Texas A&M University
Department of Mechanical Engineering

MEEN 655: Design of Nonlinear Control Systems
Exam I

March 5, 2022

Directions:

- There are three problems in this exam with one bonus sub-problem for the second problem.
- Please make sure that your answers are clearly explained; lack of clarity or legibility can result in points being deducted.
- Aggie Honor code in place. You are not allowed to consult with any other student in the class; university's policy on cheating will be *strictly* enforced.
- Good Luck.

1. This problem deals with a matrix differential equation of the form:

$$\dot{H} = [H, [H, N]],$$

where $H(t)$ and N are square matrices of dimension n; further, let $N = N^T$, and $\dot{N} = 0$.

- Given two square matrices, A, B, a bracket $[A, B]$ is defined to be

  $$[A, B] = AB - BA,$$

  and is a measure of how much their products do not commute. Consider the following differential equation:

  $$\dot{H} = [H(t), \Omega(t)],$$

  where $\Omega(t)$ is any skew-symmetric matrix.
  (Hint: Show that

  $$\frac{dH^T}{dt} = \frac{dH}{dt}^T = [H^T, \Omega(t)].$$

  - Show that if $H(t_0) = H^T(t_0)$, then $H(t) = H^T(t)$ for any $t \geq t_0$.
    (Hint: Let $X(t) := H(t) - H^T(t)$. What is the differential equation for $\frac{d(X}{dt}$? What is an obvious equilibrium?)
  - Let

    $$\dot{\Phi}(t, t_0) = \Phi(t, t_0)\Omega(t), \qquad \Phi(t_0, t_0) = I$$

    be the state-transition matrix. Show that

    $$H(t) = \Phi^T(t, t_0)H(t_0)\Phi(t, t_0)$$

    satisfies the differential equation

    $$\dot{H} = [H(t), \Omega(t)].$$

  - Show that $\Phi^T(t, t_0)\Phi(t, t_0) = I$ (in other words, $\Phi(t, t_0)$ is a rotation /unitary matrix.
    (Hint: If $Z(t) = \Phi^T(t, t_0)\Phi(t, t_0)$, show that

    $$\dot{Z} = Z(t)\Omega - \Omega(t)Z(t),$$

    and show that $Z(t) = I$ is an equilibrium.)
  - Show that $H(t)$ and $H(t_0)$ are similar, and hence, have the same set of eigenvalues.

- Suppose $H(0) = H^T(0)$ and $N = N^T$; show that $\Omega(t) := [H(t), N]$ is skew-symmetric; conclude that the solution of the double bracket equation

  $$\dot{H} = [H(t), [H(t), N]]$$

  would be symmetric and have the same set of eigenvalues as $H(t_0)$.

- Let
$$V(H) = -trace(H^T N).$$
Show that
$$\dot{V} = -\|[H,N]\|^2.$$
- For any c, show that the level set
$$S(c) := \{H : V(H) \leq c\}$$
is bounded.
- Use La Salle's theorem and conclude that the $\omega-$ limit sets of $H(t)$ are the equilibria of the double bracket system.
- Suppose
$$N = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix},$$
with $\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$; show that the $\omega-$ limit set of the trajectories must be diagonal matrices. How many such diagonal matrices would you find?
- To examine the stability, linearize the double bracket equation about any diagonal matrix in the $\omega-$ limit set. What can you conclude about the stable equilibria of the double bracket equation?

2. Consider the following nonlinear system
$$\dot{x} = Ax + \Phi(y,u) + B(\sum_{i=1}^{p} W_i(u,y)\theta_i, \qquad y = Cx,$$
where (C, A) is completely observable, (A, B) is completely controllable; the parameters $\theta_1, \ldots, \theta_p$ are constants and may not be known. The regressor vectors $W_1, \ldots, W_p$ are known and so are the matrices A, B and C. This problem is concerned with estimating the state from the output in the presence of uncertainty in the parameters $\theta_1, \ldots, \theta_p$.

   - The Kalman-Yakubovic-Popov (KYP) Lemma indicates that a strictly proper transfer function $G(s) = C(sI - \bar{A})^{-1}B$ is strictly positive real (SPR) if and only if there is a $P = P^T \succ 0$ and a real matrix L_0 satisfying
$$\bar{A}^T P + P\bar{A} = -\epsilon P - L_0^T L_0, \quad PB = C^T.$$
We will assume that this property will hold for the A, B, C matrices specified in this problem. This is a property of passive or dissipative linear systems.

We want to exploit this property to synthesize an adaptive observer of the following form:

$$\dot{w} = Aw + \Phi(y,u) + B(\underbrace{\sum_{i=1}^{p} W_i(u,y)\hat{\theta}_i}_{W^T(y,u)\hat{\theta}}) - L(y - Cw).$$

Note w is an estimate of the state x and $\hat{\theta}$ is an estimate of the parameter vector θ. The output y and the input u are measurable and the estimator is implementable. Suppose the state estimation error $e := w - x$ and the parameter estimation error is defined as $\tilde{\theta} = \hat{\theta} - \theta$, show that the error dynamics is given by

$$\dot{e} = (A + LC)e + B(W^T(y,u)\tilde{\theta}).$$

- Suppose L has been chosen so as to make $G(s)$ strictly positive real. How do you develop a scheme for parameter estimation using the Liapunov function

$$V(e,\tilde{\theta}) = e^T Pe + \tilde{\theta}^T \Gamma^{-1}\tilde{\theta},$$

where $\Gamma = \Gamma^T \succ 0$. Is your parameter adaptation scheme implementable? Note e is not known; only the output estimation error is known and used in the observer design for updating the state estimate , i.e., $\tilde{y} = Cw - y$ is known.

- Further, show that your scheme will guarantee that $e \to 0$ asymptotically by showing that $e \in \mathcal{L}_2 \cap \mathcal{L}_\infty, \tilde{\theta} \in \mathcal{L}_\infty$, i.e., e is square integrable and bounded, and $\tilde{\theta}$ is bounded.

- **Bonus:** If, in addition, suppose that there exist $\delta, \alpha > 0$ such that

$$\int_t^{t+\delta} W(\zeta)W^T(\zeta)d\zeta \succeq \alpha I \succ 0,$$

show that for any initial condition $w(0)$ and initial parameter estimate $\hat{\theta}(0)$ and for any θ, the errors decay to zero exponentially, i.e., $\|e(t)\|, \|\tilde{\theta}\| \to 0$ exponentially. (Hint: To show this result, you may assume the following result: Consider a linear time varying system

$$\dot{e} = \bar{A}e + \Omega^T(t)\tilde{\theta}, \quad \dot{\tilde{\theta}} = -\Gamma\Omega(t)Px,$$

with $\bar{A}$ being Hurwitz, $P = P^T \succ 0$ satisfying the Liapunov equation

$$\bar{A}^T P + P\bar{A} = -I,$$

and $\Gamma = \Gamma^T \succ 0$. If $\|\Omega(t)\|, \|\dot{\Omega}(t)\|$ are bounded and there exist $\delta, \alpha > 0$ satisfying

$$\int_t^{t+\delta} \Omega(\tau)\Omega^T(\tau)d\tau \succeq \alpha I \succ 0,$$

then the equilibrium $(e,\tilde{\theta}) = (0,0)$ is globally exponentially stable.)

3. Consider the following Lipschitz nonlinear system:

$$\begin{aligned}
\dot{x}_1 &= x_2 + f_1(x_1),\\
\dot{x}_2 &= x_3 + f_2(x_1, x_2),\\
\vdots & \quad \dots\\
\dot{x}_{n-1} &= x_n + f_{n-1}(x_1, x_2, \dots, x_{n-1}),\\
\dot{x}_n &= u.
\end{aligned}$$

In the above system, $x_1, x_2, \dots, x_n$ are the states of the system and u is the control input. Given that f_i's are Lipschitz function with a Lipschitz constant L, we want to explore the possibility of constructing a linear state feedback controller to stabilize the equilibrium $(x_1 = 0, \dots, x_n = 0)$ of the nonlinear system.

In connection with that, let us scale the states as follows: Let $K \neq 0$ be a scalar and define, for $i = 1, 2, \dots, n$, the scaled states

$$\xi_i(t) := \frac{x_i}{K^{i-1}}, \quad v := \frac{u}{K^n}.$$

- Show that the scaled states evolve as:

$$\begin{aligned}
\dot{\xi}_1 &= K\xi_2 + f_1(\xi_1),\\
\dot{x}i_2 &= K\xi_3 + \frac{1}{K} f_2(\xi_1, K\xi_2),\\
\vdots & \quad \dots\\
\dot{\xi}_{n-1} &= K\xi_n + \frac{1}{K^{n-1}} f_{n-1}(\xi_1, K\xi_2, \dots, K^{n-1}\xi_{n-1}),\\
\dot{x}i_n &= \frac{1}{K^n} u = v.
\end{aligned}$$

- Define

$$\begin{aligned}
g_1(\xi_1) &= f_1(\xi_1),\\
g_2(\xi_1, \xi_2) &= \frac{1}{K} f_2(\xi_1, K\xi_2),\\
\vdots & \quad \dots\\
g_i(\xi_1, \xi_2, \dots, \xi_{i-1}) &= \frac{1}{K^{i-1}} f_i(\xi_1, K\xi_2, \dots, K^{i-1}\xi_{i-1}),\\
\vdots & \quad \dots\\
g_{n-1}(\xi_1, \dots, \xi_{n-1}) &= \frac{1}{K^{n-1}} (\xi_1, K\xi_2, \dots, K^{n-1}\xi_{n-1}).
\end{aligned}$$

  Show that the scaled system can be put in the following form:

$$\dot{\xi} = K(A_c\xi + B_c v) + g(\xi),$$

where A_c, B_c are in the controllable canonical form; the i^{th} components of vectors ξ and g are respectively ξ_i and $g_i(\xi_1, \ldots, \xi_{i-1})$.

- Suppose $K > 1$. What is the Lipschitz constant of the function $g_1, \ldots, g_{n-1}$?
- We will use pole placement to construct a controller for the linear part and will show that it will work for the nonlinear system also: Suppose the desired location of the poles is $-\lambda_1, \ldots, -\lambda_n$, where $0 < 1 = \lambda_1 < \lambda_2 < \ldots < \lambda_n$. The corresponding characteristic polynomial is

  $$\Delta_{des}(s) = (s+\lambda_1)(s+\lambda_2)\cdots(s+\lambda_n) = s^n + \beta_{n-1}s^{n-1} + \beta_{n-2}s^{n-2} + \cdots + \beta_0.$$

  Consider a controller of the form

  $$\begin{aligned} v &= -\beta_0\xi_1 - \beta_1\xi_2 - \cdots - \beta_{n-1}\xi_n = -\underbrace{\begin{pmatrix}\beta_0 & \beta_1 & \cdots & \beta_{n-1}\end{pmatrix}}_{G}\underbrace{\begin{pmatrix}\xi_1\\ \xi_2\\ \vdots\\ \xi_n\end{pmatrix}}_{\xi} \\ &= -\begin{pmatrix}\beta_0 & \frac{\beta_1}{K} & \frac{\beta_2}{K^2} & \cdots & \frac{\beta_{n-1}}{K^{n-1}}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{pmatrix}\end{aligned}$$

  Show that $A_c + B_cG$ has the desired location of poles.
- Note that we have not fixed K yet. We will treat $g(\xi)$ as a perturbation to the linear system. Use exponential stability theorem or Gronwall-Bellman inequality to show that we can always choose a sufficiently large K to ensure that the equilibrium $\xi = 0$ of the nonlinear system

  $$\dot{\xi} = K(A + B_cG)\xi + g(\xi)$$

  is exponentially stable.