + g(x)$$

the scalar multiplication cf

$$(cf)(x) = cf(x) \text{ for each } x \in (0,1).$$

Then $\mathcal{C}((0,1))$ is a vector space. The zero function defined by $f(x) = 0$, for each x in ~~$\mathbb{R}$~~, is the zero vector.
$(0,1)$

4. Let $M_{m\times n}(\mathbb{R})$ denote the collection of $m \times n$ matrices (with real number entries). Together with component-wise addition and component-wise scalar multiplication, $M_{m\times n}(\mathbb{R})$ is vector space. The zero matrix (every entry is zero) is the zero vector.

scalar multiplication, $M_{m\times n}(\mathbb{R})$ is vector space.

$$M_{2\times 2}(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; a, b, c, d \text{ in } \mathbb{R} \right\}.$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix}$$

$$\alpha \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{bmatrix} \longleftarrow \text{in } M_{2\times 2}(\mathbb{R}).$$

"Non"-example

Let $H = \{p(t) \text{ polynomial} \mid \deg(p) = 3\} \cup \{0\}$. That is, the elements of H are the zero polynomial and polynomials of degree exactly 3. Is H a vector space?

$$p(t) = 1 + 2t^2 - 5t^3 \quad \text{and} \quad g(t) = 6 + 5t^3$$

are both elements of H.

It is not a vector space.

$p(t)+g(t)$ is not in H
It is not closed under addition

$p(t)+g(t) \neq 0$ and of degree 2

$\deg(p(t)) = 3$ so $p(t)$ is in H.

$\deg(g(t)) = 3$ so $g(t)$ is in H.

$$p(t)+g(t) = (1+6) + 2t^2 + (-5+5)t^3 = 7 + 2t^2 + 0t^3$$
$$= 7 + 2t^2$$

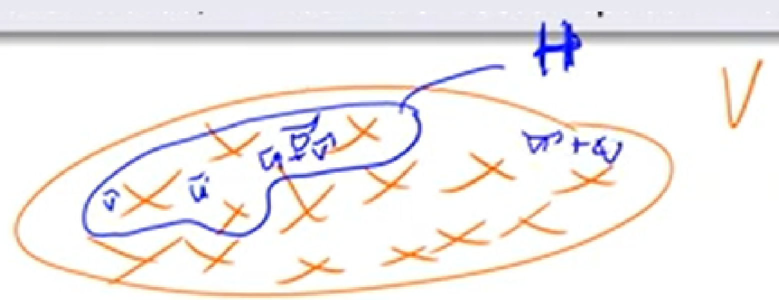
Subspaces

Definition

Let V be a vector space. A subspace of V is a subset H of V (i.e. $H \subseteq V$) that satisfies the following

1. The zero vector $\vec{0}$ is in H.
2. For every $\vec{u}$ and $\vec{v}$ in H, $\vec{u} + \vec{v}$ is also in H (closure under addition).
3. For every $\vec{u}$ in H and for every scalar c, $c\vec{u}$ is in H (closure under scalar multiplication).

In other words, H itself is a vector space which is smaller (or equal to) than V.



Example

For every vector space V

- V is a subspace of itself.
- The set $H = \{\vec{0}\}$ consisting of only the zero vector is a subspace of V (called the **zero subspace**).

Example

Let $H = \{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \mid a, b \in \mathbb{R}\}$. Show that H is a subspace of $\mathbb{R}^3$.

$\mathbb{R}^3$ is a vector space.

Since every vector in H has three entries, then is a subset of $\mathbb{R}^3$.

(1) Is $\vec{0}$ in H? If $a=b=0$, $\vec{0}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ (with the last entry 0) is in H.

(2) Is H closed under addition?

Let $\vec{u}=\begin{bmatrix}a\\b\\0\end{bmatrix}$ and $\vec{v}=\begin{bmatrix}a_1\\b_1\\0\end{bmatrix}$ be in H.

$\vec{u}+\vec{v}=\begin{bmatrix}a+a_1\\b+b_2\\0+0\end{bmatrix}=\begin{bmatrix}a+a_1\\b+b_1\\0\end{bmatrix}$. The last entry of $\vec{u}+\vec{v}$ is 0 so $\vec{u}+\vec{v}$ is in H

$\Longrightarrow$ H is closed under addition.

H is a subspace of $\mathbb{R}^3$.

(3) Is H closed under scalar multiplication?

Let $\vec{u}=\begin{bmatrix}a\\b\\0\end{bmatrix}$ be in H and α a scalar

$\alpha\vec{u}=\begin{bmatrix}\alpha a\\\alpha b\\\alpha 0\end{bmatrix}=\begin{bmatrix}\alpha a\\\alpha b\\0\end{bmatrix}$. The last entry of $\alpha\vec{u}$ is 0 so $\alpha\vec{u}$ is in H

$\Longrightarrow$ H is closed under scalar multiplication.

"Non"-examples

1. The vector space $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$ because $\mathbb{R}^2$ is not even a subset of $\mathbb{R}^3$.
2. Let $H = \{ \begin{bmatrix} 1 \\ b \\ c \end{bmatrix} \mid b, c \in \mathbb{R}\}$. Is H is subspace of $\mathbb{R}^3$?

Is $\vec{0}$ in H?

$\vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$: The first is not 1. So $\vec{0}$ is not in H.

$\Longrightarrow$ H is not a subspace of $\mathbb{R}^3$.

A subspace Spanned by a Set

Definition

Let V be a vector space and let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ be vectors in V (the $\vec{v}_i$'s could be vectors in $\mathbb{R}^n$ or polynomials, or matrices, ...). The set spanned by $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$, is

$$\mathrm{Span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p) = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_p\vec{v}_p \mid c_1, c_2, \ldots, c_p \in \mathbb{R}\}$$

i.e. it is the collection of all linear combinations of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$.
Clearly, $\mathrm{Span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p)$ is a subset of V.

Theorem

Let V be a vector space and let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ be vectors in V. Then $\mathrm{Span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p)$ is a subspace of V.
We call $H = \mathrm{Span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p)$ the subspace **spanned** (or generated) by $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$, and the set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is a **spanning** (or generating) set of H.

$p=2$, $\vec{v}_1, \vec{v}_2$ in V (vector space)

- $\vec{0} = 0\vec{v}_1 + 0\vec{v}_2$ is in $H = \mathrm{Span}(\vec{v}_1, \vec{v}_2)$
- $\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2$ and $\vec{w} = b_1\vec{v}_1 + b_2\vec{v}_2$ in H
  $\vec{u} + \vec{w} = (c_1+b_1)\vec{v}_1 + (c_2+b_2)\vec{v}_2$ still in H
  (H is closed under addition)
- $\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2$, α a scalar
  $\alpha\vec{u} = \alpha c_1\vec{v}_1 + \alpha c_2\vec{v}_2$ still in H

Example

Let H be the set of all vectors in $\mathbb{R}^4$ of the form $\begin{bmatrix} x_1 - 3x_2 \\ -x_1 + x_2 \\ x_1 \\ x_2 \end{bmatrix}$, where x_1 and x_2 are arbitrary scalars. Show that H is a subspace of $\mathbb{R}^4$.

For every vector $\vec{u}$ in H, $\vec{u}$ is of the form

$$\vec{u} = \begin{bmatrix} x_1 - 3x_2 \\ -x_1 + x_2 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 \\ x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -3x_2 \\ x_2 \\ 0 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2.$$

So every element $\vec{u} \in H$ is a linear combination of $\vec{v}_1$ and $\vec{v}_2$. It follows that $H = \mathrm{Span}(\vec{v}_1, \vec{v}_2)$. Thus by the previous Theorem, H is a subspace of $\mathbb{R}^4$.