A_∞-minimal models of matrix factorisations

Daniel Murfet

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Abstract

We study A_∞-minimal models of the differential graded algebras obtained from endomorphisms of generators of triangulated categories of matrix factorisations, giving explicit higher products for the cases of simple singularities.

1 Introduction

Let k be a characteristic zero field, and $W \in k[x_1, \ldots, x_n]$ a potential in the sense of [3, §2.2] which has the origin as its only critical point. Choose a presentation of W as a sum $W = x_1 W^1 + \cdots + x_n W^n$ with $W^i \in \mathfrak{m}$. Then the generator of the homotopy category of matrix factorisations is the stabilised diagonal

$$k^{\text{stab}} = \Big(k[x] \otimes_k \bigwedge (k\psi_1 \oplus \cdots \oplus k\psi_n), \sum_i x_i \psi_i^* + \sum_i W^i \psi_i \Big) . \tag{1.1}$$

We calculate the A_∞-minimal model $\mathscr{A}_W$ of the differential graded algebra $\operatorname{End}_{k[x]}(k^{\text{stab}})$. This is a finite-dimensional $\mathbb{Z}_2$-graded vector space, with a family of odd k-linear maps

$$\Big\{ r_q : \big(\mathscr{A}_W[1]\big)^{\otimes q} \longrightarrow \mathscr{A}_W[1] \Big\}_{q \geq 2}$$

satisfying the forward suspended A_∞-constraints [1].

2 Perturbation and Koszul complexes

Set $R = k[x_1, \ldots, x_n]$. To apply the A_∞-minimal model construction to the DG-algebra $\operatorname{End}_R(k^{\text{stab}})$, we would usually begin by finding a k-linear homotopy retract of this complex onto its cohomology. However, for our purposes it is better to first enlarge the DG-algebra to $S \otimes_k \operatorname{End}_R(k^{\text{stab}})$, where

$$S = \bigwedge (k\theta_1 \oplus \cdots \oplus k\theta_n) \tag{2.1}$$

is viewed as a DG-algebra with zero differential, and $\mathbb{Z}_2$-grading determined by $|\theta_i| = 1$. We make the tensor product $S \otimes_k \operatorname{End}_R(k^{\text{stab}})$ into a DG-algebra in the usual way, and show that it is homotopy equivalent to the following DG-algebra:

Definition 2.1. We define $\underline{\operatorname{End}}(k^{\text{stab}})$ to be the finite-dimensional algebra

$$\underline{\operatorname{End}}(k^{\text{stab}}) = R/\mathfrak{m} \otimes_k \operatorname{End}_R(k^{\text{stab}}) \cong \operatorname{End}_k\left(\bigwedge(k\psi_1 \oplus \cdots \oplus k\psi_n)\right) \tag{2.2}$$

where $\mathfrak{m} = (x_1, \ldots, x_n)$, which we view as a DG-algebra with differential zero (since $W \in \mathfrak{m}^2$ this differential on $\underline{\operatorname{End}}(k^{\text{stab}})$ is the one inherited from $\operatorname{End}_R(k^{\text{stab}})$).

The corresponding homotopy retract (see Proposition 2.2 below) is our starting point for the minimal model construction. For this we use the Koszul complex of $x_1, \ldots, x_n$,

$$K = \Big(R \otimes_k \bigwedge(k\theta_1 \oplus \cdots \oplus k\theta_n), d_K = \sum_i x_i \theta_i^*\Big). \tag{2.3}$$

and the following diagram of $\mathbb{Z}_2$-graded complexes and homotopy equivalences over k:

$$\begin{tikzcd} S \otimes_k \operatorname{End}_R(k^{\text{stab}}) \arrow[r, shift left, "\exp(-\delta)"] & K \otimes_R \operatorname{End}_R(k^{\text{stab}}) \arrow[l, shift left, "\exp(-\delta)"] \arrow[r, shift left, "\pi"] & \underline{\operatorname{End}}(k^{\text{stab}}) \arrow[l, shift left, "\sigma_\infty"] \end{tikzcd}. \tag{2.4}$$

The notation is as follows:

- The operator $\psi_j = \psi_j \wedge -$ on k^{stab} satisfies
$$[\psi_j, d_{k^{\text{stab}}}] = \Big[\psi_j, \sum_i x_i \psi_i^*\Big] = x_j \cdot 1\,,$$
where $[-,-]$ always denotes the graded commutator. That is, ψ_j is a homotopy for the action of x_j on k^{stab}. It is then easy to check that the odd operator $\alpha \mapsto \psi_j \circ \alpha$ on $\operatorname{End}_R(k^{\text{stab}})$ is also a homotopy for x_j, and where it will not cause confusion we will also write ψ_j for this operator of post-composition.

- Both $S \otimes_k \operatorname{End}_R(k^{\text{stab}})$ and $K \otimes_R \operatorname{End}_R(k^{\text{stab}})$ have the same underlying $\mathbb{Z}_2$-graded k-module, namely
$$R \otimes_k \bigwedge(k\theta_1 \oplus \cdots \oplus k\theta_n) \otimes_k \operatorname{End}_k\left(\bigwedge(k\psi_1 \oplus \cdots \oplus k\psi_n)\right).$$
On this space we define the even operator
$$\delta = \sum_{i=1}^n \psi_i \theta_i^*\,, \tag{2.5}$$
where ψ_i is the operator on $\operatorname{End}_R(k^{\text{stab}})$ discussed above, given by $\alpha \mapsto (\psi_i \wedge -) \circ \alpha$, and θ_i^* acts by contraction on S. Note that in (2.4) the complex on the very left has differential $1 \otimes d_{k^{\text{stab}}}$ while the middle complex has differential $1 \otimes d_{k^{\text{stab}}} + d_K \otimes 1$. It is easy to check that $\exp(\delta), \exp(-\delta)$ intertwines these differentials and gives an isomorphism of complexes [2, Proposition 4.11].

- The morphism of complexes $\pi : K \longrightarrow R/\mathfrak{m}$ is defined by composing the projection of K onto the submodule $R\cdot 1$ of θ-degree zero forms, with the quotient $R \longrightarrow R/\mathfrak{m}$. We also write π for the result of tensoring this morphism with $\mathrm{End}_R(k^{\mathrm{stab}})$ to obtain

$$K \otimes_R \mathrm{End}_R(k^{\mathrm{stab}}) \xrightarrow{\pi\otimes 1} R/\mathfrak{m} \otimes_R \mathrm{End}_R(k^{\mathrm{stab}}) = \underline{\mathrm{End}}(k^{\mathrm{stab}})\,.$$

- The k-linear homotopy inverse σ_∞ to π in (2.4) is defined in terms of the following ingredients. The first is the k-linear connection (viewing θ_i as a 1-form)

$$\Delta : K \longrightarrow K\,, \qquad \Delta = \sum_i \partial_{x_i}\theta_i \tag{2.6}$$

and the degree -1 (with respect to the θ-degree) k-linear operator

$$H = [d_K, \Delta]^{-1}\Delta\,. \tag{2.7}$$

We write d_{End} for the differential on $\mathrm{End}_R(k^{\mathrm{stab}})$, and $\sigma : k \longrightarrow K$ for the map which sends $\lambda \in k$ to $\lambda \cdot 1 \in K$, and define k-linear maps

$$\sigma_\infty = \sum_{m\geq 0}(-1)^m (H d_{\mathrm{End}})^m \sigma\,, \tag{2.8}$$

$$H_\infty = \sum_{m\geq 0}(-1)^m (H d_{\mathrm{End}})^m H\,. \tag{2.9}$$

Here H_∞ is an odd operator on $K \otimes_R \mathrm{End}_R(k^{\mathrm{stab}})$. These maps satisfy:

$$\pi\sigma_\infty = 1\,, \tag{2.10}$$

$$\sigma_\infty\pi = 1 - [d_K + d_{\mathrm{End}}, H_\infty]\,. \tag{2.11}$$

By inspection of (2.4), we therefore have the following:

Proposition 2.2. *There is a diagram of morphisms of $\mathbb{Z}_2$-graded k-complexes*

$$S \otimes_k \mathrm{End}_R(k^{\mathrm{stab}}) \underset{\Phi^{-1}}{\overset{\Phi}{\rightleftarrows}} \underline{\mathrm{End}}(k^{\mathrm{stab}})\,. \tag{2.12}$$

where $\Phi = \pi\exp(-\delta)$, $\Phi^{-1} = \exp(\delta)\sigma_\infty$ *and* $\widehat{H} = \exp(\delta)H_\infty\exp(-\delta)$ *satisfy*

$$\Phi\Phi^{-1} = 1\,, \qquad \Phi^{-1}\Phi = 1 - [d_{\mathrm{End}}, \widehat{H}]\,. \tag{2.13}$$

Recall that the differential on $\underline{\mathrm{End}}(k^{\mathrm{stab}})$ is zero, so that the above homotopy equivalence actually computes the cohomology of the DG-algebra on the left hand side.

3 Examples

To describe the higher multiplications it is convenient to write $[\psi_i, -]$ for the natural operator on $\mathscr{A}$ (giving ψ_i, ψ_j^* the usual anticommutation relations for fermionic creation and annhiliation operators), that is

$$\begin{aligned}[\psi_i, \psi_{j_1}^* \cdots \psi_{j_t}^*] &= \sum_{l=1}^{t} (-1)^{l-1} \psi_{j_1}^* \cdots [\psi_i, \psi_{j_l}^*] \cdots psi_{j_t}^* \\ &= \sum_{l=1}^{t} (-1)^{l-1} \delta_{i=j_l} \psi_{j_1}^* \cdots \psi_{j_t}^* \,.\end{aligned}$$

A general fact is that the higher multiplications on $\mathscr{A}$ are linear combinations of products of such operators. For example, when $n = 2$ a standard term in r_3 would look like

$$\Phi_0 \otimes \Phi_1 \otimes \Phi_2 \mapsto \lambda \cdot [\psi_1, [\psi_2, \Phi_0]] \cdot [\psi_1, \Phi_1] \cdot [\psi_2, \Phi_2] \,.$$

The coefficient λ is computed as a Feynman amplitude on a binary planar tree with with incoming fermion states $\psi_1\psi_2, \psi_1, \psi_2$ in the three leaves (respectively) and a simple list of allowed interactions, the most significant of which is a trivalent interaction vertex with an incoming fermion ψ_j and outgoing fermion θ_i and bosons $\partial_{x_i}(x^\gamma)$ whenever the polynomial W^j has a nonzero coefficient for the monomial $x^\gamma = x_1^{\gamma_1} x_2^{\gamma_2}$ (the fermions θ_i are the auxiliary spinor representation generators as in [2]). There are two other interactions which do not depend on W, and which take place only at internal edges or internal vertices (respectively).

Example 3.1. Let $W = x^d$ so that $\mathscr{A} = k \oplus k\psi^*$. Then only r_2 and r_d are nonzero and on $(\mathscr{A}[1])^{\otimes d}$ the only basis element with a nonzero value under r_d is

$$r_d(\psi^* \otimes \cdots \otimes \psi^*) = 1 \,.$$

Another way to say this is

$$r_d(\Phi_0 \otimes \cdots \otimes \Phi_{d-1}) = \prod_{i=0}^{d-1} [\psi, \Phi_i] \,.$$

This A_∞-structure is cyclic with respect to the trace form on $\mathscr{A}$ which projects onto the $k\psi^*$-summand. Note that all such products are expanded such that the index i increases from left to right.

References

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