Sec.3.3:Cramer's Rule, Volume, and Linear Transformations

Objectives:

- Use Cramer's rule to solve linear systems
- Define Inverse Formula
- Understand the geometric interpretations of determinants

Cramer's Rule

- Cramer's rule is used to solve linear systems with an invertible matrix coefficient.
- Recall that a linear system has a unique solution if and only if its coefficient matrix is invertible.

Definition

Let $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}$ be an $n \times n$ matrix. For any $\vec{b}$ in $\mathbb{R}^n$, we define

$$A_i(\vec{b}) = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_{i-1} & \vec{b} & \vec{a}_{i+1} & \cdots & \vec{a}_n \end{bmatrix}$$

Theorem (Cramer's Rule)

Let A be an $n \times n$ invertible matrix. For any $\vec{b}$ in $\mathbb{R}^n$, the unique solution $\vec{x}$ of the equation $A\vec{x} = \vec{b}$ has entries given by

$$x_i = \frac{\det(A_i(\vec{b}))}{\det(A)}, \quad \text{for } i = 1, 2, \ldots n.$$

Example

Use Cramer's rule to solve $A\vec{x} = \vec{b}$ where $A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Solution: Since $\det(A) = -1$, the equation $A\vec{x} = \vec{b}$ has a unique solution. We have

$$A_1(\vec{b}) = \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix} \text{ and } A_2(\vec{b}) = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$$

with

$$\det(A_1(\vec{b})) = 4 \text{ and } \det(A_2(\vec{b})) = -3.$$

By Cramer's Rule, we have

$$\begin{aligned} x_1 &= \frac{\det(A_1(\vec{b}))}{\det(A)} = \frac{4}{-1} = -4 \\ x_2 &= \frac{\det(A_2(\vec{b}))}{\det(A)} = \frac{-3}{-1} = 3 \end{aligned}$$

So the unique solution is $\vec{x} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$.

Example

Consider the linear system for which s is an unspecified parameter.

$$\begin{aligned} 2sx_1 + 2x_2 &= 1 \\ 5x_1 + x_2 &= -1 \end{aligned}$$

For which s the system has a unique solution. For such s, describe the solution.

Solution: The system is equivalent to the matrix equation $A\vec{x} = \vec{b}$, where $A = \begin{bmatrix} 2s & 2 \\ 5 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The system has a unique solution if and only if $\det(A) \neq 0$, i.e. $2s - 10 = 2(s-5) \neq 0$. Then $s - 5 \neq 0$, so $s \neq 5$. For the solution, we have

$$A_1(\vec{b}) = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \text{ and } A_2(\vec{b}) = \begin{bmatrix} 2s & 1 \\ 5 & -1 \end{bmatrix}$$

For such an s, the solution is (x_1, x_2) where

$$x_1 = \frac{\det(A_1(\vec{b}))}{\det(A)} = \frac{3}{2(s-5)} \text{ and } x_2 = \frac{\det(A_2(\vec{b}))}{\det(A)} = \frac{-2s-5}{2(s-5)}$$

A Formula for A^{-1}

Definition

Let A be an $n \times n$ matrix. We define the classical adjoint (or adjugate) of A to be the $n \times n$ matrix given by

$$\mathrm{Adj}A = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

where $C_{ij} = (-1)^{i+j}\det(A_{ij})$ the (i,j)-cofactor of A.
Note that

$$\mathrm{Adj}A = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}^T$$

Example

Find the adjugate matrix of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$$\mathrm{Adj}\ A = \begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix} \qquad C_{ij} = (-1)^{i+j} \det A_{ij}$$

$$C_{11} = (-1)^{1+1} \det A_{11} = d$$

$$C_{21} = (-1)^{2+1} \det A_{21} = -b$$

$$C_{12} = (-1)^{1+2} \det A_{12} = -c$$

$$C_{22} = (-1)^{2+2} \det A_{22} = a$$

$$\mathrm{Adj}\ A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Theorem (Adjoint Inverse Formula)

Let A be an invertible matrix. Then

$$A^{-1} = \frac{1}{\det(A)} \mathrm{Adj} A$$

Example

Compute the inverse of the matrix $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix}$ by using the adjoint inverse formula.

We have

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

and

$$\det(A) = 14$$

$$C_{11} = +\begin{vmatrix} -1 & 1 \\ 4 & -2 \end{vmatrix} = -2, \quad C_{12} = -\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = 3, \quad C_{13} = +\begin{vmatrix} 1 & -1 \\ 1 & 4 \end{vmatrix} = 5$$

$$C_{21} = -\begin{vmatrix} 1 & 3 \\ 4 & -2 \end{vmatrix} = 14, \quad C_{22} = +\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = -7, \quad C_{23} = -\begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = -7$$

$$C_{31} = +\begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} = 4, \quad C_{32} = -\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = 1, \quad C_{33} = +\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3$$

$$C_{23} = (-1)^{2+3} \det A_{23} = -\begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = -(8-1) = -7$$

$$A^{-1} = \frac{1}{14} \mathrm{Adj} A = \frac{1}{14} \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix}$$

So

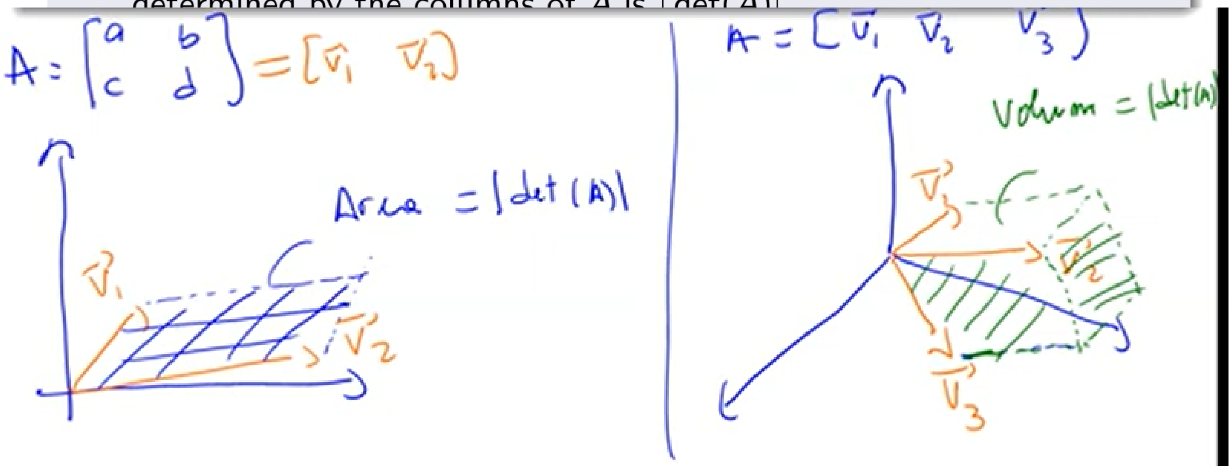
$$A^{-1} = \begin{bmatrix} -1/7 & 1 & 2/7 \\ 3/14 & -1/2 & 1/14 \\ 5/14 & -1/2 & -3/14 \end{bmatrix}$$

$$AA^{-1} = A^{-1}A = I_3$$

Determinants as Area or Volume

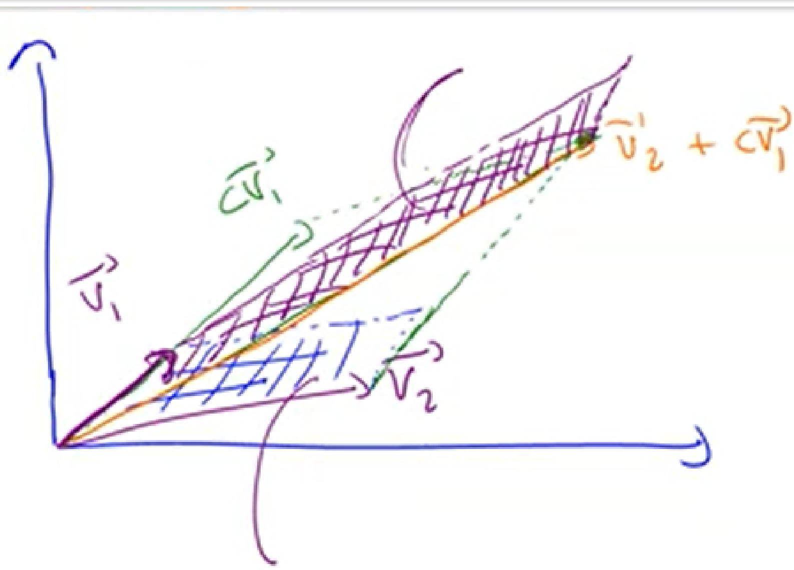
Theorem

1. If A is a 2×2 matrix, then the area of the parallelogram determined by its columns is $|\det(A)|$.
2. If A is a 3×3 matrix, then the volume of the parallelepiped determined by the columns of A is $|\det(A)|$.



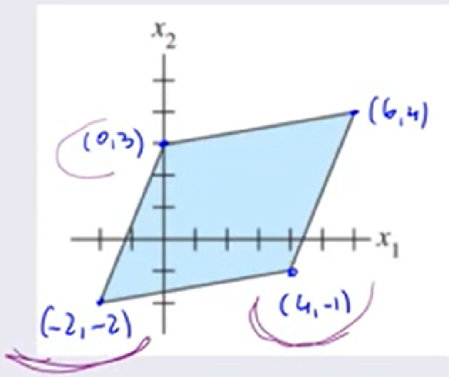
Note

Let $\vec{v}_1$ and $\vec{v}_2$ be nonzero vectors. Then for any scalar c, the area of the parallelogram determined by $\vec{v}_1$ and $\vec{v}_2$ equals the area of the parallelogram determined by $\vec{v}_1$ and $\vec{v}_2 + c\vec{v}_1$.



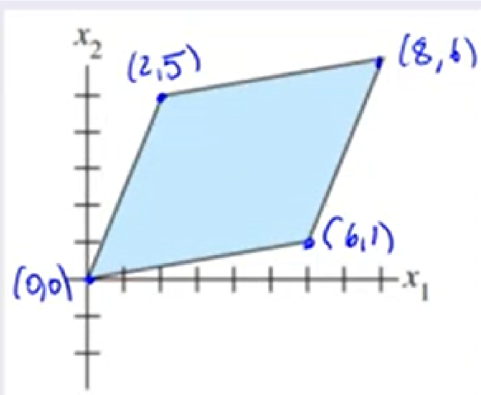
Example

Compute the area of the parallelogram determined by the points $(-2,-2), (0,3), (4,-1)$ and $(6,4)$.



$(-2,-2) - (-2,-2) = (0,0)$
$(0,3) - (-2,-2) = (2,5)$
$(4,-1) - (-2,-2) = (6,1)$
$(6,4) - (-2,-2) = (8,6)$

Solution: We can translate the parallelogram to one having the origin $(0,0)$ as a vertex. For example, subtract $(-2,-2)$ from each of the vertices. The new parallelogram has vertices $(0,0), (2,5), (6,1)$ and $(8,6)$, and it has the same area as the same area as the first parallelogram.



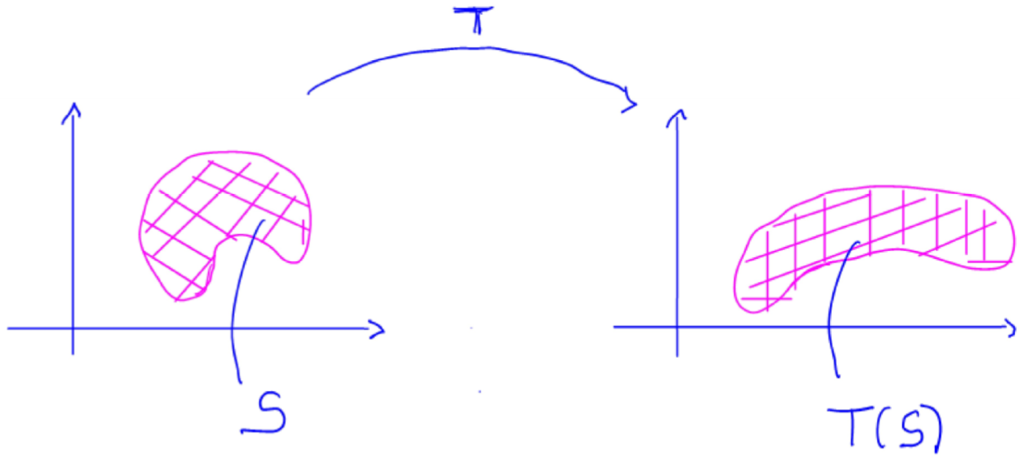
The new parallelogram is determined by the columns of

$$A = \begin{bmatrix} 2 & 6 \\ 5 & 1 \end{bmatrix}$$

We have $\det(A) = -28$, hence the area of the parallelogram is $|\det(A)| = 28$.

Linear Transformations and Determinants

If T is a linear transformation with domain $\mathbb{R}^2$ or $\mathbb{R}^3$, and if S is a set in the domain of T, then we denote by $T(S)$ the set of images of points (vectors) in S.



Theorem

1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation determined by a 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then
$$\mathrm{Area}(T(S)) = |\det(A)|\mathrm{Area}(S).$$
2. If $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation determined by a 3×3 standard matrix A, and if S is a parallelepiped in $\mathbb{R}^3$, then
$$\mathrm{Volume}(T(S)) = |\det(A)|\mathrm{Volume}(S).$$
3. In general, if S is any region of $\mathbb{R}^2$ or of $\mathbb{R}^3$, then the formulas for $\mathrm{Area}(T(S))$ and $\mathrm{Volume}(T(S))$ in (a) and (b) hold.

Example

Let S be the parallelogram determined by the vectors $\vec{v}_1 = \begin{pmatrix} -2 \\ 6 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} -2 \\ 9 \end{pmatrix}$. Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $T(\vec{x}) = A\vec{x}$ where $A = \begin{pmatrix} 4 & -7 \\ -2 & 7 \end{pmatrix}$. Compute the the area of the image $T(S)$ of S under T.

Solution

we have

$$\text{Area}(T(S)) = |A|\text{Area}(S)$$

And

$$|A| = |28 - 14| = 14 \text{ and } \text{Area}(S) = |\det \begin{pmatrix} -2 & -2 \\ 6 & 9 \end{pmatrix}| = |-18 + 12| = 6$$

So $\text{Area}(T(S)) = 14 \times 6 = 84.$