Sec.2.2: The Inverse of a Matrix

Objective: To determine the inverse of a matrix by using elementary row operations.

$$A = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$$

$$AB = \begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & -2 & -1 \\ -2 & 5 & 6 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & 0 & \\ 0 & 1 & \vec{B}_1 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -2 & 2 \\ -2 & 5 & -9 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & 0 & \\ 0 & 1 & \vec{B}_2 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -2 & -1 \\ -2 & 5 & 3 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & 0 & \\ 0 & 1 & \vec{B}_3 \end{array}\right]$$

Problem: Too many steps, solution:

$$\left[\begin{array}{cc|ccc} 1 & -2 & -1 & 2 & -1 \\ -2 & 5 & 6 & -9 & 3 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{c|c} I_2 & B \end{array}\right]$$

Recall that for a positive integer n, I_n denotes the $n \times n$ identity matrix

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}$$

and if A is an $m \times n$ matrix, then

$$I_m A = A I_n = A$$

$I_1 = \begin{bmatrix} 1 \end{bmatrix}$ $\quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Definition (Invertible Matrix)

An $n \times n$ square matrix A is said to be invertible if there exists an $n \times n$ matrix B such that

$$AB = BA = I_n$$

, Cannot invert non-square matrices

- the matrix B is called the inverse of A and it is denoted by A^{-1}.
- If A is not invertible then A is said to be **singular**.

Example

Compute AB and BA where $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ and $B = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$. Is A invertible?

$AB = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$

$BA = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$

$AB = BA = I_2$

A is invertible and $A^{-1} = B$

B is invertible and $B^{-1} = A$

The case of 2×2 Matrices

Definition

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2×2 matrix. We define the determinant of A by the quantity

det(A)

$$\det(A) = ad - bc$$

Theorem

Let $A = \begin{bmatrix} a & -b \\ -c & d \end{bmatrix}$. Then is invertible if and only if $\det(A)$ is nonzero. In this case, we have

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Rhyme: Swap the main, negate the "lame" (Do not know why lame)

Example

Is the matrix $A = \begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}$ invertible. If so, compute A^{-1}.

$\det(A) = (8)(4) - (6)(5) = 2 \neq 0$

$A^{-1} = \frac{1}{2}\begin{bmatrix} 4 & -6 \\ -5 & 8 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -\frac{5}{2} & 4 \end{bmatrix}$

To confirm, verify $AA^{-1} = A^{-1}A = I_2$

$$\begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -\frac{5}{2} & 4 \end{bmatrix} = \begin{bmatrix} 8(2) + 6(-\frac{5}{2}) & 8(-3) + 6(4) \\ 5(2) + 4(-\frac{5}{2}) & 5(-3) + 4(4) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Proposition

Let A and B be $n \times n$ matrices.

1. If A is ivertible, then A^{-1} is invertible and
$$(A^{-1})^{-1} = A.$$
2. If A and B are invertible, then AB is invertible and
$$(AB)^{-1} = B^{-1}A^{-1}.$$
3. If A is invertible, then so is A^T, and
$$(A^T)^{-1} = (A^{-1})^T$$

Theorem

Let A be an invertible $n \times n$ matrix. Then, for every $\vec{b}$ in $\mathbb{R}^n$, the equation $A\vec{x} = \vec{b}$ has a unique solution $\vec{x} = A^{-1}\vec{b}$.

Example

Use the inverse of its coefficients matrix A to solve the following system

$$\begin{aligned} 8x_1 + 6x_2 &= 1 \\ 5x_1 + 4x_2 &= -1 \end{aligned}$$

(seen above)

$A\vec{x} = \vec{b}$, A^{-1} exists.

$\underbrace{A^{-1}A}_{I_n}\vec{x} = A^{-1}\vec{b}$

$I_n\vec{x} = A^{-1}\vec{b}$

$\vec{x} = A^{-1}\vec{b}$

$A = \begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} 2 & -3 \\ -\frac{5}{2} & 4 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^{-1}\vec{b} = \begin{bmatrix} 2 & -3 \\ -\frac{5}{2} & 4 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -\frac{13}{2} \end{bmatrix} = \vec{x}$

Elementary Matrices

Definition

An elementary matrix is an $n \times n$ matrix that can be obtained by performing a single elementary row operation to identity matrix I_n.
If $\mathcal{R}$ is an elementary row operation, then we obtain an elementary matrix E by $I_n \xrightarrow{\mathcal{R}} E$.

$I_1 = \begin{bmatrix} 1 \end{bmatrix}$ $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

The following matrices E_1, E_2 and E_3 are elementary matrices.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 + 2R_3 \to R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = E_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = E_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{3R_1 \to R_1} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_3$$

Example: Product of an elementary matrix and an arbitrary matrix

Compute $E_1 A$ and $E_2 A$ where

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}+2a_{31} & a_{22}+2a_{32} & a_{23}+2a_{33} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$E_2 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}.$$

$A \; R_2 + 2R_3 \to R_3 \Rightarrow E_1 A$

$I_3 \; R_2 + 2R_3 \to R_3 \; E$

$I_n \xrightarrow{R} E$

$A \xrightarrow{R} EA$

$A \; R_2 \leftrightarrow R_3 \Rightarrow E_2 A$

$I_3 \; R_2 \leftrightarrow R_3 \Rightarrow E_2$

$\mathcal{R}$ = "Script R", $\bar{\mathcal{R}}$ = "Script R Bar"

Note

1. Let $\mathcal{R}$ denote an elementary row operation. If $I_n \xrightarrow{\mathcal{R}} E$, then for any matrix A with n rows, $A \xrightarrow{\mathcal{R}} EA$.
2. Let $I_n \xrightarrow{\mathcal{R}} E$ and let $\bar{\mathcal{R}}$ be the reverse of $\mathcal{R}$ (transforming E back to I_n). Let $I_n \xrightarrow{\bar{\mathcal{R}}} \bar{E}$. Then, by (1),

$\mathcal{R}$ = Operation that transforms I_n to E

$$E \xrightarrow{\bar{\mathcal{R}}} \bar{E}E$$

$\bar{\mathcal{R}}$ = Operation that transforms E back to I_n

Since $\bar{\mathcal{R}}$ transforms E back to I_n, we have $\bar{E}E = I_n$ and E is invertible with $E^{-1} = \bar{E}$.

3. If A can be row reduced to B by a sequence of row operations $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_k$ and $I_n \xrightarrow{\mathcal{R}_i} E_i$ then $B = E_k E_{k-1} \cdots E_2 E_1 A$.
In particular, we have

$$A \xrightarrow{\mathcal{R}_1} E_1 A \xrightarrow{\mathcal{R}_2} E_2 E_1 A \cdots \xrightarrow{\mathcal{R}_k} E_k E_{k-1} \cdots E_2 E_1 A = B$$

Fact

Let E be an elementary matrix with $I_n \xrightarrow{\mathcal{R}} E$. Then E is invertible. If $E \xrightarrow{\bar{\mathcal{R}}} I_n$ and $I_n \xrightarrow{\bar{\mathcal{R}}} \bar{E}$, then $E^{-1} = \bar{E}$.

Example

Find the inverse of the elementary matrix $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$.

(1) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 - 4R_1 \to R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$

Finding this first step

(2) To transform E back to I_3:

$E \xrightarrow{R_3 + 4R_1 \to R_3} I_3$

(3) $\therefore I_3 \xrightarrow{R_3 + 4R_1 \to R_3} E^{-1}$

(4) $\therefore \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I_3} \xrightarrow{R_3 + 4R_1 \to R_3} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}}_{E^{-1}}$

(5) Check if $EE^{-1} = E^{-1}E$
$= I_3$

Theorem

An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, in which case the sequence of elementary row operations which transform A to I_n also transform I_n into A^{-1}.

Note

If $A \xrightarrow{\mathcal{R}_1} \xrightarrow{\mathcal{R}_2} \cdots \xrightarrow{\mathcal{R}_k} I_n$, then $\left[\begin{array}{c|c} A & I_n \end{array}\right] \xrightarrow{\mathcal{R}_1} \xrightarrow{\mathcal{R}_2} \cdots \xrightarrow{\mathcal{R}_k} \left[\begin{array}{c|c} I_n & A^{-1} \end{array}\right]$

Algorithm to Find A^{-1}

Given a matrix A, to find A^{-1}

1. Start with an augmented matrix $\left[\begin{array}{c|c} A & I_n \end{array}\right]$.
2. Row reduce the matrix to reduced row echelon form.
3. If the reduced echelon form is of the form $\left[\begin{array}{c|c} I_n & B \end{array}\right]$ then $A^{-1} = B$. If the matrix is of any other form, then A is not invertible.

Example

Let $A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$. Find A^{-1} by using elementary row operations.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 0 & 0 & 0 & 1 \end{array}\right] R_3 - 2R_1 \to R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & -3 & 4 & -2 & 0 & 1 \end{array}\right] R_3 + 3R_2 \to R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 16 & -2 & 3 & 1 \end{array}\right] 1/16R_3 \to R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1/8 & 3/16 & 1/16 \end{array}\right] R_2 - 4R_3 \to R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 1/4 & -1/4 \\ 0 & 0 & 1 & -1/8 & 3/16 & 1/16 \end{array}\right] R_1 + 2R_3 \to R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & 3/8 & 1/8 \\ 0 & 1 & 0 & 1/2 & 1/4 & -1/4 \\ 0 & 0 & 1 & -1/8 & 3/16 & 1/16 \end{array}\right]$$

Hence $A^{-1} = \begin{bmatrix} 3/4 & 3/8 & 1/8 \\ 1/2 & 1/4 & -1/4 \\ -1/8 & 3/16 & 1/16 \end{bmatrix}$

If A is invertible, find A^{-1}

$$A = \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix} \qquad A^{-1} = \frac{1}{(1)(2)-(0)(4)} \begin{bmatrix} 2 & -4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & \frac{1}{2} \end{bmatrix}$$

(Plug each matrix entry into numerator)

$$[A \mid I] = \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{array}\right] \xrightarrow{\frac{R_2}{2} \to R_2} \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} \end{array}\right]$$

$$\left[\underbrace{\begin{array}{cc} 1 & 4 \\ 0 & 2 \end{array}}_{A} \middle| \underbrace{\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}}_{I_n}\right]$$

$$\xrightarrow{R_1 - 4R_2 \to R_1} \left[\underbrace{\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}}_{I_n} \middle| \underbrace{\begin{array}{cc} 1 & -2 \\ 0 & \frac{1}{2} \end{array}}_{A^{-1}}\right]$$

Let A, B, C be $n \times n$ matrices such that A is invertible, and $AB = AC$. Show that $B = C$

(note, not always true)

Since A is invertible, A^{-1} exists

$$A^{-1}AB = A^{-1}AC$$

$$I_n B = I_n C$$

$$B = C$$