Stability Analyzation and Input-State Feedback Linearization of Lorentz System

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***Abstract*—In this report, we present an analysis about the famous Lorentz system. We introduced the general model of such system and studied its system dynamics. We plotted the trajectory of the system and showed the stability of the Lorentz attractor. We also stabilized the system with input-state feedback linearization technique and used MATLAB for numerical simulations.**

Keywords—feedback linearization, chaotic behavior, Lorentz attractor, nonlinear control

I. INTRODUCTION

Often referred to as the "butterfly effect", Lorentz attractor has become one of the most attractive research highlights as a kind of chaotic behavior [1-3]. The system is a set of ordinary differential equations first demonstrated by Edward Lorenz [1] to describe the thermal convection loop. He also discovered that the system is chaotic under certain initial condition or certain set of parameters. It is the first time that the conception of chaos, which is common in nonlinear dynamical systems, unveiled its mystery. Since then, scientists have also discovered other chaotic systems that "looks like" Lorentz system but have different topology, such as Chen system [Chen & Ueta,1999], Lu system [Lu, 2002]. The discovery of these systems deepened our understanding of the connection between chaos and order.

Nowadays, Lorenz system is widely applied in the field of weather broadcasting, and as multiple kinds of control technique for chaotic systems have been proposed, Lorentz system also serves as a benchmark to verify methods of controlling chaos.

In this report, we introduced the mathematical model of a typical Lorentz system. We plotted its phase portrait to show the performance of the system. We also found out its invariant points and analyzed the stability of such a system. Finally, we applied feedback linearization technique to stabilize the system.

II. MODEL DESCRIPTION

Abundant research has been conducted to study the Lorentz system [5-8]. In this section, we give a brief introduction of the mathematical models and tools to analyze Lorentz system. The general expression of a Lorentz system in the form of ordinary differential equations is shown in (1):

$$\begin{aligned}\dot{x} &= a(y-x)\\ \dot{y} &= bx - zx - y\\ \dot{z} &= xy - cz\\ f &= f(x,y,z)\end{aligned} \tag{1}$$

The Lorenz system presents a complex dynamics depending on the values of its parameters. For example, for small values of b, the dynamics of the system is stable, while for high values of b, the dynamic behavior becomes chaotic (this will be derived later). When b = 1, bifurcation occurs.

A chaotic system is defined as dissipative when the following inequality holds:

$$\nabla \cdot f < 0 \tag{2}$$

If the system is dissipative, then we could anticipate the beautiful wings of butterfly, namely, there exists certain amount of attractor in the phase portrait of given chaotic system. In our case, we have

$$-(a+1+c) < 0 \tag{3}$$

Another important description of a chaotic system is the Lyapunov exponent λ, though we will not discuss it in this report. Generally speaking, if there is a slight difference between the two systems initially, they are likely to become separated over time (or number of iterations), and the degree of separation is described by Lyapunov exponent. A system is chaotic if at least one of Lyapunov exponents $\lambda > 0$ holds.

The equilibrium (fixed point) is given by setting the left-hand side of $f(x,y,z)$ zero, and finally we have:

$$\begin{bmatrix} x_e \\ y_e \\ z_e \end{bmatrix} = \begin{bmatrix} \pm\sqrt{c(b-1)} \\ \pm\sqrt{c(b-1)} \\ b-1 \end{bmatrix} \tag{4}$$

These 2 points are referred to as $C_{1,2}$. It is obvious that the system also has a steady-state equilibrium $C_0 = [0\ 0\ 0]^T$.

III. ANALYSIS OF THE SYSTEM DYNAMICS

The dynamics of Lorentz system is important if we wish to understand such phenomenon and use such model to analyzing real-world systems. In this section, we showed the dynamics of Lorentz system. Note that all the simulations in this report are completed with parameter a = 10, b = 28, c = 8/3, if not specified otherwise.

A. Phase Portrait

We first examined the trajectory of two Lorentz systems with different parameters. The result is shown in Figure 1 as the phase portrait of this system. The trajectory is solved with MATLAB ode45 solver.

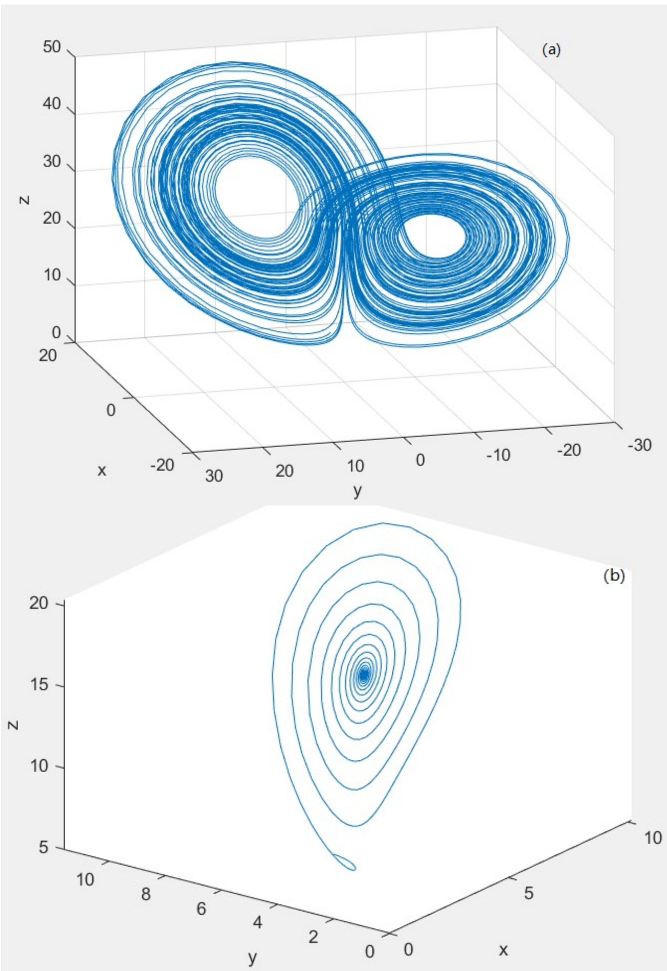


Figure 1. Phase portrait of a typical Lorentz system, with different parameters. (a). a = 10, b = 28, c = 8/3 (probably the most popular set of parameters). (b). a = 10, b = 15, c = 11/3.

B. Stability Analysis

The theory of stability is based mainly on the theory of Lyapunov. A candidate of Lyapunov function could be:

$$V = \frac{x^2}{a} + y^2 + z^2 \geq 0 \tag{5}$$

$$\dot{V} = 2\left(\frac{x\dot{x}}{a} + y\dot{y} + z\dot{z}\right)$$

$$\dot{V} = 2(b+1)xy - x^2 - y^2 - cz^2$$

$$\dot{V} = -2(x - \frac{b+1}{2}y)^2 - 2\left(1 - \frac{b+1}{2}\right)y^2 - 2cz^2 \tag{6}$$

To guarantee the negative definiteness of the derivative of Lyapunov function, we must have $b < 1$. Thus, the origin C_0 is a global attractor.

When it comes to C_1, C_2, we use Jacobian linearization method to obtain the characteristic polynomial. Let A denotes the Jacobian matrix, we have:

$$A = \frac{\partial F}{\partial X} = \begin{bmatrix} -a & a & 0 \\ -b - z & -1 & -x \\ y & x & -c \end{bmatrix} \tag{7}$$

At the neighborhood of invariant point C_1, C_2 the characteristic polynomial has the expression of:

$$c(s) = s^3 + (a + c + 1)s^2 + c(a + b)s + 2ac(b - 1) \tag{8}$$

Suppose there exists a ρ_h such that when $b < \rho_h$, the system is stable about $C_{1,2}$; and when $\rho_h > b$, the system become chaotic. Then the real parts of the eigenvalues vanish at $b = \rho_h$. Suppose the conjugate imaginary parts could be denoted as $\pm i\mu = s$. Thus, we have:

$$c(i\mu) = -i\mu^3 - (a + c + 1)\mu^2 + ic(a + b)\mu + 2ac(b - 1)$$
$$c(i\mu) = 0 \tag{9}$$

The imaginary part and the real part should cancel out; thus, we should have:

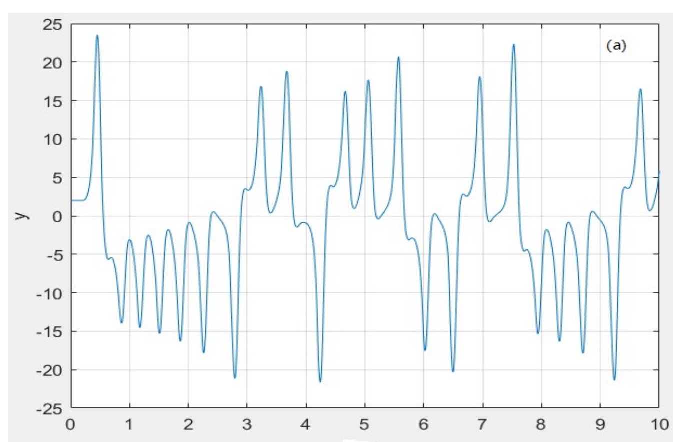
$$\mu^2 = \frac{2ac(b-1)}{a+c+1} \tag{10}$$

$$\mu^3 = \mu(a + b)c \tag{11}$$

Therefore, it follows that

$$b = \frac{a(a+c+3)}{a-c-1} = \rho_h \tag{12}$$

The result of critical value of b is about 24.74. We plotted the change of state y over time to visualize the difference between a chaotic system and a stable system, as it is shown in Figure 2:



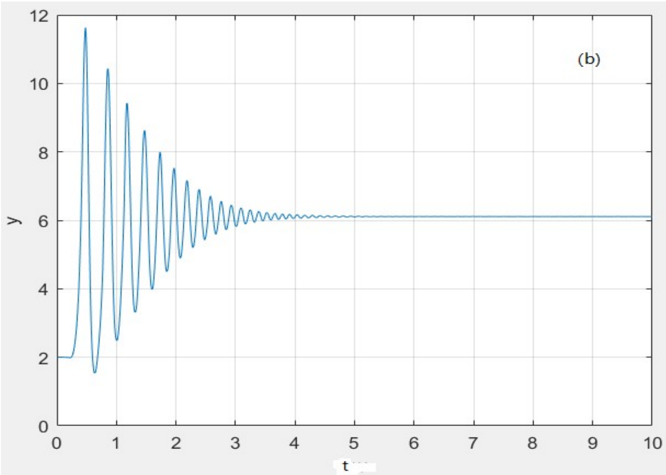


Figure 2. Trajectory of variable y with different stability conditions. (a). The system is unstable, a = 10, b = 28, c = 8/3. (b). The system is stable, a = 10, b = 15, c = 8/3.

IV. INPUT-STATE FEEDBACK LINEARIZATION

Feedback linearization technique is widely used. In this section, we try to design a controller applying input-state feedback linearization technique, as demonstrated in [4]. Consider a single-input nonlinear control system

$$\dot{x} = f(x) + g(x)u \tag{13}$$

Where $x = [x_1;\ x_2;\ x_3]$, and f(x) denotes the corresponding expression of Lorentz system stated previously. Consider a simple case: $g(x) = [0\ \ x_1\ \ 0]^T$, it is obvious that f and g are smooth vector fields on R^n.

Suppose that we did not specify the output y=h(x). Then to ensure that our system is linearizable, we must let $\{g,\ ad_g Y\}$ have involutivity. Applying the expression of Lie derivative and Lie bracket, for the transformation Y, we have:

$$L_g Y = x_1 \frac{\partial Y}{\partial x_2} = 0 \tag{14}$$

Applying (15) to the expression of $L_{ad_f g} Y$, it follows that

$$L_{ad_f g} Y = -a x_1 \frac{\partial Y}{\partial x_1} - x_1^2 \frac{\partial Y}{\partial x_3} = 0 \tag{15}$$

Therefore Y should take such form:

$$Y = \frac{1}{a} {x_1}^2 - 2x_3 + \mu \tag{16}$$

Where μ denotes some function that is independent of x. Finally, we can have the expression of the linearized canonical system:

$$z = \varphi(x) = \begin{bmatrix} Y \\ L_f Y \\ L_f^2 Y \end{bmatrix} \tag{17}$$

$$\dot{z} = A_c z + B_c v \tag{18}$$

Where we already know that

$$A_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B_c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Suppose we have $v = \gamma_1 z_1 + \gamma_2 z_2 + \gamma_3 z_3$, and place the pole at, let's say, Re(s) = -1, then we can have the value of γ_{1-3}.

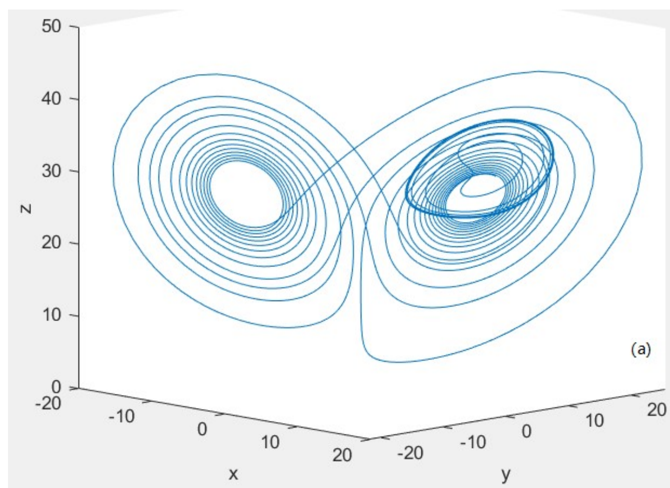
$$v = a(x) + b(x)u \tag{19}$$

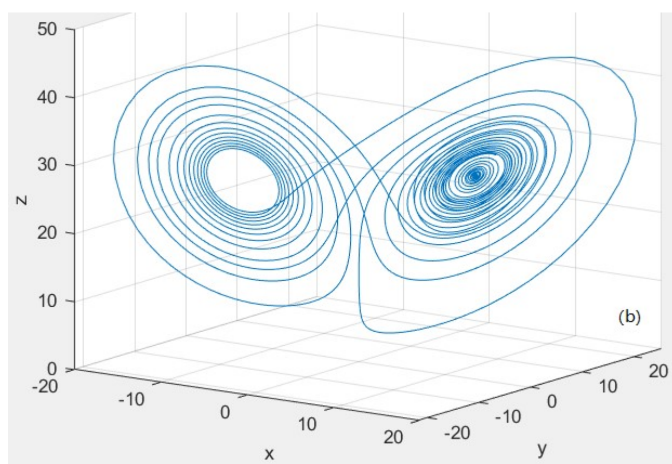
$$a(x) = L_f^n Y, \quad b(x) = L_g L_f^{n-1} Y \tag{20}$$

For our simulation, we hope to regulate the state to sinusoidal waves that oscillates around an equilibrium value with a frequency of 2π. We choose

$$\mu = \frac{2a - c}{ac} x_{equi}^2$$

with $x = A\sin(2\pi t)$. The signal, with initial amplitude A, is turned on at time t = 20. The simulation result is shown in Figure 3. Fig. 3(a) and Fig. 3(b) showed the phase portrait of system with A = 500 and A = 50, respectively. It could be seem from Fig. 3(c) and Fig. 3(d) that the system is stabilized to one of its equilibrium points C_1 at x ≈ 8.5.





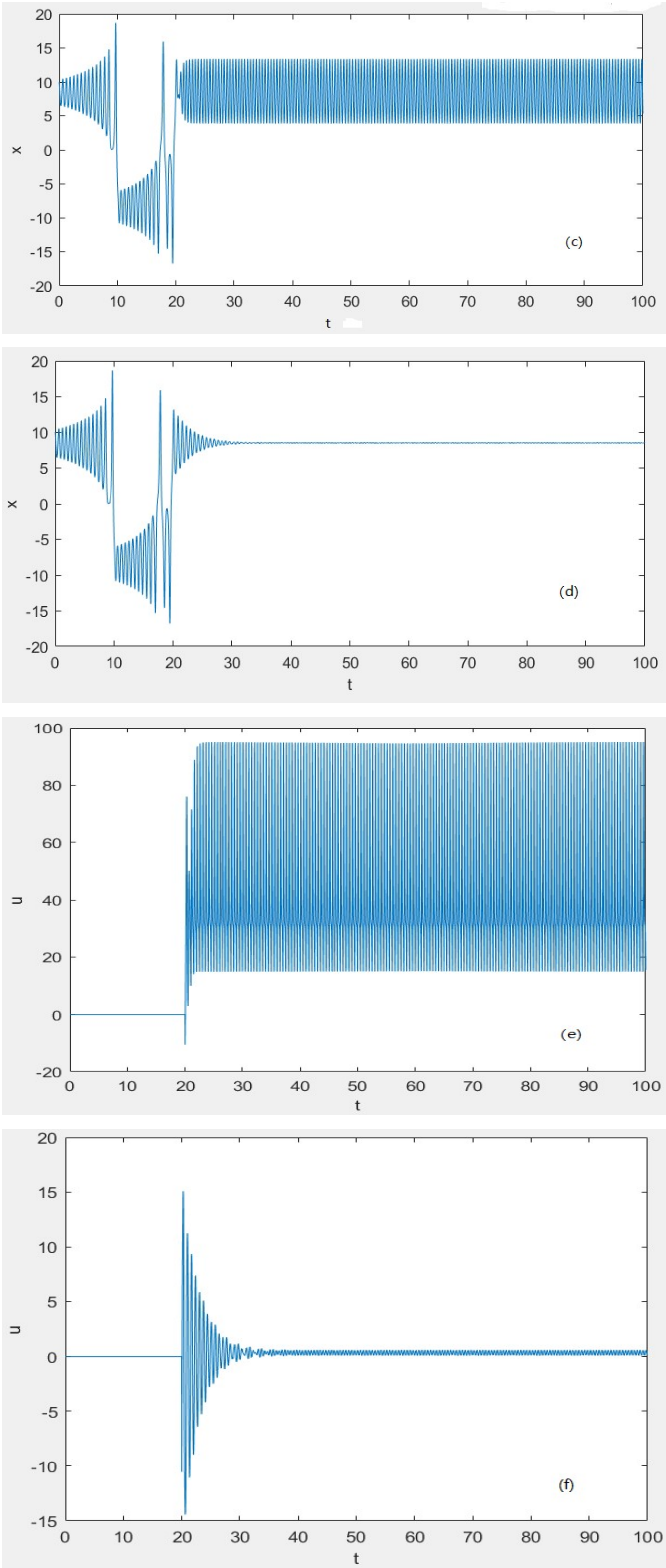


Figure 3. The input-state feedback linearization of a Lorentz system. All the states are regulated to sinusoidal form. (a). Phase portrait, A=500. (b). Phase portrait with A=50. (c). Regulated state x, A=500. (d). Regulated state x with A=50. (e). Regulated input, A=500 (f). regulated input with A=50.

From Fig .3(e) and Fig .3(f), we could observe the regulated input u. It could be seen that eventually u is also periodic, although it's not necessarily sinusoidal.

Also, from the mathematical derivation and simulation result of the control scheme, it could be noted that if controls are present on all states, then the nonlinearities can be canceled directly so that any trajectory is possible.

V. SUMMARY

In this report, we presented the mathematical model of a typical Lorentz system. We plotted its phase portrait to show the performance of the system and verified the critical radius as calculated. We also found out its invariant points and analyzed the stability of such a system near these points. Finally, we applied feedback linearization technique to stabilize the system. The result showed that the system is stabilized to one of the equilibrium points.

For future work, we can explore all the other control mechanisms to regulate the system, such as sliding plane control, time-delaying control. And the robustness of different control schemes needs to be analyzed [5]. We could also dive deep into the math behind chaos, for instance, the topology of Lorentz systems. The application of discrete Lorentz systems in signal processing is also a very promising field of study [9].

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