A_∞-minimal models of matrix factorisations

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Abstract

We study A_∞-minimal models of the differential graded algebras obtained from endomorphisms of generators of triangulated categories of matrix factorisations, giving explicit higher products for the cases of simple singularities.

1 Introduction

Let k be a commutative $\mathbb{Q}$-algebra and $W \in k[x_1, \ldots, x_n]$ a polynomial which is a potential over k in the sense of [3, §2.2]. For example, $k = \mathbb{C}$ and W has isolated critical points. The $\mathbb{Z}_2$-graded DG-category $\mathcal{A} = \mathrm{mf}(R, W)$ has for its objects finite-rank matrix factorisations of W over $R = k[x_1, \ldots, x_n]$. In this paper we study the minimal model problem for $\mathcal{A}$, the aim being to produce models that are finitely-generated and projective over k.

When k is a field, the standard minimal model theorem [?] constructs the structure of an A_∞-category on the cohomology $\mathcal{B} = H^*\mathcal{A}$ in such a way that $\mathcal{B}$ is quasi-isomorphic to $\mathcal{A}$. Moreover, this A_∞-category has finite-dimensional Hom-spaces and is therefore a good finite model of $\mathcal{A}$. The problem in this case is to have sufficient control over the inputs to the minimal model theorem that the A_∞-products on $H^*\mathcal{A}(X, X)$ can be reasonably calculated, for a given matrix factorisation X. In order to understand deformations of matrix factorisations [?, ?, ?] it is important to do this for generic X, but the special case where $X = k^{\mathrm{stab}}$, the generator of $\mathcal{A}$ studied by Dyckerhoff in the case of a single isolated singularity at the origin [Dyc11], is of particular interest.

Since one of our goals is to understand how matrix factorisation categories vary along unfoldings of singularities, we also need to consider the case where k is not a field. For example take $k = \mathbb{C}[t]$ and $R = \mathbb{C}[x_1, \ldots, x_n, t]$ so that $W = W_t(x)$ is a potential with parameter. In this case we want an A_∞-category $\mathcal{B}$ quasi-isomorphic to $\mathcal{A}$ over k, with $\mathcal{B}(a, b)$ a finitely-generated projective k-module for each pair of objects a, b.

These two desiderata, understanding the A_∞-products on $H^*\mathcal{A}(X, X)$ for generic X, and the case where k is not a field, lead us naturally to a slightly non-standard use of the minimal model theorem. Following the ideas developed in [?, ?, ?] we prove that, writing $\iota : R \longrightarrow R/(\partial_{x_1}W, \ldots, \partial_{x_n}W)$ for the quotient, there is a k-linear homotopy retract

$$(1.1) \qquad \iota^*\mathcal{A} \xrightleftharpoons{\qquad\qquad} \mathcal{A} \otimes_k \bigwedge \left(k^{\oplus n}[1]\right).$$

Suppose $W \in \mathfrak{m}^2$ where $\mathfrak{m} = (x_1, \ldots, x_n)$ and choose a presentation $W = x_1 W^1 + \cdots + x_n W^n$ with $W^i \in \mathfrak{m}$ and consider the following pair

$$(1.2) \qquad k^{\mathrm{stab}} = \left(k[x] \otimes_k \bigwedge \left(\oplus_{i=1}^n k\psi_i \right), \ d_{k^{\mathrm{stab}}} = \sum_{i=1}^n x_i \psi_i^* + \sum_{i=1}^n W^i \psi_i \right)$$

where ψ_i^* and ψ_i act respectively by contraction and wedge product on the exterior algebra underlying k^{stab}. It is easy to see that $d_{k^{\mathrm{stab}}}^2 = W$, so k^{stab} is a matrix factorisation. When k is a field and W has an isolated critical point at the origin, k^{stab} is the representative in the homotopy category of matrix factorisations of the structure sheaf of the singular point at the origin, and was first studied by Dyckerhoff [Dyc11].

The purpose of this paper is to show how to calculate the A_∞-minimal model $\mathscr{B}$ of the $\mathbb{Z}_2$-graded differential graded endomorphism algebra

$$(1.3) \qquad \mathscr{A} = \left(\mathrm{End}_{k[x]}(k^{\mathrm{stab}}), \ \partial = [d_{k^{\mathrm{stab}}}, -] \right).$$

The differential here is (throughout all commutators are graded commutators)

$$\partial = [d_{k^{\mathrm{stab}}}, -] = \left[\sum_i x_i \psi_i^* + \sum_i W^i \psi_i, - \right]$$
$$= \sum_i x_i [\psi_i^*, -] + \sum_i W^i [\psi_i, -].$$

The minimal model is a finite-dimensional $\mathbb{Z}_2$-graded vector space $\mathscr{B}$ with a family

$$\left\{ \rho_q : \left(\mathscr{B}[1] \right)^{\otimes q} \longrightarrow \mathscr{B}[1] \right\}_{q \geq 2}$$

of odd k-linear maps satisfying the forward suspended A_∞-constraints [1], and having the property that there is an A_∞-quasi-isomorphism $\mathscr{B} \longrightarrow \mathscr{A}$.

2 Background

- Commutation relations for $[\psi_i, -]$.

3 Feynman diagrams

The A_∞-products on the A_∞-algebra $\mathscr{B}$ are defined in terms of Feynman diagrams (see Definition 3.14). In this section we explain how to enumerate all the relevant diagrams and how to compute their value; in the next section we prove that the maps they give rise to do indeed compute the minimal model of $\mathscr{A}$.

The *tilde grading* is defined by $\widetilde{x} = |x| - 1$. Throughout $\otimes = \otimes_k$. Our reference for general background on Feynman diagrams is [5, Ch. 6]. The integer n is the number of variables in the ambient ring $R = k[x_1, \ldots, x_n]$. We assume in this section that $W \in \mathfrak{m}^3$.

Definition 3.1. Consider the $\mathbb{Z}_2$-graded algebra defined by the tensor product

$$\mathscr{H} = R \otimes_k \bigwedge \left(\oplus_{i=1}^n k\theta_i \right) \otimes_k \mathrm{End}_k \left(\bigwedge \left(\oplus_{i=1}^n k\psi_i \right) \right)$$

with degrees $|\theta_i| = |\psi_i| = 1$. On this space we have the homogeneous operators

$$x_i\,,\partial_i = \partial_{x_i}\,,\theta_i\,,\theta_i^*\,,\left[\psi_i\,,-\right].$$

Here ψ_i denotes the operator $\psi_i \wedge (-)$ on $\bigwedge \left(\oplus_{i=1}^n k\psi_i \right)$ and $[\psi_i, -]$ the graded commutator with this operator, defined on a homogeneous operator β by

$$\left[\psi_i, \beta\right] = \psi_i \circ \beta - (-1)^{|\beta|}\beta \circ \psi_i\,.$$

We define the $\mathbb{Z}_2$-graded k-module

(3.1) $$\mathscr{B} = \bigwedge \left(\oplus_{i=1}^n k\psi_i^* \right)$$

which we view as a submodule

$$\mathscr{B} \subset \mathrm{End}_k \left(\bigwedge \left(\oplus_{i=1}^n k\psi_i \right) \right)$$

by identifying $\psi_i^* \in \mathscr{B}$ with the operation of contraction $\psi_i^* \lrcorner (-)$ on the exterior algebra. In this way we may also identify $\mathscr{B}$ with a submodule of $\mathscr{H}$, and we write $\iota : \mathscr{B} \longrightarrow \mathscr{H}$ for the inclusion. Note that the operator $[\psi_j, -]$ on $\mathscr{H}$ defined above acts on the submodule $\mathscr{B}$ as contraction with $\psi_j = (\psi_j^*)^*$, that is,

(3.2) $$\left[\psi_j, \iota\left(\psi_{i_1}^* \wedge \cdots \wedge \psi_{i_r}^*\right)\right] = \sum_{l=1}^r (-1)^{l-1}\delta_{j,i_l}\iota\left(\psi_{i_1}^* \wedge \cdots \wedge \widehat{\psi_{i_l}^*} \wedge \cdots \wedge \psi_{i_r}^*\right).$$

We write $p : \mathscr{H} \longrightarrow \mathscr{B}$ for the natural projection (todo).

We view $\mathscr{H}$ as the tensor product of the bosonic Fock space R with creation and annihilation operators x_i, ∂_i, fermionic Fock space $\wedge(\oplus_{i=1}^n k\theta_i)$ with creation and annihilation operators θ_i, θ_i^* and the third factor, of which the calculations of the A_∞-structure only involve the submodule $\mathscr{B} = \wedge(\oplus_{i=1}^n k\psi_i^*)$ which is another fermionic Fock space. The inputs to the Feynman diagrams are tensor products of states in $\mathscr{B}$, that is, wedge products of ψ_i^*'s, on which the $[\psi_i, -]$ act as annihilation operators.

The linear maps in (3.7), (3.8), (3.9) below which determine the A_∞ products ρ_q on $\mathscr{B}$ will be defined in terms of these creation and annihilation operators on $\mathscr{H}$, and may be interpreted as particle interactions in the usual way. The interactions that appear are the following ones, depicted on the left as Feynman diagrams and on the right as operators. We call these respectively *A-, B- and C-type interactions*:

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A

ψ_j^*

θ_a $x^{\gamma - e_a}$

$$\theta_a x^{\gamma - e_a}\big[\psi_j, -\big] \in \mathrm{End}_k(\mathscr{H})$$

$$1 \le a, j \le n \,, \gamma \in \mathbb{N}^n \tag{3.3}$$

B

x_t

θ_t

$$\theta_t \partial_t \in \mathrm{End}_k(\mathscr{H})$$

$$1 \le t \le n \tag{3.4}$$

C

ψ_j^* θ_j

$$m_2\big(\big[\psi_j, -\big] \otimes \theta_j^*\big) \in \mathrm{Hom}_k(\mathscr{H}^{\otimes 2}, \mathscr{H})$$

$$1 \le j \le n \tag{3.5}$$

Here m_2 is the multiplication in $\mathscr{H}$ and, given $\gamma \in \mathbb{N}^n$, we write $x^\gamma = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$, with $e_i \in \mathbb{N}^n$ the unit vector with $x^{e_i} = x_i$. By convention if $\gamma_a = 0$ then $x^{\gamma - e_a} = 0$, which ensures that we always have $\gamma_a x^{\gamma - e_a} = \partial_a(x^\gamma)$. Finally note that bosons are depicted with dotted lines, and fermions with solid lines.

These interactions "take place" at vertices of the tree $A(T)$ for some $T \in \mathcal{T}_q$, and come with the following constraints, which will be formalised below in terms of combinatorial data called *configurations*. We write $f(\gamma)$ for the coefficient of x^γ in a polynomial $f \in R$ and recall the chosen decomposition $W = \sum_j x_j W^j$.

- A-type interactions only occur at input vertices and internal edges of T.

- B-type interactions only occur at internal edges of T.

- C-type interactions only occur at internal vertices of T, and moreover the incoming lines must come from different branches of the tree.

- There may be arbitrarily many A-type or C-type interactions per vertex of $A(T)$, but there is *precisely one* B-type interaction at each internal edge of T.

- The A-type interaction with indices a, j, γ appears with the coefficient

$$-\frac{\gamma_a}{|\gamma|} W^j(\gamma) \,. \tag{3.6}$$

where $|\gamma| = \sum_{i=1}^n \gamma_i$. Each B- and C-type interaction appears with coefficient (-1).

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As we will see, the coefficient (3.6) is the only way that W enters the rules, and thus the definition of the A_∞-products on $\mathscr{B}$. The precise definition of the Feynman rules will be given in Definition 3.3 below, but first we give an example of a tree with interactions inserted, to illustrate the various ingredients.

Example 3.2. Consider the tree $T \in \mathcal{T}_3$ whose augmented tree $A(T)$ is depicted below, together with labels for its vertices, where we label by e the vertex corresponding to the internal edge of T:
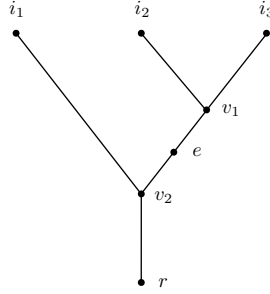
Set $W = x^3 \in \mathbb{C}[x]$ so that $W^1 = x^2$ and the only A-type interaction has $a = 1, j = 1, \gamma = 2$ and coefficient -1, as in the diagram (we write $\theta = \theta_1, \psi = \psi_1$)
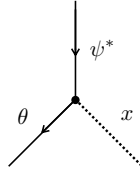
Consider the input state $\Psi = \psi^* \otimes \psi^* \otimes \psi^*$ in $\mathscr{B}^{\otimes 3}$. One possible pattern of interactions (called a configuration, below) has a single A-type interaction at the input vertex i_3, a B-type interaction at e, and two C-type interactions at v_1, v_2, as shown in the following "topological" Feynman diagram
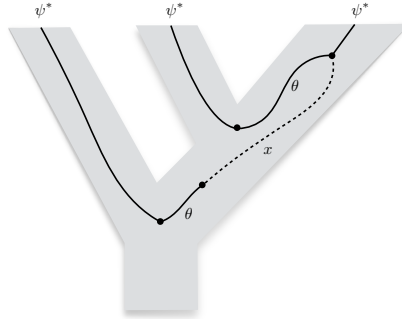
The amplitude of this Feynman diagram is the value of

$$(-1)^3 p \circ m_2\big([\psi, -] \otimes \theta^*\big) \circ \big(\iota \otimes \theta \partial_x \circ m_2([\psi, -] \otimes \theta^*)\big(\iota \otimes \theta x[\psi, -]\iota\big)\big)$$

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on the input state Ψ. Usually when evaluating such an expression we would pay attention to Koszul signs from commuting the input ψ^*'s into position (see Definition **??**). Ignoring all signs, the amplitude is

$$\pm\, p \circ m_2\big([\psi, -] \otimes \theta^*\big) \circ \big(\iota \otimes \theta\partial_x \circ m_2([\psi, -] \otimes \theta^*)(\iota \otimes \theta x[\psi, -]\iota)\big)\Big(\Psi\Big)$$

$$= \pm\, p \circ m_2\big([\psi, \psi^*] \otimes \theta^*\theta\partial_x \circ m_2([\psi, \psi^*] \otimes \theta^*\theta x[\psi, \psi^*])\big)$$

$$= \pm\, p \circ m_2\big(1 \otimes \theta^*\theta\partial_x \circ m_2(1 \otimes \theta^*\theta x)\big)$$

$$= \pm\, p \circ m_2\big(1 \otimes \partial_x(x)\big)$$

$$= \pm\, p(1) = \pm\, 1 \in \mathscr{H}\,.$$

Now we give some more precise definitions. Let $T \in \mathcal{T}_q$ be a valid plane tree with q inputs and $e_i(T)$ internal edges, and let $A(T)$ be the augmented plane tree. The definitions are all relative to the ambient integer n, but for the sake of readability we will not reflect this in the notation.

Definition 3.3. A *configuration* C of a valid plane tree T consists of the following data, for each non-root vertex v of $A(T)$:

- An integer $m(v) \geq 0$.

- A subset $J(v) \subseteq \{1, \ldots, n\}$ with $|J(v)| = m(v)$.

- If v is an input, or comes from an internal edge of T, a pair

$$(a_j(v), \gamma_j(v)) \in \{1, \ldots, n\} \times \mathbb{N}^n$$

for each $j \in J(v)$, with $\gamma_j(v)_{a_j(v)} \geq 1$.

- If v comes from an internal edge of T, an integer $t(v) \in \{1, \ldots, n\}$.

Let $\mathrm{Con}(T)$ denote the set of all configurations.

Remark 3.4. The integer $m(v)$ counts how many interactions of type A or C take place at v (there is no point counting B interactions as precisely one occurs at each internal edge). The set $J(v)$ consists of all j-indices appearing in interactions at v. The interactions of A-type (at inputs and internal edges) are defined by indices $a_j(v), \gamma_j(v)$ while at each internal edge e of T, $t(e)$ is the index of the B-type interaction.

Definition 3.5. Given a tree $T \in \mathcal{T}_q$ and configuration $C \in \mathrm{Con}(T)$ we define a decoration $D_{T,C}$ of $A(T)$ by the assignment of the modules

- $\mathscr{B}$ to the input at each non-root leaf and the output at the root, and

- $\mathscr{H}$ to each edge.

To each vertex v of $A(T)$ we associate an operator ϕ_v as follows, writing $m, J, \{(a_j, \gamma_j)\}_{j \in J}, t$ for the data associated to v by the configuration C:

- if v is an input, then ϕ_v is the linear map $\mathscr{B} \longrightarrow \mathscr{H}$ given by

$$(3.7) \qquad \phi_v = (-1)^m \prod_{j \in J} \left\{ \frac{1}{|\gamma_j|} (\gamma_j)_{a_j} W^j(\gamma_j) \theta_{a_j} x^{\gamma_j - e_{a_j}} [\psi_j, -] \right\} \circ \iota.$$

  Note that the operator under the product is even, so the order is irrelevant. If $J(v)$ is empty then the product is interpreted to be the identity operator.

- if v comes from an internal edge of T, then

$$(3.8) \qquad \phi_v = (-1)^m \prod_{j \in J} \left\{ \frac{1}{|\gamma_j|} (\gamma_j)_{a_j} W^j(\gamma_j) \theta_{a_j} x^{\gamma_j - e_{a_j}} [\psi_j, -] \right\} \circ \theta_t \partial_t.$$

- if v comes from an internal vertex of T, then

$$(3.9) \qquad \phi_v = (-1)^m m_2 \circ \prod_{j \in J} \left\{ [\psi_j, -] \otimes \theta_j^* \right\}$$

  which is a map $\mathscr{H}^{\otimes 2} \longrightarrow \mathscr{H}$. Here m_2 denotes the product on $\mathscr{H}$.

- if $v = r$ is the root, $\phi_v = p : \mathscr{H} \longrightarrow \mathscr{B}$ is the projection defined in (??).

Definition 3.6. Given $T \in \mathcal{T}_q$ and $C \in \mathrm{Con}(T)$ we define the k-linear operator

$$\mathcal{O}(T, C) : \mathscr{B}^{\otimes q} \longrightarrow \mathscr{B}$$

to be the denotation $\mathcal{O}(T, C) = \langle D_{T,C} \rangle$, defined without Koszul signs.

Remark 3.7. Let us now briefly explain the connection between configurations, the linear map $\mathcal{O}(T, C)$ and Feynman diagrams, such as the one in Example 3.2. Ultimately we will not actually use these diagrams to perform calculations, so we refrain from formulating them too precisely; however, they are very useful as a tool for understanding.

We view $\Psi_1 \otimes \cdots \otimes \Psi_q \in \mathscr{B}^{\otimes q}$ as the result of applying creation operators (meaning wedge products $\psi_i^* \wedge -$) to the vacuum (meaning the identity of the exterior algebra) in the various copies of $\mathscr{B}$. The value $\mathcal{O}(T, C)$ on Ψ may then be computed by commuting all creation operators acting on $\mathscr{H}$ (that is, the operators x_i, θ_i, ψ_i^*) leftward in the expression, which means *down* the tree. These operators commute with all creation operators and with those annihilation operators which are either of a different type (e.g. $[x_i, \theta_j^*] = 0$) or of same type but with different indices (e.g. $[x_1, \partial_2] = 0$). However, there will be a nonzero commutator every time a creation operator meets an annihilation operator of the same type and index (respectively $\partial_i, \theta_i^*, [\psi_i, -]$). We view such a commutator, e.g.

$$\theta_i \theta_i^* + \theta_i^* \theta_i = [\theta_i, \theta_i^*] = 1$$

which arises as the result of commuting the θ_i leftward to meet the θ_i^*, as generating two diagrams, corresponding to the two summands in $\theta_i^* \theta_i = 1 - \theta_i \theta_i^*$. In the first summand there in an interaction and in the second there is none.

observing that the final p will annihilate with the creation operators x_i and θ_i. The summands which are generated by this calculation can be described by Feynman diagrams. actually use these diagrams to do calculations, we would have to keep track of crossing of fermion lines (since these contribute signs)

Let $T \in \mathcal{T}_q, C \in \text{Con}(T)$ and let $I_i \subseteq \{1, \ldots, n\}$ for $1 \le i \le q$ describe the input fermions.

The following simple identity is well-known in the literature in connection with soft photons and infrared divergences in quantum field theory, see for instance [5, Ch 13] and [6, p.204]. The amplitudes for our Feynman diagrams will involve a generalisation.

Lemma 3.8. *Given a sequence $a_1, \ldots, a_m \ge 1$ of integers,*

$$(3.10) \qquad \sum_{\sigma \in \mathfrak{S}_m} \frac{1}{a_{\sigma(1)}(a_{\sigma(1)} + a_{\sigma(2)}) \cdots (a_{\sigma(1)} + \cdots + a_{\sigma(m)})} = \frac{1}{a_1 \cdots a_m}.$$

Definition 3.9. Given a sequence $a_1, \ldots, a_m \ge 1$ of integers and $a \ge 0$ we define

$$\zeta_a(a_1, \ldots, a_m) = a_1 \cdots a_m \cdot \sum_{\sigma \in \mathfrak{S}_m} \prod_{i=1}^{m} \left(a + \sum_{j=1}^{i} a_{\sigma(j)} \right)^{-1}$$

$$= \sum_{\sigma \in \mathfrak{S}_m} \frac{a_1 \cdots a_m}{(a + a_{\sigma(1)})(a + a_{\sigma(1)} + a_{\sigma(2)}) \cdots (a + a_{\sigma(1)} + \cdots + a_{\sigma(m)})}.$$

Clearly this is a symmetric function of the a_i and, by the previous lemma,

$$(3.11) \qquad \zeta_0(a_1, \ldots, a_m) = 1.$$

As we have described it above, the Feynman diagrams have as incoming and outgoing particles the fermions ψ_i^*'s, while the internal edges involve two kinds of virtual particles, the bosonic x_i's and fermionic θ_i's. A configuration C gives rise to numerous Feynman diagrams, but it will turn out that all diagrams with nonzero amplitude have the *same* pattern of connections among the virtual particles. This means that to a configuration C and edge e of T we may associate the *number* $N_C(e)$ of virtual particles entering e.

For the next two definitions we take $T \in \mathcal{T}_q$ and $C \in \text{Con}(T)$.

Definition 3.10. Given an internal edge e of T we define

$$N_C(e) = \sum_{v < e} \sum_{j \in J(v)} |\gamma_j(v)| - \sum_{z < e} m(z)$$

where v ranges over all inputs and internal edges of T which are strictly above e in $A(T)$, and z ranges over all internal vertices of T which are strictly above e.

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This counts the number of virtual particles entering v from above, since an A-type interaction with indices a, j, γ creates $|\gamma|$ virtual particles, a B-type interaction preserves the number of virtual particles, and a C-type interaction decreases the number by one.

Example 3.11. The configuration C which is described in Example 3.2 has $n = 1$ and

$$m(i_3) = m(v_1) = m(v_2) = 1,$$
$$J(i_3) = J(v_1) = J(v_2) = \{1\},$$
$$t(e) = 1.$$

At i_3 we have the pair $a = 1$ and $x^\gamma = x^2$. Consequently $N_C(e) = 1$.

Combinatorially the most important part of the A_∞-products in the minimal model for W is the rational number Z_C in the next definition. We use the following shorthand: if e is an internal edge of T assigned by the configuration C the data $m = m(v), J = J(v)$ and $\{\gamma_j\}_{j \in J}$, then we write

(3.12) $$\underline{\gamma}(e) = \left(|\gamma_j|\right)_{j \in J} = \left(|\gamma_{j_1}|, \ldots, |\gamma_{j_m}|\right)$$

for the sequence of total degrees where $J = \{j_1, \ldots, j_m\}$.

Definition 3.12. We define

$$Z(T, C) = \prod_e \frac{\zeta_{N_C(e)}\left(\underline{\gamma}(e)\right)}{N_C(e)} \in \mathbb{Q}$$

where is the product is over all internal edges e of T with $N_C(e) \neq 0$. For an edge e with $m(e) = 0$ and thus $\underline{\gamma}(e) = \emptyset$ we take $\zeta_{N_C(e)}\left(\underline{\gamma}(e)\right) = 1$.

We note that in the definition of the higher products ρ_q below, any pair (T, C) which has an internal edge e with $N_C(e) = 0$ will not contribute to the sum in any case, so we do not care about $Z(T, C)$ for such configurations.

Example 3.13. When $m(e) = 1, J(e) = \{j\}$ and we write $\gamma = \gamma_j(e), N = N_C(e)$, we have

$$\zeta_{N_C(e)}\left(\underline{\gamma}(e)\right) = \frac{|\gamma|}{N + |\gamma|}.$$

In the case where $m(e) = 2, J(e) = \{j, j'\}$ and we write $\gamma = \gamma_j(e), \gamma' = \gamma_{j'}(e)$,

$$\zeta_{N_C(e)}\left(\underline{\gamma}(e)\right) = \frac{|\gamma||\gamma'|}{N + |\gamma| + |\gamma'|} \left(\frac{1}{N + |\gamma|} + \frac{1}{N + |\gamma'|}\right).$$

In particular, for the tree and configuration of Examples 3.2, 3.11 we only have one internal edge e with $m(e) = 0$, so $Z(T, C) = \frac{1}{N_C(e)} = 1$.

Definition 3.14. We define the map $\rho_q : \mathscr{B}[1]^{\otimes q} \longrightarrow \mathscr{B}[1]$ of degree $+1$ on homogeneous elements $\Lambda_1, \ldots, \Lambda_q \in \mathscr{B}$ by

$$\rho_q(\Lambda_q, \ldots, \Lambda_1) = \sum_{T \in \mathcal{T}_q} \sum_{C \in \mathrm{Con}(T)} (-1)^{Q(T,\underline{\Lambda})} Z(T, C) \cdot \mathcal{O}(T, C)(\Lambda_1, \ldots, \Lambda_q)$$

where $\mathcal{O}(T, C)$ is from Definition 3.6 and the sign is given by

$$Q(T, \underline{\Lambda}) = e_i(T) + q + 1 + \sum_{1 \leq i < j \leq q} \widetilde{\Lambda}_i \widetilde{\Lambda}_j + \sum_{i=1}^{q} \widetilde{\Lambda}_i C_i \,.$$

Here C_i is the number of times the path from the ith input to the root enters a trivalent vertex from the left in T, $e_i(T)$ is the number of internal edges and $\widetilde{\Lambda} = |\Lambda| - 1$.

Theorem 3.15. *The operators ρ_q satisfy the forward suspended A_∞-constraints, so that $(\mathscr{B}, \{\rho_q\}_{q \geq 2})$ is an A_∞-algebra. Moreover $\mathscr{B}$ is A_∞-quasi-isomorphic to $\mathscr{A}$.*

Proof. See Section **??**. $\qquad\square$

Example 3.16. Let $W = x^d$ with $d > 2$. Then the A_∞-products on the minimal model

$$\mathscr{B} = \left(\bigwedge (k\psi^*), \{\rho_q\}_{q \geq 2} \right)$$

are given by

- ρ_2 is the forward suspension of the usual product in the exterior algebra, that is,

$$\rho_2(x_1 \otimes x_2) = (-1)^{\widetilde{x_1}\widetilde{x_2} + \widetilde{x_1} + 1} x_2 x_1 \,.$$

- For $q \notin \{2, d\}$ we have $\rho_q = 0$, and

- The tensor product $\mathscr{B}^{\otimes d}$ has dimension 2^d with basis $(\psi^*)^{i_0} \otimes \cdots \otimes (\psi^*)^{i_d}$ for $i_j \in \{0, 1\}$. The operator ρ_d is zero on all the basis elements except for

$$\rho_d(\psi^* \otimes \psi^* \otimes \cdots \otimes \psi^*) = (-1)^{d-1} \,.$$

4 Perturbation and Koszul complexes

Let k be a characteristic zero field, $R = k[x_1, \ldots, x_n]$ and let $W \in \mathfrak{m}^2$ be a potential with chosen decomposition $W = \sum_{i=1}^{n} x_i W^i$. To apply the A_∞-minimal model construction to

$$\mathscr{A} = \left(\mathrm{End}_R(k^{\mathrm{stab}}), \ \partial = [d_{k^{\mathrm{stab}}}, -] \right)$$

$$= \left(R \otimes_k \mathrm{End}_k \left(\bigwedge \left(\oplus_{i=1}^{n} k\psi_i \right) \right), \sum_i x_i [\psi_i^*, -] + \sum_i W^i [\psi_i, -] \right)$$

we first enlarge this DG-algebra to $S \otimes_k \mathscr{A}$, where S is the exterior algebra

$$(4.1) \qquad S = \bigwedge \left(\oplus_{i=1}^n k\theta_i \right)$$

viewed as a $\mathbb{Z}_2$-graded DG-algebra with grading $|\theta_i| = 1$ and zero differential. We also need to introduce the $n\mathbb{Z}_2$-graded vector space

$$(4.2) \qquad \mathscr{E} = \mathrm{End}_k \left(\bigwedge \left(\oplus_{i=1}^n k\psi_i \right) \right)$$

and the Koszul complex of $x_1, \ldots, x_n$,

$$(4.3) \qquad K = \left(R \otimes \bigwedge \left(\oplus_{i=1}^n k\theta_i \right), \ d_K = \sum_i x_i \theta_i^* \right)$$

Note that $S \otimes_k \mathscr{A}$ has the same underlying $\mathbb{Z}_2$-graded k-module as $K \otimes_R \mathscr{A}$.

Theorem 4.1. *There is a diagram of k-linear homotopy equivalences*

$$(4.4) \qquad \left(S \otimes_k \mathscr{A}, \partial \right) \underset{\exp(-\delta)}{\overset{\exp(-\delta)}{\rightleftarrows}} \left(K \otimes_R \mathscr{A}, d_K + \partial \right) \underset{\sigma_\infty}{\overset{\pi}{\rightleftarrows}} \left(\mathscr{B}, 0 \right).$$

Proof. See [2]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

The notation is as follows:

- The complexes involved here are

$$\left(S \otimes \mathrm{End}_R(k^{\mathrm{stab}}), \ \partial \right)$$
$$\left(K \otimes_R \mathrm{End}_R(k^{\mathrm{stab}}), \ d_K + \partial \right)$$
$$\left(\mathscr{E}, 0 \right).$$

- The operator $\psi_j = \psi_j \wedge -$ on k^{stab} satisfies

$$[\psi_j, d_{k^{\mathrm{stab}}}] = \left[\psi_j, \sum_i x_i \psi_i^* \right] = x_j \cdot 1,$$

  where $[-, -]$ always denotes the graded commutator. That is, ψ_j is a homotopy for the action of x_j on k^{stab}. It is then easy to check that the odd operator $\alpha \mapsto \psi_j \circ \alpha$ on $\mathrm{End}_R(k^{\mathrm{stab}})$ is also a homotopy for x_j, and where it will not cause confusion we will also write ψ_j for this operator of post-composition.

- Both $S \otimes \mathscr{A}_W$ and $K \otimes_R \mathscr{A}_W$ have the same underlying $\mathbb{Z}_2$-graded k-module, namely

$$(4.5) \qquad R \otimes \bigwedge (\oplus_{i=1}^n k\theta_i) \otimes \mathrm{End}_k \left(\bigwedge (\oplus_{i=1}^n k\psi_i) \right).$$

On this space we define the even operator

$$(4.6) \qquad \delta = \sum_{i=1}^{n} \psi_i \theta_i^* \,,$$

where ψ_i is the operator on $\mathrm{End}_R(k^{\mathrm{stab}})$ discussed above, given by $\alpha \mapsto (\psi_i \wedge -) \circ \alpha$, and θ_i^* acts by contraction on S. It is easy to check that $\exp(\delta), \exp(-\delta)$ intertwines the differentials ∂ and $d_K + \partial$ and therefore gives an isomorphism between the first two complexes in (4.4) [2, Proposition 4.11].

- The morphism of complexes $\pi : K \longrightarrow R/\mathfrak{m}$ is defined by composing the projection of K onto the submodule $R \cdot 1$ of θ-degree zero forms, with the quotient $R \longrightarrow R/\mathfrak{m}$. We also write π for the result of tensoring this morphism with $\mathrm{End}_R(k^{\mathrm{stab}})$ to obtain

$$K \otimes_R \mathrm{End}_R(k^{\mathrm{stab}}) \xrightarrow{\ \pi \otimes 1\ } R/\mathfrak{m} \otimes_R \mathrm{End}_R(k^{\mathrm{stab}}) = \underline{\mathrm{End}}(k^{\mathrm{stab}}) \,.$$

This is a morphism of complexes, because the differential on $\mathrm{End}_R(k^{\mathrm{stab}})$ vanishes on the quotient $\underline{\mathrm{End}}(k^{\mathrm{stab}})$ by the hypothesis that $W \in \mathfrak{m}^2$.

- The k-linear homotopy inverse σ_∞ to π in (4.4) is defined in terms of the following ingredients. The first is the k-linear connection (viewing θ_i as a 1-form)

$$(4.7) \qquad \nabla : K \longrightarrow K \,, \qquad \nabla = \sum_i \partial_{x_i} \theta_i$$

and the degree -1 (with respect to the θ-degree) k-linear operator

$$(4.8) \qquad H = [d_K, \nabla]^{-1} \nabla \,.$$

We write ∂ for the differential on $\mathrm{End}_R(k^{\mathrm{stab}})$, and $\sigma : k \longrightarrow K$ for the map which sends $\lambda \in k$ to $\lambda \cdot 1 \in K$, and define k-linear maps

$$(4.9) \qquad \sigma_\infty = \sum_{m \geq 0} (-1)^m (H\partial)^m \sigma \,,$$

$$(4.10) \qquad H_\infty = \sum_{m \geq 0} (-1)^m (H\partial)^m H \,.$$

Here H_∞ is an odd operator on $K \otimes_R \mathrm{End}_R(k^{\mathrm{stab}})$. These maps satisfy:

$$(4.11) \qquad \pi \sigma_\infty = 1 \,,$$

$$(4.12) \qquad \sigma_\infty \pi = 1 - [d_K + \partial, H_\infty] \,.$$

By inspection of (4.4), we therefore have the following:

Proposition 4.2. *There is a diagram of morphisms of $\mathbb{Z}_2$-graded k-complexes*

(4.13)
$$S \otimes \mathscr{A}_W \underset{\Phi^{-1}}{\overset{\Phi}{\rightleftarrows}} \mathscr{E}$$

where $\Phi = \pi \exp(-\delta), \Phi^{-1} = \exp(\delta)\sigma_\infty$ and $\widehat{H} = \exp(\delta)H_\infty \exp(-\delta)$ satisfy

(4.14)
$$\Phi\Phi^{-1} = 1\,, \qquad \Phi^{-1}\Phi = 1 - [\partial, \widehat{H}]\,.$$

Thus, Φ and Φ^{-1} are mutually inverse homotopy equivalences over k.

The endomorphism algebra $C = \mathrm{End}_k(S)$ is a Clifford algebra generated by the contraction θ_i^* and wedge product θ_i operators. It is observed in [2] that (4.13) is an isomorphism of Clifford representations (in the homotopy category) when the complex $\mathscr{E}$ is equipped with the action induced by Atiyah classes:

Definition 4.3. The Atiyah classes of $\mathscr{A}_W$ are the R-linear odd operators

$$\mathrm{At}_i = [\partial, \partial_{x_i}] : \mathrm{End}_R(k^{\mathrm{stab}}) \longrightarrow \mathrm{End}_R(k^{\mathrm{stab}})\,,$$

Lemma 4.4. *We have*

$$\mathrm{At}_i = -[\psi_i^*, -] - \sum_{q=1}^{n} \partial_{x_i}(W^q)[\psi_q, -]\,.$$

Proof. By direct calculation:

$$
\begin{aligned}
\mathrm{At}_i &= \left[[d_{k^{\mathrm{stab}}}, -], \partial_{x_i}\right] \\
&= \sum_q \left[x_q[\psi_q^*, -], \partial_{x_i}\right] + \sum_q \left[W^q[\psi_q, -], \partial_{x_i}\right] \\
&= -\sum_q \partial_{x_i}(x_q)[\psi_q^*, -] - \sum_q \partial_{x_i}(W^q)[\psi_q, -] \\
&= -[\psi_i^*, -] - \sum_q \partial_{x_i}(W^q)[\psi_q, -]\,.
\end{aligned}
$$

$\square$

5   The minimal model

We define the even operator $\delta_i = \psi_i\theta_i^*$ on the underlying module (4.5) of $S \otimes \mathscr{A}_W$, so that $\delta = \sum_i \delta_i$. Since $[\delta_i, \delta_j] = 0$ for all i, j we have

(5.1)
$$\exp(\pm\delta) = \exp(\pm\delta_1)\cdots\exp(\pm\delta_n)\,.$$

Lemma 5.1. *For $1 \leq i \leq n$ there is a commutative diagram*

(5.2)
$$
\begin{array}{ccc}
(S \otimes \mathscr{A}_W) \otimes (S \otimes \mathscr{A}_W) & \xrightarrow{\ m_2\ } & S \otimes \mathscr{A}_W \\
{\scriptstyle \delta_i \otimes 1 + 1 \otimes \delta_i + [\psi_i, -] \otimes \theta_i^*} \downarrow & & \downarrow {\scriptstyle \delta_i} \\
(S \otimes \mathscr{A}_W) \otimes (S \otimes \mathscr{A}_W) & \xrightarrow{\ m_2\ } & S \otimes \mathscr{A}_W
\end{array}
$$

Definition 5.2. We define

(5.3)
$$
\Xi_i = [\psi_i, -] \otimes \theta_i^* : S \otimes \mathscr{A}_W \longrightarrow S \otimes \mathscr{A}_W \, .
$$

and

(5.4)
$$
\Xi = \sum_i \Xi_i \, .
$$

Proposition 5.3. *There is a commutative diagram*

(5.5)
$$
\begin{array}{ccc}
(S \otimes \mathscr{A}_W) \otimes (S \otimes \mathscr{A}_W) & \xrightarrow{\ m_2\ } & S \otimes \mathscr{A}_W \\
{\scriptstyle \exp(-\delta) \otimes \exp(-\delta)} \downarrow & & \\
(S \otimes \mathscr{A}_W) \otimes (S \otimes \mathscr{A}_W) & & \downarrow {\scriptstyle \exp(-\delta)} \\
{\scriptstyle \exp(-\Xi)} \downarrow & & \\
(S \otimes \mathscr{A}_W) \otimes (S \otimes \mathscr{A}_W) & \xrightarrow{\ m_2\ } & S \otimes \mathscr{A}_W
\end{array}
$$

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