

A_∞ -minimal models of matrix factorisations

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November 2, 2016

Abstract

We study A_∞ -minimal models of the differential graded algebras obtained from endomorphisms of generators of triangulated categories of matrix factorisations, giving explicit higher products for the cases of simple singularities.

1 Introduction

Let k be a commutative \mathbb{Q} -algebra and $W \in k[x_1, \dots, x_n]$ a polynomial which is a potential over k in the sense of [3, §2.2]. For example, $k = \mathbb{C}$ and W has isolated critical points. The \mathbb{Z}_2 -graded DG-category $\mathcal{A} = \text{mf}(R, W)$ has for its objects finite-rank matrix factorisations of W over $R = k[x_1, \dots, x_n]$. In this paper we study the minimal model problem for \mathcal{A} , the aim being to produce models that are finitely-generated and projective over k .

When k is a field, the standard minimal model theorem [?] constructs the structure of an A_∞ -category on the cohomology $\mathcal{B} = H^*\mathcal{A}$ in such a way that \mathcal{B} is quasi-isomorphic to \mathcal{A} . Moreover, this A_∞ -category has finite-dimensional Hom-spaces and is therefore a good finite model of \mathcal{A} . The problem in this case is to have sufficient control over the inputs to the minimal model theorem that the A_∞ -products on $H^*\mathcal{A}(X, X)$ can be reasonably calculated, for a given matrix factorisation X . In order to understand deformations of matrix factorisations [?, ?, ?] it is important to do this for generic X , but the special case where $X = k^{\text{stab}}$, the generator of \mathcal{A} studied by Dyckerhoff in the case of a single isolated singularity at the origin [Dyc11], is of particular interest.

Since one of our goals is to understand how matrix factorisation categories vary along unfoldings of singularities, we also need to consider the case where k is not a field. For example take $k = \mathbb{C}[t]$ and $R = \mathbb{C}[x_1, \dots, x_n, t]$ so that $W = W_t(x)$ is a potential with parameter. In this case we want an A_∞ -category \mathcal{B} quasi-isomorphic to \mathcal{A} over k , with $\mathcal{B}(a, b)$ a finitely-generated projective k -module for each pair of objects a, b .

These two desiderata, understanding the A_∞ -products on $H^*\mathcal{A}(X, X)$ for generic X , and the case where k is not a field, lead us naturally to a slightly non-standard use of the minimal model theorem. Following the ideas developed in [?, ?, ?] we prove that, writing $\iota : R \longrightarrow R/(\partial_{x_1}W, \dots, \partial_{x_n}W)$ for the quotient, there is a k -linear homotopy retract

$$(1.1) \quad \iota^* \mathcal{A} \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \mathcal{A} \otimes_k \bigwedge (k^{\oplus n}[1]) .$$

Suppose $W \in \mathfrak{m}^2$ where $\mathfrak{m} = (x_1, \dots, x_n)$ and choose a presentation $W = x_1 W^1 + \dots + x_n W^n$ with $W^i \in \mathfrak{m}$ and consider the following pair

$$(1.2) \quad k^{\text{stab}} = \left(k[x] \otimes_k \bigwedge (k\psi_1 \oplus \dots \oplus k\psi_n), d_{k^{\text{stab}}} = \sum_{i=1}^n x_i \psi_i^* + \sum_{i=1}^n W^i \psi_i \right)$$

where ψ_i^* and ψ_i act respectively by contraction and wedge product on the exterior algebra underlying k^{stab} . It is easy to see that $d_{k^{\text{stab}}}^2 = W$, so k^{stab} is a matrix factorisation. When k is a field and W has an isolated critical point at the origin, k^{stab} is the representative in the homotopy category of matrix factorisations of the structure sheaf of the singular point at the origin, and was first studied by Dyckerhoff [Dyc11].

The purpose of this paper is to show how to calculate the A_∞ -minimal model of the \mathbb{Z}_2 -graded differential graded endomorphism algebra of this matrix factorisation

$$(1.3) \quad \left(\text{End}_{k[x]}(k^{\text{stab}}), \partial = [d_{k^{\text{stab}}}, -] \right).$$

The differential here is (throughout all commutators are graded commutators)

$$\begin{aligned} \partial = [d_{k^{\text{stab}}}, -] &= \left[\sum_i x_i \psi_i^* + \sum_i W^i \psi_i, - \right] \\ &= \sum_i x_i [\psi_i^*, -] + \sum_i W^i [\psi_i, -]. \end{aligned}$$

The minimal model is a finite-dimensional \mathbb{Z}_2 -graded vector space \mathcal{M}_W with a family

$$\left\{ r_q : (\mathcal{M}_W[1])^{\otimes q} \longrightarrow \mathcal{M}_W[1] \right\}_{q \geq 2}$$

of odd k -linear maps satisfying the forward suspended A_∞ -constraints [1], and having the property that there is an A_∞ -quasi-isomorphism $\mathcal{M}_W \longrightarrow \text{End}_{k[x]}(k^{\text{stab}})$.

2 Background

2.1 A-infinity algebras

The *tilde grading* is defined by $\tilde{x} = |x| - 1$.

Throughout $\otimes = \otimes_k$.

3 Perturbation and Koszul complexes

Let k be a characteristic zero field, $R = k[x_1, \dots, x_n]$ and let $W \in \mathfrak{m}^2$ be a potential with chosen decomposition $W = \sum_{i=1}^n x_i W^i$. To apply the A_∞ -minimal model construction to the DG-algebra \mathcal{A}_W from (1.3) we would usually begin by finding a k -linear homotopy

retract of this complex onto its cohomology. However, for our purposes it is better to first enlarge the DG-algebra to $S \otimes \mathcal{A}_W$, where S is the \mathbb{Z}_2 -graded vector space

$$(3.1) \quad S = \bigwedge (k\theta_1 \oplus \cdots \oplus k\theta_n)$$

with grading $|\theta_i| = 1$. We make the tensor product $S \otimes \mathcal{A}_W$ into a DG-algebra in the usual way, giving the exterior algebra S the zero differential.

Definition 3.1. We define the finite-dimensional \mathbb{Z}_2 -graded vector space

$$(3.2) \quad \mathcal{E} = R/\mathfrak{m} \otimes \text{End}_R(k^{\text{stab}}) \cong \text{End}_k \left(\bigwedge (k\psi_1 \oplus \cdots \oplus k\psi_n) \right)$$

where $\mathfrak{m} = (x_1, \dots, x_n)$.

We will show that $S \otimes \mathcal{A}_W$ is homotopy equivalent to \mathcal{E} viewed as a complex with zero differential (observe that \mathcal{E} does not depend on W). The corresponding homotopy retract (see Proposition 3.3 below) is our starting point for the minimal model construction. For this we use the Koszul complex of x_1, \dots, x_n ,

$$(3.3) \quad K = \left(R \otimes \bigwedge (k\theta_1 \oplus \cdots \oplus k\theta_n), d_K = \sum_i x_i \theta_i^* \right)$$

Theorem 3.2. *There is a diagram of \mathbb{Z}_2 -graded complexes and homotopy equivalences over k*

$$(3.4) \quad S \otimes \mathcal{A}_W \begin{array}{c} \xrightarrow{\exp(-\delta)} \\ \xleftarrow{\exp(-\delta)} \end{array} K \otimes_R \mathcal{A}_W \begin{array}{c} \xrightarrow{\pi} \\ \xleftarrow{\sigma_\infty} \end{array} \mathcal{E}.$$

Proof. See [2]. □

The notation is as follows:

- The complexes involved here are

$$\begin{aligned} & (S \otimes \text{End}_R(k^{\text{stab}}), \partial) \\ & (K \otimes_R \text{End}_R(k^{\text{stab}}), d_K + \partial) \\ & (\mathcal{E}, 0). \end{aligned}$$

- The operator $\psi_j = \psi_j \wedge -$ on k^{stab} satisfies

$$[\psi_j, d_{k^{\text{stab}}}] = [\psi_j, \sum_i x_i \psi_i^*] = x_j \cdot 1,$$

where $[-, -]$ always denotes the graded commutator. That is, ψ_j is a homotopy for the action of x_j on k^{stab} . It is then easy to check that the odd operator $\alpha \mapsto \psi_j \circ \alpha$ on $\text{End}_R(k^{\text{stab}})$ is also a homotopy for x_j , and where it will not cause confusion we will also write ψ_j for this operator of post-composition.

- Both $S \otimes \mathcal{A}_W$ and $K \otimes_R \mathcal{A}_W$ have the same underlying \mathbb{Z}_2 -graded k -module, namely

$$(3.5) \quad R \otimes \bigwedge (k\theta_1 \oplus \cdots \oplus k\theta_n) \otimes \text{End}_k \left(\bigwedge (k\psi_1 \oplus \cdots \oplus k\psi_n) \right).$$

On this space we define the even operator

$$(3.6) \quad \delta = \sum_{i=1}^n \psi_i \theta_i^*,$$

where ψ_i is the operator on $\text{End}_R(k^{\text{stab}})$ discussed above, given by $\alpha \mapsto (\psi_i \wedge -) \circ \alpha$, and θ_i^* acts by contraction on S . It is easy to check that $\exp(\delta), \exp(-\delta)$ intertwines the differentials ∂ and $d_K + \partial$ and therefore gives an isomorphism between the first two complexes in (3.4) [2, Proposition 4.11].

- The morphism of complexes $\pi : K \longrightarrow R/\mathfrak{m}$ is defined by composing the projection of K onto the submodule $R \cdot 1$ of θ -degree zero forms, with the quotient $R \longrightarrow R/\mathfrak{m}$. We also write π for the result of tensoring this morphism with $\text{End}_R(k^{\text{stab}})$ to obtain

$$K \otimes_R \text{End}_R(k^{\text{stab}}) \xrightarrow{\pi \otimes 1} R/\mathfrak{m} \otimes_R \text{End}_R(k^{\text{stab}}) = \underline{\text{End}}(k^{\text{stab}}).$$

This is a morphism of complexes, because the differential on $\text{End}_R(k^{\text{stab}})$ vanishes on the quotient $\underline{\text{End}}(k^{\text{stab}})$ by the hypothesis that $W \in \mathfrak{m}^2$.

- The k -linear homotopy inverse σ_∞ to π in (3.4) is defined in terms of the following ingredients. The first is the k -linear connection (viewing θ_i as a 1-form)

$$(3.7) \quad \Delta : K \longrightarrow K, \quad \Delta = \sum_i \partial_{x_i} \theta_i$$

and the degree -1 (with respect to the θ -degree) k -linear operator

$$(3.8) \quad H = [d_K, \Delta]^{-1} \Delta.$$

We write ∂ for the differential on $\text{End}_R(k^{\text{stab}})$, and $\sigma : k \longrightarrow K$ for the map which sends $\lambda \in k$ to $\lambda \cdot 1 \in K$, and define k -linear maps

$$(3.9) \quad \sigma_\infty = \sum_{m \geq 0} (-1)^m (H\partial)^m \sigma,$$

$$(3.10) \quad H_\infty = \sum_{m \geq 0} (-1)^m (H\partial)^m H.$$

Here H_∞ is an odd operator on $K \otimes_R \text{End}_R(k^{\text{stab}})$. These maps satisfy:

$$(3.11) \quad \pi \sigma_\infty = 1,$$

$$(3.12) \quad \sigma_\infty \pi = 1 - [d_K + \partial, H_\infty].$$

By inspection of (3.4), we therefore have the following:

Proposition 3.3. *There is a diagram of morphisms of \mathbb{Z}_2 -graded k -complexes*

$$(3.13) \quad S \otimes \mathcal{A}_W \begin{array}{c} \xrightarrow{\Phi} \\ \xleftarrow{\Phi^{-1}} \end{array} \mathcal{E}$$

where $\Phi = \pi \exp(-\delta)$, $\Phi^{-1} = \exp(\delta)\sigma_\infty$ and $\widehat{H} = \exp(\delta)H_\infty \exp(-\delta)$ satisfy

$$(3.14) \quad \Phi\Phi^{-1} = 1, \quad \Phi^{-1}\Phi = 1 - [\partial, \widehat{H}].$$

Thus, Φ and Φ^{-1} are mutually inverse homotopy equivalences over k .

The endomorphism algebra $C = \text{End}_k(S)$ is a Clifford algebra generated by the contraction θ_i^* and wedge product θ_i operators. It is observed in [2] that (3.13) is an isomorphism of Clifford representations (in the homotopy category) when the complex \mathcal{E} is equipped with the action induced by Atiyah classes:

Definition 3.4. The Atiyah classes of \mathcal{A}_W are the R -linear odd operators

$$\text{At}_i = [\partial, \partial_{x_i}] : \text{End}_R(k^{\text{stab}}) \longrightarrow \text{End}_R(k^{\text{stab}}),$$

Lemma 3.5. *We have*

$$\text{At}_i = -[\psi_i^*, -] - \sum_{q=1}^n \partial_{x_i}(W^q)[\psi_q, -].$$

Proof. By direct calculation:

$$\begin{aligned} \text{At}_i &= [[d_{k^{\text{stab}}}, -], \partial_{x_i}] \\ &= \sum_q [x_q[\psi_q^*, -], \partial_{x_i}] + \sum_q [W^q[\psi_q, -], \partial_{x_i}] \\ &= -\sum_q \partial_{x_i}(x_q)[\psi_q^*, -] - \sum_q \partial_{x_i}(W^q)[\psi_q, -] \\ &= -[\psi_i^*, -] - \sum_q \partial_{x_i}(W^q)[\psi_q, -]. \end{aligned}$$

□

4 The minimal model

We now take the homotopy retract from Proposition 3.3 as the input to the usual algorithm for the construction of an A_∞ -minimal model of the DG-algebra $S \otimes \mathcal{A}_W$. We make use of the forward suspended products for this DG-algebra, which are

$$(4.1) \quad r_q : (S \otimes \mathcal{A}_W[1])^{\otimes q} \longrightarrow S \otimes \mathcal{A}_W[1]$$

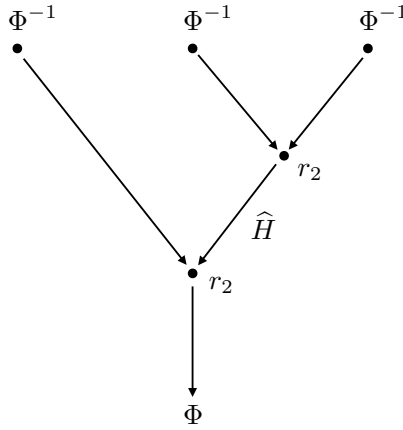
defined to be zero except for $r_1 = \partial$ and

$$(4.2) \quad r_2(\alpha \otimes \beta) = (-1)^{\tilde{\alpha}\tilde{\beta} + \tilde{\beta} + 1} \beta \alpha$$

where $\beta\alpha$ is the usual product in $S \otimes_k \mathcal{A}_W$ and $\tilde{\alpha} = |\alpha| - 1$. Applying the algorithm described in Section ?? to the data of the A_∞ -algebra $(S \otimes \mathcal{A}_W, \{r_1, r_2\})$ and the strong homotopy retract of Proposition 3.3 produces a minimal A_∞ -algebra

$$(\mathcal{E}, \{\rho_q\}_{q \geq 2}).$$

The products are defined by sums over trees, for example: one tree contributing to ρ_3 is:



For this particular tree T the associated operator is

$$\rho_T = (-1)^1 \Phi \circ r_2 \circ (\Phi^{-1} \otimes \hat{H}) \circ (1 \otimes r_2) \circ (1 \otimes \Phi^{-1} \otimes \Phi^{-1}).$$

We first describe the final algorithm which allows us to compute the operations ρ_q in terms of the Feynman rules, and then we will justify this algorithm in the remainder of the paper. Here we stick to $W \in \mathfrak{m}^3$ and consider the subalgebra of \mathcal{E} generated by the contraction operators

$$\mathcal{M} = \bigwedge (k\psi_1^* \oplus \cdots \oplus k\psi_n^*) \subseteq \mathcal{E}.$$

The formula is expressed in terms of *vacuum amplitudes* which are functionals defined for every tree T with q input vertices

$$\mathcal{O}(T) \in (\mathcal{M}^{\otimes q})^*$$

This amplitude is defined to be a sum over *configurations*

$$\mathcal{O}(T) = \sum_{c \in \mathcal{C}} \mathcal{O}(T, c).$$

We define the even operator $\delta_i = \psi_i \theta_i^*$ on the underlying module (3.5) of $S \otimes \mathcal{A}_W$, so that $\delta = \sum_i \delta_i$. Since $[\delta_i, \delta_j] = 0$ for all i, j we have

$$(4.3) \quad \exp(\pm \delta) = \exp(\pm \delta_1) \cdots \exp(\pm \delta_n).$$

Lemma 4.1. *For $1 \leq i \leq n$ there is a commutative diagram*

$$(4.4) \quad \begin{array}{ccc} (S \otimes \mathcal{A}_W) \otimes (S \otimes \mathcal{A}_W) & \xrightarrow{m_2} & S \otimes \mathcal{A}_W \\ \delta_i \otimes 1 + 1 \otimes \delta_i + [\psi_i, -] \otimes \theta_i^* \downarrow & & \downarrow \delta_i \\ (S \otimes \mathcal{A}_W) \otimes (S \otimes \mathcal{A}_W) & \xrightarrow{m_2} & S \otimes \mathcal{A}_W \end{array}$$

Definition 4.2. We define

$$(4.5) \quad \Xi_i = [\psi_i, -] \otimes \theta_i^* : S \otimes \mathcal{A}_W \longrightarrow S \otimes \mathcal{A}_W.$$

and

$$(4.6) \quad \Xi = \sum_i \Xi_i.$$

Proposition 4.3. *There is a commutative diagram*

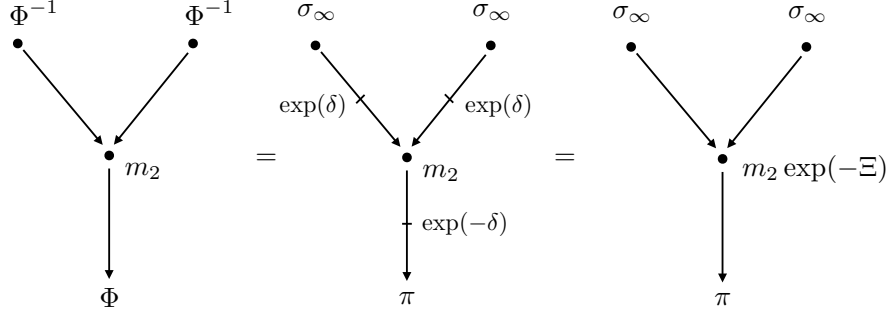
$$(4.7) \quad \begin{array}{ccc} (S \otimes \mathcal{A}_W) \otimes (S \otimes \mathcal{A}_W) & \xrightarrow{m_2} & S \otimes \mathcal{A}_W \\ \exp(-\delta) \otimes \exp(-\delta) \downarrow & & \downarrow \exp(-\delta) \\ (S \otimes \mathcal{A}_W) \otimes (S \otimes \mathcal{A}_W) & & \\ \exp(-\Xi) \downarrow & & \\ (S \otimes \mathcal{A}_W) \otimes (S \otimes \mathcal{A}_W) & \xrightarrow{m_2} & S \otimes \mathcal{A}_W \end{array}$$

Definition 4.4. ev_T is the operator associated to the tree T with σ_∞ on all input vertices, $m_2 \exp(-\Xi)$ on all internal vertices and H_∞ on internal edges.

Lemma 4.5. *For any tree T*

$$\rho_T(\alpha_1 \otimes \cdots \otimes \alpha_q) = (-1)^{S(T, \ell)} \text{ev}_{\hat{T}}(\alpha_q \otimes \cdots \otimes \alpha_1)$$

where $\ell = (\tilde{\alpha}_1, \dots, \tilde{\alpha}_q)$.



5 Notes

Remark 5.1. Varying the potential W affects only the relative “strength” of the triple-vertex interactions, which are indexed by the monomials γ . Thus we can, for example, examine the degeneration from x^{d+1} to x^d (bundles of minimal A_∞ -algebras).

6 Algorithm

Let $T \in \mathcal{T}_q$ be a valid plane tree with q inputs and $e_i(T)$ internal edges, and let $A(T)$ be the augmented plane tree. Recall that the vertices of $A(T)$ may be partitioned into the following subsets: the root, the non-root leaves (called *inputs*), the vertices coming from internal edges of T , and those internal vertices coming from internal vertices of T . The integer n is the number of variables in the ambient ring $R = k[x_1, \dots, x_n]$.

Consider the tensor product

$$\mathcal{H} = R \otimes_k \bigwedge \left(\bigoplus_{i=1}^n k\theta_i \right) \otimes_k \text{End}_k \left(\bigwedge \left(\bigoplus_{i=1}^n k\psi_i \right) \right)$$

on which we have the following natural operators:

$$x_i, \partial_i = \partial_{x_i}, \theta_i, \theta_i^*, \psi_i, \psi_i^*.$$

Definition 6.1. A *configuration* \mathcal{C} of the plane tree T consists of the following data, for each non-root vertex x of $A(T)$:

- An integer $m(x) \geq 0$.
- A subset $J(x) \subseteq \{1, \dots, n\}$ with $|J(x)| = m(x)$.
- If x is an input, or comes from an internal edge of T , a pair

$$(a_j(x), \gamma_j(x)) \in \{1, \dots, n\} \times \mathbb{N}^n$$

for each $j \in J(x)$.

- If x comes from an internal edge of T , an integer $t(x) \in \{1, \dots, n\}$.

Let $\text{Con}(T)$ denote the set of all configurations. For each vertex x of $A(T)$ coming from an internal edge of T we define

$$w(x) = \sum_{y < x} \sum_{j \in J(y)} |\gamma_j(y)| - \sum_{z < x} m(z)$$

where the sum is over all internal edges and input vertices y above x in the tree, i.e. for each x is on the unique path from y to the root, and z ranges over internal vertices. We define in addition

$$F(x) = \frac{1}{w(x)} C(w(x), \{|\gamma_j(x)|\}_{j \in J(x)}).$$

Remark 6.2. How does A sit inside \mathcal{H} ?

We define

$$\mathcal{B} = \bigwedge (k\psi_1^* \oplus \dots \oplus k\psi_n^*).$$

Definition 6.3. Given $T \in \mathcal{T}_q$ and $\mathcal{C} \in \text{Con}(T)$ we define the k -linear operator

$$\mathcal{O}^{\text{pre}}(T, \mathcal{C}) : \mathcal{B}^{\otimes q} \longrightarrow \mathcal{B}$$

by making the following assignments

- to an input vertex of T labelled with m, J and $\{(a_j, \gamma_j)\}_{j \in J}$ we assign the operator

$$(-1)^m \prod_{j \in J} \left\{ \frac{1}{|\gamma_j|} W^j(\gamma_j) \partial_{a_j}(x^{\gamma_j}) \theta_{a_j}[\psi_j, -] \right\}$$

on \mathcal{H} . Note that the operator under the product is even, so the order of the product is irrelevant.

- to each vertex coming from an internal edge of T labelled with m, J and $\{(a_j, \gamma_j)\}_{j \in J}$ and t the operator

$$(6.1) \quad (-1)^m \prod_{j \in J} \left\{ \frac{1}{|\gamma_j|} W^j(\gamma_j) \partial_{a_j}(x^{\gamma_j}) \theta_{a_j}[\psi_j, -] \right\} \circ \theta_t \partial_t$$

on \mathcal{H} .

- to an internal vertex of T labelled with m, J the operator

$$(6.2) \quad (-1)^m m_2 \circ \prod_{j \in J} \left\{ [\psi_j, -] \otimes \theta_j^* \right\}$$

which is a map $\mathcal{H}^{\otimes 2} \longrightarrow \mathcal{H} \longrightarrow \mathcal{B}$.

We define $F = \prod_{e \in E_i(T)} F(e)$ and

$$\mathcal{O}(T, \mathcal{C}) = F \cdot \mathcal{O}^{\text{pre}}(T, \mathcal{C}).$$

Definition 6.4. We define $\rho_q : \mathcal{B}[1]^{\otimes q} \longrightarrow \mathcal{B}[1]$ by

$$\rho_q(\Lambda_1 \otimes \dots \otimes \Lambda_q) = \sum_{T \in \mathcal{T}_q} (-1)^{S(T, \Lambda)} \sum_{\mathcal{C} \in \text{Con}(\hat{T})} \mathcal{O}(\hat{T}, \mathcal{C})(\Lambda_q \otimes \dots \otimes \Lambda_1).$$

7 Examples

To describe the higher multiplications it is convenient to write $[\psi_i, -]$ for the natural operator on \mathcal{A} (giving ψ_i, ψ_j^* the usual anticommutation relations for fermionic creation and annihilation operators), that is

$$\begin{aligned} [\psi_i, \psi_{j_1}^* \cdots \psi_{j_t}^*] &= \sum_{l=1}^t (-1)^{l-1} \psi_{j_1}^* \cdots [\psi_i, \psi_{j_l}^*] \cdots \psi_{j_t}^* \\ &= \sum_{l=1}^t (-1)^{l-1} \delta_{i=j_l} \psi_{j_1}^* \cdots \psi_{j_t}^* . \end{aligned}$$

A general fact is that the higher multiplications on \mathcal{A} are linear combinations of products of such operators. For example, when $n = 2$ a standard term in r_3 would look like

$$\Phi_0 \otimes \Phi_1 \otimes \Phi_2 \mapsto \lambda \cdot [\psi_1, [\psi_2, \Phi_0]] \cdot [\psi_1, \Phi_1] \cdot [\psi_2, \Phi_2] .$$

The coefficient λ is computed as a Feynman amplitude on a binary planar tree with with incoming fermion states $\psi_1 \psi_2, \psi_1, \psi_2$ in the three leaves (respectively) and a simple list of allowed interactions, the most significant of which is a trivalent interaction vertex with an incoming fermion ψ_j and outgoing fermion θ_i and bosons $\partial_{x_i}(x^\gamma)$ whenever the polynomial W^j has a nonzero coefficient for the monomial $x^\gamma = x_1^{\gamma_1} x_2^{\gamma_2}$ (the fermions θ_i are the auxiliary spinor representation generators as in [2]). There are two other interactions which do not depend on W , and which take place only at internal edges or internal vertices (respectively).

Example 7.1. Let $W = x^d$ so that $\mathcal{A} = k \oplus k\psi^*$. Then only r_2 and r_d are nonzero and on $(\mathcal{A}[1])^{\otimes d}$ the only basis element with a nonzero value under r_d is

$$r_d(\psi^* \otimes \cdots \otimes \psi^*) = 1 .$$

Another way to say this is

$$r_d(\Phi_0 \otimes \cdots \otimes \Phi_{d-1}) = \prod_{i=0}^{d-1} [\psi, \Phi_i] .$$

This A_∞ -structure is cyclic with respect to the trace form on \mathcal{A} which projects onto the $k\psi^*$ -summand. Note that all such products are expanded such that the index i increases from left to right.

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