Tropical geometry, p-adic integration, and uniformity.

David Zureick-Brown (Emory University)
Joe Rabinoff (Georgia Tech)
Eric Katz (Waterloo University)

Slides available at http://www.mathcs.emory.edu/~dzb/slides/

Special Session on Combinatorics and Algebraic Geometry

Fall Western Sectional Meeting
San Francisco State University

Oct 26, 2014

Faltings' theorem / Mordell's conjecture

Theorem (Faltings, Vojta, Bombieri)

Let X be a smooth curve over $\mathbb{Q}$ with genus at least 2. Then $X(\mathbb{Q})$ is finite.

Example

For $g \geq 2$, $y^2 = x^{2g+1} + 1$ has only finitely many solutions with $x, y \in \mathbb{Q}$.

Uniformity

Problem

1. *Given X, compute $X(\mathbb{Q})$ exactly.*
2. *Compute bounds on $\#X(\mathbb{Q})$.*

Conjecture (Uniformity)

There exists a constant $N(g)$ such that every smooth curve of genus g over $\mathbb{Q}$ has at most $N(g)$ rational points.

Theorem (Caporaso, Harris, Mazur)

Lang's conjecture $\Rightarrow$ uniformity.

Coleman's bound

Theorem (Coleman)

Let X be a curve of genus g and let $r = \operatorname{rank}_{\mathbb{Z}} \operatorname{Jac}_X(\mathbb{Q})$. Suppose $p > 2g$ is a prime of good reduction. Suppose $r < g$. Then

$$\#X(\mathbb{Q}) \leq \#X(\mathbb{F}_p) + 2g - 2.$$

Remark

1. A modified statement holds for $p \leq 2g$ or for $K \neq \mathbb{Q}$.
2. Note: this does not prove uniformity (since the first good p might be large).

Tools

p-adic integration and Riemann–Roch

Stoll's bound

Theorem (Stoll)

Let X be a curve of genus g and let $r = \operatorname{rank}_{\mathbb{Z}} \operatorname{Jac}_X(\mathbb{Q})$. Suppose $p > 2g$ is a prime of good reduction. Suppose $r < g$. Then

$$\#X(\mathbb{Q}) \leq \#X(\mathbb{F}_p) + 2r.$$

Tools

p-adic integration, Riemann–Roch, and **Clifford's theorem**

Bad reduction bound

Theorem (Lorenzini-Tucker, McCallum-Poonen)

Let X be a curve of genus g and let $r = \operatorname{rank}_{\mathbb{Z}} \operatorname{Jac}_X(\mathbb{Q})$. Suppose $p > 2g$ is a prime. Suppose $r < g$.

Let $\mathscr{X}$ be a regular proper model of X. *Then*

$$\#X(\mathbb{Q}) \leq \#\mathscr{X}^{\mathrm{sm}}(\mathbb{F}_p) + 2g - 2.$$

Remark (Still doesn't prove uniformity)

$\#\mathscr{X}^{\mathrm{sm}}(\mathbb{F}_p)$ can contain an n-gon, for n arbitrarily large.

Tools

p-adic integration and arithmetic Riemann–Roch ($\mathcal{K} \cdot \mathcal{X}_p = 2g - 2$)

Improved bad reduction bound

Theorem (Katz-ZB)

Let X be a curve of genus g and let $r = \operatorname{rank}_{\mathbb{Z}} \operatorname{Jac}_X(\mathbb{Q})$. Suppose $p > 2g$ is a prime. Let $\mathscr{X}$ be a regular proper model of X. Suppose $r < g$. Then

$$\#X(\mathbb{Q}) \leq \#\mathscr{X}^{\mathrm{sm}}(\mathbb{F}_p) + 2r.$$

Remark

Still doesn't prove uniformity.

Tools

p-adic integration and Clifford's theorem for graphs

Stoll's hyperelliptic uniformity theorem

Theorem (Stoll)

Let X be a hyperelliptic curve of genus g and let $r = \operatorname{rank}_{\mathbb{Z}} \operatorname{Jac}_X(\mathbb{Q})$. Suppose $r < g - 2$.

Let $\mathscr{X}$ be a stable proper model of X. *Then*

$$\#X(\mathbb{Q}) \leq 8(r+4)(g-1) + \max\{1, 4r\} \cdot g$$

Tools

p-adic integration on annuli

comparison of different analytic continuations of p-adic integration

Main Theorem (partial uniformity for non-hyperelliptic curves)

Theorem (Katz, Rabinoff, ZB)

Let X be any curve of genus g and let $r = \operatorname{rank}_{\mathbb{Z}} \operatorname{Jac}_X(\mathbb{Q})$. Suppose $r \leq g - 2$. Let $d = 3^{(g+1)^2}$ and let $p \geq 2g + d$. Then

$$\#X(\mathbb{Q}) \leq 2gp^{d/2} + (2g-2)(p^2+2) + 2 \cdot g^g(6g-6)(4g-4).$$

Tools

p-adic integration on annuli

comparison of different analytic continuations of p-adic integration

Rabinoff's bounds for Laurent series

Tropical canonical bundle

Comments

Corollary ((Partially) effective Manin-Mumford)

There is an effective constant $N(g)$ such that if $g(X) = g$, then

$$\#\left(X \cap \mathrm{Jac}_{X,tors}\right)(\mathbb{Q}) \leq N(g)$$

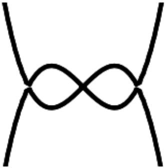
Corollary (In progress)

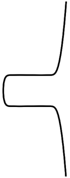
There is an effective constant $N'(g)$ such that if $g(X) = g > 3$ and X has totally degenerate, trivalent reduction mod 2, then

$$\#\left(X \cap \mathrm{Jac}_{X,tors}\right)(\mathbb{C}) \leq N'(g)$$

Models – semistable example

$$\begin{aligned} y^2 &= (x(x-1)(x-2))^3 - 5 \\ &= (x(x-1)(x-2))^3 \mod 5. \end{aligned}$$

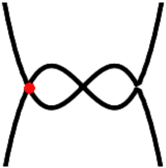


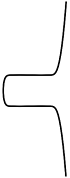


Note: no point can reduce to $(0,0)$. Local equation looks like $xy = 5$

Models – semistable example (not regular)

$$
\begin{aligned}
y^2 &= (x(x-1)(x-2))^3 - 5^{4} \\
&= (x(x-1)(x-2))^3 \mod 5
\end{aligned}
$$

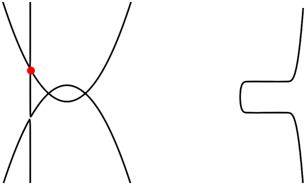




Now: $(0, 5^2)$ reduces to $(0, 0)$. Local equation looks like $xy = 5^4$

Models – semistable example

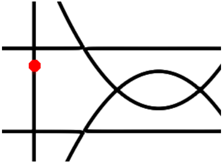
$$\begin{aligned} y^2 &= (x(x-1)(x-2))^3 - 5^{4} \\ &= (x(x-1)(x-2))^3 \mod 5 \end{aligned}$$

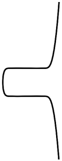


Blow up. Local equation looks like $xy = 5^3$

Models – semistable example (regular at (0,0))

$$\begin{aligned} y^2 &= (x(x-1)(x-2))^3 - 5^{4} \\ &= (x(x-1)(x-2))^3 \mod 5 \end{aligned}$$





Blow up. Local equation looks like $xy = 5$

Chabauty's method

(*p*-**adic integration**) There exists $V \subset H^0(X_{\mathbb{Q}_p}, \Omega^1_X)$ with $\dim_{\mathbb{Q}_p} V \geq g - r$ such that,

$$\int_P^Q \omega = 0 \qquad \forall P, Q \in X(\mathbb{Q}), \omega \in V$$

(**Coleman, via Newton Polygons**) Number of zeroes in a residue disc D_P is $\leq 1 + n_P$, where $n_P = \#\left(\operatorname{div}\omega \cap D_P\right)$

(**Riemann-Roch**) $\sum n_P = 2g - 2$.

(**Coleman's bound**) $\sum_{P \in X(\mathbb{F}_p)} (1 + n_P) = \#X(\mathbb{F}_p) + 2g - 2$.

Example (from McCallum-Poonen's survey paper)

Example

$$X\colon y^2 = x^6 + 8x^5 + 22x^4 + 22x^3 + 5x^2 + 6x + 1$$

1. Points reducing to $\widetilde{Q} = (0,1)$ are given by

$$\begin{aligned} x &= \quad p \cdot t, \text{ where } t \in \mathbb{Z}_p \\ y &= \quad \sqrt{x^6 + 8x^5 + 22x^4 + 22x^3 + 5x^2 + 6x + 1} = 1 + x^2 + \cdots \end{aligned}$$

2. $\displaystyle\int_{(0,1)}^{P_t} \frac{xdx}{y} = \int_0^t (x - x^3 + \cdots)dx$

Chabauty's method

(*p*-**adic integration**) There exists $V \subset H^0(X_{\mathbb{Q}_p}, \Omega^1_X)$ with $\dim_{\mathbb{Q}_p} V \geq g - r$ such that,

$$\int_P^Q \omega = 0 \qquad \forall P, Q \in X(\mathbb{Q}), \omega \in V$$

(**Coleman, via Newton Polygons**) Number of zeroes in a residue disc D_P is $\leq 1 + n_P$, where $n_P = \#\left(\operatorname{div} \omega \cap D_P\right)$

(**Riemann-Roch**) $\sum n_P = 2g - 2$.

(**Coleman's bound**) $\sum_{P \in X(\mathbb{F}_p)} (1 + n_P) = \#X(\mathbb{F}_p) + 2g - 2$.

Analytic continuation of integrals

(**Residue Discs.**)

$$P \in \mathcal{X}^{\mathrm{sm}}(\mathbb{F}_P),\ t\colon D_P \cong p\mathbb{Z}_p,\ \omega|_{D_P} = f(t)dt$$

(**Integrals on a disc.**)

$$Q, R \in D_P,\ \int_Q^R \omega := \int_{t(Q)}^{t(R)} f(t)dt.$$

(**Integrals between discs.**)

$$Q \in D_{P_1},\ R \in D_{P_2},\ \int_Q^R \omega := ?$$

Analytic continuation of integrals via Abelian varieties

(**Integrals between discs.**)

$$Q \in D_{P_1},\ R \in D_{P_2},\ \int_Q^R \omega := ?$$

(**Albanese map.**)

$$\iota\colon X \hookrightarrow \mathrm{Jac}_X,\ Q \mapsto [Q - \infty]$$

(**Abelian integrals via functorality and additivity.**)

$$\int_Q^R \iota^*\omega = \int_{\iota(Q)}^{\iota(R)} \omega = \int_{[Q-\infty]}^{[R-\infty]} \omega = \int_0^{[R-Q]} \omega = \frac{1}{n}\int_0^{n[R-Q]} \omega$$

Analytic continuation of integrals via Frobenius

(**Integrals between discs.**)

$$Q \in D_{P_1},\ R \in D_{P_2},\ \int_Q^R \omega := ?$$

(**Abelian integrals via functorality and Frobenius.**)

$$\int_Q^R \omega = \int_Q^{\phi(Q)} \omega + \int_{\phi(Q)}^{\phi(R)} \omega + \int_{\phi(R)}^R \omega$$

(**Very clever trick (Coleman)**)

$$\int_{\phi(Q)}^{\phi(R)} \omega_i = \int_Q^R \phi^* \omega = \sum_j \int_Q^R a_{ij} \omega_j$$

Comparison of integrals

Facts

1. For X with good reduction, the **Abelian** and **Coleman** integrals agree.
2. ~~A mystery~~. The associated Berkovich curve is contractable.
3. For X with bad reduction they differ.

Theorem (Stoll)

There exist linear functions $a(\omega), c(\omega)$ such that

$$\oint_Q^R \omega - \int_Q^R \omega = a(\omega)\left(\log(t(R)) - \log(t(Q))\right) + c(\omega)\left(t(Q) - t(R)\right)$$

Why bother? Integration on Annuli (a trade off)

Assumption

Assume $\mathcal{X}/\mathbb{Z}_p$ is **stable**, but not regular.

(**Residue Discs.**)

$P \in \mathcal{X}^{\mathrm{sm}}(\mathbb{F}_p)$, $t\colon D_P \cong p\mathbb{Z}_p$, $\omega|_{D_P} = f(t)dt$

(**Residue Annuli.**)

$P \in \mathcal{X}^{\mathrm{sing}}(\mathbb{F}_p)$, $t\colon D_P \cong p\mathbb{Z}_p - p^r\mathbb{Z}_p, \omega|_{D_P} = f(t, t^{-1})dt$

(**Integrals on an annulus are multivalued.**)

$$\int_Q^R \omega := \int_{t(Q)}^{t(R)} f(t, t^{-1})dt = \cdots + a(\omega)\log t + \cdots$$

(**Cover the annulus with discs**)

Each analytic continuation implicitly chooses a branch of log.

Why bother? Integration on Annuli (a trade off)

(**Abelian integrals.**) Analytically continue via Albanese.

$$\oint_Q^R \omega := \int_{t(Q)}^{t(R)} f(t, t^{-1})dt = \cdots + a(\omega) \log_{\mathrm{ab}} t + \cdots$$

(**Berkovich-Coleman integrals.**) Analytically continue via Frobenius.

$$\int_Q^R \omega := \int_{t(Q)}^{t(R)} f(t, t^{-1})dt = \cdots + a(\omega) \log_{\mathrm{Col}} t + \cdots$$

(**Stoll's theorem.**)

$$\oint_Q^R \omega - \int_Q^R \omega = a(\omega) \left(\log_{\mathrm{ab}}(r(R)) - \log_{\mathrm{ab}}(t(Q))\right) + c(\omega) \left(t(Q) - t(R)\right)$$

Stoll's comparison theorem, tropical geometry edition

Theorem (Katz, Rabinoff, ZB)

The difference $\log_{Col} - \log_{ab}$ *is the unique homomorphism that takes the value*

$$\int_\gamma \omega$$

on $Trop(\gamma)$*, where* $Trop\colon G(\mathbb{K}) \to T(\mathbb{K})/T(\mathcal{O})$.

$$\begin{tikzcd} & T \arrow[d] & \\ \Lambda \arrow[r] & G \arrow[r] \arrow[d] & \mathrm{Jac}_X \\ & B & \end{tikzcd}$$

T = torus, Λ = discrete, and B = Abelian w/ good reduction.

Proof of uniformity

(**Ignore the log and comparison terms (Stoll's idea)**)

$r < g - 2$ and linear algebra allows one to find ω with
no log term, and
same abelian and Coleman integrals.

(**Bounds on Annuli, via Rabinoff**)

$\int \omega$ is a Laurent series
$\#$ zeroes is bounded by the number of zeroes *and* poles of ω

(**Global step**)

- ω gives a section of the tropical canonical bundle on the dual graph.
- The order of the pole of ω at a node is the slope of the section of the tropical canonical bundle on the corresponding edge of the dual graph.