A_∞-minimal models of matrix factorisations

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Abstract

We study A_∞-minimal models of the differential graded algebras obtained from endomorphisms of generators of triangulated categories of matrix factorisations, giving explicit higher products for the cases of simple singularities.

1 Introduction

Let k be a commutative $\mathbb{Q}$-algebra, and $W \in k[x_1, \ldots, x_n]$ a potential in the sense of [3, §2.2] which has the origin as its only critical point. Choose a presentation of W as a sum $W = x_1 W^1 + \cdots + x_n W^n$ with $W^i \in \mathfrak{m} = (x_1, \ldots, x_n)$. Then the generator of the homotopy category of matrix factorisations of W is the stabilised diagonal

$$k^{\text{stab}} = \Big(k[x] \otimes_k \bigwedge (k\psi_1 \oplus \cdots \oplus k\psi_n),\ d_{k^{\text{stab}}} = \sum_{i=1}^n x_i \psi_i^* + \sum_{i=1}^n W^i \psi_i \Big) \tag{1.1}$$

where ψ_i^* and ψ_i act respectively by contraction and wedge product on the exterior algebra underlying k^{stab} [Dyc11]. The purpose of this paper is to calculate the A_∞-minimal model of the $\mathbb{Z}_2$-graded differential graded endomorphism algebra of k^{stab}

$$\mathscr{A}_W = \Big(\operatorname{End}_{k[x]}(k^{\text{stab}}),\ \partial = [d_{k^{\text{stab}}}, -] \Big) \tag{1.2}$$

with differential (throughout all commutators are graded commutators)

$$\partial = \Big[\sum_i x_i \psi_i^* + \sum_i W^i \psi_i, - \Big] \tag{1.3}$$

$$= \sum_i x_i [\psi_i^*, -] + \sum_i W^i [\psi_i, -] \,. \tag{1.4}$$

The minimal model is a finite-dimensional $\mathbb{Z}_2$-graded vector space $\mathscr{M}_W$ with a family

$$\Big\{ r_q : \big(\mathscr{M}_W[1]\big)^{\otimes q} \longrightarrow \mathscr{M}_W[1] \Big\}_{q \geq 2}$$

of odd k-linear maps satisfying the forward suspended A_∞-constraints [1], and having the property that there is an A_∞-quasi-isomorphism $\mathscr{A}_W \longrightarrow \mathscr{M}_W$.

2 Background

2.1 A-infinity algebras

The *tilde grading* is defined by $\widetilde{x} = |x| - 1$.

Throughout $\otimes = \otimes_k$.

3 Perturbation and Koszul complexes

Let k be a characteristic zero field, $R = k[x_1, \dots, x_n]$ and let $W \in \mathfrak{m}^2$ be a potential with chosen decomposition $W = \sum_{i=1}^n x_i W^i$. To apply the A_∞-minimal model construction to the DG-algebra $\mathscr{A}_W$ from (1.2) we would usually begin by finding a k-linear homotopy retract of this complex onto its cohomology. However, for our purposes it is better to first enlarge the DG-algebra to $S \otimes \mathscr{A}_W$, where S is the $\mathbb{Z}_2$-graded vector space

$$S = \bigwedge(k\theta_1 \oplus \cdots \oplus k\theta_n) \tag{3.1}$$

with grading $|\theta_i| = 1$. We make the tensor product $S \otimes \mathscr{A}_W$ into a DG-algebra in the usual way, giving the exterior algebra S the zero differential.

Definition 3.1. We define the finite-dimensional $\mathbb{Z}_2$-graded vector space

$$\mathscr{E} = R/\mathfrak{m} \otimes \mathrm{End}_R(k^{\mathrm{stab}}) \cong \mathrm{End}_k\Big(\bigwedge(k\psi_1 \oplus \cdots \oplus k\psi_n)\Big) \tag{3.2}$$

where $\mathfrak{m} = (x_1, \dots, x_n)$.

We will show that $S \otimes \mathscr{A}_W$ is homotopy equivalent to $\mathscr{E}$ viewed as a complex with zero differential (observe that $\mathscr{E}$ does not depend on W). The corresponding homotopy retract (see Proposition 3.3 below) is our starting point for the minimal model construction. For this we use the Koszul complex of $x_1, \dots, x_n$,

$$K = \Big(R \otimes \bigwedge(k\theta_1 \oplus \cdots \oplus k\theta_n),\ d_K = \sum_i x_i \theta_i^*\Big) \tag{3.3}$$

Theorem 3.2. *There is a diagram of $\mathbb{Z}_2$-graded complexes and homotopy equivalences over k*

$$\begin{tikzcd} S \otimes \mathscr{A}_W \arrow[r, shift left, "\exp(-\delta)"] & K \otimes_R \mathscr{A}_W \arrow[l, shift left, "\exp(-\delta)"] \arrow[r, shift left, "\pi"] & \mathscr{E} \arrow[l, shift left, "\sigma_\infty"] \end{tikzcd}. \tag{3.4}$$

Proof. See [2]. □

The notation is as follows:

- The complexes involved here are

$$\begin{gathered}\big(S \otimes \operatorname{End}_R(k^{\mathrm{stab}}),\ \partial\big)\\ \big(K \otimes_R \operatorname{End}_R(k^{\mathrm{stab}}),\ d_K + \partial\big)\\ \big(\mathscr{E}, 0\big)\,.\end{gathered}$$

- The operator $\psi_j = \psi_j \wedge -$ on k^{stab} satisfies

$$[\psi_j, d_{k^{\mathrm{stab}}}] = \big[\psi_j, \sum_i x_i \psi_i^*\big] = x_j \cdot 1\,,$$

where $[-,-]$ always denotes the graded commutator. That is, ψ_j is a homotopy for the action of x_j on k^{stab}. It is then easy to check that the odd operator $\alpha \mapsto \psi_j \circ \alpha$ on $\operatorname{End}_R(k^{\mathrm{stab}})$ is also a homotopy for x_j, and where it will not cause confusion we will also write ψ_j for this operator of post-composition.

- Both $S \otimes \mathscr{A}_W$ and $K \otimes_R \mathscr{A}_W$ have the same underlying $\mathbb{Z}_2$-graded k-module, namely

$$R \otimes \bigwedge(k\theta_1 \oplus \cdots \oplus k\theta_n) \otimes \operatorname{End}_k\big(\bigwedge(k\psi_1 \oplus \cdots \oplus k\psi_n)\big)\,. \tag{3.5}$$

On this space we define the even operator

$$\delta = \sum_{i=1}^n \psi_i \theta_i^*\,, \tag{3.6}$$

where ψ_i is the operator on $\operatorname{End}_R(k^{\mathrm{stab}})$ discussed above, given by $\alpha \mapsto (\psi_i \wedge -) \circ \alpha$, and θ_i^* acts by contraction on S. It is easy to check that $\exp(\delta), \exp(-\delta)$ intertwines the differentials ∂ and $d_K + \partial$ and therefore gives an isomorphism between the first two complexes in (3.4) [2, Proposition 4.11].

- The morphism of complexes $\pi : K \longrightarrow R/\mathfrak{m}$ is defined by composing the projection of K onto the submodule $R \cdot 1$ of θ-degree zero forms, with the quotient $R \longrightarrow R/\mathfrak{m}$. We also write π for the result of tensoring this morphism with $\operatorname{End}_R(k^{\mathrm{stab}})$ to obtain

$$K \otimes_R \operatorname{End}_R(k^{\mathrm{stab}}) \xrightarrow{\ \pi \otimes 1\ } R/\mathfrak{m} \otimes_R \operatorname{End}_R(k^{\mathrm{stab}}) = \underline{\operatorname{End}}(k^{\mathrm{stab}})\,.$$

This is a morphism of complexes, because the differential on $\operatorname{End}_R(k^{\mathrm{stab}})$ vanishes on the quotient $\underline{\operatorname{End}}(k^{\mathrm{stab}})$ by the hypothesis that $W \in \mathfrak{m}^2$.

- The k-linear homotopy inverse σ_∞ to π in (3.4) is defined in terms of the following ingredients. The first is the k-linear connection (viewing θ_i as a 1-form)

$$\Delta : K \longrightarrow K\,, \qquad \Delta = \sum_i \partial_{x_i} \theta_i \tag{3.7}$$

and the degree -1 (with respect to the θ-degree) k-linear operator

$$H = [d_K, \Delta]^{-1}\Delta\,. \tag{3.8}$$

We write ∂ for the differential on $\mathrm{End}_R(k^{\mathrm{stab}})$, and $\sigma : k \longrightarrow K$ for the map which sends $\lambda \in k$ to $\lambda \cdot 1 \in K$, and define k-linear maps

$$\sigma_\infty = \sum_{m\geq 0}(-1)^m (H\partial)^m \sigma\,, \tag{3.9}$$

$$H_\infty = \sum_{m\geq 0}(-1)^m (H\partial)^m H\,. \tag{3.10}$$

Here H_∞ is an odd operator on $K \otimes_R \mathrm{End}_R(k^{\mathrm{stab}})$. These maps satisfy:

$$\pi\sigma_\infty = 1\,, \tag{3.11}$$

$$\sigma_\infty\pi = 1 - [d_K + \partial, H_\infty]\,. \tag{3.12}$$

By inspection of (3.4), we therefore have the following:

Proposition 3.3. *There is a diagram of morphisms of $\mathbb{Z}_2$-graded k-complexes*

$$S \otimes \mathscr{A}_W \underset{\Phi^{-1}}{\overset{\Phi}{\rightleftarrows}} \mathscr{E} \tag{3.13}$$

where $\Phi = \pi\exp(-\delta), \Phi^{-1} = \exp(\delta)\sigma_\infty$ and $\widehat{H} = \exp(\delta)H_\infty\exp(-\delta)$ satisfy

$$\Phi\Phi^{-1} = 1\,, \qquad \Phi^{-1}\Phi = 1 - [\partial, \widehat{H}]\,. \tag{3.14}$$

Thus, Φ and Φ^{-1} are mutually inverse homotopy equivalences over k.

The endomorphism algebra $C = \mathrm{End}_k(S)$ is a Clifford algebra generated by the contraction θ_i^* and wedge product θ_i operators. It is observed in [2] that (3.13) is an isomorphism of Clifford representations (in the homotopy category) when the complex $\mathscr{E}$ is equipped with the action induced by Atiyah classes:

Definition 3.4. The Atiyah classes of $\mathscr{A}_W$ are the R-linear odd operators

$$\mathrm{At}_i = [\partial, \partial_{x_i}] : \mathrm{End}_R(k^{\mathrm{stab}}) \longrightarrow \mathrm{End}_R(k^{\mathrm{stab}})\,,$$

Lemma 3.5. *We have*

$$\mathrm{At}_i = -[\psi_i^*, -] - \sum_{q=1}^{n} \partial_{x_i}(W^q)[\psi_q, -]\,.$$

Proof. By direct calculation:

$$\begin{aligned}
\mathrm{At}_i &= \big[[d_{k^{\mathrm{stab}}}, -], \partial_{x_i}\big] \\
&= \sum_q \big[x_q[\psi_q^*, -], \partial_{x_i}\big] + \sum_q \big[W^q[\psi_q, -], \partial_{x_i}\big] \\
&= -\sum_q \partial_{x_i}(x_q)[\psi_q^*, -] - \sum_q \partial_{x_i}(W^q)[\psi_q, -] \\
&= -[\psi_i^*, -] - \sum_q \partial_{x_i}(W^q)[\psi_q, -]\,.
\end{aligned}$$

□

4 The minimal model

We now take the homotopy retract from Proposition 3.3 as the input to the usual algorithm for the construction of an A_∞-minimal model of the DG-algebra $S \otimes \mathscr{A}_W$. We make use of the forward suspended products for this DG-algebra, which are

$$r_q : (S \otimes \mathscr{A}_W[1])^{\otimes q} \longrightarrow S \otimes \mathscr{A}_W[1] \tag{4.1}$$

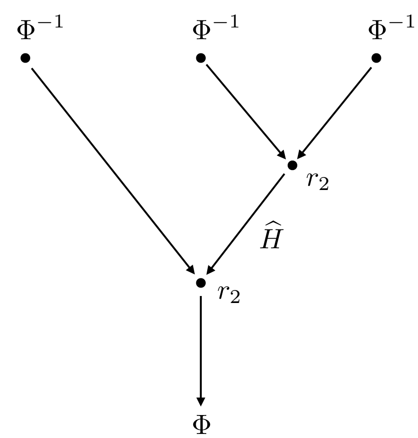
defined to be zero except for $r_1 = \partial$ and

$$r_2(\alpha \otimes \beta) = (-1)^{\widetilde{\alpha}\widetilde{\beta} + \widetilde{\beta} + 1} \beta\alpha \tag{4.2}$$

where $\beta\alpha$ is the usual product in $S \otimes_k \mathscr{A}_W$ and $\widetilde{\alpha} = |\alpha| - 1$. Applying the algorithm described in Section **??** to the data of the A_∞-algebra $\big(S \otimes \mathscr{A}_W, \{r_1, r_2\}\big)$ and the strong homotopy retract of Proposition 3.3 produces a minimal A_∞-algebra

$$\big(\mathscr{E}, \{\rho_q\}_{q \geq 2}\big)\,.$$

The products are defined by sums over trees, for example: one tree contributing to ρ_3 is:



For this particular tree T the associated operator is

$$\rho_T = (-1)^1 \Phi \circ r_2 \circ (\Phi^{-1} \otimes \widehat{H}) \circ (1 \otimes r_2) \circ (1 \otimes \Phi^{-1} \otimes \Phi^{-1}) \,.$$

We first describe the final algorithm which allows us to compute the operations ρ_q in terms of the Feynman rules, and then we will justify this algorithm in the remainder of the paper. Here we stick to $W \in \mathfrak{m}^3$ and consider the subalgebra of $\mathscr{E}$ generated by the contraction operators

$$\mathscr{M} = \bigwedge (k\psi_1^* \oplus \cdots \oplus k\psi_n^*) \subseteq \mathscr{E} \,.$$

The formula is expressed in terms of *vacuum amplitudes* which are functionals defined for every tree T with q input vertices

$$\mathcal{O}(T) \in (\mathscr{M}^{\otimes q})^*$$

This amplitude is defined to be a sum over *configurations*

$$\mathcal{O}(T) = \sum_{c \in \mathcal{C}} \mathcal{O}(T, c) \,.$$

We define the even operator $\delta_i = \psi_i \theta_i^*$ on the underlying module (3.5) of $S \otimes \mathscr{A}_W$, so that $\delta = \sum_i \delta_i$. Since $[\delta_i, \delta_j] = 0$ for all i, j we have

$$\exp(\pm\delta) = \exp(\pm\delta_1) \cdots \exp(\pm\delta_n)\,. \tag{4.3}$$

Lemma 4.1. *For $1 \le i \le n$ there is a commutative diagram*

$$\begin{tikzcd}
(S \otimes \mathscr{A}_W) \otimes (S \otimes \mathscr{A}_W) \arrow[r, "m_2"] \arrow[d, "\delta_i \otimes 1 + 1 \otimes \delta_i + [\psi_i, -] \otimes \theta_i^*"'] & S \otimes \mathscr{A}_W \arrow[d, "\delta_i"] \\
(S \otimes \mathscr{A}_W) \otimes (S \otimes \mathscr{A}_W) \arrow[r, "m_2"'] & S \otimes \mathscr{A}_W
\end{tikzcd} \tag{4.4}$$

Definition 4.2. We define

$$\Xi_i = [\psi_i, -] \otimes \theta_i^* : S \otimes \mathscr{A}_W \longrightarrow S \otimes \mathscr{A}_W\,. \tag{4.5}$$

and

$$\Xi = \sum_i \Xi_i\,. \tag{4.6}$$

Proposition 4.3. *There is a commutative diagram*

$$\begin{tikzcd}
(S \otimes \mathscr{A}_W) \otimes (S \otimes \mathscr{A}_W) \arrow[r, "m_2"] \arrow[d, "\exp(-\delta) \otimes \exp(-\delta)"'] & S \otimes \mathscr{A}_W \arrow[dd, "\exp(-\delta)"] \\
(S \otimes \mathscr{A}_W) \otimes (S \otimes \mathscr{A}_W) \arrow[d, "\exp(-\Xi)"'] & \\
(S \otimes \mathscr{A}_W) \otimes (S \otimes \mathscr{A}_W) \arrow[r, "m_2"'] & S \otimes \mathscr{A}_W
\end{tikzcd} \tag{4.7}$$

Definition 4.4. ev_T is the operator associated to the tree T with σ_∞ on all input vertices, $m_2 \exp(-\Xi)$ on all internal vertices and H_∞ on internal edges.

Lemma 4.5. *For any tree T*

$$\rho_T(\alpha_1 \otimes \cdots \otimes \alpha_q) = (-1)^{S(T,\ell)}\, \mathrm{ev}_{\widehat{T}}(\alpha_q \otimes \cdots \otimes \alpha_1)$$

where $\ell = (\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_q)$.

$$
\begin{tikzcd}
\Phi^{-1} \arrow[dr] & & \Phi^{-1} \arrow[dl] & & \sigma_\infty \arrow[dr, "\exp(\delta)"'] & & \sigma_\infty \arrow[dl, "\exp(\delta)"] & & \sigma_\infty \arrow[dr] & & \sigma_\infty \arrow[dl] \\
& m_2 \arrow[d] & & = & & m_2 \arrow[d, "\exp(-\delta)"] & & = & & m_2 \exp(-\Xi) \arrow[d] & \\
& \Phi & & & & \pi & & & & \pi &
\end{tikzcd}
$$

5 Notes

Remark 5.1. Varying the potential W affects only the relative "strength" of the triple-vertex interactions, which are indexed by the monomials γ. Thus we can, for example, examine the degeneration from x^{d+1} to x^d (bundles of minimal A_∞-algebras).

6 Examples

To describe the higher multiplications it is convenient to write $[\psi_i, -]$ for the natural operator on $\mathscr{A}$ (giving ψ_i, ψ_j^* the usual anticommutation relations for fermionic creation and annhiliation operators), that is

$$
\begin{aligned}
[\psi_i, \psi_{j_1}^* \cdots \psi_{j_t}^*] &= \sum_{l=1}^{t} (-1)^{l-1} \psi_{j_1}^* \cdots [\psi_i, \psi_{j_l}^*] \cdots psi_{j_t}^* \\
&= \sum_{l=1}^{t} (-1)^{l-1} \delta_{i=j_l} \psi_{j_1}^* \cdots \psi_{j_t}^* .
\end{aligned}
$$

A general fact is that the higher multiplications on $\mathscr{A}$ are linear combinations of products of such operators. For example, when $n = 2$ a standard term in r_3 would look like

$$
\Phi_0 \otimes \Phi_1 \otimes \Phi_2 \mapsto \lambda \cdot [\psi_1, [\psi_2, \Phi_0]] \cdot [\psi_1, \Phi_1] \cdot [\psi_2, \Phi_2] .
$$

The coefficient λ is computed as a Feynman amplitude on a binary planar tree with with incoming fermion states $\psi_1\psi_2, \psi_1, \psi_2$ in the three leaves (respectively) and a simple list of allowed interactions, the most significant of which is a trivalent interaction vertex with an incoming fermion ψ_j and outgoing fermion θ_i and bosons $\partial_{x_i}(x^\gamma)$ whenever the polynomial W^j has a nonzero coefficient for the monomial $x^\gamma = x_1^{\gamma_1} x_2^{\gamma_2}$ (the fermions θ_i are the auxiliary spinor representation generators as in [2]). There are two other interactions which do not depend on W, and which take place only at internal edges or internal vertices (respectively).

Example 6.1. Let $W = x^d$ so that $\mathscr{A} = k \oplus k\psi^*$. Then only r_2 and r_d are nonzero and on $(\mathscr{A}[1])^{\otimes d}$ the only basis element with a nonzero value under r_d is

$$r_d(\psi^* \otimes \cdots \otimes \psi^*) = 1\,.$$

Another way to say this is

$$r_d(\Phi_0 \otimes \cdots \otimes \Phi_{d-1}) = \prod_{i=0}^{d-1} [\psi, \Phi_i]\,.$$

This A_∞-structure is cyclic with respect to the trace form on $\mathscr{A}$ which projects onto the $k\psi^*$-summand. Note that all such products are expanded such that the index i increases from left to right.

References

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