A_∞-minimal models of matrix factorisations

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Abstract

We study A_∞-minimal models of the differential graded algebras obtained from endomorphisms of generators of triangulated categories of matrix factorisations, giving explicit higher products for the cases of simple singularities.

1 Introduction

Let k be a commutative $\mathbb{Q}$-algebra and $W \in k[x_1, \dots, x_n]$ a polynomial which is a potential over k in the sense of [3, §2.2]. For example, $k = \mathbb{C}$ and W has isolated critical points. The $\mathbb{Z}_2$-graded DG-category $\mathcal{A} = \mathrm{mf}(R, W)$ has for its objects finite-rank matrix factorisations of W over $R = k[x_1, \dots, x_n]$. In this paper we study the minimal model problem for $\mathcal{A}$, the aim being to produce models that are finitely-generated and projective over k.

When k is a field, the standard minimal model theorem [?] constructs the structure of an A_∞-category on the cohomology $\mathcal{B} = H^*\mathcal{A}$ in such a way that $\mathcal{B}$ is quasi-isomorphic to $\mathcal{A}$. Moreover, this A_∞-category has finite-dimensional Hom-spaces and is therefore a good finite model of $\mathcal{A}$. The problem in this case is to have sufficient control over the inputs to the minimal model theorem that the A_∞-products on $H^*\mathcal{A}(X, X)$ can be reasonably calculated, for a given matrix factorisation X. In order to understand deformations of matrix factorisations [?, ?, ?] it is important to do this for generic X, but the special case where $X = k^{\mathrm{stab}}$, the generator of $\mathcal{A}$ studied by Dyckerhoff in the case of a single isolated singularity at the origin [Dyc11], is of particular interest.

Since one of our goals is to understand how matrix factorisation categories vary along unfoldings of singularities, we also need to consider the case where k is not a field. For example take $k = \mathbb{C}[t]$ and $R = \mathbb{C}[x_1, \dots, x_n, t]$ so that $W = W_t(x)$ is a potential with parameter. In this case we want an A_∞-category $\mathcal{B}$ quasi-isomorphic to $\mathcal{A}$ over k, with $\mathcal{B}(a, b)$ a finitely-generated projective k-module for each pair of objects a, b.

These two desiderata, understanding the A_∞-products on $H^*\mathcal{A}(X, X)$ for generic X, and the case where k is not a field, lead us naturally to a slightly non-standard use of the minimal model theorem. Following the ideas developed in [?, ?, ?] we prove that, writing $\iota : R \longrightarrow R/(\partial_{x_1} W, \dots, \partial_{x_n} W)$ for the quotient, there is a k-linear homotopy retract

$$\iota^*\mathcal{A} \rightleftarrows \mathcal{A} \otimes_k \bigwedge \left(k^{\oplus n}[1]\right). \tag{1.1}$$

Suppose $W \in \mathfrak{m}^2$ where $\mathfrak{m} = (x_1, \ldots, x_n)$ and choose a presentation $W = x_1 W^1 + \cdots + x_n W^n$ with $W^i \in \mathfrak{m}$ and consider the following pair

$$k^{\text{stab}} = \Big(k[x] \otimes_k \bigwedge (k\psi_1 \oplus \cdots \oplus k\psi_n),\ d_{k^{\text{stab}}} = \sum_{i=1}^{n} x_i \psi_i^* + \sum_{i=1}^{n} W^i \psi_i \Big) \tag{1.2}$$

where ψ_i^* and ψ_i act respectively by contraction and wedge product on the exterior algebra underlying k^{stab}. It is easy to see that $d_{k^{\text{stab}}}^2 = W$, so k^{stab} is a matrix factorisation. When k is a field and W has an isolated critical point at the origin, k^{stab} is the representative in the homotopy category of matrix factorisations of the structure sheaf of the singular point at the origin, and was first studied by Dyckerhoff [Dyc11].

The purpose of this paper is to show how to calculate the A_∞-minimal model of the $\mathbb{Z}_2$-graded differential graded endomorphism algebra of this matrix factorisation

$$\Big(\operatorname{End}_{k[x]}(k^{\text{stab}}),\ \partial = [d_{k^{\text{stab}}}, -] \Big) . \tag{1.3}$$

The differential here is (throughout all commutators are graded commutators)

$$\begin{aligned} \partial = [d_{k^{\text{stab}}}, -] &= \Big[\sum_i x_i \psi_i^* + \sum_i W^i \psi_i, - \Big] \\ &= \sum_i x_i [\psi_i^*, -] + \sum_i W^i [\psi_i, -] . \end{aligned}$$

The minimal model is a finite-dimensional $\mathbb{Z}_2$-graded vector space $\mathscr{M}_W$ with a family

$$\Big\{ r_q : \big(\mathscr{M}_W[1]\big)^{\otimes q} \longrightarrow \mathscr{M}_W[1] \Big\}_{q \geq 2}$$

of odd k-linear maps satisfying the forward suspended A_∞-constraints [1], and having the property that there is an A_∞-quasi-isomorphism $\mathscr{M}_W \longrightarrow \operatorname{End}_{k[x]}(k^{\text{stab}})$.

2 Background

2.1 A-infinity algebras

The *tilde grading* is defined by $\widetilde{x} = |x| - 1$.

Throughout $\otimes = \otimes_k$.

3 Perturbation and Koszul complexes

Let k be a characteristic zero field, $R = k[x_1, \ldots, x_n]$ and let $W \in \mathfrak{m}^2$ be a potential with chosen decomposition $W = \sum_{i=1}^n x_i W^i$. To apply the A_∞-minimal model construction to the DG-algebra $\mathscr{A}_W$ from (1.3) we would usually begin by finding a k-linear homotopy retract of this complex onto its cohomology. However, for our purposes it is better to first enlarge the DG-algebra to $S \otimes \mathscr{A}_W$, where S is the $\mathbb{Z}_2$-graded vector space

$$S = \bigwedge (k\theta_1 \oplus \cdots \oplus k\theta_n) \tag{3.1}$$

with grading $|\theta_i| = 1$. We make the tensor product $S \otimes \mathscr{A}_W$ into a DG-algebra in the usual way, giving the exterior algebra S the zero differential.

Definition 3.1. We define the finite-dimensional $\mathbb{Z}_2$-graded vector space

$$\mathscr{E} = R/\mathfrak{m} \otimes \mathrm{End}_R(k^{\mathrm{stab}}) \cong \mathrm{End}_k\Big(\bigwedge (k\psi_1 \oplus \cdots \oplus k\psi_n) \Big) \tag{3.2}$$

where $\mathfrak{m} = (x_1, \ldots, x_n)$.

We will show that $S \otimes \mathscr{A}_W$ is homotopy equivalent to $\mathscr{E}$ viewed as a complex with zero differential (observe that $\mathscr{E}$ does not depend on W). The corresponding homotopy retract (see Proposition 3.3 below) is our starting point for the minimal model construction. For this we use the Koszul complex of $x_1, \ldots, x_n$,

$$K = \Big(R \otimes \bigwedge (k\theta_1 \oplus \cdots \oplus k\theta_n),\ d_K = \sum_i x_i \theta_i^* \Big) \tag{3.3}$$

Theorem 3.2. *There is a diagram of $\mathbb{Z}_2$-graded complexes and homotopy equivalences over k*

$$S \otimes \mathscr{A}_W \underset{\exp(-\delta)}{\overset{\exp(-\delta)}{\rightleftarrows}} K \otimes_R \mathscr{A}_W \underset{\sigma_\infty}{\overset{\pi}{\rightleftarrows}} \mathscr{E}\,. \tag{3.4}$$

Proof. See [2]. □

The notation is as follows:

- The complexes involved here are

$$\big(S \otimes \mathrm{End}_R(k^{\mathrm{stab}}),\ \partial \big)$$
$$\big(K \otimes_R \mathrm{End}_R(k^{\mathrm{stab}}),\ d_K + \partial \big)$$
$$\big(\mathscr{E}, 0 \big)\,.$$

- The operator $\psi_j = \psi_j \wedge -$ on k^{stab} satisfies

$$[\psi_j, d_{k^{\mathrm{stab}}}] = \big[\psi_j, \sum_i x_i \psi_i^* \big] = x_j \cdot 1\,,$$

  where $[-,-]$ always denotes the graded commutator. That is, ψ_j is a homotopy for the action of x_j on k^{stab}. It is then easy to check that the odd operator $\alpha \mapsto \psi_j \circ \alpha$ on $\mathrm{End}_R(k^{\mathrm{stab}})$ is also a homotopy for x_j, and where it will not cause confusion we will also write ψ_j for this operator of post-composition.

- Both $S \otimes \mathscr{A}_W$ and $K \otimes_R \mathscr{A}_W$ have the same underlying $\mathbb{Z}_2$-graded k-module, namely

$$R \otimes \bigwedge(k\theta_1 \oplus \cdots \oplus k\theta_n) \otimes \operatorname{End}_k\left(\bigwedge(k\psi_1 \oplus \cdots \oplus k\psi_n)\right). \tag{3.5}$$

On this space we define the even operator

$$\delta = \sum_{i=1}^{n} \psi_i \theta_i^*, \tag{3.6}$$

where ψ_i is the operator on $\operatorname{End}_R(k^{\mathrm{stab}})$ discussed above, given by $\alpha \mapsto (\psi_i \wedge -) \circ \alpha$, and θ_i^* acts by contraction on S. It is easy to check that $\exp(\delta), \exp(-\delta)$ intertwines the differentials ∂ and $d_K + \partial$ and therefore gives an isomorphism between the first two complexes in (3.4) [2, Proposition 4.11].

- The morphism of complexes $\pi : K \longrightarrow R/\mathfrak{m}$ is defined by composing the projection of K onto the submodule $R \cdot 1$ of θ-degree zero forms, with the quotient $R \longrightarrow R/\mathfrak{m}$. We also write π for the result of tensoring this morphism with $\operatorname{End}_R(k^{\mathrm{stab}})$ to obtain

$$K \otimes_R \operatorname{End}_R(k^{\mathrm{stab}}) \xrightarrow{\pi \otimes 1} R/\mathfrak{m} \otimes_R \operatorname{End}_R(k^{\mathrm{stab}}) = \underline{\operatorname{End}}(k^{\mathrm{stab}}).$$

This is a morphism of complexes, because the differential on $\operatorname{End}_R(k^{\mathrm{stab}})$ vanishes on the quotient $\underline{\operatorname{End}}(k^{\mathrm{stab}})$ by the hypothesis that $W \in \mathfrak{m}^2$.

- The k-linear homotopy inverse σ_∞ to π in (3.4) is defined in terms of the following ingredients. The first is the k-linear connection (viewing θ_i as a 1-form)

$$\Delta : K \longrightarrow K, \qquad \Delta = \sum_i \partial_{x_i} \theta_i \tag{3.7}$$

and the degree -1 (with respect to the θ-degree) k-linear operator

$$H = [d_K, \Delta]^{-1} \Delta. \tag{3.8}$$

We write ∂ for the differential on $\operatorname{End}_R(k^{\mathrm{stab}})$, and $\sigma : k \longrightarrow K$ for the map which sends $\lambda \in k$ to $\lambda \cdot 1 \in K$, and define k-linear maps

$$\sigma_\infty = \sum_{m \geq 0} (-1)^m (H\partial)^m \sigma, \tag{3.9}$$

$$H_\infty = \sum_{m \geq 0} (-1)^m (H\partial)^m H. \tag{3.10}$$

Here H_∞ is an odd operator on $K \otimes_R \operatorname{End}_R(k^{\mathrm{stab}})$. These maps satisfy:

$$\pi \sigma_\infty = 1, \tag{3.11}$$

$$\sigma_\infty \pi = 1 - [d_K + \partial, H_\infty]. \tag{3.12}$$

By inspection of (3.4), we therefore have the following:

Proposition 3.3. *There is a diagram of morphisms of $\mathbb{Z}_2$-graded k-complexes*

$$S \otimes \mathscr{A}_W \underset{\Phi^{-1}}{\overset{\Phi}{\rightleftarrows}} \mathscr{E} \tag{3.13}$$

where $\Phi = \pi \exp(-\delta), \Phi^{-1} = \exp(\delta)\sigma_\infty$ and $\widehat{H} = \exp(\delta) H_\infty \exp(-\delta)$ satisfy

$$\Phi\Phi^{-1} = 1\,, \qquad \Phi^{-1}\Phi = 1 - [\partial, \widehat{H}]\,. \tag{3.14}$$

Thus, Φ and Φ^{-1} are mutually inverse homotopy equivalences over k.

The endomorphism algebra $C = \operatorname{End}_k(S)$ is a Clifford algebra generated by the contraction θ_i^* and wedge product θ_i operators. It is observed in [2] that (3.13) is an isomorphism of Clifford representations (in the homotopy category) when the complex $\mathscr{E}$ is equipped with the action induced by Atiyah classes:

Definition 3.4. The Atiyah classes of $\mathscr{A}_W$ are the R-linear odd operators

$$\operatorname{At}_i = [\partial, \partial_{x_i}] : \operatorname{End}_R(k^{\text{stab}}) \longrightarrow \operatorname{End}_R(k^{\text{stab}})\,,$$

Lemma 3.5. *We have*

$$\operatorname{At}_i = -[\psi_i^*, -] - \sum_{q=1}^{n} \partial_{x_i}(W^q)[\psi_q, -]\,.$$

Proof. By direct calculation:

$$\begin{aligned}
\operatorname{At}_i &= \big[[d_{k^{\text{stab}}}, -], \partial_{x_i}\big] \\
&= \sum_q \big[x_q[\psi_q^*, -], \partial_{x_i}\big] + \sum_q \big[W^q[\psi_q, -], \partial_{x_i}\big] \\
&= -\sum_q \partial_{x_i}(x_q)[\psi_q^*, -] - \sum_q \partial_{x_i}(W^q)[\psi_q, -] \\
&= -[\psi_i^*, -] - \sum_q \partial_{x_i}(W^q)[\psi_q, -]\,.
\end{aligned}$$

□

4 The minimal model

We define the even operator $\delta_i = \psi_i \theta_i^*$ on the underlying module (3.5) of $S \otimes \mathscr{A}_W$, so that $\delta = \sum_i \delta_i$. Since $[\delta_i, \delta_j] = 0$ for all i, j we have

$$\exp(\pm\delta) = \exp(\pm\delta_1) \cdots \exp(\pm\delta_n)\,. \tag{4.1}$$

Lemma 4.1. *For $1 \le i \le n$ there is a commutative diagram*

$$
\begin{tikzcd}
(S \otimes \mathscr{A}_W) \otimes (S \otimes \mathscr{A}_W) \arrow[r, "m_2"] \arrow[d, "\delta_i \otimes 1 + 1 \otimes \delta_i + [\psi_i, -] \otimes \theta_i^*"'] & S \otimes \mathscr{A}_W \arrow[d, "\delta_i"] \\
(S \otimes \mathscr{A}_W) \otimes (S \otimes \mathscr{A}_W) \arrow[r, "m_2"'] & S \otimes \mathscr{A}_W
\end{tikzcd}
\tag{4.2}
$$

Definition 4.2. We define

$$\Xi_i = [\psi_i, -] \otimes \theta_i^* : S \otimes \mathscr{A}_W \longrightarrow S \otimes \mathscr{A}_W \,. \tag{4.3}$$

and

$$\Xi = \sum_i \Xi_i \,. \tag{4.4}$$

Proposition 4.3. *There is a commutative diagram*

$$
\begin{tikzcd}
(S \otimes \mathscr{A}_W) \otimes (S \otimes \mathscr{A}_W) \arrow[r, "m_2"] \arrow[d, "\exp(-\delta) \otimes \exp(-\delta)"'] & S \otimes \mathscr{A}_W \arrow[dd, "\exp(-\delta)"] \\
(S \otimes \mathscr{A}_W) \otimes (S \otimes \mathscr{A}_W) \arrow[d, "\exp(-\Xi)"'] & \\
(S \otimes \mathscr{A}_W) \otimes (S \otimes \mathscr{A}_W) \arrow[r, "m_2"'] & S \otimes \mathscr{A}_W
\end{tikzcd}
\tag{4.5}
$$

5 Trees and decorations

6 Algorithm

Let $T \in \mathcal{T}_q$ be a valid plane tree with q inputs and $e_i(T)$ internal edges, and let $A(T)$ be the augmented plane tree. Recall that the vertices of $A(T)$ may be partitioned into the following subsets: the root, the non-root leaves (called *inputs*), the vertices coming from internal edges of T, and those internal vertices coming from internal vertices of T. The integer n is the number of variables in the ambient ring $R = k[x_1, \dots, x_n]$.

Consider the tensor product

$$\mathscr{H} = R \otimes_k \bigwedge \big(\oplus_{i=1}^n k\theta_i \big) \otimes_k \operatorname{End}_k \Big(\bigwedge \big(\oplus_{i=1}^n k\psi_i \big) \Big)$$

on which we have the following homogeneous operators:

$$x_i\,, \partial_i = \partial_{x_i}\,, \theta_i\,, \theta_i^*\,, \big[\psi_i\,, -\big]$$

where ψ_i denotes the operator $\psi_i \wedge (-)$ on $\bigwedge\big(\oplus_{i=1}^n k\psi_i\big)$ and $[\psi_i, -]$ the graded commutator with this operator, defined on a homogeneous operator β by

$$\big[\psi_i, \beta\big] = \psi_i \circ \beta - (-1)^{|\beta|}\beta \circ \psi_i\,.$$

Definition 6.1. A *configuration* C of a valid plane tree T consists of the following data, for each non-root vertex v of $A(T)$:

- An integer $m(v) \geq 0$.
- A subset $J(v) \subseteq \{1, \dots, n\}$ with $|J(v)| = m(v)$.
- If v is an input, or comes from an internal edge of T, a pair

  $$(a_j(v), \gamma_j(v)) \in \{1, \dots, n\} \times \mathbb{N}^n$$

  for each $j \in J(v)$.
- If v comes from an internal edge of T, an integer $t(v) \in \{1, \dots, n\}$.

Let $\operatorname{Con}(T)$ denote the set of all configurations.

Definition 6.2. We define

$$\mathscr{B} = \bigwedge\big(k\psi_1^* \oplus \cdots \oplus k\psi_n^*\big)$$

which we view as a subalgebra

$$\mathscr{B} \subset \operatorname{End}_k\Big(\bigwedge\big(\oplus_{i=1}^n k\psi_i\big)\Big)$$

by identifying ψ_i^* with the operation of contraction $\psi_i^* \lrcorner (-)$ on the exterior algebra. In this way we may also identify $\mathscr{B}$ with a subalgebra of $\mathscr{H}$, and we write $\iota : \mathscr{B} \longrightarrow \mathscr{H}$ for the inclusion. Note that the operator $[\psi_j, -]$ on $\mathscr{H}$ defined above acts on the subalgebra $\mathscr{B}$ as contraction with $\psi_j = (\psi_j^*)^*$, that is,

$$\Big[\psi_j, \psi_{i_1}^* \wedge \cdots \wedge \psi_{i_r}^*\Big] = \sum_{l=1}^{r} (-1)^{l-1}\delta_{j,i_l}\psi_{i_1}^* \wedge \cdots \wedge \widehat{\psi_{i_l}^*} \wedge \cdots \wedge \psi_{i_r}^*\,. \tag{6.1}$$

Given $\gamma \in \mathbb{N}^n$, $f \in R$ we write $x^\gamma = \prod_{i=1}^n x_i^{\gamma_i}$ and $f(\gamma)$ for the coefficient of x^γ in f. Recall the decomposition $W = \sum_j x_j W^j$ with $W^i \in \mathfrak{m}^2$. The products in the A_∞-algebra $\mathscr{B}$ will depend on W through the coefficients $W^j(\gamma) \in k$.

Definition 6.3. Given a tree $T \in \mathcal{T}_q$ and configuration $C \in \mathrm{Con}(T)$ we define a decoration $D_{T,C}$ of $A(T)$ by the assignment of

- $\mathscr{B}$ as the input at each non-root leaf and the output at the root, and
- $\mathscr{H}$ as the label on each edge.

To each vertex v of $A(T)$ we associate an operator ϕ_v as follows, writing $m, J, \{(a_j, \gamma_j)\}_{j \in J}, t$ for the data associated to v:

- if v is an input, then ϕ_v is the linear map $\mathscr{B} \longrightarrow \mathscr{H}$ given by

$$\phi_v = (-1)^m \prod_{j \in J} \Big\{ \frac{1}{|\gamma_j|} W^j(\gamma_j) \partial_{a_j}(x^{\gamma_j}) \theta_{a_j} \big[\psi_j, -\big] \Big\} \circ \iota \,. \tag{6.2}$$

  Note that the operator under the product is even, so the order is irrelevant.

- if v comes from an internal edge of T, then

$$\phi_v = (-1)^m \prod_{j \in J} \Big\{ \frac{1}{|\gamma_j|} W^j(\gamma_j) \partial_{a_j}(x^{\gamma_j}) \theta_{a_j} \big[\psi_j, -\big] \Big\} \circ \theta_t \partial_t \,. \tag{6.3}$$

- if v comes from an internal vertex of T, then

$$\phi_v = (-1)^m m_2 \circ \prod_{j \in J} \Big\{ \big[\psi_j, -\big] \otimes \theta_j^* \Big\} \tag{6.4}$$

  which is a map $\mathscr{H}^{\otimes 2} \longrightarrow \mathscr{H}$. Here m_2 denotes the usual product on the tensor product of algebras $\mathscr{H}$.

Definition 6.4. Given a sequence $a_1, \ldots, a_m \geq 1$ of integers and $a \geq 0$ we define

$$\begin{aligned}
\tau(a, a_1, \ldots, a_m) &= a_1 \cdots a_m \sum_{\sigma \in \mathfrak{S}_m} \prod_{i=0}^{m-1} \frac{1}{a + \sum_{j=m-i}^{m} a_{\sigma(j)}} \\
&= a_1 \cdots a_m \sum_{\sigma \in \mathfrak{S}_m} \frac{1}{a + a_{\sigma(m)}} \cdots \frac{1}{a + a_{\sigma(1)} + \cdots + a_{\sigma(m)}} \,.
\end{aligned}$$

Definition 6.5. Given $T \in \mathcal{T}_q, C \in \mathrm{Con}(T)$ and vertex v of $A(T)$ coming from an internal edge of T we define

$$\omega(C, v) = \sum_{v' < v} \sum_{j \in J(v')} |\gamma_j(v')| - \sum_{z < v} m(z)$$

where the sum is over all internal edges and input vertices v' above v in the tree, i.e. for each v is on the unique path from v' to the root, and z ranges over internal vertices.

We define in addition a symmetry factor

$$S(C,v) = \frac{1}{\omega(C,v)}\tau\big(\omega(C,v), \{|\gamma_j(v)|\}_{j\in J(v)}\big) \in \mathbb{Q}\,,$$
$$S(C) = \prod_{e\in E_i(T)} S(C, v_e)$$

where $E_i(T)$ is the set of internal edges e of T, and v_e the associated vertex of $A(T)$.

Definition 6.6. Given $T \in \mathcal{T}_q$ and $C \in \mathrm{Con}(T)$ we define the k-linear operator

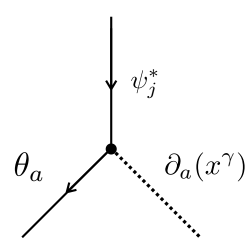
$$\mathcal{O}(T,C) : \mathscr{B}^{\otimes q} \longrightarrow \mathscr{B}$$

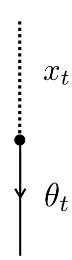
to be the denotation $\mathcal{O}(T,C) = \langle D_{T,C}\rangle$, defined without Koszul signs.

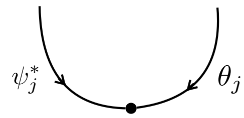
Definition 6.7. We define $\rho_q : \mathscr{B}[1]^{\otimes q} \longrightarrow \mathscr{B}[1]$ by

$$\rho_q(\Lambda_1 \otimes \cdots \otimes \Lambda_q) = \sum_{T\in\mathcal{T}_q}\sum_{C\in\mathrm{Con}(\widehat{T})} (-1)^{Q(T,\underline{\Delta})} S(C)\cdot\mathcal{O}(\widehat{T},C)(\Lambda_q\otimes\cdots\otimes\Lambda_1)\,.$$

From the physics point of view, $\mathscr{H}$ is the tensor product of the bosonic Fock space R with creation and annihilation operators x_i, ∂_i, fermionic Fock space $\wedge(\oplus_{i=1}^n k\theta_i)$ with creation and annihilation operators θ_i, θ_i^* and the third factor, of which the calculations of the A_∞-structure only involve the subalgebra $\mathscr{B} = \wedge(\oplus_{i=1}^n k\psi_i^*)$ which is another fermionic Fock space with states the products of ψ_i^*'s and annihilation operators $[\psi_i, -]$. The operators in (6.2), (6.3), (6.4) are given as products of these creation and annihilation operators, and may be interpreted as particle interactions in the usual way. The associated diagrams are the following:







Here we present incoming particle states at the top of the diagram, and read ψ_i^* as states, $[\psi_i, -]$ as the annihilation operator, θ_a as creation and θ_a^* as annihilation, x_i as creation and ∂_i as annihilation.

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