Maths Refresher

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Introduction to Infectious Disease Modelling and its Applications (Short Course)

19th June 2018

Purpose of session

1. To review key mathematical concepts used in the course
   - Exponents/logarithms
     - Logarithms to base 10
     - The constant e
     - Logarithms to base e
   - Differentiation
   - Matrices & simultaneous equations

2. To answer any specific questions from the maths refresher notes

3. To go over any exercises in the maths refresher notes

Exponents

"10 is **raised to the power** of 6"
"6 is the **exponent** of 10"

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Properties

$i.$	$a^m \text{ x } a^n = a^{m+n}$	Same "base" number *a*
$ii.$	$a^m / a^n = a^{m-n}$	
$iii.$	$(a^m)^n = a^{m \text{ x } n}$	
$iv.$	$(a/b)^n = a^n / b^n$	Everything inside brackets is a base
$v.$	$a^0 = 1$	0 exponent equals 1
$vi.$	$a^{1/n} = \sqrt[n]{a}$	Fraction exponent is a root
$vii.$	$a^{-n} = \frac{1}{a^n}$	Negative exponent is a fraction

Logarithms

Logarithms (logs) are the "opposite" of exponents

Definition:

The "**log of *x* to base *a*"** is the exponent (*b*) of *a* (the base) required to equal *x*.

$\text{Log}_a(x) = b \rightarrow x = a^b$

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Example: log to base 10

Note that 10^3 = 1000, we can say $\log_{10}(1000) = 3$

i.e. 3 is the exponent (or power) to which 10 must be raised to give 1000.

Logarithms (Examples)

What are the logs to base 10 of the following numbers?

a) 100

b) 10

c) 1

d) 0.1

e) 0.01

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$10^2 = 100$

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Logarithms (Examples)

What are the logs to base 10 of the following numbers?

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$10^2 = 100 \rightarrow \log_{10}(100) = 2$

b) 10

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$10^{-2} = 0.01 \rightarrow \log_{10}(0.01) = -2$

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Logarithms (cont.)

x	0.01	0.1	1	10	100	1000	10000
$\text{Log}_{10}\, x$	-2	-1	0	1	2	3	4

NB: Taking logs makes working with very large or very small numbers more manageable

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Note:

We cannot take a log of a negative number.

It is not possible to raise 10 to any power to obtain a negative number:

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$$0 \leftarrow \ldots, \underbrace{10^{-y}, \ldots, 10^{-20}, 10^{-19}, \ldots, 10^{-1}}_{\text{Log of a number <1 is negative}}, 10^{0}, \underbrace{10^{1}, 10^{2}, \ldots, 10^{20}, \ldots, 10^{y}}_{\text{Log of a number >1 is positive}} \ldots \rightarrow \infty$$

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The constant *e*

A "mathematical constant" (i.e. it is a fixed value) with very useful properties

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Mathematical definition:

$$e = 2.71828\ldots = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots \frac{1}{n!} + \ldots = \sum_{i=0}^{\infty} \frac{1}{i!}$$

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e raised to any two powers e.g. *r* and *t* can be written as:

$$e^{rt} = 1+rt+\frac{(rt)^2}{2!}+\frac{(rt)^3}{3!}+\frac{(rt)^4}{4!}+\frac{(rt)^5}{5!}+\ldots = \sum_{i=0}^{\infty}\frac{(rt)^i}{i!}$$

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Factorials and Combinations (an aside…)

n! ("*n* factorial") = *n* x (*n*-1) x (*n*-2) x … x 2 x 1

e.g. 4! = 4 x 3 x 2 x 1 = 24

How many ways are there to choose *r* objects from a total of *n* objects?

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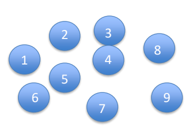

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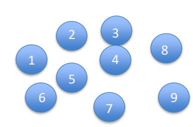

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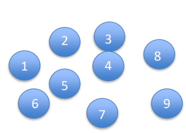
$$\frac{9x8x7x6x5x4x3x2x1}{5x4x3x2x1}$$

= 9!/5!



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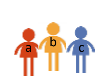

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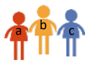

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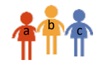

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Example 3: UK Lottery $\binom{49}{6} = \frac{49!}{6!43!} = \frac{49.48.47.46.45.44}{6.5.4.3.2.1} \approx 14 \text{ mill}$

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What does e^{rt} look like?

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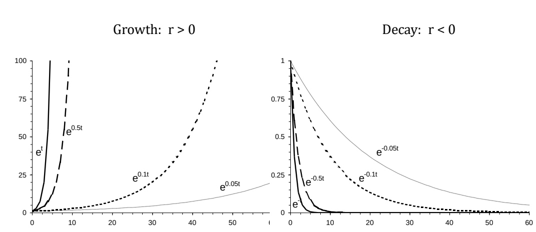
Exponential growth and decay



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Natural logarithms

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The log to base *e* of some number *x* is written as:

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x	0.14	0.37	1	2.72	7.4	20.1	54.6
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Natural logarithms (cont.)

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Differential equations

Differential equations allow us to:

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A group of differential equations that represent disease transmission is known as a "**transmission model**". A model predicts the dynamics of a disease by quantifying the rate at which events occur.

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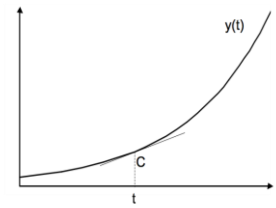
A group of differential equations that represent disease transmission is known as a "**transmission model**". A model predicts the dynamics of a disease by quantifying the rate at which events occur.

For a transmission model we are interested in:

1. the rate of change in the number of **susceptible** individuals over time
2. the rate of change in the number of **infectious** individuals over time etc.

Differential equations (cont.)



Consider an population increasing in size over time.

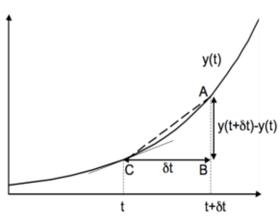
The plot shows the population size at time *t*.

What is the population growth rate at time *C* ?

The gradient at point C is the slope of the line which just touches the curve at point C

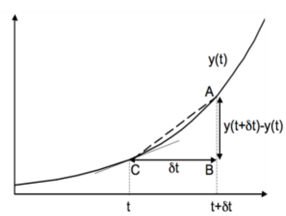
NB: The rate of change of something at a given time is the gradient at that point.

Differential equations (cont.)



Recall: a gradient is defined as the amount by which a line goes up divided by the amount it goes across.

Differential equations (cont.)



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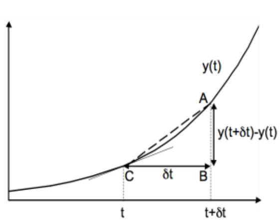
The gradient of AC = $\dfrac{y(t+\delta t) - y(t)}{\delta t}$

Differential equations (cont.)



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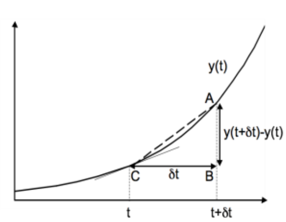

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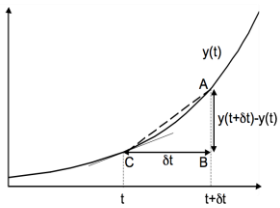
BUT if A is moved closer to C, so that δt is as small as possible, the gradient of AC eventually equals the gradient at C.

Differential equations (cont.)



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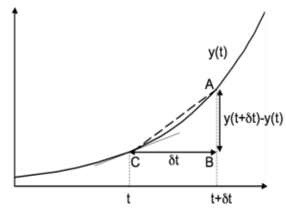
as δt approaches zero ($\delta t \to 0$)

Differential equations (cont.)



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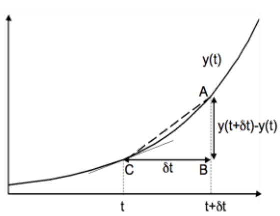

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Usual notation for the rate of change: $\frac{dy(t)}{dt}$

Other notation: $\frac{dy}{dt}$, $\dot{y}(t)$, $y'(t)$

Usual terminology: The *derivative* of *y* with respect to *t*

Differential equations (cont.)

One of the most useful results: $\frac{d}{dt}\left[e^{rt}\right] = re^{rt}$

This can be proved using the facts that

1. The derivative of any expression t^n is nt^{n-1}

2. $e^{rt} = 1 + rt + \frac{(rt)^2}{2!} + \frac{(rt)^3}{3!} + \frac{(rt)^4}{4!} + \frac{(rt)^5}{5!} + \ldots$

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Important consequence:

If you ever see an equation similar to $\frac{dy(t)}{dt} = ry(t)$

then you know that a solution of *y(t)* looks like: *y(t)* = e^{rt}

Differential equations (example 1)

Individuals in a cohort die at a constant rate *m*

$$\boxed{\text{Number alive } N(t)} \xrightarrow{m} \boxed{\text{Number dead}}$$

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The rate of change in the number of living individuals is given by:

$$\frac{dN(t)}{dt} = -mN(t)$$

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Differential equations (example 1)

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It can be shown that the number of individuals at time t is given by:

$$N(t) = N(0)e^{-mt}$$

To prove this result, differentiate $N(t)=N(0)e^{-mt}$, and check that it leads to the equation $\frac{dN(t)}{dt} = -mN(t)$

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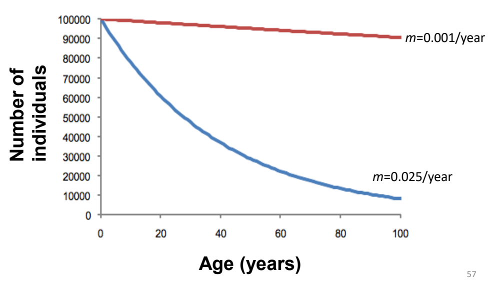
Differential equations (example 1)

Predictions of the cohort size at time *t*



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Differential equations (example 2)

Susceptible individuals in a cohort become infected at a constant rate λ

Susceptible $S(t)$ $\xrightarrow{\lambda}$ Ever infected $Z(t)$

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Differential equations (example 2)

Susceptible individuals in a cohort become infected at a constant rate λ

Susceptible $S(t)$ $\xrightarrow{\lambda}$ Ever infected $Z(t)$

The rate of change in the number of susceptible individuals is given by:

$$\frac{dS(t)}{dt} = -\lambda S(t)$$

The number of susceptible individuals at time t is given by:

$$S(t) = S(0)e^{-\lambda t}$$

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Differential equations (example 2)

Susceptible individuals in a cohort become infected at a constant rate λ

Susceptible $S(t)$ $\xrightarrow{\lambda}$ Ever infected $Z(t)$

As we are following a cohort from which no individuals are assumed to die, the number ever infected at time t is given by:

Z(t) = Number susceptible at the cohort at the start – Number susceptible at time *t*

$$Z(t) = S(0) - S(t)$$

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Differential equations (example 2)

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Matrices

Useful for denoting contact between different groups of individuals.

Matrices are a way of summarizing sets of equations.

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Useful for denoting contact between different groups of individuals.

Matrices are a way of summarizing sets of equations.

Example 1:

The equations

$$6x + 7y = 10$$

$$2x + 3y = 9$$

would be summarized as:

Variables

$$\underset{\text{Coefficients}}{\begin{pmatrix} 6 & 7 \\ 2 & 3 \end{pmatrix}} \begin{pmatrix} x \\ y \end{pmatrix} = \underset{\text{Known terms}}{\begin{pmatrix} 10 \\ 9 \end{pmatrix}}$$

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Matrices (cont.)

Example 2:

The equations

$$x + 4y + 2z = 1$$

$$5x + 7y + 13z = 7$$

$$9x + 14y + 2z = 18$$

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Matrices (cont.)

Example 2:

The equations

$$x + 4y + 2z = 1$$

$$5x + 7y + 13z = 7$$

$$9x + 14y + 2z = 18$$

would be summarized as:

$$\begin{pmatrix} 1 & 4 & 2 \\ 5 & 7 & 13 \\ 9 & 14 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \\ 18 \end{pmatrix}$$

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Matrices (cont.)

Example 3: Write down the following equations using matrix notation

$$5x + 4y = 18$$

$$5z = 14$$

$$x + 7y + 9z = -9$$

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Matrices (cont.)

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$$\begin{aligned} 5x+4y \quad &= 18 \\ 5z &= 14 \\ x+7y+9z &= -9 \end{aligned}$$

Matrices (cont.)

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$$\begin{aligned} 5x+4y \quad &= 18 \\ 5z &= 14 \\ x+7y+9z &= -9 \end{aligned}$$

Answer:

$$\begin{pmatrix} 5 & 4 & 0 \\ 0 & 0 & 5 \\ 1 & 7 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 18 \\ 14 \\ -9 \end{pmatrix}$$

Matrices (cont.)

- Dimension of matrix:
  number of rows
  X
  number of columns

$$\begin{pmatrix} 1 & 0 \\ 6 & 5 \\ 7 & 2 \end{pmatrix}$$ 3 x 2 "3 cross 2" "3 by 2"

Matrices (cont.)

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  number of rows
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- Diagonal matrix:
  matrix that has elements only on the diagonal and others are zero. Elements are called *eigenvalues*

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

Matrices (cont.)

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- Trace of a matrix: is the sum of the elements of the diagonal (tr):

$$\mathrm{tr} \begin{pmatrix} 1 & 0 & 0 \\ 6 & 5 & 3 \\ 7 & 2 & 1 \end{pmatrix} = 7$$

Matrices (cont.)

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- Determinant of matrix (det):

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Matrices (cont.)

It is usually useful to work backwards from matrix equations to:

- write down the simultaneous equations, and
- "solve" the equations to obtain the unknowns

 e.g. x, y, z

Matrices (cont.)

Example: Write down the simultaneous equations corresponding to the following matrix equations:

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 4 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 9 \end{pmatrix}$$

Matrices (cont.)

Example: Write down the simultaneous equations corresponding to the following matrix equations:

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 4 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 9 \end{pmatrix}$$

Answer:

$$2x + 0y + 0z = 2 \qquad x = 1$$
$$1x + 0y + 1z = 0 \quad \text{or} \quad x + z = 0$$
$$0x + 4y + 4z = 9 \qquad 4y + 4z = 9$$

Simultaneous Equations

$$x = 1$$
$$x + z = 0$$
$$4y + 4z = 9$$

Approach:

1. start with the simplest equation,
2. express one variable in terms of others,
3. substitute this variable into the next simplest equation,
4. if one or more variables unknown, repeat from step 2 for next equation

Solution:

Simultaneous Equations

$$x = 1$$
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1. Equation 1 is simplest

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Simultaneous Equations

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Solution:
1. Equation 1 is simplest
2. x=1
3. Substitute into Eqn 2: 1 + z = 0 → z = -1
4. Repeat for next Eqn 3: 4y + 4(-1) = 9 → y=(9+4)/4 → y=13/4

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Matrix Product

2 matrices A and B can be "multiplied".
The rule underlining the multiplication is called "**row by column**".
In order to be multiplied the
Note: number of columns of 1st matrix should equal the number of rows of 2nd

$$C = A \times B$$

$$\dim(C) = (n \times m)(m \times q) = n \times q$$

$$C = A \times B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}\cdot b_{11} + a_{12}\cdot b_{21} & a_{11}\cdot b_{12} + a_{12}\cdot b_{22} \\ a_{21}\cdot b_{11} + a_{22}\cdot b_{21} & a_{21}\cdot b_{12} + a_{22}\cdot b_{22} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 5 \\ 6 & 0 \end{pmatrix} = ?$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 5 \\ 6 & 0 \end{pmatrix} = \begin{pmatrix} 1\cdot 1 + 2\cdot 6 & 1\cdot 5 + 2\cdot 0 \\ 3\cdot 1 + 4\cdot 6 & 3\cdot 5 + 4\cdot 0 \end{pmatrix} = \begin{pmatrix} 13 & 5 \\ 27 & 15 \end{pmatrix}$$

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Matrix Product (cont.)

$$A \times B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 5 \\ 6 & 0 \end{pmatrix} = \begin{pmatrix} 13 & 5 \\ 27 & 15 \end{pmatrix}$$

$$\neq$$

$$B \times A = \begin{pmatrix} 1 & 5 \\ 6 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 16 & 22 \\ 6 & 0 \end{pmatrix}$$

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Matrix Product (cont.)

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$$D \times E = \begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 32 \\ 14 \end{pmatrix}$$

$$E \times D = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix} = ?$$

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Matrix Product: Why?

- Network theory: network use adjacency matrix (1 if there is link between 2 nodes; 0 otherwise). The matrix product give information of the number of paths of certain lengths between nodes
- Dynamical System: consider a population, divided by gender;

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$$I(t=0) = \begin{pmatrix} \text{\# HIV male} \\ \text{\# HIV female} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \text{male infecting male} & \text{female infecting male} \\ \text{female infecting female} & \text{male infecting female} \end{pmatrix}$$

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$$I(t=1) = A \times I(t=0) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$I(t=2) = A \times I(t=1) = A \times A \times I(t=0) \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 9 \\ 2 \end{pmatrix}$$

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Discussion

Any questions on:

the lecture?

the maths refresher notes?

the exercises in the maths refresher notes?

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