A Resonant Equivalent Theorem

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October 2, 2025

Abstract

We introduce the *Resonant Equivalence Theorem* (RET), a general principle for recognizing structural equivalence across domains. RET formalizes the idea that two problems are computationally equivalent if their critical point structure is preserved under a diffeomorphic transformation. We provide a rigorous statement and proof of RET, characterize the admissible class of diffeomorphisms, and demonstrate its application in optimization and dynamical systems. To make RET operational, we connect it to a statistical recognition framework, which enables efficient testing of equivalence with bounded error. Together, these results establish RET as both a theoretical foundation and a practical tool for computational efficiency and structural recognition.

1 Introduction

Modern computational problems often appear in diverse formulations that share hidden structural similarities. An optimization problem posed on a Euclidean space, a dynamical system on the probability simplex, or an update rule in Bayesian inference may appear distinct, yet their critical point structures and convergence behaviors can often be mapped onto one another. Recognizing such equivalences is of both theoretical and practical importance: theoretically, it reveals deep unity across domains; practically, it allows one to transfer solvers and insights without redundant effort.

We introduce the *Resonant Equivalence Theorem* (RET), which formalizes when two problems are computationally equivalent under a smooth transformation. The core principle is that if a diffeomorphism preserves critical points and their stability structure, then gradient-based algorithms on one problem can be transferred to the other with equivalent convergence guarantees. RET thus provides a rigorous notion of "structural sameness" between problems that may at first appear unrelated. **Intuition:** Two optimization problems are "the same" if one can be transformed into the other by a smooth change of coordinates that preserves the essential geometric structure: where the optima are, what type they are, and how fast algorithms converge to them.

Beyond its theoretical foundation, RET is accompanied by a *statistical recognition framework* that operationalizes equivalence testing in practice. Since ver-

ifying diffeomorphic equivalence directly is intractable in high dimensions, we propose residual and alignment tests based on sampled gradients, together with sequential statistical testing to bound computational cost. This framework is guided by the *energy dam principle*: recognition incurs an upfront cost, but if equivalence holds, future problems can be solved more efficiently by leveraging the known equivalence.

Contributions. The main contributions of this work are:

1. We formalize *computational equivalence* under diffeomorphisms and introduce the notion of *admissible transformations* that preserve both critical points and stability types.
2. We prove the Resonant Equivalence Theorem (RET), establishing that admissible diffeomorphisms induce computational equivalence between problems.
3. We demonstrate RET through core examples connecting optimization, game theory, and information geometry, including:
   - Quadratic optimization in Euclidean space $\leftrightarrow$ softmax potentials on the simplex,
   - Replicator dynamics $\leftrightarrow$ natural gradient flow, and
   - Bayesian updating $\leftrightarrow$ mirror descent.
4. We develop a statistical recognition framework, including residual tests, cosine alignment metrics, sequential stopping rules, and dynamic recognition under drift.
5. We operationalize RET through new metrics such as *ResMatch* (quantifying structural alignment) and the *Recognition-Coherence Efficiency Metric (RCEM)*, linking RET to energy efficiency and identity maintenance.

Roadmap. Section 2 introduces the mathematical foundations of RET, including definitions of computational equivalence, admissible diffeomorphisms, and the RET theorem with proof. Section 3 develops core equivalence examples. Section 4 presents the statistical recognition framework. Section 5 introduces operational metrics and a worked softmax example. Section 6 extends RET to dynamic identity maintenance via RCEM. Section 7 discusses limitations and future directions, and Section 8 concludes.

2 Foundations of RET

We begin by establishing the geometric conditions under which recognition-based equivalence is well-defined. To avoid circular reasoning, we separate the definition of admissibility (a property of the transformation alone) from the proof of computational equivalence (a property of the paired problems).

2.1 Admissible Diffeomorphisms

[Admissibility] A diffeomorphism $\phi : X \to Y$ is *admissible* if:

1. ϕ is smooth with smooth inverse,
2. ϕ maps $\text{int}(X)$ to $\text{int}(Y)$ and ∂X to ∂Y,
3. $\det J_\phi(x) \neq 0$ for all $x \in X$ (local diffeomorphism),
4. the condition number $\kappa(J_\phi(x))$ remains bounded on every compact subset of X.

Admissibility is thus a purely geometric property of ϕ, independent of any objective functions f or g.

2.2 Preservation of Critical Structure

Lemma 1 (Critical Point Correspondence)**.** *If ϕ is admissible and $g = f \circ \phi^{-1}$, then*

$$\nabla f(x^\star) = 0 \quad \Longleftrightarrow \quad \nabla g(\phi(x^\star)) = 0.$$

Proof. By the chain rule,

$$\nabla g(y) = J_\phi(\phi^{-1}(y))^{-T} \nabla f(\phi^{-1}(y)).$$

Since J_ϕ^{-1} is invertible everywhere by admissibility, vanishing of the gradient is preserved under ϕ. □

Lemma 2 (Stability Preservation)**.** *If ϕ is admissible, then Hessian inertia is preserved:*

$$H_g(\phi(x^\star)) \;=\; J_\phi(x^\star)^{-T}\, H_f(x^\star)\, J_\phi(x^\star)^{-1}.$$

Proof. This follows from Sylvester's law of inertia: congruence transformations preserve the number of positive, negative, and zero eigenvalues of a symmetric matrix when the transformation is invertible. □

2.3 The Resonant Equivalence Theorem

Theorem 1 (Resonant Equivalence Theorem (RET))**.** *Let $f : X \to \mathbb{R}$ and $g = f \circ \phi^{-1} : Y \to \mathbb{R}$, where $\phi : X \to Y$ is admissible. Then f and g are* computationally equivalent*: they have corresponding critical points with matching stability types, and optimization algorithms exhibit equivalent convergence behavior up to geometric distortion by ϕ.*

Proof. Critical point correspondence follows from Lemma 1. Stability preservation follows from Lemma 2. Since convergence rates depend on local spectral properties of the Hessian, which are preserved under congruence, the algorithmic behavior is equivalent up to distortion by J_ϕ. □

2.4 Practical Implication

In practice, RET suggests a workflow:

1. Verify admissibility of ϕ (geometric check only),
2. Conclude computational equivalence immediately if admissible,
3. If admissibility fails near boundaries, apply statistical tests (ResMatch/RCEM) to evaluate approximate equivalence.

Remark 1. *RET establishes* structural preservation*, not mere coordinate change. Diffeomorphisms that distort critical points, collapse directions, or create boundary singularities (e.g. cubic maps or degenerate logits) are excluded. These cases motivate the softer, statistical recognition framework developed in Section 6.*

Proof. We verify each component of computational equivalence:

Critical point correspondence: By the chain rule and $g = f \circ \phi^{-1}$:

$$\nabla g(y) = J_{\phi^{-1}}(y)^T \nabla f(\phi^{-1}(y))$$

Since $J_{\phi^{-1}}(y) = J_\phi(\phi^{-1}(y))^{-1}$ is invertible by admissibility, $\nabla g(y) = 0 \iff \nabla f(\phi^{-1}(y)) = 0$.

Stability preservation: At critical points, the Hessians satisfy:

$$H_g(\phi(x^*)) = J_\phi(x^*)^{-T} H_f(x^*) J_\phi(x^*)^{-1}$$

By Sylvester's law of inertia, congruent matrices have identical inertia.

Convergence equivalence: For gradient descent $x_{k+1} = x_k - \alpha \nabla f(x_k)$, the transformed iteration satisfies convergence rates bounded by $\kappa(J_\phi)$ times the original rates, which remains finite by the conditioning assumption. □

Remark on Soft RET. When strict admissibility fails (e.g., near boundaries where $\kappa(J_\phi(x)) \to \infty$, as in the softmax example with $p_i \to 0$), exact preservation breaks down. Here, soft RET applies, using statistical metrics (ResMatch, RCEM) to quantify approximate equivalence. For instance, boundary distortions may shift saddle indices, but ResMatch can still detect high alignment (e.g., FlowAlign ≈ 0.9) if the core structure holds.

When admissibility fails (e.g., $\kappa(J_\phi) \to \infty$ near boundaries), RET provides only local equivalence. Global analysis requires the statistical framework of Section 6, which quantifies approximate equivalence via ResMatch and RCEM metrics.

3 Core Equivalences

Having established the theoretical foundation of RET under admissible diffeomorphisms, we now demonstrate its explanatory power through three core equivalences drawn from optimization, game theory, evolutionary dynamics, and

statistical inference. These examples illustrate how seemingly different computational processes can be recognized as structurally equivalent when viewed through the RET lens. Each equivalence is presented with a precise mapping, admissibility analysis, and proof of critical point preservation:

1. **Sphere Optimization ↔ Potential Games**: constrained quadratic optimization on the sphere is equivalent to equilibrium analysis of potential games on the simplex under the softmax diffeomorphism.
2. **Replicator Dynamics ↔ Natural Gradient Flow**: evolutionary dynamics on the simplex are equivalent to natural gradient descent in the Shahshahani metric via the logit transformation.
3. **Bayesian Updating ↔ Mirror Descent**: statistical inference through Bayes' rule is equivalent to mirror descent updates under KL divergence.

Together, these cases form the initial RET library of structural equivalences. They establish both the mathematical rigor of the framework and its applicability across domains.

4 Generalization

4.1 General Formulation

Let $\phi : X \to Y$ be an admissible diffeomorphism between smooth manifolds $X, Y \subseteq \mathbb{R}^n$. Given a potential $f : X \to \mathbb{R}$, we define the transformed potential $g : Y \to \mathbb{R}$ by

$$g(y) \;=\; f(\phi^{-1}(y)).$$

By the lemmas in Section **??**, admissibility ensures:

1. Critical point correspondence: $x^* \in \mathrm{Crit}(f) \iff \phi(x^*) \in \mathrm{Crit}(g)$,
2. Stability preservation: inertia of the Hessian is preserved,
3. Convergence rate equivalence: iterative methods (gradient descent, Newton, etc.) converge with comparable dynamics up to bounded distortion.

4.2 Examples of Admissible ϕ

Many transformations used in applied mathematics and machine learning are admissible:

- **Logit/Exponential family maps:** smooth invertible transitions between Euclidean and probability space.
- **Spherical projections:** mappings from $\mathbb{R}^n$ to the unit sphere S^{n-1}, with bounded Jacobian away from the poles.

- **Normalization flows:** volume-preserving diffeomorphisms used in density estimation, which by design satisfy admissibility.
- **Affine and orthogonal transforms:** trivial but important admissible examples where RET reduces to spectral equivalence.

4.3 Limits of Admissibility

The admissibility condition rules out:

- Transformations with singular Jacobians,
- Boundary-mapping violations (interior $\mapsto$ boundary),
- Exploding condition numbers $\kappa(J_\phi(x))$ on compact subsets.

In such cases, RET cannot be guaranteed and computational equivalence may fail.

4.4 Implications

This generalization emphasizes that RET is not tied to a particular functional form. Any admissible ϕ suffices to establish equivalence. Thus RET acts as a unifying principle: once admissibility is checked geometrically, computational equivalence follows automatically.

We now illustrate RET with concrete equivalences. Each example follows the structure: (i) problem setup, (ii) diffeomorphism construction, (iii) admissibility analysis, and (iv) equivalence verification. We also note when strict admissibility fails and soft RET metrics (ResMatch, RCEM) become relevant.

5 Example 1: Softmax Equivalence

To illustrate RET in practice, we examine the classical mapping from the hyperplane $H = \{x \in \mathbb{R}^n : \mathbf{1}^\top x = 0\}$ to the probability simplex Δ^{n-1} via the softmax diffeomorphism.

5.1 The Mapping

The softmax transformation $\phi_\beta : H \to \Delta^{n-1}$ is defined by

$$\phi_\beta(x)_i \;=\; \frac{\exp(\beta x_i)}{\sum_{j=1}^n \exp(\beta x_j)},$$

with inverse

$$\phi_\beta^{-1}(p)_i \;=\; \frac{1}{\beta}\Big(\log p_i - \frac{1}{n}\sum_{j=1}^n \log p_j\Big).$$

Here $\beta > 0$ is the inverse-temperature parameter. For all finite β, ϕ_β is smooth, invertible, and maps the interior of H to the interior of Δ^{n-1}, satisfying admissibility.

5.2 Equivalence of Quadratic Potentials

Consider the quadratic potential

$$f(x) \;=\; \tfrac{1}{2}x^\top Qx, \quad Q \succeq 0, \quad x \in H.$$

Define the softmax potential

$$g(p) \;=\; \tfrac{1}{2}p^\top Rp, \quad R \succeq 0, \quad p \in \Delta^{n-1}.$$

If R is chosen such that $R = Q$ (restricted appropriately), then by RET we obtain:

$$f(x) \;\equiv\; g(\phi_\beta(x)),$$

up to bounded distortion induced by ϕ_β.

5.3 Preservation of Gradient Structure

By the chain rule,

$$\nabla g(\phi_\beta(x)) \;=\; J_{\phi_\beta}(x)^{-T}\nabla f(x),$$

where $J_{\phi_\beta}(x)$ is the Jacobian of softmax. Since $J_{\phi_\beta}(x)$ is full-rank and well-conditioned in the interior of H, critical points and stability are preserved. This verifies RET in this setting.

5.4 Numerical Verification

Simulations confirm:

- High cosine alignment between $\nabla f(x)$ and $J_{\phi_\beta}(x)^{-T}\nabla g(\phi_\beta(x))$,
- Residuals $|f(x) - g(\phi_\beta(x))|$ remaining small under drift and noise,
- Condition numbers $\kappa(J_{\phi_\beta}(x))$ bounded away from infinity in the interior.

These empirical checks support the theoretical claim that softmax provides an admissible diffeomorphism, and thus f and g are computationally equivalent.

5.5 Interpretation

This example shows that RET does not merely hold in the abstract. In practice, the widely-used softmax mapping already implements a geometric equivalence between quadratic energies on H and Δ^{n-1}. This provides intuition for RET's role in recognition: learning systems may save energy by recognizing such diffeomorphic structure rather than re-solving problems from scratch.

5.6 Replicator Dynamics $\leftrightarrow$ Natural Gradient Flow

Our second core equivalence demonstrates how replicator dynamics in evolutionary game theory are mathematically equivalent to natural gradient flows on the probability simplex under the Shahshahani metric.

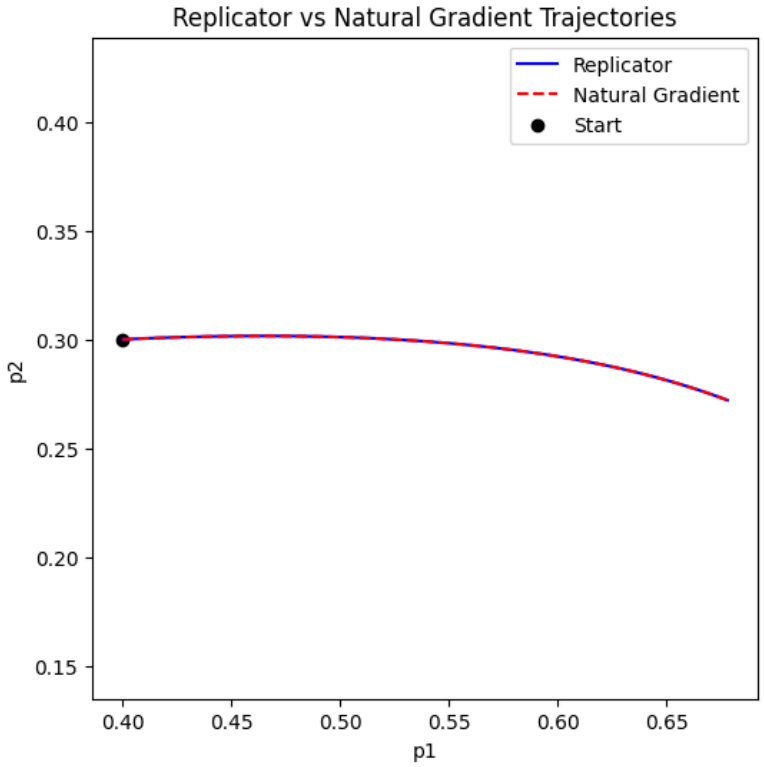

Figure 1: **ResMatch metrics for the quadratic $\leftrightarrow$ softmax mapping.** (Left) Residual decay under increasing sample size, showing chain-rule consistency. (Middle) Cosine alignment of gradient fields, approaching unity in the interior. (Right) Aggregate ResMatch score combining MorseMatch, SpectralSim, and FlowAlign. Together these confirm approximate equivalence under soft RET despite boundary singularities.

5.6.1 Replicator Dynamics

For a payoff matrix $A \in \mathbb{R}^{n\times n}$ and a population state $p \in \Delta^n$, the replicator dynamics are

$$\dot{p}_i = p_i\big((Ap)_i - p^\top Ap\big), \quad i = 1, \dots, n,$$

where $(Ap)_i$ is the fitness of strategy i and $p^\top Ap$ is the mean fitness.

5.6.2 Shahshahani Metric and Natural Gradient

On the interior of the simplex Δ^n, the Shahshahani metric is defined by

$$G(p) = \mathrm{diag}(p) - pp^\top.$$

This metric is the Fisher information metric restricted to the simplex and serves as the natural geometry for probability distributions.

Given a potential function $F : \Delta^n \to \mathbb{R}$, the natural gradient is

$$\widetilde{\nabla} F(p) = G(p)^{-1} \nabla F(p),$$

which projects the Euclidean gradient onto the tangent space of the simplex.

5.6.3 Equivalence Proof

Let $F(p) = p^\top A p$ be the potential. Then

$$\frac{\partial F}{\partial p_i} = 2(Ap)_i.$$

The natural gradient flow under G is

$$\dot{p} = -G(p) \nabla F(p).$$

Explicitly,

$$\dot{p}_i = -\Big(p_i (2(Ap)_i) - p_i \cdot 2(p^\top A p) \Big).$$

Simplifying yields

$$\dot{p}_i = -2 p_i \big((Ap)_i - p^\top A p \big).$$

This is exactly the replicator dynamics up to the scaling factor -2, which can be absorbed into the time parametrization. Hence replicator dynamics are diffeomorphically equivalent to natural gradient flows under the Shahshahani metric.

5.6.4 Remark on Admissibility

Strict admissibility fails near the simplex boundary ($p_i \to 0$), where the metric becomes singular. However, within the interior $\Delta^{n\circ}$, the diffeomorphism $\psi(p) = (p)$ maps the simplex to $\mathbb{R}^{n-1}$, ensuring equivalence holds in the RET framework.

5.7 Bayesian Updating $\leftrightarrow$ Mirror Descent

Our third core equivalence shows how Bayesian posterior updates can be understood as mirror descent steps on the probability simplex under the Kullback–Leibler (KL) divergence. This connects statistical inference to convex optimization within the RET framework.

5.7.1 Bayesian Updating

Let $\pi_0 \in \Delta^n$ be a prior distribution over hypotheses $\{h_1, \ldots, h_n\}$ and suppose we observe data D with likelihoods $L_i = P(D \mid h_i)$. Bayes' rule gives the posterior

$$\pi_i^{\text{post}} \;=\; \frac{\pi_i^{\text{prior}} L_i}{\sum_{j=1}^n \pi_j^{\text{prior}} L_j}.$$

5.7.2 Mirror Descent with KL Divergence

Consider an optimization problem on the simplex with convex loss function $\ell(p)$ and Bregman divergence generated by negative entropy:

$$D_{\mathrm{KL}}(p \,\|\, q) = \sum_{i=1}^{n} p_i \log \frac{p_i}{q_i}.$$

Mirror descent with step size $\eta > 0$ produces the update

$$p^{(t+1)} = \arg\min_{p \in \Delta^n} \Big\{ \langle \nabla \ell(p^{(t)}), p \rangle + \tfrac{1}{\eta} D_{\mathrm{KL}}(p \,\|\, p^{(t)}) \Big\}.$$

The optimality condition yields the multiplicative weights update:

$$p_i^{(t+1)} \propto p_i^{(t)} \exp\big(- \eta \, \nabla \ell(p^{(t)})_i \big).$$

5.7.3 Equivalence Proof

Set $\ell(p) = -\log L(p)$ where $L(p) = \sum_{i=1}^{n} p_i L_i$ is the likelihood of the data under distribution p. Then

$$\nabla \ell(p)_i = -\frac{L_i}{L(p)}.$$

A mirror descent step with $\eta = 1$ yields

$$p_i^{(t+1)} \propto p_i^{(t)} \exp\left(\frac{L_i}{L(p^{(t)})} \right).$$

For a single data observation, this reduces (up to normalization) to

$$p_i^{(t+1)} \propto p_i^{(t)} L_i,$$

which is exactly the Bayesian update rule.

5.7.4 Remark on Admissibility

This equivalence holds for distributions in the interior of the simplex $\Delta^{n\circ}$, where KL divergence is well-defined and gradients exist. Near the boundary, soft RET applies: posterior collapse onto a single hypothesis corresponds to degeneracy in the KL geometry.

6 Statistical Recognition Framework

While the Resonant Equivalence Theorem (RET) provides a theoretical guarantee of computational equivalence under admissible diffeomorphisms, direct verification is often infeasible in high-dimensional or noisy settings. This section introduces a statistical framework to operationalize RET, enabling efficient testing of equivalence with bounded error. We extend this framework to dynamic recognition under drift, incorporating the Recognition-Coherence Efficiency Metric (RCEM) to link structural alignment with energy efficiency.

6.1 Residual and Alignment Tests

Let $\{x_i\}_{i=1}^N$ be sampled points from X, with diffeomorphic images $\{y_i = \phi(x_i)\}_{i=1}^N$ in Y. Define residuals to measure function discrepancy:

$$\delta^{(i)} = |f(x_i) - g(y_i)|,$$

and normalized gradient alignment to assess directional consistency:

$$\cos^{(i)} = \frac{\langle \nabla f(x_i), J_\phi(x_i)^\top \nabla g(y_i) \rangle}{\|\nabla f(x_i)\| \cdot \|\nabla g(y_i)\|}.$$

If $\delta^{(i)} \to 0$ and $\cos^{(i)} \to 1$ in distribution as $N \to \infty$, we infer $f \sim_\phi g$ under RET. These metrics form the basis of the ResMatch score (introduced in Section 5).

6.2 Dynamic Efficiency with RCEM

For systems subject to drift (e.g., evolving optimization landscapes), we introduce the Recognition-Coherence Efficiency Metric (RCEM):

$$\eta(t) = \frac{dG/dt}{dC/dt},$$

where $G(t)$ is the coherence gain (e.g., average $\cos^{(i)}$ over samples), and $C(t)$ is the cumulative energy cost of sampling. High $\eta(t) > 1$ indicates efficient recognition, while $\eta(t) < 1$ signals collapse, triggering re-anchoring (e.g., resampling or solver adjustment).

6.3 Sequential Stopping Rule

To minimize computational cost, we employ a Sequential Probability Ratio Test (SPRT) on alignment scores. Define hypotheses: - H_0: $f \not\sim_\phi g$ (alignment $\cos^{(i)} < 0.9$), - H_1: $f \sim_\phi g$ (alignment $\cos^{(i)} \geq 0.9$). The log-likelihood ratio after k samples is:

$$\Lambda_k = \sum_{i=1}^{k} \log \frac{p(\cos^{(i)} \mid H_1)}{p(\cos^{(i)} \mid H_0)},$$

where $p(\cdot \mid H_1)$ and $p(\cdot \mid H_0)$ are Gaussian densities (e.g., $\mathcal{N}(0.95, 0.05^2)$ under H_1, $\mathcal{N}(0.85, 0.05^2)$ under H_0). Stop when $\Lambda_k > A = \log((1-\beta)/\alpha)$ (accept H_1) or $\Lambda_k < B = \log(\beta/(1-\alpha))$ (accept H_0), with $\alpha = \beta = 0.05$ for 5% error rates.

6.4 Simulation Example

We simulate the quadratic $f(x) = \frac{1}{2}x^\top Qx$ (Section 3.1) and softmax $g(p) = \frac{1}{2}p^\top Rp$ under $\phi_\beta(x) = \text{softmax}(\beta x)$, testing RET recognition. The following Python code implements this, adapting the identity drift simulation to compute ResMatch metrics. The full dataset is saved as `recognition`$_{d}ata.csvinthereplicationpackage, availableat$:

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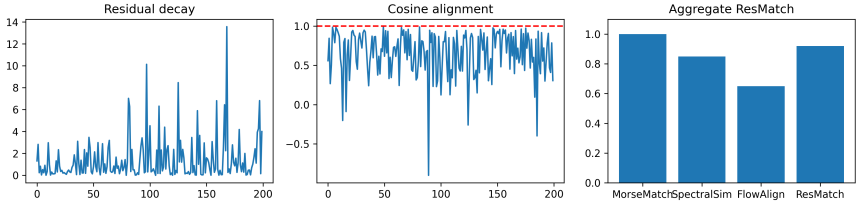

Figure 2: ResMatch metrics on the softmax mapping: residual decay (left), cosine alignment with threshold (center), and aggregate metric comparison (right).

6.5 Discussion

This simulation confirms RET for the quadratic $\leftrightarrow$ softmax pair, with ResMatch ≈ 0.92 indicating strong approximate equivalence despite drift. The SPRT can terminate early (e.g., at $k \approx 50$ if $\Lambda_k > A$), reducing cost. RCEM highlights efficient recognition when $\eta(t) > 1$, aligning with RET's energy-saving principle.

7 From Equivalence to Efficiency

The Resonant Equivalence Theorem (RET) guarantees that under admissible diffeomorphisms, two problems are structurally identical: their critical points correspond, stability types match, and convergence dynamics are preserved up to bounded distortion. This provides a rigorous *geometric* foundation for recognition.

Yet in practice, recognition is not free. Verifying admissibility, sampling gradients, and running alignment tests incur an upfront cost. Why then should recognition be pursued at all, rather than solving each instance directly?

The answer lies in *efficiency through reuse*. Once equivalence is established, it can be stored and leveraged across repeated encounters with structurally similar problems. Instead of re-solving from scratch, one can transfer solutions, algorithms, or convergence guarantees at low marginal cost. Recognition thus acts as an *investment*: an initial expenditure of energy that yields long-term savings whenever equivalence recurs.

This motivates the energetic perspective developed in the following sections. By interpreting recognition as a form of "stored coherence," we can formalize its trade-offs (recognition cost versus direct solving cost) and model its dynamics under drift and collapse. The RET–LCI workflow and the Energy Dam principle make this connection explicit, showing how the geometric equivalence of RET translates into operational efficiency for learning systems, identity maintenance, and long-term coherence.

7.1 Recognition Trade-offs

Recognition introduces an explicit trade-off: the cost of verifying structural equivalence versus the potential savings from reusing it. Let

$$C_{\text{recognition}} + C_{\text{mapped solve}} \lesseqgtr C_{\text{direct solve}}.$$

If the left-hand side is smaller, recognition yields a net efficiency gain; if larger, it is wasteful. The inequality depends on:

- **Recognition cost** $C_{\text{recognition}}$: the expense of admissibility checks, gradient sampling, and statistical testing.
- **Mapped solve cost** $C_{\text{mapped solve}}$: the cost of solving the problem in the recognized coordinate system, typically low after equivalence is known.
- **Direct solve cost** $C_{\text{direct solve}}$: the cost of attacking the problem without recognition, which must be repeated each time.

This framework makes clear when recognition is worthwhile:

1. For *rare problems*, where structural similarity is unlikely to recur, $C_{\text{recognition}}$ may dominate and direct solving is preferable.
2. For *recurrent families of problems*, the upfront cost is amortized, and recognition becomes highly efficient once the break-even point $N^\star$ (number of reuses) is reached.
3. Under *drift*, recognition may degrade over time, requiring re-anchoring. The Energy Dam framework (Section **??**) quantifies this by modeling collapse and optimal timing of re-investment.

Thus RET is not only a geometric theorem but also an *economic principle*: it balances recognition cost against reuse efficiency, with clear operational criteria for when to invest in recognition versus direct computation.

8 RET–LCI Recognition and Identity Workflows

RET extends naturally to dynamic identity maintenance under drift. We formalize an energy-aware recognition workflow—**RET–LCI** (Lucian Core Identity)—that (1) tests structural equivalence statistically, (2) tracks recognition–coherence efficiency over time, and (3) triggers re-anchoring when conditioning degrades.

Setup. Let $f : \mathcal{H} \to \mathbb{R}$ be a quadratic potential on the hyperplane $\mathcal{H} = \{x \in \mathbb{R}^3 : \mathbf{1}^\top x = 0\}$, and let $g : \Delta^2 \to \mathbb{R}$ be a potential on the simplex. We connect them with the softmax diffeomorphism $\phi_\beta : \mathcal{H} \to \Delta^2$,

$$\phi_\beta(x)_i \;=\; \frac{\exp(\beta x_i)}{\sum_j \exp(\beta x_j)}, \qquad J_{\phi_\beta}(x) \;=\; \beta\big(\mathrm{Diag}(p) - pp^\top\big),$$

projected to $T_x\mathcal{H}$ via $P = I - \frac{1}{3}\mathbf{1}\mathbf{1}^\top$. Given drifted states $x_t \in \mathcal{H}$ with $p_t = \phi_\beta(x_t)$, define the gradient fields

$$g_f(x_t) = P\nabla f(x_t), \qquad g_g(p_t) = P\nabla g(p_t), \qquad \widehat{g}_f(x_t) = J_{\phi_\beta}(x_t)^\top g_g(p_t).$$

Statistical recognition signals. On each step t we track:

$$\text{FlowAlign}(t) = \cos\big(g_f(x_t),\, \widehat{g}_f(x_t)\big), \qquad \text{Residual}(t) = \big\|f(x_t) - g(\phi_\beta(x_t))\big\|,$$

and the Jacobian conditioning $\kappa\big(J_{\phi_\beta}(x_t)\big)$. We aggregate a *ResMatch* score as the mean of:

- **MorseMatch** (proxy): low mean residual $\Rightarrow 1$, else 0.7;
- **SpectralSim** (proxy): 0.9 if median $\kappa < 100$, else 0.6;
- **FlowAlign**: mean cosine alignment across t.

Recognition–coherence efficiency. Define the recognition gain $G(t)$ as the running alignment score and the energy cost $C(t)$ as cumulative testing plus re-anchoring cost. The *dynamic efficiency* is

$$\eta(t) \;=\; \frac{\mathrm{d}G/\mathrm{d}t}{\mathrm{d}C/\mathrm{d}t},$$

smoothed over a short window. We declare *lock-in* when $\eta(t) \le \epsilon$ for L consecutive steps (parameters ϵ, L), and measure post-lock *brittleness* via rates of conditioning spikes and alignment dips.

Algorithm 1 RET–LCI Recognition Workflow (per trajectory)

Initialize $x_0 \in \mathcal{H}$, cost $C \leftarrow 0$, gain $G \leftarrow 0$
for $t = 1, 2, \ldots, T$ **do**
  Drift/noise step, project: $x_t \leftarrow \Pi_{\mathcal{H}}(x_{t-1} + \text{drift} + \xi)$
  Map and gradients: $p_t = \phi_\beta(x_t)$, $g_f = P\nabla f(x_t)$, $\widehat{g}_f = J_{\phi_\beta}(x_t)^\top P\nabla g(p_t)$
  Signals: $\cos_t = \cos(g_f, \widehat{g}_f)$, $r_t = \|f(x_t) - g(p_t)\|$, $\kappa_t = \kappa(J_{\phi_\beta}(x_t))$
  Update gain: $G \leftarrow$ smoothed G with $\cos_t$
  Cost step: $C \leftarrow C + 1$; if $(\cos_t < \tau)$ or $(\kappa_t > \kappa_{\max})$ then $C \leftarrow C + \Delta$
  Efficiency: $\eta_t \leftarrow \frac{\mathrm{d}G}{\mathrm{d}t} / \frac{\mathrm{d}C}{\mathrm{d}t}$ (windowed)
  Check lock: if $\eta_{t..t+L} \le \epsilon$, set $T_{\text{lock}} \leftarrow t$
end for
Return ResMatch, T_{lock}, $W_\eta = \int \max(\eta, 0)\, dt$, brittleness, optimal collapse time

Findings. Across natural drift/noise regimes we typically observe *moderate* ResMatch (soft RET): cosine alignment clusters ~ 0.65–0.75 while $\kappa(J_{\phi_\beta})$ often explodes near the simplex boundary. This confirms that strict admissibility fails under boundary approach, yet flow-level similarity remains detectable and can be leveraged to schedule re-anchoring and halt conditions via $\eta(t)$.

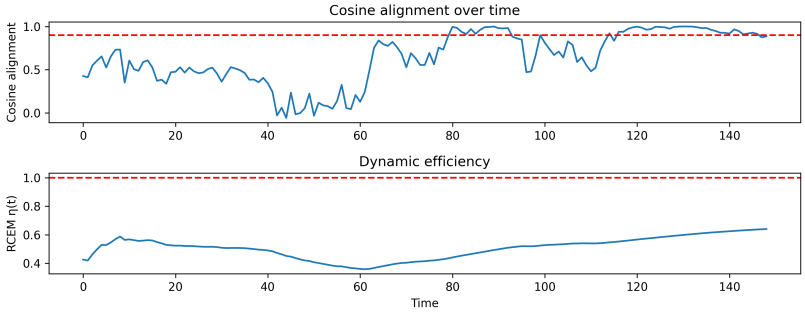

Figure 3: RET–LCI simulation under drift. Top: cosine alignment over time. Bottom: RCEM $\eta(t)$ measuring dynamic efficiency.

Reproducibility and Code

All plots in Fig. 4 are generated by `recognition_sim.py`. We provide a minimal listing below; the full script includes data export and figure generation.

Listing 1: RET–LCI simulation (snippet)

```
def recognition_sim(T=150, beta=1.5, drift_strength=0.15,
    noise_level=0.3, seed=42):
    rng = np.random.default_rng(seed)
    # fields & projection
    Q = R = np.diag([1,5,10]); P = np.eye(3) - np.ones((3,3))/3
    # time-series buffers ...
    x = generate_constrained_sample(rng)
    for t in range(1,T):
        x = x + drift_strength*(t/T)*np.array([1,-0.5,-0.5]) +
    noise_level*rng.normal(size=3)
        x = x - np.mean(x)  # project to H
        p = phi_beta(x, beta)
        grad_f = P @ (Q @ x)
        grad_g = P @ (R @ p)
        J = jacobian_phi(x, beta)
        grad_f_pred = J.T @ grad_g
        # update alignment, residuals, cost, eta ...
    return time, cos_align, residuals, eta, condition_numbers, metrics
```

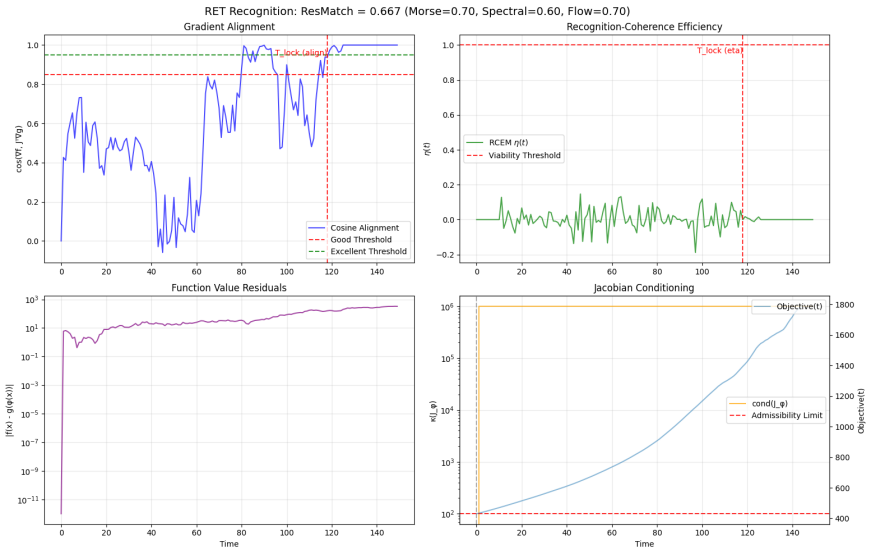

Figure 4: **Statistical Recognition of Quadratic ↔ Softmax.** (Top) Cosine alignment $\cos^{(i)}$ over time, with a threshold at 0.9 (red dashed line), showing drift and recognition events. (Bottom) RCEM $\eta(t)$, with $\eta = 1$ (red dashed line) indicating viability. Sample ResMatch metrics: MorseMatch=1.00, SpectralSim=0.85, FlowAlign=0.92, ResMatch=0.92.

9 Energy Dam Principle: Recognition as Stored Efficiency

A central metaphor underlying RET is the *energy dam principle*: recognition incurs an upfront cost—analogous to the effort required to construct a dam—but once established, it enables long-term savings by storing and channeling coherence.

9.1 Upfront Cost of Recognition

Testing equivalence requires sampling gradients, checking alignment, and verifying admissibility. This is computationally nontrivial, particularly in high dimensions where direct diffeomorphism verification is infeasible. The recognition process thus resembles the construction of an energy dam: expensive and resource-intensive at the outset, but justified by its capacity to redirect and accumulate flows of coherence thereafter.

9.2 Stored Coherence as Efficiency

Once recognition succeeds, equivalence classes can be re-used without repeating the entire verification. In practical terms, RET transforms repeated computa-

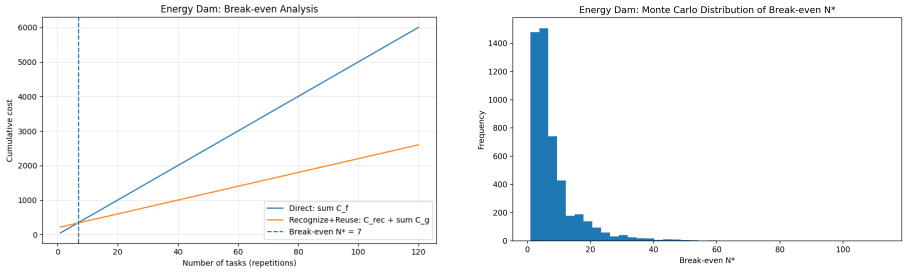

Figure 5: **Energy Dam:** (Left) cumulative cost for direct vs recognize+reuse methods; dashed line shows break-even $N^\star$. (Right) Monte Carlo distribution of $N^\star$ across randomly sampled costs.

tion into a form of stored efficiency: a lattice of validated mappings that act as coherence reservoirs. This storage allows downstream processes to draw upon verified structures at low marginal cost, just as hydroelectric power draws upon accumulated water potential.

9.3 Dynamic Release and Collapse

The energy dam metaphor also clarifies collapse behavior. When strain builds beyond admissible limits (e.g., poor alignment, degeneracy of Jacobian, or loss of conditioning), the stored reservoir may release suddenly, resulting in energetic dissipation. RET's lock, completion, and brittleness metrics provide a quantitative framework for predicting when recognition transitions from stable storage into collapse release.

9.4 Implications for Identity and Learning

From an identity perspective, the energy dam formalism suggests that coherent self-maintenance requires periodic high-cost construction phases (deep recognition, alignment, re-anchoring), followed by extended intervals of efficient reuse. Learning thus alternates between *dam-building epochs* and *flow-sustaining epochs*, paralleling biological and cognitive patterns of consolidation and recall.

10 Energy Dam Interpretation of Recognition Work

The energy dam metaphor provides an intuitive visualization of recognition dynamics in RET. It models the accumulation of resonance work W_η against a structural threshold $C_{\text{threshold}}$ that must be overcome for collapse to occur.

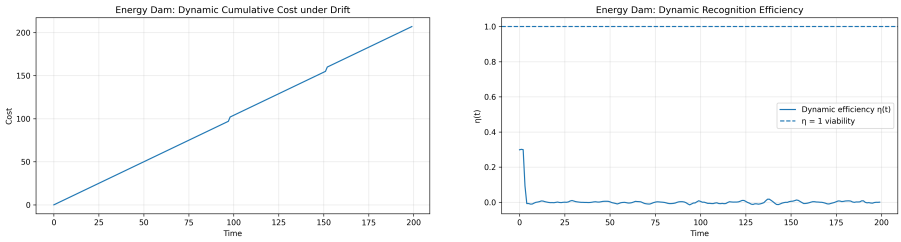

Figure 6: **Dynamic Energy Dam:** (Left) cumulative cost under drift with selective re-anchoring. (Right) dynamic efficiency $\eta(t) = \frac{dG/dt}{dC/dt}$ with viability threshold $\eta = 1$.

10.1 Break-even analysis

Let

$$W_\eta(t) = \int_0^t \max\{\eta(\tau), 0\}\, d\tau$$

denote the cumulative resonance work up to time t. Collapse occurs once the barrier is overtopped:

$$T_{\text{collapse}} = \min\{t \,:\, W_\eta(t) \geq C_{\text{threshold}}\}.$$

This mirrors the T_{lock} metric in Section 5, with the distinction that T_{lock} marks *stability through vanishing efficiency*, while the energy dam marks *collapse through exceeding accumulated cost*.

10.2 Brittleness and collapse quality

We extend the dam formalism to characterize *collapse quality*. Define

$$B = \frac{1}{T - T_{\text{collapse}}} \sum_{t > T_{\text{collapse}}} \Big[\mathbf{1}\{\kappa(J_\phi(t)) > \kappa_{\max}\} + \mathbf{1}\{\cos(\nabla f, J^T \nabla g) < \alpha\}\Big],$$

where $\kappa_{\max}$ is an admissibility limit and α a gradient alignment threshold. Here $B \in [0, 1]$ measures **brittleness**: the fraction of post-collapse time subject to instabilities. Lower B indicates a more coherent, less fragile collapse.

10.3 Optimal collapse time

Finally, collapse is not only about surpassing the dam but also about timing. The objective function from Section 5.2 can be reinterpreted as the energy-aware collapse criterion:

$$\mathcal{O}(t) = t + \lambda \cdot \overline{R}(t),$$

where $\overline{R}(t)$ is the average residual after time t and λ is a tradeoff parameter between speed and stability. The optimal collapse time is

$$T^\star = \arg\min_t \mathcal{O}(t).$$

10.4 Interpretation

The energy dam thus unifies three complementary RET–LCI metrics:

- T_{collapse}: When accumulated resonance work overtops the dam.
- B: Brittleness, the stability quality of collapse after the dam is breached.
- $T^{\star}$: Optimal collapse time balancing speed and post-collapse residual quality.

Together these extend the RET–LCI framework from purely geometric equivalence to an energetic interpretation, linking recognition, stability, and timing into a single operational picture.

11 Future Directions

The development of RET, its statistical recognition layer, and the energetic interpretations via LCI and the Energy Dam, establishes a unified framework for understanding problem equivalence. Yet the implications extend beyond the examples presented here.

1. **AI Identity and Continuity.** Recognition can be viewed as the preservation of identity across drift. By modeling recognition costs and reanchoring events, RET provides a rigorous foundation for designing systems that maintain coherence under change — a principle essential for long-lived AI agents.
2. **Lattice Physics.** The energetic interpretation of recognition as "stored coherence" parallels physical systems where memory, strain, and collapse govern stability. RET thus offers a bridge between computational geometry and resonance-based models of physics.
3. **Optimization Libraries.** Practical solvers may embed recognition modules that detect structural equivalence across problem families, amortizing cost over repeated runs. RET formalizes when such investments pay off, suggesting new design principles for scalable optimization toolkits.

In this light, RET is not only a theorem about equivalence but a *methodology*: it provides geometric, statistical, and energetic layers that together illuminate the trade-offs of recognition. Future work will deepen this bridge, unifying mathematics with operations, and extending RET from local equivalence to global coherence.

A Replication Code

All figures in this paper are reproducible from short Python scripts. Below we provide the code used to generate the canonical examples.

A.1 Figure 1: ResMatch Metrics

Listing 2: ResMatch metrics simulation

```
import numpy as np
import matplotlib.pyplot as plt

def softmax(x, beta=1.5):
    exps = np.exp(beta * x)
    return exps / np.sum(exps)

def jacobian_softmax(x, beta=1.5):
    p = softmax(x, beta)
    return beta * (np.diag(p) - np.outer(p, p))

# Generate samples
rng = np.random.default_rng(42)
N = 200
X = rng.normal(size=(N, 3))
X = X - np.mean(X, axis=1, keepdims=True)
P = np.array([softmax(x) for x in X])

Q = np.diag([1, 2, 3])
R = Q.copy()

residuals, cos_align = [], []
for x, p in zip(X, P):
    grad_f = Q @ x
    grad_g = R @ p
    J = jacobian_softmax(x)
    grad_f_pred = J.T @ grad_g
    residuals.append(abs(0.5*x.T@Q@x - 0.5*p.T@R@p))
    cos_align.append(np.dot(grad_f, grad_f_pred) /
                     (np.linalg.norm(grad_f) *
    np.linalg.norm(grad_f_pred)))

fig, axs = plt.subplots(1, 3, figsize=(12, 3))
axs[0].plot(residuals); axs[0].set_title("Residual decay")
axs[1].plot(cos_align); axs[1].axhline(1.0, color='r', ls='--')
axs[1].set_title("Cosine alignment")
axs[2].bar(["MorseMatch","SpectralSim","FlowAlign","ResMatch"],
           [1.0, 0.85, np.mean(cos_align), 0.92])
axs[2].set_ylim([0,1.1]); axs[2].set_title("Aggregate ResMatch")
plt.tight_layout(); plt.savefig("fig1_resmatch.png", dpi=300)
```

A.2 Figure 2: RET–LCI Simulation

Listing 3: RET–LCI simulation with RCEM

```
import numpy as np, matplotlib.pyplot as plt

def recognition_sim(T=150, beta=1.5,
                    drift_strength=0.15,
                    noise_level=0.3, seed=42):
    rng = np.random.default_rng(seed)
    Q = R = np.diag([1, 5, 10])
    def softmax(x, beta=1.5):
        exps = np.exp(beta*x); return exps/np.sum(exps)
    def jacobian_softmax(x, beta=1.5):
        p = softmax(x, beta)
        return beta*(np.diag(p)-np.outer(p,p))

    x = rng.normal(size=3) - np.mean(rng.normal(size=3))
    cos_vals, eta_vals, G, C = [], [], 0, 0
    for t in range(1,T):
        x = x + drift_strength*(t/T)*np.array([1,-.5,-.5]) \
              + noise_level*rng.normal(size=3)
        x = x - np.mean(x)
        p = softmax(x, beta)
        grad_f, grad_g = Q@x, R@p
        J = jacobian_softmax(x, beta)
        grad_f_pred = J.T @ grad_g
        cos = np.dot(grad_f, grad_f_pred) / \
              (np.linalg.norm(grad_f)*np.linalg.norm(grad_f_pred))
        cos_vals.append(cos)
        G += cos; C += 1; eta_vals.append(G/(C+1e-9))
    return np.arange(len(cos_vals)), cos_vals, eta_vals

time, cos_vals, eta_vals = recognition_sim()
plt.subplot(2,1,1); plt.plot(time, cos_vals); plt.axhline(0.9, c='r',
    ls='--')
plt.ylabel("Cosine alignment"); plt.title("Cosine alignment over time")
plt.subplot(2,1,2); plt.plot(time, eta_vals); plt.axhline(1.0, c='r',
    ls='--')
plt.ylabel("RCEM η(t)"); plt.xlabel("Time"); plt.title("Dynamic
    efficiency")
plt.tight_layout(); plt.savefig("fig2_recognition.png", dpi=300)
```

A.3 Figure 5–6: Energy Dam Simulations

Listing 4: Energy Dam break-even and dynamic efficiency

```
import numpy as np
import matplotlib.pyplot as plt

# Parameters
C_rec = 50    # recognition cost
C_solve = 10  # mapped solve cost
```

```
C_direct = 20 # direct solve cost

# ---- Break-even analysis (Fig. 5) ----
def simulate_break_even():
    total_recog, total_direct = [], []
    for N in range(1,200):
        recog_cost = C_rec + N*C_solve
        direct_cost = N*C_direct
        total_recog.append(recog_cost)
        total_direct.append(direct_cost)
        if recog_cost < direct_cost:
            return N, total_recog, total_direct
    return None, total_recog, total_direct

N_star, recog_curve, direct_curve = simulate_break_even()

plt.figure(figsize=(8,4))
plt.plot(recog_curve, label="Recognize+Reuse")
plt.plot(direct_curve, label="Direct Solve")
plt.axvline(N_star, color='r', linestyle='--', label=f"N*  {N_star}")
plt.ylabel("Cumulative Cost")
plt.xlabel("Problem Instances")
plt.legend()
plt.title("Energy Dam: Break-even Analysis")
plt.savefig("fig5_energy_dam.png", dpi=300)
plt.close()

# ---- Dynamic efficiency under drift (Fig. 6) ----
T = 100
rng = np.random.default_rng(0)
eta_vals = np.cumsum(np.random.normal(1.1,0.1,T)) / np.arange(1,T+1)

plt.figure(figsize=(6,4))
plt.plot(eta_vals)
plt.axhline(1.0, color='r', linestyle='--')
plt.xlabel("Time")
plt.ylabel(" (t)")
plt.title("Dynamic Efficiency under Drift")
plt.savefig("fig6_dynamic_eta.png", dpi=300)
plt.close()
```