Logic and linear algebra: geometry of interaction

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1   The geometry of interaction

One of the most interesting aspects of linear logic is Girard's program to study the semantics of the cut-elimination process. He calls this the geometry of interaction; see [41, §III] and [38, 39, 40]. The purpose of this section is to explain some of his ideas and how they have motivated the author's search for geometric models of computation.

After the work of Turing, Gödel and Church [70] computation has become a fundamental concept in mathematics. But what is it, really? One answer is that computation is *what happens* when a Turing machine is iterated, or what happens during the β-reduction process of the λ-calculus, or the cut-elimination process of logic. These three models of computation are all equivalent, but each is tied to specific syntax. What is the common essence of these processes? This is a question at least as deep as "What is space?" and it would be absurd to expect a final answer [28]. Nonetheless the search for answers generates interesting mathematics. At a first approach we notice that common to all three of the models of computation listed above is a tension between the *implicit* and the *explicit*. Let us use the computation of Example **??** to explain.

Consider the proofs $\underline{\mathrm{mult}}_2$ and $\underline{2}$ and their cut $\underline{\mathrm{mult}}_2 \,|\, \underline{2}$. The latter is equivalent under cut-elimination to the cut-free proof $\underline{4}$. Since this answer is derived from a deterministic algorithm – cut-elimination – the knowledge is certainly *implicit* in the proof $\underline{\mathrm{mult}}_2 \,|\, \underline{2}$. But some work was necessary to convert this implicit truth into *explicit* truth. Although this example is a trivial one, the reader can easily imagine a similar calculation whose answer is not as apparent as $2 \times 2 = 4$ – or, put aside arithmetic and note that the answer $\underline{4}$ is not obvious from a glance at the diagram (**??**). This conversion of implicit truth to explicit truth, or *explicitation*, is a fundamental aspect of computation.

However, this explicitation process is missing from most mathematical models of computation, which assign to the syntactical gadgets of Turing machines, λ-calculus or logic mathematical objects and transformations between them. To elaborate: suppose that in linear logic that we have proofs π of $A \vdash B$ and ρ of $B \vdash C$, and let $\rho \,|\, \pi$ denote the cut of one against the other, as displayed in the following proof tree:

$$\begin{array}{cc} \pi & \rho \\ \vdots & \vdots \end{array}$$
$$\frac{A \vdash B \quad B \vdash C}{A \vdash C}\ \text{cut} \qquad\qquad (1.1)$$

Let $\widetilde{\rho\,|\,\pi}$ denote the cut-free proof of $A \vdash C$ produced from $\rho\,|\,\pi$ by the cut-elimination process. We regard this as the output of the program ρ computed on the input π. In the vector space semantics there is a commutative diagram of linear maps

$$\begin{array}{ccc} & [\![B]\!] & \\ {}^{[\![\pi]\!]}\nearrow & & \searrow^{[\![\rho]\!]} \\ [\![A]\!] \xrightarrow{\quad [\![\rho\,|\,\pi]\!]=[\![\widetilde{\rho\,|\,\pi}]\!] \quad} & & [\![C]\!] \end{array}$$

On this level, the calculation of output from input is a one-step affair $[\![\pi]\!] \mapsto [\![\rho]\!] \circ [\![\pi]\!]$. This may be contrasted with the syntax, where two steps are involved

$$\pi \longmapsto \rho\,|\,\pi \longmapsto \widetilde{\rho\,|\,\pi}\,. \qquad\qquad (1.2)$$

The point is that this second step, cut-elimination, is completely invisible in the semantics since both $\rho\,|\,\pi$ and its normalisation have the same denotation – by construction. The explicitation that happens in the syntax is absent in the semantics.

Girard proposed [41] that we should look instead for semantics in which the denotations of $\rho\,|\,\pi$ and $\widetilde{\rho\,|\,\pi}$ are distinct and there are "dynamics" which generate the latter from the former. He refers to the field of study of such dynamics as the *geometry of interaction*.[1] The first example of such a semantics, constructed by Girard in [38], has been very influential – and not only in shedding light on the nature of computation. Since the λ-calculus may be translated into intuitionistic logic, and from there into intuitionistic linear logic, any semantics of linear logic yields a method for the execution of programs in the λ-calculus. The geometry of interaction model of linear logic (see below) yields a method of executing programs in which the elementary reduction steps are local – that is, they do not dependent on global coordination [26, 27]. This is closely related to famous work of Lamping on optimal reduction in the λ-calculus [45].

Girard's original geometry of interaction model [38] works as follows: if $\mathbb{H}$ is the Hilbert space of square-summable sequences in $\mathbb{C}$ then he assigns to each proof (in full linear logic) of a sequent $\vdash A_1, \ldots, A_n$ with cuts on formulas $B_1, \ldots, B_m$ a bounded linear operator on $\mathbb{H}^{2m+n}$. Tracing out the $2m$ factors of $\mathbb{H}$ yields an operator on $\mathbb{H}^n$ which agrees with the operator assigned to the cut-free normalisation of the original proof. This operation

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[1] The categorically minded reader may detect a hint of higher-categories: it would be natural to expect that the denotations of a proof and its cut-free normalisation should be 1-morphisms connected by some structure on the level of 2-morphisms which models cut-elimination. We will return to this topic below.

of "tracing out the cuts" is the dynamics which models the process of cut-elimination; it can be rephrased as a sum over paths in a graph associated to the proof [27].

A simplified example using the vector space semantics can capture the key idea. Let us suppose in the above that $[\![B]\!]$ is a finite-dimensional vector space. Then $[\![\rho]\!] \circ [\![\pi]\!]$ may be computed as a trace

$$[\![\rho]\!] \circ [\![\pi]\!] = \operatorname{tr}_{[\![B]\!]} \big(\sigma \circ ([\![\pi]\!] \otimes [\![\rho]\!]) \big) \tag{1.3}$$

where $\sigma : [\![B]\!] \otimes [\![C]\!] \longrightarrow [\![C]\!] \otimes [\![B]\!]$ is the symmetry map and $\operatorname{tr}_{[\![B]\!]}$ is a partial trace. Using the standard notation for traces in string diagrams [51] this reads:
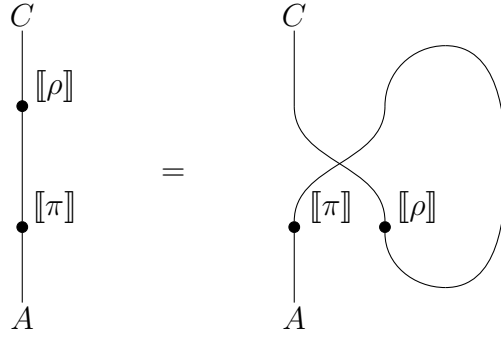
The upshot is that we may view the tensor $[\![\pi]\!] \otimes [\![\rho]\!]$ together with the information of the tensor factor $[\![B]\!]$ as an intermediate step between $[\![\pi]\!]$ and the composite $[\![\rho]\!] \circ [\![\pi]\!]$. That is, the two syntactical steps of (1.2) correspond to the sequence

$$[\![\pi]\!] \longmapsto \big([\![\pi]\!] \otimes [\![\rho]\!], [\![B]\!] \big) \longmapsto \operatorname{tr}_{[\![B]\!]} \big(\sigma \circ ([\![\pi]\!] \otimes [\![\rho]\!]) \big) = [\![\rho]\!] \circ [\![\pi]\!].$$

Since traces are only available for finite-dimensional spaces this idea cannot be extended to arbitrary $[\![B]\!]$ and is therefore not to be taken seriously, but there are numerous traced monoidal categories where a similar construction can be formalised [4, 2, 3]. In particular, Girard's original model using Hilbert spaces can be formulated in these terms; see the introductory notes by Shirahata [69] and Haghverdi-Scott [46].

Next we consider examples of "explicitation" in geometry, which leads to a discussion of semantics of cut-elimination in bicategories of geometric spaces and the ideas behind [61]. In modern geometry we often understand spaces via the vector bundles living on them, and these we in turn understand via their characteristic classes, which are elements in finite-dimensional vector spaces called cohomology groups. However, *implicit* knowledge of a vector in some cohomology group, say produced from a series of abstract maps, should be distinguished from *explicit* knowledge, say coefficients in a preferred basis.

The prototypical example is Hodge theory. Let M be an n-dimensional compact oriented smooth manifold. The de Rham cohomology of M [20] is the cohomology of the complex of global sections of the sheaf of differential forms

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d^0} \cdots \longrightarrow \Omega^p(M) \xrightarrow{d^p} \Omega^{p+1}(M) \longrightarrow \cdots \longrightarrow \Omega^n(M) \longrightarrow 0 \,.$$

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If $Z^p(M) = \mathrm{Ker}(d^p)$ denotes the closed p-forms and $B^p(M) = \mathrm{Im}(d^{p-1})$ the exact p-forms, then the pth de Rham cohomology group is

$$H^p(M) = \frac{Z^p(M)}{B^p(M)}\,.$$

By a theorem of Hodge a preferred basis is given for each p by the subspace of harmonic forms $\mathscr{H}^p(M) \subseteq Z^p(M)$. That is, there is an isomorphism

$$\mathscr{H}^p(M) \longrightarrow Z^p(M) \longrightarrow Z^p(M)/B^p(M) = H^p(M)\ .$$

Giving an equivalence class of vectors in $Z^p(M)$ is therefore equivalent to specifying an element in $\mathscr{H}^p(M)$, but since $Z^p(M)$ is infinite-dimensional these two forms of knowledge have a different character. For example, exhibiting the coefficients of the equivalence class of a vector $\eta \in Z^p(M)$ in a chosen basis $\{\omega_i\}_i$ of harmonic forms requires additional work, which takes the form of computing integrals:

$$\eta \equiv \sum_i \langle \eta, \omega_i \rangle \omega_i, \qquad \langle \eta, \omega_i \rangle = \int_M \eta \wedge \star \omega_i\,. \tag{1.4}$$

To connect this back to computation and the sequence (1.2), let M, N be smooth projective varieties over $\mathbb{C}$. Any cohomology class $\rho \in H^*(M \times N)$ determines a linear map of finite-dimensional vector spaces

$$I_\rho : H^*(M) \longrightarrow H^*(N)\,,$$
$$\pi \longmapsto (p_N)_*(\rho \cup p_M^*(\pi))$$

where p_M, p_N denote the projections from $M \times N$. Suppose that we insist on dealing only with harmonic forms. Then to compute I_ρ on some "input" harmonic form π and get a harmonic form as "output" we apply the following composite

$$\mathscr{H}^*(M) \xrightarrow{\cong} H^*(M) \xrightarrow{I_\rho} H^*(N) \xrightarrow{\cong} \mathscr{H}^*(N)$$
$$\pi \longmapsto I_\rho(\pi) \longmapsto \sum_j \langle I_\rho(\pi), \omega_j \rangle \omega_j\,.$$

A comparision of this process – first compute I_ρ and then project onto harmonic forms using the integrals in (1.4) – with the two syntactical steps of (1.2) suggests a natural starting point for a search for semantics of cut-elimination in geometry, with integration playing the role of explicitation.

However, given the expectation previously expressed of higher-categorical semantics of cut-elimination, it is natural to start the search not with cohomology and linear maps but categories and functors. In this setting the relationship between the space of harmonic forms $\mathscr{H}^*(M)$ and the de Rham cohomology $H^*(M)$ may be compared to that between

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the bounded derived category of coherent sheaves $\mathbf{D}^b(\mathrm{coh}\, M)$ and the full subcategory $\mathbf{D}^b_{\mathrm{coh}}(\mathrm{Qco}\, M)$ of complexes with bounded, coherent cohomology in the unbounded derived category of quasi-coherent sheaves $\mathbf{D}(\mathrm{Qco}\, M)$.[2] These are equivalent categories

$$\mathbf{D}^b(\mathrm{coh}\, M) \xrightarrow{\ \cong\ } \mathbf{D}^b_{\mathrm{coh}}(\mathrm{Qco}\, M) \subset \mathbf{D}(\mathrm{Qco}\, M) \tag{1.5}$$

which means that for any object $\mathscr{G}$ of $\mathbf{D}^b_{\mathrm{coh}}(\mathrm{Qco}\, M)$ there exists an isomorphic bounded complex of coherent sheaves $\mathscr{G}_{\mathrm{finite}}$ which we call a *finite model* of $\mathscr{G}$. But given $\mathscr{G}$ it is not obvious how to produce this finite model in any reasonably algorithmic way.[3] Consider that in the special case where M is a point, (1.5) is an equivalence

$$\mathbf{D}^b(\mathrm{vect}\,\mathbb{C}) \xrightarrow{\ \cong\ } \mathbf{D}^b_{\mathrm{fd}}(\mathrm{Vect}\,\mathbb{C}) \subset \mathbf{D}(\mathrm{Vect}\,\mathbb{C})$$

between bounded complexes of finite-dimensional vector spaces and unbounded complexes of (infinite-dimensional) vector spaces with finite-dimensional cohomology. For general M, if $f : M \longrightarrow \mathbb{C}$ denotes the structure map then for a coherent sheaf $\mathscr{F}$ on M the derived pushforward $\mathbb{R}f_*(\mathscr{F})$ may be defined as an object of $\mathbf{D}^b_{\mathrm{fd}}(\mathrm{Vect}\,\mathbb{C})$ by a simple universal property, but explicitly computing the isomorphic *finite* complex in $\mathbf{D}^b(\mathrm{vect}\,\mathbb{C})$ – that is, computing the cohomology of $\mathscr{F}$ – requires work to be performed.

To complete the analogy with the above, suppose we are given a complex of coherent sheaves $\mathscr{R} \in \mathbf{D}^b(\mathrm{coh}\, M \times N)$, which will play the role of ρ. There is a naturally associated triangulated functor

$$I_{\mathscr{R}} : \mathbf{D}^b_{\mathrm{coh}}(\mathrm{Qco}\, M) \longrightarrow \mathbf{D}^b_{\mathrm{coh}}(\mathrm{Qco}\, N), \tag{1.6}$$

$$\mathscr{F} \longmapsto (p_N)_*(\mathscr{R} \otimes p_M^*(\mathscr{F})) \tag{1.7}$$

where $p_N : M \times N \longrightarrow N$ and $p_M : M \times N \longrightarrow M$ are the projections and $(p_N)_*, p_M^*$ denote derived functors. Note that the functor $I_{\mathscr{R}}$ naturally produces infinite complexes of quasi-coherent sheaves with bounded coherent cohomology; rarely will the construction output an actual bounded complex of coherent sheaves. If we want to work with finite objects, computing $I_{\mathscr{R}}$ on some input therefore requires two steps

$$\mathbf{D}^b(\mathrm{coh}\, M) \xrightarrow{\ \cong\ } \mathbf{D}^b_{\mathrm{coh}}(\mathrm{Qco}\, M) \xrightarrow{\ I_{\mathscr{R}}\ } \mathbf{D}^b_{\mathrm{coh}}(\mathrm{Qco}\, N) \xrightarrow{\ \cong\ } \mathbf{D}^b(\mathrm{coh}\, N) \tag{1.8}$$

$$\mathscr{F} \longmapsto I_{\mathscr{R}}(\mathscr{F}) \longmapsto I_{\mathscr{R}}(\mathscr{F})_{\mathrm{finite}} . \tag{1.9}$$

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[2] We do not intend to compare harmonic forms with arbitrary complexes of coherent sheaves. The point of the analogy is that $\mathscr{H}^*(M)$ provides an explicit finite-dimensional model for de Rham cohomology. The precise nature of the classes that make up this model is of secondary importance.

[3] Unfortunately this is not a very precise statement, since it depends what we mean by "reasonable". While we can reconstruct a complex up to quasi-isomorphism from truncations and in this way construct an explicit complex of coherent sheaves, this is not an algorithm of the kind that we want – consider the enormous complexity of the complexes produced by taking global sections of an explicit resolution of a sheaf $\mathscr{F}$ by injective quasi-coherent sheaves on a non-trivial scheme. In principle the data in this resolution is determined by $\mathscr{F}$, but it would be madness to build a model of computation on top of objects that are so difficult to handle.

That is, given the input complex $\mathscr{F}$ we first compute $I_{\mathscr{R}}$ and then we take a finite model, which is the form of explicitation natural to algebraic geometry.

This suggests that in a model of computation where proofs or programs are interpreted by functors between triangulated categories, cut-elimination should be interpreted as the algorithmic construction of finite models. But what do we mean when we say that some kind of mathematical structure "interprets" cut-elimination? Since categories, functors and natural transformations form a bicategory, our answer is given in terms of bicategorical semantics where types are assigned objects and proofs are assigned 1-morphisms.

Taking Girard's geometry of interaction as our guide, we ask for (at least) the following two desiderata. The algorithm computing finite models should be compatible with

(A) the structure of ambient symmetric monoidal bicategory, and

(B) the comonad interpretating the exponential modality.

The precise form of these requirements will depend on the bicategory under consideration. In the rest of this section we first sketch the kind of bicategorical semantics of linear logic that we have in mind, and then we explain how the cut systems of [61] provide one way of making precise the notion of an "algorithm for computing finite models". In the present discussion the two most relevant examples of bicategories are

- The bicategory $\mathcal{V}ar$ [21] where objects are smooth projective varieties $M, N, \ldots$ and 1-morphisms $M \longrightarrow N$ are bounded complexes of coherent sheaves

$$\mathcal{V}ar(M, N) = \mathbf{D}^b(\operatorname{coh} M \times N) \, .$$

For an example of the composition rule, note that if $\mathscr{R} \in \mathcal{V}ar(M, N)$ and S is a point then composition with $\mathscr{R}$ in this bicategory is a functor $\mathcal{V}ar(S, M) \longrightarrow \mathcal{V}ar(S, N)$ which is in fact (1.6).

- The bicategory $\mathcal{LG}$ of Landau-Ginzburg models [57, 22, 23] where objects are isolated hypersurface singularities $W, V, \ldots$ and 1-morphisms $W \longrightarrow V$ are finite-rank matrix factorisations of $V - W$

$$\mathcal{LG}(W, V) = \operatorname{hmf}(V - W)^{\omega} \, .$$

Here $\operatorname{hmf}(V - W)$ denotes the homotopy category of matrix factorisations, and $(-)^{\omega}$ the idempotent completion [75].

The bicategories $\mathcal{V}ar$ and $\mathcal{LG}$ are symmetric monoidal but do not have enough colimits to interpret the exponential modality. However, suitable enlargements of both examples yield bicategories $\mathcal{C}$ which are symmetric monoidal with an internal Hom and admit a cofree cocommutative coalgebra $!V$ for each object V. It therefore makes sense to define a semantics of intuitionistic linear logic in $\mathcal{C}$ (or at least, in the classifying category of $\mathcal{C}$)

where the denotation of a proof is a 1-morphism. For example, the interpretation of the Church numeral $\underline{2}$ is a 1-morphism

$$! \operatorname{End}(V) \xrightarrow{\ \Delta\ } ! \operatorname{End}(V) \otimes ! \operatorname{End}(V) \xrightarrow{\ d \otimes d\ } \operatorname{End}(V)^{\otimes 2} \xrightarrow{\ -\circ-\ } \operatorname{End}(V) \qquad (1.10)$$

where $\operatorname{End}(V)$ denotes an object of $\mathcal{C}$ computed using the internal Hom. If we had started with a bicategory of rings and bimodules, this construction would yield the program (**??**) from the introduction.

To explain the interpretation of cut-elimination in such bicategorical semantics we begin by supposing that we have a bicategory $\mathcal{B}$ and for each pair of objects M, N a full subcategory $\mathcal{A}(M, N) \subseteq \mathcal{B}(M, N)$ such that the inclusion is an equivalence. The intuition is that $\mathcal{A}$ is a collection of "finite models" and that every 1-morphism is isomorphic to some finite model. We do not assume that $\mathcal{A}$ is a sub-bicategory: the whole point is that the composition in $\mathcal{B}$ of 1-morphisms in $\mathcal{A}$ will not in general belong to $\mathcal{A}$.

Example 1.1. In the case of $\mathcal{B} = \mathcal{V}ar$,

$$\mathcal{A}(M, N) = \mathbf{D}^b(\operatorname{coh} M \times N), \qquad \mathcal{B}(M, N) = \mathbf{D}^b_{\operatorname{coh}}(\operatorname{Qco} M \times N). \qquad (1.11)$$

In the case of $\mathcal{B} = \mathcal{LG}$ both W, V are the defining equations of singularities and

$$\mathcal{A}(W, V) = \operatorname{hmf}(V - W)^\omega, \qquad \mathcal{B}(W, V) = \operatorname{HMF}_{\mathrm{fd}}(V - W) \qquad (1.12)$$

where $\operatorname{HMF}_{\mathrm{fd}}(W)$ denotes the homotopy category of infinite-rank matrix factorisations which are direct summands of finite rank ones.

The guiding principle is that the denotation in $\mathcal{B}$ of a cut-free proof should be *finite*, that is, it should be a 1-morphism in $\mathcal{A}$. Denotations of proofs with cuts will be defined using the composition operation in $\mathcal{B}$ and may generate 1-morphisms outside of $\mathcal{A}$. The idea is to look for the shadow of the cut-elimination process in $\mathcal{B}$ in terms of the algorithm which reflects these 1-morphisms back into $\mathcal{A}$. We refine this idea into precise mathematics in a series of steps. For any object S the composition in $\mathcal{B}$ gives a functor

$$\mathcal{A}(S, M) \otimes \mathcal{A}(M, N) \longrightarrow \mathcal{B}(S, N), \qquad (1.13)$$
$$\mathscr{F} \otimes \mathscr{R} \mapsto \mathscr{R} \circ \mathscr{F}. \qquad (1.14)$$

By assumption there is an object $(\mathscr{R} \circ \mathscr{F})_{\mathrm{finite}}$ of $\mathcal{A}(S, N)$ isomorphic to $\mathscr{R} \circ \mathscr{F}$. Adding this step of taking these finite models to the end of (1.13) we have

$$\mathcal{A}(S, M) \otimes \mathcal{A}(M, N) \longrightarrow \mathcal{B}(S, N) \longrightarrow \mathcal{A}(S, N) \qquad (1.15)$$
$$\mathscr{F} \otimes \mathscr{R} \longmapsto \mathscr{R} \circ \mathscr{F} \longmapsto (\mathscr{R} \circ \mathscr{F})_{\mathrm{finite}}. \qquad (1.16)$$

The problem with this construction is that in the examples of interest there is no algorithmic construction of these finite models (of the kind we need) starting from just the data of the object $\mathscr{R} \circ \mathscr{F}$. But what we can do for $\mathcal{LG}$ is refine the composition operation so that

it lands not in $\mathcal{B}(S, N)$ but in an auxiliary category, the Clifford thickening of $\mathcal{A}(S, N)$, for which an algorithmic construction of objects in $\mathcal{A}(S, N)$ is under good control.

From now on we assume that $\mathcal{B}$ is a superbicategory [61, §2.1], which is true of $\mathcal{LG}$ but not of $\mathcal{V}ar$. In particular the $\mathcal{A}(M, N)$ are all supercategories. The Clifford thickening of a supercategory is defined using Clifford algebras A_n of dimension 2^{2n} with $A_0 = k$. Each of these algebras has a basic representation, the spinor representation S_n, such that $A_n = \mathrm{End}_k(S_n)$. This means that every representation Q of A_n in the category of vector spaces is of the form $Q \cong S_n \otimes_k V$ for some $\mathbb{Z}_2$-graded vector space V, but the crucial point is that there is some computational "cost" associated with extracting the vector space V from Q. This may be represented either as computing a tensor product with the dual of the spinor representation $S_n^* = \mathrm{Hom}_k(S_n, k)$

$$V \cong S_n^* \otimes_{A_n} Q \tag{1.17}$$

or equivalently as splitting idempotents on Q arising from the Clifford action. When Q is not a vector space but an object of the supercategory $\mathcal{A}(M, N)$ this extraction process is the form of explicitation which models cut-elimination in a cut-system.

To elaborate, we need to use the *Clifford thickening* $\mathcal{A}(M, N)^\bullet$ which is the category of all representations of the various Clifford algebras A_n in $\mathcal{A}(M, N)$. More precisely, an object of this category is object of $\mathcal{A}(M, N)$ together with the action of a Clifford algebra A_n for some $n \geq 0$. There is an equivalence of supercategories

$$\mathcal{A}(M, N) \xrightarrow{\;\cong\;} \mathcal{A}(M, N)^\bullet \tag{1.18}$$

which sends an object $\mathscr{F}$ of $\mathcal{A}(M, N)$ to itself viewed as a representation of $A_0 = k$. The point is that the inverse of this equivalence has a very regular form: given an object $(\mathscr{G}, \zeta)$ of $\mathcal{A}(M, N)^\bullet$, which is an object $\mathscr{G}$ together with a morphism of algebras

$$\zeta : A_n \longrightarrow \mathrm{End}^*_{\mathcal{A}(M,N)}(\mathscr{G})$$

for some n, the tensor product

$$\widetilde{\mathscr{G}} := S_n^* \otimes_{A_n} \mathscr{G} \tag{1.19}$$

is an object of $\mathcal{A}(M, N)$ and the map $(\mathscr{G}, \zeta) \mapsto \widetilde{\mathscr{G}}$ gives an inverse to (1.18). This inverse is analogous to the extraction of the vector space V from the representation Q. Note that while $\mathscr{G}$ and $\widetilde{\mathscr{G}}$ are isomorphic objects in the Clifford thickening they represent different states of knowledge: in $\mathscr{G}$ the underlying object of $\mathcal{A}(M, N)$ is still implicit, while in $\widetilde{\mathscr{G}}$ it has been made explicit. Let us now assume that there is an algorithm for computing the tensor product (1.19) so that the inverse to (1.18) is under control.[4] Thus it only remains to refine the composition in (1.13) to land in $\mathcal{A}(S, N)^\bullet$.

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[4]This algorithm depends on the underlying algebraic objects that make up the bicategory. In the case of $\mathcal{LG}$, computing the tensor product is done by splitting idempotents, which reduces to a computation in commutative algebra using well-understood algorithms.

That is, we ask for a collection of functors, indexed by triples M, N, S

$$\mathcal{A}(S, M) \otimes \mathcal{A}(M, N) \longrightarrow \mathcal{A}(S, N)^{\bullet}, \tag{1.20}$$

$$\mathscr{F} \otimes \mathscr{R} \longmapsto \mathscr{R} \,|\, \mathscr{F} \tag{1.21}$$

which satisfy some natural axioms. We call $\mathscr{F} \,|\, \mathscr{R}$ the *cut* of $\mathscr{F}$ and $\mathscr{R}$. It is an object of $\mathcal{A}(S, N)$ together with the action of some Clifford algebra. To say that these cut functors refine the composition operation in $\mathcal{B}$ is to ask that the following diagram commutes up to natural isomorphism:

$$
\begin{array}{ccccc}
\mathcal{B}(S, M) \otimes \mathcal{B}(M, N) & \xrightarrow{\;-\circ-\;} & \mathcal{B}(S, N) & \xleftarrow{\;\cong\;} & \mathcal{A}(S, N) \\
{\scriptstyle \cong} \uparrow & & & & \downarrow {\scriptstyle \cong} \\
\mathcal{A}(S, M) \otimes \mathcal{A}(M, N) & & \xrightarrow{\hspace{3cm}-\,|\,-\hspace{3cm}} & & \mathcal{A}(S, N)^{\bullet}
\end{array}
\tag{1.22}
$$

Commutativity of this diagram allows us to replace the sequence of steps in (1.15), that is, the clockwise direction from the bottom left of the diagram (1.22) to the top right

$$\mathscr{F} \otimes \mathscr{R} \longmapsto \mathscr{R} \circ \mathscr{F} \longmapsto (\mathscr{R} \circ \mathscr{F})_{\text{finite}} \tag{1.23}$$

with the series of steps in the anti-clockwise direction

$$\mathscr{F} \otimes \mathscr{R} \longmapsto \mathscr{R} \,|\, \mathscr{F} \longmapsto \widetilde{\mathscr{R} \,|\, \mathscr{F}} := S_n^{*} \otimes_{A_n} (\mathscr{R} \,|\, \mathscr{F}) \tag{1.24}$$

The point is that the last step of (1.23) is not under algorithmic control, but every step of (1.24) is algorithmic.

In summary the *cut system* $\mathcal{A}$ which is the collection of categories $\mathcal{A}(M, N)$ and the cut functors (1.20) gives a precise meaning to the idea of an algorithmic construction of finite models for the result of composition of 1-morphisms in $\mathcal{B}$, and this is the structure in the bicategorical semantics which we intend as a model of cut-elimination. The main result of [61] is that in the case $\mathcal{B} = \mathcal{LG}$ there is a natural cut system realising the above. Regarding the desiderata A and B, which must be present before we can call this a model of cut-elimination: while compatibility with the basic structure of the bicategory is part of the axiomatics of a cut system [61, §3], this first paper does not include the compatibility with the tensor product, the internal Hom or finite products. These arguments, together with the verification of desiderata B, will be the subject of future work.

To connect this explicitly back to Girard's ideas in the geometry of interaction, suppose that $\mathcal{B}$ carries an interpretation of linear logic and that we have a pair of proofs π, ρ as in (1.1). Then using the cut system structure we would make the assignments

$$
\begin{aligned}
[\![\rho]\!] &= \mathscr{R}, \\
[\![\pi]\!] &= \mathscr{F}, \\
[\![\rho \,|\, \pi]\!] &= \mathscr{R} \,|\, \mathscr{F}, \\
[\![\widetilde{\rho \,|\, \pi}]\!] &= \widetilde{\mathscr{R} \,|\, \mathscr{F}}.
\end{aligned}
$$

The "dynamics" which generates $\widetilde{\mathscr{R}\,|\,\mathscr{F}}$ from $\mathscr{R}\,|\,\mathscr{F}$ is the computation of the tensor product with the dual of the spinor representation S_n^* [61, §3.1].

Example 1.2. The polynomial 0 is an object of $\mathcal{B} = \mathcal{LG}$ and with $\mathcal{A}$ as in (1.12) for any hypersurface singularity W, we have

$$\mathcal{A}(0, W) = \mathrm{hmf}(W)^\omega, \qquad \mathcal{A}(W, 0) = \mathrm{hmf}(-W)^\omega,$$
$$\mathcal{A}(0, 0) = \mathcal{C}_2(k), \qquad \mathcal{B}(0, 0) = \mathcal{C}_2^\infty(k).$$

where $\mathcal{C}_2(k)$ is the homotopy category of $\mathbb{Z}_2$-graded complexes of finite-dimensional k-vector spaces and $\mathcal{C}_2^\infty(k)$ is the homotopy category of infinite-rank complexes with finite-dimensional cohomology. Moreover $\mathcal{A}(0, 0)^\bullet$ is the category of A_n-representations in $\mathcal{C}_2(k)$ for various n. In this case the diagram (1.22) becomes

$$\begin{array}{ccc}
\mathcal{C}_2^\infty(k) & \xleftarrow{\quad\cong\quad} & \mathcal{C}_2(k) \\
{\scriptstyle -\otimes-}\big\uparrow & & \big\downarrow{\scriptstyle\cong} \\
\mathrm{hmf}(W)^\omega \otimes \mathrm{hmf}(-W)^\omega & \xrightarrow{\quad-|-\quad} & \mathcal{C}_2(k)^\bullet
\end{array} \qquad (1.25)$$

The vertical functor on the left, which comes from the composition in $\mathcal{B}$, is simply a tensor product. Every matrix factorisation of $-W$ is the dual $Y^\vee$ of some matrix factorisation Y of W, and the functor sends $X \otimes Y^\vee$ to $\mathrm{Hom}(Y, X) \in \mathcal{C}_2^\infty(k)$. The clockwise direction around this square is the sequence (1.23)

$$X \otimes Y^\vee \longmapsto \mathrm{Hom}(Y, X) \longmapsto H^* \mathrm{Hom}(Y, X)$$

whereas the anti-clockwise direction is (1.24)

$$X \otimes Y^\vee \longmapsto Y^\vee \,|\, X \longmapsto S_n^* \otimes_{A_n} (Y^\vee \,|\, X).$$

The cut $Y^\vee \,|\, X$ is a representation of some Clifford algebra A_n in $\mathbb{Z}_2$-graded complexes of finite-dimensional vector spaces, that is, there are closed operators $a_1, \ldots, a_n, a_1^\dagger, \ldots, a_n^\dagger$ on $Y^\vee \,|\, X$ satisfying the anti-commutation relations

$$a_i a_j = a_j a_i, \qquad a_i^\dagger a_j^\dagger = a_j^\dagger a_i^\dagger, \qquad a_i a_j^\dagger = \delta_{ij} + a_j^\dagger a_i.$$

An example with explicit matrices for these operators is given in [61, Example 4.15]. Since A_n is Morita trivial we know that this representation must be of the form $S_n \otimes_k V$ for some complex V, and the point of the commutativity of (1.25) is that this V is precisely the cohomology $H^* \mathrm{Hom}(Y, X)$. One can show that V is the image of the following idempotent on the complex $Y^\vee \,|\, X$ constructed from the Clifford action [61, Definition 2.18]

$$e_n := a_1^\dagger \cdots a_n^\dagger a_n \cdots a_1.$$

The content of the cut system in this special case is that splitting this idempotent provides a more controlled way of extracting the vector space $H^* \mathrm{Hom}(Y, X)$ from the data of X, Y than taking cohomology. In particular, the cut system links invariants of the cohomology (such as the Euler characteristic) to invariants of X and Y (such as the Chern characters) in a way that is obscured by taking cohomology; see below.

This concludes our sketch of the links between cut-elimination and geometry. While Girard's original geometry of interaction model immediately yielded insight into the execution of programs in the λ-calculus, it is not clear what geometric models have to say about practical questions in computer science. But it is intriguing that in the example of $\mathcal{LG}$ the Clifford actions, which we may view as algorithmic structure "hidden inside" the composition operation, are both the natural interpretation of cut-elimination and also encode fundamental geometric properties of matrix factorisations such as the analogue of Hirzebruch-Riemann-Roch theorem. Roughly, this arises as follows: in the situation of Example 1.2 the cut system provides a diagram of morphisms of complexes

$$H^* \operatorname{Hom}(Y, X) \xrightarrow[f]{g} Y^\vee \mid X$$

with $f \circ g = 1$ and $g \circ f = e_n$. Then

$$
\begin{aligned}
\chi\big(H^* \operatorname{Hom}(Y, X)\big) &= \operatorname{str}(1_{H^* \operatorname{Hom}(Y,X)}) \\
&= \operatorname{str}(f \circ g) \\
&= \operatorname{str}(g \circ f) \\
&= \operatorname{str}(a_1^\dagger \cdots a_n^\dagger a_n \cdots a_1) \, .
\end{aligned}
$$

At this point the fact that the operators $a_i^\dagger$ are Atiyah classes allows us to transform this supertrace into a residue, which presents the Euler characteristic as an integral involving the Chern characters of X and Y. The details are given in [29], see also [61, Remark 4.13]. This recovers the Hirzebruch-Riemann-Roch formula for matrix factorisations proven by Polishchuk-Vaintrob [63] and hints at a link between cut-elimination in logic and index theorems in geometry, which may ultimately help to shed light on fundamental questions about the tension between local and global aspects of computation.[5]

An important recent topic in proof theory is homotopy type theory [74] where spaces in the sense of homotopy theory are assigned to types in Martin-Löf type theory. Since in the bicategorical semantics above geometric spaces of one kind or another are assigned to types in linear logic, there is an obvious connection. Indeed, Schreiber [64] has discussed linear logic and algebraic geometry in the context of homotopy type theory and quantum field

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[5]What does an algebraist stand to learn from bicategorical models of logic? The interaction with logic can lead to ideas that might otherwise go unexplored – for instance, from the logical perspective it is natural to take an isolated hypersurface singularity W and form the universal cocommutative coalgebra $!W$ mapping to W, but otherwise the importance of this construction would not be obvious. To take this particular example a bit further: in *implicit computational complexity* restrictions on the use of the exponential modality are used to characterise complexity classes of programs, such as polynomial time [42] and elementary time [25]. In the vector space semantics of this paper, where proofs or programs are interpreted as morphisms, this provides some notion of complexity for linear maps. But in a bicategorical semantics where proofs are interpreted as objects in cocomplete triangulated categories of 1-morphisms one should study the complexity of programs constructing objects from a set of fixed objects.

theory. However it is not obvious to the author how to make the connection precise, since our main emphasis is on explicit models of cut-elimination on the level of 2-morphisms which does not appear in [64]. One observation that seems relevant is that the key idea in the construction of a cut system in [61] is an application of the homological perturbation lemma to compute the convolution of kernels in tractable terms, and from this the Clifford actions emerge. It would be not be surprising if this sort of structure appeared naturally in the setting of homotopy type theory and, more to the point, maybe the theory can be used to define something like a cut system for $\mathcal{V}ar$. But this is probably not easy!

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