Texas A&M University
Department of Mechanical Engineering

MEEN 655: Design of Nonlinear Control Systems
Homework 4

February 24, 2022

1. **No connection between eigenvalues and stability for Linear Time Varying Systems:** Consider the following linear time-varying system:

$$\underbrace{\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix}}_{\dot{x}(t)} = \underbrace{\begin{pmatrix} -1 + \alpha \cos wt \sin wt & \alpha \cos^2 wt + w \\ -\alpha \sin^2 wt - w & -1 - \alpha \cos wt \sin wt \end{pmatrix}}_{A(t)} \underbrace{\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}}_{x(t)}$$

   - Show that the state-transition matrix for the linear time-varying system is given by:

   $$\Phi(t, t_0) = e^{-(t-t_0)} \begin{pmatrix} \cos wt & \sin wt \\ -\sin wt & \cos wt \end{pmatrix} \begin{pmatrix} 1 & \alpha(t - t_0) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos wt_0 & -\sin wt_0 \\ \sin wt_0 & \cos wt_0 \end{pmatrix}$$

   (Hint: It is sufficient to show the following two conditions are satisfied:

   $$\frac{d\Phi(t, t_0)}{dt} = A(t)\Phi(t, t_0), \quad \Phi(t_0, t_0) = I.$$

   This will ensure that the solution, $x(t)$, will satisfy:

   $$x(t) = \Phi(t, t_0)x(t_0).$$

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   - Show that the equilibrium, $x = 0$, is exponentially stable by showing that for some $M, \lambda > 0$,

   $$\|x(t)\| \leq M\|x(t_0)\|e^{-\lambda(t-t_0)}.$$

   - What is the characteristic polynomial of $A(t)$?
   (Hint: For a 2×2 matrix, $A(t)$, the characteristic polynomial may be expressed as

   $$\det(\lambda I - A(t)) = \lambda^2 - \lambda\, tr(A(t)) + \det(A(t)).$$

   Here, $tr(A)$ is the trace of A, i.e., the sum of all its diagonal elements.)

- Show that the eigenvalues of $A(t)$ are $-1 \pm \sqrt{-\alpha w - w^2}$, i.e., they are independent of t.
- If $w^2 + \alpha w + 1 < 0$, show that one of the eigenvalues is positive. When will the roots of $w^2 + \alpha w + 1$ be real and distinct? Can you find a set of (α, w) for which one of the eigenvalues of $A(t)$ is positive? Do you see the discrepancy between eigenvalues of $A(t)$ and stability?

2. Consider another LTV system:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} -1 + 1.5\cos^2 t & 1 - 1.5\sin t \cos t \\ -1 - 1.5 \sin t \cos t & -1 + 1.5 \sin^2 t \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

- Show that

$$\Phi(t, 0) = \begin{pmatrix} e^{0.5t} \cos t & e^{-t} \sin t \\ -e^{0.5t} \sin t & e^{-t} \cos t \end{pmatrix}.$$

  Notice that $A(t)$ is periodic with a period π; however, a solution to the differential equation may not be periodic. For example, what is the solution corresponding to an initial condition $x_1(0) = 1$ and $x_2(0) = 0$? Is it periodic?
- What are the eigenvalues of $A(t)$? Do you see the discrepancy about inferring stability of equilibrium, $x = 0$, from the eigenvalues of $A(t)$?

3. Consider a gradient dynamical system associated with a smooth function $U(x, y)$:

$$\dot{x} = -\frac{\partial U(x, y)}{\partial x}, \quad \dot{y} = -\frac{\partial U(x, y)}{\partial y}.$$

Show that the gradient dynamical system cannot have periodic trajectories.

4. Use backstepping technique from last homework to design a controller to regulate the state of the following non-linear system

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1^2 - x_1^3 \\ \dot{x}_2 &= x_3. \\ \dot{x}_3 &= u, \end{aligned}$$

using the following steps:

- Consider the system:

$$\dot{x}_1 = v_1 + x_1^2 - x_1^3.$$

  Consider $V_1(x_1) = \frac{1}{2}x_1^2$. Show that by choosing $v_1 = -(1 + \mu)x_1^2 + x_1^3$ for some $\mu > 0$, the closed loop equilibrium $x_1 = 0$ of the first-order system is asymptotically stable.

- Now define $z_2 := x_2 - v_1$, and consider the second order system:

$$\begin{aligned}\dot{x}_1 &= x_2 + x_1^2 - x_1^3, \\ \dot{x}_2 &= v_2.\end{aligned}$$

Show that this system can be recast as:

$$\begin{aligned}\dot{x}_1 &= z_2 - \mu x_1, \\ \dot{z}_2 &= v_2 - (\mu + \frac{\partial v_1}{\partial x_1})(z_2 - \mu x_1).\end{aligned}$$

Construct a second Lyapunov function $V_2(x_1, z_2) = \frac{1}{2}x_1^2 + \frac{1}{2}z_2^2$. Show that

$$\frac{dV_2}{dt} = x_1(z_2 - \mu x_1) + z_2(v_2 - (\mu + \frac{\partial v_1}{\partial x_1})(z_2 - \mu x_1).$$

By picking

$$v_2 - (\mu + \frac{\partial v_1}{\partial x_1})(z_2 - \mu x_1) = -x_1 - \mu z_2,$$

show that

$$\frac{dV_2}{dt} = -\mu(x_1^2 + z_2^2).$$

Show that $(x_1 = 0, z_2 = 0)$ is a stable equilibrium of the closed loop second-order system with control input

$$v_2 = (\mu + \frac{\partial v_1}{\partial x_1})(z_2 - \mu x_1) - x_1 - \mu z_2.$$

- Now consider the original third order non-linear system; set $z_3 := x_3 - v_2$. Show the nonlinear system can be recast as:

$$\begin{aligned}\dot{x}_1 &= z_2 - \mu x_1, \\ \dot{z}_2 &= z_3 - x_1 - \mu z_2, \\ \dot{z}_3 &= u - \frac{\partial v_2}{\partial x_1}(z_2 - \mu x_1) - \frac{\partial v_2}{\partial z_2}(z_2 - x_1 - \mu z_2).\end{aligned}$$

Consider a Liapunov function

$$V(x_1, z_2, z_3) = \frac{1}{2}(x_1^2 + z_2^2 + z_3^2).$$

How would you choose u so that $\frac{dV}{dt}$ is negative definite? Show that $x_1, x_2, x_3 \to 0$ asymptotically.