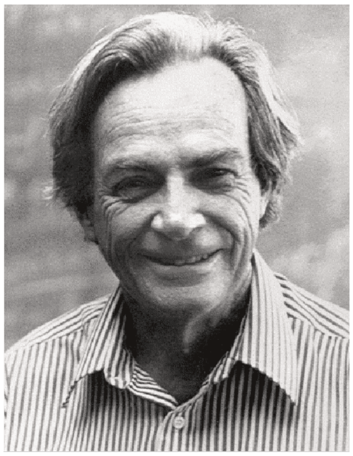
3 Determinants

INTRODUCTORY EXAMPLE

Random Paths and Distortion



In his autobiographical book *Surely You're Joking, Mr. Feynman*, the Nobel Prize–winning physicist Richard Feynman tells of observing ants in his Princeton graduate school apartment. He studied the ants' behavior by providing paper ferries to sugar suspended on a string where the ants would not accidentally find it. When an ant would step onto a paper ferry, Feynman would transport the ant to the food and then back. After the ants learned to use the ferry, he relocated the return landing. The colony soon confused the outbound and return ferry landings, indicating that their "learning" consisted of creating and following trails. Feynman confirmed this conjecture by laying glass slides on the floor. Once the ants established trails on the glass slides, he rearranged the slides and therefore the trails on them. The ants followed the repositioned trails and Feynman could direct the ants where he wished.

Suppose Feynman had decided to conduct additional investigations using a globe built of wire mesh on which an ant must follow individual wires and choose between going left and right at each intersection. If several ants and an equal number of food sources are placed on the globe, how likely is it that each ant would find its own food source rather than encountering another ant's trail and following it to a shared resource?

Note

The solution to the ant-path problem (and two other applications) can be found in a June 2005, *Mathematical Monthly* article by Arthur Benjamin and Naomi Cameron.

In order to record the actual routes of the ants and to communicate the results to others, it is convenient to use a rectangular map of the globe. There are many ways to create such maps. One simple way is to use the longitude and latitude on the globe as x and y coordinates on the map. As is the case with all maps, the result is not a faithful representation of the globe. Features near the "equator" look much the same on the globe and the map, but regions near the "poles" of the globe are distorted. Images of polar regions are much larger than the images of similar sized regions near the equator. To fit in with its surroundings on the map, the image of an ant near one of the poles should be larger than one near the equator. How much larger?

Surprisingly, both the ant–path and the area distortion problems are best answered through the use of the determinant, the subject of this chapter. Indeed, the determinant has so many uses that a summary of the applications known in the early

1900's filled a four volume treatise by Thomas Muir. With changes in emphasis and the greatly increased sizes of the matrices used in modern applications, many uses that were important then are no longer critical today. Nevertheless, the determinant still plays an important role.

In This Chapter

Beyond introducing the determinant in Section 3.1, this chapter presents two important ideas. Section 3.2 derives an invertibility criterion for a square matrix that plays a pivotal role in Chapter 5. Section 3.3 shows how the determinant measures the amount by which a linear transformation changes the area of a figure. When applied locally, this technique answers the question of a map's expansion rate near the poles. This idea plays a critical role in multivariable calculus in the form of the Jacobian.

3.1 INTRODUCTION TO DETERMINANTS

The Determinant of a 3×3 Matrix

Recall from Section 2.2 that a 2×2 matrix is invertible if and only if its determinant is nonzero. To extend this useful fact to larger matrices, a definition for the determinant of an $n \times n$ matrix is needed. We can discover the definition for the 3×3 case by watching what happens when an invertible 3×3 matrix A is row reduced.

Consider $A = [a_{ij}]$ with $a_{11} \neq 0$. If we multiply the second and third rows of A by a_{11} and then subtract appropriate multiples of the first row from the other two rows, we find that A is row equivalent to the following two matrices:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11}a_{21} & a_{11}a_{22} & a_{11}a_{23} \\ a_{11}a_{31} & a_{11}a_{32} & a_{11}a_{33} \end{bmatrix} \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{21}a_{12} & a_{11}a_{23} - a_{21}a_{13} \\ 0 & a_{11}a_{32} - a_{31}a_{12} & a_{11}a_{33} - a_{31}a_{13} \end{bmatrix} \tag{1}$$

Since A is invertible, either the (2, 2)-entry or the (3, 2)-entry on the right in (1) is nonzero. Let us suppose that the (2, 2)-entry is nonzero. (Otherwise, we can make a row interchange before proceeding.) Multiply row 3 by $a_{11}a_{22} - a_{21}a_{12}$, and then to the new row 3 add $-(a_{11}a_{32} - a_{31}a_{12})$ times row 2. This will show that

$$A \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{21}a_{12} & a_{11}a_{23} - a_{21}a_{13} \\ 0 & 0 & a_{11}\Delta \end{bmatrix}$$

where

$$\Delta = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{13}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{22}a_{13}a_{31} \tag{2}$$

Since A is invertible, Δ must be nonzero. The converse is true, too, as we will see in Section 3.2. We call Δ in (2) the **determinant** of the 3×3 matrix A.

Recall that the determinant of a 2×2 matrix, $A = [a_{ij}]$, is the number

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}$$

For a 1×1 matrix—say, $A = [a_{11}]$—we define $\det A = a_{11}$. To generalize the definition of the determinant to larger matrices, use 2×2 determinants to rewrite the 3×3 determinant Δ described above. Since the terms in Δ can be grouped as

$$\begin{aligned} \Delta &= (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32}) - (a_{12}a_{21}a_{33} - a_{12}a_{31}a_{23}) + (a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}) \\ &= a_{11} \cdot \det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \cdot \det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \cdot \det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \end{aligned}$$

For brevity, write

$$\Delta = a_{11} \cdot \det A_{11} - a_{12} \cdot \det A_{12} + a_{13} \cdot \det A_{13} \tag{3}$$

where A_{11}, A_{21}, and A_{13} are obtained from A by deleting the first row and one of the three columns.

The Determinant of an $n \times n$ Matrix

For any square matrix A, let A_{ij} denote the submatrix formed by deleting the ith row and jth column of A. For instance, let

$$A = \begin{bmatrix} 1 & -2 & 5 & 0 \\ 2 & 0 & 4 & -1 \\ 3 & 1 & 0 & 7 \\ 0 & 4 & -2 & 0 \end{bmatrix}$$

Use Figure A to investigate A_{ij} for different values of i and j.

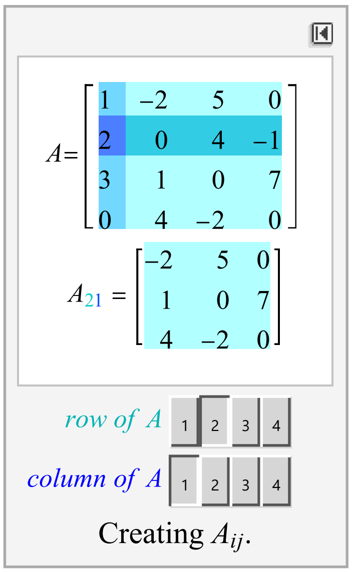


FIGURE A ◆

We can now give a *recursive* definition of a determinant. When $n = 3$, $\det A$ is defined using determinants of the 2×2 submatrices A_{1j}, as in (3) above. When $n = 4$, $\det A$ uses determinants of the 3×3 submatrices A_{1j}. In general, an $n \times n$ determinant is defined by determinants of $(n-1) \times (n-1)$ submatrices.

DEFINITION The Determinant

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j} \det A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A. In symbols

$$\begin{aligned} \det A &= a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j} \end{aligned}$$

EXAMPLE 1

EXAMPLE 2

Theorem 1 is helpful for computing the determinant of a matrix that contains many zeros. For example, if a row is mostly zeros, then the cofactor expansion across that row has many terms that are zero, and the cofactors in those terms need not be calculated. The same approach works with a column that contains many zeros.

EXAMPLE 3

Compute $\det A$, where

$$A = \begin{bmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$$

SOLUTION

The matrix in Example 3 was nearly triangular. The method in that example is easily adapted to prove the following theorem.

THEOREM 2 Determinants of Triangular Matrices

If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A.

The strategy in Example 3 of looking for zeros works extremely well when an entire row or column consists of zeros. In such a case, the cofactor expansion along such a row or column is a sum of zeros! So the determinant is zero. Unfortunately, most cofactor expansions are not so quickly evaluated.

NUMERICAL NOTE

By today's standards, a 25×25 matrix is small. Yet it would be impossible to calculate a 25×25 determinant by cofactor expansion. In general, a cofactor expansion requires over $n!$ multiplications, and $25!$ is approximately 1.5×10^{25}.

If a computer performs one trillion multiplications per second, it would have to run for over 500,000 years to compute a 25×25 determinant by this method. Fortunately, there are faster methods, as we'll soon discover.

Exercises 19–38 explore important properties of determinants, mostly for the 2×2 case. The results from Exercises 33–36 will be used in the next section to derive the analogous properties for $n\times n$ matrices.

PRACTICE PROBLEM

Compute $\begin{vmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{vmatrix}$.

SOLUTION TO PRACTICE PROBLEM

Take advantage of the zeros. Begin with a cofactor expansion down the third column to obtain a 3×3 matrix, which may be evaluated by an expansion down its first column.

$$\begin{vmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{vmatrix} = (-1)^{1+3}(2)\begin{vmatrix} 0 & 3 & -4 \\ -5 & -8 & 3 \\ 0 & 5 & -6 \end{vmatrix} = 2(-1)^{2+1}(-5)\begin{vmatrix} 3 & -4 \\ 5 & -6 \end{vmatrix} = 10(-18+20) = 20$$

The $(-1)^{2+1}$ in the next-to-last calculation came from the (2, 1)-position of the -5 in the 3×3 determinant.

3.1 EXERCISES

Compute the determinants in Exercises 1–8 using a cofactor expansion across the first row. In Exercises 1–4, also compute the determinant by a cofactor expansion down the second column.

1. $\begin{vmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{vmatrix}$

2. $\begin{vmatrix} 0 & 4 & 1 \\ 5 & -3 & 0 \\ 2 & 3 & 1 \end{vmatrix}$

3. $\begin{vmatrix} 2 & -2 & 3 \\ 3 & 1 & 2 \\ 1 & 3 & -1 \end{vmatrix}$

4. $\begin{vmatrix} 1 & 2 & 4 \\ 3 & 1 & 1 \\ 2 & 4 & 2 \end{vmatrix}$

5. $\begin{vmatrix} 2 & 3 & -3 \\ 4 & 0 & 3 \\ 6 & 1 & 5 \end{vmatrix}$

6. $\begin{vmatrix} 5 & -2 & 2 \\ 0 & 3 & -3 \\ 2 & -4 & 7 \end{vmatrix}$

7. $\begin{vmatrix} 4 & 3 & 0 \\ 6 & 5 & 2 \\ 9 & 7 & 3 \end{vmatrix}$

8. $\begin{vmatrix} 4 & 1 & 2 \\ 4 & 0 & 3 \\ 3 & -2 & 5 \end{vmatrix}$

Compute the determinants in Exercises 9–14 by cofactor expansions. At each step, choose a row or column that involves the least amount of computation.

9. $\begin{vmatrix} 4 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 3 & 0 & 0 & 0 \\ 8 & 3 & 1 & 7 \end{vmatrix}$

10. $\begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 2 \\ 2 & -4 & -3 & 5 \\ 2 & 0 & 3 & 5 \end{vmatrix}$

11. $\begin{vmatrix} 3 & 5 & -6 & 4 \\ 0 & -2 & 3 & -3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 3 \end{vmatrix}$

12. $\begin{vmatrix} 3 & 0 & 0 & 0 \\ 7 & -2 & 0 & 0 \\ 2 & 6 & 3 & 0 \\ 3 & -8 & 4 & -3 \end{vmatrix}$

13. $\begin{vmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{vmatrix}$

14. $\begin{vmatrix} 6 & 3 & 2 & 4 & 0 \\ 9 & 0 & -4 & 1 & 0 \\ 8 & -5 & 6 & 7 & 1 \\ 2 & 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 & 0 \end{vmatrix}$

The expansion of a 3×3 determinant can be remembered by the following device. Write a second copy of the first two columns to the right of the matrix, and compute the determinant by multiplying entries on six diagonals:

$$\begin{array}{ccccc} & & - & - & - \\ \left[\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}\right] & \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix} & & & \\ & & + & + & + \end{array}$$

Add the downward diagonal products and subtract the up- ward products. Use this method to compute the determinants in Exercises 15–18. ***Warning:*** *This trick does not generalize in any reasonable way to* 4×4 *or larger matrices.*

15. $\begin{vmatrix} 1 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -2 \end{vmatrix}$

16. $\begin{vmatrix} 0 & 3 & 1 \\ 4 & -5 & 0 \\ 3 & 4 & 1 \end{vmatrix}$

17. $\begin{vmatrix} 2 & -3 & 3 \\ 3 & 2 & 2 \\ 1 & 3 & -1 \end{vmatrix}$

18. $\begin{vmatrix} 1 & 3 & 4 \\ 2 & 3 & 1 \\ 3 & 3 & 2 \end{vmatrix}$

In Exercises 19–24, explore the effect of an elementary row operation on the determinant of a matrix. In each case, state the row operation and describe how it affects the determinant.

19. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} c & d \\ a & b \end{bmatrix}$

20. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a+kc & b+kd \\ c & d \end{bmatrix}$

21. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a & b \\ kc & kd \end{bmatrix}$

22. $\begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 5+3k & 4+2k \end{bmatrix}$

23. $\begin{bmatrix} a & b & c \\ 3 & 2 & 1 \\ 4 & 5 & 6 \end{bmatrix}, \begin{bmatrix} 3 & 2 & 1 \\ a & b & c \\ 4 & 5 & 6 \end{bmatrix}$

24. $\begin{bmatrix} 1 & 0 & 1 \\ -3 & 4 & -4 \\ 2 & -3 & 1 \end{bmatrix}, \begin{bmatrix} k & 0 & k \\ -3 & 4 & -4 \\ 2 & -3 & 1 \end{bmatrix}$

Compute the determinants of the elementary matrices given in Exercises 25–30. (See Section 2.2.)

25. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{bmatrix}$

26. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

27. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{bmatrix}$

28. $\begin{bmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

29. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}$

30. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Use Exercises 25–28 to answer the questions in Exercises 31 and 32. Give reasons for your answers.

31. What is the determinant of an elementary row replacement matrix?

32. What is the determinant of an elementary scaling matrix with k on the diagonal?

In Exercises 33–36, verify that $\det EA = (\det E)(\det A)$, where E is the elementary matrix shown and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

33. $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

34. $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

35. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

36. $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$

37. Let $A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$. Write $5A$. Is $\det 5A = 5 \det A$?

38. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and let k be a scalar. Find a formula that relates $\det kA$ to k and $\det A$.

In Exercises 39 and 40, A is an $n \times n$ matrix. Mark each statement True or False. Justify each answer.

39.

a. An $n \times n$ determinant is defined by determinants of $(n-1) \times (n-1)$ submatrices.
b. The (i, j)-cofactor of a matrix A is the matrix A_{ij} obtained by deleting from A its ith row and jth column.

40.

a. The cofactor expansion of $\det A$ down a column is equal to the cofactor expansion along a row.
b. The determinant of a triangular matrix is the sum of the entries on the main diagonal.

41. Let $\mathbf{u} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Compute the area of the parallelogram determined by $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u} + \mathbf{v}$, and $\mathbf{0}$, and compute the determinant of $[\ \mathbf{u}\ \ \mathbf{v}\ ]$. How do they compare? Replace the first entry of $\mathbf{v}$ by an arbitrary number x, and repeat the problem. Draw a picture and explain what you find.

42. Let $\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} c \\ 0 \end{bmatrix}$, where a, b, and c are positive (for simplicity). Compute the area of the parallelogram determined by $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u} + \mathbf{v}$, and $\mathbf{0}$, and compute the determinants of the matrices $[\ \mathbf{u}\ \ \mathbf{v}\ ]$ and $[\ \mathbf{v}\ \ \mathbf{u}\ ]$. Draw a picture and explain what you find.

43. **[M]** Construct a random 4×4 matrix A with integer entries between -9 and 9. How is $\det A^{-1}$ related to $\det A$? Experiment with random $n \times n$ integer matrices for $n = 4$, 5, and 6, and make a conjecture. *Note:* In the unlikely event that you encounter a matrix with a zero determinant, reduce it to echelon form and discuss what you find.

44. **[M]** Is it true that $\det AB = (\det A)(\det B)$? To find out, generate random 5×5 matrices A and B, and compute $\det AB - (\det A \det B)$. Repeat the calculations for three other pairs of $n \times n$ matrices, for various values of n. Report your results.

45. **[M]** Is it true that $\det(A + B) = \det A + \det B$? Experiment with four pairs of random matrices as in Exercise 44, and make a conjecture.

46. **[M]** Construct a random 4×4 matrix A with integer entries between -9 and 9, and compare $\det A$ with $\det A^T$, $\det(-A)$, $\det(2A)$, and $\det(10A)$. Repeat with two other random 4×4 integer matrices, and make conjectures about how these determinants are related. (Refer to Exercise 36 in Section 2.1.) Then check your conjectures with several random 5×5 and 6×6 integer matrices. Modify your conjectures, if necessary, and report your results.