A_∞-minimal models of matrix factorisations

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Abstract

We study A_∞-minimal models of the differential graded algebras obtained from endomorphisms of generators of triangulated categories of matrix factorisations, giving explicit higher products for the cases of simple singularities.

1 Introduction

Let k be a characteristic zero field, and $W \in k[x_1, \dots, x_n]$ a potential in the sense of [3, §2.2] which has the origin as its only critical point. Choose a presentation of W as a sum $W = x_1 W^1 + \cdots + x_n W^n$ with $W^i \in \mathfrak{m}$. Then the generator of the homotopy category of matrix factorisations is the stabilised diagonal

$$k^{\mathrm{stab}} = \Big(k[x] \otimes_k \bigwedge (k\psi_1 \oplus \cdots \oplus k\psi_n), \sum_i x_i \psi_i^* + \sum_i W^i \psi_i \Big). \tag{1.1}$$

We calculate the A_∞-minimal model $\mathscr{A}_W$ of the differential graded algebra $\mathrm{End}_{k[x]}(k^{\mathrm{stab}})$. This is a finite-dimensional $\mathbb{Z}_2$-graded vector space, with a family of odd k-linear maps

$$\Big\{ r_q : \big(\mathscr{A}_W[1]\big)^{\otimes q} \longrightarrow \mathscr{A}_W[1] \Big\}_{q \ge 2}$$

satisfying the forward suspended A_∞-constraints [1].

2 Perturbation and Koszul complexes

Set $R = k[x_1, \dots, x_n]$. To apply the A_∞-minimal model construction to the DG-algebra $\mathrm{End}_R(k^{\mathrm{stab}})$, we would usually begin by finding a k-linear homotopy retract of this complex onto its cohomology. However, for our purposes it is better to first enlarge the DG-algebra to $S \otimes_k \mathrm{End}_R(k^{\mathrm{stab}})$, where

$$S = \bigwedge (k\theta_1 \oplus \cdots \oplus k\theta_n) \tag{2.1}$$

is viewed as a DG-algebra with zero differential, and $\mathbb{Z}_2$-grading determined by $|\theta_i| = 1$. We show $S \otimes_k \mathrm{End}_R(k^{\mathrm{stab}})$ is homotopy equivalent to the finite-dimensional DG-algebra

$$\underline{\mathrm{End}}(k^{\mathrm{stab}}) = R/\mathfrak{m} \otimes_k \mathrm{End}_R(k^{\mathrm{stab}}) \cong \mathrm{End}_k\left(\bigwedge(k\psi_1 \oplus \cdots \oplus k\psi_n)\right)$$

where $\mathfrak{m} = (x_1, \ldots, x_n)$, and the corresponding homotopy retract is our starting point for the minimal model construction.

For this we will use the Koszul complex over R of $x_1, \ldots, x_n$,

$$K = \left(R \otimes_k \bigwedge(k\theta_1 \oplus \cdots \oplus k\theta_n), d_K = \sum_i x_i \theta_i^*\right). \tag{2.2}$$

and the following diagram of $\mathbb{Z}_2$-graded complexes over k:

$$S \otimes_k \mathrm{End}_R(k^{\mathrm{stab}}) \underset{\exp(-\delta)}{\overset{\exp(-\delta)}{\rightleftarrows}} K \otimes_R \mathrm{End}_R(k^{\mathrm{stab}}) \underset{\sigma_\infty}{\overset{\pi}{\rightleftarrows}} \underline{\mathrm{End}}(k^{\mathrm{stab}}). \tag{2.3}$$

The notation is as follows:

- The operator $\psi_j = \psi_j \wedge -$ on k^{stab} satisfies

  $$[\psi_j, d_{k^{\mathrm{stab}}}] = \left[\psi_j, \sum_i x_i \psi_i^*\right] = x_j \cdot 1,$$

  where $[-,-]$ always denotes the graded commutator. That is, ψ_j is a homotopy for the action of x_j on k^{stab}. It is then easy to check that the odd operator $\alpha \mapsto \psi_j \circ \alpha$ on $\mathrm{End}_R(k^{\mathrm{stab}})$ is also a homotopy for x_j, and where it will not cause confusion we will also write ψ_j for this operator of post-composition.

- Both $S \otimes_k \mathrm{End}_R(k^{\mathrm{stab}})$ and $K \otimes_R \mathrm{End}_R(k^{\mathrm{stab}})$ have the same underlying $\mathbb{Z}_2$-graded k-module, namely

  $$R \otimes_k \bigwedge(k\theta_1 \oplus \cdots \oplus k\theta_n) \otimes_k \mathrm{End}_k\left(\bigwedge(k\psi_1 \oplus \cdots \oplus k\psi_n)\right).$$

  On this space we define the even operator

  $$\delta = \sum_{i=1}^n \psi_i \theta_i^*, \tag{2.4}$$

  where ψ_i is the operator on $\mathrm{End}_R(k^{\mathrm{stab}})$ discussed above, given by $\alpha \mapsto (\psi_i \wedge -) \circ \alpha$, and θ_i^* acts by contraction on S. Note that in (2.3) the complex on the very left has differential $1 \otimes d_{k^{\mathrm{stab}}}$ while the middle complex has differential $1 \otimes d_{k^{\mathrm{stab}}} + d_K \otimes 1$. It is easy to check that $\exp(\delta), \exp(-\delta)$ intertwines these differentials and gives an isomorphism of complexes [2, Proposition 4.11].

- The morphism of complexes $\pi : K \longrightarrow R/\mathfrak{m}$ is defined by composing the projection of K onto the submodule $R\cdot 1$ of θ-degree zero forms, with the quotient $R \longrightarrow R/\mathfrak{m}$. We also write π for the result of tensoring this morphism with $\mathrm{End}_R(k^{\mathrm{stab}})$ to obtain

$$K \otimes_R \mathrm{End}_R(k^{\mathrm{stab}}) \xrightarrow{\pi\otimes 1} R/\mathfrak{m} \otimes_R \mathrm{End}_R(k^{\mathrm{stab}}) = \underline{\mathrm{End}}(k^{\mathrm{stab}})\,.$$

By [2] there is a diagram of k-linear homotopy equivalences
and an odd k-linear map

$$H_\infty : K \otimes_R \mathrm{End}_R(k^{\mathrm{stab}}) \longrightarrow K \otimes_R \mathrm{End}_R(k^{\mathrm{stab}})$$

with

$$\sigma_\infty = \sum_{s\geq 0} \sum_{p_1,\ldots,p_s} (-1)^{\binom{s+1}{2}} \frac{1}{s!} \mathrm{At}_{p_1} \cdots \mathrm{At}_{p_s}\, \theta_{p_1} \cdots \theta_{p_s} \pmod{\mathfrak{m}}$$

3 Examples

To describe the higher multiplications it is convenient to write $[\psi_i, -]$ for the natural operator on $\mathscr{A}$ (giving ψ_i, ψ_j^* the usual anticommutation relations for fermionic creation and annhiliation operators), that is

$$\begin{aligned}[\psi_i, \psi_{j_1}^* \cdots \psi_{j_t}^*] &= \sum_{l=1}^{t} (-1)^{l-1} \psi_{j_1}^* \cdots [\psi_i, \psi_{j_l}^*] \cdots psi_{j_t}^* \\ &= \sum_{l=1}^{t} (-1)^{l-1} \delta_{i=j_l} \psi_{j_1}^* \cdots \psi_{j_t}^* \,.\end{aligned}$$

A general fact is that the higher multiplications on $\mathscr{A}$ are linear combinations of products of such operators. For example, when $n = 2$ a standard term in r_3 would look like

$$\Phi_0 \otimes \Phi_1 \otimes \Phi_2 \mapsto \lambda \cdot [\psi_1, [\psi_2, \Phi_0]] \cdot [\psi_1, \Phi_1] \cdot [\psi_2, \Phi_2] \,.$$

The coefficient λ is computed as a Feynman amplitude on a binary planar tree with with incoming fermion states $\psi_1\psi_2, \psi_1, \psi_2$ in the three leaves (respectively) and a simple list of allowed interactions, the most significant of which is a trivalent interaction vertex with an incoming fermion ψ_j and outgoing fermion θ_i and bosons $\partial_{x_i}(x^\gamma)$ whenever the polynomial W^j has a nonzero coefficient for the monomial $x^\gamma = x_1^{\gamma_1} x_2^{\gamma_2}$ (the fermions θ_i are the auxiliary spinor representation generators as in [2]). There are two other interactions which do not depend on W, and which take place only at internal edges or internal vertices (respectively).

Example 3.1. Let $W = x^d$ so that $\mathscr{A} = k \oplus k\psi^*$. Then only r_2 and r_d are nonzero and on $(\mathscr{A}[1])^{\otimes d}$ the only basis element with a nonzero value under r_d is

$$r_d(\psi^* \otimes \cdots \otimes \psi^*) = 1 \,.$$

Another way to say this is

$$r_d(\Phi_0 \otimes \cdots \otimes \Phi_{d-1}) = \prod_{i=0}^{d-1} [\psi, \Phi_i] \,.$$

This A_∞-structure is cyclic with respect to the trace form on $\mathscr{A}$ which projects onto the $k\psi^*$-summand. Note that all such products are expanded such that the index i increases from left to right.

References

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