Texas A&M University
Department of Mechanical Engineering

MEEN 655: Design of Nonlinear Control Systems
Homework 5

March 5, 2022

1. Consider a nonlinear system
$$\dot{x} = f(x, u).$$
Suppose there is a smooth control input $u = h(x)$ that renders the equilibrium $x = 0$ of the closed loop system
$$\dot{x} = f(x, h(x))$$
exponentially stable. You may assume that f is Lipschitz in its arguments. Show that the extended system
$$\begin{aligned}\dot{x} &= f(x, z),\\ \dot{z} &= g(x, z) + v\end{aligned}$$
is also smoothly exponentially stabilizable, i.e., there is a smooth function $v = \psi(x, z)$ such that the equilibrium $(x = 0, z = 0)$ of the closed loop system
$$\begin{aligned}\dot{x} &= f(x, z),\\ \dot{z} &= g(x, z) + \psi(x, z)\end{aligned}$$
is exponentially stable.
(Hint: Use the converse Lyapunov theorem and use a procedure similar to backstepping problems in HWs 3 and 4). Also note that by setting $g(x, z) \equiv 0$, we can see that the property of smooth stabilizability is not lost when integrators are added in the system.)

2. Consider the linear time varying system:
$$\dot{x}(t) = A(t)x(t) + B(t)u(t).$$
Let
$$\frac{d\Phi(t, t_0)}{dt} = A(t)\Phi(t, t_0), \qquad \Phi(t_0, t_0) = I.$$

Suppose for some L_A, L_B, and for all t, we have $\|A(t)\| \leq L_A$ and $\|B(t)\| \leq L_B$; furthermore, the controllability grammian is uniformly positive definite, i.e., for some $\delta, \alpha, \lambda > 0$,

$$W_c(t, t+\delta) := \int_t^{t+\delta} e^{4\lambda(t-\tau)} \Phi(t,\tau) B(\tau) B^T(\tau) \Phi^T(t,\tau) d\tau \succ \alpha I.$$

- Show that
$$\|W_c(t,t+\delta)\| \leq \frac{L_B^2}{2(2\lambda + L_A)} [1 - e^{-2(2\lambda+L_A)\delta}].$$
- Show that
$$\frac{2(2\lambda + L_A)}{L_B^2} \|x(t)\|^2 \leq x^T(t) W_c^{-1}(t,t+\delta) x(t) \leq \frac{1}{\alpha} \|x(t)\|^2.$$
- Show that $W_c(t, t+\delta)$ (for short W_c) satisfies:
$$\frac{dW_c}{dt} = 4\lambda W_c + A W_c + W_c A^T - B(t) B^T(t) + e^{-4\lambda\delta} \Phi(t,t+\delta) B(t+\delta) B(t+\delta)^T \Phi^T(t,t+\delta).$$
- Show further that
$$\begin{aligned} \frac{dW_c^{-1}}{dt} &= -4\lambda W_c^{-1} - (A - B^T W_c^{-1})^T W_c^{-1} - W_c^{-1}(A - B^T W_c^{-1}) - W_c^{-1} B B^T W_c^{-1} \\ &\quad - e^{-4\lambda\delta} W_c^{-1} \Phi^T(t,t+\delta) B(t+\delta) B^T(t+\delta) \Phi^T(t,t+\delta) W_c^{-1}. \end{aligned}$$
- Show that the feedback control law
$$u = -B^T(t) W_c^{-1}(t, t+\delta) x(t)$$
exponentially stabilizes the linear system. (Hint: Try $V(x,t) = x^T(t) W_c^{-1}(t,t+\delta)x(t)$.)
- What is an estimate of the rate of convergence?

3. Suppose the equilibrium $x = 0$ of the linear time-varying system
$$\dot{x}(t) = A(t)x(t)$$
is exponentially stable. Suppose $\lim_{t\to\infty} \|B(t)\| = 0$, show that the equilibrium of the perturbed linear system
$$\dot{x}(t) = (A(t) + B(t))x(t)$$
is exponentially stable.

4. Consider the problem of estimating the state of a linear system:

$$\dot{x} = Ax, \quad y = Cx,$$

where A, B, C are known constants and the initial condition, $x(0) = \theta$ is unknown. One can express

$$y(t) = \underbrace{C\Phi(t,0)}_{W^T(t)}\theta,$$

where $\Phi(t,\tau) = e^{A(t-\tau)}$. Suppose the best estimate of θ after observing the output on $[0,t]$ is $\hat{\theta}(t)$. Let

$$\hat{y}(t) = W^T(t)\hat{\theta}(t)$$

be the output estimate; the parameter estimation error is $\tilde{\theta}(t) := \hat{\theta}(t) - \theta(t)$, and the output estimation error is $e := y(t) - \hat{y}(t) = W^T(t)\tilde{\theta}(t)$. We want to construct an adaptive parameter estimation scheme to estimate the initial condition, θ. For this we use gradient estimation scheme to drive $e \to 0$ asymptotically.

- Suppose $\phi(\tilde{\theta}) = \frac{1}{2}\|e(\tilde{\theta})\|^2$. The gradient dynamical system that seeks the minimum value of ϕ is

  $$\dot{\tilde{\theta}} = -\nabla\phi(\tilde{\theta}) = -W(t)e(t) = -W(t)W^T(t)\tilde{\theta}(t).$$

  (This can be implemented as:

  $$\dot{\hat{\theta}} = -W(t)e(t).$$

)
- Show the following are equivalent:
  (a) The LTV system

  $$\dot{\tilde{\theta}} = -W(t)W^T(t)\tilde{\theta}(t), \quad e(t) = W^T(t)\tilde{\theta}(t)$$

  is UCO if and only if the LTV

  $$\dot{\tilde{\theta}} = 0, \qquad e(t) = W^T(t)\tilde{\theta}$$

  is UCO.
  (b) Show that the latter LTV is UCO if and only if the pair (C, A) is completely observable.
  (c) Show that the equilibrium $\tilde{\theta} = 0$ is exponentially stable if (C, A) is completely observable.
  (d) Pick any linear system with no more than 3 states and corroborate the observation in the previous part.

5. Consider a dynamical system of the form:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -Dx_2 - \nabla\phi(x_1),$$

where $x_1, x_2 \in \Re^n$ are the position and velocity vectors of a mechanical system. Suppose $D = D^T \succ 0$, and $\phi(x_1)$ represents the potential energy of this system with isolated extrema; you may assume $\phi(x_1)$ to be a sufficiently smooth function.

   (a) What are the equilibria of this system?

   (b) What would the corresponding Jacobi linearization be?

   (c) What can you say about the eigenvalues of the Jacobian?

   (d) What can you say about the stability of the equilibria?

   (e) Can a periodic orbit exist for this system?

6. **Bendixson's & DuLac's criteria:** A simplified version of divergence theorem for a plane states the following: If $\mathcal{D}$ is a simply connected region of $\Re^2$ with Γ being its boundary. The curve Γ is parametrized by the arc length s and let $\mathcal{A} \subset \mathcal{D}$ be the area enclosed within Γ. Let $\mathbf{n}(s)$ be the outward normal to Γ at s. Let $\mathbf{f}(x_1, x_2)$ be a 2-d vector field defined on $\mathcal{D}$ with $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ as its components. Then

$$\int_\Gamma \mathbf{f} \cdot \mathbf{n} \, ds = \int \int_\mathcal{A} div(\mathbf{f}) \, dx_1 dx_2.$$

   - Consider a planar nonlinear dynamical system

     $$\dot{x}_1 = f_1(x_1, x_2), \quad \dot{x}_2 = f_2(x_1, x_2).$$

     Suppose $div(\mathbf{f})(x_1, x_2) > 0$ on A, then show that the planar dynamical system cannot have a closed orbit.
     (Hint: If it had a periodic orbit Γ, the tangent to Γ is specified by $\mathbf{f}$. What would $\mathbf{f} \cdot \mathbf{n}$ be then? Use simplified divergence theorem to arrive at a contradiction).

   - Consider a planar nonlinear dynamical system:

     $$\dot{x}_1 = f_1(x_1, x_2), \quad \dot{x}_2 = f_2(x_1, x_2),$$

     where f_1, f_2 are smooth functions. Let S be a simply connected region and $B(x_1, x_2)$ be a smooth function such that

     $$div(Bf) := \frac{\partial B(x_1, x_2) f_1(x_1, x_2)}{\partial x_1} + \frac{\partial B(x_1, x_2) f_2(x_1, x_2)}{\partial x_2}$$

     does not change sign or is identically zero on S. Show that S cannot have a closed orbit (trajectory) of the planar dynamical system.

7. Consider a planar dynamical nonlinear system:

$$\dot{x} = -y + xy, \quad \dot{y} = x + \frac{1}{2}(x^2 - y^2).$$

- Show that equilibria for the above nonlinear system are: $(0,0), (-2,0), (1, \pm\sqrt{3})$.
- A sketch of phase portrait of the above system is shown in figure 1. Consider the

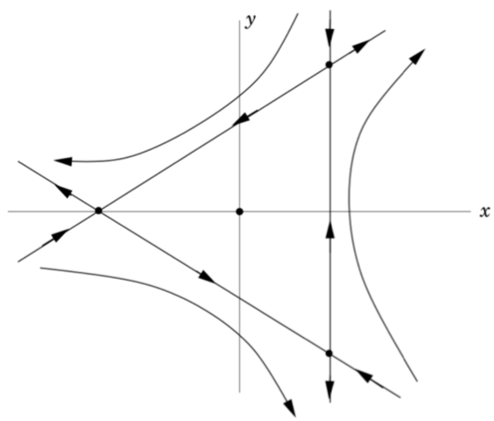

Figure 1: Sketch of Phase Portrait of the Planar Dynamical System

triangle with vertices $A = (-2, 0)$, $B = (1, \sqrt{3})$ and $C = (1, -\sqrt{3})$. The equation of line connecting A and B is

$$L_{AB}(x,y) := y - \frac{1}{\sqrt{3}}(x+2) = 0.$$

Similarly, the line connecting A and C has the equation:

$$L_{AC}(x,y) = y + \frac{1}{\sqrt{3}}(x+2) = 0.$$

The line connecting B and C has the equation $L_{BC}(x,y) := x - 1 = 0$. Show that the sets $L_{AB} = 0, L_{AC} = 0$ and $L_{BC} = 0$ are positively invariant, i.e., if you start on them, you will remain on the respective line segment, and conclude that the triangle ΔABC is positively invariant.

- Consider the function

$$V(x,y) = -\frac{1}{2}(x^2 + y^2) + \frac{1}{2}(xy^2 - \frac{x^3}{3}),$$

and argue the plausibility of existence of a closed orbit of this system.
(Hint: For flows starting sufficiently close to the origin, who that the rate of change of V is even smaller (in order, say cubic)), while the dominating terms of V are still quadratic. Approximate cubic terms to zero and deal with dominating quadratic terms. Support your argument with a phase portrait around equilibrium).