MEEN 655. MIDTERM I

Mingfan Dong

1.1). $\frac{dH}{dt} = H\Omega - \Omega H$ $\qquad \frac{dH^T}{dt} = H^T\Omega - \Omega H^T$ by definition

I)

$(\frac{dH}{dt})^T = \Omega^T H^T - H^T \Omega^T$ since $-\Omega^T = \Omega$ (Ω is skew-symmetric)

we have $(\frac{dH}{dt})^T = \frac{dH^T}{dt}$ proved.

1.2). Let $X(t) = H(t) - H^T(t)$ Note that ~~X(t)~~ X is also skew symmetric.

I).

$[X(t)]^T = H^T(t) - H(t) = -X(t)$ holds.

$\frac{dX}{dt} = [X, \Omega]$ Since Ω can be any skew-symmetric matrix, Let $\Omega = X$. then

$\frac{dX}{dt} = [X, X] = 0$. An obvious equilibrium shall be $X(t_0) = H(t_0) - \cancel{H(t_0)}\, H^T(t_0) = 0$.

And since $\dot{X} = 0$, we know that for $t \geq t_0$, $X(t) = 0$ holds.

1.3). $\dot{H}(t) = \dot{\phi}^T(t,t_0) H(t_0) \phi(t,t_0) + \phi^T(t,t_0) H(t_0) \dot{\phi}(t,t_0)$

I).

$= [\phi(t,t_0)\Omega(t)]^T H(t_0) \phi(t,t_0) + \phi^T(t,t_0) H(t_0) \phi(t,t_0) \Omega(t)$

$= \phi^T(t,t_0) H(t_0) \phi(t,t_0) \Omega(t) + \Omega^T(t) \phi^T(t,t_0) H(t_0) \phi(t,t_0)$

$= H(t)\Omega(t) + \Omega^T(t) H(t)$

$= H(t)\Omega(t) - \Omega(t) H(t)$ since Ω is skew-symmetric

$= [H, \Omega]$ holds

1.4). Let $Z(t) = \phi^T(t,t_0)\phi(t,t_0)$ then

I).
$$\dot{Z}(t) = \dot{\phi}^T(t,t_0)\,\phi(t,t_0) + \phi^T(t,t_0)\,\dot{\phi}(t,t_0)$$
$$= \Omega^T(t)\,\phi^T(t_0)\phi(t,t_0) + \phi^T(t,t_0)\phi(t,t_0)\,\Omega(t)$$
$$= Z(t)\Omega(t) + \Omega^T(t)Z(t)$$
$$= Z(t)\Omega(t) - \Omega(t)Z(t) = [Z,\Omega] \quad \text{holds}$$

$\dot{Z}(\hat{t}) = f(\hat{t}) = 0$ denotes an equilibrium. In this case, $Z(\hat{t}) = I$ lead to $\dot{Z}(\hat{t}) = 0$, therefore it is an equilibrium. Thus $\phi^T(t,t_0)\phi(t,t_0) = I$.

1.5). $\phi^T\phi = I \qquad \phi^T = \phi^{-1} \qquad (\phi = \phi(t,t_0))$. rotation matrix.

I). which is surfice to say that ϕ is ~~rotatory matrix~~

Since we have defined that $H(t) = \phi^T H(t_0)\phi$,

Now that we know $\phi^T = \phi^{-1}$ exists, for ϕ is some rotation matrix. Then $H(t) = \phi^{-1}H(t_0)\phi$. By definition we knew that $H(t)$ is similar to $H(t_0)$.

And hence, $H(t)$ and $H(t_0)$ have the same set of eigenvalues. (This is the property of similar matrix.)

1. II). $\Omega(t) = H(t)N - NH(t)$.

$$[\Omega(t)]^T = N^T H^T - H^T N^T = NH^T - H^T N$$

(Since we already proved that if $H(t_0) = H^T(t_0)$, then $H(t) = H^T(t)$ for any $t \geq t_0$) ~~Then~~ As we know, $H(t) = H^T(t)$ $t \geq t_0 = 0$.

Therefore $[\Omega(t)]^T = NH^T - H^T N = NH - HN$. ①

By definition $\Omega(t) = [H(t), N] = HN - NH$ ②.

Since ② = −①, we know that $[\Omega(t)]^T = -\Omega(t)$, $\Omega(t)$ is skew-symmetric ~~to~~ could be proved.

→ And since we already proved the symmetry of $H(t)$ in part I). of this problem, it follows that the solution of the double bracket equation is symmetric.

As it is defined, $H(t) = \phi^T(t, t_0) H(t_0) \phi(t, t_0)$ holds.

And since $\phi^T \phi = I$, $\phi^{-1} = \phi^T$ exists, we know that $\phi^T \phi = 1$ (ϕ is rotation matrix).

Thus $H(t) = \phi^{-1} H(t_0) \phi \Longrightarrow H(t)$ is similar to $H(t_0)$

Therefore, $H(t)$ and $H(t_0)$ have the same set of eigenvalues.

1. III) As we know, $tr(BA) = tr(AB)$ $\quad tr := trace$

Since $N = N^T$ and $\dot{N} = 0$. And we already proved that $H(t) = H^T(t)$ for $t \geq t_0$ holds, we know that

$$-\frac{dV}{dt} = tr(\dot{H}^T N) = tr(\dot{H} N) = tr([H,[H,N]]\, N).$$

Note that $tr(A[B,C]) = tr([A,B]C)$

And that $[N,H] = -[H,N] = [H,N]^T$

$$tr([H,[H,N]]\, N) = tr([N,H]\,[H,N]) = tr([H,N]^T [H,N])$$

$$= \| [H(t), N] \|^2$$

which follows the definition of Frobenius norm.

~~Q~~ 1. IV). To prove that $S(c) := \{H : V(H) \le c\}$ is bounded, We must prove that $\|H(t) - H(t_0)\| \le r$ holds for some "radius" r. That is, $H(t) \in [H(t_0) - r, H(t_0) + r]$. Which indicates that $H(t)$ is bounded. The fact that $H(t)$ is bounded is equivalent to the fact that Frobenius norm of $H(t)$ is bounded, that is, $\sum_{i,j=1}^{n} h_{ij}^2(t) \le M$ holds for some $M > 0$. $\|H\|_F^2 = \sigma_1^2(H) + \sigma_2^2(H) + \cdots + \sigma_n^2(H)$, where $\sigma_1, \sigma_2, \cdots, \sigma_n$ are eigenvalues of $H(t)$.

$$\|H(t)\|_F^2 = Tr(H^T H) = Tr(U\Sigma^2 U^T) = Tr(\Sigma^2 U^T U) = Tr(\Sigma^2)$$
$$= \lambda_1^2(H(t)) + \lambda_2^2(H(t)) + \cdots + \lambda_n^2(H(t)).$$

Notice that $H(t)$ ~~are~~ is similar to H_0, so $H(t)$ and H_0 have the same set of eigenvalues. Thus we know that

$$\|H(t)\|_F^2 = \lambda_1^2(H(0)) + \lambda_2^2(H(0)) + \cdots + \lambda_n^2(H(0)) = \|H(0)\|_F^2.$$

Therefore the ~~For~~ Frobenius norm of $H(t)$ is bounded, Hence $H(t)$ must be bounded.

Since $\dot{V} \le 0$, for $t \ge t_0$, ~~then~~ $V(H(t)) \le \underline{V(H(t_0))}$ let this $= c$.

~~Suppose~~ Then $\|H(t) - H(t_0)\| \le r$ holds since $H(t)$ is bounded.

1. II) The ω-limit set of points are the set of points that the system of equilibrium approach as $t \to \infty$.

LaSalle's Theorem: Let $S \subseteq \mathbb{R}^n$ be compact invariant set. If there exists a differentiable function $V: S \to \mathbb{R}$ s.t. $\dot{V}(x) \leq 0$ $\forall x \in S$. If the set $\{x \in S \mid \dot{V}(x) = 0\}$ contains no ~~only $x(t)=0$~~ trajectories other than $x(t)=0$, then $x(t)=0$ is locally asymptotically stable. Moreover, all trajectories starting in S converges to zero.

In this case, $V = -\mathrm{trace}(H^T N)$. ~~$\dot{V}(t) \leq 0$~~ $\dot{V} = -\|[H, N]\|^2$. as time increases, V converges to a finite value since H is bounded, thus the time derivative of V must go to zero. Therefore, every solution of $\dot{H} = [H, [H, N]]$ converges to a connected component of the set of equilibria points as $t \to \infty$, namely $[H_\infty, N] = 0$ must hold. Since the only trajectory to ensure $\dot{V}(t) = 0$ is $[H|_{t\to\infty}, N] = 0$, we can say that the ω-limit set (described above) coincides with the equilibria of the double bracket system.

~~ⅡⅠ~~. 1. Ⅵ). This is to say that as $t \to \infty$, $H(t)$ would become diagonal.

As we know, the solutions of double bracket equation is characterized by $[H_\infty, N] = 0$. ~~Now suppose that~~ ~~$N = diag(\mu_1, \cdots, \mu_n)$ with~~ As we know,

$N = diag(\lambda_1, \cdots, \lambda_n)$ with $\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$.

$[H_\infty, N] = 0$ leads to $\lambda_i h_{ij} = \lambda_j h_{ij}$ for $i, j = 1, \cdots, n$.

$[H_\infty] = [h_{ij}]$. which is $(\lambda_i - \lambda_j) h_{ij} = 0$.

Therefore $h_{ij} = 0$ when $i \neq j$, because the eigenvalues of N are all ~~diff~~ distinctive. Thus, H_∞ is a diagonal matrix with the same eigenvalues as H_0. ~~dut~~ (Similarity of $H(t)$ and $H(0)$).

Suppose that the eigenvalues of H_0 is $\mu_1, \mu_2, \cdots, \mu_n$

Then the diagonal matrices must have the form of

$H_\infty = \pi\, diag(\mu_1, \mu_2, \cdots, \mu_n)\, \pi'$. where π is permutation matrix. Therefore total number of such diagonal ~~matrix~~ matrices should be $n!$.

1. III). $\dot{H} = H^2 N - 2HNH + NH^2$ suppose H^* denotes equilibrium (diagonal)

$H = \tilde{H} + H^*$

Then $H^2 N = [H^{*2} + H^*\tilde{H} + \tilde{H}H^* + \tilde{H}^2] N$ $O(\tilde{H}^2)$. negligible.

$HNH = H^*NH^* + \tilde{H}NH^* + H^*N\tilde{H} + \tilde{H}N\tilde{H}$ $O(\tilde{H}^2)$

$NH^2 = N[H^{*2} + H^*\tilde{H} + \tilde{H}H^* + \tilde{H}^2]$ $O(\tilde{H}^2)$

$\dot{\tilde{H}} = \dot{H}$ (since $\dot{H}^* = 0$).

$= \underbrace{H^{*2}N - 2H^*NH^* + NH^{*2}}_{①} + ②$ $① = 0$.

$② = [H^*\tilde{H}N + \tilde{H}H^*N - 2\tilde{H}\underbrace{NH^*}_{H^*N} - 2\underbrace{H^*N}_{NH^*}\tilde{H} + NH^*\tilde{H} + N\tilde{H}H^*]$

$\cancel{= H^*\tilde{H}N \quad \tilde{H}H^*N = H^*N\tilde{H} +}$

$② = \dot{\tilde{H}} = H^*\tilde{H}N - \tilde{H}H^*N - H^*N\tilde{H} + N\tilde{H}H^*$ is the differential equation.

Note that eigenvalues of N: $\lambda_1, \cdots, \lambda_n$

H^*: $\mu_1, \mu_2, \cdots, \mu_n$.

$\left(\begin{array}{l} Ne_i = \lambda_i e_i \\ H^* e_j = \mu_j e_j \end{array} \right)$.

Linearization: $L(\tilde{H}) := ②$

$L[e_i e_j^T] = (\mu_i e_i)(\lambda_j e_j^T) = \mu_i \lambda_j e_i e_j^T$

$L(\tilde{H}) = [\mu_i \lambda_j - \mu_j \lambda_j - \mu_i \lambda_i + \lambda_i \mu_j] e_i e_j^T$

$= -(\mu_i - \mu_j)(\lambda_i - \lambda_j) e_i e_j^T$

1. VII). (Continued).

$\dot{\tilde{H}} = \mathcal{L}(\tilde{H})$. $\quad \mathcal{L}[\tilde{H}_{ij}] = \xi_{ij} \tilde{H}_{ij}$

$\tilde{H}(0) = \tilde{H}_{ij} \Rightarrow \tilde{H}(t) = e^{\xi_{ij} t} \tilde{H}_{ij}(0)$

suppose that ξ_{ij} is an eigenvalue of $\mathcal{L}[\tilde{H}_{ij}]$.
And $\tilde{H}_{ij}$ is the corresponding eigenvector.

The stable equilibria of the flow $\dot{H} = [H, [H, N]]$. converges as $t \to \infty$ with an exponential bound on the rate of convergence.

$$\dot{\tilde{H}} = \xi_{ij} \tilde{H}(t) = \xi_{ij} e^{\xi_{ij} t} \tilde{H}_{ij}(0).$$

$$\mathcal{L}[\tilde{H}(t)] = e^{\xi_{ij} t} \mathcal{L}[\tilde{H}_{ij}] = \xi_{ij} e^{\xi_{ij} t} \tilde{H}_{ij}$$

($\mathcal{L}$: differential operator, $\mathcal{L}$ is a linear transformation.)

2. I). The error dynamics is given by

$$\dot{e} = \dot{w} - \dot{x} = Aw + \Phi(y_w, u) + BW^T(y_w, u)\hat{\theta} - L(Cx - Cw)$$

$$- Ax - \Phi(y_x, u) - BW^T(y_x, u)\theta$$

$$= A(w-x) + LC(w-x) + \underline{\Phi(y_w, u) - \Phi(y_x, u)} \quad ①$$

$$+ \underline{BW^T(y_w, u)\hat{\theta} - BW^T(y_x, u)\theta} \quad ②$$

$$= (A+LC)e + ① + ②.$$

Since the output y and the input u are measurable, then ① is of fixed value, and $W^T(y_w, u) \approx W^T(y_x, u)$ as $y_w \to y_x$, for Φ, W^T are known functions, and Φ, W^T is assumed to follow Lipschitz condition.

Thus

$$\dot{e} = (A+LC)e + ① + ②$$

$$\approx (A+LC)e + BW^T(y, u)\tilde{\theta}$$

For $\underbrace{\|\Phi(w) - \Phi(x)\|}_{①} \le \gamma_1 \|e\|$ $\quad \|W^T(w) - W^T(x)\| \le \gamma_2 \|e\|$.

$$② = BW^T(y_x, u)\tilde{\theta} + \underline{B[W^T(y_w, u) - W^T(y_x, u)]\hat{\theta}}$$

$$\hookrightarrow \le B\gamma_2 \|e\| \hat{\theta} \quad (\text{neglected})$$

2. II) $V = e^T P e + \tilde{\theta}^T \Gamma^{-1} \tilde{\theta}$

$$\dot{V} = e^T[(A+LC)^T P + P(A+LC)]e + e^T P[f(w) - f(x)]$$ 4
$$+[f(w) - f(x)]Pe + e^T PB[F(w)\hat{\theta} - F(x)\theta]$$
$$+[F(w)\hat{\theta} - F(x)\theta]B^T Pe + 2\tilde{\theta}^T \Gamma^{-1} \dot{\tilde{\theta}}$$

where $f(x) = \Phi(y, u)$. $F(x) = W^T(y, u)$.

Since for any matrix X, Y and $\varepsilon > 0$, $\varepsilon(\frac{1}{\varepsilon}X - Y)^T(\frac{1}{\varepsilon}X - Y) \geq 0$ holds, then $X^T Y + Y^T X \leq \frac{1}{\varepsilon} X^T X + \varepsilon Y^T Y$.

Thus for any positive number $\varepsilon_1, \varepsilon_2$ we have

$$e^T P[f(w) - f(x)] + [f(w) - f(x)]Pe \leq \frac{1}{\varepsilon_1} e^T P P e + \varepsilon_1 \| f(\hat{x}) - f(x) \|^2$$
$$\leq \frac{1}{\varepsilon_1} e^T P P e + \varepsilon_1 \gamma_1^2 \|e\|^2$$

(We already assumed that f, F are Lipschitz continuous).

And $e^T PB[F(w)\hat{\theta} - F(x)\theta] + [F(w)\hat{\theta} - F(x)\theta]B^T Pe$

$$= e^T PB[F(w) - F(x)]\theta + [F(w) - F(x)]\theta B^T Pe + 2[F(w)\tilde{\theta}]^T B^T Pe$$
$$\leq \frac{1}{\varepsilon_2} e^T PBB^T Pe + \varepsilon_2 \| F(w)\theta - F(x)\theta \|^2 + 2[F(w)\tilde{\theta}]^T B^T Pe$$
$$\leq \frac{1}{\varepsilon_2} e^T PBB^T Pe + \varepsilon_2 \gamma_2^2 \gamma_3^2 \|e\|^2 + 2[F(w)\tilde{\theta}]^T B^T Pe$$

(Assuming $\|\theta\| \leq \gamma_3$, which is to say θ is bounded)

2. II). (Continued)

Therefore, $\dot{V} \le e^T[(A+LC)^T P + P(A+LC)]e + \frac{1}{\varepsilon_1} e^T P P e$

$$+ \varepsilon_1 \gamma_1^2 \|e\|^2 + \frac{1}{\varepsilon_2} e^T P B B^T P e + \varepsilon_2 \gamma_2^2 \gamma_3^2 \|e\|^2$$

$$\underline{+ 2\left[F(w)\tilde{\theta}\right]^T B^T P e + 2\tilde{\theta}^T \Gamma^{-1} \dot{\tilde{\theta}}} \quad (3)$$

Now we determine the parameters adaption law by setting (3) $=0$. To satisfy this equation, we have

$$\dot{\tilde{\theta}} = -\Gamma F(w)^T B^T P e = -\Gamma F(w)^T C^T e$$
$$= -\Gamma W C^T e$$

(☆. Actually, the error dynamic term proved in I). showed that we don't have to ~~take terms~~ consider terms such as (4), marked in "〰")

Since $\tilde{y}$ is known, $\tilde{y} = \underbrace{Cw}_{} - \underline{y}_{Cx}$, then the corresponding $\tilde{x} = C^{-1}\tilde{y}$.

$$\tilde{y} = Ce$$
$$\tilde{x} = e = C^{-1}\tilde{y}.$$

2. Ⅲ). Since we already let ③ $= 0$.

If $\dot{V} < 0$, we must have

⑤ $[A+LC]^T P + P[A+LC] + \frac{1}{\varepsilon_1} P^2 + \varepsilon_1 \gamma_1^2 + \frac{1}{\varepsilon_2} PBB^T P + \varepsilon_2 \gamma_2^2 \gamma_3^2 < 0$.

For $\dot{V} \leq e^T ⑤ e$

$$\dot{V} \leq -\beta e^T e \quad (\beta > 0).$$

$$= -\beta \|e\|^2$$

$$V(t) \leq V(0) - \beta \int_0^t e^T e \, dt$$

Since $V(t) \in L_\infty$ and $V(0)$ is finite, this ~~impli~~ showed that $e \in L_2$. Therefore $e \in L_2 \cap L_\infty$.

(From the error dynamic equation, the controller must let it be stablized to the origin, so that

$$\lim_{t \to \infty} \|e(t)\| = 0 \quad \text{holds.}$$
)

It is natural that $\tilde{\theta} = \hat{\theta} - \theta$ is bounded, for we assumed that $\|\theta\| \leq \gamma_3$ holds.

$$\text{Therefore } \dot{e} = (A+LC)\underbrace{\overset{\text{bounded}}{e}}_{\cancel{\text{boun}}} + BW^T(y,u)\overset{\text{bounded}}{\tilde{\theta}}.$$

$$\dot{e} \in L_\infty.$$

Thus this scheme guarantee that $e \to 0$ asymptotically.

2. Bonus). $e = C^{-1}\tilde{y}$ $\quad \tilde{y} = Cw - y = \hat{y} - y$

Now we have $\bar{A} = A + LC$

$$\begin{cases} \dot{e} = \bar{A}e + BW^T\tilde{\theta} \\ \dot{\tilde{\theta}} = -\Gamma W B^T P e \end{cases}$$

let $\Omega(t) = WB^T$ $\quad \Omega^T = BW^T$

Since (A, B) is completely controllable, and (A, C) completely ~~stab~~ observable, $G(s) = C(sI - \bar{A})^{-1}B$ must have ~~eigenvalues~~ characteristic roots with real part < 0. which is to say that $\bar{A}$ is Hurwitz.

clearly $\|\Omega(t)\|$ is bounded, $\|\dot{\Omega}(t)\| = \|\dot{W}B^T\|$ is also bounded

since regressor vectors and matrices A, B, C are known.

As $\int_t^{t+\delta} \Omega(\tau)\Omega^T(\tau)\,d\tau = \int_t^{t+\delta} W(\tau)\underline{B^TB}W^T(\tau)\,d\tau \geq \alpha I > 0$ holds by assumption, it is clear that $[\dot{e}, \tilde{\theta}]^T = [0, 0]$ is globally exponentially stable.

~~Note that B^TB~~ B^TB is finite (actually B should be constant matrix).

3. I). Just plug in the equation $x_i = K^{i-1}\xi_i$.

We have $\dot{\xi}_1 = K\xi_2 + f_1(\xi_1)$

$$\cancel{\dot{\xi}_2 = \frac{\dot{x}_2}{K} = \frac{\quad}{K^2}}$$

$$\dot{x}_2 = K\dot{\xi}_2 = x_3 + f_2(x_1, x_2) = K^2\xi_3 + f_2(\xi_1, K\xi_2)$$

Thus $\dot{\xi}_2 = K\xi_3 + \frac{1}{K} f_2(\xi_1, K\xi_2)$ holds.

Likewise we could prove that for $i = 1, 2, \cdots, n-1$,

$$\dot{\xi}_i = K\xi_{i+1} + \frac{1}{K^{i-1}} f_i(\xi_1, K\xi_2, \cdots, K^{i-1}\xi_i).$$

for $i = n$, $x_n = u$ $\quad v = \frac{u}{K^n}$. $\quad \xi_n = \frac{x_n}{K^{n-1}}$

Thus we have $\xi_n = \frac{x_n}{K^{n-1}} = \frac{u}{K^{n-1}} = Kv$

(I do believe that there is some typo in Problem 3.)

For instance, $\dot{\xi}_{n-1} = \frac{\dot{x}_{n-1}}{K^{n-2}}$

$$\dot{x}_{n-1} = x_n + f_{n-1}(x_1, x_2, \cdots, x_{n-1})$$

$$K^{n-2}\dot{\xi}_{n-1} = K^{n-1}\xi_n + f_{n-1}(\xi_1, K\xi_2, \cdots, K^{n-2}\xi_{n-1})$$

$$\dot{\xi}_{n-1} = K\xi_n + \frac{1}{K^{n-2}} f_{n-1}(\xi_1, K\xi_2, \cdots, K^{n-2}\xi_{n-1}).$$

~~slightly different~~ should be the correct expression.

3.II). Assume that $g_i(\xi_1, \xi_2, \cdots, \xi_i) = \frac{1}{k^{i-1}} f_i(\xi_1, k\xi_2, \cdots, k^{i-1}\xi_i)$

Then $\dot{\xi} = \left[\begin{array}{c} \dot{\xi}_i \\ \hline kv \end{array}\right]_{n\times 1}$ Therefore $B_c = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \hline 1 \end{array}\right] \begin{array}{l} (n-1)\times 1 \\ 1\times 1 \end{array}$

~~The~~ Since we now have

$$\begin{cases} \dot{\xi}_1 = k\xi_2 + g_1(\xi_1) \\ \dot{\xi}_2 = k\xi_3 + g_2(\xi_1, k\xi_2) \\ \vdots \\ \dot{\xi}_{n-1} = k\xi_n + g_{n-1}(\xi_1, k\xi_2, \cdots, k^{n-2}\xi_{n-1}) \\ \dot{\xi}_n = kV. \end{cases}$$

We know that $A_c = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}_{n\times n}$ $g(\xi) = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_{n-1} \\ 0 \end{bmatrix}$

It could be verifed that $(A_c. B_c)$ is controllable,

for $\text{rank}[M] = n$.

$[M] = [B_c, A_cB_c, \cdots, A_c^{n-1}B_c]$.

3.III). Since f_i is Lipschitz functions with constant L,

$$\| f_i(y) - f_i(x) \| \leq L \| y - x \| \text{ holds.}$$

Note that $g_i = \frac{1}{K^{i-1}} f_i$. Substituting f with g gives us

$$\| K^{i-1} g_i(y) - K^{i-1} g_i(x) \| \leq L \| y - x \| \qquad K > 1 :$$

$$\| g_i(y) - g_i(x) \| \leq \frac{L}{K^{i-1}} \| y - x \|$$

therefore the Lipschitz constant for g_i is $\frac{L}{K^{i-1}}$, $i = 1, 2, \cdots, n-1$

3.IV).

$$[A_c + B_c G] = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} [\beta_0, \beta_1, \cdots, \beta_{n-1}] \cdot (-1)$$

$$= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -\beta_0 & -\beta_1 & \cdots & & -\beta_{n-1} \end{bmatrix}_{n \times n}$$

let $M = A_c + B_c G$.

~~$\det(A_c + B_c G) = s^n + \beta_{n-1} s^{n-1} + \beta_{n-2} s^{n-2} + \cdots + \beta_0$ holds~~

Therefore ~~$\det(A_c + B_c G) = \Delta(s)$~~ they have the same set of eigenvalues, namely, they ~~has it~~ have the same location poles as desired.

$$\det[M - sI] = s^n + \beta_{n-1} s^{n-1} + \beta_{n-2} s^{n-2} + \cdots + \beta_0 \text{ holds.}$$

This could be proved easily.

3. II). Since $g(\xi)$ is treated as a perturbation, we could say that $\|g(\xi)\| \le \alpha\|\xi\|$. That is, $g(\xi)$ is a Lipschitz function, with constant $L=\alpha$.

Let $M = A_c + B_c G$.

$$\dot{\xi} = KM\xi + g(\xi)$$
$$\le KM\|\xi\| + \alpha\|\xi\|$$
$$= (KM + \alpha I)\|\xi\|.$$

Since we know that all the eigenvalues of M is smaller than -1, ~~(As it is shown~~ then all the eigenvalues of KM should be $-K\lambda_1, -K\lambda_2, \cdots, -K\lambda_n$ $(1<\lambda_1<\cdots<\lambda_n)$

~~We~~ To guarantee that $(KM+\alpha I)$ is Hurwitz is to guarantee that $\xi=0$ is asympototically stable with $g(\xi)$.

Therefore we need to let $-K\lambda_1 + \alpha < 0$.

$$K\lambda_1 > \alpha \qquad \underline{K > \frac{\alpha}{\lambda_1}}$$