

A_∞ -minimal models of matrix factorisations

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Abstract

We study A_∞ -minimal models of the differential graded algebras obtained from endomorphisms of generators of triangulated categories of matrix factorisations, giving explicit higher products for the cases of simple singularities.

1 Introduction

Let k be a characteristic zero field, and $W \in k[x_1, \dots, x_n]$ a potential in the sense of [?, §2.2] which has the origin as its only critical point. Choose a presentation of W as a sum $W = x_1 W^1 + \dots + x_n W^n$ with $W^i \in \mathfrak{m} = (x_1, \dots, x_n)$. Then the generator of the homotopy category of matrix factorisations of W is the stabilised diagonal

$$(1.1) \quad k^{\text{stab}} = \left(k[x] \otimes_k \bigwedge (k\psi_1 \oplus \dots \oplus k\psi_n), d_{k^{\text{stab}}} = \sum_{i=1}^n x_i \psi_i^* + \sum_{i=1}^n W^i \psi_i \right)$$

where ψ_i^* and ψ_i act respectively by contraction and wedge product on the exterior algebra underlying k^{stab} [?]. The purpose of this paper is to calculate the A_∞ -minimal model of the \mathbb{Z}_2 -graded differential graded endomorphism algebra of k^{stab}

$$(1.2) \quad \mathcal{A}_W = \left(\text{End}_{k[x]}(k^{\text{stab}}), \partial = [d_{k^{\text{stab}}}, -] \right)$$

with differential (throughout all commutators are graded commutators)

$$(1.3) \quad \partial = \left[\sum_i x_i \psi_i^* + \sum_i W^i \psi_i, - \right]$$

$$(1.4) \quad = \sum_i x_i [\psi_i^*, -] + \sum_i W^i [\psi_i, -].$$

The minimal model is a finite-dimensional \mathbb{Z}_2 -graded vector space \mathcal{M}_W with a family

$$\left\{ r_q : (\mathcal{M}_W[1])^{\otimes q} \longrightarrow \mathcal{M}_W[1] \right\}_{q \geq 2}$$

of odd k -linear maps satisfying the forward suspended A_∞ -constraints [?], and having the property that there is an A_∞ -quasi-isomorphism $\mathcal{A}_W \longrightarrow \mathcal{M}_W$.

2 Background

2.1 A-infinity algebras

The *tilde grading* is defined by $\tilde{x} = |x| - 1$.

Throughout $\otimes = \otimes_k$.

3 Perturbation and Koszul complexes

Let k be a characteristic zero field, $R = k[x_1, \dots, x_n]$ and let $W \in \mathfrak{m}^2$ be a potential with chosen decomposition $W = \sum_{i=1}^n x_i W^i$. To apply the A_∞ -minimal model construction to the DG-algebra \mathcal{A}_W from (??) we would usually begin by finding a k -linear homotopy retract of this complex onto its cohomology. However, for our purposes it is better to first enlarge the DG-algebra to $S \otimes \mathcal{A}_W$, where S is the \mathbb{Z}_2 -graded vector space

$$(3.1) \quad S = \bigwedge (k\theta_1 \oplus \dots \oplus k\theta_n)$$

with grading $|\theta_i| = 1$. We make the tensor product $S \otimes \mathcal{A}_W$ into a DG-algebra in the usual way, giving the exterior algebra S the zero differential.

Definition 3.1. We define the finite-dimensional \mathbb{Z}_2 -graded vector space

$$(3.2) \quad \mathcal{E} = R/\mathfrak{m} \otimes \text{End}_R(k^{\text{stab}}) \cong \text{End}_k \left(\bigwedge (k\psi_1 \oplus \dots \oplus k\psi_n) \right)$$

where $\mathfrak{m} = (x_1, \dots, x_n)$.

We will show that $S \otimes \mathcal{A}_W$ is homotopy equivalent to \mathcal{E} viewed as a complex with zero differential (observe that \mathcal{E} does not depend on W). The corresponding homotopy retract (see Proposition ?? below) is our starting point for the minimal model construction. For this we use the Koszul complex of x_1, \dots, x_n ,

$$(3.3) \quad K = \left(R \otimes \bigwedge (k\theta_1 \oplus \dots \oplus k\theta_n), d_K = \sum_i x_i \theta_i^* \right)$$

Theorem 3.2. *There is a diagram of \mathbb{Z}_2 -graded complexes and homotopy equivalences over k*

$$(3.4) \quad S \otimes \mathcal{A}_W \begin{array}{c} \xrightarrow{\exp(-\delta)} \\ \xleftarrow{\exp(-\delta)} \end{array} K \otimes_R \mathcal{A}_W \begin{array}{c} \xrightarrow{\pi} \\ \xleftarrow{\sigma_\infty} \end{array} \mathcal{E}.$$

Proof. See [?]. □

The notation is as follows:

- The complexes involved here are

$$\begin{aligned} & (S \otimes \text{End}_R(k^{\text{stab}}), \partial) \\ & (K \otimes_R \text{End}_R(k^{\text{stab}}), d_K + \partial) \\ & (\mathcal{E}, 0). \end{aligned}$$

- The operator $\psi_j = \psi_j \wedge -$ on k^{stab} satisfies

$$[\psi_j, d_{k^{\text{stab}}}] = [\psi_j, \sum_i x_i \psi_i^*] = x_j \cdot 1,$$

where $[-, -]$ always denotes the graded commutator. That is, ψ_j is a homotopy for the action of x_j on k^{stab} . It is then easy to check that the odd operator $\alpha \mapsto \psi_j \circ \alpha$ on $\text{End}_R(k^{\text{stab}})$ is also a homotopy for x_j , and where it will not cause confusion we will also write ψ_j for this operator of post-composition.

- Both $S \otimes \mathcal{A}_W$ and $K \otimes_R \mathcal{A}_W$ have the same underlying \mathbb{Z}_2 -graded k -module, namely

$$(3.5) \quad R \otimes \bigwedge (k\theta_1 \oplus \cdots \oplus k\theta_n) \otimes \text{End}_k \left(\bigwedge (k\psi_1 \oplus \cdots \oplus k\psi_n) \right).$$

On this space we define the even operator

$$(3.6) \quad \delta = \sum_{i=1}^n \psi_i \theta_i^*,$$

where ψ_i is the operator on $\text{End}_R(k^{\text{stab}})$ discussed above, given by $\alpha \mapsto (\psi_i \wedge -) \circ \alpha$, and θ_i^* acts by contraction on S . It is easy to check that $\exp(\delta), \exp(-\delta)$ intertwines the differentials ∂ and $d_K + \partial$ and therefore gives an isomorphism between the first two complexes in (??) [?, Proposition 4.11].

- The morphism of complexes $\pi : K \rightarrow R/\mathfrak{m}$ is defined by composing the projection of K onto the submodule $R \cdot 1$ of θ -degree zero forms, with the quotient $R \rightarrow R/\mathfrak{m}$. We also write π for the result of tensoring this morphism with $\text{End}_R(k^{\text{stab}})$ to obtain

$$K \otimes_R \text{End}_R(k^{\text{stab}}) \xrightarrow{\pi \otimes 1} R/\mathfrak{m} \otimes_R \text{End}_R(k^{\text{stab}}) = \underline{\text{End}}(k^{\text{stab}}).$$

This is a morphism of complexes, because the differential on $\text{End}_R(k^{\text{stab}})$ vanishes on the quotient $\underline{\text{End}}(k^{\text{stab}})$ by the hypothesis that $W \in \mathfrak{m}^2$.

- The k -linear homotopy inverse σ_∞ to π in (??) is defined in terms of the following ingredients. The first is the k -linear connection (viewing θ_i as a 1-form)

$$(3.7) \quad \Delta : K \rightarrow K, \quad \Delta = \sum_i \partial_{x_i} \theta_i$$

and the degree -1 (with respect to the θ -degree) k -linear operator

$$(3.8) \quad H = [d_K, \Delta]^{-1} \Delta.$$

We write ∂ for the differential on $\text{End}_R(k^{\text{stab}})$, and $\sigma : k \rightarrow K$ for the map which sends $\lambda \in k$ to $\lambda \cdot 1 \in K$, and define k -linear maps

$$(3.9) \quad \sigma_\infty = \sum_{m \geq 0} (-1)^m (H\partial)^m \sigma,$$

$$(3.10) \quad H_\infty = \sum_{m \geq 0} (-1)^m (H\partial)^m H.$$

Here H_∞ is an odd operator on $K \otimes_R \text{End}_R(k^{\text{stab}})$. These maps satisfy:

$$(3.11) \quad \pi \sigma_\infty = 1,$$

$$(3.12) \quad \sigma_\infty \pi = 1 - [d_K + \partial, H_\infty].$$

By inspection of (??), we therefore have the following:

Proposition 3.3. *There is a diagram of morphisms of \mathbb{Z}_2 -graded k -complexes*

$$(3.13) \quad S \otimes \mathcal{A}_W \begin{array}{c} \xrightarrow{\Phi} \\ \xleftarrow{\Phi^{-1}} \end{array} \mathcal{E}$$

where $\Phi = \pi \exp(-\delta)$, $\Phi^{-1} = \exp(\delta) \sigma_\infty$ and $\hat{H} = \exp(\delta) H_\infty \exp(-\delta)$ satisfy

$$(3.14) \quad \Phi \Phi^{-1} = 1, \quad \Phi^{-1} \Phi = 1 - [\partial, \hat{H}].$$

Thus, Φ and Φ^{-1} are mutually inverse homotopy equivalences over k .

The endomorphism algebra $C = \text{End}_k(S)$ is a Clifford algebra generated by the contraction θ_i^* and wedge product θ_i operators. It is observed in [?] that (??) is an isomorphism of Clifford representations (in the homotopy category) when the complex \mathcal{E} is equipped with the action induced by Atiyah classes:

Definition 3.4. The Atiyah classes of \mathcal{A}_W are the R -linear odd operators

$$\text{At}_i = [\partial, \partial_{x_i}] : \text{End}_R(k^{\text{stab}}) \rightarrow \text{End}_R(k^{\text{stab}}),$$

Lemma 3.5. *We have*

$$\text{At}_i = -[\psi_i^*, -] - \sum_{q=1}^n \partial_{x_i}(W^q)[\psi_q, -].$$

Proof. By direct calculation:

$$\begin{aligned}
\text{At}_i &= [[d_{k^{\text{stab}}}, -], \partial_{x_i}] \\
&= \sum_q [x_q[\psi_q^*, -], \partial_{x_i}] + \sum_q [W^q[\psi_q, -], \partial_{x_i}] \\
&= -\sum_q \partial_{x_i}(x_q)[\psi_q^*, -] - \sum_q \partial_{x_i}(W^q)[\psi_q, -] \\
&= -[\psi_i^*, -] - \sum_q \partial_{x_i}(W^q)[\psi_q, -].
\end{aligned}$$

□

4 The minimal model

We now take the homotopy retract from Proposition ?? as the input to the usual algorithm for the construction of an A_∞ -minimal model of the DG-algebra $S \otimes \mathcal{A}_W$. We make use of the forward suspended products for this DG-algebra, which are

$$(4.1) \quad r_q : (S \otimes \mathcal{A}_W[1])^{\otimes q} \longrightarrow S \otimes \mathcal{A}_W[1]$$

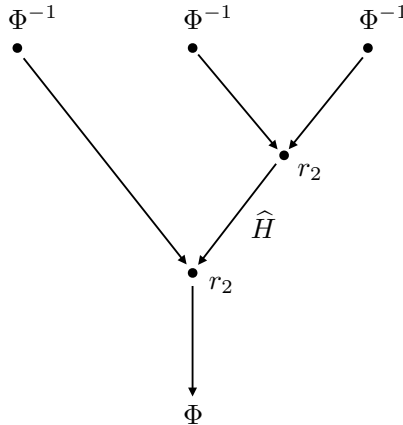
defined to be zero except for $r_1 = \partial$ and

$$(4.2) \quad r_2(\alpha \otimes \beta) = (-1)^{\tilde{\alpha}\tilde{\beta} + \tilde{\beta} + 1} \beta \alpha$$

where $\beta\alpha$ is the usual product in $S \otimes_k \mathcal{A}_W$ and $\tilde{\alpha} = |\alpha| - 1$. Applying the algorithm described in Section ?? to the data of the A_∞ -algebra $(S \otimes \mathcal{A}_W, \{r_1, r_2\})$ and the strong homotopy retract of Proposition ?? produces a minimal A_∞ -algebra

$$(\mathcal{E}, \{\rho_q\}_{q \geq 2}).$$

The products are defined by sums over trees, for example: one tree contributing to ρ_3 is:



For this particular tree T the associated operator is

$$\rho_T = (-1)^1 \Phi \circ r_2 \circ (\Phi^{-1} \otimes \widehat{H}) \circ (1 \otimes r_2) \circ (1 \otimes \Phi^{-1} \otimes \Phi^{-1}).$$

We first describe the final algorithm which allows us to compute the operations ρ_q in terms of the Feynman rules, and then we will justify this algorithm in the remainder of the paper. Here we stick to $W \in \mathfrak{m}^3$ and consider the subalgebra of \mathcal{E} generated by the contraction operators

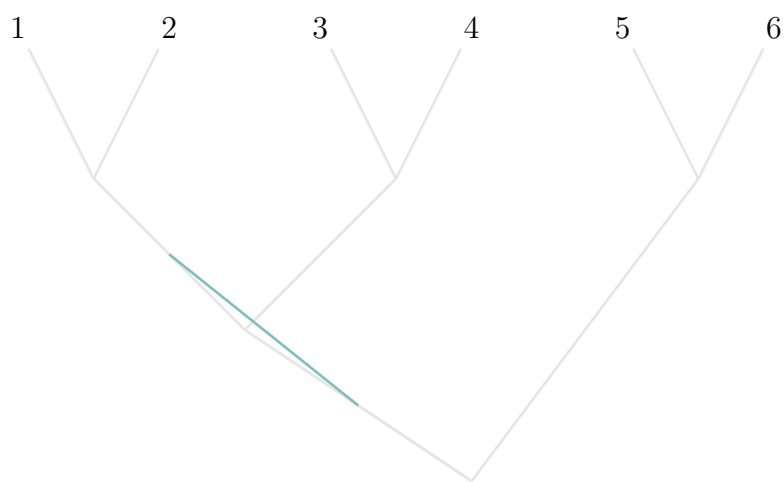
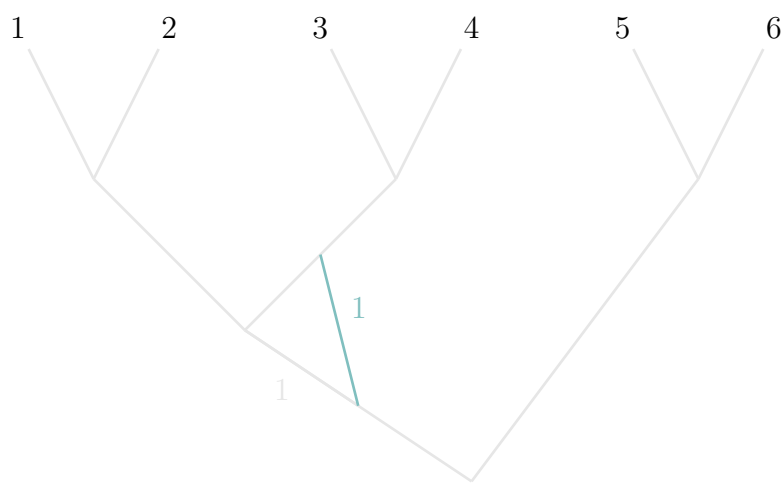
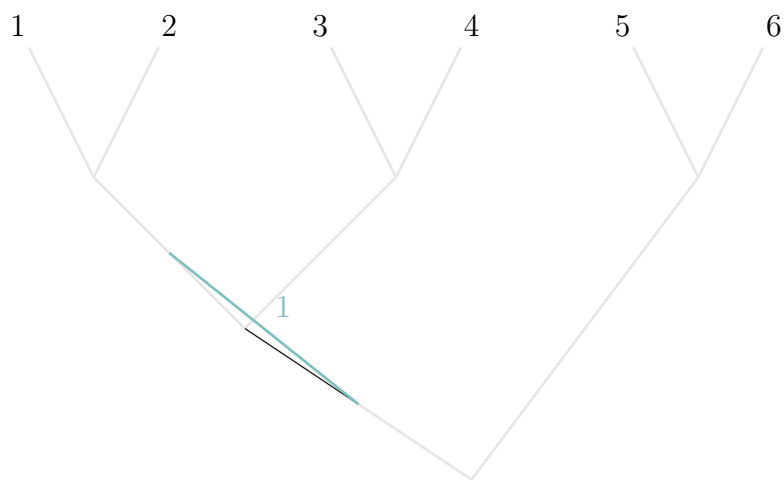
$$\mathcal{M} = \bigwedge (k\psi_1^* \oplus \cdots \oplus k\psi_n^*) \subseteq \mathcal{E}.$$

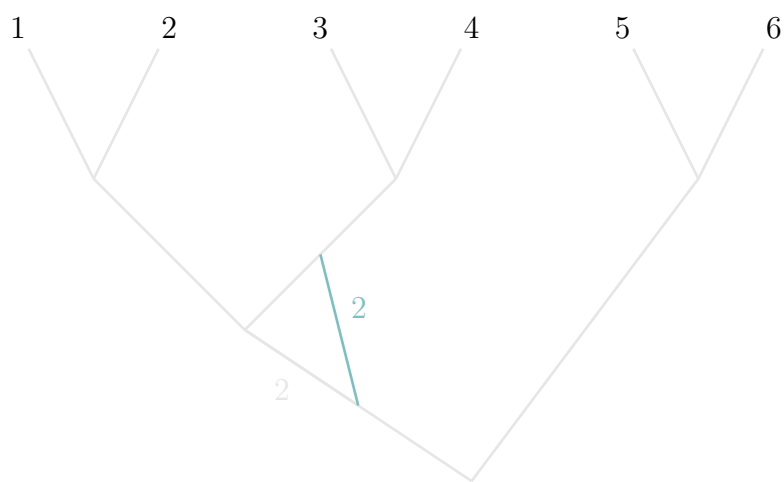
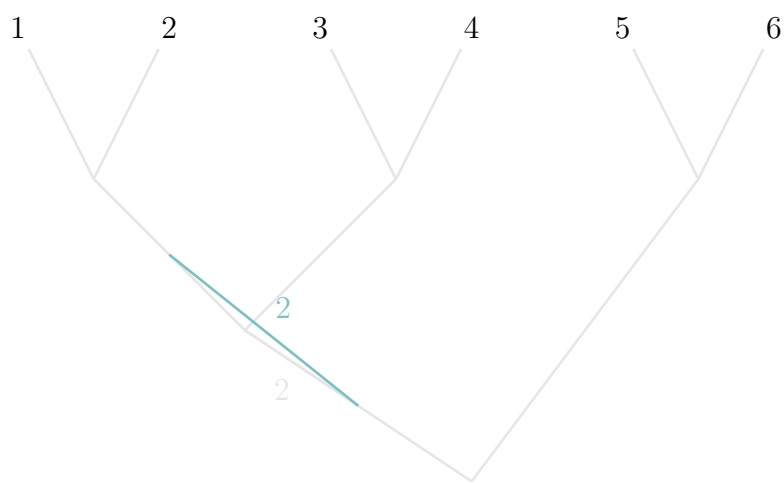
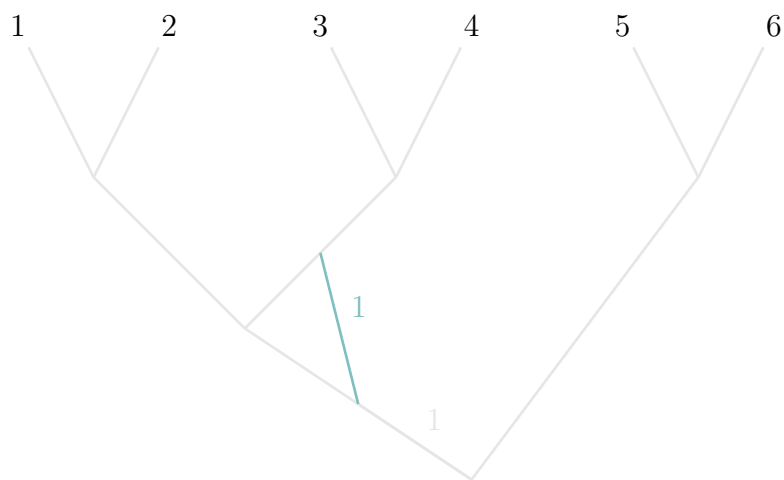
The formula is expressed in terms of *vacuum amplitudes* which are functionals defined for every tree T with q input vertices

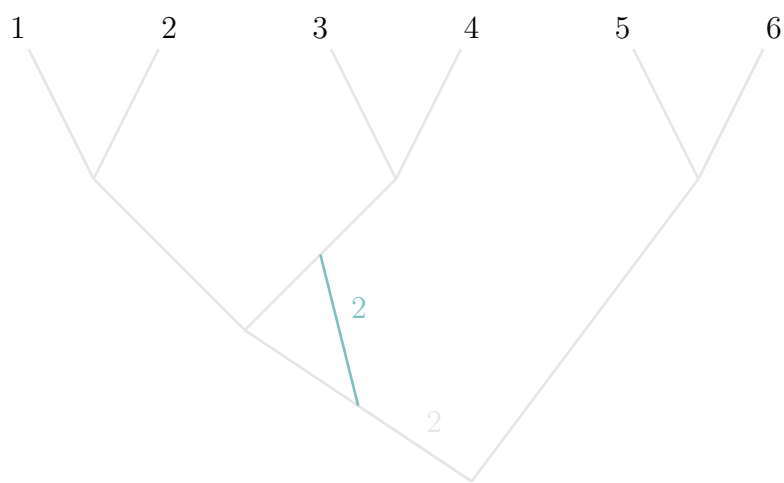
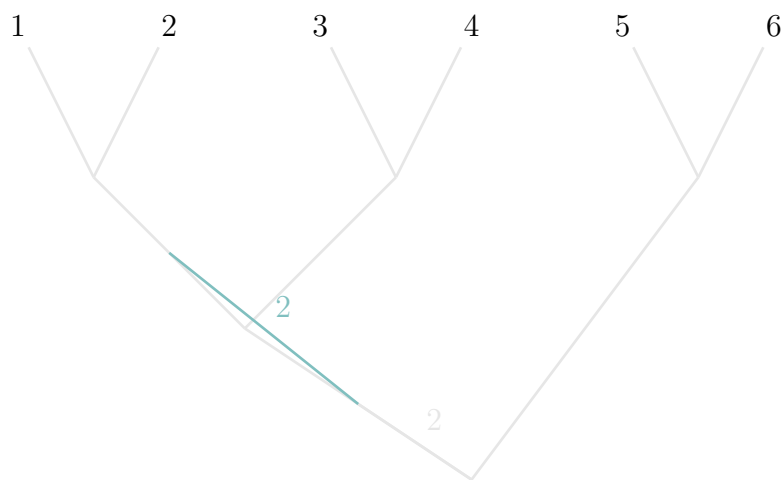
$$\mathcal{O}(T) \in (\mathcal{M}^{\otimes q})^*$$

This amplitude is defined to be a sum over *configurations*

$$\mathcal{O}(T) = \sum_{c \in \mathcal{C}} \mathcal{O}(T, c).$$







We define the even operator $\delta_i = \psi_i \theta_i^*$ on the underlying module (??) of $S \otimes \mathcal{A}_W$, so that $\delta = \sum_i \delta_i$. Since $[\delta_i, \delta_j] = 0$ for all i, j we have

$$(4.3) \quad \exp(\pm \delta) = \exp(\pm \delta_1) \cdots \exp(\pm \delta_n).$$

Lemma 4.1. *For $1 \leq i \leq n$ there is a commutative diagram*

$$(4.4) \quad \begin{array}{ccc} (S \otimes \mathcal{A}_W) \otimes (S \otimes \mathcal{A}_W) & \xrightarrow{m_2} & S \otimes \mathcal{A}_W \\ \delta_i \otimes 1 + 1 \otimes \delta_i + [\psi_i, -] \otimes \theta_i^* \downarrow & & \downarrow \delta_i \\ (S \otimes \mathcal{A}_W) \otimes (S \otimes \mathcal{A}_W) & \xrightarrow{m_2} & S \otimes \mathcal{A}_W \end{array}$$

Definition 4.2. We define

$$(4.5) \quad \Xi_i = [\psi_i, -] \otimes \theta_i^* : S \otimes \mathcal{A}_W \longrightarrow S \otimes \mathcal{A}_W.$$

and

$$(4.6) \quad \Xi = \sum_i \Xi_i.$$

Proposition 4.3. *There is a commutative diagram*

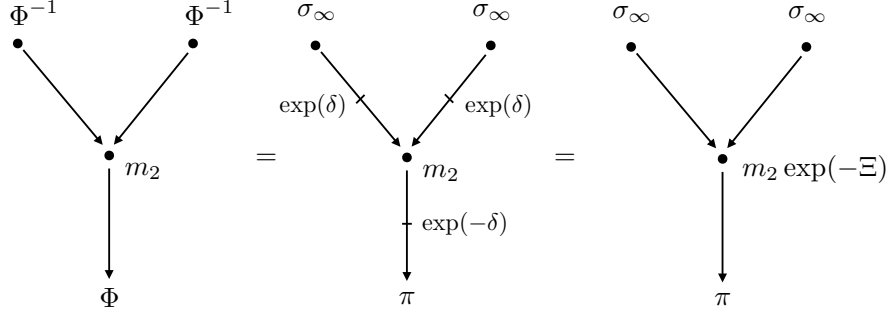
$$(4.7) \quad \begin{array}{ccc} (S \otimes \mathcal{A}_W) \otimes (S \otimes \mathcal{A}_W) & \xrightarrow{m_2} & S \otimes \mathcal{A}_W \\ \exp(-\delta) \otimes \exp(-\delta) \downarrow & & \downarrow \exp(-\delta) \\ (S \otimes \mathcal{A}_W) \otimes (S \otimes \mathcal{A}_W) & & \\ \exp(-\Xi) \downarrow & & \\ (S \otimes \mathcal{A}_W) \otimes (S \otimes \mathcal{A}_W) & \xrightarrow{m_2} & S \otimes \mathcal{A}_W \end{array}$$

Definition 4.4. ev_T is the operator associated to the tree T with σ_∞ on all input vertices, $m_2 \exp(-\Xi)$ on all internal vertices and H_∞ on internal edges.

Lemma 4.5. *For any tree T*

$$\rho_T(\alpha_1 \otimes \cdots \otimes \alpha_q) = (-1)^{S(T, \ell)} \text{ev}_{\hat{T}}(\alpha_q \otimes \cdots \otimes \alpha_1)$$

where $\ell = (\tilde{\alpha}_1, \dots, \tilde{\alpha}_q)$.



5 Notes

Remark 5.1. Varying the potential W affects only the relative “strength” of the triple-vertex interactions, which are indexed by the monomials γ . Thus we can, for example, examine the degeneration from x^{d+1} to x^d (bundles of minimal A_∞ -algebras).

6 Examples

To describe the higher multiplications it is convenient to write $[\psi_i, -]$ for the natural operator on \mathcal{A} (giving ψ_i, ψ_j^* the usual anticommutation relations for fermionic creation and annihilation operators), that is

$$\begin{aligned} [\psi_i, \psi_{j_1}^* \cdots \psi_{j_t}^*] &= \sum_{l=1}^t (-1)^{l-1} \psi_{j_1}^* \cdots [\psi_i, \psi_{j_l}^*] \cdots \psi_{j_t}^* \\ &= \sum_{l=1}^t (-1)^{l-1} \delta_{i=j_l} \psi_{j_1}^* \cdots \psi_{j_t}^* . \end{aligned}$$

A general fact is that the higher multiplications on \mathcal{A} are linear combinations of products of such operators. For example, when $n = 2$ a standard term in r_3 would look like

$$\Phi_0 \otimes \Phi_1 \otimes \Phi_2 \mapsto \lambda \cdot [\psi_1, [\psi_2, \Phi_0]] \cdot [\psi_1, \Phi_1] \cdot [\psi_2, \Phi_2] .$$

The coefficient λ is computed as a Feynman amplitude on a binary planar tree with with incoming fermion states $\psi_1, \psi_2, \psi_1, \psi_2$ in the three leaves (respectively) and a simple list of allowed interactions, the most significant of which is a trivalent interaction vertex with an incoming fermion ψ_j and outgoing fermion θ_i and bosons $\partial_{x_i}(x^\gamma)$ whenever the polynomial W^j has a nonzero coefficient for the monomial $x^\gamma = x_1^{\gamma_1} x_2^{\gamma_2}$ (the fermions θ_i are the auxiliary spinor representation generators as in [?]). There are two other interactions which do not depend on W , and which take place only at internal edges or internal vertices (respectively).

Example 6.1. Let $W = x^d$ so that $\mathcal{A} = k \oplus k\psi^*$. Then only r_2 and r_d are nonzero and on $(\mathcal{A}[1])^{\otimes d}$ the only basis element with a nonzero value under r_d is

$$r_d(\psi^* \otimes \cdots \otimes \psi^*) = 1.$$

Another way to say this is

$$r_d(\Phi_0 \otimes \cdots \otimes \Phi_{d-1}) = \prod_{i=0}^{d-1} [\psi, \Phi_i].$$

This A_∞ -structure is cyclic with respect to the trace form on \mathcal{A} which projects onto the $k\psi^*$ -summand. Note that all such products are expanded such that the index i increases from left to right.

References

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