

Sporadic Cubic Torsion

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Mazur's Theorem

Let E/\mathbb{Q} be an elliptic curve.

Theorem (Mazur, 1978)

$E(\mathbb{Q})_{\text{tors}}$ is isomorphic to one of the following groups.

$$\begin{array}{ll} \mathbb{Z}/N\mathbb{Z}, & \text{for } 1 \leq N \leq 10 \text{ or } N = 12, \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2N\mathbb{Z}, & \text{for } 1 \leq N \leq 4. \end{array}$$

Modular curves:

- $Y_1(N)$ parametrizes (E, P) with $P \in E[N]$ (of exact order N);
- $Y_1(M, N)$ parametrizes containments $\mathbb{Z}/M\mathbb{Z} \oplus \mathbb{Z}/N\mathbb{Z} \subset E(K)_{\text{tors}}$.

Mazur:

$Y_1(N)(\mathbb{Q}) \neq \emptyset$ and $Y_1(2, 2N)(\mathbb{Q}) \neq \emptyset$ iff N are as above.

Modular curves via Tate normal form

Example ($N = 9$)

$E(K) \supset \mathbb{Z}/9\mathbb{Z}$ if and only if there exists $t \in K$ such that E is isomorphic to

$$y^2 + (t - rt + 1)xy + (rt - r^2t)y = x^3 + (rt - r^2t)x^2$$

where r is $t^2 - t + 1$. The torsion point is $(0, 0)$.

Example ($N = 11$)

$E(K) \supset \mathbb{Z}/11\mathbb{Z}$ if and only if there exist $a, b \in K$ such that

$$a^2 + (b^2 + 1)a + b;$$

in which case E is isomorphic to

$$y^2 + (s - rs + 1)xy + (rs - r^2s)y = x^3 + (rs - r^2s)x^2$$

where r is $ba + 1$ and s is $-b + 1$.

Rational Points on $X_1(N)$ and $X_1(2, 2N)$

Let $X_1(N)$ and $X_1(M, N)$ be smooth compactifications of $Y_1(N)$ and $Y_1(M, N)$.

We can restate Mazur's Theorem as follows.

Theorem (Mazur, 1978)

- $X_1(N)$ and $X_1(2, 2N)$ have **genus 0** for **exactly** the N in Mazur's Theorem.
- In particular, there are **infinitely many** E/\mathbb{Q} with such torsion structures.
- If $g(X)$ is **greater than 0**, then $X(\mathbb{Q})$ consists **only of cusps**.

Minimalism

The *simplest* thing that could happen does for these modular curves.

Higher Degree Torsion

Let K/\mathbb{Q} have degree d .

Theorem

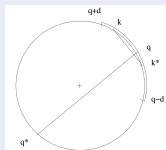
If $p \mid \#E(K)_{\text{tors}}$, then:

$$(\text{Merel, 1996}) \quad p \leq d^{3d^2}$$

$$(\text{Oesterlé}) \quad p \leq (3^{d/2} + 1)^2 \text{ (if } p > 3)$$

Proof: **formal immersions** on $\text{Sym}^{(d)} X_1(p)$.

Expository reference: Darmon, Rebello (Clay 2006)



Problem: Classify possibilities for $E(K)_{\text{tors}}$ for K/\mathbb{Q} of degree d .

Quadratic Torsion

Theorem (Kamienny–Kenku–Momose, 1980's)

*Let E be an elliptic curve over a quadratic number field K .
Then $E(K)_{tors}$ is one of the following groups.*

$$\begin{aligned} &\mathbb{Z}/N\mathbb{Z}, && \text{for } 1 \leq N \leq 16 \text{ or } N = 18, \\ &\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2N\mathbb{Z}, && \text{for } 1 \leq N \leq 6, \\ &\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3N\mathbb{Z}, && \text{for } 1 \leq N \leq 2, \text{ or} \\ &\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}. \end{aligned}$$

- The corresponding modular curves all have $g(X) \leq 2$.
- Each admits a **degree 2 map** $X \rightarrow \mathbb{P}^1$.
- This guarantees that $\text{Sym}^{(2)} X(\mathbb{Q})$ is infinite.
- i.e., each has infinitely many quadratic points.

Sporadic Points

Let X/\mathbb{Q} be a curve and let $P \in \overline{\mathbb{Q}}$. The **degree** of P is $[\mathbb{Q}(P) : \mathbb{Q}]$.

The set of degree d points of X is infinite if (and only if)

- X admits a degree d map $X \rightarrow \mathbb{P}^1$;
- X admits a degree d map $X \rightarrow E$, where $\text{rank } E(\mathbb{Q}) > 0$; or
- Jac_X contains a positive rank abelian subvariety such that ...

Most $\overline{\mathbb{Q}}$ points on curves arise in this fashion.

- We call outliers **isolated**.
- **Cusps and CM** points are often isolated on modular curves.
- An isolated point is **sporadic** if it is not cuspidal or CM.

See Bianca Viray's CNTA talk, linked [here](#).

Cubic Torsion

Theorem (Jeon–Kim–Schweizer, 2004)

Let E be an elliptic curve over a cubic number field K . Then the subgroups which arise as $E(K)_{\text{tors}}$ infinitely often are exactly the following.

$$\begin{array}{ll} \mathbb{Z}/N\mathbb{Z}, & \text{for } 1 \leq N \leq 20, N \neq 17, 19, \text{ or} \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2N\mathbb{Z}, & \text{for } 1 \leq N \leq 7. \end{array}$$

Minimalist conjecture

Conjecture

A modular curve X admits a non cuspidal, non CM point of degree d if and only if

- *X admits a degree d map $X \rightarrow \mathbb{P}^1$; or*
- *X admits a degree d map $X \rightarrow E$, where $\text{rank } E(\mathbb{Q}) > 0$; or*
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Theorem (Najman, 2014)

The elliptic curve [162b1](#) has a 21-torsion point over $\mathbb{Q}(\zeta_9)^+$.

Theorem (Parent)

The largest prime that can divide $E(K)_{\text{tors}}$ in the cubic case is $p = 13$.

Classification of Cubic Torsion

Theorem (Etropolski–Morrow–ZB–Derickx–van Hoeij)

The only torsion subgroups which appear for an elliptic curve over a cubic field are

$$\begin{aligned} \mathbb{Z}/N\mathbb{Z}, & \quad \text{for } 1 \leq N \leq 21, N \neq 17, 19, \text{ and} \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2N\mathbb{Z}, & \quad \text{for } 1 \leq N \leq 7. \end{aligned}$$

The only sporadic point is the elliptic curve 162b1 over $\mathbb{Q}(\zeta_9)^+$.

Najman's example

explained

Theorem (Najman, 2014)

The elliptic curve 162b1 has a 21-torsion point over $\mathbb{Q}(\zeta_9)^+$.

- Let $H := \rho_{E,21}(G_{\mathbb{Q}})$.
- Then H contains an **index 3** subgroup H' such that $H' \subset \langle \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \rangle$
- Thus there is a degree 3 map

$$X_{H'} \rightarrow X_H$$

and an induced map

$$X_H \rightarrow \mathrm{Sym}^{(3)} X_{H'} \rightarrow \mathrm{Sym}^3 X_1(21)$$

Sporadic points on $X_1(N)$ with rational j -invariant

Bourdon–Gill–Rouse–Watson (2020)

The odd degree isolated points on $X_1(N)$ with rational j -invariant are

$$j = -3^2 \cdot 5^6/2^3, \text{ or } 3^3 \cdot 13/2^2$$

The first is the Najman cubic example, and the second corresponds to a degree 8 point on $X_1(28)$, found by Najman and González-Jiménez.

Bourdon–Hashimoto–Keller–Klagsbrun–Lowry–Duda–Morrison–Najman–Shukla, with Derickx–Van Hoeij (2023)

Strong evidence that the other other isolated $j \in \mathbb{Q}$ are

$$j = -7 \cdot 11^3 \text{ or } 7 \cdot 137^3 \cdot 2083^3 \quad (\text{from } X_0(37)(\mathbb{Q})).$$

Rouse–Sutherland–Zureick–Brown–Voight

Conjectural classification of $X_H(\mathbb{Q})$ for prime power level.

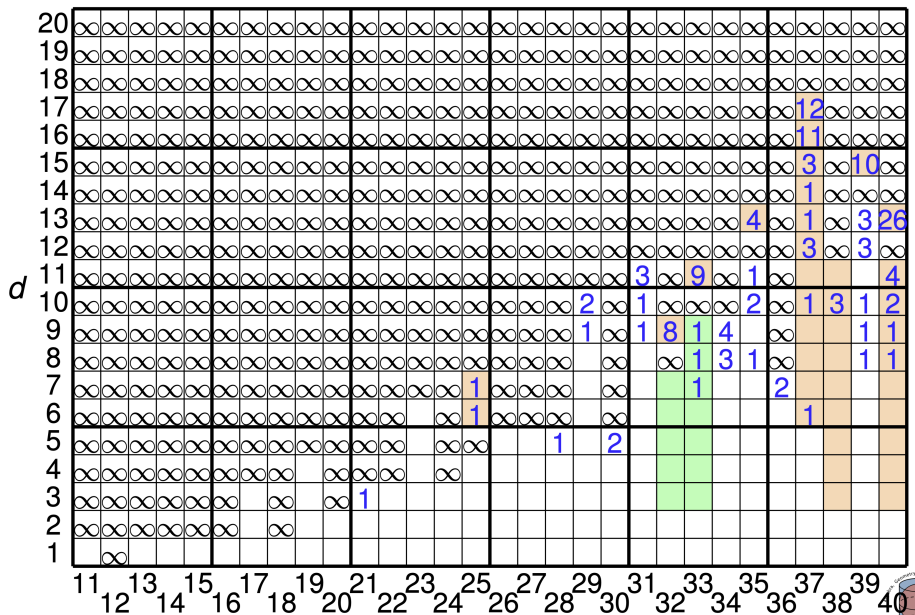
See Jeremy Rouse's CNTA talk, linked [here](#)

Mazur - Rational Isogenies of Prime Degree (1978)

Let N be a positive integer. Examples of elliptic curves over \mathbf{Q} possessing rational cyclic N -isogenies are known for the following values of N :

N	g	v	N	g	v	N	g	v
10	0	∞	11	1	3	27	1	1
12	0	∞	14	1	2	37	2	2
13	0	∞	15	1	4	43	3	1
16	0	∞	17	1	2	67	5	1
18	0	∞	19	1	1	163	13	1
25	0	∞	21	1	4			

More Sporadic Points on $X_1(N)$, via Derickx–van Hoeij



Classification of Cubic Torsion

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The only sporadic point is the elliptic curve 162b1 over $\mathbb{Q}(\zeta_9)^+$.

Good fortune – many small level ranks are zero

Let

$$S_0 = \{1, \dots, 36, 38, \dots, 42, 44, \dots, 52, 54, 55, 56, 59, 60, 62, 63, 64, 66, 68, \\ 69, 70, 71, 72, 75, 76, 78, 80, 81, 84, 87, 90, 94, 95, 96, 98, 100, 104, 105, \\ 108, 110, 119, 120, 126, 132, 140, 144, 150, 168, 180\},$$

$$S_1 = \{1, \dots, 21, 24, 25, 26, 27, 30, 33, 35, 36, 42, 45\}.$$

Theorem (Etropolski–Morrow–ZB–Derickx–van Hoeij)

- 1 $\text{rank } J_0(N)(\mathbb{Q}) = 0$ *if and only if* $N \in S_0$.
- 2 $\text{rank } J_1(N)(\mathbb{Q}) = 0$ *if and only if* $N \in S_0 - \{63, 80, 95, 104, 105, 126, 144\}$.
- 3 $\text{rank } J_1(2, 2N)(\mathbb{Q}) = 0$ *if and only if* $N \in S_1$.

Strategy

Previous work

- (Parent) handles $p > 13$ (via formal immersions).
- (Momose) $N = 27, 64$.
- (Wang) $N = 77, 91, 143, 169$
- (Bruin–Najman) $N = 40, 49, 55$

This leaves

- (rank 0) $N = 21, 22, 24, 25, 26, 28, 30, 32, 33, 35, 36, 39, 45$
- (rank 1) $N = 65, 121$

Rank 0

“Direct” analysis: $J(\mathbb{Q})$ is finite, and in principle it is a straightforward Riemann–Roch computation to compute the preimages of the Abel–Jacobi map:

$$X^{(d)}(\mathbb{Q}) \xrightarrow{\iota} J(\mathbb{Q})$$

Mordell–Weil Sieve: For a finite set S of primes of good reduction, we compare the images of α and β :

$$\begin{array}{ccc} X^{(d)}(\mathbb{Q}) & \xrightarrow{\iota} & J(\mathbb{Q}) \\ \downarrow & & \downarrow \alpha \\ \prod_{p \in S} X^{(d)}(\mathbb{F}_p) & \xrightarrow{\beta} & \prod_{p \in S} J(\mathbb{F}_p) \end{array}$$

Big obstacle: we need to know $J(\mathbb{Q})$!

Minutiae

Level	Genus	Method of proof	Genus of quotient
32	17	Maps to another curve in this table	$g(X_1(2, 16)) = 5$
36	17	Maps to another curve in this table	$g(X_1(2, 18)) = 7$
22	6	Local methods at $p = 3$ (§6.1)	N/A
25	12	Local methods at $p = 3$	N/A
21	5	Direct analysis over \mathbb{Q} (§6.2)	N/A
26	10	Direct analysis over \mathbb{F}_3	N/A
30	9	Direct analysis over \mathbb{Q} on $X_0(30)$ (§6.4)	$g(X_0(30)) = 3$
33	21	Direct analysis over \mathbb{Q} on $X_0(33)$	$g(X_0(33)) = 3$
35	25	Direct analysis over \mathbb{Q} on $X_0(35)$	$g(X_0(35)) = 3$
39	33	Direct analysis over \mathbb{Q} on $X_0(39)$	$g(X_0(39)) = 3$
(2,16)	5	Hecke bound + direct analysis over \mathbb{F}_3 (§6.5)	N/A
(2,18)	7	Hecke bound + direct analysis over \mathbb{F}_5	N/A
28	10	Hecke bound + direct analysis over \mathbb{F}_3 (§6.6)	N/A
24	5	Hecke bound + additional argument (§4.13) + direct analysis over \mathbb{F}_5	N/A
45	41	Hecke bound + direct analysis over \mathbb{Q} on $X_H(45)$ (§6.7)	$g(X_H(45)) = 5$
65	121	Formal immersion criteria (§7.3)	$g(X_0(65)) = 5$
121	526	Formal immersion criteria (§7.1)	$g(X_0(121)) = 6$

Formal immersions

- Classically, one takes p so large that any points of $X_1(p)^{(d)}(\mathbb{Q})$ reduces to a cusp mod 3
- (possible by the Hasse bound).
- *formal immersion criterion* \Rightarrow the diagonal map is injective

$$\begin{array}{ccccc} X^{(d)}(\mathbb{Q}) & \xrightarrow{\iota} & J(\mathbb{Q}) & \longrightarrow & A(\mathbb{Q}) \\ \uparrow & & & \nearrow & \\]\infty[& & & & \end{array}$$

Maarten's insight

- This doesn't really have anything to do with modular forms
- (just differentials).
- For small N , if you understand what is going on well enough, you can modify the “criterion” to any individual case you need.

Thank you!

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