Take home Final, MEEN655

Minfan Dong

1. $\dot{x} = Ax + Bu$. $\quad A = \frac{\partial f}{\partial x}(0,0)$; $\quad B = \frac{\partial f}{\partial u}(0,0)$

① NL: $\dot{x} = f(x,u)$. $\quad f(0,0) = 0$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$ is easily verified. And $B = 0$, for there is no linear term in $u = u(x_1, x_2)$.

②. eigenvalues of A are $0, -5, 0$. The stability information cannot be inferred, for $\lambda_1 = \lambda_2 = 0$ fall on the imaginary axis. (Indeterminate).

③. $\dot{x}_3(t) = -5x_3(t)$ ~~$+u(t)$~~. $\quad$ ~~$u(t) = u(x_1(t), x_2(t))$~~

<u>Using Jacobian linearization,</u> we know that

$\dot{x}_1 = x_2 \qquad \dot{x}_2 = 0 \qquad$ (at $t \to 0$)

~~Thus~~ And since $x_2(0) = 0$, we know that $x_2(t) = 0 = \dot{x}_1$. Thus $x_1(t) = 0$, because $\dot{x}_1(t) = 0$ and $x_1(0) = 0$.

Therefore we have $\dot{x}_3(t) = -5x_3(t)$, $\quad x_3(t) = e^{-5t} \cdot \underline{C_1}$ some constant coeff.

$\lim_{t\to\infty} x_3(t) = \lim_{t\to\infty}(e^{-5t} \cdot C_1) = 0$ holds.

The manifold $M_s := \{(0, 0, x_3),\ x_3 \in \mathbb{R}\}$ is the stable manifold.

④. $\dot{x}_3 = -5x_3 + u$

$= -5h(x_1, x_2) + \alpha x_1^2 + \beta x_1 x_2 + \gamma x_2^2$ LHS.

[we already know that $x_3 = h(x_1, x_2)$]

$$\frac{dh}{dt} = \frac{\partial h}{\partial x_1}\frac{\partial x_1}{\partial t} + \frac{\partial h}{\partial x_2}\frac{\partial x_2}{\partial t}$$

$$= \frac{\partial h}{\partial x_1}\dot{x}_1 + \frac{\partial h}{\partial x_2}\dot{x}_2$$

$$= \frac{\partial h}{\partial x_1}(x_2 - x_1^3 + x_1 x_2^2 - 2x_2 x_3) + \frac{\partial h}{\partial x_2}(x_1^3 + x_1 x_3) \quad \text{RHS}$$

$$= \frac{\partial h}{\partial x_1}(x_2 - x_1^3 + x_1 x_2^2 - 2x_2 h(x_1, x_2)) + \frac{\partial h}{\partial x_2}(x_1^3 + x_1 h(x_1, x_2))$$

Due to the definition of invariance, we know that $\dot{x}_3 = \dot{h}(x_1, x_2)$ must hold. Therefore LHS = RHS, QED.

⑤. For quadratic approximation, we know that $h(x_1, x_2) \approx h_2(x_1, x_2)$

$h(x_1, x_2) = Ax_1^2 + Bx_1 x_2 + Cx_2^2$

As we know, $\dot{x}_3 = -5(Ax_1^2 + Bx_1 x_2 + Cx_2^2)$
$+ \alpha x_1^2 + \beta x_1 x_2 + \gamma x_2^2$

And from ④ we know that $\frac{dh}{dt} = \dot{x}_3$

$\frac{\partial h}{\partial x_1} = 2Ax_1 + Bx_2$ $\quad \frac{\partial h}{\partial x_2} = 2Cx_2 + Bx_1$, Therefore,

⑤. (Continued)

$$(-5A+\alpha)x_1^2 + (\beta - 5B)x_1x_2 + (\gamma - 5C)x_2^2$$

$$= (2Ax_1 + Bx_2)\cdot x_2 \cancel{+(2Cx_2+Bx_1)}$$

Thus, $\begin{cases} -5A+\alpha = 0. & \text{for } x_1^2 \\ \beta - 5B = 2A & \text{for } x_1x_2 \\ \gamma - 5C = B & \text{for } x_2^2 \end{cases}$

~~QED~~.

⑥. On the center manifold, $x_3 = h(x_1, x_2) \approx h_2(x_1, x_2)$

$$\cancel{\left[\begin{array}{c|c} \dot{x}_1 & 0 \\ \dot{x}_2 & \\ \hline \dot{x}_3 & \end{array}\right]}$$

$$A = \left[\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right]$$

where $\det(A_1 - \lambda I) = 0$ gives $\lambda = 0$, or $Re(\lambda) = 0$, and $\det(A_2 - \lambda I)$ gives $Re(\lambda) < 0$.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\overset{A_1}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + f_3(x_1, x_2),$$

where we know

$$f_3 = \begin{bmatrix} -x_1^3 + x_1x_2^2 - 2x_2(Ax_1^2 + Bx_1x_2 + Cx_2^2) \\ x_1^3 + x_1(Ax_1^2 + Bx_1x_2 + Cx_2^2) \end{bmatrix}$$

Since $x_3 = h(x_1, x_2) \approx h_2(x_1, x_2)$.

$A_1 = J_2$

⑦. $\dot{V} = \frac{\partial V}{\partial y_1}\dot{y}_1 + \frac{\partial V}{\partial y_2}\dot{y}_2$

$= \lambda y_1^3 y_2 + y_2 \dot{y}_2 = \lambda y_1^3 y_2 + y_2(-\lambda y_1^3 + \mu y_1^2 y_2)$

$= \mu y_1^2 y_2^2$

Thus to guarantee exponential stability of the equilibrium (asymptotically stable)

$(0,0)$, $\dot{V} < 0$ must hold. Thus $\mu < 0$.

⑧. Transforming the system to canonical form.

$x = y + h_3(y)$. $\quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

The result ~~should~~ is shown in section ⑨ – ⑰.

⑨. It is natural that for homogenous polynomials of degree 3, all possible terms are y_1^3, $y_1^2 y_2$, $y_1 y_2^2$, y_2^3.

And since $h_3(y)$ is an 2×1 matrix, then we know that

$h_3^1(y) = a_1 y_1^3 + b_1 y_1^2 y_2 + c_1 y_1 y_2^2 + d_1 y_2^3$

$h_3^2(y) = a_2 y_1^3 + b_2 y_1^2 y_2 + c_2 y_1 y_2^2 + d_2 y_2^3$

Since $h_3^1(y)$ and $h_3^2(y)$ are independent with each other, It's natural that we express

⑨. (Continued)

$$h_3(y) = \begin{bmatrix} h_3^1(y_1, y_2) \\ h_3^2(y_1, y_2) \end{bmatrix} = \text{span}\left\{ \begin{pmatrix} y_1^3 \\ 0 \end{pmatrix} \begin{pmatrix} y_1^2 y_2 \\ 0 \end{pmatrix} \begin{pmatrix} y_1 y_2^2 \\ 0 \end{pmatrix} \begin{pmatrix} y_2^3 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ y_1^3 \end{pmatrix} \begin{pmatrix} 0 \\ y_1^2 y_2 \end{pmatrix} \begin{pmatrix} 0 \\ y_1 y_2^2 \end{pmatrix} \begin{pmatrix} 0 \\ y_2^3 \end{pmatrix} \right\}$$

$$= \text{span}\{ \phi_{30}^1 \ \phi_{30}^2 \ \phi_{21}^1 \ \phi_{21}^2 \ \phi_{12}^1 \ \phi_{12}^2 \ \phi_{03}^1 \ \phi_{03}^2 \}$$

⑩. We know that $\begin{cases} \dot{x} = J_2 x + f_3(x) \\ x = T(y) = y + h_3(y). \end{cases} \Longrightarrow LHS = RHS$ (*)

$$J_2(x) + f_3(x) = RHS = J_2(y + h_3(y)) + f_3(y + h_3(y))$$

$$\dot{x} = \dot{T}(y) = \frac{\partial T}{\partial y}\frac{\partial y}{\partial t} \qquad \dot{y} = J_2 y + g_3(y) + O_4(y)$$

$$LHS = \dot{x} = \left(I + \frac{\partial h_3}{\partial y}\right) \cdot \dot{y}$$

$$= \left(I + \frac{\partial h_3}{\partial y}\right)\left(J_2 y + g_3(y) + O_4(y)\right).$$

$\because$ LHS = RHS $\therefore$ The condition above must hold.

By definition of Lie Bracket, we know that $L_{J_2}(h_3)$

$$[J_2 y, h_3(y)] = \frac{\partial h_3}{\partial y} J_2 y - \frac{\partial (J_2 y)}{\partial y} h_3 = \frac{\partial h_3}{\partial y} J_2 y - J_2 h_3(y)$$

Since (*) must hold, we have (ignoring $O_4(y)$).

$$\cancel{J_2 y} + J_2 h_3(y) + f_3(y + h_3(y)) = \cancel{J_2 y} + \frac{\partial h_3}{\partial y} J_2 y + g_3(y) + \frac{\partial h_3}{\partial y} g_3(y)$$

⑩ (Continued) Thus we have

$$\frac{\partial h_3}{\partial y} J_2 y - J_2 h_3(y) = \underline{f_3(y+h_3(y))} - \frac{\partial h_3}{\partial y} g_3(y) - g_3(y)$$

$$= f_3(y) - g_3(y) + O_4(y).$$

Taylor expansion of $f_3(y+h_3(y))$ gives $f_3(y) + O_4(y)$. Also,

$\frac{\partial h_3}{\partial y} g_3(y) \in O_4(y)$ because h_3 is supposed to have order 3.

⑪. $L_J(\phi) = [J_2 y, \phi(y)] = \frac{\partial \phi}{\partial y} J_2 y - J_2 \phi$

For $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ $\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$ we know that $\frac{\partial \phi}{\partial y} = \begin{bmatrix} \frac{\partial \phi_1}{\partial y_1} & \frac{\partial \phi_1}{\partial y_2} \\ \frac{\partial \phi_2}{\partial y} & \frac{\partial \phi_2}{\partial y_2} \end{bmatrix}$

Thus it's easy to calculate that

$$L_J(\phi^1_{30}) = 3\phi^1_{21} \qquad \phi^1_{30} = \begin{pmatrix} y_1^3 \\ 0 \end{pmatrix} \qquad \phi^1_{21} = \begin{pmatrix} y_1^2 y_2 \\ 0 \end{pmatrix}$$

For example, For $L_J(\phi^2_{30})$, we have

$$\frac{\partial \phi^2_{30}}{\partial y} J_2 y - J_2 \phi^2_{30} = \begin{pmatrix} 0 & 0 \\ 3y_1^2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ y_1^3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 3y_1^2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - \begin{pmatrix} y_1^3 \\ 0 \end{pmatrix} = -\phi^1_{30} + \begin{pmatrix} 0 \\ 3y_1^2 y_2 \end{pmatrix} = -\phi^1_{30} + 3\phi^2_{21}$$

(12). From the result in (11), it's easy to know that

$$L_J([\phi^1_{30}\ \phi^1_{21}\ \phi^1_{12}\ \phi^1_{03}\ \phi^2_{30}\ \phi^2_{21}\ \phi^2_{12}\ \phi^2_{03}])^{[\phi]}$$

$$= \begin{bmatrix} L_J(\phi^1_{30}) \\ L_J(\phi^1_{21}) \\ \vdots \\ L_J(\phi^2_{03}) \end{bmatrix}^T = \begin{bmatrix} 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \phi^1_{30} \\ \phi^1_{21} \\ \vdots \\ \phi^2_{03} \end{bmatrix}$$

$$= Q^T \phi^T \implies L_J([\phi]) = \phi Q$$

rank(Q) = 6, because Q have 6 pivots.

$$Q = \text{span}\left\{ \underset{a}{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}} \underset{b}{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}} \underset{c}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}} \underset{d}{\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 3 \\ 0 \\ 0 \end{bmatrix}} \underset{e}{\begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}} \underset{f}{\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}} \right\}$$

$$= \text{span}\left\{ \underbrace{a, b, c, (-d), (e+b), \frac{(f+a)}{2}} \right\} \quad [Q_1]$$

Thus, ϕQ_1 are

the basis vectors for the range space of L_J,

$$\{\phi^1_{21}, \phi^1_{12}, \phi^1_{03}, \phi^1_{30} - 3\phi^2_{21}, \phi^2_{12}, \phi^2_{03}\}$$

QED.

(13). "Reversing" the solution process, it's easy to know that

$$L_J(\phi^1_{30}) = 3\phi^1_{21} \Longrightarrow L_J(\tfrac{1}{3}\phi^1_{30}) = \phi^1_{21} \Longrightarrow h_{21} = \tfrac{1}{3}\phi^1_{30}$$

Likewise, we could know that $h_{12} = \frac{1}{2}\phi^1_{21}$, $h_{03} = \phi^1_{12}$

(Note that constant ~~term~~ coefficient does not affect the linearity of this transformation).

~~For example, ϕ^1_{12} could be expressed as linear combination of~~

For the last three equations, $L_J(g_{12}) = \phi^2_{12}$

ϕ^2_{12} is a linear combination of $\frac{1}{2}L_J(\phi^2_{21}) + \frac{1}{6}L_J(\phi^1_{30})$

$$= \frac{1}{2}(2\phi^2_{12} - \phi^1_{21}) + \frac{3}{6}\phi^1_{21}$$

Thus, $g_{12} = \frac{1}{2}\phi^2_{21} + \frac{1}{6}\phi^1_{30}$.

Likewise we could prove that

$$g_{03} = \frac{1}{2}\phi^1_{21} + \phi^2_{12}, \quad g_d = -\phi^2_{30}.$$

(14). $$F_3(x_1, x_2) = \frac{p_{30}}{6}\begin{pmatrix} x_1^3 \\ 0 \end{pmatrix} + \frac{q_{30}}{6}\begin{pmatrix} 0 \\ x_1^3 \end{pmatrix} + \frac{p_{21}}{2}\begin{pmatrix} x_1^2 x_2 \\ 0 \end{pmatrix} + \frac{q_{21}}{2}\begin{pmatrix} 0 \\ x_1^2 x_2 \end{pmatrix}$$

$$+ \frac{p_{12}}{2}\begin{pmatrix} x_1 x_2^2 \\ 0 \end{pmatrix} + \frac{q_{12}}{2}\begin{pmatrix} 0 \\ x_1 x_2^2 \end{pmatrix} + \frac{p_{03}}{6}\begin{pmatrix} x_2^3 \\ 0 \end{pmatrix} + \frac{q_{03}}{6}\begin{pmatrix} 0 \\ x_2^3 \end{pmatrix}$$

$$= \frac{p_{30}}{6}\phi^1_{30} + \frac{q_{30}}{6}\phi^2_{30} + \frac{p_{21}}{2}\phi^1_{21} + \frac{q_{21}}{2}\phi^2_{21} + \frac{p_{12}}{2}\phi^1_{12} + \frac{q_{12}}{2}\phi^2_{12} + \frac{p_{03}}{6}\phi^1_{03} + \frac{q_{03}}{6}\phi^2_{03}$$

(14) (Continued)

$$F_3(x_1,x_2) = \frac{p_{30}}{6}(\phi^1_{30} - 3\phi^2_{21}) + \frac{(q_{21}+p_{30})}{2}\phi^2_{21} + \frac{q_{30}}{6}\phi^2_{30} + \frac{p_{21}}{2}\phi^1_{21} + \frac{p_{12}}{2}\phi^1_{12}$$

$$+ \frac{q_{12}}{2}\phi^2_{12} + \frac{p_{03}}{6}\phi^1_{03} + \frac{q_{03}}{6}\phi^2_{03}$$

The term h_3 in the transformation $x = y + h_3(y)$ must solve for

$$\cancel{L_J(h_3(y))} \quad L_J(h_3(y)) = \frac{p_{30}}{6}(\phi^1_{30} - 3\phi^2_{21}) + \frac{p_{21}}{2}\phi^1_{21} + \frac{p_{12}}{2}\phi^1_{12}$$

$$+ \frac{p_{03}}{6}\phi^1_{03} + \frac{q_{12}}{2}\phi^2_{12} + \frac{q_{03}}{6}\phi^2_{03}$$

Note that from (13), we know the expression of $L_J(h_{21}), \cdots$.

Thus according to the linearity of transformation $L_J(h_3(y))$, it follows that

$$L_J(h_3(y)) = L_J(g_d)\cdot\frac{p_{30}}{6} + \frac{p_{21}}{2}L_J(h_{21}) + \frac{p_{12}}{2}L_J(h_{12})$$

$$+ \frac{p_{03}}{6}L_J(h_{03}) + \frac{q_{12}}{2}L_J(g_{12}) + \frac{q_{03}}{6}L_J(g_{03})$$

Thus $L_J(h_3(y)) = L_J\Big(\underbrace{\left[\frac{p_{30}}{6}g_d + \frac{p_{21}}{2}h_{21} + \frac{p_{12}}{2}h_{12} + \frac{p_{03}}{6}h_{03} + \frac{q_{12}}{2}g_{12} + \frac{q_{03}}{6}g_{03}\right]}_{RHS}\Big)$

$h_3(y) = RHS$. <u>QED.</u>

(15). From (10) we know that

$$L_J(h_3(y)) = f_3(y) - g_3(y) + O_4(y)$$

And we know that $f_3(y) = F_3(y)$ from (14). Ignoring

(15). (Continued)

Ignoring the $O_4(y)$ terms, we will have the equation:

$$g_3(y) = f_3(y) - L_J(h_3(y)) = F_3(y) - L_J(h_3(y))$$

Therefore it's easy to know that

$$g_3(y) = \frac{q_{30}}{6}\phi_{30}^2 + \frac{q_{21}+p_{30}}{2}\phi_{21}^1$$

Thus, $\dot{y} = J_2 y + g_3(y) + O_4(y)$ holds after transformation $x = T(y)$. Unfolding the ~~exp~~ equation above gives us the desired expression. QED

(16). Unfolding the equation in (15) gives us that

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = \frac{q_{30}}{6} y_1^3 + \frac{q_{21}+p_{30}}{2} y_1^2 y_2 + O_4(y_1, y_2). \end{cases}$$

Therefore, $-\lambda = \frac{q_{30}}{6}$, $\mu = \frac{q_{21}+p_{30}}{2}$. We can also express them with $\{\alpha, \beta, \gamma\}$ or $\{A, B, C\}$.

From equation in part (14) it's easy to know that

$$F_3(x_1, x_2) = \frac{x_1^3}{6}\begin{pmatrix} p_{30} \\ q_{30} \end{pmatrix} + \frac{x_1^2 x_2}{2}\begin{pmatrix} p_{21} \\ q_{21} \end{pmatrix} + \frac{x_1 x_2^2}{2}\begin{pmatrix} p_{12} \\ q_{12} \end{pmatrix} + \frac{x_2^3}{6}\begin{pmatrix} p_{03} \\ q_{03} \end{pmatrix}$$

Thus we have $q_{30} \cdot \frac{1}{6} = \lambda = (1+A) \cdot -1$

(16). (Continued)

And $\frac{P_{30}}{2} = \frac{P_{30}}{6} \cdot 3 = -3$ $\quad \frac{q_{21}}{2} = B$

$$\mu = -3 + B.$$

From (4) we know that $A = \frac{1}{5}\alpha$ $\quad B = \frac{1}{5}\beta - \frac{2}{25}\alpha.$

(17). To ensure ~~stability~~ that $u = \alpha x_1^2 + \beta x_1 x_2 + \gamma x_2^2$ is stablizing, ~~in the~~ we must let $-\lambda = 1 + A < 0$. in the system after transformation. This gives us that $A < -1$, $A = \frac{1}{5}\alpha$. namely, $\alpha < -5$.

The restriction from Lyaponov function ~~let~~ leads to $\mu < 0$. Thus $B < 3$.

when ~~$\beta - \frac{2}{5}\alpha < 0$~~, $\frac{1}{5}\beta - \frac{2}{25}\alpha < 3$

$$\Rightarrow \beta - \frac{2}{5}\alpha < 15, \quad 0 < 15$$

Therefore when $\beta - \frac{2}{5}\alpha < 0$, and $\alpha < -5$

It is guaranteed that $u = \alpha x_1^2 + \beta x_1 x_2 + \gamma x_2^2$ stablizes the system.

QED

2. ①. $$\frac{dW_{k,l}}{dt} = \frac{\partial W_{k,l}}{\partial x_1}\dot{x}_1 + \frac{\partial W_{k,l}}{\partial x_2}\dot{x}_2$$

$$= k x_1^{k-1} x_2^{l} \dot{x}_1 + l x_1^{k} x_2^{l-1} \dot{x}_2$$

(plug in the expression of our reduced system)

$$= k x_1^{k-1} x_2^{l}\left[x_2 - x_1^3 + x_1 x_2^2 - 2x_2(Ax_1^2 + Bx_1x_2 + Cx_2^2)\right]$$

$$+ l x_1^{k} x_2^{l-1}\left[x_1^3 + x_1(Ax_1^2 + Bx_1x_2 + Cx_2^2)\right]$$

Then by definition of $W_{k,l}$, we would have

$$\frac{dW_{k,l}}{dt} = k\left[W_{k-1,l+1} - W_{k+2,l} - 2AW_{k+1,l+1} - (2B-1)W_{k,l+2} - 2CW_{k-1,l+3}\right]$$

$$+ l\left[W_{k+3,l-1}(1+A) + BW_{k+2,l} + CW_{k+1,l+1}\right]$$

QED.

②. Suppose $V = z^T Q z$. $z^T = [x_1^2 \;\; x_1 \;\; x_1x_2 \;\; x_2 \;\; x_2^2]$

Then, $Q = [q_{ij}]_{5\times5}$.

$V = ① + ② + ③ + ④ + ⑤$. where $⑤ = x_2^2 A_5$

$① = x_1^2 \cdot A_1$, $② = x_1 A_2$ $③ = x_1x_2 A_3$ $④ = x_2 A_4$

$A_1 = (q_{11}x_1^2 + q_{12}x_1 + q_{13}x_1x_2 + q_{14}x_2 + q_{15}x_2^2)$

$\vdots$

$A_5 = (q_{51}x^2 + q_{52}x_1 + q_{53}x_1x_2 + q_{54}x_2 + q_{55}x_2^2)$

2. ② (continued)

Comparing terms of $V_2+V_3+V_4$ v.s. $[q_{ij}]_{5\times5}$,

we will find that Q where the remaining q satisfies that

$$Q = \begin{bmatrix} P_{40} & P_{30} & P_{31} & \underline{q_{14}} & \underline{q_{15}} \\ P_{30} & P_{20} & \underline{q_{23}} & P_{11} & \underline{q_{25}} \\ P_{31} & \underline{q_{32}} & \underline{q_{33}} & \underline{q_{34}} & P_{13} \\ \underline{q_{41}} & P_{11} & \underline{q_{43}} & P_{02} & P_{03} \\ \underline{q_{51}} & \underline{q_{52}} & P_{13} & P_{03} & P_{04} \end{bmatrix}$$

$P_{21} = \frac{1}{2}(q_{14}+q_{23}+q_{32}+q_{41})$

$P_{12} = \frac{1}{2}(q_{25}+q_{34}+q_{43}+q_{52})$

$P_{22} = q_{33}+q_{15}+q_{51}$

Also, $Q = Q^T$ must hold.

Therefore, to simplify the expression we let

$P_{21} = q_{14}+q_{23}$ $\quad P_{12} = q_{25}+q_{34}$ $\quad P_{22} = q_{33}+2q_{15}$

let $q_{15}=0$. $q_{23}=0$. $q_{34}=0$. ~~Thus $P_{21} = q_{14}$~~

Thus $P_{21}=q_{14}$ $\quad P_{12}=q_{25}$ $\quad P_{22}=q_{33}$, and we have

$$Q = \begin{bmatrix} P_{40} & P_{30} & P_{31} & P_{21} & 0 \\ P_{30} & P_{20} & 0 & P_{11} & P_{12} \\ P_{31} & 0 & P_{22} & 0 & P_{13} \\ P_{21} & P_{11} & 0 & P_{02} & P_{03} \\ 0 & P_{12} & P_{13} & P_{03} & P_{04} \end{bmatrix}$$

2. ② (continued).

Q must be positive definite matrix so that $z^T Q z > 0$. holds.

Therefore, we have LMI conditions below:

~~det~~ $\det(Q) > 0$.

(Real part)

All the eigenvalues of Q must have $\underline{Re}(\lambda_i) > 0$.

~~Real part~~

If we specify more values of $[q_{ij}]_{5\times5}$, then maybe we can have more specific relations between coefficients in Q, namely ~~$[q_{ij}]_{5\times5}$~~ q_{ij} in $[q_{ij}]_{5\times5}$, or P_{mn}, $0 \le m \le 4$, $0 \le n \le 4$.

~~m,n~~ $(m+n=4)$

I don't know if there could be any relationship between ε and the coefficients P_{mn} $(m+n=4,\ 0 \le m, n \le 4)$ here. But it doesn't hurt to set

$$V - [\varepsilon - (x_1^2 + x_2^2)] \geq 0. \quad \text{for } V \text{ must } \geq 0.$$

$$\Longrightarrow z^T Q z - [\varepsilon - (x_1^2 + x_2^2)] \geq 0. \quad \left(\begin{array}{l} \varepsilon - (x_1^2 + x_2^2) \geq 0 \\ \text{~~for~~ must hold} \end{array} \right).$$

2. ③. $V = z^T Q z$ $\quad \dot{V} = \dot{z}^T Q z + z^T Q \dot{z}$

$$= 2 z^T Q \dot{z}$$

[It is not likely that we specify $\dot{z}$ to be $\frac{\partial z}{\partial x_i} \dot{x}_i$ and expand the whole equation, ~~and~~ very time consuming and messy, although it is possible to find some specific relationship between ε and P_{mn}. ?]

$\dot{V}$ is negative definite $\rightarrow$ $\dot{V} < 0$, $\quad -\dot{V} > 0$.

Therefore, It's ~~same~~ safe to have such LMI condition: $-\dot{V} - (x_1^2 + x_2^2)(\varepsilon - (x_1^2 + x_2^2)) \geq 0$.

For $x_1^2 + x_2^2 \in [0, \varepsilon]$. This LMI is strong enough to guarantee that $\dot{V}$ is negative definite.

(If we want to determine A, B, C, then maybe the only way is to ~~arithmetically~~ expand $\dot{V}$?)

Also, we ~~can~~ notice that $2 z^T Q \dot{z}$ could be written as ~~$z^T Q f + f^T Q z$~~

2. ③ (Continued)

$$\begin{bmatrix} 2x_1 & 1 & x_2 & 0 & 0 \\ 0 & 0 & x_1 & 1 & 2x_2 \end{bmatrix} = Z_{x,y}^T$$

$$\dot{V} = 2Z^T Q \dot{z} \qquad f = \dot{z}(x,y) = Z_x \dot{x} + Z_y \dot{y}$$

$$f = \begin{bmatrix} 2x_1 & 0 \\ 1 & 0 \\ x_2 & x_1 \\ 0 & 1 \\ 0 & 2x_2 \end{bmatrix} \begin{bmatrix} -x_1^3 + (1-2B)x_1x_2^2 - 2Ax_1^2x_2 - 2Cx_2^3 + x_2 \\ (1+A)x_1^3 + Bx_1^2x_2 + Cx_1x_2^2 \end{bmatrix}$$

$$f = \begin{bmatrix} -2x_1^4 + 2(1-2B)x_1^2x_2^2 - 4Ax_1^3x_2 - 4Cx_1x_2^3 + 2x_1x_2 \\ -x_1^3 + (1-2B)x_1x_2^2 - 2Ax_1^2x_2 - 2Cx_2^3 + x_2 \\ (1+A)x_1^4 + (C-2A)x_1^2x_2^2 + x_2^2 - 2Cx_2^4 + (B-1)x_1^3x_2 + (1-2B)x_1x_2^3 \\ (1+A)x_1^3 + Bx_1^2x_2 + Cx_1x_2^2 \\ 2(1+A)x_1^3x_2 + 2Bx_1^2x_2^2 + 2Cx_1x_2^3 \end{bmatrix}$$

2. ③.

P.S. I ~~don't~~ wonder if it's applicable to also set $V(x) = x^T M x$, $x^T = [x_1 \ x_2]$

But in this case V_3, V_4 is just neglected, for $x^T x$ is quadratic.

Therefore it seems unnecessary to use (or further, unlikely) to use such equation below:

$$\dot{V} = x^T \overset{M}{Q} f + f^T \underset{M}{Q} x = 2 Z^T(x) Q \dot{Z}(x)$$

I'll stop there -- I really don't know how to get the exact ~~relatio~~ constraints for all the coefficients in $V = V_2 + V_3 + V_4$ without massive calculation.

I already wrote the LMI conditions that I have thought of in previous pages.

3. ①. Jacobian Linearization:

The input is supposed to be $u = e$ (voltage).

$$\begin{cases} \dfrac{di}{dt} = \dot{x}_1 = \dfrac{x_0+b-x}{L_0} e - \dfrac{R(x_0+b-x)}{L_0} i - \dfrac{iv}{x_0+b-x} \\ \dfrac{dx}{dt} = \dot{x}_2 = v \\ \dfrac{dv}{dt} = \dot{x}_3 = \dfrac{L_0 i^2}{2(x_0+b-x)^2 m_p} - \dfrac{\mu_p}{m_p} v - \dfrac{k_s x}{m_p} + g. \end{cases}$$

In this case, $y = h(x) = x_2$ $\qquad X = [i, x, v]^T$

$\dot{X} = f(x,u) \longrightarrow \dot{X} = AX + Bu.$ $\qquad y = CX.$ $(D = [0])$

$$A = \frac{\partial f}{\partial x}(x,u)\Big|_{\text{equilibrium}} \qquad B = \frac{\partial f}{\partial u}(x,u)\Big|_{\text{equilibrium}}$$

An equilibrium: $u = e = 0$, $\dot{X} = 0$

$\longrightarrow i = 0.$ $v = 0.$ $\qquad x = \dfrac{m_p g}{k_s}$

(I feel like this should be x_0?)

~~but~~ so, $\dfrac{m_p g}{k_s} = x_{initial} = x_{init}$ [but let's avoid this]

~~the demo~~

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial i} & \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial i} & \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial v} \\ \frac{\partial f_3}{\partial i} & \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial v} \end{bmatrix} \qquad B = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \\ \partial f_3/\partial u \end{bmatrix} \qquad C = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}^T$$

3. ①. (Continued) $X_{init} = 1/6$

$$A = \begin{bmatrix} \frac{-R(X_0+b-X_{init})}{L_0} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -\frac{k_s}{m_p} & \frac{-\mu_p}{m_p} \end{bmatrix} = \begin{bmatrix} -\frac{1}{6} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -60 & -200 \end{bmatrix}$$

$$B = [(X_0+b-X_{init})/L_0 \quad 0 \quad 0]^T = [\tfrac{1}{30} \quad 0 \quad 0]^T.$$

state-space model: $\begin{cases} \dot{X} = AX + Bu \\ y = CX \end{cases}$

$\text{rank}([B \ AB \ A^2B]) = 1$, $\text{rank}([C; CA; CA^2]) = 2$

Neither controllable nor observable. Try to use Kalman Decomposition to determine invariant subspaces, but it seems that (Controllable, observable) subspace V_{co} is ~~nullspace, nullset~~ a null set. So I think we have to find another way.

(used matlab function
SS(A, B, C, D) = sys
sys1 = SS(sys, 'minimal'))

$\longrightarrow$ observable : X, v
controllable : $\dot{x}$ ($\dot{x}^2$)

3. ①. (Continued). To calculate V, we must have

$$X_{des} = 0.1 + 0.02\cos(2\pi t)$$

$$V_{des} = -0.04\pi \sin(2\pi t)$$

$$i_{des} = \sqrt{\frac{2(X_0 + b - X_{des})^2}{L_0}\left(m_p \frac{dV_{des}}{dt} + \mu_p V_{des} + k_s X_{des} - m_p g\right)}$$

And for linear feedback controller,

$$u = -KX + V = e$$

$$e = V - KX = V - [k_i \;\; k_x \;\; k_v]\begin{bmatrix} i \\ x \\ v \end{bmatrix}$$

And the transfer function would thus be

$$G(s) = C(sI - A + BK)^{-1}B$$

~~eigen~~ characteristic polynomials $= \det[\lambda I - (A - BK)]$

All the real parts of the eigenvalues must be < 0, to ensure stability. This gives some restriction for K.

plug in the constant values. we have

$$i_{des} \approx 0.04714\sqrt{0.0513\cos(2\pi t) - 0.5 - \pi\sin(2\pi t)}$$

3. ① (continued) Let $K = [k_1 \ k_2 \ k_3]$

~~or~~ ~~$\det(A - BK)$~~ $\operatorname{eig}(A - B^* K) = \begin{bmatrix} -k_1/30 - 1/6 \\ -100 - 2\sqrt{5}\cdot\sqrt{497} \\ 2\sqrt{5}\cdot\sqrt{497} - 100 \end{bmatrix}$.

$-k_1/30 - 1/6 < 0.$

$k_1 > -5.$

Therefore, we only need to let $k_1 > -5$. No other confinement is found of k_2, k_3.

In reality, ~~we~~ we found that i_{des} somes times seems imaginary, this is because i_{des} is not directly controllable. Thus we may find ~~$I = I^2$~~ $I = i^2$, which is proportional to ~~F_{mag}~~ F_{mag} (F_{mag} controllable), and design our system as follows.

[I put simulink results in ~~a not~~ another pdf..... this is because I don't really ~~wa~~ want to draw ~~to~~ the whole diagram here, on paper...]

3. ②. $\dot{X} = f(X) + g(X)u \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} z \\ x \\ v \end{bmatrix}$

relative degree $\quad y = h(x) = x_2$

$$\frac{\partial h}{\partial x} = [0 \ 1 \ 0]$$

$$\dot{y} = \frac{\partial h}{\partial x}\dot{x} = L_f h + \underline{L_g h} u = \underline{L_f h}$$

$$\ddot{y} = L_f^2 h + \underline{L_g L_f h} u = \underline{L_f^2 h}$$

$$\dddot{y} = L_f^3 h + \underline{L_g L_f^2 h} u = L_f^3 h + \underline{\frac{L_0 x_1}{(x_0+b-x_2)^2 m_p}} u$$

relative degree $r = 3 = n$. $\quad$ ⊛

$L_f h = x_3. \qquad \underline{L_f^2 h = \frac{L_0 x_1^2}{2(x_0+b-x_2)^2 m_p} - \frac{\mu_p}{m_p} x_3 - \frac{k_s}{m_s} x_2 + g}$

$L_g h = 0. \qquad L_g L_f h = 0.$

<u>Therefore the system is linearizable,</u> ~~no~~.

$$f(X) = \begin{bmatrix} -\frac{R(x_0+b-x_2)}{L_0} x_1 - \frac{x_1 x_3}{x_0+b-x_2} \\ x_3 \\ \frac{L_0 x_1^2}{2(x_0+b-x_2)^2 m_p} - \frac{k_s}{m_p} x_2 - \frac{\mu_p}{m_p} x_3 + g \end{bmatrix} \qquad g(x) = \begin{bmatrix} \frac{x_0+b-x_2}{L_0} \\ 0 \\ 0 \end{bmatrix}$$

And Thus, $T(x) = \xi(x) = [h(x) \ L_f h(x) \ L_f^2 h(x)]^T$

3. ② (continued)

$$\xi(x) = [x_2 \quad x_3 \quad \circledast]^T.$$

$$\dot{\xi}(x) = [x_3 \quad \circledast \quad \star]^T.$$

This should satisfy the canonical form of state space.

$$\dot{\xi}(x) = \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \end{bmatrix} = A_c \xi + B_c v$$

$$A_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B_c = [0 \quad 0 \quad 1]^T$$

For $v = -k_1\xi_1 - k_2\xi_2 - k_3\xi_3$, we have

$$\dot{\xi} = A_{ck}\xi \qquad A_{ck} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_1 & -k_2 & -k_3 \end{bmatrix}$$

must be Hurwitz

$$\star = \frac{\partial \circledast}{\partial x_1}\dot{x}_1 + \frac{\partial \circledast}{\partial x_2}\dot{x}_2 + \frac{\partial \circledast}{\partial x_3}\dot{x}_3$$

$$= -\frac{40x_1^2}{0.2 - x_2} - 60x_3 - 200\circledast - 16x_1^2x_3/(0.2-x_2)^3.$$

$$-\lambda^2(k_3+\lambda) - \lambda k_2 - k_1 = 0$$

$$\lambda^3 + k_3\lambda^2 + k_2\lambda + k_1 = 0.$$

There is a general solution for order 3 polynomial.

Now we should have

$$v = -\xi_1 - 3\xi_2 - \xi_3 \cdot 3$$

pole placed at $Re(\lambda) = -1$

We can find some confinements regarding k_1, k_2, k_3.

But now let's just suppose that $k_1 = 1$ $k_2 = 3 = k_3$

3. ③. In this case, the dynamics become

$e = iR$

$$\begin{cases} \dot{x} = v \\ \dot{v} = \frac{1}{m_p}\left(u - \frac{\mu_p}{m_p} v - \frac{k_s}{m_p} x\right) \end{cases}$$

state-space:

$$A = \begin{bmatrix} 0 & 1 \\ -60 & -200 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k_s}{m_p} & -\frac{\mu_p}{m_p} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ \frac{1}{m_p} \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \end{bmatrix} \qquad C = [1 \quad 0] \text{ for } y = x$$

let $V = U - KX$. $\quad K = [k_1 \quad k_2]$ (in the simulation randomly chosed to be $[1 \quad 1]$)

$= -k_1 x - k_2 v + F$

$x_{des} = 0.1 + 0.02\cos(2\pi t)$. $\quad V_{des} = -0.04\pi \cancel{\cos} \sin(2\pi t)$

Suppose F is constant (step input)

Then $V = (F - 0.1k_1) - \underline{0.02k_1 \cos(2\pi t)}$

$\underline{+ 0.04\pi k_2 \sin(2\pi t)}$. $\quad \cancel{\sin(2\pi t + \varphi)}$

$V = (F - 0.1k_1) + A_1 \sin(2\pi t + \varphi)$.

The controller is designed in matlab simulink.

3. ③. Last but not least, if F is constant ($F = u$) then the general solution of the equation should be

$$x(t) = \cancel{A_{constant}}$$

$$= Amp\, e^{-\frac{\mu}{2m}t} \cos(\omega t + \varphi) + x_0 \quad \omega^2 = \omega_0^2 - \left(\frac{\mu}{2m}\right)^2.$$

$\omega_0 = \sqrt{\frac{k}{m}}$ is the natural frequency of the spring.

The expression is more complex when $F = F(t)$. (may not be analytical)

$$\left[\text{In this case } x_0 = \frac{F_{mag} + mg}{m_p}\right]$$

The point is, in this case $e^{-\frac{\mu}{2m}t}$ term cannot be neglected? so I wonder how to limit ~~and~~ (push and stretch, mainly stretch) so that we can finally reach x_{des}? (Luckily I found another solution written in the previous page).

[This page is negligible]