# Texas A&M University
# Department of Mechanical Engineering

## MEEN 655: Design of Nonlinear Control Systems
## Homework 3

February 11, 2022

1. *Gradient Dyanmical System:* Let $V : \Re^n \to \Re$ be a $C^1$ map. Consider the autonomous nonlinear system:
$$\dot{x} = -\nabla V(x).$$
   - Suppose $x = 0$ is an *isolated* local minimum, show that $x = 0$ is an asymptotically stable equilibrium.
   - Furthermore, if $V(x)$ is convex in $x$, then show that $x = 0$ is a globally asymptotically stable equilibrium.
   - If $x = 0$ is an isolated local maximum of $V(x)$, then show that it is an unstable equilibrium.

2. Mechanical systems, such as robotic manipulators with rigid links, can be modeled using the following canonical system of equations involving joint variables $q(t) \in \Re^n$ and joint velocities, $\dot{q}(t) \in \Re^n$:
$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + \nabla U(q) = \tau,$$
where $M(q) = M^T(q) \succ 0$ (i.e., $M(q)$ is a positive definite matrix), $C(q, \dot{q})$ is Corialis matrix and $U(q)$ is the potential energy. Compliance with the principle of conservation of energy implies that $\Omega := \dot{M} - 2C$ must be skew-symmetric, i.e. $\Omega + \Omega^T = 0$.
   - When $\tau = 0$, show that
$$\frac{d}{dt}(\frac{1}{2}\dot{q}^T M(q)\dot{q} + U(q)) = 0,$$
i.e., total energy of the system is conserved.
   - Suppose $U(q)$ is an isolated local minimum, show that $(q = 0, \dot{q} = 0)$ is a stable equilibrium.

- Suppose $\nabla U(0) = 0$, but $q = 0$ is not a local minimum. With a PD control, we want to stabilize the equilibrium $(q = 0, \dot{q} = 0)$; consider the control law:

$$\tau = -K_p q - K_d \dot{q},$$

where $K_p = K_p^T,\ K_d = K_d^T$ are positive definite matrices that need to be chosen (control parameters). Show that one can pick some positive definite $K_p, K_d$ so that the closed loop system has an asymptotically stable equilibrium at $(q = 0, \dot{q} = 0)$. (Hint: Try the Liapunov function candidate:

$$V(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + U(q) - U(0) + \frac{1}{2}q^T K_p q,$$

and notice from Taylor's expansion of $U$:

$$U(q) - U(0) = \frac{1}{2}q^T H q + O_3(q),$$

where $H$ is the Hessian of $U$ at $q = 0$ and is *not positive semi-definite.* )

3. Consider an $n-$ dimensional autonomous nonlinear system:

$$\dot{x} = f(x), \quad f(0) = 0.$$

Suppose there exist constant, symmetric positive definite matrices $P$ and $Q$ that satisfy for all $x \in \Re^n$:

$$\left(\frac{\partial f}{\partial x}\right)^T P + P\left(\frac{\partial f}{\partial x}\right) = -Q.$$

Show that $x = 0$ is globally asymptotically stable using the following steps:

- Using

$$\int_0^1 \frac{\partial f}{\partial x}(\sigma x) x d\sigma = f(x),$$

show that

$$f(x)^T P x + x^T P f(x) = -x^T Q x.$$

- Show that $f(x) = 0 \implies x = 0$. (Hint: what happens if $x \neq 0$ and $f(x) = 0$?).
- Show that $\widehat{V} = f(x)^T P f(x)$ is a positive definite function of $x$.
- Show that $\frac{d}{dt}\widehat{V}$ is negative definite.
- Show that $\widehat{V}$ is radially unbounded.
- Conclude that $x = 0$ is a globally asymptotically stable equilibrium.

4. *Luenberger-like Observer design for Lipschitz nonlinear systems:* Consider a Lipschitz nonlinear system of the form:

$$\dot{x} = Ax + f(x), \quad y = Cx,$$

where $f(x)$ is Lipschitz with a constant $\gamma$, i.e., $\forall x, y \in \Re^n$,

$$\|f(x) - f(y)\| \leq \gamma \|x - y\|.$$

Suppose $(A, C)$ is observable; then Hautus' theorem indicates that there is a $L$ such that $A - LC$ is Hurwitz. Find a bound on $\gamma$ so that estimates of the state of the nonlinear system converge to their true values irrespective of what the true initial conditions are:

$$\dot{w} = Aw + L(y - Cw) + f(w).$$

(Hint: If $e = x - w$ is the state estimation error, what is the differential equation for $\dot{e}$? Consider a Liapunov equation of the form

$$(A - LC)^T P + P(A - LC) = -I,$$

and set up the Liapunov function $V(e) = e^T Pe$. )

5. Consider a nonlinear system of the following form:

$$\dot{x} = Ax + Bu, \quad y = Cx + Du.$$

Consider an output feedback control of the form $u = -\phi(y)$, where $0 \leq y^T \phi(y)$. Suppose there exists a positive definite $P$, positive constant $\epsilon$ and matrices $L$ and $W$ satisfying the following:

$$A^T P + PA = -\epsilon P - L^T L, \quad PB = C^T - L^T W, \quad W^T W = D + D^T.$$

Show that

- Let $V(x) = x^T Px$. Show that

$$\dot{V} = -\epsilon x^T Px - (Lx)^T (Lx) + u^T (C - W^T L)x + x^T (C^T - L^T W)u.$$

- Using $y = Cx + Du$ and completion of squares, show that one may express

$$V = -\epsilon x^T Px - (Lx + Wu)^T (Lx + Wu) + 2u^T y.$$

- Using the fact that $u = -\phi(y)$ and $y^T \phi(y) \geq 0$, show that

$$\dot{V} \leq -\epsilon \lambda_{min}(P) \|x\|^2.$$

- Show that all conditions of the exponential stability theorem are met and conclude that $x = 0$ is exponentially stable.

6. Consider a two dimensional nonlinear system in the following form (referred to as strict feedback form):
$$\dot{x}_1 = x_2 + f_1(x_1), \quad \dot{x}_2 = f_2(x_1, x_2) + u.$$
The objective is to regulate the state to zero, (i.e., make $(x_1 = 0, x_2 = 0)$ an exponentially stable equilibrium of the closed loop system).

   In order to do so, consider a simpler system:
$$\dot{x}_1 = v + f_1(x_1),$$
where the "synthetic" input is $v$ (in place of the state $x_2$). Consider a Liapunov function candidate:
$$V_1(x_1) = \frac{1}{2}x_1^2,$$
and show that
$$\frac{dV_1}{dt} = -\mu_1 x_1^2$$
if $v = -\mu x_1 - f_1(x_1)$.

7. We want to treat $v$ as a desired value for $x_2$ and want to ensure that $x_2(t) - v(t) \to 0$ asymptotically. In this task, let
$$z_2 := x_2 - v = x_2 + \mu x_1 + f_1(x_1).$$
   - Show that the governing equations for the nonlinear system can be represented as:
$$\begin{aligned} \dot{x}_1 &= z_2 - \mu x_1, \\ \dot{z}_2 &= u + \underbrace{f_2(x_1, x_2) + \mu + \frac{\partial f_1}{\partial x_1}(x_2 + f_1(x_1))}_{\phi_2(x_1, z_2)}. \end{aligned}$$
   - Use the Liapunov function candidate
$$V_2(x_1, z_2) = V_1(x_1) + \frac{1}{2}z_2^2,$$
show that
$$\frac{dV_2}{dt} = x_1(-\mu x_1 + z_2) + z_2(u + \phi_2(x_1, z_2)).$$
   - Show that
$$u = -\phi_2(x_1, z_2) - \mu x_1 - z_2,$$
will ensure that $(x_1 = 0, z_2 = 0)$ is an asymptotically stable equilibrium of the closed loop system.