Texas A&M University
Department of Mechanical Engineering

MEEN 655: Design of Nonlinear Control Systems
Homework #2

February 8, 2022

1. Let $\Phi(t, t_0)$ denote the state transformation matrix of A, i.e.,

$$\frac{d\Phi(t, t_0)}{dt} = A\Phi(t, t_0), \qquad \Phi(t_0, t_0) = I.$$

Let $Q = Q^T \succ 0$. Let P be defined as

$$P(t) := \int_{t_0}^{t} \Phi(t, \tau) Q \Phi^T(t, \tau) d\tau.$$

(a) Noting that

$$\frac{d}{dt}\int_{f(t)}^{g(t)} H(t, u)du = H(t, g(t))g'(t) - H(t, f(t))f'(t) + \int_{f(t)}^{g(t)} \frac{dH}{dt}(t, u)du,$$

show that

$$\frac{dP}{dt} = AP + PA^T + Q.$$

(b) Show that for any matrix $F \in \Re^{n\times m}$ and a vector $v \in \Re^m$,

$$\|FV\|^2 := (Fv)^T(Fv) = v^T F^T F v \leq \lambda_{max}(F^T F)v^T v.$$

Note that the maximum singular value, $\sigma_{max}(F)$ of F is defined to be $\sqrt{\lambda_{max}(F^T F)}$. Further show that

$$\|Fv\| \leq \sigma(F)\|v\|.$$

Show that For any $F, G \in \Re^{n\times m}$,

(i) $\sigma(F) \geq 0$ and $\sigma(F) = 0 \iff F = 0$,

(ii) $\sigma(F) + \sigma(G) \geq \sigma(F + G)$, and

(iii) for any $\alpha \in \Re$,

$$\sigma(\alpha F) = |\alpha|\sigma(F).$$

(iv) Show that there is a $v \in \Re^m$ such that $\|Fv\| = \sigma(F)\|v\|$. (Hint: What would you get with the eigen vector corresponding to the maximum eigenvalue of $F^T F$?) Properties (i), (ii) and (iii) render $\sigma(F)$ to be a norm for matrices.

(v) Property (iv) suggests the following alternative definition of a norm:

$$\|F\| = \max\{\|Fv\| : \|v\| = 1\},$$

or equivalently,

$$\|F\| = \max_{v \neq 0} \frac{\|Fv\|}{\|v|}.$$

Show that this definition is also proper, in that, it satisfies all the three properties of a norm we discussed in the class. (One may think of v as an input and Fv as an output, and this norm may be thought of as the maximum gain of F. Hence, it is referred to as the induced norm of F).

(v) If $R \in \Re^m \times p$, show that

$$\sigma(FR) \leq \sigma(F)\sigma(R).$$

Property (v) makes $\sigma(F)$ an induced norm. For this reason, one often expresses the last inequality as

$$\|FR\| \leq \|F\|\|R\|.$$

(c) Show that $P = P^T \succ 0$. (Hint: Show that for any $v \neq 0$, we obtain $v^T PV > 0$.)

(d) If A is Hurwitz, we know that for some $M, \lambda > 0$,

$$\|\Phi(t, t_0)\| \leq Me^{-\lambda(t-t_0)}.$$

Show that $P^* = \lim_{t\to\infty} P(t)$ is well defined and that it satisfies

$$AP^* + P^* A^T + Q = 0.$$

(First show that for any vectors u, v

$$|u^T P(t)v| \leq M^2\|u\|\|v\|\|Q\|\frac{1}{2\lambda}.$$

Then conclude that every element of $P(t)$ is bounded. Moreover, if $t_2 \geq t_1$, then $P(t_2) - P(t_1) \succeq 0$, i.e., it is accumulating. Conclude by observing that If a bounded, accumulating (or increasing) function has a limit; in this case, P^*.)

(e) Show that A is Hurwitz if and only if A^T is Hurwitz. (Hint: It suffices to show that the characteristic polynomial of A and A^T are the same).

(f) Show that if A is Hurwitz, then for any $Q = Q^T \succ 0$, there is a $P = P^T \succ 0$ satisfying

$$A^T P + PA + Q = 0.$$

(g) If

$$\dot{x} = Ax + Bu$$

is completely controllable, then there exists a $P = P^T \succ 0$ satsifying

$$A^T P + PA + BB^T = 0.$$

(h) If

$$\dot{x} = Ax, \qquad y = Cx$$

is completely observable, then there is a $P = P^T \succ 0$ satisfying

$$AP + PA^T + C^T C = 0.$$

2. Consider a $n-$ dimensional smooth autonomous nonlinear system

$$\dot{x} = Jx + h_2(x) + h_3(x) + \dots$$

with an equilibrium at $x = 0$; the vector $h_k(x)$ consists of all homogeneous monomial terms of order k. Suppose every eigenvalue of J has a negative real part less than $-\alpha$, $\alpha > 0$. We want to show that for some $M, \delta > 0$,

$$\|x(t_0)\| < \delta \implies \|x(t)\| \leq M\|x(t_0)\|e^{-\alpha(t-t_0)}.$$

We will accomplish in the following steps:

- If the eigenvalues of J have a real part $< -\alpha$, what can you say about the eigenvalues of $J + \alpha I$? Why?
- When does there exist a $P = P^T \succ 0$ satisfying

$$(J + \alpha I)^T P + P(J + \alpha I) = -I.$$

  Does it hold here?
- Consider $z(t) = e^{\alpha(t-t_0)}x(t)$. Show that

$$\dot{z} = (J + \alpha I)z + e^{-\alpha(t-t_0)}h_2(z) + e^{-2\alpha(t-t_0)}h_3(z) + \dots$$

- Consider a Liapunov function $V(z) := z^T P z$ and show that

$$\frac{dV}{dt} \leq -\|z\|^2 + 2\lambda_{max}(P)\|z\|\phi(z),$$

  where $\lim_{\|z\| \to 0} \frac{\phi(z)}{\|z\|} = 0$.

- Show that $z = 0$ is a uniformly asymptotically stable equilibrium using the definitions of Liapunov stability.
- Conclude then that $x = 0$ is a uniformly exponentially stable equilibrium of the nonlinear system.

3. Consider an autonomous $n-$ dimensional nonlinear system

$$\dot{x} = (A - r_A(x)I_n)x,$$

where

$$r_A(x) = x^T A x,$$

and $A = A^T$ has distinct set of n real eigenvalues, $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n$.

- Show that $\{x : \|x\| = 1\}$ is a positively invariant manifold for this system.
- Show that at equilibrium $\|x\| \in \{0, 1\}$
- What are the possible values of $r_A(x)$ at equilibrium? What are the equilibria?
- Which of the equilibria is stable? (Consider the Jacobian corresponding to equilibria, $x^* \neq 0$).