

A_∞ -minimal models of matrix factorisations

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Abstract

We study A_∞ -minimal models of the differential graded algebras obtained from endomorphisms of generators of triangulated categories of matrix factorisations, giving explicit higher products for the cases of simple singularities.

1 Introduction

Let k be a characteristic zero field, and $W \in k[x_1, \dots, x_n]$ a potential in the sense of [3, §2.2] which has the origin as its only critical point. Choose a presentation of W as a sum $W = x_1 W^1 + \dots + x_n W^n$ with $W^i \in \mathfrak{m}$. Then the generator of the homotopy category of matrix factorisations of W is the stabilised diagonal

$$(1.1) \quad k^{\text{stab}} = \left(k[x] \otimes_k \bigwedge (k\psi_1 \oplus \dots \oplus k\psi_n), \ d_{k^{\text{stab}}} = \sum_{i=1}^n x_i \psi_i^* + \sum_{i=1}^n W^i \psi_i \right)$$

where ψ_i^* and ψ_i act respectively by contraction and wedge product on the exterior algebra underlying k^{stab} . The purpose of this paper is to calculate the A_∞ -minimal model of the \mathbb{Z}_2 -graded differential graded algebra

$$(1.2) \quad \mathcal{A}_W = \left(\text{End}_{k[x]}(k^{\text{stab}}), \ \partial = [d_{k^{\text{stab}}}, -] \right)$$

with differential (throughout all commutators are graded commutators)

$$(1.3) \quad \partial = \left[\sum_i x_i \psi_i^* + \sum_i W^i \psi_i, - \right]$$

$$(1.4) \quad = \sum_i x_i [\psi_i^*, -] + \sum_i W^i [\psi_i, -].$$

The minimal model is a finite-dimensional \mathbb{Z}_2 -graded vector space \mathcal{M}_W with a family

$$\left\{ r_q : (\mathcal{M}_W[1])^{\otimes q} \longrightarrow \mathcal{M}_W[1] \right\}_{q \geq 2}$$

of odd k -linear maps satisfying the forward suspended A_∞ -constraints [1], and having the property that there is an A_∞ -quasi-isomorphism $\mathcal{A}_W \longrightarrow \mathcal{M}_W$.

2 Perturbation and Koszul complexes

Set $R = k[x_1, \dots, x_n]$ and let W be a potential as above. To apply the A_∞ -minimal model construction to \mathcal{A}_W , we would usually begin by finding a k -linear homotopy retract of this complex onto its cohomology. However, for our purposes it is better to first enlarge the DG-algebra to $S \otimes_k \mathcal{A}_W$, where S is the \mathbb{Z}_2 -graded vector space

$$(2.1) \quad S = \bigwedge (k\theta_1 \oplus \dots \oplus k\theta_n)$$

with grading determined by $|\theta_i| = 1$. We make the tensor product $S \otimes_k \mathcal{A}_W$ into a DG-algebra in the usual way, giving S zero differential.

Definition 2.1. We define the finite-dimensional \mathbb{Z}_2 -graded vector space

$$(2.2) \quad \mathcal{M} = R/\mathfrak{m} \otimes_k \text{End}_R(k^{\text{stab}}) \cong \text{End}_k \left(\bigwedge (k\psi_1 \oplus \dots \oplus k\psi_n) \right)$$

where $\mathfrak{m} = (x_1, \dots, x_n)$.

We will show that $S \otimes_k \mathcal{A}_W$ is homotopy equivalent to \mathcal{M} viewed as a complex with zero differential (observe that \mathcal{M} does not depend on W). The corresponding homotopy retract (see Proposition 2.2 below) is our starting point for the minimal model construction. For this we use the Koszul complex of x_1, \dots, x_n ,

$$(2.3) \quad K = \left(R \otimes_k \bigwedge (k\theta_1 \oplus \dots \oplus k\theta_n), d_K = \sum_i x_i \theta_i^* \right)$$

and the following diagram of \mathbb{Z}_2 -graded complexes and homotopy equivalences over k :

$$(2.4) \quad \begin{array}{ccccc} S \otimes_k \mathcal{A}_W & \xrightarrow{\exp(-\delta)} & K \otimes_R \mathcal{A}_W & \xrightarrow{\pi} & \mathcal{M} \\ & \xleftarrow{\exp(-\delta)} & & \xleftarrow{\sigma_\infty} & \end{array}$$

The notation is as follows:

- The complexes involved here are

$$\begin{aligned} & (S \otimes_k \text{End}_R(k^{\text{stab}}), \partial = [d_{k^{\text{stab}}}, -]) \\ & (K \otimes_R \text{End}_R(k^{\text{stab}}), d_K + \partial) \\ & (\mathcal{M}, 0). \end{aligned}$$

- The operator $\psi_j = \psi_j \wedge -$ on k^{stab} satisfies

$$[\psi_j, d_{k^{\text{stab}}}] = [\psi_j, \sum_i x_i \psi_i^*] = x_j \cdot 1,$$

where $[-, -]$ always denotes the graded commutator. That is, ψ_j is a homotopy for the action of x_j on k^{stab} . It is then easy to check that the odd operator $\alpha \mapsto \psi_j \circ \alpha$ on $\text{End}_R(k^{\text{stab}})$ is also a homotopy for x_j , and where it will not cause confusion we will also write ψ_j for this operator of post-composition.

- Both $S \otimes_k \mathcal{A}_W$ and $K \otimes_R \mathcal{A}_W$ have the same underlying \mathbb{Z}_2 -graded k -module, namely

$$R \otimes_k \bigwedge (k\theta_1 \oplus \cdots \oplus k\theta_n) \otimes_k \text{End}_k \left(\bigwedge (k\psi_1 \oplus \cdots \oplus k\psi_n) \right).$$

On this space we define the even operator

$$(2.5) \quad \delta = \sum_{i=1}^n \psi_i \theta_i^*,$$

where ψ_i is the operator on $\text{End}_R(k^{\text{stab}})$ discussed above, given by $\alpha \mapsto (\psi_i \wedge -) \circ \alpha$, and θ_i^* acts by contraction on S . It is easy to check that $\exp(\delta), \exp(-\delta)$ intertwines the differentials ∂ and $d_K + \partial$ and therefore gives an isomorphism between the first two complexes in (2.4) [2, Proposition 4.11].

- The morphism of complexes $\pi : K \longrightarrow R/\mathfrak{m}$ is defined by composing the projection of K onto the submodule $R \cdot 1$ of θ -degree zero forms, with the quotient $R \longrightarrow R/\mathfrak{m}$. We also write π for the result of tensoring this morphism with $\text{End}_R(k^{\text{stab}})$ to obtain

$$K \otimes_R \text{End}_R(k^{\text{stab}}) \xrightarrow{\pi \otimes 1} R/\mathfrak{m} \otimes_R \text{End}_R(k^{\text{stab}}) = \underline{\text{End}}(k^{\text{stab}}).$$

This is a morphism of complexes, because the differential on $\text{End}_R(k^{\text{stab}})$ vanishes on the quotient $\underline{\text{End}}(k^{\text{stab}})$ by the hypothesis that $W \in \mathfrak{m}^2$.

- The k -linear homotopy inverse σ_∞ to π in (2.4) is defined in terms of the following ingredients. The first is the k -linear connection (viewing θ_i as a 1-form)

$$(2.6) \quad \Delta : K \longrightarrow K, \quad \Delta = \sum_i \partial_{x_i} \theta_i$$

and the degree -1 (with respect to the θ -degree) k -linear operator

$$(2.7) \quad H = [d_K, \Delta]^{-1} \Delta.$$

We write ∂ for the differential on $\text{End}_R(k^{\text{stab}})$, and $\sigma : k \longrightarrow K$ for the map which sends $\lambda \in k$ to $\lambda \cdot 1 \in K$, and define k -linear maps

$$(2.8) \quad \sigma_\infty = \sum_{m \geq 0} (-1)^m (H\partial)^m \sigma,$$

$$(2.9) \quad H_\infty = \sum_{m \geq 0} (-1)^m (H\partial)^m H.$$

Here H_∞ is an odd operator on $K \otimes_R \text{End}_R(k^{\text{stab}})$. These maps satisfy:

$$(2.10) \quad \pi \sigma_\infty = 1,$$

$$(2.11) \quad \sigma_\infty \pi = 1 - [d_K + \partial, H_\infty].$$

By inspection of (2.4), we therefore have the following:

Proposition 2.2. *There is a diagram of morphisms of \mathbb{Z}_2 -graded k -complexes*

$$(2.12) \quad S \otimes_k \mathcal{A}_W \begin{array}{c} \xrightarrow{\Phi} \\ \xleftarrow{\Phi^{-1}} \end{array} \mathcal{M}.$$

where $\Phi = \pi \exp(-\delta)$, $\Phi^{-1} = \exp(\delta)\sigma_\infty$ and $\widehat{H} = \exp(\delta)H_\infty \exp(-\delta)$ satisfy

$$(2.13) \quad \Phi\Phi^{-1} = 1, \quad \Phi^{-1}\Phi = 1 - [\partial, \widehat{H}].$$

Thus, Φ and Φ^{-1} are mutually inverse homotopy equivalences over k .

The endomorphism algebra $C = \text{End}_k(S)$ is a Clifford algebra generated by the contraction θ_i^* and wedge product θ_i operators. It is observed in [2] that (??) is an isomorphism of Clifford representations (in the homotopy category) when the complex \mathcal{M} is equipped with the action induced by Atiyah classes:

Definition 2.3. The Atiyah classes of \mathcal{A}_W are the R -linear odd operators

$$\text{At}_i = [\partial, \partial_{x_i}] : \text{End}_R(k^{\text{stab}}) \longrightarrow \text{End}_R(k^{\text{stab}}),$$

Lemma 2.4. *We have*

$$\text{At}_i = -[\psi_i^*, -] - \sum_{q=1}^n \partial_{x_i}(W^q)[\psi_q, -].$$

Proof. By direct calculation:

$$\begin{aligned} \text{At}_i &= [[d_{k^{\text{stab}}}, -], \partial_{x_i}] \\ &= \sum_q [x_q[\psi_q^*, -], \partial_{x_i}] + \sum_q [W^q[\psi_q, -], \partial_{x_i}] \\ &= -\sum_q \partial_{x_i}(x_q)[\psi_q^*, -] - \sum_q \partial_{x_i}(W^q)[\psi_q, -] \\ &= -[\psi_i^*, -] - \sum_q \partial_{x_i}(W^q)[\psi_q, -]. \end{aligned}$$

□

3 The minimal model

We now take the homotopy retract from Proposition 2.2 as the input to the usual algorithm for the construction of an A_∞ -minimal model of the DG-algebra $S \otimes \text{End}_R(k^{\text{stab}})$.

4 Examples

To describe the higher multiplications it is convenient to write $[\psi_i, -]$ for the natural operator on \mathcal{A} (giving ψ_i, ψ_j^* the usual anticommutation relations for fermionic creation and annihilation operators), that is

$$\begin{aligned} [\psi_i, \psi_{j_1}^* \cdots \psi_{j_t}^*] &= \sum_{l=1}^t (-1)^{l-1} \psi_{j_1}^* \cdots [\psi_i, \psi_{j_l}^*] \cdots \psi_{j_t}^* \\ &= \sum_{l=1}^t (-1)^{l-1} \delta_{i=j_l} \psi_{j_1}^* \cdots \psi_{j_t}^* . \end{aligned}$$

A general fact is that the higher multiplications on \mathcal{A} are linear combinations of products of such operators. For example, when $n = 2$ a standard term in r_3 would look like

$$\Phi_0 \otimes \Phi_1 \otimes \Phi_2 \mapsto \lambda \cdot [\psi_1, [\psi_2, \Phi_0]] \cdot [\psi_1, \Phi_1] \cdot [\psi_2, \Phi_2] .$$

The coefficient λ is computed as a Feynman amplitude on a binary planar tree with with incoming fermion states $\psi_1 \psi_2, \psi_1, \psi_2$ in the three leaves (respectively) and a simple list of allowed interactions, the most significant of which is a trivalent interaction vertex with an incoming fermion ψ_j and outgoing fermion θ_i and bosons $\partial_{x_i}(x^\gamma)$ whenever the polynomial W^j has a nonzero coefficient for the monomial $x^\gamma = x_1^{\gamma_1} x_2^{\gamma_2}$ (the fermions θ_i are the auxiliary spinor representation generators as in [2]). There are two other interactions which do not depend on W , and which take place only at internal edges or internal vertices (respectively).

Example 4.1. Let $W = x^d$ so that $\mathcal{A} = k \oplus k\psi^*$. Then only r_2 and r_d are nonzero and on $(\mathcal{A}[1])^{\otimes d}$ the only basis element with a nonzero value under r_d is

$$r_d(\psi^* \otimes \cdots \otimes \psi^*) = 1 .$$

Another way to say this is

$$r_d(\Phi_0 \otimes \cdots \otimes \Phi_{d-1}) = \prod_{i=0}^{d-1} [\psi, \Phi_i] .$$

This A_∞ -structure is cyclic with respect to the trace form on \mathcal{A} which projects onto the $k\psi^*$ -summand. Note that all such products are expanded such that the index i increases from left to right.

References

- [1] C. I. Lazaroiu, *Generating the superpotential on a D-brane category: I*, [arXiv:hep-th/0610120].

- [2] D. Murfet, *Computing with cut systems*, [[arXiv:1402.4541](#)].
- [3] N. Carqueville and D. Murfet, *Adjunctions and defects in Landau-Ginzburg models*, Adv. Math. **289** (2016), 480–566.
- [4] T. Dyckerhoff and D. Murfet, *Pushing forward matrix factorisations*, Duke Math. J. **162** (2013), 1249–1311.