Texas A&M University
Department of Mechanical Engineering

MEEN 655: Design of Nonlinear Control Systems
Final Exam (Take-home part)

April 19, 2022

Directions:

- There are three problems in this exam with one bonus sub-problem for one of them.
- Please make sure that your answers are clearly explained; lack of clarity or legibility can result in points being deducted.
- Aggie Honor code in place. You are not allowed to consult with any other student in the class; university's policy on cheating will be *strictly* enforced.
- Good Luck.

1. Consider a nonlinear system:

$$\begin{aligned}\dot{x}_1 &= x_2 - x_1^3 + x_1 x_2^2 - 2x_2 x_3,\\ \dot{x}_2 &= x_1^3 + x_1 x_3,\\ \dot{x}_3 &= -5x_3 + u.\end{aligned}$$

Consider a *non-linear* control law of the form

$$u = \alpha x_1^2 + \beta x_1 x_2 + \gamma x_2^2.$$

Determine the conditions on α, β, γ for the equilibrium $(x_1, x_2, x_3) = (0, 0, 0)$ to be asymptotically stable.

We will explore a solution to this problem using the center manifold approach using Poincare normal form in this problem. The crux of this approach is to identify (i) a reduced order system, and (ii) transform it to the canonical associated with the Jordan form for this system. Transformation to the canonical form will simplify the stability analysis.

Please follow the steps given below:

- Show that the Jacobi Linearization of the above (both open and closed loop) nonlinear system about the equilibrium $(0, 0, 0)$ is:

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{pmatrix} = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & -5\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}$$

- Can you infer the stability of the equilibrium $(0, 0, 0)$? What are the eigenvalues of the Jacobian??
- Show that if $x_1(0) = 0, x_2(0) = 0$, the closed loop system will evolve in such a way that $x_1(t) = 0, x_2(t) = 0$; moreover, $x_3(t) \to 0$ as $t \to \infty$. The manifold, $\mathcal{M}_s := \{(0, 0, x_3), x_3 \in \Re\}$ is referred to as the stable manifold.
- Essentially, if $x_1 = 0, x_2 = 0$, then $x_3 \to 0$ asymptotically. What happens if $x_1(0), x_2(0) \neq 0$? Depending on how $x_1(t), x_2(t)$ evolve $x_3(t)$ will correspondingly evolve. The center manifold theorem asserts the existence of an invariant "center" manifold (surface) of the form $x_3 - h(x_1, x_2) = 0$ that is tangential to the subspace spanned by the eigenvectors corresponding to eigenvalues of the Jacobian with "zero" real part, i.e., $h(0, 0) = 0$, $\frac{\partial h}{\partial x_i}(0, 0) = 0$, $i = 1, 2$.

  In essence, locally, we can express

$$h(x_1, x_2) = h_2(x_1, x_2) + h_3(x_1, x_2) + \ldots,$$

where $h_k(x_1, x_2)$ is a k^{th} degree homogeneous polynomial. We will seek to determine a polynomial approximation to this manifold by first trying to determine $h_2(x_1, x_2)$. Note that the general form for h_2 is:

$$h_2(x_1, x_2) = Ax_1^2 + Bx_1x_2 + Cx_2^2.$$

For the manifold $x_3 - h(x_1, x_2) = 0$ to be invariant, show that $\dot{x}_3 - \frac{dh}{dt} = 0$, i.e.,

$$-5h(x_1, x_2) + \alpha x_1^2 + \beta x_1x_2 + \gamma x_2^2 = \frac{\partial h}{\partial x_1}(x_2 - x_1^3 + x_1x_2^2 + 2x_2h(x_1, x_2)) + \frac{\partial h}{\partial x_2}(x_1^3 + x_1h(x_1, x_2)).$$

- For the quadratic approximation for $h(x_1, x_2)$, compare the coefficients of quadratic monomials and show that the following equations must hold:

  $$\alpha - 5A = 0, \quad \beta - 5B = 2A, \quad \gamma - 5C = B.$$

  In other words,

  $$A = \frac{1}{5}\alpha, \quad B = \frac{1}{5}\beta - \frac{2}{25}\alpha, \quad C = \frac{1}{5}\gamma - \frac{1}{25}\beta + \frac{2}{125}\alpha.$$

  Hence, a quadratic approximation of the center manifold is $x_3 = Ax_1^2 + Bx_1x_2 + Cx_2^2$, where A, B, C are computed above.

- On the center manifold, show that the dynamics of the system is given by:

  $$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{J_2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} -x_1^3 + x_1x_2^2 - 2x_2(Ax_1^2 + Bx_1x_2 + Cx_2^2), \\ x_1^3 + x_1(Ax_1^2 + Bx_1x_2 + Cx_2^2) \end{pmatrix}}_{f_3(x_1, x_2)}$$

  The center manifold theorem asserts that the stability of equilibrium $(x_1 = 0, x_2 = 0)$ of the above reduced system implies the stability of the equilibrium $(x_1 = 0, x_2 = 0, x_3 = 0)$ of the original nonlinear system. (How does this compare to Problem 1 from Homework 6?)

- Henceforth, we may focus on finding the conditions of stability for this nonlinear system by transforming it to the following canonical form:

  $$\dot{y}_1 = y_2, \quad \dot{y}_2 = -\lambda y_1^3 + \mu y_1^2 y_2 + O_4(y_1, y_2),$$

  where $O_4(y_1, y_2)$ are terms of quartic or higher order. Note that this canonical form shares the same Jordan structure for the linear part, but the nonlinear part is signficantly simplified, i.e., a lot of cubic terms have been cancelled out. An alternate way to represent the same canonical form:

  $$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = J_2 \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ -\lambda y_1^3 + \mu y_1^2 y_2 \end{pmatrix}}_{g_3(y_1, y_2)}$$

Suppose $\lambda > 0$, consider the Liapunov function candidate

$$V(y_1, y_2) = \frac{1}{4}\lambda y_1^4 + \frac{1}{2}y_2^2.$$

What conditions on μ ascertain stability of the equilibrium $(0,0)$?

- From hereon, we will focus on transforming the system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -x_1^3 + x_1x_2^2 - 2x_2(Ax_1^2 + Bx_1x_2 + Cx_2^2), \\ x_1^3 + x_1(Ax_1^2 + Bx_1x_2 + Cx_2^2) \end{pmatrix}$$

to

$$\dot{y}_1 = y_2, \quad \dot{y}_2 = -\lambda y_1^3 + \mu y_1^2 y_2 + O_4(y_1, y_2),$$

with an appropriate coordinate transformation. This seems reasonable because the order of the terms is odd and at most 3. In particular, we have

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{J_2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -x_1^3 - 2Ax_1^2x_2 - (2B-1)x_1x_2^2 - 2Cx_2^3 \\ x_1^3(1+A) + Bx_1^2x_2 + Cx_1x_2^2 \end{pmatrix}$$

We will consider near-identity coordinate transformations of the form

$$\underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{x} = T(y_1, y_2) = \underbrace{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}}_{y} + \underbrace{\begin{pmatrix} h_{31}(y_1, y_2) \\ h_{32}(y_1, y_2) \end{pmatrix}}_{h_3(y)}.$$

Note that we seek two homogeneous polynomials $h_{31}(y_1, y_2)$ and $h_{32}(y_1, y_2)$ of degree 3.

- Show that *any* $h_3(y)$ can be expressed as a linear combination of the following *eight* basis vectors:

$$\phi_{30}^1 = \begin{pmatrix} y_1^3 \\ 0 \end{pmatrix}, \quad \phi_{21}^1 = \begin{pmatrix} y_1^2 y_2 \\ 0 \end{pmatrix}, \quad \phi_{12}^1 = \begin{pmatrix} y_1 y_2^2 \\ 0 \end{pmatrix}, \quad \phi_{03}^1 = \begin{pmatrix} y_2^3 \\ 0 \end{pmatrix},$$

and

$$\phi_{30}^2 = \begin{pmatrix} 0 \\ y_1^3 \end{pmatrix}, \quad \phi_{21}^2 = \begin{pmatrix} 0 \\ y_1^2 y_2 \end{pmatrix}, \quad \phi_{12}^2 = \begin{pmatrix} 0 \\ y_1 y_2^2 \end{pmatrix}, \quad \phi_{03}^2 = \begin{pmatrix} 0 \\ y_2^3 \end{pmatrix}.$$

- Suppose we want to transform a nonlinear system

$$\dot{x} = J_2 x + f_3(x),$$

to

$$\dot{y} = J_2 y + g_3(y) + O_4(y)$$

using a transformation $x = T(y) = y + h_3(y)$; show that h_3 must satisfy the following condition:

$$J_2(y + h_3(y)) + f_3(y + h_3(y)) = (I + \frac{\partial h_3}{\partial y})(J_2 y + g_3(y) + O_4(y)).$$

Show that the match up to cubic terms is exact if

$$L_{J_2}(h_3) := [J_2 y, h_3(y)] = \frac{\partial h_3}{\partial y} J_2 y - J_2 h_3(y) = f_3(y) - g_3(y) + O_4(y).$$

(From hereon, we will simply use $L_J(\phi) = [J_2 y, \phi(y)]$ and drop the subscript 2 from L_{J_2})

- With J_2 being a constant, the operator $L_J(\phi)$ is linear in ϕ. We have previously identified the basis vectors for all cubic homogeneous polynomial vectors. It is sufficient to understand how the basis vectors are mapped by a linear operator. Show that

$$\begin{aligned}
L_J(\phi^1_{30}) &= 3\phi^1_{21}, \\
L_J(\phi^1_{21}) &= 2\phi^1_{12}, \\
L_J(\phi^1_{12}) &= \phi^1_{03}, \\
L_J(\phi^1_{03}) &= \mathbf{0}, \\
L_J(\phi^2_{30}) &= -\phi^1_{30} + 3\phi^2_{21}, \\
L_J(\phi^2_{21}) &= -\phi^1_{21} + 2\phi^2_{12}, \\
L_J(\phi^2_{12}) &= -\phi^1_{12} + \phi^2_{03}, \\
L_J(\phi^2_{03}) &= -\phi^1_{03}.
\end{aligned}$$

- Show that

$$\begin{aligned}
L_J([\ \phi^1_{30}\ \ \phi^1_{21}\ \ \phi^1_{12}\ \ \phi^1_{03}\ \ \phi^2_{30}\ \ \phi^2_{21}\ \ \phi^2_{12}\ \ \phi^2_{03}\ ]) &= \\
[\ \phi^1_{30}\ \ \phi^1_{21}\ \ \phi^1_{12}\ \ \phi^1_{03}\ \ \phi^2_{30}\ \ \phi^2_{21}\ \ \phi^2_{12}\ \ \phi^2_{03}\ ]\, Q,
\end{aligned}$$

where

$$Q := \begin{pmatrix}
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}$$

Show that the rank of Q is 6 and the range of Q is spanned by the columns of

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Essentially, the basis vectors for the range space of L_J will be

$$\{\phi^1_{21}, \phi^1_{12}, \phi^1_{03}, \phi^2_{12}, \phi^2_{03}, \phi^1_{30} - 3\phi^2_{21}\}$$

- Show that the following equations have a solution:

$$\begin{aligned}
L_J(h_{21}) &= \phi^1_{21} \implies h_{21} = \frac{1}{3}\phi^1_{30}. \\
L_J(h_{12}) &= \phi^1_{12} \implies h_{12} = \frac{1}{2}\phi^1_{21}. \\
L_J(h_{03}) &= \phi^1_{03} \implies h_{03} = \phi^1_{12}. \\
L_J(g_{12}) &= \phi^1_{12} \implies g_{12} = \frac{1}{6}\phi^1_{30} + \frac{1}{2}\phi^2_{21}. \\
L_J(g_{03}) &= \phi^2_{03} \implies g_{03} = \frac{1}{2}\phi^1_{21} + \phi^2_{12}, \\
L_J(g_d) &= \phi^1_{30} - 3\phi^2_{21} \implies g_d = -\phi^2_{30}.
\end{aligned}$$

Also note:

$$L_J(\phi^1_{03}) = 0, \quad L_J(\phi^1_{12} + \phi^2_{03}) = 0.$$

This essentially implies that not all the terms can be cancelled in $f_3(x_1, x_2)$. The terms that do not get cancelled define the canonical form. From the solutions above, the terms that can get cancelled are $\phi^1_{30}, \phi^1_{21}, \phi^1_{12}, \phi^1_{03}, .\phi^2_{12}, \phi^2_{03}$; the other two terms cannot be cancelled, i.e., ϕ^2_{30} and ϕ^2_{21}; hence, the canonical form has only these two terms.

- Suppose

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\frac{x_1^3}{6}\begin{pmatrix} p_{30} \\ q_{30} \end{pmatrix} + \frac{x_1^2 x_2}{2}\begin{pmatrix} p_{21} \\ q_{21} \end{pmatrix} + \frac{x_1 x_2^2}{2}\begin{pmatrix} p_{12} \\ q_{12} \end{pmatrix} + \frac{x_2^3}{6}\begin{pmatrix} p_{03} \\ q_{03} \end{pmatrix}}_{F_3(x_1,x_2)} + O_4(x_1, x_2).$$

Show that

$$F_3(x_1, x_2) = \frac{p_{30}}{6}(\phi^1_{30} - 3\phi^2_{21}) + \frac{p_{12}}{2}\phi^1_{21} + \frac{p_{12}}{2}\phi^1_{12} + \frac{p_{03}}{6}\phi^1_{03} + \frac{q_{12}}{2}\phi^2_{12} + \frac{q_{03}}{6}\phi^2_{03} + \frac{q_{30}}{6}\phi^2_{30} + \frac{(q_{21} + p_{30})}{2}\phi^2_{21}.$$

The term h_3 in the transformation $x = y + h_3(y)$ that tries to eliminate six cubic terms must solve for

$$L_J(h_3(y)) = \frac{p_{30}}{6}(\phi^1_{30} - 3\phi^2_{21}) + \frac{p_{12}}{2}\phi^1_{21} + \frac{p_{12}}{2}\phi^1_{12} + \frac{p_{03}}{6}\phi^1_{03} + \frac{q_{12}}{2}\phi^2_{12} + \frac{q_{03}}{6}\phi^2_{03}$$

Show that

$$h_3(y) \quad = \quad \frac{p_{30}}{6}g_d + \frac{p_{21}}{2}h_{21} + \frac{p_{12}}{2}h_{12} + \frac{p_{03}}{6}h_{03} + \frac{q_{12}}{2}g_{12} + \frac{q_{03}}{6}g_{03}.$$

- Show that this choice of $h_3(y)$ along with the transformation $x = y + h_3(y)$ yields:

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \frac{y_1^3}{6}\begin{pmatrix} 0 \\ q_{30} \end{pmatrix} + \frac{y_1^2 y_2}{2}\begin{pmatrix} 0 \\ q_{21} + p_{30} \end{pmatrix} + O_4(y_1, y_2).$$

- Show that it is possible to cancel all cubic nonlinearities other than ϕ^2_{30}, ϕ^2_{21}; show that the resulting nonlinear system takes the form

$$\dot{y}_1 = y_2, \quad \dot{y}_2 = \lambda y_1^3 + \mu y_1^2 y_2 + O_4(y_1, y_2).$$

What are λ, μ?

- Show that if

$$\alpha < -5, \quad \beta - \frac{2}{5}\alpha < 0,$$

the controller

$$u = \alpha x_1^2 + \beta x_1 x_2 + \gamma x_2^2$$

is stabilizing. As an example, $u = -10x_1^2 - 6x_1x_2$ stabilizes the nonlinear system.

2. We will explore the same problem as before, but will use sum-of-squares approach. To make matters simple, we will only consider the reduced order system:

$$\begin{aligned}\dot{x}_1 &= x_2 - x_1^3 + x_1 x_2^2 - 2x_2(Ax_1^2 + Bx_1x_2 + Cx_2^2)\\ \dot{x}_2 &= x_1^3 + x_1(Ax_1^2 + Bx_1x_2 + Cx_2^2).\end{aligned}$$

Note the asymptotic stability of its equilibrium $(0,0)$ is equivalent to the asymptotic stability of equilibrium $(0,0,0)$ of the original closed loop system:

$$\begin{aligned}\dot{x}_1 &= x_2 - x_1^3 + x_1 x_2^2 - 2x_2x_3,\\ \dot{x}_2 &= x_1^2 + x_1x_3,\\ \dot{x}_3 &= -5x_3 + \underbrace{\alpha x_1^2 + \beta x_1 x_2 + \gamma x_2^2}_{u}.\end{aligned}$$

- Suppose $W_{k,l} = x_1^k x_2^l$. Show that

$$\begin{aligned}\frac{dW_{k,l}}{dt} &= kx_1^{k-1}x_2^l(x_2 - x_1^3 + x_1x_2^2 - 2x_2(Ax_1^2 + Bx_1x_2 + Cx_2^2)\\ &\quad + lx_1^k x_2^{l-1}(x_1^3 + x_1(Ax_1^2 + Bx_1x_2 + Cx_2^2)\\ &= k[W_{k-1,l+1} - W_{k+2,l} - 2AW_{k+1,l+1} - (2B-1)W_{k,l+2} - 2CW_{k-1,l+3}]\\ &\quad + l[W_{k+3,l-1}(1+A) + BW_{k+2,l} + CW_{k+1,l+1}]\end{aligned}$$

- Let us explore quartic functions that could potentially serve as Lyapunov functions. Show that a general homogeneous quartic, cubic and quadratic functions are given by:

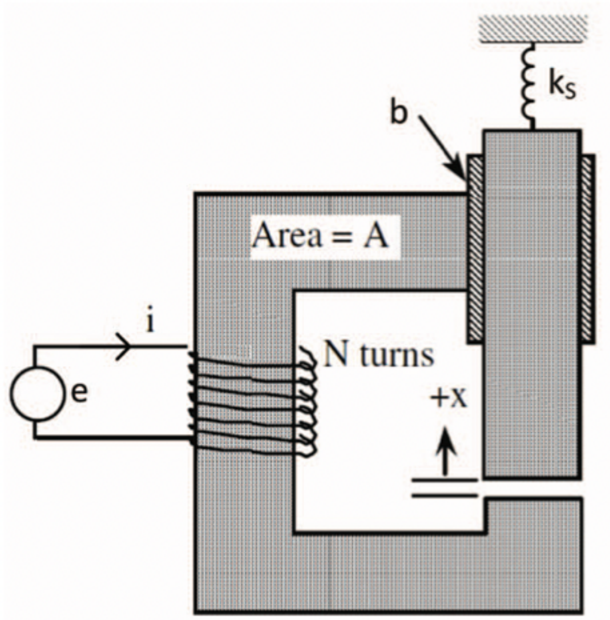
$$\begin{aligned}V_4(x_1,x_2) &= p_{40}x_1^4 + 2p_{31}x_1^3x_2 + p_{22}x_1^2x_2^2 + 2p_{13}x_1x_2^3 + p_{04}x_2^4\\ V_3(x_1,x_2) &= 2p_{30}x_1^3 + 2p_{21}x_1^2x_2 + 2p_{12}x_1x_2^2 + 2p_{03}x_2^3,\\ V_2(x_1,x_2) &= p_{20}x_1^2 + 2p_{11}x_1x_2 + p_{02}x_2^2.\end{aligned}$$

  What LMI condition guarantees that $V = V_4 + V_3 + V_2$ is positive definite on $x_1^2 + x_2^2 \leq \epsilon$ for a specified ϵ?

- What LMI condition guarantees that $\dot{V}$ is negative definite on $x_1^2 + x_2^2 \leq \epsilon$?

- **Bonus:** Use SOSTOOLS or CPLEX or JuMP/Julia or ORTOOLS to solve for feasibility of LMI conditions above.

3. Consider an axial reluctance actuator [1]shown below:



A simple dynamical model for this actuator is given below:

$$
\begin{aligned}
e &= iR + \frac{L_0}{(x_0+b-x)}\frac{di}{dt} + iv\frac{L_0}{(x_0+b-x)^2}, \\
v &= \frac{dx}{dt} \\
F_{mag} = \frac{L_0 i^2}{2(x_0+b-x)^2} &= m_p\frac{dv}{dt} + \mu_p v + k_s x - m_p g
\end{aligned}
$$

The output to be controlled is $y = x$. You may assume the following parameters:

$$m_p = 0.125kg,\ \mu_p == 25Ns/m,\ k_s = 7.5N/m,\ x_0 + b = 0.2m,\ R = 5\ \Omega,\ L_0 = 1H.$$

Suppose all the parameters are known, and you want to design a controller that regulates the output to the desired trajectory $x_{des}(t) = 0.1 + 0.02\cos(2\pi t)$. For now, you can assume that you have access to every state.

- Find a Jacobi linearization of the nonlinear system and design a linear state feedback controller to accomplish the desired objective. How well does this linear controller perform on the nonlinear system in terms of accuracy in tracking & control effort?

[1]for details, see R. K. Govindarajan and G. Venkataramanan, "Servo control of solenoid actuators using augmented feedback linearization," 2017 IEEE 18th Workshop on Control and Modeling for Power Electronics (COMPEL), 2017, pp. 1-6, doi: 10.1109/COMPEL.2017.8013284.

- Find a feedback linearizing controller for the nonlinear system. How well does this controller perform for the same nonlinear system in terms of accuracy in tracking & control effort?
- Suppose you neglect the dynamics of electrical part of the system, i.e., neglect L_0 for designing the controller, and set $i = e/R$ for designing the controller. How well does this controller work in terms of accuracy in tracking & control effort?