2. $\mathbb{R}$ is a convex set.

Sps $f(x) = x^2$, then $f''(x) = 2 > 0$

3. Sps $f(\mathbf{x})$ and $g(\mathbf{x})$ are convex functions

Then $f(\lambda \mathbf{x}^2 + (1-\lambda)\mathbf{x}^1) + g(\lambda \mathbf{x}^2 + (1-\lambda)\mathbf{x}^1)$

$$\leq \lambda f(\mathbf{x}^2) + (1-\lambda) f(\mathbf{x}^1) + \lambda g(\mathbf{x}^2) + (1-\lambda) g(\mathbf{x}^1)$$

$$= \lambda (f(\mathbf{x}^2) + g(\mathbf{x}^2)) + (1-\lambda)(f(\mathbf{x}^1) + g(\mathbf{x}^1))$$

Similar to the case that functions are more than two.

4.

$$\nabla f(\mathbf{x}) = \begin{bmatrix} -400 x_1 (x_2 - x_1^2) - 2(1-x_1) \\ 200(x_2 - x_1^2) \end{bmatrix}$$

$$H_f = \begin{bmatrix} 1200 x_1^2 - 400 x_2 + 2 & -400 x_1 \\ -400 x_1 & 200 \end{bmatrix}$$

5. $\nabla f(\mathbf{x}) = \begin{bmatrix} 2x_1 + 2x_2 \\ 2x_1 + 6x_2 \end{bmatrix}$ $\quad H_f = \begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix}$

$$f(\mathbf{x}) = \frac{1}{2} \mathbf{x}^T \begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix} \mathbf{x}$$

$$|H_f - \lambda I| = \begin{vmatrix} 2-\lambda & 2 \\ 2 & 6-\lambda \end{vmatrix} = (2-\lambda)(6-\lambda) - 4$$

$$= \lambda^2 - 8\lambda + 8$$

$$\lambda^2 - 8\lambda + 8 = 0 \Rightarrow \lambda_{1,2} = \frac{8 \pm \sqrt{32}}{2} = 4 \pm 2\sqrt{2}$$

$\therefore$ H is not positive-definite. So is $f(\mathbf{x})$

6. (i)

$$\nabla f(\mathbf{x}) = \begin{bmatrix} 2x_1 + 2x_2 + 4x_3 + 4 \\ 2x_1 + 6x_2 + 2x_3 - 2 \\ 4x_1 + 2x_2 + 10x_3 + 3 \end{bmatrix}$$

$$H_f(\mathbf{x}) = \begin{bmatrix} 2 & 2 & 4 \\ 2 & 6 & 2 \\ 4 & 2 & 10 \end{bmatrix} \Rightarrow A = H_f(\mathbf{x}), \quad b = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}, \quad c = 0$$

(ii)

$$\nabla f(\mathbf{x}) = \begin{bmatrix} 10x_1 + 12x_2 - 16x_3 - 2 \\ 12x_1 + 20x_2 - 26x_3 - 4 \\ -16x_1 - 26x_2 + 34x_3 - 6 \end{bmatrix} \qquad H_f(\mathbf{x}) = \begin{bmatrix} 10 & 12 & -16 \\ 12 & 20 & -26 \\ -16 & -26 & 34 \end{bmatrix}$$

$$\Rightarrow A = H_f(\mathbf{x}), \quad b = \begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix}, \quad c = 0$$

(iii)

$$\nabla f(\mathbf{x}) = \begin{bmatrix} 2x_1 - 4x_2 + 6x_3 \\ -4x_1 + 10x_2 - 10x_3 \\ 6x_1 - 10x_2 + 16x_3 \end{bmatrix} \qquad H_f(\mathbf{x}) = \begin{bmatrix} 2 & -4 & 6 \\ -4 & 10 & -10 \\ 6 & -10 & 16 \end{bmatrix}$$

$$\Rightarrow A = H_f(\mathbf{x}), \quad \mathbb{b} = \mathbb{0}, \quad c = 0.$$

7. By 1.7.6. (iii), $A = \begin{bmatrix} 2 & -4 & 6 \\ -4 & 10 & -10 \\ 6 & -10 & 16 \end{bmatrix}$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & -4 & 6 \\ -4 & 10-\lambda & -10 \\ 6 & -10 & 16-\lambda \end{vmatrix} = (2-\lambda)\left[(10-\lambda)(16-\lambda) - 100\right] - 4\left[-4(16-\lambda) + 60\right] + 6\left[40 - 6(10-\lambda)\right]$$

$$= -\lambda^3 + 28\lambda^2 - 60\lambda - 16$$

$-\lambda^3 + 28\lambda^2 - 60\lambda - 16 = 0$ has a negative root.

Thus A is not positive definite. $f(\mathbf{x})$ neither.

8. By 1.7.4

$$\nabla f(\mathbf{x}) = \begin{bmatrix} -400x_1(x_2 - x_1^2) - 2(1-x_1) \\ 200(x_2 - x_1^2) \end{bmatrix}$$

$$H_f = \begin{bmatrix} 1200x_1^2 - 400x_2 + 2 & -400x_1 \\ -400x_1 & 200 \end{bmatrix}$$

Thus $\nabla f(\mathbf{x}^0) = \begin{bmatrix} -215.6 \\ -88 \end{bmatrix}$ $\quad H_f(\mathbf{x}^0) = \begin{bmatrix} 1330 & -480 \\ -480 & 200 \end{bmatrix}$

$$f(\mathbf{x}^0 + \Delta\mathbf{x}) \approx f(\mathbf{x}^0) + \nabla^T f(\mathbf{x}^0)\Delta\mathbf{x} + \tfrac{1}{2}\Delta\mathbf{x}^T H_f(\mathbf{x}^0)\Delta\mathbf{x}$$

$$= 24.2 - 88 + 100$$

$$= 36.2$$

10. By 1.7.9, $\nabla^T f(\mathbf{x}^0) = [-215.6 \quad -88]$

For $\mathbf{u}^1$, $\nabla^T f(\mathbf{x}^0)\mathbf{u}^1 = -215.6$

For $\mathbf{u}^2$, $\nabla^T f(\mathbf{x}^0)\mathbf{u}^2 = -88$

For $\mathbf{u}^3$. $\nabla^T f(\mathbf{x}^0)\mathbf{u}^3 = -\dfrac{759\sqrt{2}}{5}$

Function value decreases more faster along the $\mathbf{u}^1$ direction.

11. Since all the directional derivaties are negative.
all the directions are descent direction.

12. Sps $\mathbf{u}$ is the descent direction and $f(\mathbf{x})$ is a function with gradiant $\nabla f(\mathbf{x}^0)$ at $\mathbf{x}^0$.

Then $\nabla^T f(\mathbf{x}^0)\mathbf{u} = |\nabla^T f(\mathbf{x}^0)|\,|\mathbf{u}|\cos\theta$ where θ is the angle b/w $\nabla f(\mathbf{x}^0)$ and $\mathbf{u}$.

Since $\min(\cos\theta) = -1$ and $\cos\pi = -1$

$u = -\nabla f(x^0)$ is the descent direction we want.

13.14

(i)

$$A = \begin{bmatrix} 2 & 2 & 4 \\ 2 & 6 & 2 \\ 4 & 2 & 10 \end{bmatrix}$$

positive - definite

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 2 & 4 \\ 2 & 6-\lambda & 2 \\ 4 & 2 & 10-\lambda \end{vmatrix} = -\lambda^3 + 18\lambda^2 - 68\lambda + 8$$

$$|A - \lambda I| = 0 \Rightarrow \lambda_1 \approx 12.69 \quad v_1 \approx [0.46 \quad 0.43 \quad 1]^T$$

$$\lambda_2 \approx 5.19 \quad v_2 \approx [-0.11 \quad -2.18 \quad 1]^T$$

$$\lambda_3 \approx 0.12 \quad v_3 \approx [-2.77 \quad 0.60 \quad 1]^T$$

(ii)

$$A = \begin{bmatrix} 10 & 12 & -16 \\ 12 & 20 & -26 \\ -16 & -26 & 34 \end{bmatrix}$$

positive - definite

$$|A - \lambda I| = \begin{vmatrix} 10-\lambda & 12 & -16 \\ 12 & 20-\lambda & -26 \\ -16 & -26 & 34-\lambda \end{vmatrix} = -\lambda^3 + 64\lambda^2 - 144\lambda + 8$$

$$|A - \lambda I| = 0 \Rightarrow \lambda_1 \approx 61.67 \quad v_1 \approx [-0.49 \quad -0.76 \quad 1]^T$$

$$\lambda_2 \approx 2.28 \quad v_2 \approx [4.00 \quad -1.24 \quad 1]^T$$

$$\lambda_3 \approx 0.06 \quad v_3 \approx [0.13 \quad 1.23 \quad 1]^T$$

(iii)

$$A = \begin{bmatrix} 2 & -4 & 6 \\ -4 & 10 & -10 \\ 6 & -10 & 16 \end{bmatrix}$$

Since $|A - \lambda I| = 0$ has both positive and negative roots, A is indefinite

$$|A-\lambda I| = \begin{vmatrix} 2-\lambda & -4 & 6 \\ -4 & 10-\lambda & -10 \\ 6 & -10 & 16-\lambda \end{vmatrix} = -\lambda^3 + 28\lambda^2 - 60\lambda - 16$$

$$|A-\lambda I| = 0 \Rightarrow \begin{array}{lll} \lambda_1 \approx 25.64 & \quad & v_1 \approx [0.38 \quad -0.74 \quad 1]^T \\ \lambda_2 \approx 2.60 & & v_2 \approx [0.21 \quad 1.47 \quad 1]^T \\ \lambda_3 \approx -0.24. & & v_3 \approx [-3.09 \quad -0.23 \quad 1]^T \end{array}$$

15.