Mathematics S 4

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Today

In these slides we aim to explain the following:

1. Geometric interpretation of derivatives (Tangents and normal lines to curves)
2. Linear Approximations and Differentials
3. Mean value theorems
4. Increasing and decreasing functions
5. Maxima, minima and concavity
6. Applied Optimization Problems
7. L'Hôpital's Rule

If the function $y = f(x)$ is represented by a curve, then $f'(x) = \frac{dy}{dx}$ is the slope of the tangent line at any point of the curve of the form $(x, f(x))$. This is also known as the first derivative of our function or the rate of change of our function with respect to the independent variable x. The equation of the tangent in the point $(a, f(a))$ is given by:

Definition

he equation of the tangent is by equating slopes and is

$$T \equiv y - f(a) = f'(a)(x - a).$$

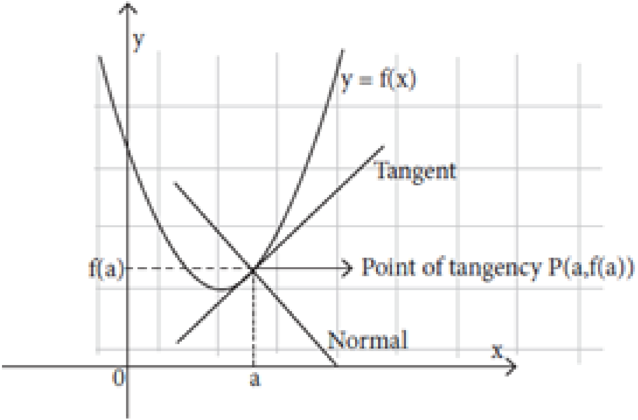


Figure: The tangent and Normal line to the curve at a point $(a, f(a))$.

Observe from the above figure that the normal line is perpendicular to the tangent line in our point. Thus their slope satisfy the perpendicularity condition i.e. the product of their slope is equal to -1. This means that the equation of our Normal line is given by

Definition

If $y = f(x)$ and the point on the curve of y is given by $(a, f(a))$, then the equation of the Normal line in this point is given by

$$N \equiv y - f(a) = -\frac{1}{f'(a)}(x - a).$$

Example

Find, at the given point, the equation of the tangent and the equation of the normal lines on the curve of $y = x^2 - 1$ at $x = 1$.

We have been given $y = x^2 - 1$ and $x = 1$. Our point is $(a, f(a)) = (1, 0)$ and the slope is given at any point by $f'(x) = 2x$ which yields the slope of our tangent at $x = 1$ to be $f'(1) = 2$. Now the tangent will be given by

$$\begin{aligned} y - 0 &= 2(x-1) \\ &= 2x - 2. \end{aligned}$$

Since the normal line is perpendicular to the tangent we have that

$$N \equiv y - 0 = -\frac{1}{2}(x-2).$$

to get

$$N \equiv y = -\frac{1}{2}x + 1.$$

REMEMBER that we can also have particular tangents such as those that are horizontal (i.e. where the first derivatives vanish) and those that are vertical (i.e the firs derivative is not defined or is infinite.) See example 9.15 in REB book pge 185.

We have just seen how derivatives allow us to compare related quantities that are changing over time. In this section, we examine another application of derivatives: the ability to approximate functions locally by linear functions. Linear functions are the easiest functions with which to work, so they provide a useful tool for approximating function values. In addition, the ideas presented in this section are generalized later in the text when we study how to approximate functions by higher-degree (Later application.) polynomials

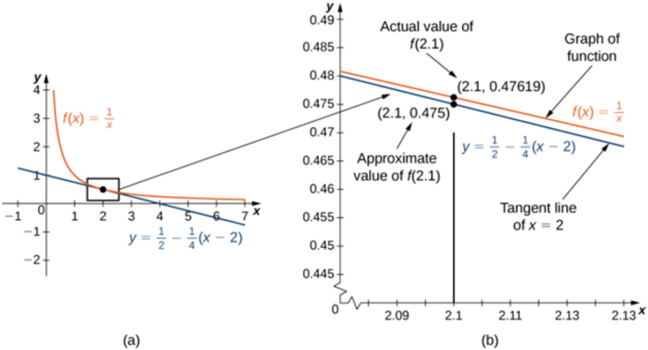
Consider a function f that is differentiable at a point $x = a$. Recall that the tangent line to the graph of f at a is given by the equation

$$y = f(a) + f'(a)(x - a).$$

For example, consider the function $f() = \frac{1}{x}$ at $a = 2$. Since f is differentiable at $x = 2$ and $f'(x) = -\frac{1}{x^2}$, we see that $f'(2) = -\frac{1}{4}$. Therefore, the tangent line to the graph of f at $a = 2$ is given by the equation

$$y = \frac{1}{2} - \frac{1}{4}(x - 2).$$

Now, how to approximate for example $f(2.1)$? We first observe the following figures



y
4
3
2
1
0
−1
−2
−1
1
2
3
4
5
6
7
x
f(x) = 1/x
y = 1/2 − 1/4(x − 2)
(a)
y
0.49
0.485
0.48
0.475
0.47
0.465
0.46
0.455
0.45
0.445
0
Actual value of f(2.1)
(2.1, 0.47619)
(2.1, 0.475)
Approximate value of f(2.1)
Graph of function
f(x) = 1/x
y = 1/2 − 1/4(x − 2)
Tangent line of x = 2
2.09
2.1
2.11
2.13
2.13
x
(b)

The above figure (a) shows a graph of $f(x) = 1/x$ along with the tangent line to f at $x = 2$. Note that for x near 2, the graph of the tangent line is close to the graph of f. As a result, we can use the equation of the tangent line to approximate $f(x)$ for x near 2. For example, if $x = 2.1$, the y value of the corresponding point on the tangent line is

$$y = \frac{1}{2} - \frac{1}{4}(2.1 - 2) = 0.475$$

The actual value of $f(2.1)$ is given by $f(2.1) = \frac{1}{2.1} \approx 0.47619$. Therefore, the tangent line gives us a fairly good approximation of $f(2.1)$.

In Figure (b))however, note that for values of x far from 2, the equation of the tangent line does not give us a good approximation. For example, if $x = 10$, the $y-$value of the corresponding point on the tangent line is

$$y = \frac{1}{2} - \frac{1}{4}(10 - 2) = -1.5$$

whereas the value of the function at $x = 10$ is $f(10) = 0.1$. In general, for a differentiable function f, the equation of the tangent line to f at $x = a$ can be used to approximate $f(x)$ for x near a. Therefore, we can write

$$f(x) \approx f(a) + f'(a)(x - a)$$

We call the linear function

$$L(x) \approx f(a) + f'(a)(x - a)$$

the **linear approximation**, or **tangent line approximation**, of f at $x = a$. This function L is also known as the linearization of f at $x = a$. To show how useful the linear approximation can be, we look at how to find the linear approximation for $f = \sqrt{x}$ at $x = 9$.

Example

Find the linear approximation of $f(x) = \sqrt{x}$ at $x = 9$ and use the approximation to estimate $\sqrt{9.1}$.

Since we are looking for the linear approximation at $x = 9$. We know the linear approximation is given by

$$L(x) \approx f(9) + f'(9)(x - a)$$

We need to find $f(9)$ and $f'(9)$.

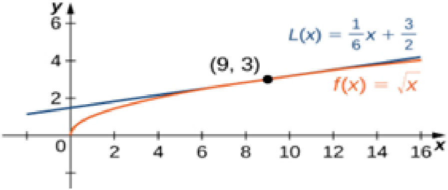
$$\begin{aligned} f(x) &= \sqrt{9} &\Rightarrow\quad& f(9) = 3 \\ f'(x) &= \frac{1}{2\sqrt{x}} &\Rightarrow\quad& f'(9) = \frac{1}{6}. \end{aligned}$$

Therefore, the linear approximation is given by

$$L(x) \approx 3 + \frac{1}{6}(x - 9).$$

Using the linear approximation, we can estimate $\sqrt{9.1}$ by writing

$$\sqrt{9.1} = f(9.1) \approx 3 + \frac{1}{6}(9.1 - 9) \approx 3.0167.$$



We have seen that linear approximations can be used to estimate function values. They can also be used to estimate the amount a function value changes as a result of a small change in the input. To discuss this more formally, we define a related concept: differentials. Differentials provide us with a way of estimating the amount a function changes as a result of a small change in input values. When we first looked at derivatives, we used the Leibniz notation dy/dx to represent the derivative of y with respect to x. Although we used the expressions dy and dx in this notation, they did not have meaning on their own. Here we see a meaning to the expressions dy and dx. Suppose $y = f(x)$ is a differentiable function. Let dx be an independent variable that can be assigned any nonzero real number, and define the dependent variable dy by

$$dy = f'(x)dx. \tag{1.1}$$

It is important to notice that dy is a function of both x and dx. The expressions dy and dx are called **differentials**. We can divide both sides of 1.1 by dx and get

$$\frac{dy}{dx} = f'(x). \tag{1.2}$$

This is the familiar expression we have used to denote a derivative. The equation 1.1 is known as the differential form of 1.2.

Example

For each of the function $f(x) = x^2 + 2x$ find dy and evaluate when $x = 3$ and $dx = 0.1$.

The key step is calculating the derivative. When we have that, we can obtain dy directly. Since $f(x) = x^2 + 2x$, then $f'(x) = 2x + 2$ and therefore

$$dy = (2x + 2)dx.$$

when $x = 3$ and $dx = 0.1$ we have that

$$dy = (2 \cdot 3 + 2)(0.1) = 0.8.$$

We now connect differentials to linear approximations. Differentials can be used to estimate the change in the value of a function resulting from a small change in input values. Consider a function f that is differentiable at point a. Suppose the input x changes by a small amount. We are interested in how much the output y changes.

If x changes from a to $a+dx$, then the change in x is dx (also denoted $\triangle x$), and the change in y is given by

$$\triangle y = f(a+dx) - f(a)$$

Instead of calculating the exact change in y, however, it is often easier to approximate the change in y by using a linear approximation. For x near $a, f(x)$ can be approximated by the linear approximation

$$L(x) = f(a) + f'(a)(x-a)$$

Therefore, if dx is small,

$$f(a+dx) \approx L(a+dx) = f(a) + f'(a)(a+dx-a)$$

That is

$$f(a+dx)-f(a) \approx L(a+dx)-f(a) = f'(a)dx.$$

In other words, the actual change in the function f if x increases from a to $a+dx$ is approximately the difference between $L(a+dx)$ and $f(a)$, where $L(x)$ is the linear approximation of f at a. By definition of $L(x)$, this difference is equal to $f'(a)dx$. In summary,

$$\triangle y = f(a+dx)-f(a) \approx L(a+dx)-f(a) = f'(a)dx = dy.$$

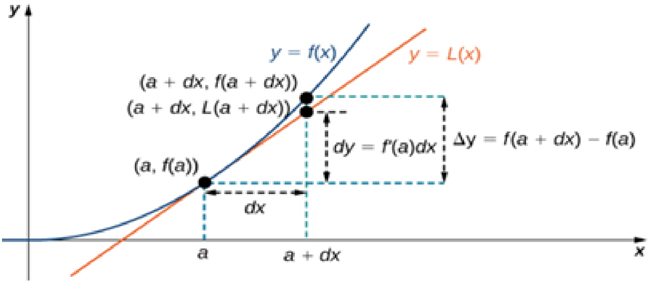


Figure: The differential $dy = f'(a)dx$ is used to approximate the actual change in y if x increases from a to adx.

Given a particular function, we are often interested in determining the largest and smallest values of the function. This information is important in creating accurate graphs. Finding the maximum and minimum values of a function also has practical significance because we can use this method to solve optimization problems, such as maximizing profit, minimizing the amount of material used in manufacturing an aluminum can, or finding the maximum height a rocket can reach. In this section, we look at how to use derivatives to find the largest and smallest values for a function.

The following definition should be used

Definition

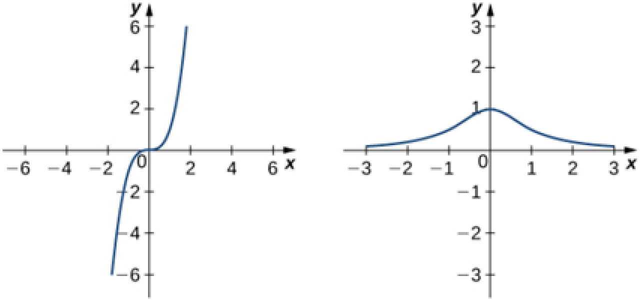
Let f be a function defined over an interval I and let $c \in I$. We say f has an absolute maximum on I at c if $f(c) \geq f(x)$ for all $x \in I$. We say f has an absolute minimum on I at c if $f(c) \leq f(x)$ for all $x \in I$. If f has an absolute maximum on I at c or an absolute minimum on I at c, we say f has an absolute extremum on I at c.

Before proceeding, let's note two important issues regarding this definition. First, the term absolute here does not refer to absolute value. An absolute extremum may be positive, negative, or zero. Second, if a function f has an absolute extremum over an interval I at c, the absolute extremum is $f(c)$.

The real number c is a point in the domain at which the absolute extremum occurs. For example, consider the function $f(x) = 1/(x^2 + 1)$ over the interval $(-\infty, \infty)$. Since

$$f(0) = 1 \geq \frac{1}{x^2 + 1} = f(x)$$

for all real numbers x, we say f has an absolute maximum over $(-\infty, \infty)$ at $x = 0$. The absolute maximum is $f(0) = 1$. A function may have both an absolute maximum and an absolute minimum, just one extremum, or neither. The following figures show several functions and some of the different possibilities regarding absolute extrema.

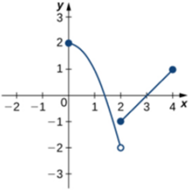


$f(x) = x^3$ on $(-\infty, \infty)$
No absolute maximum
No absolute minimum

(a)

$f(x) = \frac{1}{x^2+1}$ on $(-\infty, \infty)$
Absolute maximum of 1 at $x = 0$
No absolute minimum

(b)

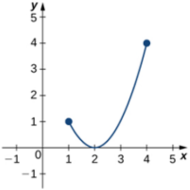


$f(x) = \begin{cases} 2 - x^2 & 0 \le x < 2 \\ x - 3 & 2 \le x \le 4 \end{cases}$

Absolute maximum of 2 at $x = 0$

No absolute minimum

(d)

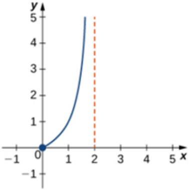


$f(x) = (x - 2)^2$ on $[1, 4]$

Absolute maximum of 4 at $x = 4$

Absolute minimum of 0 at $x = 2$

(e)



$f(x) = \frac{x}{2 - x}$ on $[0, 2)$

No absolute maximum

Absolute minimum of 0 at $x = 0$

(f)

Consider the function f shown in the figure on the following slide. The graph can be described as two mountains with a valley in the middle. The absolute maximum value of the function occurs at the higher peak, at $x = 2$. However, $x = 0$ is also a point of interest. Although $f(0)$ is not the largest value of f, the value $f(0)$ is larger than $f(x)$ for all x near 0. We say f has a local maximum at $x = 0$. Similarly, the function f does not have an absolute minimum, but it does have a local minimum at $x = 1$ because $f(1)$ is less than $f(x)$ for x near 1.

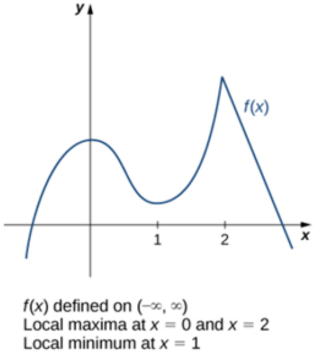


Figure: This function f has two local maxima and one local minimum. The local maximum at $x = 2$ is also the absolute maximum.

Definition

A function f has a local maximum at c if there exists an open interval I containing c such that I is contained in the domain of f and $f(c) \geq f(x)$ for all $x \in I$. A function f has a local minimum at c if there exists an open interval I containing c such that I is contained in the domain of f and $f(c) \leq f(x)$ for all $x \in I$. A function f has a local extremum at c if f has a local maximum at c or f has a local minimum at c.

Note that if f has an absolute extremum at c and f is defined over an interval containing c, then $f(c)$ is also considered a local extremum.

Definition

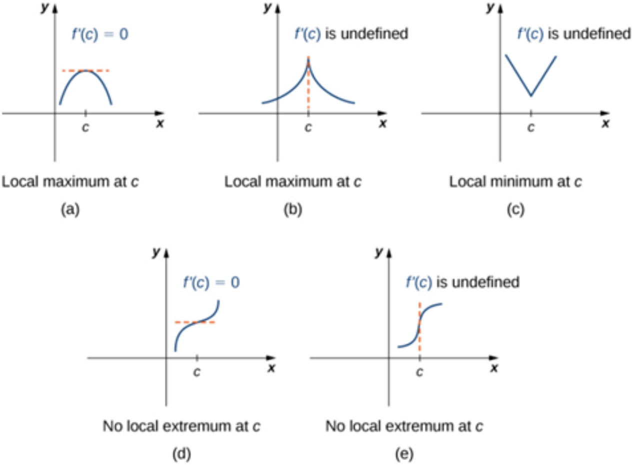
Let c be an interior point in the domain of f. We say that c is a critical or stationary point of f if $f'(c) = 0$ or $f'(c)$ is undefined.

As mentioned earlier, if f has a local extremum at a point $x = c$, then c must be a critical point of f. This fact is known as Fermat's theorem.

Theorem

If f has a local extremum at c and f is differentiable at c, then $f'(c) = 0$.

However to locate different critical points requires some techniques such as those of using a graphing utility, or even a table of sign should be used to identify different types of critical points. More over, the following graphs should be of importance when trying to locate and classify these points.


y
x
c
f'(c) = 0
Local maximum at c
(a)
f'(c) is undefined
Local maximum at c
(b)
f'(c) is undefined
Local minimum at c
(c)
f'(c) = 0
No local extremum at c
(d)
f'(c) is undefined
No local extremum at c
(e)

Example

For each of the following functions, find all critical (stationary) points. Use a graphing utility to determine whether the function has a local extremum at each of the critical points.

a. $f(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x$

b. $g(x) = (x^2 - 1)^3$

The derivative for the first function is $f'(x) = x^2 - 5x + 4$ and is defined for all real numbers x. Therefore, we only need to find the values for x where $f'(x) = 0$. Since $f(x) = x^2 - 5x + 4 = (x-4)(x-1)$, the critical points are $x = 1$ and $x = 4$. For the second function we have that the derivative is given by the Chain Rule

$$g'(x) = 3(x^2 - 1)^2 \cdot 2x = 6x(x^2 - 1)^2.$$

For critical points we have $g'(x) = 0$ which gives critical points at $x = 0$ and $x = \pm 1$.

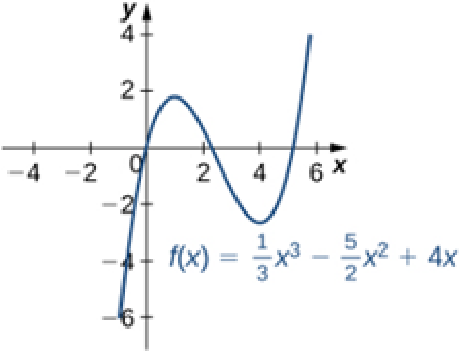


Figure: This function has a local maximum and a local minimum.

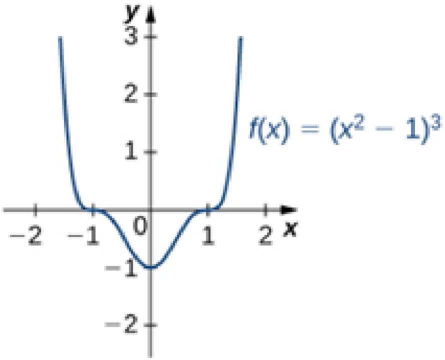


Figure: This function has three critical points: $x = 0, x = 1$, and $x = -1$. The function has a local (and absolute) minimum at $x = 0$, but does not have extrema at the other two critical points.

Now let's look at how to use this strategy to find the absolute maximum and absolute minimum values for continuous functions in a given interval.

Example

For each of the following functions, find the absolute maximum and absolute minimum over the specified interval and state where those values occur.

a. $f(x) = -x^2 + 3x - 2$ over $[1, 3]$.

b. $x^2 - 3x^{3/2}$ over $[0, 2]$.

We need to proceed as follows:

Step1:Evaluate f at the endpoints $x = 1$ and $x = 3$.

$$f(1) = 0 \quad and \quad f(3) = -2$$

Step2: Since $f'(x) = -2x + 3, f'$ is defined for all real numbers x. Therefore, there are no critical points where the derivative is undefined. It remains to check where $f'(x) = 0$. Since $f'(x) = -2x + 3 = 0$ at $x = \frac{3}{2}$ and $x = \frac{3}{2}$ is in the interval $[1, 3], f\left(\frac{3}{2}\right)$ is a a candidate for local extremum of f over our interval. When evaluating

$$f\left(\frac{3}{2}\right) = \frac{1}{4}.$$

We summarize our results in the following table

x	$f(x)$	Conclusion
1	0	
$\frac{3}{2}$	$\frac{1}{4}$	absolute maximum
3	-2	absolute minimum

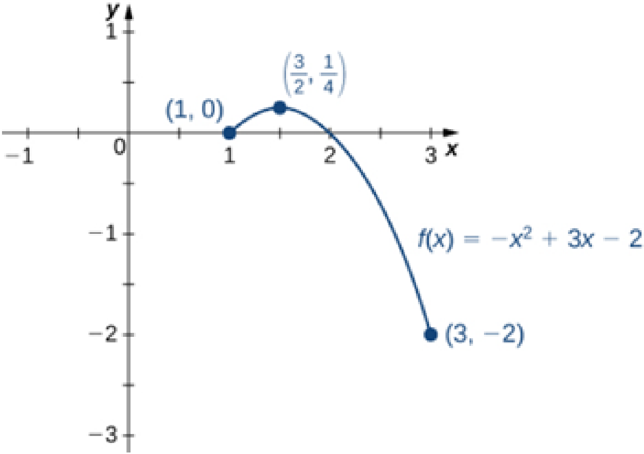


Figure: This function has both an absolute maximum and an absolute minimum.

The Mean Value Theorem is one of the most important theorems in calculus. We look at some of its implications at the end of this section. First, let's start with a special case of the Mean Value Theorem, called Rolle's theorem.
Informally, Rolle's theorem states that if the outputs of a differentiable function f are equal at the endpoints of an interval, then there must be an interior point c where $f'(c) = 0$. which is the first derivative of our function.

Theorem

Let f be a continuous function over the closed interval $[a, b]$ and differentiable over the open interval (a, b) such that $f(a) = f(b)$. There then exists at least one $c \in (a, b)$ such that $f'(c) = 0$.

We illustrate this theorem in the following figures

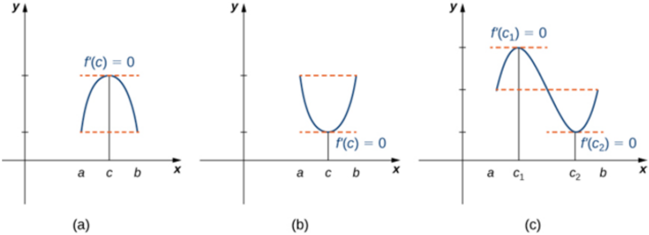


Figure: If a differentiable function f satisfies $f(a) = f(b)$, then its derivative must be zero at some point(s) between a and b.

We illustrate the theorem using an example

Example

For each of the following functions, verify that the function satisfies the criteria stated in Rolle's theorem and find all values c in the given interval where $f'(c) = 0$.

$$f(x) = x^2 + 2x \quad in \quad [-2, 0].$$

Since f is a polynomial, it is continuous and differentiable everywhere. In addition, $f(-2) = 0 = f(0)$. Therefore, f satisfies the criteria of Rolle's theorem. We conclude that there exists at least one value $c \in (-2, 0)$ such that $f'(c) = 0$. Since $f'(x) = 2x + 2 = 2(x + 1)$, we see that $f'(c) = 2(c + 1) = 0$ implies $c = -1$ as shown in the following graph.

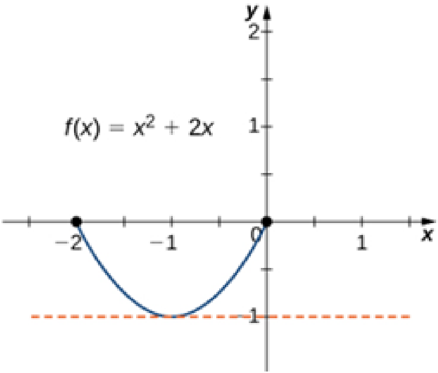


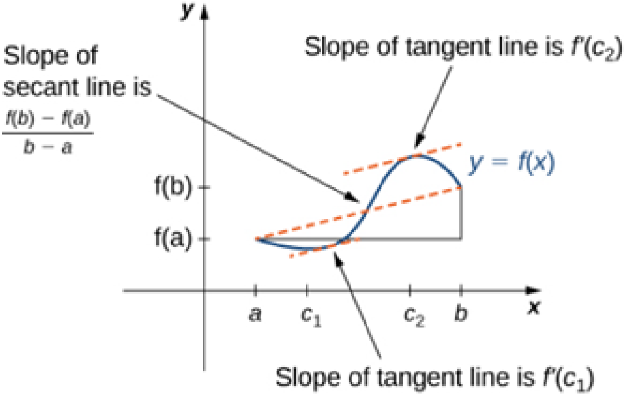
Figure: This function is continuous and differentiable over $[-2, 0], f'(c) = 0$ when $c = -1$.

Rolle's theorem is a special case of the Mean Value Theorem. In Rolle's theorem, we consider differentiable functions f defined on a closed interval $[a, b]$ with $f(a) = f(b)$. The Mean Value Theorem generalizes Rolle's theorem by considering functions that do not necessarily have equal value at the endpoints. Consequently, we can view the Mean Value Theorem as a slanted version of Rolle's theorem.

Theorem

Let f be continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b). Then, there exists at least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



y
Slope of tangent line is $f'(c_2)$
Slope of secant line is $\frac{f(b)-f(a)}{b-a}$
$y = f(x)$
f(b)
f(a)
a
c_1
c_2
b
x
Slope of tangent line is $f'(c_1)$

In the next example, we show how the Mean Value Theorem can be applied

Example

For $f(x) = \sqrt{x}$ over the interval $[0, 9]$, show that f satisfies the hypothesis of the Mean Value Theorem, and therefore there exists at least one value $c \in (0, 9)$ such that $f'(c)$ is equal to the slope of the line connecting $(0, f(0))$ and $(9, f(9))$. Find these values c guaranteed by the Mean Value Theorem.

We know that $f(x) = \sqrt{x}$ is continuous over $[0, 9]$ and differentiable over $(0, 9)$. Therefore, f satisfies the hypotheses of the Mean Value Theorem, and there must exist at least one value $c \in (0, 9)$ such that $f'(c)$ slope of the line connecting $(0, f(0))$ and $(9, f(9))$.

To determine which value(s) of c are guaranteed, first calculate the derivative of f. The derivative $f'(x) = \frac{1}{2\sqrt{x}}$. The slope of the line connecting $(0, f(0))$ and $(9, f(9))$ is given by

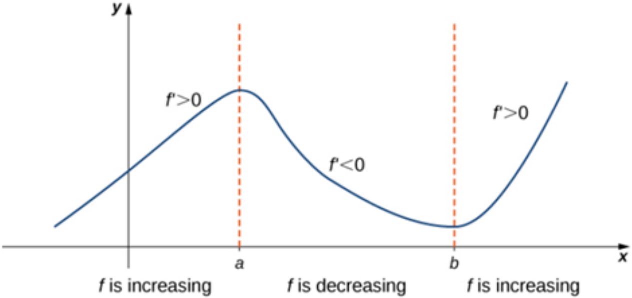
$$\frac{f(9) - f(0)}{9 - 0} = \frac{\sqrt{9} - \sqrt{0}}{9} = \frac{1}{3}$$

We want to find a c such that $f'(c) = \frac{1}{3}$. That is, we want to find c such that

$$\frac{1}{2\sqrt{c}} = \frac{1}{3}.$$

Solving for c we get that $c = \frac{9}{4}$. At this point, the slope of the tangent line equals the slope of the line joining the endpoints.

The third corollary of the Mean Value Theorem discusses when a function is increasing and when it is decreasing. Observe the following figure



Recall that a function f is increasing over I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, whereas f is decreasing over I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$. Using the Mean Value Theorem, we can show that if the derivative of a function is positive, then the function is increasing; if the derivative is negative, then the function is decreasing as it can be seen in the above figure.

Corollary

Let f be continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b).

i. *If $f'(x) > 0$ for all $x \in (a, b)$, then f is an increasing function over $[a, b]$.*
ii. *If $f'(x) < 0$ for all $x \in (a, b)$, then f is an decreasing function over $[a, b]$.*

- For other clarifications, please refer to REB book(from page 184 up to 187).
- Exercises are also found in the attached material called EXERCISES!!!.

THANK YOU!!!