9.8 (a) The system is first expressed as an augmented matrix:

$$\begin{bmatrix} 10 & 2 & -1 & 27 \\ -3 & -6 & 2 & -61.5 \\ 1 & 1 & 5 & -21.5 \end{bmatrix}$$

Forward elimination: a_{21} is eliminated by multiplying row 1 by –3/10 and subtracting the result from row 2. a_{31} is eliminated by multiplying row 1 by 1/10 and subtracting the result from row 3.

$$\begin{bmatrix} 10 & 2 & -1 & 27 \\ 0 & -5.4 & 1.7 & -53.4 \\ 0 & 0.8 & 5.1 & -24.2 \end{bmatrix}$$

a_{32} is eliminated by multiplying row 2 by 0.8/(–5.4) and subtracting the result from row 3.

$$\begin{bmatrix} 10 & 2 & -1 & 27 \\ 0 & -5.4 & 1.7 & -53.4 \\ 0 & 0 & 5.351852 & -32.1111 \end{bmatrix}$$

Back substitution:

$$x_3 = \frac{-32.1111}{5.351852} = -6$$

$$x_2 = \frac{-53.4 - 1.7(-6)}{-5.4} = 8$$

$$x_1 = \frac{27 - (-1)(-6) - 2(8)}{10} = 0.5$$

(b) Check:

$$10(0.5) + 2(8) - (-6) = 27$$

$$-3(0.5) - 6(8) + 2(-6) = -61.5$$

$$0.5 + 8 + 5(-6) = -21.5$$

9.10 (a) The determinant can be computed as:

$$A_1 = \begin{vmatrix} 2 & -1 \\ -2 & 0 \end{vmatrix} = 2(0) - (-1)(-2) = -2$$

$$A_2 = \begin{vmatrix} 1 & -1 \\ 5 & 0 \end{vmatrix} = 1(0) - (-1)(5) = 5$$

$$A_3 = \begin{vmatrix} 1 & 2 \\ 5 & -2 \end{vmatrix} = 1(-2) - 2(5) = -12$$

$$D = 0(-2) - (-3)5 + 7(-12) = -69$$

(b) Cramer's rule

$$x_1 = \frac{\begin{vmatrix} 2 & -3 & 7 \\ 3 & 2 & -1 \\ 2 & -2 & 0 \end{vmatrix}}{D} = \frac{-68}{-69} = 0.985507$$

$$x_2 = \frac{\begin{vmatrix} 0 & 2 & 7 \\ 1 & 3 & -1 \\ 5 & 2 & 0 \end{vmatrix}}{D} = \frac{-101}{-69} = 1.463768$$

$$x_3 = \frac{\begin{vmatrix} 0 & -3 & 2 \\ 1 & 2 & 3 \\ 5 & -2 & 2 \end{vmatrix}}{D} = \frac{-63}{-69} = 0.913043$$

(c) The system is first expressed as an augmented matrix:

$$\begin{bmatrix} 0 & -3 & 7 & 2 \\ 1 & 2 & -1 & 3 \\ 5 & -2 & 0 & 2 \end{bmatrix}$$

Forward elimination: First, we pivot by switching rows 1 and 3:

$$\begin{bmatrix} 5 & -2 & 0 & 2 \\ 1 & 2 & -1 & 3 \\ 0 & -3 & 7 & 2 \end{bmatrix}$$

Multiply row 1 by 1/5 = 0.2 and subtract from row 2 to eliminate a_{21}. Because a_{31} already equals zero, it does not have to be eliminated.

$$\begin{bmatrix} 5 & -2 & 0 & 2 \\ 0 & 2.4 & -1 & 2.6 \\ 0 & -3 & 7 & 2 \end{bmatrix}$$

Pivot:

$$\begin{bmatrix} 5 & -2 & 0 & 2 \\ 0 & -3 & 7 & 2 \\ 0 & 2.4 & -1 & 2.6 \end{bmatrix}$$

Multiply row 2 by 2.4/(–3) = –0.8 and subtract from row 3 to eliminate a_{32}.

$$\begin{bmatrix} 5 & -2 & 0 & 2 \\ 0 & -3 & 7 & 2 \\ 0 & 0 & 4.6 & 4.2 \end{bmatrix}$$

Back substitution:

$$x_3 = \frac{4.2}{4.6} = 0.913043$$
$$x_2 = \frac{2 - 7(0.913043)}{-3} = 1.463768$$
$$x_1 = \frac{2 - 0(0.913043) - (-2)(1.463768)}{5} = 0.985507$$

(d) Check:

$$-3(1.463768) + 7(0.913043) = 2$$
$$(0.985507) + 2(1.463768) - (0.913043) = 3$$
$$5(0.985507) - 2(1.463768) = 2$$

9.12 The system is first expressed as an augmented matrix:

$$\begin{bmatrix} 2 & 1 & -1 & 1 \\ 5 & 2 & 2 & -4 \\ 3 & 1 & 1 & 5 \end{bmatrix}$$

Normalize the first row and then eliminate a_{21} and a_{31},

$$\begin{bmatrix} 1 & 0.5 & -0.5 & 0.5 \\ 0 & -0.5 & 4.5 & -6.5 \\ 0 & -0.5 & 2.5 & 3.5 \end{bmatrix}$$

Normalize the second row and eliminate a_{12} and a_{32},

$$\begin{bmatrix} 1 & 0 & 4 & -6 \\ 0 & 1 & -9 & 13 \\ 0 & 0 & -2 & 10 \end{bmatrix}$$

Normalize the third row and eliminate a_{13} and a_{23},

$$\begin{bmatrix} 1 & 0 & 0 & 14 \\ 0 & 1 & 0 & -32 \\ 0 & 0 & 1 & -5 \end{bmatrix}$$

Thus, the answers are $x_1 = 14$, $x_2 = -32$, and $x_3 = -5$. Check:

$$2(14) + (-32) - (-5) = 1$$
$$5(14) + 2(-32) + 2(-5) = -4$$
$$3(14) + (-32) + (-5) = 5$$

9.13 (a) The system is first expressed as an augmented matrix:

$$\begin{bmatrix} 1 & 1 & -1 & -3 \\ 6 & 2 & 2 & 2 \\ -3 & 4 & 1 & 1 \end{bmatrix}$$

Forward elimination: a_{21} is eliminated by multiplying row 1 by 6/1 = 6 and subtracting the result from row 2. a_{31} is eliminated by multiplying row 1 by –3/1 = –3 and subtracting the result from row 3.

$$\begin{bmatrix} 1 & 1 & -1 & -3 \\ 0 & -4 & 8 & 20 \\ 0 & 7 & -2 & -8 \end{bmatrix}$$

a_{32} is eliminated by multiplying row 2 by 7/(–4) = –1.75 and subtracting the result from row 3.

$$\begin{bmatrix} 1 & 1 & -1 & -3 \\ 0 & -4 & 8 & 20 \\ 0 & 0 & 12 & 27 \end{bmatrix}$$

Back substitution:

$$x_3 = \frac{27}{12} = 2.25$$

$$x_2 = \frac{20 - 8(2.25)}{-4} = -0.5$$

$$x_1 = \frac{-3 - (-1)(2.25) - 1(-0.5)}{1} = -0.25$$

(b) The system is first expressed as an augmented matrix:

$$\begin{bmatrix} 1 & 1 & -1 & -3 \\ 6 & 2 & 2 & 2 \\ -3 & 4 & 1 & 1 \end{bmatrix}$$

Forward elimination: First, we pivot by switching rows 1 and 2:

$$\begin{bmatrix} 6 & 2 & 2 & 2 \\ 1 & 1 & -1 & -3 \\ -3 & 4 & 1 & 1 \end{bmatrix}$$

Multiply row 1 by 1/6 = 0.16667 and subtract from row 2 to eliminate a_{21}. Multiply row 1 by –3/6 = –0.5 and subtract from row 3 to eliminate a_{31}.

$$\begin{bmatrix} 6 & 2 & 2 & 2 \\ 0 & 0.66667 & -1.33333 & -3.33333 \\ 0 & 5 & 2 & 2 \end{bmatrix}$$

Pivot:

$$\begin{bmatrix} 6 & 2 & 2 & 2 \\ 0 & 5 & 2 & 2 \\ 0 & 0.66667 & -1.33333 & -3.33333 \end{bmatrix}$$

Multiply row 2 by 0.66667/5 = 0.133333 and subtract from row 3 to eliminate a_{32}.

$$\begin{bmatrix} 6 & 2 & 2 & 2 \\ 0 & 5 & 2 & 2 \\ 0 & 0 & -1.6 & -3.6 \end{bmatrix}$$

Back substitution:

$$x_3 = \frac{-3.6}{1.6} = 2.25$$
$$x_2 = \frac{2-2(2.25)}{5} = -0.5$$
$$x_1 = \frac{2-2(2.25)-2(-0.5)}{6} = -0.25$$

(c) The system is first expressed as an augmented matrix:

$$\begin{bmatrix} 1 & 1 & -1 & -3 \\ 6 & 2 & 2 & 2 \\ -3 & 4 & 1 & 1 \end{bmatrix}$$

Normalize the first row, and then eliminate a_{21} and a_{31},

$$\begin{bmatrix} 1 & 1 & -1 & -3 \\ 0 & -4 & 8 & 20 \\ 0 & 7 & -2 & -8 \end{bmatrix}$$

Normalize the second row and eliminate a_{12} and a_{32},

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & -5 \\ 0 & 0 & 12 & 27 \end{bmatrix}$$

Normalize the third row and eliminate a_{13} and a_{23},

$$\begin{bmatrix} 1 & 0 & 0 & -0.25 \\ 0 & 1 & 0 & -0.5 \\ 0 & 0 & 1 & 2.25 \end{bmatrix}$$

10.2 (a) The coefficient a_{21} is eliminated by multiplying row 1 by $f_{21} = -3/10 = -0.3$ and subtracting the result from row 2. a_{31} is eliminated by multiplying row 1 by $f_{31} = 1/10 = 0.1$ and subtracting the result from row 3. The factors f_{21} and f_{31} can be stored in a_{21} and a_{31}.

$$\begin{bmatrix} 10 & 2 & -1 \\ -0.3 & -5.4 & 1.7 \\ 0.1 & 0.8 & 5.1 \end{bmatrix}$$

a_{32} is eliminated by multiplying row 2 by $f_{32} = 0.8/(-5.4) = -0.14815$ and subtracting the result from row 3. The factor f_{32} can be stored in a_{32}.

$$\begin{bmatrix} 10 & 2 & -1 \\ -0.3 & -5.4 & 1.7 \\ 0.1 & -0.14815 & 5.351852 \end{bmatrix}$$

Therefore, the LU decomposition is

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ -0.3 & 1 & 0 \\ 0.1 & -0.14815 & 1 \end{bmatrix} \qquad [U] = \begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0 & 5.351852 \end{bmatrix}$$

These two matrices can be multiplied to yield the original system. For example, using MATLAB to perform the multiplication gives

```
>> L=[1 0 0;-.3 1 0;0.1 -.14815 1];
>> U=[10 2 -1;0 -5.4 1.7;0 0 5.351852];
>> L*U
ans =
   10.0000    2.0000   -1.0000
   -3.0000   -6.0000    2.0000
    1.0000    1.0000    5.0000
```

(b) Forward substitution: $[L]\{D\} = \{B\}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.3 & 1 & 0 \\ 0.1 & -0.14815 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 27 \\ -61.5 \\ -21.5 \end{Bmatrix}$$

Solving yields $d_1 = 27$, $d_2 = –53.4$, and $d_3 = –32.1111$.

Back substitution:

$$\begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0 & 5.351852 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 27 \\ -53.4 \\ -32.1111 \end{Bmatrix}$$

$$x_3 = \frac{-32.1111}{5.351852} = -6$$

$$x_2 = \frac{-53.4 - 1.7(-6)}{-5.4} = 8$$

$$x_1 = \frac{27 - (-1)(-6) - 2(8)}{10} = 0.5$$

(c) Forward substitution: $[L]\{D\} = \{B\}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.3 & 1 & 0 \\ 0.1 & -0.14815 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 12 \\ 18 \\ -6 \end{Bmatrix}$$

Solving yields $d_1 = 12$, $d_2 = 21.6$, and $d_3 = –4$.

Back substitution:

$$\begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0 & 5.351852 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 12 \\ 21.6 \\ -4 \end{Bmatrix}$$

$$x_3 = \frac{-4}{5.351852} = -0.7474$$

$$x_2 = \frac{21.6 - 1.7(-0.7474)}{-5.4} = -4.23529$$

$$x_1 = \frac{12 - (-1)(-0.7474) - 2(-4.23529)}{10} = 1.972318$$

10.3 (a) The coefficient a_{21} is eliminated by multiplying row 1 by $f_{21} = -2/8 = -0.25$ and subtracting the result from row 2. a_{31} is eliminated by multiplying row 1 by $f_{31} = 2/8 = 0.25$ and subtracting the result from row 3. The factors f_{21} and f_{31} can be stored in a_{21} and a_{31}.

$$\begin{bmatrix} 8 & 4 & -1 \\ -0.25 & 6 & 0.75 \\ 0.25 & -2 & 6.25 \end{bmatrix}$$

a_{32} is eliminated by multiplying row 2 by $f_{32} = -2/6 = -0.33333$ and subtracting the result from row 3. The factor f_{32} can be stored in a_{32}.

$$\begin{bmatrix} 8 & 4 & -1 \\ -0.25 & 6 & 0.75 \\ 0.25 & -0.33333 & 6.5 \end{bmatrix}$$

Therefore, the *LU* decomposition is

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ -0.25 & 1 & 0 \\ 0.25 & -0.33333 & 1 \end{bmatrix} \qquad [U] = \begin{bmatrix} 8 & 4 & -1 \\ 0 & 6 & 0.75 \\ 0 & 0 & 6.5 \end{bmatrix}$$

Forward substitution: $[L]\{D\} = \{B\}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.25 & 1 & 0 \\ 0.25 & -0.33333 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 11 \\ 4 \\ 7 \end{Bmatrix}$$

Solving yields $d_1 = 11$, $d_2 = 6.75$, and $d_3 = 6.5$.

Back substitution:

$$\begin{bmatrix} 8 & 4 & -1 \\ 0 & 6 & 0.75 \\ 0 & 0 & 6.5 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 11 \\ 6.75 \\ 6.5 \end{Bmatrix}$$

$$x_3 = \frac{6.5}{6.5} = 1$$

$$x_2 = \frac{6.75 - 0.75(1)}{6} = 1$$

$$x_1 = \frac{11 - (-1)(1) - 4(1)}{8} = 1$$

(b) The first column of the inverse can be computed by using $[L]\{D\} = \{B\}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.25 & 1 & 0 \\ 0.25 & -0.33333 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

This can be solved for $d_1 = 1$, $d_2 = 0.25$, and $d_3 = -0.16667$. Then, we can implement back substitution

$$\begin{bmatrix} 8 & 4 & -1 \\ 0 & 6 & 0.75 \\ 0 & 0 & 6.5 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0.25 \\ -0.16667 \end{Bmatrix}$$

to yield the first column of the inverse

$$\{X\} = \begin{Bmatrix} 0.099359 \\ 0.0448718 \\ -0.025641 \end{Bmatrix}$$

For the second column use $\{B\}^{\mathrm{T}} = \{0\ 1\ 0\}$ which gives $\{D\}^{\mathrm{T}} = \{0\ 1\ 0.33333\}$. Back substitution then gives $\{X\}^{\mathrm{T}} = \{-0.073718\ 0.160256\ 0.051282\}$.

For the third column use $\{B\}^{\mathrm{T}} = \{0\ 0\ 1\}$ which gives $\{D\}^{\mathrm{T}} = \{0\ 0\ 1\}$. Back substitution then gives $\{X\}^{\mathrm{T}} = \{0.028846\ -0.01923\ 0.153846\}$.

Therefore, the matrix inverse is

$$[A]^{-1} = \begin{bmatrix} 0.099359 & -0.073718 & 0.028846 \\ 0.044872 & 0.160256 & -0.019231 \\ -0.025641 & 0.051282 & 0.153846 \end{bmatrix}$$

We can verify that this is correct by multiplying $[A][A]^{-1}$ to yield the identity matrix. For example, using MATLAB,

```
>> A=[8 4 -1;-2 5 1;2 -1 6];
>> AI=[0.099359 -0.073718 0.028846;
0.044872 0.160256 -0.019231;
-0.025641 0.051282 0.153846]
>> A*AI
ans =
    1.0000   -0.0000   -0.0000
    0.0000    1.0000   -0.0000
         0         0    1.0000
```

10.6 First, we compute the LU decomposition. The coefficient a_{21} is eliminated by multiplying row 1 by $f_{21} = -3/10 = -0.3$ and subtracting the result from row 2. a_{31} is eliminated by multiplying row 1 by $f_{31} = 1/10 = 0.1$ and subtracting the result from row 3. The factors f_{21} and f_{31} can be stored in a_{21} and a_{31}.

$$\begin{bmatrix} 10 & 2 & -1 \\ -0.3 & -5.4 & 1.7 \\ 0.1 & 0.8 & 5.1 \end{bmatrix}$$

a_{32} is eliminated by multiplying row 2 by $f_{32} = 0.8/(-5.4) = -0.148148$ and subtracting the result from row 3. The factor f_{32} can be stored in a_{32}.

$$\begin{bmatrix} 10 & 2 & -1 \\ -0.3 & -5.4 & 1.7 \\ 0.1 & -0.148148 & 5.351852 \end{bmatrix}$$

Therefore, the LU decomposition is

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ -0.3 & 1 & 0 \\ 0.1 & -0.148148 & 1 \end{bmatrix} \qquad [U] = \begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0 & 5.351852 \end{bmatrix}$$

The first column of the inverse can be computed by using $[L]\{D\} = \{B\}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.3 & 1 & 0 \\ 0.1 & -0.148148 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

This can be solved for $d_1 = 1$, $d_2 = 0.3$, and $d_3 = -0.055556$. Then, we can implement back substitution

$$\begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0 & 5.351852 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0.3 \\ -0.055556 \end{Bmatrix}$$

to yield the first column of the inverse

$$\{X\} = \begin{Bmatrix} 0.110727 \\ -0.058824 \\ -0.0103806 \end{Bmatrix}$$

For the second column use $\{B\}^T = \{0\ 1\ 0\}$ which gives $\{D\}^T = \{0\ 1\ 0.148148\}$. Back substitution then gives $\{X\}^T = \{0.038062\ -0.176471\ 0.027682\}$.

For the third column use $\{B\}^T = \{0\ 0\ 1\}$ which gives $\{D\}^T = \{0\ 0\ 1\}$. Back substitution then gives $\{X\}^T = \{0.00692\ 0.058824\ 0.186851\}$.

Therefore, the matrix inverse is

$$[A]^{-1} = \begin{bmatrix} 0.110727 & 0.038062 & 0.006920 \\ -0.058824 & -0.176471 & 0.058824 \\ -0.010381 & 0.027682 & 0.186851 \end{bmatrix}$$

We can verify that this is correct by multiplying $[A][A]^{-1}$ to yield the identity matrix. For example, using MATLAB,

```
>> A=[10 2 -1;-3 -6 2;1 1 5];
>> AI=[0.110727 0.038062 0.006920;
-0.058824 -0.176471 0.058824;
-0.010381 0.027682 0.186851];
>> A*AI
ans =
    1.0000   -0.0000   -0.0000
    0.0000    1.0000   -0.0000

   -0.0000    0.0000    1.0000
```

11.6

$$l_{11} = \sqrt{8} = 2.828427$$

$$l_{21} = \frac{20}{2.828427} = 7.071068$$

$$l_{22} = \sqrt{80 - 7.071068^2} = 5.477226$$

$$l_{31} = \frac{15}{2.828427} = 5.303301$$

$$l_{32} = \frac{50 - 7.071068(5.303301)}{5.477226} = 2.282177$$

$$l_{33} = \sqrt{60 - 5.303301^2 - 2.282177^2} = 5.163978$$

Thus, the Cholesky decomposition is

$$[L] = \begin{bmatrix} 2.828427 & & \\ 7.071068 & 5.477226 & \\ 5.303301 & 2.282177 & 5.163978 \end{bmatrix}$$