IDA PAPER P-2249

HETEROGENEOUS POINT FIRE AND AREA FIRE ATTRITION PROCESSES THAT EXPLICITLY CONSIDER VARIOUS TYPES OF MUNITIONS AND LEVELS OF COORDINATION

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October 1989

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PREFACE

This paper was prepared under IDA contract MDA 903 89 C 0003, Task Order T-I6-682, Net Assessment Methodologies and Critical Data Elements for Strategic and Theater Force Comparisons, for the Capabilities Assessment Division of the Force Structure, Resource, and Assessment Directorate (J-8) of the Joint Chiefs of Staff, and has been written in partial fulfillment of the above Task Order.

This paper describes attrition structures that (1) consider both area and point fire, (2) consider various levels of coordination among shooters, (3) allow explicit consideration of the use of multiple types of munitions, (4) limit the maximum density of targets for area fire, and (5) allow meaningful allocations of fire for point fire. The precise forms of the resulting attrition equations are given.

The author is grateful to Dr. Peter S. Brooks, Dr. Frederic A. Miercort, and Ms. Eleanor L. Schwartz for their quite helpful reviews of this paper. Mrs. Marcia Kostelnick also contributed her valuable time and efforts to preparing the typed manuscript.

ABSTRACT

This paper describes attrition structures that (1) consider both area fire and point fire, (2) consider various levels of coordination among the shooters for both area fire and point fire, (3) allow the explicit consideration of the use of various types of munitions by various types of shooters against various types of targets for both area fire and point fire, (4) prevent unreasonably high numbers of kills by assuming a maximum density of targets in the target area for area fire, and (5) allow meaningful allocations of fire by various types of shooters using various types of munitions against various types of targets for point fire. For each type of fire (area or point) and for each relevant level of coordination except one (shoot-look-shoot fire), the precise form of the corresponding attrition equation is given. A companion paper discusses some details concerning shoot-look-shoot fire.

CONTENTS

A. INTRODUCTION

1. Purpose

This paper describes unilateral attrition structures that (1) consider both area fire and point fire, (2) consider various levels of coordination among the shooters for both area fire and point fire, (3) allow the explicit consideration of the use of various types of munitions by various types of shooters against various types of targets for both area fire and point fire, (4) prevent unreasonably high numbers of kills by assuming a maximum density of targets in the target area for area fire, and (5) allow meaningful allocations of fire by various types of shooters using various types of munitions against various types of targets for point fire. A general method to convert this unilateral attrition into bilateral attrition is also discussed.

For each type of fire (area or point) and for each relevant level of coordination (except for shoot-look-shoot fire), this paper gives the precise form of the corresponding attrition equation. (Formulas concerning shoot-look-shoot fire are discussed in Reference [14].) This paper does not, however, give a formal statement of specific assumptions for these attrition equations to hold, nor does it give rigorous proofs that these equations follow from such assumptions. Thus, the purpose of this paper is to describe and document these equations sufficiently well so that (1) they can be compared and contrasted with each other and better understood in their own right, (2) they can be used in models, and (3) future research either can determine specific assumptions needed for these equations to hold and rigorously derive these equations from such assumptions (if possible), or can determine why these equations do not follow from relevant sets of assumptions (otherwise). To assist such future research, overall descriptions and relatively extensive plausibility arguments are given for the specific forms of these equations.

All of the attrition equations discussed here are difference equations that allow the time interval to be sufficiently long that multiple kills can occur within that interval. This is the type of attrition equation most commonly used in large-scale deterministic conflict models. See Section A of Chapter V of Reference [8] for a more thorough discussion of this topic.

2. Background

Research has been done on all of the topics discussed in this paper, but those results were neither as general nor as integrated as the results reported here.

A seminal (perhaps *the* seminal) paper in this area is Reference [1]. That paper addresses a number of issues concerning unilateral point and area fire attrition processes, but it also leaves a number of issues unanswered. For example, while it explicitly considers one-on-one probabilities of detection for point fire, it does not provide a tractable method for computing attrition when these probabilities of detection depend on both the types of shooting weapons and the types of targets involved. Also concerning point fire, it does not consider meaningful allocations of fire, it does not explicitly consider munitions, and it considers only two levels of coordination among shooters. Concerning area fire, it does not consider multiple types of either shooters ot targets, it does not explicitly consider munitions, and it considers only uncoordinated fire.

Reference [2] discusses computationally tractable approximations for the case where the one-on-one probabilities of detection depend on both the type of shooter and the type of target for point fire. Reference [3] discusses how to incorporate meaningful allocations of fire into the results of Reference [2], and it discusses how to convert multiple unilateral attrition assessments into bilateral attrition (see also Reference [4]). However, Reference [3] considers only uncoordinated point fire; it does not consider area fire and it does not explicitly consider munitions. Reference [5] considers uniformly coordinated point fire, but in a purely homogeneous setting; also, it does not consider area fire and does not consider munitions. Reference [6] considers heterogeneous area fire, but it (implicitly) assumes that there is no coordination among the shooters and it does not consider munitions. (Heterogeneity here means that multiple types of weapons on the side in question can be distinctly simulated; homogeneity means that only one, perhaps notional, type of weapon can be simulated on the given side.) Reference [7] considers an area fire structure with two types of shooters, two types of targets, and (essentially) two levels of coordination among the shooters. However, Reference [7] assumes a "zero-or-one cookie- cutter" type of attrition, it docs not consider multiple types of munitions for shooting weapons, and it does not adjust the area over which targets are located if (due, say, to other actions in the model) the number of targets is increased.

Reference [8] considers four levels of coordination in a consistent manner within a heterogeneous point-fire structure. Reference [8] achieves this consistency, maintains relative simplicity, and produces results which appear (in some cases) to rigorously follow from certain sets of assumptions by assuming (implicitly) that the one-on-one probability of detection is unity for all types of shooters and targets. It is frequently quite appropriate to assume that these one-on-one probabilities of detection are all unity in point fire for the following reasons. First, precise data for these probabilities are essentially impossible to obtain, even generally relevant data frequently do not exist. Second, just making the assumption that these probabilities are not very small frequently is essentially equivalent to making the assumption that these probabilities are unity. For example, suppose it is assumed that the probability that a particular shooter detects a particular target is at least 0.1, and suppose that there are 50 or more targets present. Then the probability that a shooter detects one or more targets (and so is able to fire at some target) is at least 0.995. Thus, in this example, if there are 50 or more targets present, there is essentially no difference between using a one-on-one probability of detection of 0.1 and using a one-on- one probability of detection of 1.0. Third, using one-on-one probabilities of detection of 1.0 greatly simplifies a significant number of cases. Fourth, engagement rates can be set directly through inputs to the point-fire attrition structures described below, so there is no need for non-unity one-on-one probabilities of detection just to obtain non-unity engagement rates. Finally, engagement rates can be adjusted indirectly through the use of false targets in the point-fire attrition structures described below, so there is no need for non-unity one-on-one probabilities of detection just to lower the engagement rates when facing small numbers of targets.

The results presented below draw upon and extend the results reported in these references.

3. Organization

Section B, below, briefly describes the levels of coordination considered in this paper, and it constructs a taxonomy for attrition equations based on these levels of coordination, on whether point fire or area fire is being addressed, and on whether multiple types of munitions are being explicitly considered. Section C introduces some notation needed for the remainder of the paper. Section D discusses point fire attrition structures (with and without the explicit consideration of munitions), and Section E does the same for area fire. Sections D and E both consider heterogeneous shooters versus heterogeneous targets. For ease of comparison, the appendix states some corresponding equations for homogeneous shooters versus homogeneous targets.

Sections D and E describe distinct attrition processes in that each formula in each of those sections assumes that all of the shooters operate at the same level of coordination and use the same type of fire (point or area). Section F considers cases in which some shooters operate at one level of coordination while other shooters operate at a different level of coordination, and some shooters use point fire while others use area fire. Section G discusses how to use the unilateral results of Sections D, E, and F to obtain bilateral attrition. Section G is based on Section B of Reference [3] and Section V.C of Reference [8]; it is included here for completeness only. Section H presents some ideas for implementation and future research.

B. A TAXONOMY FOR ATTRITION EQUATIONS

The taxonomy described here is based on three characteristics: (1) whether point fire or area fire is being used, (2) whether or not munitions are being explicitly considered, and (3) the level of coordination that is being considered.

The distinction between point fire and area fire here follows an intuitive structure. If a weapon is firing point fire, then it must be engaging a particular target Weapons may be able to make more than one engagement per time period. However, on any one of its engagements, a shooting weapon can engage and be able to kill only one target, where the probability of killing that target can depend on the type of shooter, type of target, and (if munitions are being addressed) the type of munition involved. Since, in general, shooters are capable of engaging any one of several types of targets, some allocation-of-fire rule is needed to determine how to allocate their engagements over the various types of targets. Further, since several shooters might be able to engage any of several targets, some level-of-coordination rule is also needed to determine the degree of coordination among these shooters.

Conversely, if a weapon is firing area fire, then it is firing into a general area, but not at a particular target. Weapons may be able to fire more than once per time period. Each time a weapon fires into the general area, all of the targets (if any) within the appropriate lethal area of that salvo can be killed, where the size of that lethal area and the probability of kill can depend on the type of shooter, type of targets, and (if munitions are being addressed) type of munition involved. Since, in area fire, shooters are not engaging particular targets, allocation of fire rules are not needed if the targets are all located in the same general area. However, some level-of-coordination rule is needed to determine the degree of coordination among the various shooters that are firing into the general area in question.

The distinction between whether or not munitions are being explicitly considered is, in a sense, purely formal. That is, if the number of types of munitions being addressed is set equal to one, then each of the formulas presented below in which multiple types of munitions are addressed reduces to the corresponding formula in which munitions are not explicitly considered. The reasons for giving both sets of formulas (with and without munitions) are: (1) to facilitate comparisons with previous work that does not consider multiple types of munitions and (2) to facilitate the incorporation of the work presented here into models that do not consider multiple types of munitions.

Several different levels of coordination are considered in this taxonomy. The first and lowest level considered is no coordination at all. For point fire, each shooter for each of its possible engagements selects a target to attack independent of its selection for any other of its engagements and independent of the selections of all other shooters in all of their engagements. For area fire, each shooter for each of its possible salvos selects a target point in the general target area independent of its selection of target points for any other of its salvos and independent of the selections of target points by all other shooters for all of their salvos.

The second level of coordination considered here applies only to point fire.

This level of coordination is to preallocate shooters in that each shooter on each of its possible engagements is assigned to attack one and only one type of target, but there is no coordination among the shooters other than this.

For example, suppose that there are two types of shooters, three shooters of each type, two types of targets, four targets of the first type and two of the second type, that each shooter can make one (point-fire) engagement, and that multiple types of munitions are not being considered. Suppose that the allocation of fire for this case is to be that, on average, two-thirds of each type of shooter will fire at the first type of target and one-third of the second type of target. Then this average can be achieved in the completely uncoordinated case if each shooter selects one of the six targets to attack according to a uniform distribution independently of the selection of the other shooters. This means that with probability $(2/3)^6$ no shooters will attack targets of type two. Indeed, with the very small (but non-zero) probability of $(1/6)^5$, all of the shooters will attack the same target, leaving all of the other five targets unengaged. However, with preallocated fire in this example, exactly two shooters of each type are assigned to attack some target of type one, and exactly one shooter of each type is assigned to attack some target of type two. Since no coordination is assumed beyond this assignment, this means that with probability 1/4 both of the shooters of type one that are assigned to attack a target of type one will attack the same target.

The third level of coordination considered here applies to both point and area fire. If multiple types of munitions are not being considered, this level of coordination assumes the following. For point fire, each shooter of each type coordinates with all other shooters of that type to allocate their fire in a uniform manner over the targets of the type they are assigned to attack, but no shooters of different types can coordinate their fire. For area fire, each shooter of each type coordinates with all other shooters of that type to allocate their fire in a uniform manner over the target area, but no shooters of different types can coordinate their fire. When multiple types of munitions are addressed, there are two possible levels of coordination here for both point and area fire. In the first of these levels, weapons of the same type only coordinate when they are using munitions of the same type. In the second of these two levels, weapons of the same type always coordinate no matter what munitions they are using. (Again, there is no coordination among weapons of different types.)

Continuing with the example above, under this coordination-within-weapon-type case for point fire, the two shooters of type one that are assigned to engage targets of type one must engage different targets. The same applies to the two shooters of type two that are assigned to engage these targets. Since there are four targets of type one in this example, it is possible that each of these four shooters will engage a separate target. However, this "perfect distribution" will only occur with probability 1/6 since this level of coordination assumes that there is no coordination among shooters of different types.

The fourth level of coordination considered here also applies to both point and area fire. This level of coordination assumes that all shooters of all types (using any type of munition) coordinate with each other to allocate this fire in some uniform manner over the targets of the type they are assigned to attack (for point fire) or over the target area (for area fire). Applied to the example above, this level of coordination for point fire might allow each of the six shooters (three of type one and three of type two) to shoot at one of the six targets (four of type one and two of type two) in such a way that no two shooters shoot at the same target (and so all of the targets are engaged).

The fifth and highest level of coordination included in this taxonomy applies only to point fire. This level of coordination assumes that shooters can engage targets using some type of shoot-look-shoot firing structure. Theoretically, there are several such structures, some less tractable than others (and with some being quite intractable). None of these structures will be discussed in detail in this paper-shoot-look-shoot fire is included in this taxonomy for completeness only. For more information concerning shoot-look-shoot fire, see References [8] and [14].

The point of this discussion is not to provide a definitive discussion of various levels of coordination in point and area fire. Instead, the point is to provide a setting for the taxonomy listed on Table 1.

The point of Table 1 is to provide motivation and structure for the remainder of this paper. The categorization across the top of Table 1 separates the attrition equations being considered into either point fire or area fire equations. For each, either multiple types of munitions for each type of shooting weapon can be explicitly considered or not. Again, note that each equation that considers multiple types of munitions reduces to the corresponding equation that does not do so if each type of weapon being considered uses only one type of munition. The left side of Table 1 lists the five levels of coordination discussed above.

Coordination level 1 is relatively straightforward. The four cases (point fire and area fire, each without or with explicit consideration of munitions) are denoted by P1, PM1, A1, and AM1 on Table 1.

Coordination level 2 applies only to point fire since it involves allocations of fire in a manner not relevant to area fire. In uncoordinated point fire, if shooters of type one are allocated evenly between targets of type one and all other types of targets, then the probability that any given shooter of type one selects a target of type one to attack is 0.5. For example, if there are two shooters of type one, the probability that no target of type one is attacked by a shooter of type one is 0.25. Conversely, in preallocated fire, if shooters of type one are allocated evenly between targets of type one and all other types of targets, then exactly one-half of the shooters of type one will select targets of type one to attack. For example, if there are two shooters of type one, exact1y one of them will attack a target of type one.

The various possibilities for implementing preallocated fire can be quite complex. For example, even if multiple types of munitions are not considered, several types of preallocation are possible. First, some types of shooters but not others could be preallocated. Second, shooters that are preallocated might be preallocated only against some types of targets but be uncoordinated against other types of targets. Third, if a shooter can make multiple engagements, then that shooter might be able to coordinate its own engagements at a higher level than it could coordinate with other shooters of the same type. Other possibilities may exist. For simplicity, only one type of preallocation is described for the munitions-not-considered case. This preallocation is to preallocate all potential engagements by all shooters of the same type but to assume uncoordinated fire among different types of shooters; this preallocation is denoted by P2 on Table 1. The situation is potentially more complex when multiple types of munitions are considered. Again, for simplicity, only one type of preallocation will be described here. In particular, the direct extension of P2 to the corresponding equation that considers munitions, denoted by PM2 on Table I, will be discussed in Section D, below.

Table 1. A Taxonomy for Attrition Equations

Coordination Assumptions	Point-fire Equations Are Munitions Considered?		Area-fire Equations Are Munitions Considered?	
	No	Yes	No	Yes
1) Uncoordinated Fire	P1	PM1	A1	AM1
2) Preallocated fire	P2	PM2	n/a	n/a
3) *Coordinated Fire* within Shooter Types	P3		A3	
3.1) But only within Munition Types		PM3.1		AM3.1
3.2) And Across all Munition Types		PM3.2		AM3.2
4) Coordinated Fire Across ail Shooter (and Munition) Types				
4.1) Uniform Fire by Numbers of Engagements or Salvos	P4.1	PM4.1	A4	AM4
4.2) Proportional fire by Potential Kills	P4.2	PM4.2	n/a	n/a
5) Shoot-Look-Shoot Fire	P5	PM5	n/a	n/a

The third and fourth levels of coordination on Table 1 concern coordination among shooters. Exactly what is meant by coordination here will be discussed in Section D (for point fire) and Section E (for area fire). However, it should be noted that, given any particular definition of coordination, many structures are still possible. For example, some types of shooters (using some types of munitions) might coordinate with some other types of shooters (using some other types of munitions) but not with yet other types of shooters (using yet other types of munitions). When multiple types of munitions are not being addressed, the bounding cases of considering only coordination among shooters of the same type (denoted by P3 for point fire and by A3 for area fire) and considering coordination among all shooters of all types (two methods of coordination, denoted by P4.1 and P4.2, are considered for point fire; one method, denoted by A4, is considered for area fire) are listed on Table 1. If multiple types of munitions are being addressed, the third level of coordination is subdivided into two sublevels: coordination only among those shooters of the same type when they are using the same type of munition (denoted on Table 1 by PM3.1 for point fire and by AM3.1 for area fire), and coordination among all shooters of the same type no matter what type of munition they are using (denoted by PM3.2 for point fire and by AM3.2 for area fire). Clearly, the case that is analogous to assuming that all shooters of all types coordinate with each other when multiple types of munitions are not being considered is to assume that, when such munitions are considered, then all shooters of all types coordinate with each other no matter what types of munitions they are using. Thus, the consideration of munitions naturally extends P4.1 to PM4.1, P4.2 to PM4.2, and A4 to AM4 in the taxonomy of Table 1.

Of all of the theoretically possible versions of shoot-look-shoot attrition, one approach appears to be relatively promising from a computational viewpoint while still being fully heterogeneous in types of shooters, types of targets, and (optionally) types of munitions. This approach is to preallocate the shooters to types of targets (so that the shooters that are allocated to a given type of target can shoot at and only at targets of that type), and then to assume that perfect shoot-look-shoot attrition applies for targets of each type. This is the approach recommended in Section B.6.d of Chapter V of Reference [8]- see that reference for details. When implemented, this approach would give a point-fire- shoot-look-shoot attrition mechanism for the case in which munitions are not considered, and would give a corresponding mechanism for the case in which multiple types of munitions are considered. The attrition mechanisms for these cases are denoted by P5 and PM5, respectively, on Table 1. There is no area fire equivalent to shoot-look-shoot point fire.

Formulas for shoot-look-shoot attrition will not be given in this paper, Reference [14] discusses the computation of shoot-look-shoot attrition as denoted by P5 and PM5 on Table 1. Formulas for all of the other entries on Table 1 are given below. In order to state these formulas it is necessary to define some relevant notation.

C. NOTATION

1. Notation Common to all Types of Fire

The following notation will be used by all of the attrition structures considered here.

I = the (input) number of types of shooters being considered; $I \in \{1,2,\ldots\}$.

s_i = the (input) number of shooters of type i for $i = 1,\ldots,I$, $s_i \in [0,\infty)$.

J = the (input) number of types of targets being considered; $J \in \{1,2,\ldots\}$.

t_j = the (input) number of targets of type j for $j = 1,\ldots,J$; $t_j \in [0,\infty)$.

v_j = the (input) fraction of targets of type j that are vulnerable to both point fire and area fire for $j = 1,\ldots,J$; $v_j \in [0,1]$.

Δt_j = the (calculated) number of targets of type j that are killed in the attrition process being considered for $j = 1,\ldots,J$.

2. Notation Concerning Point Fire

a. General Point-Fire Notation

The following notation is used in point fire attrition equations whether or not multiple types of munitions are being considered.

z = the (input) number of point-fire combat zones where $1/z$ of the shooters are assumed to be attacking $1/z$ of the targets in each of these z zones; $z \in (0,\infty)$.

u_j = the (input) fraction of targets of type j that are vulnerable to point fire but not to area fire for $j = 1,\ldots,J$; $u_j \in [0,1-v_j]$.

$\bar{t}_j = \dfrac{(u_j + v_j)t_j}{z}$ – the (calculated) number of targets of type j per combat zone that are vulnerable to point fire in the attrition process being considered for $j = 1,\ldots,J$.

e_i – the average number of point-fire engagements that a shooter of type i makes per time period for $i = 1,\ldots,I$; $e_i \in [0,\infty)$.

$\bar{s}_i = \dfrac{e_i s_i}{z}$ – the (calculated) average number of point-fire engagements per combat zone that are made by all shooters of type i during the time period in question for $i = 1,\ldots,I$.

If multiple types of munitions are not being explicitly considered, then ei is an input to the attrition calculation. If multiple types of munitions are being addressed, then ej either can be an input or can be calculated from other inputs to the attrition calculations as described in Section 2.c, below.

b. Point-Fire Notation Without Munitions

The following notation is used in point-fire attrition equations when multiple types of munitions are not being addressed.

p_{ij} – the (input) probability of kill per engagement by a shooter of type i when that shooter is making a point-fire engagement against a target of type j for $i = 1,\ldots,I$, $j = 1,\ldots,J$; $p_{ij} \in [0,1]$.

$a_{ij} = a_{ij}(\bar{t})$ – the average fraction of engagements that shooters of type i make against targets of type j (out of all of the point-fire engagements made by those type-i shooters) when the target force, $\bar{t}$, is $\{\bar{t}_1,\ldots,\bar{t}_J\}$, where $i = 1,\ldots,I$ and $j = 1,\ldots,J$.

Allocations of fire can be computed in many ways. See (Chapters III and IV of [8] for a discussion of a relatively wide variety of methods to compute such allocations. For the purpose of this paper, assume that these allocations are computed by the method described in Section B of (Chapter HI of [8]. (This method is used to determine allocations of fire in IDAGAM, INBATIM, TACWAR, JCS FPM, and IDAPLAN, all of which are dynamic combat models.) Discussions of various aspects of this method can be found in Chapter n of [9], on pages 98 through 100 of [10], on pages 31 and 32 of [11], on pages 53 and 54 of [12] (see also pages 42 and 43 of [12]), and on pages 4 through 8 of [4].) This method uses the following inputs.

t_j^* – the (input) number of targets of type j in a typical target force, where this target force must contain a strictly positive number of targets of each type for $j = 1,\ldots,J$; $t_j^* \in (0,\infty)$.

a_{ij}^* = the (input) fraction of point-fire engagements that shooters of type i would make, on average, against targets of type j (out of all of the point-fire engagements made by shooters of that type) when the target force consists of t_j^* weapons of type j', where $i = 1,\ldots,I$, $j = 1,\ldots,J$, and $j' = 1,\ldots,J$; $a_{ij}^* \in [0,1]$.

The allocations of fire a_{ij} are then calculated by the formula

$$a_{ij} = a_{ij}(\bar{t}) = \begin{cases} \dfrac{a_{ij}^*\bar{t}_j / t_j^*}{\sum_{j'=1}^{J} a_{ij'}^*\bar{t}_{j'} / t_{j'}^*}, & a_{ij}^*\bar{t}_j / t_j^* > 0, \\ 0, \textit{otherwise}. \end{cases}$$

for $i = 1,\ldots,I$ and $j = 1,\ldots,J$. Note that, according to this formula, $\sum_{j=1}^{J} a_{ij} = 1$ whether or not $\sum_{j=1}^{J} a_{ij}^* = 1$.

See the aforementioned references for discussions concerning this method for allocating fire in combat models.

c. Point-Fire Notation With Munitions

The following notation is used in point-fire attrition equations when multiple types of munitions are being addressed.

M = the (input) number of types of munitions being considered; $M \in \{1,2,\ldots\}$.

P_{imj} = the (input) probability of kill per engagement by a shooter of type i when that shooter is making a point-fire engagement using munitions of type m against a target of type j for $i=1,\ldots,I$, $m=1,\ldots,M$, and $j=1,\ldots,J$; $p_{imj} \in [0,1]$.

$a_{ij} = a_{ij}(\bar{t})$ = the average fraction of point-fire engagements that shooters of type i make against targets of type j using any type of munition (out of all of the point-fire engagements made by that type-i shooter) when the target force, $\tilde{t}$ is $\{\tilde{t}_1,\ldots,\tilde{t}_J\}$, where $i=1,\ldots,I$ and $j=1,\ldots,J$; $a_{ij} \in [0,1]$. Note that:

$$\sum_j a_{ij} = 1$$

for all relevant i.

b_{imj} = the average fraction of point-fire engagements by shooters of type i against targets of type j that are made using munitions of type m, where $i=1,\ldots,I$, $m=1,\ldots,M$, and $j=1,\ldots,J$; $b_{imj} \in [0,1]$. Note that;

$$\sum_m b_{imj} = 1$$

for all relevant i and j.

$c_{imj} = c_{imj}(\tilde{t})$ = the average fraction of point-fire engagements by shooters of type i that are made using munitions of type m against targets of type j (out of all of the point-fire engagements made by that type-i shooter) when the target force, $\tilde{t}$ is $\{\tilde{t}_1,\ldots,\tilde{t}_J\}$, where $i=1,\ldots,I$, $m=1,\ldots,M$, and $j=1,\ldots,J$; $c_{imj} \in [0,1]$. Note that:

$$c_{imj} = a_{ij} b_{imj},$$

and so

$$a_{ij} = \sum_m c_{imj},$$

and

$$b_{imj} = \frac{c_{imj}}{\sum_{m'} c_{im'j}},$$

for all relevant i, m, and j.

$\tilde{a}_{imj} = \tilde{a}_{imj}(\tilde{t})$ – average fraction of point-fire engagements by shooters of type i using munitions of type m that are made against targets of type j when the target force, $\tilde{t}$ is $\{\tilde{t}_1,\ldots,\tilde{t}_J\}$, where $i=1,\ldots,I$, $m=1,\ldots,M$, and $j=1,\ldots,J$; $\tilde{a}_{imj} \in [0,1]$. Note that:

$$\tilde{a}_{imj} = \frac{c_{imj}}{\sum_{j'} c_{imj'}},$$

and

$$\sum_j \tilde{a}_{imj} = 1,$$

for all relevant i, m, and j.

$\tilde{b}_{im}$ = the average fraction of point-fire engagements that shooters of type i make using munitions of type m against any target (out of all of the point-fire engagements made by those type-i shooters), where $i=1,\ldots,I$ and $m=1,\ldots,M$; $\tilde{b}_{im} \in [0,1]$. Note that:

$$\tilde{b}_{im} = \sum_j c_{imj},$$

$$\sum_m \tilde{b}_{im} = 1,$$

and

$$\tilde{a}_{imj}\tilde{b}_{im} = a_{ij}b_{imj} = c_{imj}$$

for all relevant i, m, and j.

$\hat{a}_{ij} = \hat{a}_{ij}(\tilde{t})$ – the average number of point-fire engagements that a shooter of type i makes per time period against targets of type j using any type of munition when the target force, $\tilde{t}$ is $\{\tilde{t}_1,\ldots,\tilde{t}_J\}$, where $i=1,\ldots,I$, and $j=1,\ldots,J$; $\hat{a}_{ij} \in [0,\infty)$. Note that:

$$\hat{a}_{ij} = e_i a_{ij},$$

for all relevant, i and j.

$\hat{b}_{im}$ – the average number of point-fire engagements that a shooter of type i makes per time period using munitions of type m against any target, where $i=1,\ldots,I$ and $m=1,\ldots,M$; $\hat{b}_{im} \in [0,\infty)$. Note that:

$$\hat{b}_{im} = e_i\tilde{b}_{im}$$

for all relevant i and m.

$\hat{c}_{imj} = \hat{c}_{imj}(\tilde{t})$ – average number of point-fire engagements that a shooter of type i makes per time period using munitions of type m against targets of type j when the target force, $\tilde{t}$ is $\{\tilde{t}_1,\ldots,\tilde{t}_J\}$, where $i=1,\ldots,I$, $m=1,\ldots,M$, and $j=1,\ldots,J$; $\hat{c}_{imj} \in [0,\infty)$. Note that:

$$\hat{c}_{imj} = e_i c_{imj},$$

and

$$\sum_m \sum_j \widehat{c}_{imj} = e_i ,$$

for all relevant i, m, and j.

The reason for introducing this multiply redundant notation is four-fold. First, of course, some of it is needed in order to express the attrition equations categorized in the taxonomy of Table 1. Second, any particular model could (in general) use any particular form of this notation to express its attrition structure. By presenting the relationships among multiple forms of this notation, the attrition structures of such models can be more easily compared with each other and with the attrition equations described below. For example, Reference [13] defines and uses terms that correspond to $\widehat{b}$ in order to compute attrition due to aircraft on close-air support and interdiction missions. Thus, defining and relating $\widehat{b}$ to the other notation used here simplifies comparing the attrition equations of [13] to the structure presented here. Third, this notation may be useful for suggesting ideas for future research. Finally, input data may be available (or more easily obtained) in particular forms. By presenting the relationships among multiple forms for these data, the form most appropriate for the data in question can be used to determine input values, and other values can then be calculated from these inputs as needed.

In particular, to exercise the attrition mechanisms discussed here, data are needed for exactly one of the following sets of terms. For all relevant i, m, and j, data must be obtained either for

1) e_i, $a_{ij}\left(\tilde{t}\right)$, and b_{imj}, or for
2) e_i, $\tilde{b}_{im}$, and $\tilde{a}_{imj}\left(\tilde{t}\right)$ or for
3) e_i and $c_{imj}\left(\tilde{t}\right)$ or for
4) $\widehat{a}_{ij}\left(\tilde{t}\right)$ and b_{imj}, or for
5) $\widehat{b}_{im}$ and $\tilde{a}_{imj}\left(\tilde{t}\right)$ or for
6) $\widehat{c}_{imj}\left(\tilde{t}\right)$.

Given data values for any one of these six sets of terms, values for all of the other terms can be directly computed as indicated in the formulas above.

3. Notation Concerning Area Fire

a. General Area-Fire Notation

The following notation is used in area-fire attrition equations whether or not multiple types of munitions are being considered.

$\ddot{u}_j$ = the (input) fraction of targets of type j that are vulnerable to area fire but not to point fire for $j = 1,\ldots,J$; $\ddot{u}_j \in \left[0,1 - v_j - u_j\right]$.

$\ddot{z}$ = the (input) number of area-fire combat zones where $1/\ddot{z}$ of the shooters are assumed to be attacking $1/\ddot{z}$ of the targets in each of these $\ddot{z}$ zones; $\ddot{z} \in \left(0,\infty\right)$.

$\ddot{t}_j = \dfrac{(\ddot{u}_j + v_j)t_j}{\ddot{z}}$ – the (calculated) number of targets of type j per combat zone j that are vulnerable to area fire in the attrition process being considered for $j = 1,\ldots,J$;

d_j – the (input) average size of the area needed by the defending side to effectively operate a system (i.e., target) of type j for $j = 1,\ldots,J$; $d_j \in (0,\infty)$. For simplicity, it is assumed that these operating areas are strictly positive and do not overlap.

$h = \sum_j d_j \ddot{t}_j$ – the (calculated) total size of the area per combat zone needed by the defending side to effectively operate all of its systems that are potentially vulnerable to area fire. If desired, a simple extension here would be to assume that the targets always occupy an area that is at least of size h'. Adhere h' is an input, and then h would be calculated by the formula:

$$h = \max\left\{h', \sum_j d_j \ddot{t}_j\right\}.$$

H – the geographical area of size $\ddot{z}h$ in which the targets vulnerable to area fire are located.

G – the geographical area into which the shooters are attacking using area fire.

$g = g'/\ddot{z}$, where g' is the (input) size of G; for simplicity assume $g' > 0$, so $g \in (0,\infty)$.

$\ddot{e}_i$ = the average number of area-fire salvos that a shooter of type i fires per time period for $i = 1,\ldots,I$; $\ddot{e}_i \in [0,\infty)$.

$\ddot{s}_i = \dfrac{\ddot{e}_i s_i}{\ddot{z}}$ – the (calculated) number of area fire salvos that are made by all shooters of type i per combat zone during the time period in question for $i = 1,\ldots,I$.

If multiple types of munitions are not being explicitly considered, $\ddot{e}_i$ is an input to the attrition calculations. If multiple types of munitions are being addressed, then $\ddot{e}_i$ either can be an input or can be calculated as described in Section 3.c, below.

A fundamental assumption made here is that

$G \subset H$ if and only if $g \le h$,

$G = H$ if and only if $g = h$, and

$H \subset G$ if and only if $g \ge h$.

That is, if $g \ge h$ then all of the vulnerable targets are inside of the area being attacked by the shooters, while if $g \le h$ then the area being attacked by the shooters is a subset of the area containing the vulnerable targets.

There are several advantages to this assumption. First, it is reasonable that there be an upper bound on the density of targets, i.e., it is unreasonable to assume that all of the targets are always located in an area of fixed size no matter how many targets there are. In a dynamic model, the number of targets being considered can (in general) vary both up and down over time either as resources are added or moved or as allocations are changed. Thus, even if the ratio of the initial number of targets to the initial target area is within a maximum plausible density, it is appropriate to automatically ensure that this ratio does not exceed such a plausible maximum over time due to the addition or movement of resources or to their reassignments. Second, it is reasonable that there be an upper bound on the number of targets any fixed number of shooters can kill. The assumption here will be used (as described below) to place such an upper bound on the number of targets killed. (For comparison, the simple homogeneous Lanchester linear difference equation

$$\Delta t = \min\{kst, t\}$$

has no such upper bound in that Δt goes to infinity as t goes to infinity for any fixed (nonzero) values of s and k. Accordingly, it is generally unreasonable to use such Lanchester linear equations in dynamic combat models.) Finally, while it would be easy to assume that the smaller of G and H is not necessarily contained in the larger, it might be quite difficult to obtain relevant data concerning such an assumption. Accordingly, for simplicity and for reasonableness in data requirements, it is assumed that either $G \subset H$ or $H \subset G$.
Given this assumption concerning G and H, let

$$\dot{t}_j = \begin{cases} \ddot{t}_j \min\{1, g/h\}, h > 0; \\ 0, h = 0. \end{cases}$$

so that $\dot{t}_j$ is the number of targets of type j per combat zone that are vulnerable to area fire into G for $j = 1, \ldots, J$.

b. Area-Fire Notation Without Munitions

The following notation is used in area-fire attrition equations when multiple types of munitions are not being explicitly considered. Let $\ddot{a}_{ij}$, $\ddot{p}_{ij}$, and $\bar{a}_i$ be defined as follows.

$\ddot{a}_{ij}$, $\ddot{p}_{ij}$: For each j, a salvo by a shooter of type i creates an area of (input) size $\ddot{a}_{ij}$ such that if a target of type j is inside of that area then it is killed with (input) probability $\ddot{p}_{ij}$ (otherwise, it survives that salvo), where $i = 1, \ldots, I$ and $j = 1, \ldots, J$; $\ddot{a}_{ij} \in [0, \infty)$ and $\ddot{p}_{ij} \in [0,1]$.

$\bar{a}_i$ – the (input) size of the area for a salvo by a shooter of type i that is used to coordinate fire for those cases in which those shooters can coordinate their fire; it is reasonable (but not necessary) to set $\bar{a}_i$ so that

$$\min_j \ddot{a}_{ij} \le \bar{a}_i \le \max_j \ddot{a}_{ij};$$

for simplicity, the formulas below require that $\bar{a}_i$ be set so that

$$0 \le \bar{a}_i \le g$$

for $i = 1,\ldots,I$.

c. Area-Fire Notation With Munitions

The following notation is used in area-fire attrition equations when multiple types of munitions are being addressed.

M – the (input) number of types of munitions being considered; $M \in \{1,2,\ldots\}$.

$\ddot{a}_{imj}$, $\ddot{p}_{imj}$: For each j, a salvo by a shooter of type i firing munitions of type m creates an area of (input) size $\ddot{a}_{imj}$ such that if a target of type j is inside of that area then it is killed with (input) probability p_{imj} (otherwise, it survives that salvo), where $i = 1,\ldots,I$, $m = 1,\ldots,M$, and $j = 1,\ldots,J$;

$$\ddot{a}_{imj} \in [0,\infty) \text{ and } \ddot{p}_{imj} \in [0,1].$$

$\bar{a}_{im}$ = the (input) size of the area for a salvo by a shooter of type i firing munitions of type m that is used to coordinate fire for those cases in which those shooters can coordinate their fire; it is reasonable (but not necessary) to set $\bar{a}_{im}$ so that

$$\min_j \ddot{a}_{imj} \le \bar{a}_{im} \le \max_j \ddot{a}_{imj};$$

for simplicity, the formulas below require that $\bar{a}_{im}$ be set so that

$$0 \le \mathrm{a}_{\mathrm{im}} \le \mathrm{g}$$

for $i = 1,\ldots,I$ and $m = 1,\ldots,M$.

$\ddot{b}_{im}$ – the average fraction of area-fire salvos by shooters of type i that are made using munitions of type m, where $i = 1,\ldots,I$ and $m = 1,\ldots,M$; $\ddot{b}_{im} \in [0,1]$. Note that

$$\sum_m \ddot{b}_m = 1$$

for all relevant i .

$\hat{e}_{im}$ – the average number of area-fire salvos that a shooter of type i makes per time period using munitions of type m, where $i = 1,\ldots,I$ and $m = 1,\ldots,M$; $\hat{e}_{im} \in [0,\infty)$.

Either $\hat{e}_{im}$ for all relevant i and m or $\ddot{e}_i$ and $\ddot{b}_{im}$ for all relevant i and m are needed as inputs to these area-fire attrition calculations. If values for $\ddot{e}_i$ and $\ddot{b}_{im}$ are input, then $\hat{e}_{im}$ can be calculated by

$$\hat{e}_{im} = \ddot{e}_i \ddot{b}_{im},$$

If values for $\hat{e}_{im}$ are input, then $\ddot{e}_i$ and $\ddot{b}_{im}$ can be calculated by

$$\ddot{e}_i = \sum_m \hat{e}_{im} ,$$

and

$$\ddot{b}_{im} = \widehat{e}_{im} / \ddot{e}_i \ .$$

4. Notation Concerning Functions

For any non-negative number x, let $\lfloor x \rfloor$ denote the largest integer less than or equal to x and let $\langle x \rangle$ be the fractional part of x so that

$$x = \lfloor x \rfloor + \langle x \rangle .$$

To prevent anomalies in the equations below, it will frequently be necessary to consider the term

$$\min\{1, x\} .$$

Such anomalies could potentially occur, for example, in point fire if (perhaps due to previous attrition) the number of targets of some type is strictly between zero and one, and in area fire if the lethal area of a single shooter, $\ddot{a}_{ij}$, or $\ddot{a}_{imj}$, exceeds the target area h. For simplicity in writing the equations below, let

$$\tilde{l}\,(x) = \min\{1, x\} .$$

D.POINT-FIRE ATTRITION EQUATIONS

1. Point-Fire Equations That do not Consider Munitions

Throughout this section it is assumed that if a shooter of type i attacks a target of type j, then it kills that target (i.e., fires a lethal shot at that target) with probability p_{ij}, and that these killing processes are mutually independent of each other and of all target selection processes. Two or more shooters might attack and fire lethal shots at the same target, which results in only one target being killed.

a. A General Form for Calculating Point-Fire Attrition

Three of the attrition equations presented below can be put into the general form:

$$\Delta t_j = \begin{cases} z\tilde{t}_j \left[1 - \prod_{i=1}^{I} \left(1 - p_{ij}\tilde{l}\left(x_{ij}\right)\right)^{\left\lfloor \tilde{s}_i a_{ij} / x_{ij}\tilde{t}_j \right\rfloor} \left(1 - \left\langle \frac{\tilde{s}_i a_{ij}}{x_{ij}\tilde{t}_j} \right\rangle p_{ij}\tilde{l}\left(x_{ij}\right) \right) \right]_j, \tilde{t}_j > 0, \\ 0, \tilde{t}_j = 0, \end{cases}$$

where the specification of x_{ij} differs for each of the three equations. The reasons for presenting this general form are:

(1) to help relate the equations presented below to each other and to the point-fire equations that explicitly consider munitions in Section 2 below, and

(2) to note that it may be possible to develop and relate other attrition equations to those considered here by specifying x_{ij} in other ways.

It should also be noted that there may be several ways to specify x_{ij} that are structurally consistent (i.e., Δt_j is properly bounded and moves in the correct direction as parameters are changed) but are not appropriate in that the resulting attrition equations do not follow from a consistent set of physically realizable assumptions. Future research might address alternative specifications and develop reasonable interpretations for x_{ij}.

b. Uncoordinated Fire

Briefly, uncoordinated point fire assumes: (1) that each shooter of type i on each of its engagements selects one target (from among those vulnerable) to attack, where the probability that it selects a particular target of type j is $a_{ij}/\tilde{t}_j$, and (2) that the target selection processes are all mutually independent, so that, for example, two or more different shooters can select the same target to attack and can fire lethal shots at that target, which results in only one target being killed. The lack of coordination among shooters is reflected in the mutually independent selection of targets by all shooters on all of their engagements. Sec Section B.3 of

Chapter V of Reference [8] for details concerning these uncoordinated fire assumptions. The assumption of uncoordinated point fire yields

<u>Equation P1</u>:

$$\Delta t_j = \begin{cases} z\tilde{t}_j\left[1-\prod_{i=1}^{I}\left(1-p_{ij}\tilde{l}\left(\frac{a_{ij}}{\tilde{t}_j}\right)\right)^{\lfloor\tilde{s}_i\rfloor}\left(1-\langle\tilde{s}_i\rangle p_{ij}\tilde{l}\left(\frac{a_{ij}}{\tilde{t}_j}\right)\right)\right]_j, \tilde{t}_j > 0, \\ 0, \tilde{t}_j = 0, \end{cases}$$

Note that setting $x_{ij} = a_{ij}/\tilde{t}_j$ in the general form above produces Equation P1. Note also that Equation P1 is identical to the attrition equation assuming uncoordinated fire as defined in B.3 of Chapter V of Reference [8] whenever, for each i, either $\tilde{s}_i$ is an integer or $\tilde{s}_i < 1$. If, for some i, $\tilde{s}_i$ is not an integer and is greater than one, then these equations differ only in how to interpolate results between $\lfloor\tilde{s}_i\rfloor$ and $\lfloor\tilde{s}+1_i\rfloor$. The choice is arbitrary-any reasonable interpolation would do. The advantages of P1 over the corresponding interpolation in Reference [8] are:

(1) the interpolation for $\tilde{s}_i > 1$ suggested here is consistent with the linear interpolation used by both equations for $0 < \tilde{s}_i < 1$,

(2) Equation P1 is consistent with the general form given above, and (3) Equation P1 might be more computationally efficient than the corresponding equation in [8] since P1 takes a real to an integer power while [8] takes a real to a real power.

However, these differences are very minor and only affect cases involving nonintegral numbers of shooters-in essence, these two equations are identical.

c. Preallocated Fire

Briefly, preallocated fire assumes:

(1) that (the fraction) a_{ij} of the $\tilde{s}_i$ engagements by shooters of type i are directed against targets of type j, where a_{ij} is a deterministic quantity,

(2) that each shooter on each of its engagements selects, according to a uniform distribution, one target (from among those of the type it is directed against) to attack, and

(3) that all of die target selection processes are mutually independent. (For comparison, in uncoordinated fire the expected number of engagements by shooters of type i against targets of type j is $\tilde{s}_i a_{ij}$, but this expected number is an average of many possible realizations. Conversely, here it is assumed that, through coordination, *this expected* number of engagements is achieved as close as integer constraints allow.) The assumption of preallocated fire yields

<u>Equation P2:</u>

$$\Delta t_j = \begin{cases} z\tilde{t}_j\left[1-\prod_{i=1}^{I}\left(1-p_{ij}\tilde{l}\left(\frac{1}{\tilde{t}_j}\right)\right)^{\lfloor \tilde{s}_i a_{ij} \rfloor}\left(1-\langle \tilde{s}_i a_{ij}\rangle p_{ij}\tilde{l}\left(\frac{1}{\tilde{t}_j}\right)\right)\right]_j, \tilde{t}_j > 0, \\ 0, \tilde{t}_j = 0, \end{cases}$$

Note that setting $x_{ij} = 1/\tilde{t}_j$ for all i in the general form given in Section a above produces Equation P2. Note also that Equation P2 is identical to Equation 19 of Reference [1] if:

(a) none of the numbers of targets by type (t_j here, R_j in [1]) are less than one,

(b) all of the probabilities of detection (d_{ij}) in [1] are one, and

(c) the number of engagements by each type of shooter assigned to each type of target ($\tilde{s}_i a_{ij}$ here, $B(i, j)$ in [1]) are all integers and

$$\tilde{s}_i a_{ij} = B(i, j),$$

for all i and j. Thus, the only fundamental difference between Equation P2 and Equation (19) of [1] is that (19) allows probabilities of detection to be less than one, while this paper assumes that all such *one-on-one* probabilities of detection are (essentially) one.

d. Coordinated Fire Within Shooter Types, Uncoordinated Fire Across Shooter Types

As in Section c above, this level of coordination also assumes that shooters by type are preallocated to targets by type so that a_{ij} of the $\tilde{s}_i$ engagements by shooters of type i are directed against targets of type j, but (unlike Section c) it does not assume that, given this preallocation, the fires are uncoordinated. Instead, this level of coordination assumes that each shooter of any given type coordinates with all of the other shooters of that same type that are directed against the same type of target in order to distribute their fire as evenly as possible over the particular targets of that type. However, shooters of different types (even though they are preallocated against the same type of target) are assumed to be unable to coordinate their fire at particular targets. These assumptions yield

Equation P3:

$$\Delta t_j = \begin{cases} z\tilde{t}_j\left[1-\prod_{i=1}^{I}\left(1-p_{ij}\right)^{\lfloor \tilde{s}_i a_{ij}/\tilde{t}_j \rfloor}\left(1-\langle \tilde{s}_i a_{ij}/\tilde{t}_j\rangle p_{ij}\right)\right]_j, \tilde{t}_j > 0, \\ 0, \tilde{t}_j = 0, \end{cases}$$

Note that setting $x_{ij} = 1$ for all i and j in the general form given in Section a above produces Equation P3.

The rationale behind Equation P3 is roughly as follows. Since shooters are preallocated against targets by type, attrition to the t_j targets of type j depends only on the engagements against those targets by the $\tilde{s}_i a_{ij}$ shooters of type i for all relevant i. For each relevant i and j, let S_{ij} denote the event that a particular target of type j survives an encounter involving $\tilde{s}_i a_{ij}$ shooters of type i and $\tilde{t}_j$ targets of type j. Then, since the event S_{ij} is homogeneous in shooter type and target type, the results of Reference [5] apply, and

$$\text{Prob}\{S_{ij}\} = (1 - p_{ij})^{\lfloor \tilde{s}_i a_{ij} / \tilde{t}_j \rfloor} \left(1 - \left\langle \tilde{s}_i a_{ij} / \tilde{t}_j \right\rangle p_{ij}\right).$$

Since shooters do not coordinate across shooter types, the overall probability that a particular target survives is the product of the probabilities that it survives each type of shooter. Thus, the probability that a particular target of type j survives all of the types of shooters is

$$\prod_{i=1}^{I} \text{Prob}\{S_{ij}\}.$$

Further, since each target of the same type has, a priori, the same probability of survival,

$$\Delta t_j = \tilde{t}_j \left[1 - \prod_{i=1}^{I} \text{Prob}\{S_{ij}\}\right].$$

Making the appropriate substitution gives Equation P3 (for $z = 1$ and $\tilde{t}_j > 0$).

e. Coordinated Fire Across all Shooter Types

The assumption that all of the shooters can coordinate their fire with each other allows many different allocations of fire. For consistency, for (relative) simplicity, and for useful implementation, shooters by type are again assumed to be preallocated to targets by type so that $\tilde{a}_{imj}$ of the $\tilde{s}_i \tilde{b}_{im}$ possible engagements by shooters of type i are directed against the m vulnerable targets of type j.

Given this preallocation, what the shooting side clearly would like to do (short of having a shoot-look-shoot capability) is to coordinate fire in such a manner as to maximize the number of targets of each type that are killed. Given an integral number of shooters (engagements) and an integral number of targets, the problem of finding an optimal integral allocation of specific shooters to specific targets could be quite difficult. For the purposes here, all of these integer restrictions are ignored, and no optimization problem is solved. Instead, two heuristic methods to coordinate fire are presented. In the first method, the numbers of possible engagements by type of shooter are distributed in a relatively even manner over the targets of the designated type without explicit regard to the probable results of those engagements. If there are more than enough shooters to engage all of the targets, then the shooters attack in evenly distributed layers with no coordination among layers. (This method has a direct analogue in area fire.) In the second method, the shooters are assumed to be further preallocated against individual targets of the type in question, where this further preallocation depends on both the numbers of shooters by type and the probabilities of kill of each type of shooter against the type of target in question. (This method does not have an analogue in area fire.) These two methods are discussed, in turn, below.

(1) Even Distribution of Fire by Numbers of Engagements

For this method, the following additional assumptions are made. Let $\bar{q}_j$ denote the total number of engagements that have been allocated against targets of type j, so that

$$\bar{q}_j = \sum_{i=1}^{I} \sum_{m=1}^{M} \tilde{s}_i a_{ij} \ .$$

Let q_{ij} denote the fraction of these engagements that are made by shooters of type i, so that

$$q_{ij} = \begin{cases} \tilde{s}_i a_{ij} / \bar{q}_j \ , \bar{q}_j > 0; \\ 0, \bar{q}_j = 0. \end{cases}$$

If $\bar{q}_j$ is less than or equal to $\tilde{t}_j$, then this level of coordination assumes that all engagements are made against different targets. If $\bar{q}_j$ is greater than $\tilde{t}_j$, then this level of coordination assumes that the shooters attack in layers where:

(1) the fraction of engagements by shooters of type m in each layer is q_{imj},

(2) the total number of engagements made by each layer except (perhaps) for the last layer is $\tilde{t}_j$ (the last layer can make fewer than $\tilde{t}_j$ engagements) and each engagement in the same layer is made against a different target, and

(3) subject to the restriction that no target is engaged *more* than once by the shooters in any one layer, the shooters randomly select a target to engage such that the selections of which targets are engaged by which shooters are mutually independent among all layers. (Note that, if $\bar{q}_j$ is less than or equal to $\tilde{t}_j$, then this is the special case of multiple-possible-layer attack that consists of only one layer.)

Since each of these layers has the same mix of shooters engaging the same type of target and since events arc independent across layers, an average probability of kill can be used. Let $\tilde{p}_j$ denote this average, so that

$$\tilde{p}_j = \sum_{i=1}^{I} p_{ij} q_{ij} \ .$$

These coordination assumptions yield

Equation PM4,1

$$\Delta t_j = z\tilde{t}_j \left[1 - \left(1 - \tilde{p}_j\right)^{\lfloor \bar{q}_j / \tilde{t}_j \rfloor} \left(1 - \left\langle \frac{\bar{q}_j}{\tilde{t}_j} \right\rangle \tilde{p}_j \right) \right],$$

where $\tilde{p}_j$ and $\tilde{q}_j$ are as defined just above.

Several points should be noted concerning Equation P4.1.

First, this attrition equation cannot be obtained from the general form presented in Section a, above, by setting x_{ij} to suitable quantities. It is, however, a homogeneous special case of that general form.

Second, note that the first three levels of coordination discussed above have the property that, as the level of coordination is increased (holding all else constant), the amount of attrition increases (or, in degenerate cases, remains constant). This property carries over here if

$$\sum_{i=1}^{I} \tilde{s}_i a_{ij} \leq \tilde{t}_j .$$

That is, if this inequality holds then Δt_j as computed by Equation P4.1 is greater than or equal to Δt_j as computed by Equation P3 (when using the same parameters in each case). However, if

$$\sum_{i=1}^{I} \tilde{s}_i a_{ij} > \tilde{t}_j$$

then Equation P3 can produce higher attrition than Equation P4.1.

Finally, note that, unlike Equations P1, P2, and P3, Equation P4.1 is not monotonically non-decreasing in the number of shooters. That is, the attrition equations presented here are in the form

$$\Delta t_j = f_j\left(s, t, \underline{P}\right)$$

for some parameter set $\underline{P}$. The attrition equations for the first three levels of coordination above have the property that, if $s_i' \geq s_i$ for all i, then

$$\mathrm{f}_{\mathrm{j}}\left(\mathrm{s}', \mathrm{t}, \underline{\mathrm{P}}\right) \geq \mathrm{f}_{\mathrm{j}}\left(\mathrm{s}, \mathrm{t}, \underline{\mathrm{P}}\right).$$

However, Equation P4.1 does not have this property. That is, there exist cases in which $s_i' \geq s_i$ for all i yet

$$f_j\left(s', t, \underline{P}\right) < f_j\left(s, t, \underline{P}\right)$$

according to Equation P4.1.

Equation P4.2, below, is also subject to this non-monotonicity in the number of shooters, but it appears to be less sensitive to this anomaly than Equation P4.1. This characteristic will be discussed further following the presentation of Equation P4.2, next.

(2) Proportional Distribution of Fire by Potential Kills

For this method of coordination, shooters are assumed to be further preallocated in the following manner.

For each relevant i and j, let w_{ij} be such that $0 \leq w_{ij} \leq 1$ and

$$\sum_{i=1}^{I} w_{ij} = 1 .$$

Given w_{ij}, shooters are assumed to be assigned to targets such that:

(1) all of the $\tilde{s}_i a_{ij}$ engagements by shooters of type i that are to be allocated against targets of type j are allocated only against $w_{ij} \tilde{t}_j$ targets of type j, and

(2) no shooters of different types ever shoot at the same target.

That is, the $\tilde{t}_j$ targets of type j are subdivided into I disjoint groups of size $w_{ij}\tilde{t}_j$ and shooters of type i fire at (and only at) the i-th such group. With this structure, each shooter versus target interaction is homogeneous in both shooter and target types. Thus (given that the shooting side does not have a shoot-look-shoot capability) it is optimal for the shooters to distribute their fire as evenly as possible over the targets in each of these homogeneous interactions. Accordingly, the resultant attrition in this case would be given by;

$$\Delta t_j = z\tilde{t}_j \left[1 - \sum_{\substack{i=1 \\ w_{ij}>0}}^{I} w_{ij} \left(1 - p_{ij}\right)^{\lfloor \tilde{s}_i a_{ij} / w_{ij} \tilde{t}_j \rfloor} \left(1 - \left\langle \frac{\tilde{s}_i a_{ij}}{w_{ij} \tilde{t}_j} \right\rangle p_{ij} \right) \right]$$

which can be approximated by:

$$\Delta t_j = z\tilde{t}_j \left[1 - \sum_{\substack{i=1 \\ w_{ij}>0}}^{I} w_{ij} \left(1 - p_{ij}\right)^{\tilde{s}_i a_{ij} / w_{ij} \tilde{t}_j} \right].$$

This formulation gives rise to the optimization problem:

$$\max_{w_{ij}} \Delta t_j$$

such that $0 \le w_{ij} \le 1$ and

$$\sum_{i=1}^{I} w_{ij} = 1.$$

Dropping momentarily the notation irrelevant to this optimization problem (such as the subscript over target types) and using the approximate form for Δt_j above, this optimization problem becomes:

$$\max_{w_i} \sum_{\substack{i=1 \\ w_i>0}}^{I} w_i \left(1 - p_i\right)^{s_i / w_i t}$$

such that $0 \le w_i \le 1$ and

$$\sum_{i=1}^{I} w_i = 1;$$

while, with the original form for Δt_j, this optimization problem becomes:

$$\max_{w_i} \sum_{\substack{i=1 \\ w_i>0}}^{I} w_i \left(1 - p_i\right)^{\lfloor s_i / w_i t \rfloor} \left(1 - \left\langle s_i / w_i t \right\rangle p_i \right)$$

such that $0 \le w_i \le 1$ and

$$\sum_{i=1}^{I} w_i = 1.$$

Future research could address this latter optimization problem (or, if it is too difficult or intractable, the former problem could be addressed). For the time being, reasonable but not necessarily optimal values for w_{ij} will simply be postulated.

The attrition structure above is essentially complete once values for w_{ij} are specified. Until the appropriate optimization problems are adequately solved, assume that values for w_{ij} are given as follows. Let

$$\overline{w}_j = \sum_{i=1}^{I} \tilde{s}_i a_{ij} p_{ij},$$

and

$$w_{ij} = \begin{cases} \tilde{s}_i a_{ij} p_{ij} / \overline{w}_j, \overline{w}_j > 0; \\ 0, \overline{w}_j = 0. \end{cases}$$

In a sense, $\tilde{s}_i a_{ij} p_{ij}$ is the number of potential kills that shooters of type i can make against targets of type j. Thus, this specification of w_{ij} sets the number of targets assigned to shooters of type i in proportion to the number of potential kills that those shooters can make against those targets. This specification of w_{ij} yields

<u>EquationP4.2</u>:

$$\Delta t_j = \begin{cases} z\tilde{t}_j \left[1 - \sum_{i=1, w_{ij}>0}^{I} w_{ij} \left(1 - p_{ij}\right)^{\left\lfloor \tilde{s}_i a_{ij} / w_{ij} \tilde{t}_j \right\rfloor} \left(1 - \left\langle \frac{\tilde{s}_i a_{ij}}{w_{ij} \tilde{t}_j} \right\rangle \right) \right], \overline{w}_j = 0 \; and \; \tilde{t}_j = 0, \\ 0, \overline{w}_j = 0 \; or \; \tilde{t}_j = 0, \end{cases}$$

where w_{ij} and $\overline{w}_j$ are as defined just above.

Like Equation P4.1, Equation P4.2 cannot be obtained from the general form presented in Section a by setting x_{ij} to suitable quantities.

f. Shoot-Look-Shoot Fire

The point of this section is to remind the reader that still higher levels of coordination are (theoretically) possible. In particular, there are many variations of shoot- look-shoot firing processes that can be considered. While none of these processes may be computationally attractive, preallocated shoot-look-shoot with no upper bound on the number of engagements per target may be, relatively speaking, the most tractable of these shoot-look-shoot processes. In anticipation of the detailed specification of this process by Reference [14], unbounded preallocated shoot-look-shoot attrition that does not explicitly consider munitions is labeled P5 here.

2. Point-Fire Equations That Explicitly Consider Munitions

The point of this section is to extend the point-fire equations presented in Section 1 to equations that explicitly consider multiple types of munitions. For each of the levels of coordination considered in Section 1, one or more corresponding levels are considered here and the corresponding attrition equations that explicitly consider munitions are given. The goal here is essentially just to present these equations, not to provide either a detailed discussion of assumptions that might underlie these equations or a mathematical derivation of these equations from such assumptions.

Throughout this section it is assumed that if a shooter of type i attacks a target of type j using munitions of type m, then it kills that target (i.e., fires a lethal shot at that target) with probability p_{imj}, and that the killing processes are mutually independent of each other and of all target selection processes. Two or more shooters might attack and fire lethal shots at the same target, which results in only one target being killed.

a. A General Form for Calculating Point-Fire Attrition

A general form for calculating point-fire attrition when explicitly considering munitions that correspond to the general form presented in Section 1.a, above, is:

$$\Delta t_j = \begin{cases} z\tilde{t}_j \left[1 - \prod_{i=1}^{I} \prod_{m=1}^{M} \left(1 - p_{imj} \tilde{l}\left(x_{imj}\right)\right)^{\left\lfloor \frac{\tilde{s}_i c_{imj}}{x_{imj} \tilde{t}_j} \right\rfloor} \left(1 - \left\langle \frac{\tilde{s}_i c_{imj}}{x_{imj} \tilde{t}_j} \right\rangle p_{imj} \tilde{l}\left(x_{imj}\right)\right) \right], \tilde{t}_j > 0, \\ 0, \tilde{t}_j = 0, \end{cases}$$

where the specification of x_{imj} differs as described below. In particular, three of the following attrition equations that consider munitions can be put into this general form by specifying x_{imj} in three different ways. As in Section 1, the reasons for presenting this general form are:

(1) to help relate the equations presented below to each other and to the point-fire equations that do not consider munitions described above, and

(2) to note that it may be possible to develop and relate other attrition equations to those considered here by specifying x_{imj} in other ways. Also as in Section 1, it should be noted that there may be several ways to specify x_{imj} that are structurally consistent (i.e., Δt_j is properly bounded and moves in the correct direction as parameters are changed) but are not appropriate in that the resulting attrition equations do not follow from a consistent set of physically realizable assumptions. No such alternative specifications of x_{imj} are discussed here.

b. Uncoordinated Fire

Uncoordinated fire here assumes:

(1) that each shooter of type i on each of its engagements selects one target (from among those vulnerable) to attack, where the probability that it selects a particular target of type j to attack in an engagement in which the shooter is using

munitions of type m is $\tilde{a}_{imj}/\tilde{t}_j$, and

(2) that the target selection processes are all mutually independent.

This assumption of uncoordinated point fire yields

Equation PM1:

$$\Delta t_j = \begin{cases} z\tilde{t}_j \left[1 - \prod_{i=1}^{I} \prod_{m=1}^{M} \left(1 - p_{imj} \tilde{l} \left(\frac{\tilde{a}_{imj}}{\tilde{t}_j} \right) \right)^{\lfloor \tilde{s}_i \tilde{b}_{im} \rfloor} \left(1 - \left\langle \tilde{s}_i \tilde{b}_{im} \right\rangle p_{imj} \tilde{l} \left(\frac{\tilde{a}_{imj}}{\tilde{t}_j} \right) \right) \right], \tilde{t}_j > 0, \\ 0, \tilde{t}_j = 0, \end{cases}$$

Note that setting $x_{imj} = \tilde{a}_{imj}/\tilde{t}_j$ in the general form above produces Equation PM1.

Note also that setting $M = 1$ (so that multiple types of munitions are not being addressed) essentially converts Equation PM1 into Equation P1.

c. Preallocated Fire

Preallocated fire here assumes:

(1) that $\tilde{a}_{imj}$ of the $\tilde{s}_i \tilde{b}_{im}$ possible engagements by shooters of type i with munitions of type m are directed against targets of type j, where $\tilde{a}_{imj}$ is a deterministic quantity,

(2) each shooter in each of its engagements selects, according to a uniform distribution, one target (from among those of the type it is directed against) to attack, and

(3) that the target selection processes are all mutually independent. This assumption of preallocated fire yields

Equation PM2:

$$\Delta t_j = \begin{cases} z\tilde{t}_j \left[1 - \prod_{i=1}^{I} \prod_{m=1}^{M} \left(1 - p_{imj} \tilde{l} \left(\frac{1}{\tilde{t}_j} \right) \right)^{\lfloor \tilde{s}_i c_{imj} \rfloor} \left(1 - \left\langle \tilde{s}_i c_{imj} \right\rangle p_{imj} \tilde{l} \left(\frac{1}{\tilde{t}_j} \right) \right) \right], \tilde{t}_j > 0, \\ 0, \tilde{t}_j = 0, \end{cases}$$

Note that setting $x_{imj} = 1/\tilde{t}_j$ in the general form above produces Equation PM2. Note also that setting $M = 1$ essentially converts Equation PM2 into Equation P2.

d. Partially or Completely Coordinated Fire Within Shooter Types, Uncoordinated Fire Across Shooter Types

(1) Coordinated Fire Only Within Both Shooter and Munition Types

This level of coordination also assumes that shooters by type are preallocated to targets by type so that $\tilde{a}_{imj}$ of the $\tilde{s}_i\tilde{b}_{im}$ possible engagements by shooters of type i with munitions of type m are directed against targets of type j, but it does not assume that, given this preallocation, the fires are uncoordinated. Instead, this level of coordination assumes that each shooter of any given type when using munitions of any given type coordinates with all other shooters of that type using munitions of that type in order to distribute their fire as evenly as possible over the targets of that type. However, shooters of the same type using different types of munitions and shooters of different types (no matter what munitions they are using) are assumed to be unable to coordinate with each other. These assumptions yield

Equation PM3.1:

$$\Delta t_j = \begin{cases} z\tilde{t}_j\left[1-\prod_{i=1}^{I}\prod_{m=1}^{M}\left(1-p_{imj}\right)^{\left\lfloor \frac{\tilde{s}_i c_{imj}}{\tilde{t}_j}\right\rfloor}\left(1-\left\langle\frac{\tilde{s}_i c_{imj}}{\tilde{t}_j}\right\rangle p_{imj}\right)\right], \tilde{t}_j > 0, \\ 0, \tilde{t}_j = 0, \end{cases}$$

Note that setting $x_{imj} = 1$ in die general form above produces Equation PM3.1. Note also that setting $M = 1$ essentially converts Equation PM3.1 into Equation P3.

(2) Coordinated Fire Within Shooter Types but Across all Munitions Used by Each Type of Shooter

As above, this level of coordination assumes that shooters by type are preallocated to targets by type so that $\tilde{a}_{imj}$ of the $\tilde{s}_i\tilde{b}_{im}$ possible engagements by shooters of type i with munitions of type m are directed against targets of type j. It also assumes that each shooter of any given type coordinates with all other shooters of that type, no matter what munitions they are using.

This coordination of munitions of different types raises essentially the same problems as discussed in Section 1.e, above, concerning the coordination of shooters of different types, and either (or both) of the two heuristic methods for considering such coordination discussed there could also be used here to address these coordination problems. For simplicity, only the second of these two methods is presented here. In particular, a procedure for considering coordination among the use of various types of munitions that corresponds to the second of the two heuristic methods given in Section 1.e for considering coordination among types of shooters is as follows. Let

$$\hat{w}_{ij} = \sum_{m=1}^{M}\tilde{s}_i c_{imj} p_{imj}$$

and

$$\tilde{w}_{imj} = \begin{cases} \tilde{s}_i c_{imj} p_{imj} / \hat{w}_{ij}, \hat{w}_{ij} > 0; \\ 0, \hat{w}_{ij} = 0. \end{cases}$$

Given $\tilde{w}_{imj}$, the shooters of type i are assumed to be further preallocated against the targets of type j such that:

(1) all of the $\tilde{s}_i c_{imj}$ engagements by shooters of type i with munitions of type m that are assigned against targets of type j are allocated only against $\tilde{w}_{imj}\tilde{t}_j$ targets of that type, and

(2) no shooters of the same type using munitions of different types attack the same target Further, the $\tilde{s}_i c_{imj}$ engagements by shooters of type i using munitions of type m are assumed to be distributed as evenly as possible over the $\tilde{w}_{imj}\tilde{t}_j$ targets they are allocated against These assumptions yield

Equation PM3.2:

$$\Delta t_j = \begin{cases} z\tilde{t}_j \left[1 - \prod_{i=1}^{I} \left(\sum_{m=1, \tilde{w}_{imj}>0}^{M} \tilde{w}_{imj} \left(1 - p_{imj}\right)^{\left\lfloor \frac{\tilde{s}_i c_{im}}{\tilde{w}_{imj}\tilde{t}_j} \right\rfloor} \left(1 - \left\langle \frac{\tilde{s}_i c_{im}}{\tilde{w}_{imj}\tilde{t}_j} \right\rangle p_{imj} \right) \right) \right], \hat{w}_{ij} > 0 \text{ and } \tilde{t}_j > 0, \\ 0, \hat{w}_{ij} = 0 \text{ or } \tilde{t}_j = 0, \end{cases}$$

where $\tilde{w}_{imj}$ and $\hat{w}_{ij}$ are as defined just above.

Note that, unlike Equation PM3.1, Equation PM3.2 is not a special case of the general form presented above. However, like Equation PM3.1, setting $M = 1$ essentially converts Equation PM3.2 into Equation P3.

e. Coordinated Fire Across all Shooter Types

The characteristics, problems, and potential anomalies discussed in Section 1.e, above, all directly extend to the consideration of multiple types of munitions—and the structure proposed there also extends directly here. In particular, again assume that the shooters by type are preallocated to the targets by type so that $\tilde{a}_{imj}$ of the $\tilde{s}_i\tilde{b}_{im}$ possible engagements by shooters of type i with munitions of type m are directed only against targets of type j.

Given this preallocation, two heuristic methods for coordinating fire are presented below, where these methods are the direct extension of the two methods presented in Section I.e. In the first method, the numbers of possible engagements by type of shooter and type of munition are distributed in a relatively even manner over the ***targets*** of the designated type without regard to the probable results of those engagements. If there are more than enough shooters to engage all of the targets, then the shooters attack in evenly distributed layers with no coordination between layers. (This case has a direct analogue in area fire). In the second method, the shooters are assumed to be further preallocated against individual targets of the type in question, where this further preallocation depends on the numbers of engagements that each type of shooter can make with each type of munition and on the probabilities of kill of those shooters with those munitions against that type of target (This method does not have an analogue in area fire.) These two methods are discussed, in turn, below.

(1) **Even Distribution of Fire by Numbers of Engagements**

For this method, the following additional assumptions are made. Let $\bar{q}_j$ denote the total number of engagements that have been allocated against targets of type j, so that here

$$\bar{q}_j = \sum_{i=1}^{I} \sum_{m=1}^{M} \tilde{s}_i c_{imj} .$$

Let q_{imj} denote the fraction of these engagements that are made by shooters of type i using munitions of type m, so that here

$$q_{imj} = \begin{cases} \tilde{s}_i c_{imj} / \bar{q}_j , \bar{q}_j > 0, \\ 0, \bar{q}_j = 0. \end{cases}$$

If $\bar{q}_j$ is less than or equal to $\tilde{t}_j$, then this level of coordination assumes that all engagements are made against different targets. If $\bar{q}_j$ is greater than $\tilde{t}_j$, then this level of coordination assumes that the shooters attack in layers where:

(1) the fraction of engagements by shooters of type i using munitions of type m in each layer is q_{imj},

(2) the total number of engagements made by each layer except (perhaps) for the last layer is $\tilde{t}_j$ (the last layer can have fewer than $\tilde{t}_j$ engagements) and each engagement in the same layer is made against a different target, and

(3) subject to the restriction that no target is engaged more than once by the shooters in any one layer, the shooters randomly select a target to engage such that the selections of which targets are engaged by which shooters using which types of munitions are mutually independent among all layers.

Since each of these layers consists of the same distribution of shooters using the same distribution of munitions against the same type of target and since events are independent across layers, an average probability of kill can be used. Let $\tilde{p}_j$ denote this average, so that here

$$\tilde{p}_j = \sum_{i=1}^{I} \sum_{m=1}^{M} p_{imj} q_{imj} .$$

These coordination assumptions yield

<u>Equation PM4.1</u>:

$$\Delta t_j = z\tilde{t}_j \left[1 - \left(1 - \tilde{p}_j\right)^{\left\lfloor \frac{\bar{q}_j}{\tilde{t}_j} \right\rfloor} \left(1 - \left\langle \frac{\bar{q}_j}{\tilde{t}_j} \right\rangle \tilde{p}_j \right) \right],$$

where $\tilde{p}_j$ and $\bar{q}_j$ are as defined just above. Thus, the only difference between Equation P4.1 and Equation PM4.1 is the definition of $\tilde{p}_j$ and $\bar{q}_j$ and setting $M = 1$ makes these definitions equivalent.

(2) **Proportional Distribution of Fire by Potential Kills**

For this method of coordination, shooters are assumed to be further preallocated in the following manner.

Here, let

$$\overline{w}_j = \sum_{i=1}^{I} \sum_{m=1}^{M} \tilde{s}_i c_{imj} p_{imj}$$

so that $\overline{w}_j$ is a measure of the number of potential kills that all of the assigned shooters can make against targets of type j, and let

$$w_{imj} = \begin{cases} \overline{s}_i c_{imj} p_{imj} / \overline{w}_j, \overline{w}_j > 0; \\ 0, \overline{w}_j = 0. \end{cases}.$$

Given w_{imj}, the shooters of type i are assumed to be further preallocated against targets of type j such that:

(1) all of the $\tilde{s}_i c_{imj}$ engagements by shooters of type i with munitions of type m that are assigned against targets of type j are allocated only against $w_{imj} \tilde{t}_j$ targets of that type, and

(2) for all relevant i and m, no shooter of type i using munitions of type m attacks the same target as a shooter of type i' using munitions of type m' if $i \neq i'$ or $m \neq m'$. Further, the $\tilde{s}_i c_{imj}$ engagements by shooters of type i using munitions of type m are assumed to be distributed as evenly as possible over the targets they are allocated against.

These assumptions yield

<u>Equation PM4.2</u>:

$$\Delta t_j = \begin{cases} z\tilde{t}_j \left[1 - \sum_{i=1, w_{imj}>0}^{I} \sum_{m=1}^{M} w_{imj} \left(1 - p_{imj}\right)^{\left\lfloor \frac{\tilde{s}_i c_{imj}}{w_{imj} \tilde{t}_j} \right\rfloor} \left(1 - \left\langle \frac{\tilde{s}_i c_{imj}}{w_{imj} \tilde{t}_j} \right\rangle p_{imj} \right) \right], \overline{w}_j > 0 and \tilde{t}_j > 0, \\ 0, \overline{w}_j = 0 \; and \; \tilde{t}_j = 0, \end{cases}$$

where w_{imj} and $\overline{w}_j$ are as defined just above.

Note that setting $M = 1$ essentially converts Equation PM4.2 into Equation P4.2.

f. Shoot-Look-Shoot Fire

As stated in Section 1.f, above, there are many types of shoot-look-shoot fire. The particular type proposed there is to preallocate this fire, but then to place no upper bound on the number of engagements per target given this preallocation. In addition to being more computationally tractable than other types of shoot-look-shoot fire, this type of fire has the advantage that it has a simple and straightforward extension to cases in which multiple types of munitions are explicitly considered. In anticipation of the detailed specification of such a process by Reference [14], unbounded preallocated shoot-look-shoot attrition that considers multiple types of munitions is labeled PM5 here.

E. AREA-FIRE ATTRITION EQUATIONS

As with point fire above, the goal here is to describe various forms of area fire in somewhat general terms and to postulate specific attrition equations that correspond to these general descriptions. Future research is needed if it is desired to convert these general descriptions into specific sets of assumptions and to rigorously derive the resultant attrition equations from these assumptions.

The descriptions (in Sections 1 and 2) below will typically be presented as if the number of area-fire combat zones, $\ddot{z}$, is one. (Occasionally, combat zones will be explicitly mentioned.) This general omission of combat zones will considerably simplify the wording of the following discussions at no real loss in generality, since the extension to multiple combat zones is clear. In particular, multiple combat zones assume that $1/\ddot{z}$ of the shooters and $1/\ddot{z}$ of the targets are in each of $\ddot{z}$ combat zones, with no interaction among various zones. Thus, a description of combat in one zone suffices. The formal statement of the attrition equations (and any ancillary equations or notation) will include $\ddot{z}$ as appropriate.

Of the total number of targets of type j, t_j, the fraction v_j are vulnerable to both area fire and point fire while the fraction $\ddot{u}_j$ are vulnerable to area fire only. Thus, the total number of vulnerable targets per combat zone, $\ddot{t}_j$, is given by

$$\ddot{t}_j = \frac{(\ddot{u}_j + v_j)t_j}{\ddot{z}_j}.$$

Throughout this section these vulnerable targets are assumed to be uniformly distributed over a target area, H, of size $\ddot{z}h$, where

$$h = \sum_{j=1}^{J} d_j \ddot{t}_j .$$

The shooters are assumed to be firing into an attack area, G, of size $\ddot{z}g$. As stated in Section C.3.a, it is assumed that $G \subset H$ if $g \le h$ and $H \subset G$ if $h \le g$. Thus, the number of targets that are both vulnerable and are in the area being attacked (per combat zone), $\dot{t}_j$, is given by

$$\dot{t}_j = \begin{cases} \ddot{t}_j \min(1, g/h), h > 0; \\ 0, h = 0. \end{cases}$$

1. Area-Fire Equations That do not Consider Munitions

a. Uncoordinated Fire

The uncoordinated area-fire attrition equation stated below is postulated to follow from the following general assumptions. Each salvo by each shooter of type i results in J (overlapping) lethal areas, where the size of the j-th lethal area is $\ddot{a}_{ij}$ (These lethal areas can be pictured as being concentric circles whose radii depend on the type of shooter and type of potential target in that a target of a particular type can be killed by a salvo from a shooter of a particular type only if that target is within the corresponding radius of the center of the impact area of that salvo.) If $\ddot{a}_{ij} \le g$ then the j-th lethal area is contained in G, and if $g \le \ddot{a}_{ij}$ then G is contained in the j-th lethal area. (This statement assumes that $\ddot{z} = 1$, the extension to general $\ddot{z}$ is obvious.) If a target of type j is in the j-th lethal area of a salvo by a shooter of type i, then it is killed with probability $\ddot{p}_{ij}$ by that salvo (otherwise, it survives that salvo). The locations of the j-th lethal area of different salvos are uniformly distributed and are mutually independent in the sense that if $A_{1j},\ldots,A_{Nj}$ are the lethal areas covered by N such salvos, if $\hat{a}_{nj} = \ddot{a}_{ij}$ when n and i are such that the n-th salvo is fired by a shooter of type i, and if x is a randomly generated point according to a uniform distribution over G, then

$$\text{Prob}\left\{ x \in \bigcap_{n=1}^{N} A_{nj} \right\} = \prod_{n=1}^{N} \min\left\{1, \frac{\hat{a}_{nj}}{g}\right\} = \prod_{n=1}^{N} \tilde{l}\left(\frac{\hat{a}_{nj}}{g}\right).$$

The uniform distribution and mutual independence (i.e., lack of coordination) of these salvos, combined with the uniform distribution of targets and the assumptions concerning G and H, lead to

Equation A1:

$$\Delta t_j = \ddot{z}\ddot{t}_j \left[1 - \prod_{i=1}^{I} \left(1 - \ddot{p}_{ij}\tilde{l}\left(\frac{\ddot{a}_{ij}}{g}\right)\right)^{\lfloor \ddot{s}_i \rfloor} \left(1 - \ddot{p}_{ij}\tilde{l}\left(\frac{\ddot{a}_{ij}}{g}\right)\langle \ddot{s}_i \rangle \right) \right].$$

Note that "edge effects" are being ignored here in that it is assumed that:

(1) the lethal areas of all salvos fall entirely within G,

(2) $G \subset H$ if $g \le h$ and $H \subset G$ if $h \le g$, and

(3) if x is a randomly generated point according to a uniform distribution over G, then

$$\text{Prob}\{x \in A_{ij}\} = \tilde{l}\left(\frac{\ddot{a}_{ij}}{g}\right)$$

where A_{ij} is the j-th lethal area covered by a salvo from a shooter of type i.

b. Coordinated Fire Within Shooter Types, Uncoordinated Fire Across Shooter Types

If shooters of the same type can coordinate their area fire then, given that the targets are randomly and uniformly distributed over the target area, the shooters want to cover the attack area as evenly as possible. If $\ddot{a}_{ij}$ did not depend on j, then the assumption that "edge effects" can be ignored would yield a simple formula for such uniform coverage. The assumption that "edge effects" car be ignored will be kept throughout this paper, and the following approach is suggested to handle cases in which $\ddot{a}_{ij}$ varies with j.

Shooters of type i are assumed to coordinate their area fire based on a planning size, denoted by $\bar{a}_i$ in Section C.3.b, above. Since $\bar{a}_i$ is independent of j and since "edge effects" are being ignored, it is easy to plan uniform coverage based on this planning size. The impact of planning fire based on $\bar{a}_i$ is assessed as follows.

Suppose, for a particular j, that $\ddot{a}_{ij} = \bar{a}_i/2$. Then, while a salvo by a shooter of type i is planned to cover an area of size $\bar{a}_i$, its lethal area with respect to targets of type j is only $\bar{a}_i/2$. Thus, the probability that such a salvo kills a randomly (uniformly) located target in its planned attack area (of size $\bar{a}_i$ is

$$\ddot{p}_{ij}/2 = \ddot{p}_{ij}\left(\ddot{a}_{ij}/\bar{a}_i\right).$$

Conversely, for a particular j, suppose that $\ddot{a}_{ij} = 2\bar{a}_i$. Then the attacker would plan on using two salvos by shooters of type i in order to provide single coverage of an area of size $\ddot{a}_{ij}$. Yet, since $\ddot{a}_{ij} = 2\bar{a}_i$, two such salvos (if coordinated) would provide double coverage of that area. The probability that a target of type j is killed in such a double-covered area is

$$1-\left(1-\ddot{p}_{ij}\right)^2 = 1-\left(1-\ddot{p}_{ij}\right)^{\left(\ddot{a}_{ij}/\bar{a}_i\right)}.$$

In general, it is assumed that if shooters of type i coordinate their area fire such that each shooter of type i plans to attack an area of size $\bar{a}_i$, and that no shooters of that type plan to attack the same area until the whole attack area has been covered by that type of shooter, then the probability that a target of type j located in particular type-i shooters' attack area is killed by a salvo from that shooter, $\bar{p}_{ij}$, is given by

$$\bar{p}_{ij} = \begin{cases} 1-\left(1-\ddot{p}_{ij}\right)^{\lfloor \ddot{a}_{ij}/\bar{a}_i \rfloor}\left(1-\ddot{p}_{ij}\left\langle \ddot{a}_{ij}/\bar{a}_i \right\rangle\right) \\ 0, \bar{a}_i = 0, \end{cases}$$

where $\ddot{a}_{ij} = \min\{\ddot{a}_{ij}, g\}$.

This structure for coordinating fire within shooter types combined with the assumption of no coordination among different types of shooters leads to

Equation A3:

$$\Delta t_j = \ddot{z}\dot{t}_j\left[1-\prod_{i=1}^{I}\left(1-\bar{p}_{ij}\right)^{\lfloor \ddot{s}_i\bar{a}_i/g \rfloor}\left(1-\bar{p}_{ij}\left\langle \ddot{s}_i\bar{a}_i/g \right\rangle\right)\right]$$

where $\bar{p}_{ij}$ is as defined just above.

This equation is labeled A.3 instead of A.2, even though it is the second area fire equation, for consistency with the corresponding underlying assumptions concerning point fire. The second equation for point fire, preallocated point fire, has no analogue in area fire.

c. Coordinated Fire Across all Shooter Types

Many of the characteristics, problems, and potential anomalies discussed in Section D.1.e concerning coordinated point fire have analogues concerning coordination in area fire. In particular, if all fire can be coordinated, then the attacker may want to coordinate its fire to maximize some measure of the number of **targets** killed, and it is not clear how to perform this maximization. One of the heuristic techniques proposed in Section D.1.e, preallocation of shooters to a subset of the targets of a particular type in proportion to particular kills, does not have an analogue in area fire because area fire does not allocate shooters to targets. The other heuristic technique proposed in Section D.1.e, independent layers of fire with each layer being the same mix of shooters, does have a direct analogue in area fire – this analogue is proposed below.

Assume that the shooters coordinate their fire in the following manner. Let $\tilde{q}$ denote the total (planning) area that can be covered by all of the shooters per combat zone, so that

$$\tilde{q} = \sum_{i=1}^{I} \ddot{s}_i \bar{a}_i \ .$$

Let $\ddot{q}_i$ denote the fraction of this area that is due to shooters of type i, so that

$$\ddot{q}_i = \begin{cases} \ddot{s}_i \bar{a}_i / \tilde{q} \, , \tilde{q} = 0; \\ 0, \tilde{q} = 0. \end{cases}$$

If $\tilde{q}$ is less than or equal to g, then each salvo fires into its own planning area. If $\tilde{q}$ is greater than g, then the shooters are assumed to fire in layers such that:

(1) the fraction of the area covered by each layer that is due to shooters of type i is $\ddot{q}_i$,

(2) each layer, except (perhaps) for the last layer, exactly covers the attacked area of size g with no overlap among the planned attack areas of different salvos in the same layer, (i.e., in each layer, each salvo fires into its own planning area), and

(3) the areas covered by shooters of different types in different layers are mutually independent in the sense that knowledge that an independently randomly generated point in G is contained in the planning area of a salvo by a shooter of a particular type in a particular layer gives no information concerning the coverage of that point by any other layer.

Since each of these layers has the same mix of shooters and since events are independent across layers, an average probability of kill can be used. Let $\hat{p}_j$ denote this average, so that

$$\hat{p}_j = \sum_{i=1}^{I} \bar{p}_{ij} \ddot{q}_i$$

where, as above,

$$\bar{p}_{ij} = \begin{cases} 1-\left(1-\ddot{p}_{ij}\right)^{\lfloor \ddot{a}_{ij}/\bar{a}_i \rfloor}\left(1-\ddot{p}_{ij}\left\langle \ddot{a}_{ij}/\bar{a}_i \right\rangle\right) \\ 0, \bar{a}_i = 0, \end{cases}$$

and $\ddot{a}_{ij} = \min\{\dot{a}_{ij}, g\}$.

These coordination assumptions yield

Equation A4:

$$\Delta t_j = \ddot{z}\dot{t}_j \left[1-\left(1-\hat{p}_j\right)^{\lfloor \tilde{q}/g \rfloor}\left(1-\hat{p}_j\langle \tilde{q}/g \rangle\right)\right]$$

where $\hat{p}_j$ and $\tilde{q}$ are as defined just above.

Note that, if $\tilde{q}$ is less than or equal to g, then the attrition produced by Equation A4 must be greater than or (in degenerate cases) equal to the attrition produced by Equation A3 (when using the same parameters in each case). This inequality may not hold if $\tilde{q}$ is greater than g. It is possible to guarantee that increasing coordination would not decrease attrition for any one particular type of target (or any particular weighted average of targets) by replacing Equation A4 with Equation A4', when the attrition produced by Equation A4' would be, for all types of targets, the attrition produced either by Equation A3 or by Equation A4 – whichever produced the higher attrition for the particular type of target (or weighted average of targets) in question.

2. Area-Fire Equations That Explicitly Consider Munitions

The goal of this section is to extend the area-fire equations presented in Section 1 (that do not consider munitions) to equations that explicitly consider multiple types of munitions. For each of the levels of coordination considered in Section 1, one or more corresponding levels are considered here and the corresponding attrition equations that explicitly consider munitions are given. As with point fire, the goal here is essentially just to present these equations, not to provide either a detailed discussion of assumptions that might underlie these equations or a rigorous derivation of these equations from such assumptions.

a. Uncoordinated Fire

The uncoordinated area-fire attrition equation stated below is postulated to follow from the following general assumptions. Each salvo by a shooter of type i using munitions of type m creates J (overlapping) lethal areas, where the size of the j-th lethal area is $\ddot{a}_{imj}$. (These lethal areas can be pictured as being concentric circles whose radii depend on the type of shooter, type of munition, and type of potential target in that a target of a particular type can be killed by a salvo from a shooter of a particular type using a munition of a particular type only if that target is within the corresponding radius of the center of the impact area of that salvo.) If $\ddot{a}_{imj} \le g$ then the j-th lethal area is contained in G, and if $\ddot{a}_{imj} \ge g$ then G is contained in the j-th lethal area. (This statement assumes that $\ddot{z} = 1$, the extension to general $\ddot{z}$ is obvious.) If a target of type j is in the j-th lethal area of a salvo by a shooter of type i using munitions of type m, then it is killed with probability $\ddot{p}_{imj}$ by that salvo (otherwise, it survives that salvo). The locations of the j-th lethal areas of different salvos are uniformly distributed and mutually independent in the sense that if $A_{1j},\ldots,A_{Nj}$ are the lethal areas covered by N such salvos, if $\hat{a}_{ij} = \ddot{a}_{imj}$ where n, and m are such that the n-th salvo is fired by a shooter of type i using munitions of type m, and if x is a randomly generated point according to a uniform distribution over G, then

$$\text{Prob}\left\{ x \in \bigcap_{n=1}^{N} A_{nj} \right\} = \prod_{n=1}^{N} \min\left\{ 1, \frac{\hat{a}_{nj}}{g} \right\} = \prod_{n=1}^{N} \tilde{l}\left(\frac{\hat{a}_{nj}}{g} \right).$$

The uniform distribution and mutual independence (i.e., level of coordination) of these salvos, combined with the uniform distribution of targets and the assumptions concerning G and H, lead to

<u>Equation AM1</u>:

$$\Delta t_j = \ddot{z}\dot{t}_j \left[1 - \prod_{i=1}^{I} \prod_{m=1}^{M} \left(1 - \frac{\ddot{p}_{imj}}{g} \tilde{l}\left(\frac{\ddot{a}_{imj}}{g} \right) \right)^{\lfloor \ddot{s}_i \ddot{b}_{im} \rfloor} \left(1 - \ddot{p}_{imj} \tilde{l}\left(\frac{\ddot{a}_{imj}}{g} \right) \langle \ddot{s}_i \ddot{b}_{im} \rangle \right) \right]$$

Note that, as in Section 1 above, all “edge effects” are ignored here. Note also that, if $M = 1$, then Equation AM1 essentially reduces to Equation A1.

b. Partially or Completely Coordinated Fire Within Shooter Types, Uncoordinated Fire Across Shooter Types

1) Coordinated Fire Only Within Both Shooter and Munition Types

This level of coordination assumes that shooters of the same type attempt to coordinate their fire when using munitions of the same type, but they cannot (or do not) coordinate when using munitions of different types, and shooters of different types cannot (or do not) coordinate.

When shooters do coordinate, they are assumed to use essentially the same coordination techniques as described in Section 1.b, above – the distinction here is that the size of the planning area, $\bar{a}_{im}$, depends on the type of munition being used as well as on the type of shooter. That is, it is assumed here that each shooter of type i when using munitions of type m plans to attack an area of size $\bar{a}_{im}$, and no two shooters of the same type plan to attack the same area with the same type of munition until the whole attack area has been covered by that type of shooter using that type of munition. The resulting probability of kill of a target of type j in the planning area of a shooter of type i using munitions of type m is assumed to be given by

$$\bar{p}_{imj} = \begin{cases} 1-\left(1-\ddot{p}_{imj}\right)^{\left\lfloor \frac{\ddot{a}_{imj}}{\bar{a}_{im}} \right\rfloor}\left(1-\ddot{p}_{imj}\left\langle \frac{\ddot{a}_{imj}}{\bar{a}_{im}} \right\rangle\right), \bar{a}_{im} > 0, \\ 0, \bar{a}_{im} = 0, \end{cases}$$

where $\ddot{a}_{imj} = \min\{\ddot{a}_{imj}, g\}$.

This structure for coordinating fire among shooters of the same type using munitions of the same type, with uncoordinated fire otherwise, leads to

Equation AM3.1:

$$\Delta t_j = \ddot{z}\dot{t}_j\left[1-\prod_{i=1}^{I}\prod_{m=1}^{M}\left(1-\bar{p}_{imj}\right)^{\left\lfloor \ddot{s}_i\ddot{b}_{im}\bar{a}_{im}/g \right\rfloor}\left(1-\bar{p}_{imj}\left\langle \frac{\ddot{s}_i\ddot{b}_{im}\bar{a}_{im}}{g} \right\rangle\right)\right]$$

where $\bar{p}_{imj}$ is as defined just above. Note that, if $M = 1$, then Equation AM3.1 essentially reduces to Equation A3.

2) Coordinated Fire Within Shooter Types but Across all Munitions Used by Each Type of Shooter

Sections D.1.e and D.2.e give two heuristic methods for considering coordination among various types of shooters in point fire. For simplicity, only one of these methods was used in Section D.2.d(2) to consider coordination among shooters of the same type when using munitions of different types in point fire. The method used there (further preallocation of shooters to targets based on potential kills) does not extend to area fire. However, the other method (distributing the shooters evenly over the targets in uncoordinated layers) extends directly to area fire. Accordingly, this other method is used here to consider coordination among shooters of the same type when using munitions of different types in area fire.

Assume that the shooters of each type coordinate their fire with other shooters of the same type in the following manner. Let $\tilde{q}_i'$ denote the total (planning) area that can be covered by all of the shooters of type i per combat zone, so that

$$\tilde{q}_i' = \sum_{m=1}^{M}\ddot{s}_i\ddot{b}_{im}\bar{a}_{im}\,.$$

Let $\ddot{q}_{im}'$ denote the fraction of this area that is covered by munitions of type m, so that

$$\ddot{q}_{im}' = \begin{cases} \ddot{s}_i\ddot{b}_{im}\bar{a}_{im}/\tilde{q}', \tilde{q}_i' = 0, \\ 0, \tilde{q}_i' = 0. \end{cases}$$

Assume, for each i, that the shooters of type i fire in layers such that:

(1) the fraction of the area covered by each layer that is due to munitions of type m is $\ddot{q}_{im}'$,

(2) each layer, except (perhaps) for the last layer, exactly covers the attacked area of size g with no overlap among the planned attack areas of different salvos in the same layer, and

(3) the areas covered by munitions of different types in different layers are mutually independent in the sense that knowledge that an independently randomly generated point in G is contained in the planning area of a salvo using a particular type of munition gives no information concerning the coverage of that point by any other layer.

Since each of these layers uses the same mix of munitions and since events are independent across layers, an average probability of kill can be used. Let $\widehat{p}'_{ij}$ denote this average for shooters of type i, so that

$$\widehat{p}'_{ij} = \sum_{m=1}^{M} \overline{p}_{imj}\ddot{q}'_{im}$$

where, as above

$$\overline{p}_{imj} = \begin{cases} 1-\left(1-\ddot{p}_{imj}\right)^{\left\lfloor \frac{\ddot{a}'_{imj}}{\overline{a}_{im}} \right\rfloor}\left(1-\ddot{p}_{imj}\left\langle \frac{\ddot{a}'_{imj}}{\overline{a}_{im}} \right\rangle\right), \overline{a}_{im} > 0, \\ 0, \overline{a}_{im} = 0, \end{cases}$$

and $\ddot{a}'_{imj} = \min\{\ddot{a}_{imj}, g\}$.

These coordination assumptions yield

Equation AM3.2:

$$\Delta t_j = \ddot{z}\dot{t}_j\left[1-\prod_{i=1}^{I}\left(1-\widehat{p}_{ij}\right)^{\left\lfloor \frac{\tilde{q}'}{g} \right\rfloor}\left(1-\widehat{p}'_{ij}\left\langle \frac{\tilde{q}'_i}{g} \right\rangle\right)\right],$$

where $\widehat{p}'_{ij}$ and $\tilde{q}'_{ij}$ are as defined just above. Like Equation AM3.1, if $M = 1$ then Equation AM3.2 also essentially reduces to Equation A3.

c. Coordinated Fire Across all Shooter Types

The characteristics, problems, and procedures concerning coordination among all shooters when munitions are not considered all extend to the corresponding case here that explicitly considers multiple types of munitions. The corresponding heuristic procedure to address this coordination is as follows.

Let $\tilde{q}$ denote the total (planning) area that can be covered by all of the shooters per combat zone, so that here

$$\tilde{q} = \sum_{i=1}^{I}\sum_{m=1}^{M} \ddot{s}_i \ddot{b}_{im} \overline{a}_{im}.$$

Let $\ddot{q}_{im}$ denote the fraction of this area that is due to shooters of type i using munitions of type m, so that

$$\ddot{q}_{im} = \begin{cases} \ddot{s}_i \ddot{b}_{im} \overline{a}_{im} / \tilde{q}, \tilde{q} > 0, \\ 0, \tilde{q} = 0. \end{cases}$$

The shooters are assumed to fire in layers such that:

(1) the fraction of the area covered by each layer that is due to shooters of type i using munitions of type m is $\ddot{q}_{im}$;

(2) each layer, except (perhaps) for the last layer, exactly covers the attacked area of size g with no overlap among the planned attack areas of different salvos in the same layer, and

(3) the areas covered by different salvos in different layers are mutually independent in the sense that knowledge that an independently randomly generated point in G is contained in the planning area of a salvo by a shooter of a particular type using a munition of a particular type gives no information concerning the coverage of that point by any other layer.

Since each of these layers has the same mix of shooters using the same mix of munitions and since events are independent across layers, an average probability of kill can be used. Let $\widehat{p}_j$ denote this average so that here

$$\widehat{p}_j = \sum_{i=1}^{I} \sum_{m=1}^{M} \overline{p}_{imj} \ddot{q}_{im} ,$$

where, as above,

$$\overline{p}_{imj} = \begin{cases} 1 - \left(1 - \ddot{p}_{imj}\right)^{\left\lfloor \frac{\ddot{a}'_{imj}}{\overline{a}_{im}} \right\rfloor} \left(1 - \ddot{p}_{imj} \left\langle \frac{\ddot{a}'_{imj}}{\overline{a}_{im}} \right\rangle \right), \overline{a}_{im} > 0, \\ 0, \overline{a}_{im} = 0, \end{cases}$$

and $\ddot{a}'_{imj} = \min\left(\ddot{a}_{imj}, g\right)$.

These coordination assumptions yield

<u>Equation AM4</u>:

$$\Delta t_j = \ddot{z}\dot{t}_j \left[1 - \left(1 - \widehat{p}_j\right)^{\left\lfloor \frac{\tilde{q}}{g} \right\rfloor} \left(1 - \widehat{p}_j \left\langle \frac{\tilde{q}}{g} \right\rangle \right) \right]$$

where $\widehat{p}_j$ and $\tilde{q}$ are as defined just above.

Note that the only difference between Equation A4 and Equation AM4 is the difference in the definitions of $\widehat{p}_j$ and $\tilde{q}$, and if $M = 1$ these definitions are equivalent.

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