

# Linear logic and recurrent neural networks

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## 1 Introduction

An old problem in artificial intelligence is *program induction*: given a set of input-output pairs, the problem is to produce a program which generalises the given pairs. There is a large literature on this problem [?, ?] but it has received renewed attention in the past few years due to the advances in deep learning [?, ?, ?]. The general theme of much of this recent work has been to use an RNN or LSTM controllers coupled to additional computational elements, such as memory or a library of special functions.

Our approach is to augment an RNN controller with a space of functional programs based on the polynomial semantics of intuitionistic linear logic [?]. It is clear in principle that linear logic is well-suited for coupling to neural networks, and the aim of this paper is to demonstrate that in the simplest nontrivial case this coupling is actually of some practical value. Conceptually, one of the fundamental contributions of linear logic is that it demonstrates how to represent a general class of programs in terms of matrices, and multiplication of matrices.

## 2 Architecture

The architecture is a modified RNN. As usual we use “weight” as a synonym for variable, or more precisely, the variables which we will vary during gradient descent. At time  $t \geq 1$  we denote the hidden state by  $h^{(t)}$  and the input by  $x^{(t)}$ . We denote by  $\sigma$  the function

$$\begin{aligned}\sigma : \mathbb{R}^k &\longrightarrow \mathbb{R}^k \\ \sigma(X)_i &= \frac{1}{2}(X_i + |X_i|).\end{aligned}$$

An RNN is defined by its *evolution equation* which expresses  $h^{(t+1)}$  as a function of  $h^{(t)}, x^{(t)}$  and its *output equation* which gives the output  $y^{(t)}$  as a function of  $h^{(t)}$ .

**Definition 2.1.** A standard Elman-style RNN [1] is defined by weight matrices  $H, U, B$  and the evolution equation

$$(2.1) \quad h^{(t+1)} = \sigma(Hh^{(t)} + Ux^{(t+1)} + B)$$

where  $h^{(0)}$  is some specified initial state.

Our augmented RNN adds a new term to the evolution equation, in the style of recent multiplicative RNNs [5, 7, 6]. The key intuition is that the evolution equation maps inputs to transformations of the internal state. To explain, suppose the input vector space is  $\mathcal{I}$  and the hidden state space is  $\mathcal{H}$ , so that  $x^{(t)} \in \mathcal{I}$  and  $h^{(t)} \in \mathcal{H}$  for all  $t$ . If our sequences are of a fixed length  $T$  then the RNN computes a function of input sequences

$$\mathbf{x} = (x^{(1)}, \dots, x^{(T)}) \in \mathcal{I}^{\oplus T}.$$

The value of the RNN on  $\mathbf{x}$  is computed by applying (2.1) for  $0 \leq t \leq T - 1$  to compute the vector  $h^{(T)}$ , and then feeding this into a second decoder neural network to output the final value [8]. For the moment we do not specify the decoder. The point is that the evolution equation implicitly determines a piecewise-linear function

$$\begin{aligned} \mathcal{I} &\longrightarrow \text{Func}(\mathcal{H}, \mathcal{H}), \\ x &\mapsto \sigma(H(-) + Ux + B), \end{aligned}$$

which specifies how we may view inputs as transforming the internal state of the RNN. Modifications of the RNN architecture can be viewed as allowing more complicated (for example, piecewise-polynomial of higher degree) transformations of the internal state.

For example, compositional matrix-space models in NLP [6] add a new term built from an auxiliary learned linear mapping of the input space  $\mathcal{I}$  into the space of linear operators on the state space  $\text{End}_{\mathbb{R}}(\mathcal{H})$ . Our approach is similar, but we allow a particular class of non-linear maps  $\mathcal{I} \longrightarrow \text{End}_{\mathbb{R}}(\mathcal{H})$  selected by linear logic. The class of maps we consider are the denotations of proofs  $\pi$  in intuitionistic linear logic of type

$$(2.2) \quad \mathbf{bint}_{\alpha} = !(\alpha \multimap \alpha) \multimap (!(\alpha \multimap \alpha) \multimap (\alpha \multimap \alpha)).$$

A full explanation of linear logic is not necessary. We mention only that under the Curry-Howard correspondence a proof  $\pi$  of the above type can be thought of as a functional program taking two inputs (of type  $\alpha \multimap \alpha$ ) and giving an output of the same type.

The *denotation* of such a proof is a polynomial function

$$\llbracket \pi \rrbracket : \text{End}_{\mathbb{R}}(\mathcal{H}) \times \text{End}_{\mathbb{R}}(\mathcal{H}) \longrightarrow \text{End}_{\mathbb{R}}(\mathcal{H}).$$

where  $\text{End}_{\mathbb{R}}(V)$  denotes the space of linear operators on  $V$ .<sup>1</sup> For every binary sequence  $S \in \{0, 1\}^*$  there is a corresponding proof  $t_S$  of type  $\mathbf{bint}_{\alpha}$ , and for example the denotation of  $t_{01101}$  is the polynomial function

$$(2.3) \quad \llbracket t_{01101} \rrbracket(X, Y) = XY YXY.$$

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<sup>1</sup>To explain what we mean by a “polynomial function”, suppose  $\mathcal{H} = \mathbb{R}^r$  so that  $\text{End}_{\mathbb{R}}(\mathcal{H})$  is the space of matrices  $M_r(\mathbb{R})$ . Then for all  $1 \leq i, j \leq r$ , the denotation  $\llbracket \pi \rrbracket_{i,j}$  is a function of pairs of matrices  $X, Y \in M_r(\mathbb{R})$  which is a polynomial in the entries  $X_{ab}, Y_{cd}$ .

It is these polynomial functions that we will couple into the RNN.

Let  $\Pi$  be a finite sequence of proofs  $\Pi = (\pi_1, \dots, \pi_k)$  of  $\mathbf{bint}_\alpha$ . When we say that we allow the RNN controller to call this library of functional programs, what we actually mean is that it can predict a linear combination  $p^{(t)}$  of the polynomial programs  $\llbracket \pi_i \rrbracket$  to be applied to its internal state, as follows:

**Definition 2.2.** The *linear logic* RNN associated to  $\Pi$  has weight matrices  $H, U, B$  as in the standard RNN, together with new weight matrices  $V, Q, C$ , and an evolution equation defined in two parts:

$$(2.4) \quad p^{(t+1)} = \sigma(Qh^{(t)} + C),$$

$$(2.5) \quad h^{(t+1)} = \sigma\left(\sum_i p_i^{(t+1)} \llbracket \pi_i \rrbracket(Vx^{(t)}, Vx^{(t+1)})(h^{(t)}) + Hh^{(t)} + Ux^{(t+1)} + B\right).$$

Here  $V$  is a linear map  $\mathcal{I} \rightarrow \text{End}_{\mathbb{R}}(\mathcal{H})$  and  $p^{(t)} \in \mathbb{R}^k$  for all  $t$ .

Note that the RNN evolution equation for  $h^{(t+1)}$  involves both the current input  $x^{(t+1)}$  and the previous input  $x^{(t)}$ . We do this in order to allow the RNN to learn local transformations of an input sequence, for instance, the transformation of a sequence  $a_0a_1a_2a_3 \dots$  which swaps every pair of symbols, to get  $a_1a_0a_3a_2 \dots$  might use  $\llbracket t_{10} \rrbracket$ . More complicated transformations could be learned by adapting the equation to a function of  $x^{(i)}, \dots, x^{(t+1)}$  with a more complicated type than  $\mathbf{bint}_\alpha$ .

**Remark 2.3.** To make it explicit, suppose  $\mathcal{I} = \mathbb{R}^{n_I}$  and  $\mathcal{H} = \mathbb{R}^{n_H}$ , then  $H \in M_{n_H \times n_H}(\mathbb{R})$ ,  $U \in M_{n_H \times n_I}(\mathbb{R})$ ,  $B \in M_{n_H \times 1}(\mathbb{R})$ ,  $V \in M_{n_H^2 \times n_I}(\mathbb{R})$ ,  $Q \in M_{k \times n_H}(\mathbb{R})$ ,  $C \in \mathbb{R}^k$ . Since  $n_H^2$  may be large, we can act on a subspace by choosing a decomposition  $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$  and letting  $\rho : \mathcal{H} \rightarrow \mathcal{H}_1, \iota : \mathcal{H}_1 \rightarrow \mathcal{H}$  be respectively be the projection to and inclusion of the subspace  $\mathcal{H}_1$ . Then we replace the evolution equation by

$$(2.6) \quad h^{(t+1)} = \sigma\left(\sum_i p_i^{(t+1)} \iota \llbracket \pi_i \rrbracket(Vx^{(t)}, Vx^{(t+1)})(\rho h^{(t)}) + Hh^{(t)} + Ux^{(t+1)} + B\right).$$

with  $\llbracket \pi \rrbracket$  now a function  $\text{End}_{\mathbb{R}}(\mathcal{H}_1) \times \text{End}_{\mathbb{R}}(\mathcal{H}_1) \rightarrow \text{End}_{\mathbb{R}}(\mathcal{H}_1)$ .

If we write  $n'_H = \dim(\mathcal{H}_1)$  then the number of weights in the linear logic RNN is

$$(2.7) \quad n_H^2 + n_H n_I + n_H + (n'_H)^2 n_I + k n_H + k,$$

where  $k$  is the number of proofs we are indexing over.

## A Background on CPUs

Recall that an assembly program for an ordinary CPU looks like

```
LOAD R1, A
ADD R3, R1, R2
STORE C, R3
```

Where R1,R2,R3 stand for the first three registers of the CPU and A,B,C are numbers representing addresses in memory. Thus series of instructions will result in the CPU fetching a number from memory location A and storing it in R1, adding this number to a previously stored number in R2 with the result being stored in R3, and finally writing that register out to the memory address C. In the analogy between a CPU and a vanilla RNN we think of the number read from A as the current input  $x^{(t)}$  and the previously stored value in R2 as (part of) the internal state  $h^{(t-1)}$ .

Recent work [?, ?] on coupling memory to neural networks takes as its starting point the first of the above instructions `LOAD R1, A` and makes it “differentiable” by having the RNN controller predict at time  $t$  both the memory address A and the register R3 to write to (in this case for example, as a mask on the vector giving the internal state  $h^{(t+1)}$ ). The same differentiable interpretation can be given of the `STORE` command. This is done by adding suitable terms to the update equation (2.1).

In contrast our focus is on the third command, the `ADD`. We increase the expressive power of the update equation by allowing it to predict at time  $t$  an operation  $p^{(t)}$  (a vector specifying a point in a space of “programs”) which is to be performed on the input and internal state. This could be used with a differentiable memory, but for the sake of simplicity we do not try this. In order to preserve our analogy with CPU instructions even without `LOAD` and `STORE`, we could imagine a CPU with a command

```
ADD R3, A, R2
```

which at once reads the number from address A, adds it to the stored value in R2 and writes the result to R3. Note that without a `LOAD` instruction, the only way a value could have gotten into R2 in a previous time step is as the result of another `ADD`.

**Remark A.1.** Although the architecture takes some inspiration from normal CPUs, there is an important distinction: on a normal CPU the program is given as a series of instructions prior to the beginning of execution. In contrast, in the model we have described each command is *predicted* at runtime from the current internal state. Perhaps we can understand the process intuitively as follows: we are co-learning a part of  $H$ , call it  $H_0$ , which generates some part of the internal state  $h_0^{(1)}, h_0^{(2)}, \dots$  giving a path through the state space on which the weight matrix  $P$  picks out the right program to run at each time step. The overall algorithm is distributed amongst the weights of  $H_0$  and  $P$ .

This also suggests an alternative algorithm: we do not predict  $p^{(t)}$  at each time step, rather we have some fixed number of time steps  $T$  and matrices of weights  $p^{(1)}, \dots, p^{(T)}$  which are learned by gradient descent.

## B Old stuff

The operation to be performed is given by a sequence of vectors

$$(B.1) \quad p_i^{(t)} = \sigma(P_i h^{(t-1)}) \in \mathbb{R}^{n_P}, 1 \leq i \leq m$$

in a way that we will now explain. In outline, we think of the entries of  $p_i^{(t)}$  as telling us the coefficients of monomials in the entries of  $x^{(t)}$  and  $h^{(t-1)}$  to use in the modified update equation. For simplicity let us only consider monomials of degree 2 in what follows, since the general case is similar. We set

$$(B.2) \quad y^{(t)} = Ex^{(t)} \oplus h^{(t-1)} \in \mathbb{R}^{n_Y}$$

where  $E$  is another matrix of weights. See Example ?? for an explanation.

Let  $F$  denote a linear map  $F : M_{n_Y}(\mathbb{R}) \longrightarrow \mathbb{R}^{n_Y^2}$  which reads off the entries of a matrix into a column vector. In TensorFlow we can represent this using *reshape*. Writing  $Y = y^{(t)}$  observe that  $YY^T$  is an  $n_Y \times n_Y$  matrix with  $(i, j)$ -entry  $Y_i Y_j$ . We choose  $n_P = n_Y^2$  and compute the entry-wise multiplication

$$p_i^{(t)} \odot F(YY^T) = p_i^{(t)} \odot F(y^{(t)}(y^{(t)})^T) \in \mathbb{R}^{n_P}.$$

Finally,  $q^{(t)}$  is the column vector whose  $i$ th row is  $p_i^{(t)} \odot F(YY^T)$ , that is,

$$q^{(t)} = \begin{pmatrix} p_1^{(t)} \odot F(y^{(t)}(y^{(t)})^T) \\ \vdots \\ p_m^{(t)} \odot F(y^{(t)}(y^{(t)})^T) \end{pmatrix}.$$

In summary, we view  $x^{(t)}, h^{(t-1)}$  as respectively the differentiable analogues of **A**, **R2** and the sequence  $p_1^{(t)}, \dots, p_m^{(t)}$  as the analogue of the command **ADD**. The output of the command is the vector  $q^{(t)}$ . We incorporate this output into the update equation as follows:

$$(B.3) \quad h^{(t+1)} = \sigma(Vq^{(t)} + Hh^{(t)} + Ux^{(t+1)}).$$

Thus  $V$  is the differentiable analogue of the register **R3**. The weights are  $P_i$  from (B.1),  $E$  from (B.2) and  $V, H, U$  from (B.3). This architecture is easily generalised to polynomials of higher degree, by adding additional terms.

**Example B.1.** Let us consider how the system might reproduce the program that repeats every digit of an input binary sequence, e.g.

$$(B.4) \quad 0110 \longmapsto 00111100.$$

We take the inputs  $x \in \mathbb{R}^2$  with  $e_1 = (1, 0)$  standing for the binary digit 1 and  $e_0 = (0, 1)$  standing for 0. We suppose that the system has learned the embedding matrix  $E$  such that  $A = E(e_1)$  and  $B = E(e_0)$  are matrices in  $M_n(\mathbb{R}_{>0})$  with the property that the subgroup

they generate under multiplication is a free group on two letters. This condition just means that the map

$$\Psi : \{0, 1\}^* \longrightarrow M_n(\mathbb{R})$$

from binary sequences to matrices, defined inductively for  $s \in \{0, 1\}$  by

$$\Psi(sS) = \begin{cases} B\Psi(S) & s = 0 \\ A\Psi(S) & s = 1 \end{cases}$$

is injective. The space of matrices  $\mathcal{H} = M_n(\mathbb{R})$  is the internal state of our RNN. To extract output from the RNN we apply a series of fully-connected layers with the final internal state  $h^{(T)}$  as input, and we think of this series of layers as approximating a function  $\Psi' : M_n(\mathbb{R}) \longrightarrow \{0, 1\}^*$  with the property that  $\Psi' \circ \Psi = 1$ , that is, which can read off from a product of matrices  $ABA$  the corresponding binary sequence 101. So, in order to approximate the function (B.4) our RNN needs to take the inputs

$$x^{(1)} = B, x^{(2)} = A, x^{(3)} = A, x^{(4)} = B$$

and produce the final internal state

$$h^{(T)} = BBAAAABB \in M_n(\mathbb{R}).$$

This can be done if we assume that in the update equation (B.3) has weights  $H, U = 0$  and  $V$  is chosen so that

$$Vq^{(t)} = (x^{(t)})^2 h^{(t-1)}.$$

Note that the right hand side is a cubic polynomial in the entries of  $x^{(t)}, h^{(t-1)}$  so we actually need the generalised form of (B.3).

## References

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