A_∞-minimal models of matrix factorisations

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Abstract

We study A_∞-minimal models of the differential graded algebras obtained from endomorphisms of generators of triangulated categories of matrix factorisations, giving explicit higher products for the cases of simple singularities.

1 Introduction

Let k be a commutative $\mathbb{Q}$-algebra and $W \in k[x_1, \ldots, x_n]$ a polynomial which is a potential over k in the sense of [3, §2.2]. For example, $k = \mathbb{C}$ and W has isolated critical points. The $\mathbb{Z}_2$-graded DG-category $\mathcal{A} = \mathrm{mf}(R, W)$ has for its objects finite-rank matrix factorisations of W over $R = k[x_1, \ldots, x_n]$. In this paper we study the minimal model problem for $\mathcal{A}$, the aim being to produce models that are finitely-generated and projective over k.

When k is a field, the standard minimal model theorem [?] constructs the structure of an A_∞-category on the cohomology $\mathcal{B} = H^*\mathcal{A}$ in such a way that $\mathcal{B}$ is quasi-isomorphic to $\mathcal{A}$. Moreover, this A_∞-category has finite-dimensional Hom-spaces and is therefore a good finite model of $\mathcal{A}$. The problem in this case is to have sufficient control over the inputs to the minimal model theorem that the A_∞-products on $H^*\mathcal{A}(X, X)$ can be reasonably calculated, for a given matrix factorisation X. In order to understand deformations of matrix factorisations [?, ?, ?] it is important to do this for generic X, but the special case where $X = k^{\mathrm{stab}}$, the generator of $\mathcal{A}$ studied by Dyckerhoff in the case of a single isolated singularity at the origin [Dyc11], is of particular interest.

Since one of our goals is to understand how matrix factorisation categories vary along unfoldings of singularities, we also need to consider the case where k is not a field. For example take $k = \mathbb{C}[t]$ and $R = \mathbb{C}[x_1, \ldots, x_n, t]$ so that $W = W_t(x)$ is a potential with parameter. In this case we want an A_∞-category $\mathcal{B}$ quasi-isomorphic to $\mathcal{A}$ over k, with $\mathcal{B}(a, b)$ a finitely-generated projective k-module for each pair of objects a, b.

These two desiderata, understanding the A_∞-products on $H^*\mathcal{A}(X, X)$ for generic X, and the case where k is not a field, lead us naturally to a slightly non-standard use of the minimal model theorem. Following the ideas developed in [?, ?, ?] we prove that, writing $\iota: R \longrightarrow R/(\partial_{x_1}W, \ldots, \partial_{x_n}W)$ for the quotient, there is a k-linear homotopy retract

$$(1.1) \qquad \iota^*\mathcal{A} \underset{\longleftarrow}{\overset{\longrightarrow}{\rule{2cm}{0pt}}} \mathcal{A} \otimes_k \bigwedge \left(k^{\oplus n}[1] \right).$$

Suppose $W \in \mathfrak{m}^2$ where $\mathfrak{m} = (x_1, \ldots, x_n)$ and choose a presentation $W = x_1 W^1 + \cdots + x_n W^n$ with $W^i \in \mathfrak{m}$ and consider the following pair

$$(1.2) \qquad k^{\mathrm{stab}} = \left(k[x] \otimes_k \bigwedge (k\psi_1 \oplus \cdots \oplus k\psi_n), \ d_{k^{\mathrm{stab}}} = \sum_{i=1}^n x_i \psi_i^* + \sum_{i=1}^n W^i \psi_i \right)$$

where ψ_i^* and ψ_i act respectively by contraction and wedge product on the exterior algebra underlying k^{stab}. It is easy to see that $d_{k^{\mathrm{stab}}}^2 = W$, so k^{stab} is a matrix factorisation. When k is a field and W has an isolated critical point at the origin, k^{stab} is the representative in the homotopy category of matrix factorisations of the structure sheaf of the singular point at the origin, and was first studied by Dyckerhoff [Dyc11].

The purpose of this paper is to show how to calculate the A_∞-minimal model of the $\mathbb{Z}_2$-graded differential graded endomorphism algebra of this matrix factorisation

$$(1.3) \qquad \left(\mathrm{End}_{k[x]}(k^{\mathrm{stab}}), \ \partial = [d_{k^{\mathrm{stab}}}, -] \right).$$

The differential here is (throughout all commutators are graded commutators)

$$\partial = [d_{k^{\mathrm{stab}}}, -] = \left[\sum_i x_i \psi_i^* + \sum_i W^i \psi_i, - \right]$$
$$= \sum_i x_i [\psi_i^*, -] + \sum_i W^i [\psi_i, -].$$

The minimal model is a finite-dimensional $\mathbb{Z}_2$-graded vector space $\mathscr{M}_W$ with a family

$$\left\{ r_q : \left(\mathscr{M}_W[1] \right)^{\otimes q} \longrightarrow \mathscr{M}_W[1] \right\}_{q \geq 2}$$

of odd k-linear maps satisfying the forward suspended A_∞-constraints [1], and having the property that there is an A_∞-quasi-isomorphism $\mathscr{M}_W \longrightarrow \mathrm{End}_{k[x]}(k^{\mathrm{stab}})$.

2 Background

2.1 A-infinity algebras

The *tilde grading* is defined by $\widetilde{x} = |x| - 1$.

Throughout $\otimes = \otimes_k$.

3 Perturbation and Koszul complexes

Let k be a characteristic zero field, $R = k[x_1, \ldots, x_n]$ and let $W \in \mathfrak{m}^2$ be a potential with chosen decomposition $W = \sum_{i=1}^n x_i W^i$. To apply the A_∞-minimal model construction to the DG-algebra $\mathscr{A}_W$ from (1.3) we would usually begin by finding a k-linear homotopy

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retract of this complex onto its cohomology. However, for our purposes it is better to first enlarge the DG-algebra to $S \otimes \mathscr{A}_W$, where S is the $\mathbb{Z}_2$-graded vector space

$$(3.1) \qquad S = \bigwedge(k\theta_1 \oplus \cdots \oplus k\theta_n)$$

with grading $|\theta_i| = 1$. We make the tensor product $S \otimes \mathscr{A}_W$ into a DG-algebra in the usual way, giving the exterior algebra S the zero differential.

Definition 3.1. We define the finite-dimensional $\mathbb{Z}_2$-graded vector space

$$(3.2) \qquad \mathscr{E} = R/\mathfrak{m} \otimes \mathrm{End}_R(k^{\mathrm{stab}}) \cong \mathrm{End}_k \left(\bigwedge(k\psi_1 \oplus \cdots \oplus k\psi_n) \right)$$

where $\mathfrak{m} = (x_1, \ldots, x_n)$.

We will show that $S \otimes \mathscr{A}_W$ is homotopy equivalent to $\mathscr{E}$ viewed as a complex with zero differential (observe that $\mathscr{E}$ does not depend on W). The corresponding homotopy retract (see Proposition 3.3 below) is our starting point for the minimal model construction. For this we use the Koszul complex of $x_1, \ldots, x_n$,

$$(3.3) \qquad K = \left(R \otimes \bigwedge(k\theta_1 \oplus \cdots \oplus k\theta_n), \ d_K = \sum_i x_i \theta_i^* \right)$$

Theorem 3.2. *There is a diagram of $\mathbb{Z}_2$-graded complexes and homotopy equivalences over k*

$$(3.4) \qquad S \otimes \mathscr{A}_W \xrightarrow[\exp(-\delta)]{\exp(-\delta)} K \otimes_R \mathscr{A}_W \xrightarrow[\sigma_\infty]{\pi} \mathscr{E} \ .$$

Proof. See [2]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The notation is as follows:

- The complexes involved here are

$$\left(S \otimes \mathrm{End}_R(k^{\mathrm{stab}}), \ \partial \right)$$
$$\left(K \otimes_R \mathrm{End}_R(k^{\mathrm{stab}}), \ d_K + \partial \right)$$
$$\left(\mathscr{E}, 0 \right).$$

- The operator $\psi_j = \psi_j \wedge -$ on k^{stab} satisfies

$$\left[\psi_j, d_{k^{\mathrm{stab}}} \right] = \left[\psi_j, \sum_i x_i \psi_i^* \right] = x_j \cdot 1,$$

where $[-, -]$ always denotes the graded commutator. That is, ψ_j is a homotopy for the action of x_j on k^{stab}. It is then easy to check that the odd operator $\alpha \mapsto \psi_j \circ \alpha$ on $\mathrm{End}_R(k^{\mathrm{stab}})$ is also a homotopy for x_j, and where it will not cause confusion we will also write ψ_j for this operator of post-composition.

- Both $S \otimes \mathscr{A}_W$ and $K \otimes_R \mathscr{A}_W$ have the same underlying $\mathbb{Z}_2$-graded k-module, namely

$$(3.5) \qquad R \otimes \bigwedge(k\theta_1 \oplus \cdots \oplus k\theta_n) \otimes \mathrm{End}_k\left(\bigwedge(k\psi_1 \oplus \cdots \oplus k\psi_n)\right).$$

On this space we define the even operator

$$(3.6) \qquad \delta = \sum_{i=1}^{n} \psi_i \theta_i^*,$$

where ψ_i is the operator on $\mathrm{End}_R(k^{\mathrm{stab}})$ discussed above, given by $\alpha \mapsto (\psi_i \wedge -) \circ \alpha$, and θ_i^* acts by contraction on S. It is easy to check that $\exp(\delta), \exp(-\delta)$ intertwines the differentials ∂ and $d_K + \partial$ and therefore gives an isomorphism between the first two complexes in (3.4) [2, Proposition 4.11].

- The morphism of complexes $\pi : K \longrightarrow R/\mathfrak{m}$ is defined by composing the projection of K onto the submodule $R \cdot 1$ of θ-degree zero forms, with the quotient $R \longrightarrow R/\mathfrak{m}$. We also write π for the result of tensoring this morphism with $\mathrm{End}_R(k^{\mathrm{stab}})$ to obtain

$$K \otimes_R \mathrm{End}_R(k^{\mathrm{stab}}) \xrightarrow{\ \pi \otimes 1\ } R/\mathfrak{m} \otimes_R \mathrm{End}_R(k^{\mathrm{stab}}) = \underline{\mathrm{End}}(k^{\mathrm{stab}}).$$

This is a morphism of complexes, because the differential on $\mathrm{End}_R(k^{\mathrm{stab}})$ vanishes on the quotient $\underline{\mathrm{End}}(k^{\mathrm{stab}})$ by the hypothesis that $W \in \mathfrak{m}^2$.

- The k-linear homotopy inverse σ_∞ to π in (3.4) is defined in terms of the following ingredients. The first is the k-linear connection (viewing θ_i as a 1-form)

$$(3.7) \qquad \Delta : K \longrightarrow K, \qquad \Delta = \sum_i \partial_{x_i} \theta_i$$

and the degree -1 (with respect to the θ-degree) k-linear operator

$$(3.8) \qquad H = [d_K, \Delta]^{-1} \Delta.$$

We write ∂ for the differential on $\mathrm{End}_R(k^{\mathrm{stab}})$, and $\sigma : k \longrightarrow K$ for the map which sends $\lambda \in k$ to $\lambda \cdot 1 \in K$, and define k-linear maps

$$(3.9) \qquad \sigma_\infty = \sum_{m \geq 0} (-1)^m (H\partial)^m \sigma,$$

$$(3.10) \qquad H_\infty = \sum_{m \geq 0} (-1)^m (H\partial)^m H.$$

Here H_∞ is an odd operator on $K \otimes_R \mathrm{End}_R(k^{\mathrm{stab}})$. These maps satisfy:

$$(3.11) \qquad \pi \sigma_\infty = 1,$$

$$(3.12) \qquad \sigma_\infty \pi = 1 - [d_K + \partial, H_\infty].$$

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By inspection of (3.4), we therefore have the following:

Proposition 3.3. *There is a diagram of morphisms of $\mathbb{Z}_2$-graded k-complexes*

$$(3.13) \qquad\qquad S \otimes \mathscr{A}_W \xrightarrow{\;\;\Phi\;\;} \mathscr{E}$$
$$\xleftarrow[\Phi^{-1}]{}$$

where $\Phi = \pi \exp(-\delta), \Phi^{-1} = \exp(\delta)\sigma_\infty$ and $\widehat{H} = \exp(\delta)H_\infty \exp(-\delta)$ satisfy

$$(3.14) \qquad\qquad \Phi\Phi^{-1} = 1\,, \qquad \Phi^{-1}\Phi = 1 - [\partial, \widehat{H}]\,.$$

Thus, Φ and Φ^{-1} are mutually inverse homotopy equivalences over k.

The endomorphism algebra $C = \mathrm{End}_k(S)$ is a Clifford algebra generated by the contraction θ_i^* and wedge product θ_i operators. It is observed in [2] that (3.13) is an isomorphism of Clifford representations (in the homotopy category) when the complex $\mathscr{E}$ is equipped with the action induced by Atiyah classes:

Definition 3.4. The Atiyah classes of $\mathscr{A}_W$ are the R-linear odd operators

$$\mathrm{At}_i = [\partial, \partial_{x_i}] : \mathrm{End}_R(k^{\mathrm{stab}}) \longrightarrow \mathrm{End}_R(k^{\mathrm{stab}})\,,$$

Lemma 3.5. *We have*

$$\mathrm{At}_i = -[\psi_i^*, -] - \sum_{q=1}^{n} \partial_{x_i}(W^q)[\psi_q, -]\,.$$

Proof. By direct calculation:

$$\begin{aligned}
\mathrm{At}_i &= \big[[d_{k^{\mathrm{stab}}}, -], \partial_{x_i}\big] \\
&= \sum_q \big[x_q[\psi_q^*, -], \partial_{x_i}\big] + \sum_q \big[W^q[\psi_q, -], \partial_{x_i}\big] \\
&= -\sum_q \partial_{x_i}(x_q)[\psi_q^*, -] - \sum_q \partial_{x_i}(W^q)[\psi_q, -] \\
&= -[\psi_i^*, -] - \sum_q \partial_{x_i}(W^q)[\psi_q, -]\,.
\end{aligned}$$

$\square$

4 Trees and decorations

Example 4.1.

5   Feynman diagrams

The integer n is the number of variables in the ambient ring $R = k[x_1, \ldots, x_n]$.

Definition 5.1. Consider the $\mathbb{Z}_2$-graded algebra defined by the tensor product

$$\mathscr{H} = R \otimes_k \bigwedge \left(\oplus_{i=1}^n k\theta_i \right) \otimes_k \operatorname{End}_k \left(\bigwedge \left(\oplus_{i=1}^n k\psi_i \right) \right)$$

with degrees $|\theta_i| = |\psi_i| = 1$. On this space we have the homogeneous operators

$$x_i, \partial_i = \partial_{x_i}, \theta_i, \theta_i^*, \left[\psi_i, - \right].$$

Here ψ_i denotes the operator $\psi_i \wedge (-)$ on $\bigwedge \left(\oplus_{i=1}^n k\psi_i \right)$ and $[\psi_i, -]$ the graded commutator with this operator, defined on a homogeneous operator β by

$$\left[\psi_i, \beta \right] = \psi_i \circ \beta - (-1)^{|\beta|} \beta \circ \psi_i.$$

We define the $\mathbb{Z}_2$-graded k-module

$$\mathscr{B} = \bigwedge \left(\oplus_{i=1}^n k\psi_i^* \right)$$

which we view as a submodule

$$\mathscr{B} \subset \operatorname{End}_k \left(\bigwedge \left(\oplus_{i=1}^n k\psi_i \right) \right)$$

by identifying $\psi_i^* \in \mathscr{B}$ with the operation of contraction $\psi_i^* \lrcorner (-)$ on the exterior algebra. In this way we may also identify $\mathscr{B}$ with a submodule of $\mathscr{H}$, and we write $\iota : \mathscr{B} \longrightarrow \mathscr{H}$ for the inclusion. Note that the operator $[\psi_j, -]$ on $\mathscr{H}$ defined above acts on the submodule $\mathscr{B}$ as contraction with $\psi_j = (\psi_j^*)^*$, that is,

$$(5.1) \qquad \left[\psi_j, \iota \left(\psi_{i_1}^* \wedge \cdots \wedge \psi_{i_r}^* \right) \right] = \sum_{l=1}^r (-1)^{l-1} \delta_{j,i_l} \iota \left(\psi_{i_1}^* \wedge \cdots \wedge \widehat{\psi_{i_l}^*} \wedge \cdots \wedge \psi_{i_r}^* \right).$$

We write $p : \mathscr{H} \longrightarrow \mathscr{B}$ for the natural projection (todo).

We view $\mathscr{H}$ as the tensor product of the bosonic Fock space R with creation and annihilation operators x_i, ∂_i, fermionic Fock space $\wedge(\oplus_{i=1}^n k\theta_i)$ with creation and annihilation operators θ_i, θ_i^* and the third factor, of which the calculations of the A_∞-structure only involve the submodule $\mathscr{B} = \wedge(\oplus_{i=1}^n k\psi_i^*)$ which is another fermionic Fock space. The inputs to the Feynman diagrams are tensor products of states in $\mathscr{B}$, that is, wedge products of ψ_i^*'s, on which the $[\psi_i, -]$ act as annihilation operators.

The linear maps in (5.6), (5.7), (5.8) below which determine the A_∞ products ρ_q on $\mathscr{B}$ will be defined in terms of these creation and annihilation operators on $\mathscr{H}$, and may be interpreted as particle interactions in the usual way. The interactions that appear are the following ones, depicted on the left as Feynman diagrams and on the right as operators. We call these respectively *A-, B- and C-type interactions*:

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A

$$\theta_a x^{\gamma - e_a} \big[\psi_j, - \big] \in \mathrm{End}_k(\mathscr{H})$$

$$1 \le a, j \le n \,, \gamma \in \mathbb{N}^n$$

(5.2)

B

$$\theta_t \partial_t \in \mathrm{End}_k(\mathscr{H})$$

$$1 \le t \le n$$

(5.3)

C

$$m_2 \big(\big[\psi_j, - \big] \otimes \theta_j^* \big) \in \mathrm{Hom}_k(\mathscr{H}^{\otimes 2}, \mathscr{H})$$

$$1 \le j \le n$$

(5.4)

Here m_2 is the multiplication in $\mathscr{H}$ and, given $\gamma \in \mathbb{N}^n$, we write $x^\gamma = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$, with $e_i \in \mathbb{N}^n$ the unit vector with $x^{e_i} = x_i$. By convention if $\gamma_a = 0$ then $x^{\gamma - e_a} = 0$. Finally, note that in the Feynman diagrams, bosonic particles are depicted with dotted lines, and fermionic particles with solid lines.

These interactions "take place" at vertices of the tree $A(T)$ for some $T \in \mathcal{T}_q$, and come with the following constraints, which will be formalised below in terms of combinatorial data called *configurations*. We write $f(\gamma)$ for the coefficient of x^γ in a polynomial $f \in R$ and recall the chosen decomposition $W = \sum_j x_j W^j$.

- A-type interactions only occur at input vertices and internal edges of T.

- B-type interactions only occur at internal edges of T, and moreover there is *precisely one* at each internal edge.

- C-type interactions only occur at internal vertices of T.

- The A-type interaction with indices a, j, γ appears with the coefficient

(5.5)
$$- \frac{\gamma_a}{|\gamma|} W^j(\gamma)$$

where $|\gamma| = \sum_i \gamma_i$. Each B- and C-type interaction appears with coefficient (-1).

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As we will see, the coefficient (5.5) is the only way that W enters the rules, and thus the definition of the A_∞-products on $\mathscr{B}$. The precise definition of the Feynman rules will be given in Definition 5.3 below, but first we give an example of a tree with interactions inserted, to illustrate the various ingredients.

Example 5.2. Consider the tree $T \in \mathcal{T}_3$ whose augmented tree $A(T)$ is depicted below, together with labels for its vertices, where we label by e the vertex corresponding to the internal edge of T:
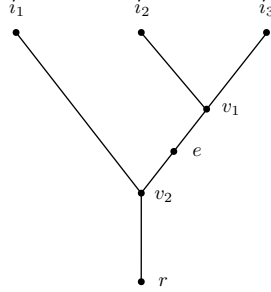
Set $W = x^3 \in \mathbb{C}[x]$ so that $W^1 = x^2$ and the only A-type interaction has $a = 1, j = 1, \gamma = 2$ and coefficient 1, as in the diagram (we write $\theta = \theta_1, \psi = \psi_1$)
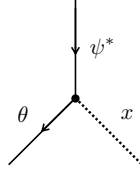
Consider the input state $\Psi = \psi^* \otimes \psi^* \otimes \psi^*$ in $\mathscr{B}^{\otimes 3}$. One possible pattern of interactions (called a configuration, below) has a single A-type interaction at the input vertex i_3, a B-type interaction at e, and two C-type interactions at v_1, v_2, as shown in
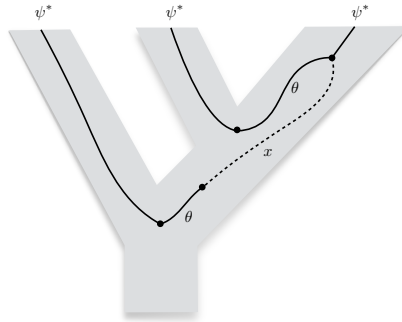
This "topological" Feynman diagram computes the value of the linear map

$$(-1)^3 p \circ m_2\big([\psi, -] \otimes \theta^*\big) \circ \big(\iota \otimes \theta \partial_x \circ m_2([\psi, -] \otimes \theta^*)\big(\iota \otimes \theta x[\psi, -]\iota\big)\big)$$

on the input state Ψ, which may be checked is $-1 \in \mathscr{B}$.

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Now we give a precise definition of the Feynman rules. Let $T \in \mathcal{T}_q$ be a valid plane tree with q inputs and $e_i(T)$ internal edges, and let $A(T)$ be the augmented plane tree.

Definition 5.3. A *configuration* C of a valid plane tree T consists of the following data, for each non-root vertex v of $A(T)$:

- An integer $m(v) \geq 0$.

- A subset $J(v) \subseteq \{1, \ldots, n\}$ with $|J(v)| = m(v)$.

- If v is an input, or comes from an internal edge of T, a pair

$$(a_j(v), \gamma_j(v)) \in \{1, \ldots, n\} \times \mathbb{N}^n$$

for each $j \in J(v)$.

- If v comes from an internal edge of T, an integer $t(v) \in \{1, \ldots, n\}$.

Let $\mathrm{Con}(T)$ denote the set of all configurations.

Remark 5.4. The integer $m(v)$ counts *how many* interactions take place at v. The set $J(v)$ consists of all j-indices appearing in interactions at v. The interactions of A-type (at inputs and internal edges) are defined by indices $a_j(v), \gamma_j(v)$ while at each internal edge e of T, $t(e)$ is the index of the C-type interaction.

Definition 5.5. Given a tree $T \in \mathcal{T}_q$ and configuration $C \in \mathrm{Con}(T)$ we define a decoration $D_{T,C}$ of $A(T)$ by the assignment of the modules

- $\mathscr{B}$ to the input at each non-root leaf and the output at the root, and

- $\mathscr{H}$ to each edge.

To each vertex v of $A(T)$ we associate an operator ϕ_v as follows, writing $m, J, \{(a_j, \gamma_j)\}_{j \in J}, t$ for the data associated to v by the configuration C:

- if v is an input, then ϕ_v is the linear map $\mathscr{B} \longrightarrow \mathscr{H}$ given by

(5.6) $$\phi_v = (-1)^m \prod_{j \in J} \left\{ \frac{(\gamma_j)_a}{|\gamma_j|} W^j(\gamma_j) \theta_{a_j} x^{\gamma_j - e_{a_j}} \left[\psi_j, - \right] \right\} \circ \iota \,.$$

Note that the operator under the product is even, so the order is irrelevant.

- if v comes from an internal edge of T, then

(5.7) $$\phi_v = (-1)^m \prod_{j \in J} \left\{ \frac{(\gamma_j)_a}{|\gamma_j|} W^j(\gamma_j) \theta_{a_j} x^{\gamma_j - e_{a_j}} \left[\psi_j, - \right] \right\} \circ \theta_t \partial_t \,.$$

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- if v comes from an internal vertex of T, then

(5.8)
$$\phi_v = (-1)^m m_2 \circ \prod_{j \in J} \left\{ [\psi_j, -] \otimes \theta_j^* \right\}$$

which is a map $\mathscr{H}^{\otimes 2} \longrightarrow \mathscr{H}$. Here m_2 denotes the product on $\mathscr{H}$.

- if $v = r$ is the root, $\phi_v = p : \mathscr{H} \longrightarrow \mathscr{B}$ is the projection defined in (**??**).

Finally, there is a symmetry factor.

Definition 5.6. Given a sequence $a_1, \ldots, a_m \geq 1$ of integers and $a \geq 0$ we define

$$\tau(a, a_1, \ldots, a_m) = a_1 \cdots a_m \sum_{\sigma \in \mathfrak{S}_m} \prod_{i=0}^{m-1} \frac{1}{a + \sum_{j=m-i}^{m} a_{\sigma(j)}}$$

$$= a_1 \cdots a_m \sum_{\sigma \in \mathfrak{S}_m} \frac{1}{a + a_{\sigma(m)}} \cdots \frac{1}{a + a_{\sigma(1)} + \cdots + a_{\sigma(m)}} .$$

Definition 5.7. Given $T \in \mathcal{T}_q, C \in \mathrm{Con}(T)$ and an internal edge e of T we define

$$\omega(C, e) = \sum_{e' < e} \sum_{j \in J(e')} |\gamma_j(e')| - \sum_{z < e} m(z)$$

where the sum is over all internal edges and input vertices e' above e in $A(T)$ i.e. for each e is on the unique path from e' to the root, and z ranges over internal vertices.

We define in addition a symmetry factor

$$S(C, e) = \frac{1}{\omega(C, e)} \tau\big(\omega(C, e), \{|\gamma_j(e)|\}_{j \in J(e)}\big) \in \mathbb{Q},$$

$$S(C) = \prod_{e \in E_i(T)} S(C, e)$$

where $E_i(T)$ is the set of internal edges e of T.

Definition 5.8. Given $T \in \mathcal{T}_q$ and $C \in \mathrm{Con}(T)$ we define the k-linear operator

$$\mathcal{O}(T, C) : \mathscr{B}^{\otimes q} \longrightarrow \mathscr{B}$$

to be the denotation $\mathcal{O}(T, C) = \langle D_{T,C} \rangle$, defined without Koszul signs.

Definition 5.9. We define $\rho_q : \mathscr{B}[1]^{\otimes q} \longrightarrow \mathscr{B}[1]$ by

$$\rho_q(\Lambda_1 \otimes \cdots \otimes \Lambda_q) = \sum_{T \in \mathcal{T}_q} \sum_{C \in \mathrm{Con}(T)} (-1)^{Q(T, \underline{\Lambda})} S(C) \cdot \mathcal{O}(T, C)(\Lambda_q \otimes \cdots \otimes \Lambda_1) .$$

where

$$Q(T, \underline{\Lambda}) = sign .$$

Theorem 5.10. *The operators ρ_q satisfy the forward suspended A_∞-constraints, so that $(\mathscr{B}, \{\rho_q\}_{q \geq 2})$ is an A_∞-algebra, and it is quasi-isomorphic to $\mathrm{End}_k(k^{\mathrm{stab}})$.*

6 The minimal model

We define the even operator $\delta_i = \psi_i \theta_i^*$ on the underlying module (3.5) of $S \otimes \mathscr{A}_W$, so that $\delta = \sum_i \delta_i$. Since $[\delta_i, \delta_j] = 0$ for all i, j we have

$$\text{(6.1)} \qquad \exp(\pm\delta) = \exp(\pm\delta_1) \cdots \exp(\pm\delta_n) \,.$$

Lemma 6.1. *For $1 \leq i \leq n$ there is a commutative diagram*

$$\text{(6.2)}$$

$$
\begin{array}{ccc}
(S \otimes \mathscr{A}_W) \otimes (S \otimes \mathscr{A}_W) & \xrightarrow{\;\; m_2 \;\;} & S \otimes \mathscr{A}_W \\
{\scriptstyle \delta_i \otimes 1 + 1 \otimes \delta_i + [\psi_i, -] \otimes \theta_i^*} \Big\downarrow & & \Big\downarrow {\scriptstyle \delta_i} \\
(S \otimes \mathscr{A}_W) \otimes (S \otimes \mathscr{A}_W) & \xrightarrow{\;\; m_2 \;\;} & S \otimes \mathscr{A}_W
\end{array}
$$

Definition 6.2. We define

$$\text{(6.3)} \qquad \Xi_i = [\psi_i, -] \otimes \theta_i^* : S \otimes \mathscr{A}_W \longrightarrow S \otimes \mathscr{A}_W \,.$$

and

$$\text{(6.4)} \qquad \Xi = \sum_i \Xi_i \,.$$

Proposition 6.3. *There is a commutative diagram*

$$\text{(6.5)}$$

$$
\begin{array}{ccc}
(S \otimes \mathscr{A}_W) \otimes (S \otimes \mathscr{A}_W) & \xrightarrow{\;\; m_2 \;\;} & S \otimes \mathscr{A}_W \\
{\scriptstyle \exp(-\delta) \otimes \exp(-\delta)} \Big\downarrow & & \\
(S \otimes \mathscr{A}_W) \otimes (S \otimes \mathscr{A}_W) & & \Big\downarrow {\scriptstyle \exp(-\delta)} \\
{\scriptstyle \exp(-\Xi)} \Big\downarrow & & \\
(S \otimes \mathscr{A}_W) \otimes (S \otimes \mathscr{A}_W) & \xrightarrow{\;\; m_2 \;\;} & S \otimes \mathscr{A}_W
\end{array}
$$

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