Texas A&M University
Department of Mechanical Engineering

MEEN 655: Design of Nonlinear Control Systems
Homework 1

February 3, 2022

1. Use MATLAB or any of your favorite scientific packages, draw the phase portraits for the following systems:

   - $$\dot{x}_1 = x_2, \quad \dot{x}_2 = \frac{1}{m}[-\alpha x_2 - \beta x_2^3 - k x_1^3],$$
     where α, β, k, m are positive constants.
   - $$\dot{x}_1 = x_2, \quad \dot{x}_2 = +\alpha x_2 - \alpha x_2^3 - x_1,$$
     where $\alpha > 0$.

2. Suppose $f : \Re \to \Re$ satisfies the following properties:

   - $f''(x)$ is continuous everywhere.
   - $f'(x) \neq 0$ everywhere.
   - For some $\beta \in (0,1)$, $|f(x)f''(x)| \leq \beta |f'(x)|^2$ for everywhere.

   Define Newton-Raphson iterate, $T : \Re \to \Re$ as follows:

   $$T(x) = x - \frac{f(x)}{f'(x)}.$$

   - Show that T is a contraction using Lagrange's Mean Value Theorem.
   - Show that any fixed point of $T(x)$ is a root of $f(x)$.
   - Set up an iteration to find a root of $f(x) = xtan(x)$ in the interval $(\frac{\pi}{2}, \frac{3\pi}{2})$, as:

   $$x_{n+1} = T(x_n), \qquad \frac{\pi}{2} < x_0 < \frac{3\pi}{2}.$$

- Is there a subinterval in which the conditions of the contracting mapping theorem hold?

3. In the contraction mapping theorem, the domain of T, namely, $\mathcal{U}$, should be compact (i.e., closed and *bounded*). I forgot to mention about the boundedness in my lecture notes. Here is a counterexample when boundedness is not satisfied: Let $T(x) := x + \frac{1}{x}$. Define the domain $\mathcal{U} := [1, \infty)$.
   - Show that $T : \mathcal{U} \to \mathcal{U}$.
   - Show that T is a contraction on $\mathcal{U}$.
   - Does this function have a fixed point in $\mathcal{U}$?

4. In nonlinear control applications, sliding mode control is often used. Essentially, the closed loop equations can be expressed as:
$$\dot{e} = -\lambda e + S, \quad \dot{S} = -\mu\, sgn(S),$$
where λ, μ are constants. Does this nonlinear system have solutions in the sense of Caratheodory? (You may use the following definition of *sgn* function:
$$sgn(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$
)

5. Consider the following nonlinear system specified by four positive constants a, b, c, d:
$$\dot{x} = ax - bxy, \quad \dot{y} = dy - cxy.$$
   - What are the equilibria?
   - Can you assess their stability using Liapunov's indirect theorem?
   - Draw phase portraits in the vicinity of each equilibria and compare them with the phase portraits of corresponding linearized systems.

6. A moving plate capacitor microphone can be modeled using the following governing equations:
$$R\dot{Q} + \frac{Qx}{\epsilon A} = V_{in},$$
$$m\ddot{x} + b\dot{x} + k(x - D) = -\frac{Q^2}{2\epsilon A},$$
where Q is the charge on the parallel plate capacitor, x is the gap between the stationary and moving plates, A is the area of the plates, D is the standstill gap between plates when the spring is not deformed, k is the spring constant, b is the damping coefficient, ϵ is the permissivity of air and m is the mass of the moving plate.

(a) Set up the equations for finding equilibria corresponding to a constant input $V_{in} = V^*$.

(b) What are the possible number of equilibria one can attain physically?

(c) What parameters govern the number of equilibria? Can you find any critical values of the parameters corresponding to a change in the number of equilibria? (Express the equation for equilibria as

$$k(D - x) = \frac{\gamma V_0^2}{x^2},$$

for some constant γ. Use root locus technique to analyze the number of roots and the critical value of V_0:

$$1 + \frac{\gamma V_0^2}{kx^2(x - D)} = 0.$$

)

(d) Suppose the system has multiple equilibria. Pick an equilibrium and perform Jacobi linearization.

(e) Is the equilibrium stable? Justify mathematically and provide a corresponding physical explanation.