# Mathematics S 4

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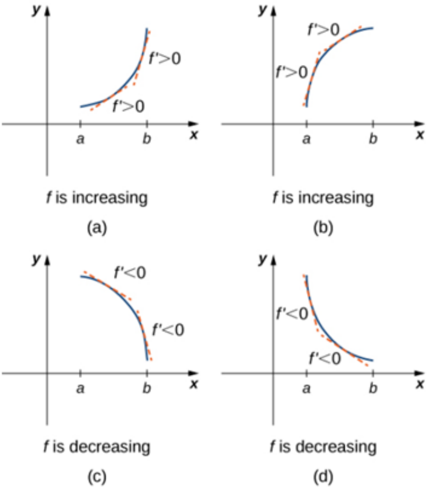
Today

In these slides we aim to explain the following:

1. Derivatives and shape of a graph
2. Applied Optimization Problems
3. L'Hôpital's Rule

Earlier in this chapter we stated that if a function $f$ has a local extremum at a point $c$, then $c$ must be a critical point of $f$. However, a function is not guaranteed to have a local extremum at a critical point. For example, $f(x) = x^3$ has a critical point at $x = 0$ since $f'(x) = 3x^2$ is zero at $x = 0$, but $f$ does not have a local extremum at $x = 0$. Using the results from the previous section, we are now able to determine whether a critical point of a function actually corresponds to a local extreme value. In this section, we also see how the second derivative provides information about the shape of a graph by describing whether the graph of a function curves upward or curves downward.

## Let us have a look on these figures



We realize that both functions are increasing over the interval $(a, b)$. At each point $x$, the derivative $f'(x) > 0$. Both functions are decreasing over the interval $(a, b)$. At each point $x$, the derivative $f'(x) < 0$.

- A continuous function $f$ has a local maximum at point $c$ if and only if $f$ switches from increasing to decreasing at point $c$.
- Similarly, f has a local minimum at $c$ if and only if $f$ switches from decreasing to increasing at $c$.
- If $f$ is a continuous function over an interval $I$ containing $c$ and differentiable over $I$, except possibly at $c$, the only way $f$ can switch from increasing to decreasing (or vice versa) at point $c$ is if $f'$ changes sign as $x$ increases through $c$. If $f$ is differentiable at $c$, the only way that $f'$ can change sign as $x$ increases through $c$ is if $f'(c) = 0$.

Therefore, for a function $f$ that is continuous over an interval $I$ containing $c$ and differentiable over $I$, except possibly at $c$, the only way $f$ can switch from increasing to decreasing (or vice versa) is if $f'(c) = 0$ or $f(c)$ is undefined. Consequently, to locate local extrema for a function $f$, we look for points $c$ in the domain of $f$ such that $f(c) = 0$ or $f'(c)$ is undefined. **Note that f need not have a local extrema at a critical point.**

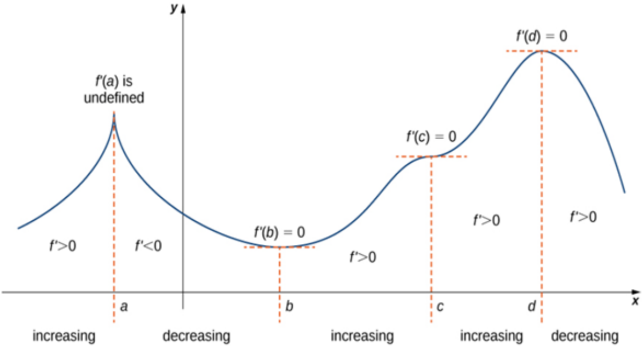



Figure: The function $f$ has four critical points: $a, b, c$, and $d$. The function $f$ has local maxima at $a$ and $d$, and a local minimum at $b$. The function $f$ does not have a local extremum at $c$. The sign of $f'$ changes at all local extrema.

Using the above figure, we summarize the main results regarding local extrema.

- If a continuous function $f$ has a local extremum, it must occur at a critical point $c$.
- The function has a local extremum at the critical point $c$ if and only if the derivative $f'$ switches sign as $x$ increases through $c$.
- Therefore, to test whether a function has a local extremum at a critical point $c$, we must determine the sign of $f(x)$ to the left and right of $c$.

This result is known as **the first derivative test**.

## Theorem

*Suppose that $f$ is a continuous function over an interval $I$ containing a critical point $c$. If $f$ is differentiable over $I$, except possibly at point $c$, then $f(c)$ satisfies one of the following descriptions:*

i. *If $f'$ changes sign from positive when $x < c$ to negative when $x > c$, then $f(c)$ is a local maximum of $f$.*

ii. *If $f'$ changes sign from negative when $x < c$ to positive when $x > c$, then $f(c)$ is a local minimum of $f$.*

iii. *If $f'$ has the same sign for $x < c$ and $x > c$, then $f(c)$ is neither a local maximum nor a local minimum of $f$.*

This can also be done using the table of sign of the derivative. But different strategies can also be used. We propose to use the following.

We can summarize the first derivative test as a strategy for locating local extrema. Consider a function $f$ that is continuous over an interval $I$.

a. Find all critical points of $f$ and divide the interval $I$ into smaller intervals using the critical points as endpoints.
b. Analyze the sign of $f'$ in each of the subintervals. If $f'$ is continuous over a given subinterval (which is typically the case), then the sign of $f'$ in that subinterval does not change and, therefore, can be determined by choosing an arbitrary test point $x$ in that subinterval and by evaluating the sign of $f'$ at that test point. Use the sign analysis to determine whether $f$ is increasing or decreasing over that interval.
c. Use **First Derivative Test** and the results of step 2 to determine whether $f$ has a local maximum, a local minimum, or neither at each of the critical points.

We illustrate our strategy by examples

## Example

Use the first derivative test to find the location of all local extrema for

a. $f(x) = x^3 - 3x^2 - 9x - 1$.
b. $f(x) = 5x^{1/3} - x^{5/3}$.

For the first one we have the following
**Step1**:The derivative is $f'(x) = 3x^2 - 6x - 9$. To find the critical points, we need to find where $f'(x) = 0$. Factoring the polynomial, we conclude that the critical points must satisfy

$$3(x^2 - 2x - 3) = 3(x - 3)(x + 1) = 0$$

Therefore, the critical points are $x = 3, -1$. Now divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, -1), (-1, 3)$ and $(3, \infty)$.

**Step2**:Since $f'$ is a continuous function, to determine the sign of $f'(x)$ over each subinterval, it suffices to choose a point over each of the intervals $(-\infty, -1)$, $(-1, 3)$ and $(3, \infty)$ and determine the sign of $f'$ at each of these points. For example, let's choose $x = -2$, $x = 0$, and $x = 4$ as test points.

| Interval | Test point | $f'$ of test point | conclusion |
|---|---|---|---|
| $(-\infty, -1)$ | $x = -2$ | $(+)(-)(-) = +$ | $f$ is increasing. |
| $(-1, 3)$ | $x = 0$ | $(+)(-)(+) = -$ | $f$ is decreasing. |
| $(3, \infty)$ | $x = 4$ | $(+)(+)(+) = +$ | $f$ is increasing. |

**Step3**: Since $f'$ switches sign from positive to negative as $x$ increases through 1, $f$ has a local maximum at $x = -1$. Since $f'$ switches sign from negative to positive as $x$ increases through 3, $f$ has a local minimum at $x = 3$.

For the second question we have
**Step1**:The derivative is

$$f'(x) = \frac{5}{3}x^{-2/3} - \frac{5}{3}x^{2/3} = \frac{5(1 - x^{4/3})}{3x^{2/3}}.$$

The derivative $f' = 0$ when $1 - x^{4/3} = 0$, and this is for $x = \pm 1$. The derivative $f'(x)$ is undefined at $x = 0$. Therefore, we have three critical points: $x = 0, x = 1$, and $x = -1$. Consequently, divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, -1), (-1, 0), (0, 1)$, and $(1, \infty)$.
**Step2**: Since $f'$ is continuous over each subinterval, it suffices to choose a test point $x$ in each of the intervals from step 1 and determine the sign of $f'$ at each of these points.

The points $x = -2, x = -\frac{1}{2}, \frac{1}{2}$, and $x = 2$ are test points for these intervals.

| Interval | Test point | $f'$ of test point | conclusion |
|---|---|---|---|
| $(-\infty, -1)$ | $x = -2$ | $\frac{(+)(-)}{(+)} = -$ | $f$ is decreasing. |
| $(-1, 0)$ | $x = -\frac{1}{2}$ | $\frac{(+)(+)}{(+)} = +$ | $f$ is increasing. |
| $(0, 1)$ | $x = \frac{1}{2}$ | $\frac{(+)(+)}{(+)} = +$ | $f$ is increasing. |
| $(1, \infty)$ | $x = 2$ | $\frac{(+)(-)}{(+)} = -$ | $f$ is decreasing |

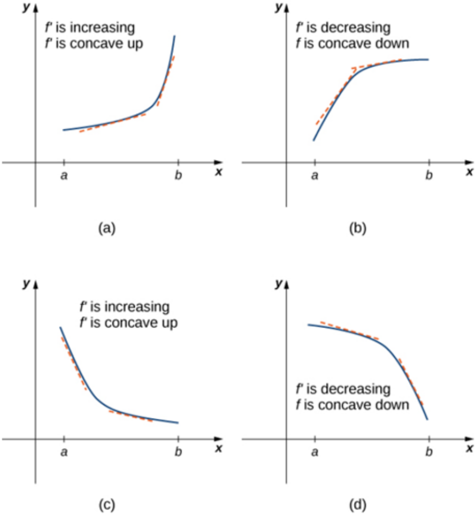
**Step3**:Since $f$ is decreasing over the interval $(-\infty, -1)$ and increasing over the interval $(-1, 0)$, $f$ has a local minimum at $x = -1$. Since $f$ is increasing over the interval $(-1, 0)$ and the interval $(0, 1)$, $f$ does not have a local extremum at $x = 0$. Since $f$ is increasing over the interval $(0, 1)$ and decreasing over the interval $(1, \infty)$, $f$ has a local maximum at $x = 1$.

We now know how to determine where a function is increasing or decreasing. However, there is another issue to consider regarding the shape of the graph of a function. If the graph curves, does it curve upward or curve downward? This notion is called the **concavity of the function** Figure 4.34(a) shows a function f with a graph that curves upward. As x increases, the slope of the tangent line increases. Thus, since the derivative increases as $x$ increases, $f'$ is an increasing function. We say this function $f$ is concave up. Figure 4.34(b) shows a function $f$ that curves downward. As $x$ increases, the slope of the tangent line decreases. Since the derivative decreases as $x$ increases, $f'$ is a decreasing function. We say this function $f$ is concave down.

## Definition

Let $f$ be a function that is differentiable over an open interval $I$. If $f'$ is increasing over $I$, we say $f$ is concave up over $I$. If $f'$ is decreasing over $I$, we say $f$ is concave down over $I$.

In general, without having the graph of a function $f$, how can we determine its concavity? By definition, a function $f$ is concave up if $f'$ is increasing. From Corollary 3, we know that if $f'$ is a differentiable function, then $f'$ is increasing if its derivative $f''(x) > 0$. Therefore, a function $f$ that is twice differentiable is concave up when $f''(x) > 0$. Similarly, a function $f$ is concave down if $f'$ is decreasing. We know that a differentiable function $f'$ is decreasing if its derivative $f''(x) < 0$. Therefore, a twice-differentiable function $f$ is concave down when $f''(x) < 0$. Applying this logic is known as **the concavity test**.


y
f' is increasing
f' is concave up
a
b
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(a)
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f' is decreasing
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(b)
y
f' is increasing
f' is concave up
a
b
x
(c)
y
f' is decreasing
f is concave down
a
b
x
(d)

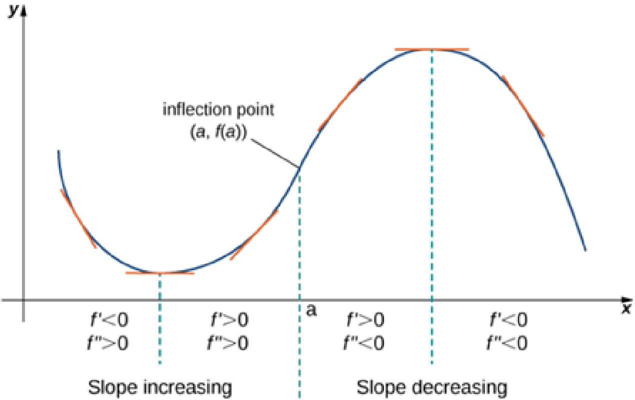
For concavity we shall use the following theorem

**Theorem**

*Let $f$ be a function that is twice differentiable over an interval $I$.*

a. *If $f''(x) > 0$ for all $x \in I$, then $f$ is concave up over $I$.*
b. *If $f''(x) < 0$ for all $x \in I$, then $f$ is concave down over $I$.*

We conclude that we can determine the concavity of a function $f$ by looking at the second derivative of $f$. In addition, we observe that a function f can switch concavity in the following figure. However, a continuous function can switch concavity only at a point $x$ if $f''(x) = 0$ or $f''(x)$ is undefined.


y
x
inflection point
(a, f(a))
a
f'<0
f''>0
f'>0
f''>0
f'>0
f''<0
f'<0
f''<0
Slope increasing
Slope decreasing

## Definition

If $f$ is continuous at a and $f$ changes concavity at $a$, the point $(a, f(a))$ is an inflection point of $f$.

## Example

For the function $f(x) = x^3 - 6x^2 + 9x + 30$, determine all intervals where $f$ is concave up and all intervals where $f$ is concave down. List all inflection points for $f$. Use a graphing utility to confirm your results.

To determine concavity, we need to find the second derivative $f''(x)$. The first derivative is $f'(x) = 3x^2 = 12x + 9$, so the second derivative is $f''(x) = 6x - 12$. If the function changes concavity, it occurs either when $f''(x) = 0$ or $f''(x)$ is undefined. Since $f''$ is defined for all real numbers $x$, we need only find where $f''(x) = 0$.

Solving the equation $6x - 12 = 0$, we see that $x = 2$ is the only place where $f$ could change concavity.
We now test points over the intervals $(-\infty, 2)$ and $(2, \infty)$ to determine the concavity of $f$. The points $x = 0$ and $x = 3$ are test points for these intervals.

| Interval | Test point | $f''$ at the test point | Conclusion |
|---|---|---|---|
| $(-\infty, 2)$ | $x = 0$ | Negative sign | $f$ is concave down |
| $(2, \infty)$ | $x = 3$ | Positive sign | $f$ is concave up |

We conclude that $f$ is concave down over the interval $(-\infty, 2)$ and concave up over the interval $(2, \infty)$. Since $f$ changes concavity at $x = 2$, the point $(2, f(2)) = (2, 32)$ is an inflection point.

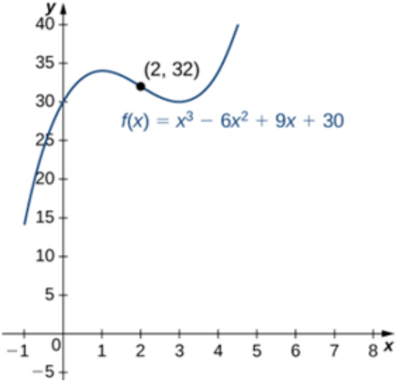



Figure: The given function has a point of inflection at $(2, 32)$ where the graph changes concavity.

One common application of calculus is calculating the minimum or maximum value of a function. For example, companies often want to minimize production costs or maximize revenue. In manufacturing, it is often desirable to minimize the amount of material used to package a product with a certain volume. In this section,

- We show how to set up these types of minimization and maximization problems
- We solve them by using the tools developed in this chapter.

The basic idea of the optimization problems that follow is the same. We have a particular quantity that we are interested in maximizing or minimizing. However, we also have some auxiliary condition that needs to be satisfied.

We start by the following example

**Example**

A rectangular garden is to be constructed using a rock wall as one side of the garden and wire fencing for the other three sides (see the figure3 on the following slide). Given 100 ft of wire fencing, determine the dimensions that would create a garden of maximum area. What is the maximum area?

Let $x$ denote the length of the side of the garden perpendicular to the rock wall and $y$ denote the length of the side parallel to the rock wall. Then the area of the garden is

$$A = x \cdot y.$$

We want to find the maximum possible area subject to the constraint that the total fencing is 100 ft. From figure3, the total amount of fencing used will be $2x + y$. Therefore, the constraint equation is $2x + y = 100$.

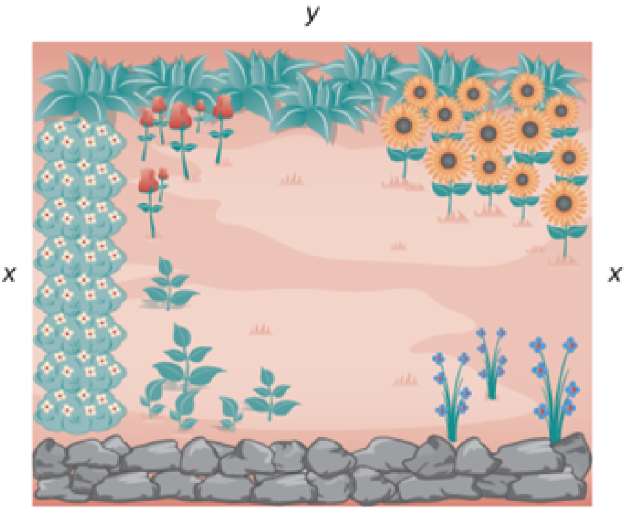



Figure: We want to determine the measurements $x$ and $y$ that will create a garden with a maximum area using 100 ft of fencing.

Solving this equation for $y$, we have $y = 100 - 2x$. Thus, we can write the area as

$$A(x) = x \cdot (100 - 2x) = 100x - 2x^2.$$

Before trying to maximize the area function $A(x) = 100x - 2x^2$, we need to determine the domain under consideration. To construct a rectangular garden, we certainly need the lengths of both sides to be positive. Therefore, we need $x > 0$ and $y > 0$. Since $y = 100 - 2x$, if $y > 0$, then $x < 50$. Therefore, we are trying to determine the maximum value of $A(x)$ for $x$ over the open interval $(0, 50)$. We do not know that a function necessarily has a maximum value over an open interval. However, we do know that a continuous function has an absolute maximum (and absolute minimum) over a closed interval.

Therefore, let's consider the function $A(x) = 100x - 2x^2$ over the closed interval $[0, 50]$. If the maximum value occurs at an interior point, then we have found the value $x$ in the open interval $(0, 50)$ that maximizes the area of the garden. Therefore, we consider the following problem:

$$\max A(x) = 100x - 2x^2 \quad over \quad [0, 50].$$

As mentioned earlier, since A is a continuous function on a closed, bounded interval, by the extreme value theorem, it has a maximum and a minimum. These extreme values occur either at endpoints or critical points. At the endpoints, $A(x) = 0$.

Since the area is positive for all $x$ in the open interval $(0, 50)$, the maximum must occur at a critical point. Differentiating the function $A(x)$, we obtain

$$A'(x) = 100 - 4x.$$

Therefore, the only critical point is $x = 25$. We conclude that the maximum area must occur when $x = 25$. Then we have

$$y = 100 - 2x = 100 - 2(25) = 50.$$

To maximize the area of the garden, let $x = 25$ ft and $y = 50ft$. The area of this garden is $1250ft^2$. The strategy of studying the sign of the first derivative can also be used to make sure that we have a maximum at $x = 25$.

Now let's look at a general strategy.

1. Introduce all variables. If applicable, draw a figure and label all variables.
2. Determine which quantity is to be maximized or minimized, and for what range of values of the other variables (if this can be determined at this time).
3. Write a formula for the quantity to be maximized or minimized in terms of the variables. This formula may involve more than one variable.
4. Write any equations relating the independent variables in the formula from step 3. Use these equations to write the quantity to be maximized or minimized as a function of one variable.
5. Identify the domain of consideration for the function in step 4 based on the physical problem to be solved.
6. Locate the maximum or minimum value of the function from step 4. This step typically involves looking for critical

Now let's apply this strategy to maximize the volume of an open-top box given a constraint on the amount of material to be used.

**Example**

An open-top box is to be made from a 24 in. by 36 in. piece of cardboard by removing a square from each corner of the box and folding up the flaps on each side. What size square should be cut out of each corner to get a box with the maximum volume?

**Step 1**: Let $x$ be the side length of the square to be removed from each corner. Then, the remaining four flaps can be folded up to form an open-top box. Let $V$ be the volume of the resulting box.
**Step 2**: We are trying to maximize the volume of a box. Therefore, the problem is to maximize $V$.

**Step 3**: As mentioned in step 2, are trying to maximize the volume of a box. The volume of a box is $V = L \cdot W \cdot H$, where $L$, $W$, and $H$ are the length, width, and height, respectively.

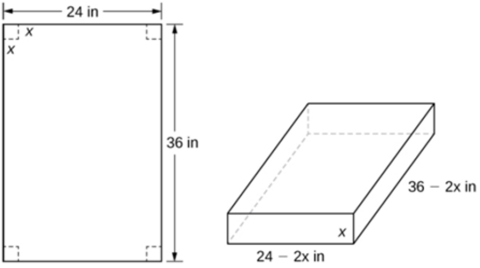



Figure: A square with side length $x$ inches is removed from each corner of the piece of cardboard. The remaining flaps are folded to form an open-top box.

**Step 4**: From Figure4, we see that the height of the box is $x$ inches, the length is $36 - 2x$ inches, and the width is $24 - 2x$ inches. Therefore, the volume of the box is

$$V(x) = x(36 - 2x)(24 - 2x) = 4x^3 - 120x^2 + 864x.$$

**Step 5**: To determine the domain of consideration, let's examine Figure4. Certainly, we need $x > 0$. Furthermore, the side length of the square cannot be greater than or equal to half the length of the shorter side, 24 in.; otherwise, one of the flaps would be completely cut off. Therefore, we are trying to determine whether there is a maximum volume of the box for $x$ over the open interval $(0, 12)$.

Since $V$ is a continuous function over the closed interval $[0, 12]$, we know V will have an absolute maximum over the closed interval. Therefore, we consider V over the closed interval $[0, 12]$ and check whether the absolute maximum occurs at an interior point.

**Step 6**: Since $V(x)$ is a continuous function over the closed, bounded interval $[0, 12]$, $V$ must have an absolute maximum (and an absolute minimum). Since $V(x) = 0$ at the endpoints and $V(x) > 0$ for $0 < x < 12$, the maximum must occur at a critical point. The derivative is

$$V'(x) = 12x^2 - 240x + 864.$$

To find the critical points, we need to solve the equation

$$12x^2 - 240x + 864 = 0$$

Dividing both sides of this equation by 12, the problem simplifies to solving the equation

$$x^2 - 20x + 72 = 0$$

to get

$$x = \frac{20 \pm \sqrt{112}}{2} = 10 \pm 2\sqrt{7}.$$

Since $10 + 2\sqrt{7}$ is not in the domain of consideration, the only critical point we need to consider is $10 - 2\sqrt{7}$. Therefore, the volume is maximized if we let $x = 10 - 2\sqrt{7}$ in. The maximum volume is $V(10 - 2\sqrt{7}) = 640 + 4487 \approx 1825in^3$.

## Example

An island is 2 mi due north of its closest point along a straight shoreline. A visitor is staying at a cabin on the shore that is 6 mi west of that point. The visitor is planning to go from the cabin to the island. Suppose the visitor runs at a rate of 8 mph and swims at a rate of 3 mph. How far should the visitor run before swimming to minimize the time it takes to reach the island?

**Step 1**: Let $x$ be the distance running and let $y$ be the distance swimming (Figure 4.66). Let $T$ be the time it takes to get from the cabin to the island.
**Step 2**: The problem is to minimize T.

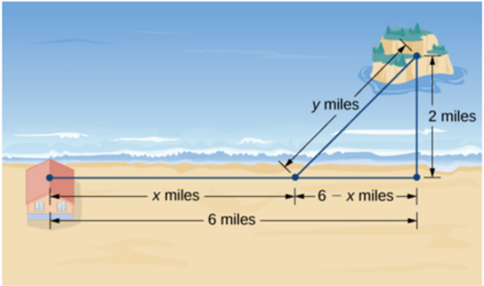



Figure: How can we choose $x$ and $y$ to minimize the travel time from the cabin to the island?.

**Step 3**: To find the time spent traveling from the cabin to the island, add the time spent running and the time spent swimming. Since $Distance = Rate \times Time(D = R \times T)$, the time spent running is

$$T_{running} = \frac{D_{running}}{R_{running}} = \frac{x}{8}$$

and the time spent swimming is

$$T_{swimming} = \frac{D_{swimming}}{R_{swimming}} = \frac{y}{3}.$$

Thus

$$T = \frac{x}{8} + \frac{y}{3}.$$

**Step 4**: From Figure5, the line segment of $y$ miles forms the hypotenuse of a right triangle with legs of length 2 mi and $6 - x$ mi. Therefore, by the Pythagorean theorem, $2^2 + (6-x)^2 = y^2$, and we obtain $y = \sqrt{2^2 + (6-x)^2}$. Thus, the total time spent traveling is given by the function

$$T = \frac{x}{8} + \frac{\sqrt{4 + (6-x)^2}}{3}.$$

**Step 5**: From the figure, we see that $0 \leq x \leq 6$. Therefore, $[0, 6]$ is the domain of consideration.
**Step 6**: Since $T(x)$ is a continuous function over a closed, bounded interval, it has a maximum and a minimum. Let's begin by looking for any critical points of T over the interval$[0, 6]$

The derivative is

$$T'(x) = \frac{1}{8} - \frac{6-x}{3\sqrt{4+(6-x)^2}}.$$

If $T'(x) = 0$, then

$$\frac{1}{8} = \frac{6-x}{3\sqrt{4+(6-x)^2}}$$

Therefore

$$3\sqrt{4+(6-x)^2} = 8(6-x) \tag{2.1}$$

Squaring both sides to get

$$64(6-x)^2 = 9(4+(6-x)^2)$$

Which implies

$$55(6-x)^2 = 36$$

We conclude that if $x$ is a critical point, then $x$ satisfies

$$(6-x)^2 = \frac{36}{55} \Leftrightarrow x = 6 \pm \frac{6}{\sqrt{55}}.$$

Since $6 + \frac{6}{\sqrt{55}}$ is not in the domain, it is not a possibility for a critical point. On the other hand, $6 - \frac{6}{\sqrt{55}}$ is in the domain. Since we squared both sides of2.1 to arrive at the possible critical points, it remains to verify that this value satisfies this equation. Checking, we see that it satisfies, thus it is the only critical point.

To justify that the time is minimized for this value of $x$, we just need to check the values of $T(x)$ at the endpoints $x = 0$ and $x = 6$, and compare them with the value of $T(x)$ at the critical point $x = 6 - \frac{6}{\sqrt{55}}$ We find that $T(0) \approx 2.108h$ and $T(6) \approx 1.417h$, whereas $T(6 - \frac{6}{\sqrt{55}}) \approx 1.368h$.

In the previous examples, we considered functions on closed, bounded domains. Consequently, by the extreme value theorem, we were guaranteed that the functions had absolute extrema. Let's now consider functions for which the domain is neither closed nor bounded.

**Example**

A rectangular box with a square base, an open top, and a volume of $216in^3$ is to be constructed. What should the dimensions of the box be to minimize the surface area of the box? What is the minimum surface area?

**Step 1**: Draw a rectangular box and introduce the variable $x$ to represent the length of each side of the square base; let $y$ represent the height of the box. Let S denote the surface area of the open-top box.

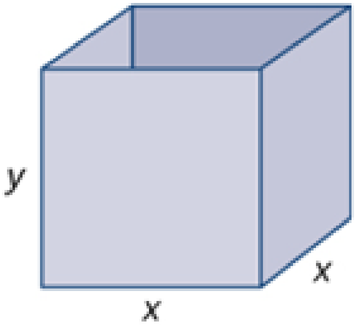



Figure: We want to minimize the surface area of a square-based box with a given volume.

**Step 2**: We need to minimize the surface area. Therefore, we need to minimize $S$.
**Step 3**: Since the box has an open top, we need only determine the area of the four vertical sides and the base. The area of each of the four vertical sides is $x \cdot y$. The area of the base is $x^2$. Therefore, the surface area of the box is

$$S = 4xy + x^2$$

**Step 4**: Since the volume of this box is $x^2y$ and the volume is given as $216in^3$, the constraint equation is

$$x^2y = 216$$

and solving this constraint yields $y = \frac{216}{x^2}$.

Therefore, we can write the surface area as a function of $x$ only:

$$S(x) = 4x\left(\frac{216}{x^2}\right) + x^2 = \frac{864}{x} + x^2.$$

**Step 5**: Since we are requiring that $x^2y = 216$, we cannot have $x = 0$. Therefore, we need $x > 0$. On the other hand, $x$ is allowed to have any positive value. Note that as $x$ becomes large, the height of the box $y$ becomes correspondingly small so that $x^2y = 216$. Similarly, as $x$ becomes small, the height of the box becomes correspondingly large. We conclude that the domain is the open, unbounded interval $(0, \infty)$.

**Step 6**: Note that as $x \to 0^+$, $S \to \infty$ and also as $x \to \infty$, $S \to \infty$. Since $S$ is a continuous function that approaches infinity at the ends, it must have an absolute minimum at some $x \in (0, \infty)$. This minimum must occur at a critical point of $S$. The derivative is

$$S'(x) = -\frac{864}{x^2} + 2x.$$

Therefore $S'(x) = 0$ if $\frac{864}{x^2} = 2x$. Solving this equation for $x$, we obtain $x^3 = 432$, so $x = 6\sqrt[3]{2}$. Since this is the only critical point of $S$, the absolute minimum must occur at this value. From this $x$ we get that $y = 3\sqrt[3]{2}$, With these dimensions, the surface area is

$$S = 4xy + x^2 = 108\sqrt[3]{2}in^2.$$

In this section, we examine a powerful tool for evaluating limits. This tool, known as L'Hôpital's rule, uses derivatives to calculate limits. With this rule, we will be able to evaluate many limits we have not yet been able to determine. Instead of relying on numerical evidence to conjecture that a limit exists, we will be able to show definitively that a limit exists and to determine its exact value. L'Hôpital's rule can be used to evaluate limits involving the quotient of two functions. Consider

$$\lim_{x \to a} \frac{f(x)}{g(x)}.$$

If $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2 \neq 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L_1}{L_2}.$$

However, what if $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$? We call this one of the **indeterminate forms** of type $\frac{0}{0}$. This is considered an indeterminate form because we cannot determine the exact behavior of $\frac{f(x)}{g(x)}$ as $x \to a$ without further analysis. We have seen examples of this earlier in the text. To overcame this situation we used different techniques but one easy way of doing so is to use the following theorem known as l'Hôpital's rule.

## Theorem

*Suppose $f$ and $g$ are differentiable functions over an open interval containing $a$, except possibly at $a$. If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

*assuming the limit on the right exists or is $\infty$ or $-\infty$. This result also holds if we are considering one-sided limits, or if $a = \infty$ or $-\infty$.*

## Example

Calculate by means of l'Hôpital's rule

$$\lim_{x\to 2} \frac{x^2 - 4}{x - 2}$$

Remember that we can evaluate this limit by factoring and cancelling, that is

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{(x-2)(x+2)}{x-2}=\lim_{x\to 2}(x+2)=4.$$

We have of course the IF of the form $0/0$, therefore we can apply the rule to get

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{\frac{d}{dx}(x^2-4)}{\frac{d}{dx}(x-2)}=\lim_{x\to 2}2x=4.$$

We can also use l'Hôpital's rule to evaluate limits of quotients $\frac{f(x)}{g(x)}$ when $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$ as $x \to a$. Limits of this form are classified as indeterminate forms of type $\infty/\infty$. Again, note that we are not actually dividing $\infty$ by $\infty$. Since $\infty$ is not a real number, that is impossible; rather, $\infty/\infty$ is used to represent a quotient of limits, each of which is $\infty$ or $-\infty$.

**Theorem**

*Suppose $f$ and $g$ are differentiable functions over an open interval containing $a$, except possibly at $a$. If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

*assuming the limit on the right exists or is $\infty$ or $-\infty$. This result also holds if we are considering one-sided limits, or if $a = \infty$ or $-\infty$.*

## Example

Calculate

$$\lim_{x\to\infty} \frac{3x+5}{2x+1}.$$

Since $3x+5$ and $2x+1$ are first-degree polynomials with positive leading coefficients, $\lim_{x\to\infty}(3x+5)=\infty$ and $\lim_{x\to\infty}(2x+1)=\infty$. Therefore, we apply L'Hôpital's rule and obtain

$$\lim_{x\to\infty} \frac{3x+5}{2x+1} = \lim_{x\to\infty} \frac{3}{2} = \frac{3}{2}.$$

Observe that also this value can be obtained when using the leading term rule since we have that $x\to\infty$.

- For other clarifications, please refer to REB book(from page 189 Task 9.8, page 195 Task 9.9, 196 Task 9.10) and different examples from page 188 up to 200.
- Exercises are also found in the attached material called EXERCISES!!!.

THANK YOU!!!