A_∞-minimal models of matrix factorisations

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Abstract

We study A_∞-minimal models of the differential graded algebras obtained from endomorphisms of generators of triangulated categories of matrix factorisations.

Contents

1 Introduction

Let k be a commutative $\mathbb{Q}$-algebra and $W \in k[x_1, \dots, x_n]$ a polynomial which is a potential over k in the sense of [4, §2.2]. For example, $k = \mathbb{C}$ and W has isolated critical points. The $\mathbb{Z}_2$-graded DG-category $\mathcal{A} = \mathrm{mf}(R, W)$ has for its objects finite-rank matrix factorisations of W over $R = k[x_1, \dots, x_n]$. In this paper we study the minimal model problem for $\mathcal{A}$, the aim being to produce models that are finitely-generated and projective over k.

When k is a field, the standard minimal model theorem [?] constructs the structure of an A_∞-category on the cohomology $\mathcal{B} = H^*\mathcal{A}$ in such a way that $\mathcal{B}$ is quasi-isomorphic to $\mathcal{A}$. Moreover, this A_∞-category has finite-dimensional Hom-spaces and is therefore a good finite model of $\mathcal{A}$. The problem in this case is to have sufficient control over the inputs to the minimal model theorem that the A_∞-products on $H^*\mathcal{A}(X, X)$ can be reasonably calculated, for a given matrix factorisation X. In order to understand deformations of matrix factorisations [?, ?, ?] it is important to do this for generic X, but the special case where $X = k^{\mathrm{stab}}$, the generator of $\mathcal{A}$ studied by Dyckerhoff in the case of a single isolated singularity at the origin [Dyc11], is of particular interest.

Since one of our goals is to understand how matrix factorisation categories vary along unfoldings of singularities, we also want to consider the case where k is not a field. For example take $k = \mathbb{C}[t]$ and $R = \mathbb{C}[x_1, \ldots, x_n, t]$ so that $W = W_t(x)$ is a potential with a parameter t. In this case we want an A_∞-category $\mathcal{B}$ quasi-isomorphic to $\mathcal{A}$ over k, with $\mathcal{B}(a, b)$ a finitely-generated projective k-module for each pair of objects a, b.

These two desiderata, understanding the A_∞-products on $H^*\mathcal{A}(X, X)$ for generic X, and the case where k is not a field, lead us naturally to a slightly non-standard use of the minimal model theorem. Following the ideas developed in [?, ?, ?] we prove that, writing $\iota : R \longrightarrow R/(\partial_{x_1} W, \ldots, \partial_{x_n} W)$ for the quotient, there is a k-linear homotopy retract

$$(1.1) \qquad \iota^*\mathcal{A} \rightleftarrows \mathcal{A} \otimes_k \bigwedge \left(k^{\oplus n}[1]\right).$$

1.1 The special case of the generator

Suppose that k is a field and that the potential W belongs to $\mathfrak{m}^3$ where $\mathfrak{m} = (x_1, \ldots, x_n)$, choose a presentation $W = \sum_i x_i W^i$ with $W^i \in \mathfrak{m}^2$ and consider the following pair

$$(1.2) \qquad k^{\mathrm{stab}} = \Big(k[x] \otimes_k \bigwedge \big(\oplus_{i=1}^n k\psi_i \big),\ d_{k^{\mathrm{stab}}} = \sum_{i=1}^n x_i \psi_i^* + \sum_{i=1}^n W^i \psi_i \Big)$$

where ψ_i^*, ψ_i denote respectively the operators of contraction $\psi_i^* \lrcorner (-)$ and wedge product $\psi_i = \psi_i \wedge (-)$ on the exterior algebra. Clearly $(d_{k^{\mathrm{stab}}})^2 = W$ so that k^{stab} is a matrix factorisation of W. When k is a field and W has an isolated critical point at the origin, k^{stab} is the representative in the homotopy category of matrix factorisations of the structure sheaf of the singular point at the origin, and was first studied by Dyckerhoff [Dyc11]. If the origin is the only singular point of the zero locus of W, or we work with power series instead of the polynomial ring, k^{stab} is a split generator of the homotopy category of matrix factorisations [?, Theorem ?].

The purpose of this paper is to show how to calculate the A_∞-minimal model $\mathscr{B}$ of the $\mathbb{Z}_2$-graded differential graded endomorphism algebra

$$(1.3) \qquad \mathscr{A} = \Big(\operatorname{End}_{k[x]}(k^{\mathrm{stab}}),\ \partial = [d_{k^{\mathrm{stab}}}, -] \Big).$$

The differential here is (throughout all commutators are graded commutators)

$$\partial = [d_{k^{\mathrm{stab}}}, -] = \Big[\sum_i x_i \psi_i^* + \sum_i W^i \psi_i, - \Big]$$
$$= \sum_i x_i [\psi_i^*, -] + \sum_i W^i [\psi_i, -]\,.$$

The minimal model is a finite-dimensional $\mathbb{Z}_2$-graded vector space $\mathscr{B}$ with a family

$$\Big\{ \rho_q : \big(\mathscr{B}[1]\big)^{\otimes q} \longrightarrow \mathscr{B}[1] \Big\}_{q \geq 2}$$

of odd k-linear maps satisfying the forward suspended A_∞-constraints [1], and having the property that there is an A_∞-quasi-isomorphism $\mathscr{B} \longrightarrow \mathscr{A}$.

2 Background

2.1 Algebra

Let k be a commutative ring. Given $\mathbb{Z}_2$-graded k-modulea M, N and $\phi \in \mathrm{Hom}_k(M, N)$ we say ϕ is *even* (resp. *odd*) if $\phi(M_i) \subseteq N_i$ (resp. $\phi(M_i) \subseteq N_{i+1}$) for all $i \in \mathbb{Z}_2$. This makes $\mathrm{Hom}_k(M, N)$ into a $\mathbb{Z}_2$-graded k-module, having even maps in degree zero and odd maps in degree one. Given two homogeneous operators ψ, ϕ the *graded commutator* is

$$[\phi, \psi] = \phi\psi - (-1)^{|\phi||\psi|}\psi\phi\,. \tag{2.1}$$

In this note all operators are given a $\mathbb{Z}_2$-grading and the commutator always denotes the graded commutator. We often make implicit use of the graded Jacobi identity

$$(-1)^{|\alpha||\gamma|}\big[\alpha, [\beta, \gamma]\big] + (-1)^{|\alpha||\beta|}\big[\beta, [\gamma, \alpha]\big] + (-1)^{|\gamma||\beta|}\big[\gamma, [\alpha, \beta]\big] = 0\,. \tag{2.2}$$

Among the most important examples are the exterior algebras

$$M = \bigwedge \big(\oplus_{i=1}^r k\varepsilon_i \big)$$

where $F = \oplus_{i=1}^r k\varepsilon_i$ denotes a free k-module of rank r with basis $\varepsilon_1, \ldots, \varepsilon_r$. We give F a $\mathbb{Z}_2$-grading by assigning $|\varepsilon_i| = 1$, that is, $F \cong k^{\oplus r}[1]$. The inherited $\mathbb{Z}_2$-grading on $\bigwedge F$ is the reduction mod 2 of the usual $\mathbb{Z}$-grading on the exterior algebra, e.g. $|\varepsilon_1\varepsilon_2| = 0$.

We define odd operators $\varepsilon_j \wedge (-), \varepsilon_j^* \lrcorner (-)$ on $\bigwedge F$ by wedge product and contraction, respectively, where contraction is defined by the formula

$$\varepsilon_j^* \lrcorner \Big(\varepsilon_{i_1} \cdots \varepsilon_{i_s} \Big) = \sum_{l=1}^{s} (-1)^{l-1} \delta_{j, i_l} \varepsilon_{i_1} \wedge \cdots \wedge \widehat{\varepsilon_{i_l}} \wedge \cdots \wedge \varepsilon_{i_s}\,.$$

Often we will simply write ε_j for $\varepsilon_j \wedge (-)$ and ε_j^* for $\varepsilon_j^* \lrcorner (-)$. Clearly with this notation, as operators on $\bigwedge F$, we have the commutator

$$\big[\varepsilon_i, \varepsilon_j^*\big] = \delta_{ij} \cdot 1\,. \tag{2.3}$$

2.2 Trees and denotations

2.3 A_∞-algebras and minimal models

The *tilde grading* on a $\mathbb{Z}_2$-graded k-module is defined by $\widetilde{x} = |x| - 1$.

2.4 Matrix factorisations

3 Feynman diagrams

The A_∞-products on $\mathscr{B}$ are defined in terms of Feynman diagrams (see Definition 3.16). In this section we explain how to enumerate all the relevant diagrams and how to compute their value; in the next section we prove that the maps they give rise to do indeed compute the minimal model of $\mathscr{A}$. Our reference for general background on Feynman diagrams is [6, Ch. 6] or [7, §4.4], although we need only superficial aspects of the theory there.

Throughout $\otimes = \otimes_k$. The integer n is the number of variables in the ambient ring $R = k[x_1, \ldots, x_n]$. We assume in this section that $W \in \mathfrak{m}^3$. This is not a real restriction, as we will prove in Section **??** that if $W \in \mathfrak{m}^2$ is written in the form

$$W = W'(x_1, \ldots, x_r) + \sum_{i=r+1}^{n} \lambda_i x_i^2\,, \quad \text{with} \quad W' \in \mathfrak{m}^3 \tag{3.1}$$

and if $\mathscr{B}'$ denotes the minimal model of W' as constructed below, then as A_∞-algebras

$$\mathscr{B} \cong \mathscr{B}' \otimes_k C(Q) \tag{3.2}$$

where $C(Q)$ is the Clifford algebra of the quadratic form $Q = \sum_{i=r+1}^{n} \lambda_i x_i^2$, with no higher products (i.e. viewed simply as a $\mathbb{Z}_2$-graded algebra). We may therefore restrict without loss of generality to $W \in \mathfrak{m}^3$ in what follows.

Definition 3.1. Consider the $\mathbb{Z}_2$-graded k-algebra defined by the tensor product

$$\mathscr{H} = R \otimes_k \bigwedge \big(\oplus_{i=1}^{n} k\theta_i \big) \otimes_k \operatorname{End}_k \Big(\bigwedge \big(\oplus_{i=1}^{n} k\psi_i \big) \Big) \tag{3.3}$$

with degrees $|x_i| = 0$ and $|\theta_i| = |\psi_i| = 1$. On this space we have homogeneous operators

$$x_i\,, \partial_i = \partial_{x_i}\,, \theta_i\,, \theta_i^*\,, \big[\psi_i\,, -\big]\,. \tag{3.4}$$

Here ψ_i denotes the operator $\psi_i \wedge (-)$ on $\bigwedge \big(\oplus_{i=1}^{n} k\psi_i \big)$ and $[\psi_i, -]$ the graded commutator with this operator, defined on a homogeneous operator β by

$$\big[\psi_i, \beta\big] = \psi_i \circ \beta - (-1)^{|\beta|} \beta \circ \psi_i\,.$$

We define the $\mathbb{Z}_2$-graded k-module

$$\mathscr{B} = \bigwedge \big(\oplus_{i=1}^{n} k\psi_i^* \big) \tag{3.5}$$

with $|\psi_i^*| = 1$ which we view as a graded submodule

$$\mathscr{B} \subset \mathrm{End}_k\Big(\bigwedge \big(\oplus_{i=1}^n k\psi_i \big)\Big)$$

by identifying $\psi_i^* \in \mathscr{B}$ with the operation of contraction $\psi_i^* \lrcorner (-)$ on the exterior algebra. In this way we may also identify $\mathscr{B}$ with a submodule of $\mathscr{H}$, and we write $\sigma : \mathscr{B} \longrightarrow \mathscr{H}$ for the inclusion. Note that the operator $[\psi_j, -]$ on $\mathscr{H}$ defined above acts on the submodule $\mathscr{B}$ as contraction with $\psi_j = (\psi_j^*)^*$, that is,

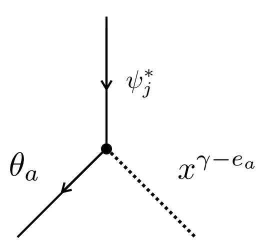
$$\Big[\psi_j, \sigma\big(\psi_{i_1}^* \wedge \cdots \wedge \psi_{i_r}^*\big)\Big] = \sum_{l=1}^{r} (-1)^{l-1} \delta_{j,i_l} \sigma\big(\psi_{i_1}^* \wedge \cdots \wedge \widehat{\psi_{i_l}^*} \wedge \cdots \wedge \psi_{i_r}^*\big)\,. \tag{3.6}$$

We write $p : \mathscr{H} \longrightarrow \mathscr{H}$ for the k-algebra endomorphism of $\mathscr{H}$ sending x_i, θ_i to zero for $1 \le i \le n$ and which is the identity on $\mathrm{End}_k(\wedge(\oplus_i k\psi_i))$.

We view $\mathscr{H}$ as the tensor product of the bosonic Fock space R with creation and annihilation operators x_i, ∂_i, fermionic Fock space $\wedge(\oplus_{i=1}^n k\theta_i)$ with creation and annihilation operators θ_i, θ_i^* and the third factor, of which the calculations of the A_∞-structure only involve the submodule $\mathscr{B} = \wedge(\oplus_{i=1}^n k\psi_i^*)$ which is another fermionic Fock space. The inputs to the Feynman diagrams are tensor products of states in $\mathscr{B}$, that is, wedge products of ψ_i^*'s, on which the $[\psi_i, -]$ act as annihilation operators.

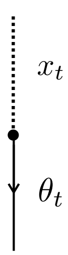
The linear maps in (3.12), (3.13), (3.14) below which determine the A_∞ products ρ_q on $\mathscr{B}$ will be defined in terms of these creation and annihilation operators on $\mathscr{H}$, and may be interpreted as particle interactions in the usual way. The interactions that appear are the following ones, depicted on the left as Feynman diagrams and on the right as operators. We call these respectively *A-, B- and C-type interactions*:

A



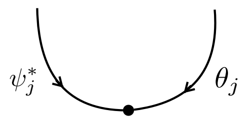
$$\theta_a x^{\gamma - e_a} \big[\psi_j, -\big] \in \mathrm{End}_k(\mathscr{H}) \qquad 1 \le a, j \le n\,, \gamma \in \mathbb{N}^n \tag{3.7}$$

B



$$\theta_t \partial_t \in \mathrm{End}_k(\mathscr{H}) \qquad 1 \le t \le n \tag{3.8}$$

C



$$m_2\big([\psi_j, -] \otimes \theta_j^*\big) \in \mathrm{Hom}_k(\mathscr{H}^{\otimes 2}, \mathscr{H}) \qquad 1 \leq j \leq n \tag{3.9}$$

Here m_2 is the multiplication in $\mathscr{H}$ and, given $\gamma \in \mathbb{N}^n$, we write $x^\gamma = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$, with $e_i \in \mathbb{N}^n$ the unit vector with $x^{e_i} = x_i$. By convention if $\gamma_a = 0$ then $x^{\gamma - e_a} = 0$, which ensures that we always have $\gamma_a x^{\gamma - e_a} = \partial_a(x^\gamma)$. Finally note that bosons are depicted with dotted lines, and fermions with solid lines.

These interactions "take place" at vertices of the tree $A(T)$ for some $T \in \mathcal{T}_q$, and come with the following constraints, which will be formalised below in terms of combinatorial data called *configurations*. We write $f(\gamma)$ for the coefficient of x^γ in a polynomial $f \in R$ and recall the chosen decomposition $W = \sum_j x_j W^j$.

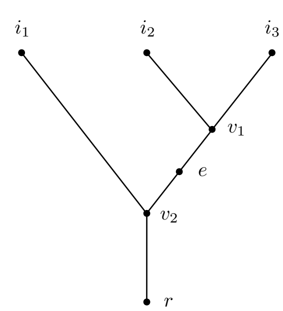
- A-type interactions only occur at input vertices and internal edges of T.
- B-type interactions only occur at internal edges of T.
- C-type interactions only occur at internal vertices of T, and moreover the incoming lines must come from different branches of the tree.
- There may be arbitrarily many A-type or C-type interactions per vertex of $A(T)$, but there is *precisely one* B-type interaction at each internal edge of T.
- The A-type interaction with indices a, j, γ appears with the coefficient

$$-\frac{\gamma_a}{|\gamma|} W^j(\gamma)\,. \tag{3.10}$$

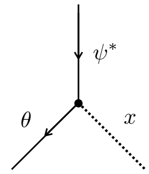
  where $|\gamma| = \sum_{i=1}^n \gamma_i$. Each B- and C-type interaction appears with coefficient (-1).

As we will see, the coefficient (3.10) is the only way that W enters the rules, and thus the definition of the A_∞-products on $\mathscr{B}$. The precise definition of the Feynman rules will be given in Definition 3.3 below, but first we give an example of a tree with interactions inserted, to illustrate the various ingredients.

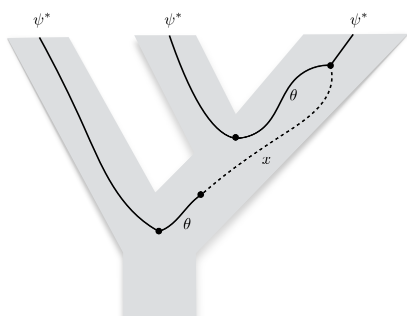
Example 3.2. Consider the tree $T \in \mathcal{T}_3$ whose augmented tree $A(T)$ is depicted below, together with labels for its vertices, where we label by e the vertex corresponding to the internal edge of T:



Set $W = x^3 \in \mathbb{C}[x]$ so that $W^1 = x^2$ and the only A-type interaction has $a = 1, j = 1, \gamma = 2$ and coefficient -1, as in the diagram (we write $\theta = \theta_1, \psi = \psi_1$)



Consider the input state $\Psi = \psi^* \otimes \psi^* \otimes \psi^*$ in $\mathscr{B}^{\otimes 3}$. One possible pattern of interactions (called a configuration, below) has a single A-type interaction at the input vertex i_3, a B-type interaction at e, and two C-type interactions at v_1, v_2, as shown in the following "topological" Feynman diagram



The *value* of this diagram is the value of the linear map

$$(-1)^3 p \circ m_2\big([\psi, -] \otimes \theta^*\big) \circ \big(\sigma \otimes \theta\partial_x \circ m_2([\psi, -] \otimes \theta^*)\big(\sigma \otimes \theta x[\psi, -]\sigma\big)\big) \tag{3.11}$$

on the input state Ψ. Usually when evaluating such an expression we would pay attention to Koszul signs from commuting the input ψ^*'s into position (see Definition **??**). Ignoring all signs, the value is

$$\begin{aligned}
&\pm\, p \circ m_2\big([\psi, -] \otimes \theta^*\big) \circ \big(\sigma \otimes \theta\partial_x \circ m_2([\psi, -] \otimes \theta^*)\big(\sigma \otimes \theta x[\psi, -]\sigma\big)\big)\Big(\Psi\Big) \\
&\quad = \pm\, p \circ m_2\big([\psi, \psi^*] \otimes \theta^*\theta\partial_x \circ m_2([\psi, \psi^*] \otimes \theta^*\theta x[\psi, \psi^*])\big) \\
&\quad = \pm\, p \circ m_2\big(1 \otimes \theta^*\theta\partial_x \circ m_2(1 \otimes \theta^*\theta x)\big) \\
&\quad = \pm\, p \circ m_2\big(1 \otimes \partial_x(x)\big) \\
&\quad = \pm\, p(1) = \pm\, 1 \in \mathscr{H}\,.
\end{aligned}$$

Now we give some more precise definitions. Let $T \in \mathcal{T}_q$ be a valid plane tree with q inputs and $e_i(T)$ internal edges, and let $A(T)$ be the augmented plane tree. The definitions are all relative to the ambient integer n, but for the sake of readability we will not reflect this in the notation.

Definition 3.3. A *configuration* C of a valid plane tree T consists of the following data, for each non-root vertex v of $A(T)$:

- An integer $m(v) \geq 0$.
- A subset $J(v) \subseteq \{1, \ldots, n\}$ with $|J(v)| = m(v)$.
- If v is an input, or comes from an internal edge of T, a pair

  $$(a_j(v), \gamma_j(v)) \in \{1, \ldots, n\} \times \mathbb{N}^n$$

  for each $j \in J(v)$, with $\gamma_j(v)_{a_j(v)} \geq 1$.
- If v comes from an internal edge of T, an integer $t(v) \in \{1, \ldots, n\}$.

Let $\mathrm{Con}(T)$ denote the set of all configurations.

Remark 3.4. The integer $m(v)$ counts how many interactions of type A or C take place at v (there is no point counting B interactions as precisely one occurs at each internal edge). The set $J(v)$ consists of all j-indices appearing in interactions at v. The interactions of A-type (at inputs and internal edges) are defined by indices $a_j(v), \gamma_j(v)$ while at each internal edge e of T, $t(e)$ is the index of the B-type interaction.

Note that a configuration may contain no A-type or C-type interactions, that is, we may have $m(v) = 0$ for every non-root vertex v of $A(T)$. Then the configuration is just the assignment of an integer $1 \leq t(e) \leq n$ to every internal edge e of T. For the unique tree $T \in \mathcal{T}_2$ with two inputs, there are no internal edges and precisely one configuration with $m(v) = 0$ for all v.

Definition 3.5. Given a tree $T \in \mathcal{T}_q$ and configuration $C \in \mathrm{Con}(T)$ we define a decoration $D_{T,C}$ of $A(T)$ by the assignment of the modules

- $\mathscr{B}$ to the input at each non-root leaf, and
- $\mathscr{H}$ to each edge and the output at the root.

To each vertex v of $A(T)$ we associate an operator ϕ_v as follows, writing m, J, $\{(a_j, \gamma_j)\}_{j \in J}, t$ for the data associated to v by the configuration C:

- if v is an input, then ϕ_v is the linear map $\mathscr{B} \longrightarrow \mathscr{H}$ given by

  $$\phi_v = (-1)^m \prod_{j \in J} \Big\{ \frac{1}{|\gamma_j|} (\gamma_j)_{a_j} W^j(\gamma_j) \theta_{a_j} x^{\gamma_j - e_{a_j}} \big[\psi_j, -\big] \Big\} \circ \sigma \,. \tag{3.12}$$

  Note that the operator under the product is even, so the order is irrelevant. If $J(v)$ is empty then the product is interpreted to be the identity operator.

- if v comes from an internal edge of T, then

$$\phi_v = (-1)^m \prod_{j \in J} \Big\{ \frac{1}{|\gamma_j|} (\gamma_j)_{a_j} W^j(\gamma_j) \theta_{a_j} x^{\gamma_j - e_{a_j}} \big[\psi_j, -\big] \Big\} \circ \theta_t \partial_t \,. \tag{3.13}$$

- if v comes from an internal vertex of T, then

$$\phi_v = (-1)^m m_2 \circ \prod_{j \in J} \Big\{ \big[\psi_j, -\big] \otimes \theta_j^* \Big\} \tag{3.14}$$

  which is a map $\mathscr{H}^{\otimes 2} \longrightarrow \mathscr{H}$. Here m_2 denotes the product on $\mathscr{H}$.

- if $v = r$ is the root, $\phi_v = p : \mathscr{H} \longrightarrow \mathscr{H}$ is the projection of Definition 3.1.

The denotation $\langle D_{T,C} \rangle$ of this decoration is *a priori* a linear map $\mathscr{B}^{\otimes q} \longrightarrow \mathscr{H}$, but since the only operators in the decoration acting on the third tensor factor of $\mathscr{H}$ in (**??**) are the commutators $[\psi_i, -]$ under which $\mathscr{B}$ is closed, $\langle D_{T,C} \rangle$ actually has its image contained in the submodule $\mathscr{B} \subseteq \mathscr{H}$.

Definition 3.6. Given $T \in \mathcal{T}_q$ and $C \in \mathrm{Con}(T)$ we define the k-linear operator

$$\mathcal{O}(T, C) : \mathscr{B}^{\otimes q} \longrightarrow \mathscr{B}$$

to be the denotation $\mathcal{O}(T, C) = \langle D_{T,C} \rangle$, defined without Koszul signs.

Example 3.7. The configuration C which is described in Example 3.2 has $n = 1$ and

$$\begin{gathered} m(i_3) = m(v_1) = m(v_2) = 1 \,, \\ J(i_3) = J(v_1) = J(v_2) = \{1\} \,, \\ t(e) = 1 \,. \end{gathered}$$

At i_3 we have the pair $a = 1$ and $x^\gamma = x^2$. The operator $\mathcal{O}(T, C)$ is precisely (3.11).

Remark 3.8. Let us now briefly explain the connection between configurations, the linear map $\mathcal{O}(T, C)$ and Feynman diagrams, such as the one in Example 3.2. Ultimately we will not actually use these diagrams to perform calculations, so we refrain from formulating them too precisely; however, they are very useful as a tool for understanding.

We view $\Psi_1 \otimes \cdots \otimes \Psi_q \in \mathscr{B}^{\otimes q}$ as the result of applying creation operators (meaning wedge products $\psi_i^* \wedge -$) to the vacuum (meaning the identity of the exterior algebra) in the various copies of $\mathscr{B}$. The value of $\mathcal{O}(T, C)$ on Ψ may be computed by commuting all creation operators acting on $\mathscr{H}$ (that is, the operators x_i, θ_i, ψ_i^*) leftward in the expression, which means *down* the tree. These operators commute with all other creation operators and with those annihilation operators that are either of a different type (e.g. $[x_i, \theta_j^*] = 0$) or of same type but with different indices (e.g. $[x_1, \partial_2] = 0$). However, there will be a nonzero commutator every time a creation operator meets an annihilation operator of the same type and index (respectively $\partial_i, \theta_i^*, [\psi_i, -]$). We view such a commutator, say

$$\theta_i \theta_i^* + \theta_i^* \theta_i = [\theta_i, \theta_i^*] = 1\,,$$

which arises as the result of commuting the θ_i leftward to meet the θ_i^*, as generating two diagrams, corresponding to the two choices of summands in $\theta_i^* \theta_i = 1 - \theta_i \theta_i^*$. Choosing the first summand means there is an interaction (drawn as a line marked θ_i from the original position of the θ_i to the position of the θ_i^* in the tree) while choosing the second summand means there is no interaction (the θ_i sails on, to meet its partner further down the tree). If a particular series of choices leads to an x_i or θ_i meeting the bottom of the tree then this diagram does not contribute to $\mathcal{O}(T, C)(\Psi)$ since p projects out such elements of $\mathscr{H}$.

We can depict a series of choices by linking the creation and annihilation operators,

$$p \circ m_2\big([\psi, -] \otimes \theta^*\big) \circ \big(\iota \otimes \theta\partial_x \circ m_2([\psi, -] \otimes \theta^*)\big(\iota \otimes \theta x[\psi, -]\iota\big)\big)\Big(\psi^* \otimes \psi^* \otimes \psi^*\Big)$$

or by drawing these linkages, marked with the type and index of the "particle" on the tree T, joining the two positions determined by the configuration as in Example 3.2.

Note that in this example one of the other series of "choices" would involve commuting the rightmost θ past the first θ^* to meet with the leftmost θ^*. In the corresponding Feynman diagram the θ fermion would travel from the A-type interaction vertex all the way to the bottom of the tree. However, this diagram gives a zero contribution to $\mathcal{O}(T, C)(\Psi)$, for at least two reasons: firstly, in the calculation we see parallel identical fermion lines which vanish (since $\theta^2 = 0$), and secondly the leftmost θ may in this case be commuted left to give zero on p, after the "guard" θ^* is cancelled by the other θ.

Remark 3.9. In the genuine Feynman diagrams of quantum field theory (as compared to the toy version considered here) particle lines labelled by momenta p satisfying Einstein's equation $p^2 = m^2$ with m the mass for the given field are "on the mass shell" or simply "on-shell" and they represent real particles. Those lines with momenta that do not satisfy this equation are "off-shell" and are called *virtual* particles.

In topological string theory this terminology is used in the following way: the space of states is now a complex $(\mathscr{H}, Q)$ and elements of $\mathrm{Ker}(Q)$ are described as "on-shell" while all other elements are "off-shell" [8, 2]. Following this point of view we think of our Feynman diagrams as having incoming and outgoing states the real particles ψ_i^*'s, while the internal edges involve the virtual particles x_i and θ_i. For us this is purely convenient terminology, and is not intended to have any deeper significance. For the experts we note that the ψ_i^*'s are not actually cycles; see Remark 4.8 (todo: weird).

The following simple identity is well-known in the literature in connection with soft photons and infrared divergences in quantum field theory, see for instance [6, Ch 13] and [7, p.204]. The amplitudes for our Feynman diagrams will involve a generalisation.

Lemma 3.10. *Given a sequence $a_1, \ldots, a_m \geq 1$ of integers,*

$$\sum_{\sigma \in \mathfrak{S}_m} \frac{1}{a_{\sigma(1)}(a_{\sigma(1)} + a_{\sigma(2)}) \cdots (a_{\sigma(1)} + \cdots + a_{\sigma(m)})} = \frac{1}{a_1 \cdots a_m} \,. \tag{3.15}$$

Definition 3.11. Given a sequence $a_1, \ldots, a_m \geq 1$ of integers and $a \geq 0$ we define

$$\begin{aligned} \zeta_a(a_1, \ldots, a_m) &= a_1 \cdots a_m \cdot \sum_{\sigma \in \mathfrak{S}_m} \prod_{i=1}^{m} \Big(a + \sum_{j=1}^{i} a_{\sigma(j)} \Big)^{-1} \\ &= \sum_{\sigma \in \mathfrak{S}_m} \frac{a_1 \cdots a_m}{(a + a_{\sigma(1)})(a + a_{\sigma(1)} + a_{\sigma(2)}) \cdots (a + a_{\sigma(1)} + \cdots + a_{\sigma(m)})} \,. \end{aligned}$$

Clearly this is a symmetric function of the a_i and, by the previous lemma,

$$\zeta_0(a_1, \ldots, a_m) = 1 \,. \tag{3.16}$$

A configuration C gives rise to numerous Feynman diagrams, but it will turn out that all diagrams with nonzero amplitude have the *same* pattern of connections among the virtual particles. This means that to a configuration C and edge e of T we may associate the *number* $N_C(e)$ of virtual particles entering e.

For the next two definitions we take $T \in \mathcal{T}_q$ and $C \in \mathrm{Con}(T)$.

Definition 3.12. Given an internal edge e of T we define

$$N_C(e) = \sum_{v < e} \sum_{j \in J(v)} |\gamma_j(v)| - \sum_{z < e} m(z)$$

where v ranges over all inputs and internal edges of T which are strictly above e in $A(T)$, and z ranges over all internal vertices of T which are strictly above e.

This counts the number of virtual particles entering v from above, since an A-type interaction with indices a, j, γ creates $|\gamma|$ virtual particles, a B-type interaction preserves the number of virtual particles, and a C-type interaction decreases the number by one.

Example 3.13. In the situation of Example 3.7, $N_C(e) = 1$.

Combinatorially the most important part of the A_∞-products in the minimal model for W is the rational number Z_C in the next definition. We use the following shorthand: if e is an internal edge of T assigned by the configuration C the data $m = m(v)$, $J = J(v)$ and $\{\gamma_j\}_{j \in J}$, then we write

$$\underline{\gamma}(e) = \big(|\gamma_j|\big)_{j \in J} = \big(|\gamma_{j_1}|, \ldots, |\gamma_{j_m}|\big) \tag{3.17}$$

for the sequence of total degrees where $J = \{j_1, \ldots, j_m\}$.

Definition 3.14. We define

$$Z(T,C) = \prod_e \frac{\zeta_{N_C(e)}\big(\underline{\gamma}(e)\big)}{N_C(e)} \in \mathbb{Q}$$

where is the product is over all internal edges e of T with $N_C(e) \neq 0$. For an edge e with $m(e) = 0$ and thus $\underline{\gamma}(e) = \emptyset$ we take $\zeta_{N_C(e)}\big(\underline{\gamma}(e)\big) = 1$. If there are no internal edges then we take $Z(T,C) = 1$.

We note that in the definition of the higher products ρ_q below, any pair (T,C) which has an internal edge e with $N_C(e) = 0$ will not contribute to the sum in any case, so we do not care about $Z(T,C)$ for such configurations.

Example 3.15. When $m(e) = 1, J(e) = \{j\}$ and we write $\gamma = \gamma_j(e), N = N_C(e)$, we have

$$\zeta_{N_C(e)}\big(\underline{\gamma}(e)\big) = \frac{|\gamma|}{N + |\gamma|}\,.$$

In the case where $m(e) = 2, J(e) = \{j, j'\}$ and we write $\gamma = \gamma_j(e), \gamma' = \gamma_{j'}(e)$,

$$\zeta_{N_C(e)}\big(\underline{\gamma}(e)\big) = \frac{|\gamma||\gamma'|}{N + |\gamma| + |\gamma'|}\left(\frac{1}{N + |\gamma|} + \frac{1}{N + |\gamma'|}\right)\,.$$

In particular, for the tree and configuration of Examples 3.2, 3.7 we only have one internal edge e with $m(e) = 0$, so $Z(T,C) = \frac{1}{N_C(e)} = 1$.

Definition 3.16. We define the map $\rho_q : \mathscr{B}[1]^{\otimes q} \longrightarrow \mathscr{B}[1]$ of degree $+1$ on homogeneous elements $\Lambda_1, \ldots, \Lambda_q \in \mathscr{B}$ by

$$\rho_q(\Lambda_q, \ldots, \Lambda_1) = \sum_{T \in \mathcal{T}_q} \sum_{C \in \mathrm{Con}(T)} (-1)^{Q(T,\underline{\Lambda})} Z(T,C) \cdot \mathcal{O}(T,C)(\Lambda_1, \ldots, \Lambda_q)$$

where $\mathcal{O}(T,C)$ is from Definition 3.6 and the sign is given by

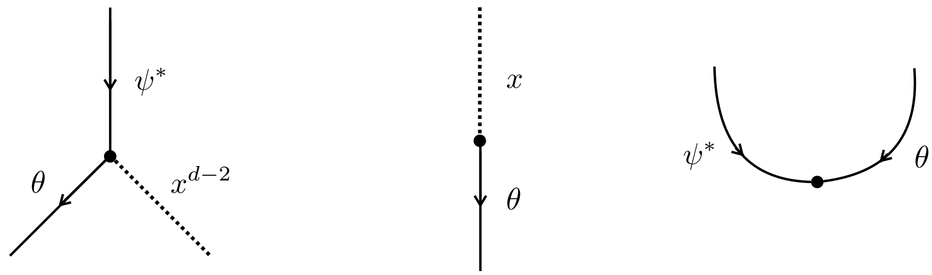
$$Q(T,\underline{\Lambda}) = e_i(T) + q + 1 + \sum_{1 \le i < j \le q} \widetilde{\Lambda}_i \widetilde{\Lambda}_j + \sum_{i=1}^{q} \widetilde{\Lambda}_i C_i\,.$$

Here C_i is the number of times the path from the ith input to the root enters a trivalent vertex from the left in T, $e_i(T)$ is the number of internal edges and $\widetilde{\Lambda} = |\Lambda| - 1$.

Theorem 3.17. *The operators ρ_q satisfy the forward suspended A_∞-constraints, so that $(\mathscr{B}, \{\rho_q\}_{q \ge 2})$ is an A_∞-algebra. Moreover $\mathscr{B}$ is A_∞-quasi-isomorphic to $\mathscr{A}$.*

Proof. See Section **??**. ∎

Example 3.18. The potential $W = x^d$ for $d > 2$ has the following Feynman rules:



Let us compute the A_∞-products ρ_q on $\mathscr{B}$:

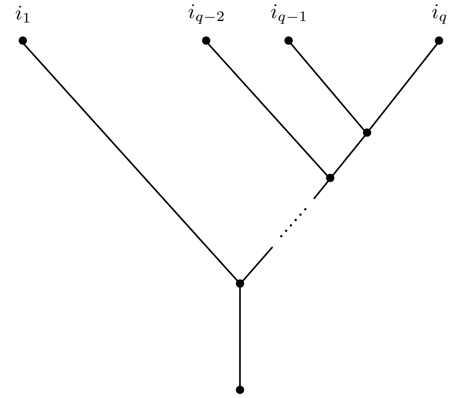
- For $q = 2$ there is only one tree $T \in \mathcal{T}_2$. We claim that since T has no internal edge the only configuration C with $\mathcal{O}(T, C) \neq 0$ is the unique C with no A-type or C-type interactions (see Remark 3.4). To see this, note that an A-type interaction would create a polynomial x^{d-2} which cannot be eliminated by derivatives at internal edges, and therefore annihilates with p. Since there can be no A-type interaction to create a θ, the θ^* in a C-type interaction will end up acting on the identity in the exterior algebra, and giving zero as well. Since this C is the only contributor to the sum, and $\langle D_{T,C} \rangle = p \circ m_2 \circ (\sigma \otimes \sigma)$, we have

$$\rho_2(\Lambda_2, \Lambda_1) = (-1)^{\widetilde{\Lambda_1}\widetilde{\Lambda_2} + \widetilde{\Lambda_1} + 1} \Lambda_1 \cdot \Lambda_2 \tag{3.18}$$

  where $(-) \cdot (-)$ denotes the usual product in the exterior algebra. This is just the forward suspension of the product in the exterior algebra.

- For $q > 2$ let C be a configuration with $\mathcal{O}(T, C) \neq 0$. Then C must have at least one A-type interaction, since otherwise the ∂_x in the B-type interactions are applied to $1 \in R$. Moreover, these A-type interactions can only be inserted at input vertices, since at an internal edge (3.13) would contribute $\theta^2 = 0$.

  For the same reason, the θ coming out of the B-type interaction at an internal edge e must be consumed in a C-type interaction at the vertex v immediately after e on the path to the root: if not, it will annihilate with the θ emitted at the next internal edge, or, if v is adjacent to the root, it will annihilate with p. This means that every edge e must be the right-hand branch at v, from which we deduce that T is the tree in which all internal edges lie on the path from the rightmost input to the root:



This implies that there is *precisely* one A-type interaction in C, and it takes place at the vertex marked i_q in the diagram. But since T contains $q-2$ internal edges, if $\mathcal{O}(T,C) \neq 0$ then we must have $q = d$. That is, $\rho_q = 0$ unless $q \in \{2, d\}$.

- Finally, let us describe $\rho_d : \mathscr{B}^{\otimes d} \longrightarrow \mathscr{B}$ on the basis of tensors $(\psi^*)^{s_0} \otimes \cdots \otimes (\psi^*)^{s_d}$ where $s_j \in \{0,1\}$ for $1 \le j \le d$. We require a copy of ψ^* at each input $i_1, \ldots, i_{q-2}$ in order to meet the θ emitted by B-type interactions, a copy of ψ^* at i_{q-1} to meet the θ from the A-type interaction and a ψ^* at i_q to initiate that A-type interaction. Hence ρ_d is zero on all the basis elements except for

$$\rho_d(\psi^* \otimes \psi^* \otimes \cdots \otimes \psi^*) = (-1)^{Q(T,\underline{\Lambda})} \cdot 1\,.$$

To compute the sign, observe that $e_i(T) = d-2$ and $\Lambda_i = \psi^*$ is odd so $\widetilde{\Lambda}_i$ is even, whence $Q(T,\underline{\Lambda}) = (d-2) + d + 1 = 1$.

In summary: the underlying $\mathbb{Z}_2$-graded k-module of $\mathscr{B}$ is the exterior algebra $\bigwedge(k\psi^*)$ and the A_∞-products ρ_q are zero unless $q \in \{2, d\}$. The product ρ_2 is the forward suspended version of the usual product on the exterior algebra, while

$$\rho_d(\psi^* \otimes \psi^* \otimes \cdots \otimes \psi^*) = -1\,.$$

4 The proof

4.1 The homotopy retract

Let k be a characteristic zero field, $R = k[x_1, \ldots, x_n]$ and let $W \in \mathfrak{m}^3$ be a potential with chosen decomposition $W = \sum_{i=1}^n x_i W^i$. We aim to construct the minimal model of the following DG-algebra

$$\begin{aligned}\mathscr{A} &= \Big(\operatorname{End}_R(k^{\text{stab}}),\ \partial = [d_{k^{\text{stab}}}, -] \Big)\\ &= \Big(R \otimes_k \operatorname{End}_k \Big(\bigwedge \big(\oplus_i k\psi_i \big) \Big), \partial = \sum_i x_i[\psi_i^*, -] + \sum_i W^i[\psi_i, -] \Big)\,.\end{aligned}$$

Recall the module $\mathscr{H}$ of Definition 3.1 which we recall here for the reader's convenience, writing $F = \oplus_i k\psi_i$ for compactness

$$\mathscr{H} = R \otimes_k \bigwedge \big(\oplus_i k\theta_i \big) \otimes_k \operatorname{End}_k \Big(\bigwedge F \Big) .$$

If we extend $\mathscr{A}$ by the exterior algebra on the θ's we get the DG-algebra

$$\mathscr{A} \otimes_k \bigwedge \big(\oplus_i k\theta_i \big) = \big(\mathscr{H}, \partial \big) .$$

The Koszul complex of the sequence $x_1, \ldots, x_n$ is

$$K = \Big(R \otimes_k \bigwedge \big(\oplus_i k\theta_i \big), \; d_K = \sum_i x_i \theta_i^* \Big) \tag{4.1}$$

which may be extended by $\operatorname{End}_k \big(\bigwedge F \big)$ to give

$$K \otimes_k \operatorname{End}_k \big(\bigwedge F \big) = \big(\mathscr{H}, d_K \big) .$$

Definition 4.1. The morphism of complexes $\pi : K \longrightarrow R/\mathfrak{m}$ is defined by composing the projection of K onto the submodule $R \cdot 1$ of θ-degree zero forms, with the quotient $R \longrightarrow R/\mathfrak{m}$. We also write π for the following tensor product

$$\big(\mathscr{H}, d_K + \partial \big) \xrightarrow{\pi \otimes 1} \Big(\operatorname{End}_k \big(\bigwedge F \big), 0 \Big) .$$

where we use that $R/\mathfrak{m} \otimes_R \operatorname{End}_R(k^{\mathrm{stab}})$ has zero differential by the hypothesis that $W \in \mathfrak{m}^2$.

Definition 4.2. We write σ for the inclusion $\operatorname{End}_k \big(\bigwedge F \big) \longrightarrow \mathscr{H}$ defined by

$$\sigma(\Phi) = 1 \otimes 1 \otimes \Phi .$$

Definition 4.3. Let L_{ψ_i} denote the odd operator on $\operatorname{End}_k \big(\bigwedge F \big)$ defined by

$$L_{\psi_i}(\alpha) = \psi_i \circ \alpha = (\psi_i \wedge (-)) \circ \alpha ,$$

so that $L_{\psi_i}(\alpha)(x) = \psi_i \wedge \alpha(x)$.

Throughout $\sum_i$ stands for $\sum_{i=1}^n$.

Theorem 4.4. *Using the following even homogeneous operators on* $\mathscr{H}$*,*

$$\delta = \sum_i L_{\psi_i} \theta_i^* , \qquad \sigma_\infty = \sum_{m \geq 0} (-1)^m (H\partial)^m \sigma , \tag{4.2}$$

and the following odd homogeneous operators

$$\partial = \sum_i x_i[\psi_i^*, -] + \sum_i W^i[\psi_i, -]\,, \tag{4.3}$$

$$\nabla = \sum_i \partial_i \theta_i\,, \qquad H = [d_K, \nabla]^{-1}\nabla\,, \tag{4.4}$$

$$H_\infty = \sum_{m\geq 0} (-1)^m (H\partial)^m H\,. \tag{4.5}$$

there is a diagram of k-linear homotopy equivalences

$$\big(\mathscr{H}, \partial\,\big) \underset{\exp(\delta)}{\overset{\exp(-\delta)}{\rightleftarrows}} \big(\mathscr{H}, d_K + \partial\,\big) \underset{\sigma_\infty}{\overset{\pi}{\rightleftarrows}} \Big(\operatorname{End}_k\big(\textstyle\bigwedge F\big), 0\Big)\,. \tag{4.6}$$

More precisely, $\exp(-\delta), \exp(\delta)$ *are mutually inverse as k-linear maps,* $\pi \circ \sigma_\infty = 1$ *and*

$$1_{\mathscr{H}} - \big[d_K + \partial, H_\infty\big] = \sigma_\infty \circ \pi\,. \tag{4.7}$$

Proof. For the duration of the proof let $\mathscr{E} = \operatorname{End}_k\big(\bigwedge F\big)$. It is easy to check that $\exp(\delta), \exp(-\delta)$ intertwines the differentials ∂ and $d_K + \partial$ and therefore gives an isomorphism between the first two complexes in (4.6) [3, Proposition 4.11]. The operator $\psi_j = \psi_j \wedge -$ on k^{stab} satisfies

$$[\psi_j, d_{k^{\mathrm{stab}}}] = \big[\psi_j, \sum_i x_i \psi_i^*\big] = x_j \cdot 1\,.$$

That is, ψ_j is a homotopy for the action of x_j on k^{stab}. It is then easy to check that the odd operator $\alpha \mapsto \psi_j \circ \alpha$ on $\operatorname{End}_R(k^{\mathrm{stab}})$ is also a homotopy for x_j. $\square$

It is the homotopy retract (4.6) that we will ultimately feed into the minimal model theorem, taking as our DG-algebra $(\mathscr{H}, \partial)$ equipped with the projection $\pi\exp(-\delta)$ injection $\exp(\delta)\sigma_\infty$ and the homotopy $\exp(\delta)H_\infty\exp(-\delta)$. In broad outlines, these operators lead to the three types of Feynman diagram interactions as follows:

- A-type interactions arise from H_∞, more precisely the $H\partial$ factors.
- B-type interactions arise from the "odd man out" H at the right end of H_∞.
- C-type interactions arise from $\exp(\pm\delta)$.

We address each of these in turn before, in Section **??**, finishing the proof.

4.2 Deriving A-type and B-type interactions

The A-type and B-type interactions arise from σ_∞ and H_∞ from (4.2),(4.5). The main task is to understand $(H\partial)^m$. We have for $m > 0$, as an operator on $\mathscr{H}$

$$\big(H\partial\big)^m = \prod_{i=1}^{m} \sum_{j_i=1}^{n} H \circ \Big(x_{j_i}[\psi^*_{j_i}, -] + W^{j_i}[\psi_{j_i}, -] \Big) .$$

Prior to Definition 3.6 we observed that in calculating $\langle D_{T,C} \rangle$ it suffices to consider operators on the $\mathbb{Z}_2$-graded k-module of $\mathscr{H}$ given by

$$\mathscr{H}' = R \otimes_k \bigwedge \big(\oplus_{i=1}^n k\theta_i \big) \otimes_k \mathscr{B} . \tag{4.8}$$

In the next lemma we calculate $(H\partial)^m$ on $\mathscr{H} \cap \operatorname{Ker}(\nabla)$, where $\nabla = \sum_i \partial_i \theta_i$. Before we do this we introduce some notation: in the calculation we use a $\mathbb{Z}$-grading on $\mathscr{H}'$ defined as follows: $\mathscr{H}'_a$ is the k-submodule spanned by tensors of the form

$$f \otimes \omega \otimes \kappa \text{ where } f \in R\,, \omega \in \bigwedge(\oplus_i k\theta_i), \kappa \in \mathscr{B} \text{ and } |f|_{\mathbb{Z}} + |\omega|_{\mathbb{Z}} = a\,.$$

Here $|-|_{\mathbb{Z}}$ denotes the $\mathbb{Z}$-grading with $|x_i|_{\mathbb{Z}} = 1$ and $|\theta_i|_{\mathbb{Z}} = 1$. To be perfectly clear: we have $|\theta_1\theta_2|_{\mathbb{Z}} = 2$, as compared to $|\theta_1\theta_2| = 0$ in the $\mathbb{Z}_2$-grading used elsewhere in the text.

Lemma 4.5. *If $f \in R$ and $\omega \in \bigwedge(\oplus_{i=1}^n k\theta_i)$ are homogeneous then*

$$[d_K, \nabla](f \otimes \omega) = (|f|_{\mathbb{Z}} + |\omega|_{\mathbb{Z}}) f \otimes \omega\,. \tag{4.9}$$

Proposition 4.6. *Restricted to $\mathscr{H}'_a \cap \operatorname{Ker}(\nabla)$, $(H\partial)^m$ is equal to*

$$\sum_{\substack{1 \le j_1 < \cdots < j_m \le n \\ 1 \le z_1 < \cdots < z_m \le n \\ \gamma_1, \ldots, \gamma_m \in \mathbb{N}^n \setminus \{0\}}} \sum_{\nu \in \mathfrak{S}_m} \frac{\zeta_a(|\gamma_1|, \ldots, |\gamma_m|)}{|\gamma_1| \cdots |\gamma_m|} \prod_{i=1}^{m} W^{j_i}(\gamma_i) \partial_{z_{\nu(i)}}(x^{\gamma_i}) \theta_{z_{\nu(i)}} [\psi_{j_i}, -]\,. \tag{4.10}$$

Proof. On this submodule of $\mathscr{H}$ the operators $[\psi^*_i, -]$ act as zero, and so $(H\partial)^m$ takes a simpler form. If $\underline{j}$ ranges over sequences of length m in $\{1, \ldots, n\}^m$ and $\underline{\gamma}$ over sequences of length m in $\mathbb{N}^n$ then

$$\begin{aligned}
\big(H\partial\big)^m|_{\mathscr{H}'} &= \sum_{\underline{j}} \prod_{i=1}^{m} H \circ \Big(W^{j_i}[\psi_{j_i}, -] \Big) \\
&= \sum_{\underline{j}} \prod_{i=1}^{m} H \circ \Big(\sum_{\gamma_i} W^{j_i}(\gamma_i) x^{\gamma_i} [\psi_{j_i}, -] \Big) \\
&= \sum_{\underline{j}, \underline{\gamma}} \prod_{i=1}^{m} H \circ \Big(W^{j_i}(\gamma_i) x^{\gamma_i} [\psi_{j_i}, -] \Big) \\
&= \sum_{\underline{j}, \underline{\gamma}} \prod_{i=1}^{m} [d_K, \nabla]^{-1} \nabla \circ \Big(W^{j_i}(\gamma_i) x^{\gamma_i} [\psi_{j_i}, -] \Big) .
\end{aligned}$$

Note that W^i contains no constant term, since by hypothesis $W \in \mathfrak{m}^2$, so in fact we may assume that all γ_i are nonzero. Our convention is that products of operators expand from left to right as i increases. Lemma 4.5

Now we consider the scalar factors produced by the $[d_K, \nabla]^{-1}$ when we apply $(H\partial)^m$ to a tensor $f \otimes \omega \otimes \Psi$. Observe how the sum of the polynomial and θ-degree changes with each application of $H\partial$. Multiplying with x^{γ_i} increases the polynomial degree by $|\gamma_i|$, while applying ∇ decreases the polynomial degree and increases the θ-degree by one, so overall the ith copy of $H\partial$ (reading from left to right) increases the sum of the two degrees from

$$|f|_{\mathbb{Z}} + |\omega|_{\mathbb{Z}} + \sum_{t>i} |\gamma_t| \qquad \text{to} \qquad |f|_{\mathbb{Z}} + |\omega|_{\mathbb{Z}} + \sum_{t\geq i} |\gamma_t|\,.$$

If we write $a = |f| + |\omega|$ then the scalar factor contributed by the copies of $[d_K, \nabla]^{-1}$ in $(H\partial)^m$ applied to $f \otimes \omega \otimes \Psi$ is therefore

$$\frac{1}{(a+|\gamma_m|)(a+|\gamma_m|+|\gamma_{m-1}|)\cdots(a+|\gamma_m|+\cdots+|\gamma_1|)}\,.$$

This leads us to the un-symmetrised version of ζ from Definition 3.11: given a sequence $a_1, \ldots, a_m \geq 1$ of integers and $a \geq 0$ we define

$$\zeta'_a(a_1, \ldots, a_m) = \frac{1}{(a+a_m)(a+a_{m-1}+a_m)\cdots(a+a_1+\cdots+a_m)}\,. \tag{4.11}$$

Let $\mathscr{H}'_a \subseteq \mathscr{H}'$ denote the submodule of elements of degree a, when we take the $\mathbb{Z}$-grading induced by the usual $\mathbb{Z}$-grading on R and the exterior algebra in the θ's (so, the grading does not count ψ's). Then we have computed that

$$(H\partial)^m|_{\mathscr{H}'_a} = \sum_{\underline{j},\underline{\gamma}} \zeta'_a(|\gamma_1|, \ldots, |\gamma_m|) \prod_{i=1}^m \nabla \circ \big(W^{j_i}(\gamma_i) x^{\gamma_i}[\psi_{j_i}, -]\big)\,. \tag{4.12}$$

For a k-linear operator T and an element $x \in \operatorname{Ker}(\nabla)$ since $\nabla^2 = 0$ we have

$$\nabla T \cdots \nabla T(x) = [\nabla, T] \cdots [\nabla, T](x)\,.$$

Using $[\nabla, f] = \sum_q \partial_q(f)\theta_q$ we therefore have

$$\begin{aligned}
\big(H\partial)^m|_{\mathscr{H}'_a \cap \operatorname{Ker}(\nabla)} &= \sum_{\underline{j},\underline{\gamma}} \zeta'_a(|\gamma_1|, \ldots, |\gamma_m|) \prod_{i=1}^m \Big[\nabla, W^{j_i}(\gamma_i) x^{\gamma_i}[\psi_{j_i}, -]\Big] \\
&= \sum_{\underline{j},\underline{\gamma},\underline{z}} \zeta'_a(|\gamma_1|, \ldots, |\gamma_m|) \prod_{i=1}^m W^{j_i}(\gamma_i) \partial_{z_i}(x^{\gamma_i}) \theta_{z_i}[\psi_{j_i}, -]
\end{aligned}$$

where $\underline{j}, \underline{z}$ both range over $\{1, \dots, n\}^m$. Since $[\psi_j, -]^2 = 0$ the only nonzero contributions are from sequences $\underline{j} = (j_1, \dots, j_m)$ without repeats and similarly for $\underline{z}$. Thus

$$\begin{aligned}
&= \sum_{\substack{j_1<\cdots<j_m\\ z_1<\cdots<z_m}} \sum_{\rho,\nu\in\mathfrak{S}_m} \sum_{\underline{\gamma}} \zeta'_a(|\gamma_1|, \dots, |\gamma_m|) \prod_{i=1}^m W^{j_{\rho(i)}}(\gamma_i)\partial_{z_{\nu(i)}}(x^{\gamma_i})\theta_{z_{\nu(i)}}[\psi_{j_{\rho(i)}}, -]\\
&= \sum_{\substack{j_1<\cdots<j_m\\ z_1<\cdots<z_m}} \sum_{\rho,\nu\in\mathfrak{S}_m} \sum_{\underline{\gamma}} \zeta'_a(|\gamma_1|, \dots, |\gamma_m|) \prod_{i=1}^m W^{j_i}(\gamma_{\rho^{-1}(i)})\partial_{z_{\nu_{\rho^{-1}(i)}}}(x^{\gamma_{\rho^{-1}(i)}})\theta_{z_{\nu_{\rho^{-1}(i)}}}[\psi_{j_i}, -]\\
&= \sum_{\substack{j_1<\cdots<j_m\\ z_1<\cdots<z_m}} \sum_{\rho,\nu\in\mathfrak{S}_m} \sum_{\underline{\gamma}} \zeta'_a(|\gamma_{\rho(1)}|, \dots, |\gamma_{\rho(m)}|) \prod_{i=1}^m W^{j_i}(\gamma_i)\partial_{z_{\nu(i)}}(x^{\gamma_i})\theta_{z_{\nu(i)}}[\psi_{j_i}, -]\\
&= \sum_{\substack{j_1<\cdots<j_m\\ z_1<\cdots<z_m}} \sum_{\nu\in\mathfrak{S}_m} \sum_{\underline{\gamma}} \frac{\zeta_a(|\gamma_1|, \dots, |\gamma_m|)}{|\gamma_1|\cdots|\gamma_m|} \prod_{i=1}^m W^{j_i}(\gamma_i)\partial_{z_{\nu(i)}}(x^{\gamma_i})\theta_{z_{\nu(i)}}[\psi_{j_i}, -]
\end{aligned}$$

as claimed. □

Remark 4.7. Since $\nabla\sigma = 0$ and $\nabla^2 = 0$ the proposition applies to calculate σ_∞, H_∞. In the former case we simply append σ and use (3.16) to see that $(H\partial)^m\sigma$ is given under the same summation over $\underline{j}, \underline{z}, \underline{\gamma}, \nu$ as in (4.10) by

$$\sum_{\underline{j},\underline{z},\underline{\gamma},\nu} \frac{1}{|\gamma_1|\cdots|\gamma_m|}\Big\{\prod_{i=1}^m W^{j_i}(\gamma_i)\partial_{z_{\nu(i)}}(x^{\gamma_i})\theta_{z_{\nu(i)}}[\psi_{j_i}, -]\Big\} \circ \sigma\,. \tag{4.13}$$

In the case of H_∞ we observe that $H = [d_K, \nabla]^{-1}\nabla$ sends $\mathscr{H}'_a$ into $\mathscr{H}'_a \cap \mathrm{Ker}(\nabla)$, and we have that $(H\partial)^m H$ on $\mathscr{H}'_a$ is given by

$$\sum_{\underline{j},\underline{z},\underline{\gamma},\nu} \sum_{t=1}^n \frac{1}{a}\frac{\zeta_a(|\gamma_1|, \dots, |\gamma_m|)}{|\gamma_1|\cdots|\gamma_m|}\Big\{\prod_{i=1}^m W^{j_i}(\gamma_i)\partial_{z_{\nu(i)}}(x^{\gamma_i})\theta_{z_{\nu(i)}}[\psi_{j_i}, -]\Big\} \circ \partial_t\theta_t\,. \tag{4.14}$$

Remark 4.8. Continuing from Remark 3.9 we note that in contrast to the way we usually think about the minimal model construction in the setting of topological string theory [?] the elements Ψ of $\mathscr{B}$ are not immediately identified with cohomology classes of $\mathscr{B}$.

Of course $\sigma_\infty(\Psi)$ is a cycle for ∂, but this is already given by a complicated sum which contributes multiple Feynman diagrams in the calculus of Definition **??**. That is to say, we have found it useful for purposes of calculation to have the incoming states in our Feynman diagrams to be elements of $\mathscr{H}$ that are *not* cycles. Note that as we vary the potential W the elements of $\mathscr{H}$ that are cycles will change - this makes it awkward to take as a starting point the cohomology of $\mathscr{H}$. In our approach the subspace $\mathscr{B}$ remains fixed, and even in the case of quadratic terms it is only the projection to $\mathscr{B}$ that changes.

4.3 Deriving C-type interactions

Let us briefly recall some of the notation from earlier: we write $F = \oplus_i k\psi_i$ and use the operators L_{ψ_i} of Definition 4.3 and δ of (4.2). We set $\delta_i = L_{\psi_i}\theta_i^*$ so that $\delta = \sum_i \delta_i$. To be explicit, given $\omega_i \in \bigwedge(\oplus_i k\theta_i)$ and $\kappa_i \in \mathrm{End}_k(\bigwedge F)$

$$\delta_i(\omega \otimes \kappa) = L_{\psi_i}\big(\theta_i^*(\omega) \otimes \kappa\big) = (-1)^{|\omega|+1}\theta_i^*(\omega) \otimes \psi_i \circ \kappa\,.$$

Lemma 4.9. *For $1 \le i \le n$ there is a commutative diagram*

$$\begin{tikzcd} \mathscr{H} \otimes_k \mathscr{H} \arrow[r, "m_2"] \arrow[d, "\delta_i\otimes 1 + 1\otimes\delta_i + [\psi_i,-]\otimes\theta_i^*"'] & \mathscr{H} \arrow[d, "\delta_i"] \\ \mathscr{H} \otimes_k \mathscr{H} \arrow[r, "m_2"'] & \mathscr{H} \end{tikzcd} \tag{4.15}$$

Proof. By direction calculation: given $\omega_j \in \bigwedge(\oplus_i k\theta_i)$ and $\kappa_j \in \bigwedge(\oplus_i k\psi_i)$

$$\begin{aligned} &m_2\big(\delta_i \otimes 1 + 1 \otimes \delta_i\big)\big(\omega_1 \otimes \kappa_1 \otimes \omega_2 \otimes \kappa_2\big) \\ &= m_2\big(\delta_i(\omega_1 \otimes \kappa_1) \otimes \omega_2 \otimes \kappa_2\big) \\ &+ m_2\big(\omega_1 \otimes \kappa_1 \otimes \delta_i(\omega_2 \otimes \kappa_2)\big) \\ &= (-1)^{|\omega_1|+1} m_2\big(\theta_i^*(\omega_1) \otimes \psi_i \circ \kappa_1 \otimes \omega_2 \otimes \kappa_2\big) \\ &+ (-1)^{|\omega_2|+1} m_2\big(\omega_1 \otimes \kappa_1 \otimes \theta_i^*(\omega_2) \otimes \psi_i \circ \kappa_2\big) \\ &= (-1)^{|\omega_1|+1+(|\kappa_1|+1)|\omega_2|}\theta_i^*(\omega_1)\omega_2 \otimes \psi_i \circ \kappa_1 \circ \kappa_2 \\ &+ (-1)^{|\omega_2|+1+(|\omega_2|+1)|\kappa_1|}\omega_1\theta_i^*(\omega_2) \otimes \kappa_1 \circ \psi_i \circ \kappa_2 \end{aligned}$$

Similarly one computes

$$\begin{aligned} &m_2\big([\psi_i, -] \otimes \theta_i^*\big)\big(\omega_1 \otimes \kappa_1 \otimes \omega_2 \otimes \kappa_2\big) \\ &= (-1)^{|\kappa_1|} m_2\big(\omega_1 \otimes [\psi_i, \kappa_1] \otimes \theta_i^*(\omega_2) \otimes \kappa_2\big) \\ &= (-1)^{|\kappa_1|+(|\kappa_1|+1)(|\omega_2|+1)}\omega_1\theta_i^*(\omega_2) \otimes [\psi_i, \kappa_1] \circ \kappa_2\,. \end{aligned}$$

Adding these together yields that $\delta_i \otimes 1 + 1 \otimes \delta_i + [\psi_i, -] \otimes \theta_i^*$ on $\omega_1 \otimes \kappa_1 \otimes \omega_2 \otimes \kappa_2$ is

$$\begin{aligned} &= (-1)^{|\omega_1|+1+(|\kappa_1|+1)|\omega_2|}\theta_i^*(\omega_1)\omega_2 \otimes \psi_i \circ \kappa_1 \circ \kappa_2 \\ &+ (-1)^{|\kappa_1|+(|\kappa_1|+1)(|\omega_2|+1)}\omega_1\theta_i^*(\omega_2) \otimes \psi_i \circ \kappa_1 \circ \kappa_2 \\ &= (-1)^{|\omega_1|+|\omega_2|+|\kappa_1||\omega_2|+1}\theta_i^*(\omega_1\omega_2) \otimes \psi_i \circ \kappa_1 \circ \kappa_2 \\ &= (-1)^{|\kappa_1||\omega_2|}\delta_i\big(\omega_1\omega_2 \otimes \kappa_1 \circ \kappa_2\big) \\ &= \delta_i \circ m_2\big(\omega_1 \otimes \kappa_1 \otimes \omega_2 \otimes \kappa_2\big) \end{aligned}$$

as claimed. $\square$

Definition 4.10. We define $\Xi_i = [\psi_i, -] \otimes \theta_i^* : \mathscr{H}^{\otimes 2} \longrightarrow \mathscr{H}^{\otimes 2}$ and $\Xi = \sum_i \Xi_i$.

Lemma 4.11. *As operators on $\mathscr{H} \otimes_k \mathscr{H}$, for all $1 \leq i, j \leq n$*

$$\big[\delta_i \otimes 1, 1 \otimes \delta_j\big] = 0 \tag{4.16}$$
$$\big[\delta_i \otimes 1, \Xi_j\big] = 0 \tag{4.17}$$
$$\big[1 \otimes \delta_i, \Xi_j\big] = 0\,. \tag{4.18}$$

Proposition 4.12. *There is a commutative diagram*

$$\begin{tikzcd} \mathscr{H} \otimes \mathscr{H} \arrow[r, "m_2"] \arrow[d, "\exp(-\delta)\otimes\exp(-\delta)"'] & \mathscr{H} \arrow[dd, "\exp(-\delta)"] \\ \mathscr{H} \otimes \mathscr{H} \arrow[d, "\exp(-\Xi)"'] & \\ \mathscr{H} \otimes \mathscr{H} \arrow[r, "m_2"'] & \mathscr{H} \end{tikzcd} \tag{4.19}$$

Proof. By Lemma 4.11 we have

$$\exp\big(-\big[\delta_i \otimes 1 + 1 \otimes \delta_i + \Xi_i\big]\big) = \exp(-\Xi_i) \circ \big\{\exp(-\delta_i) \otimes \exp(-\delta_i)\big\}$$

and therefore

$$\exp\big(-\big[\delta \otimes 1 + 1 \otimes \delta + \Xi\big]\big) = \exp(-\Xi) \circ \big\{\exp(-\delta) \otimes \exp(-\delta)\big\}\,.$$

By Lemma 4.9 then

$$\begin{aligned} \exp(-\delta) m_2 &= \sum_{m \geq 0} (-1)^m \frac{1}{m!} \delta^m m_2 \\ &= \sum_{m \geq 0} (-1)^m \frac{1}{m!} m_2 \big\{\delta \otimes 1 + 1 \otimes \delta + \Xi\big\}^m \\ &= m_2 \exp\big(-\big[\delta \otimes 1 + 1 \otimes \delta + \Xi\big]\big) \\ &= m_2 \exp(-\Xi) \circ \big\{\exp(-\delta) \otimes \exp(-\delta)\big\}\,, \end{aligned}$$

which completes the proof. □

Observe finally that

$$\exp(-\Xi) = \sum_{m \geq 0} (-1)^m \sum_{1 \leq j_1 < \cdots < j_m \leq n} \prod_{i=1}^{m} [\psi_{j_i}, -] \otimes \theta_{j_i}^*\,. \tag{4.20}$$

4.4 Atiyah classes and idempotents

The endomorphism algebra $C = \mathrm{End}_k(S)$ is a Clifford algebra generated by the contraction θ_i^* and wedge product θ_i operators. It is observed in [3] that (4.20) is an isomorphism of Clifford representations (in the homotopy category) when the complex $\mathscr{E}$ is equipped with the action induced by Atiyah classes:

Definition 4.13. The Atiyah classes of $\mathscr{A}_W$ are the R-linear odd operators

$$\mathrm{At}_i = [\partial, \partial_i] : \mathrm{End}_R(k^{\mathrm{stab}}) \longrightarrow \mathrm{End}_R(k^{\mathrm{stab}}) \, ,$$

Lemma 4.14. *We have*

$$\mathrm{At}_i = -[\psi_i^*, -] - \sum_{q=1}^{n} \partial_i(W^q)[\psi_q, -] \, .$$

Proof. By direct calculation:

$$\begin{aligned} \mathrm{At}_i &= \big[[d_{k^{\mathrm{stab}}}, -], \partial_i\big] \\ &= \sum_q \big[x_q[\psi_q^*, -], \partial_i\big] + \sum_q \big[W^q[\psi_q, -], \partial_i\big] \\ &= -\sum_q \partial_i(x_q)[\psi_q^*, -] - \sum_q \partial_i(W^q)[\psi_q, -] \\ &= -[\psi_i^*, -] - \sum_q \partial_i(W^q)[\psi_q, -] \, . \end{aligned}$$

$\square$

Lemma 4.15. *The induced Clifford algebra structure on $\mathscr{E}$ is given by*

$$\gamma_i^\dagger = \mathrm{At}_i = -[\psi_i^*, -] \, , \qquad \gamma_i = -\psi_i \, .$$

Lemma 4.16. *The idempotent $e = \gamma_1^\dagger \cdots \gamma_n^\dagger \gamma_n \cdots \gamma_1$ is the projection onto $\mathscr{B} \subseteq \mathscr{E}$.*

4.5 Putting it all together

Definition 4.17. Let ξ_q denote the product $\xi_q : \mathscr{E}[1]^{\otimes q} \longrightarrow \mathscr{E}$ induced by the homotopy retract (**??**) and the minimal model construction, defining an A_∞-structure on $\mathscr{E}$.

Proposition 4.18. *The linear map ξ_q is the sum over $T \in \mathcal{T}_q$ of $e_i(T)$ multiplied by the denotation of the decoration of $A(T)$ defined by the assignment of modules*

- *$\mathscr{E}[1]$ to each leaf including the root, and*
- *$\mathscr{H}[1]$ to each edge.*

To each vertex v of $A(T)$ we associate an operator ϕ_v as follows:

- *if v is an input, then $\phi_v = \sigma_\infty$.*
- *if v comes from an internal edge of T, then $\phi_v = H_\infty$.*
- *if v comes from an internal vertex of T, then*

$$\phi_v = m_2 \circ \exp(-\Xi)\,.$$

By inspection of (4.6), we therefore have the following:

Proposition 4.19. *There is a diagram of morphisms of $\mathbb{Z}_2$-graded k-complexes*

$$S \otimes \mathscr{A}_W \underset{\Phi^{-1}}{\overset{\Phi}{\rightleftarrows}} \mathscr{E} \tag{4.21}$$

where $\Phi = \pi \exp(-\delta), \Phi^{-1} = \exp(\delta)\sigma_\infty$ and $\widehat{H} = \exp(\delta)H_\infty \exp(-\delta)$ satisfy

$$\Phi\Phi^{-1} = 1\,, \qquad \Phi^{-1}\Phi = 1 - [\partial, \widehat{H}]\,. \tag{4.22}$$

Thus, Φ and Φ^{-1} are mutually inverse homotopy equivalences over k.

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