Landau-Ginzburg semantics of linear logic

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1 Introduction

The bicategory of Landau-Ginzburg models $\mathcal{LG}$ has as objects isolated hypersurface singularities, and as 1-morphisms matrix factorisations of differences of potentials. In [?] this bicategory was studied in connection with topological field theory with defects and shown to have various good properties, and in [?] it was shown how to make composition in this bicategory constructive. In this paper we extend this work by adding *moduli spaces* of matrix factorisations, and showing how the resulting bicategory hosts a model (called a semantics) of linear logic.

2 Matrix factorisations

- Definition
- Basic examples $y^d - x^d$ and morphisms
- Bicategorical structure: fusion
- The cut system
- Denotations $[\![\mathbf{x}]\!] = (k[x], W(x))$ in multiplicative fragment
- Define indexed objects

3 Moduli spaces

- Simplest kind of moduli space over poly ring, universal property
- Example $y^d - x^d$
- Moduli of MFs with null-homotopies
- Moduli of MFs with Clifford actions and homotopies

- Denotation $[\![!A]\!]$ as an indexed object, A multiplicative

3.1 Bundles on moduli space

- Universal MF
- Bundles on moduli spaces $\mathrm{Hom}(X, Y)$
- The denotation of 2, a MF on $\mathscr{M} \times \mathbb{A}^n$

3.2 Full moduli space

- Moduli of MFs over non-affine spaces
- Full definition: over f.g. algebra, with homotopies, with Clifford action

4 Moduli potentials

- Definition of moduli potential
- Example of $[\![\mathbf{int}_A]\!]$
- Definition of kernels
- Examples of kernels
- Moduli of kernels
- Denotation $[\![A]\!]$, A an arbitrary formula
- Example of $[\![!\mathbf{int}_A]\!]$

5 Promotion

- Spacelike moduli potentials
- Define lifting
- Theorem: lifting vs dereliction

6 Landau-Ginzburg semantics

- Bicategories of indexed objects
- 2-morphisms of kernels
- Lifting as something involving indices
- Definition of denotations of proofs
- Assignment of 2-morphisms to cut-elimination steps
- Theorem: it works

7 Examples

- The proof 2 for $(k[x], x^N)$
- The proof 2 for space-like thing, as in cutsys53.

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