Explicit Modular Approaches to Generalized Fermat Equations

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Emory University Colloquium

February 14, 2011

Basic Problem (Solving Diophantine Equations)

Let $f_1, \ldots, f_m \in \mathbb{Z}[x_1, ..., x_n]$ be polynomials and let R be a ring (e.g., $R = \mathbb{Z}$, $\mathbb{Q}$).

Problem

Describe the set

$$\left\{(a_1, \ldots, a_n) \in R^n : \forall i, f_i(a_1, \ldots, a_n) = 0\right\}.$$

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Fact

Solving diophantine equations is hard.

Fermat's Last Theorem

Theorem (Wiles; Taylor-Wiles 1995)

The only integer solutions to the equation

$$x^n + y^n = z^n, n \geq 3$$

satisfy $xyz = 0$.

Template for the proof of FLT

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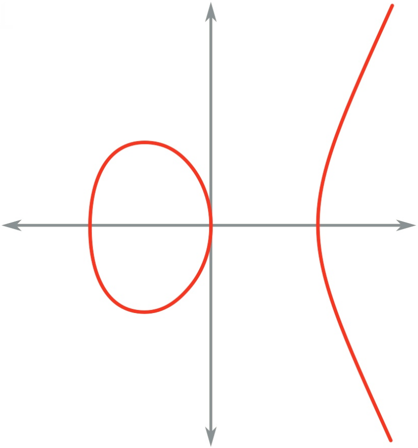
Step 4: Prove that every elliptic curve over $\mathbb{Q}$ is modular.

Modularity is now a theorem

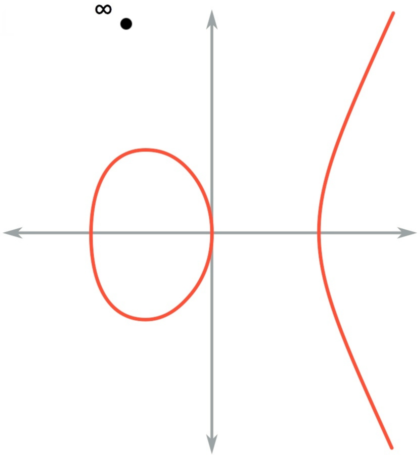
Theorem (Wiles 1995; Breuil-Conrad-Diamond-Taylor 2002)

Every elliptic curve over $\mathbb{Q}$ is modular.



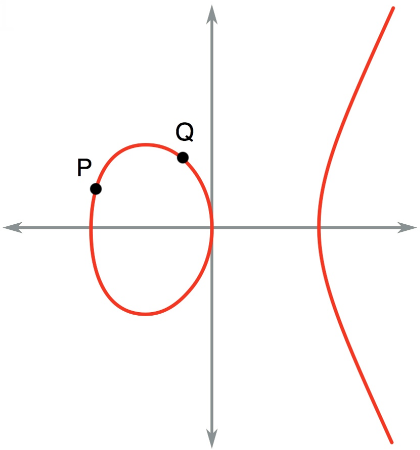
$$E: y^2 = x^3 + ax + b$$

Elliptic Curves - point at infinity



$$E: zy^2 = x^3 + axz^2 + bz^3$$
$$\infty = [0 : 1 : 0]$$

Elliptic Curves – addition



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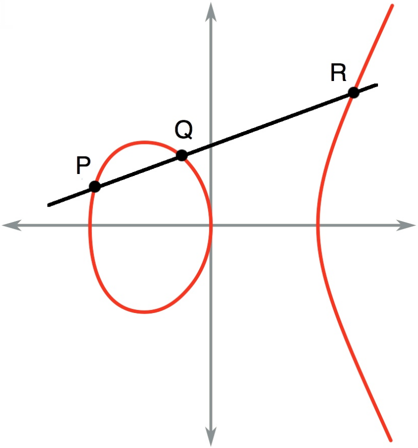

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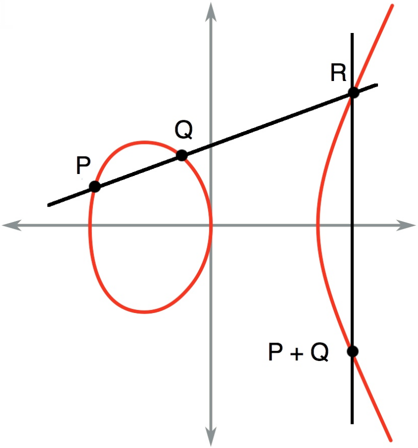

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$$P + Q = (x_2, -y_2) \in \mathbb{Q}^2$$

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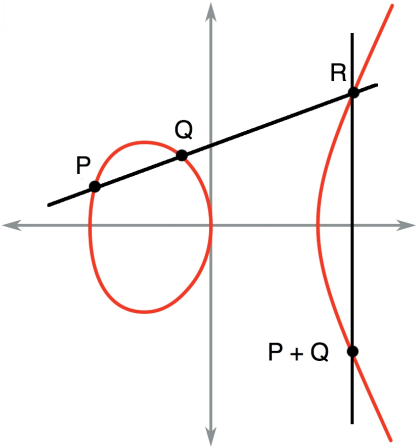
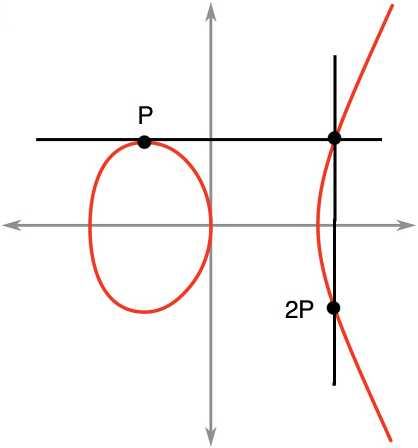


$$E: y^2 = x^3 + ax + b$$

$$E(\mathbb{Q}) \times E(\mathbb{Q}) \to E(\mathbb{Q})$$

$$(P\ ,\ Q) \quad \mapsto P + Q$$

Elliptic Curves - duplication



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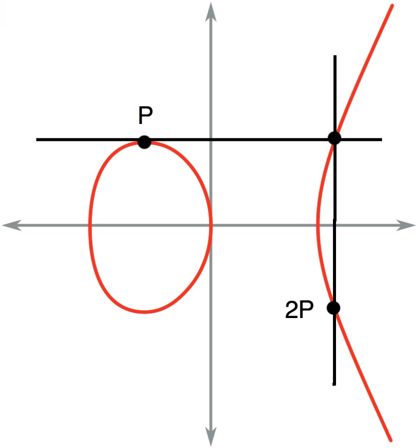

$$E\colon y^2 = x^3 + ax + b$$
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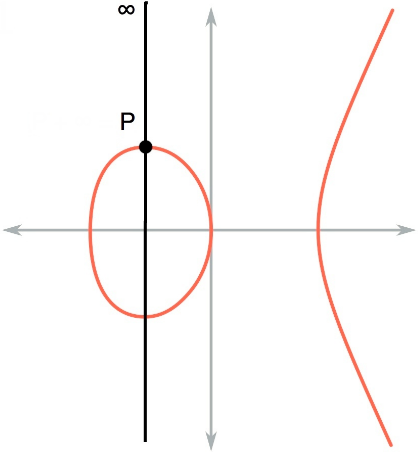
$$2P = (x_3, y_3) \in \mathbb{Q}^2$$

Elliptic Curves – identity



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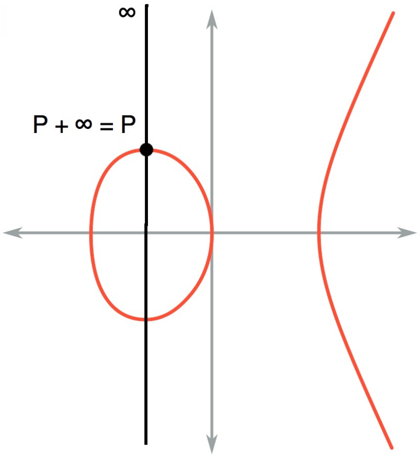

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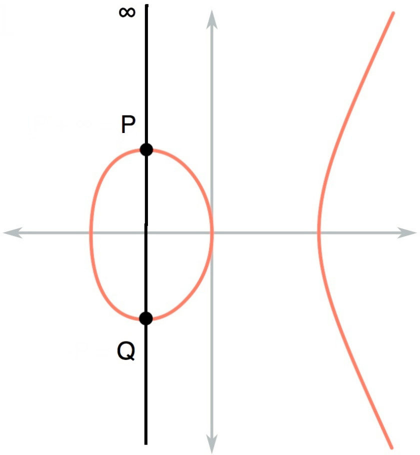

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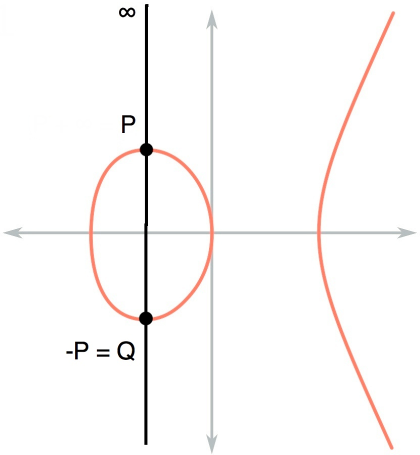

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Elliptic Curves – torsion subgroup

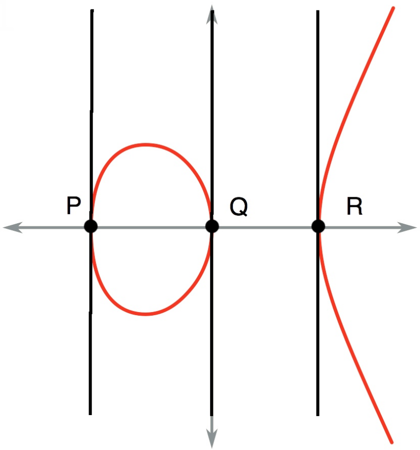
Let $n \in \mathbb{Z}$ be an integer.

Definition

The *n-torsion subgroup* $E[n]$ of E is defined to be

$$\ker\left(E \xrightarrow{[n]} E\right) := \{P \in E : nP := P + \ldots + P = \infty\}.$$

Elliptic Curves – two torsion



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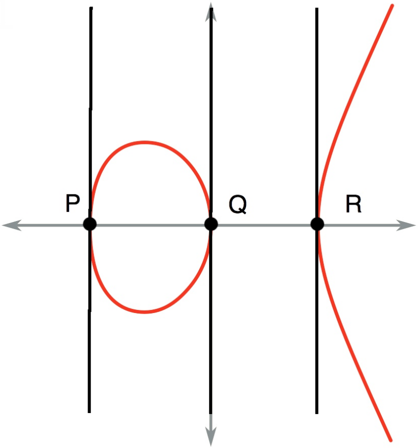

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Elliptic Curves – structure of torsion

Let E be given by the equation $y^2 = f(x) = x^3 + ax + b$.

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- $E[n](\mathbb{Q})$ may be smaller, e.g.,

$$E[2](\mathbb{Q}) \cong \begin{cases} \{\infty\} & \text{if } f(x) \text{ has 0 rational roots} \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } f(x) \text{ has 1 rational roots} \\ (\mathbb{Z}/2\mathbb{Z})^2, & \text{if } f(x) \text{ has 3 rational roots} \end{cases}$$

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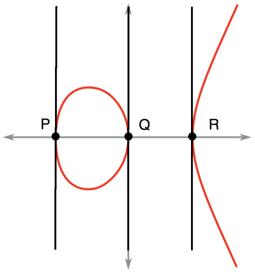

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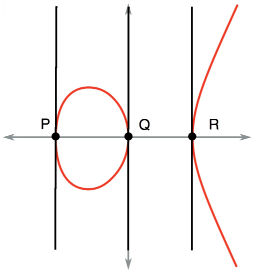
Definition

The mod n Galois representation associated to E is the homomorphism

$$G_{\mathbb{Q}} \to Aut(E[n]) \cong \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z}).$$



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Galois Representations: examples

Example

Suppose that $E(\mathbb{Q})[2] \cong (\mathbb{Z}/2\mathbb{Z})^2$. (E.g., $E\colon y^2 = x(x-1)(x-\lambda)$ with $\lambda \in \mathbb{Q}$.) Then

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$$\begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}.$$

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Definition

- A **modular function** is a complex analytic function $f : \mathcal{H} \to \mathbb{C}$ which is invariant under the action of a congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ such that f is holomorphic at ∞.
- A **modular form of weight** $2k$ is a complex analytic function $f : \mathcal{H} \to \mathbb{C}$ such that $f(z)(dz)^k$ is invariant under the action of a congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ such that f is holomorphic at ∞.

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Step 4: The output of step 3 turns out to be empty!

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Theorem (Bugeaud, Mignotte, Siksek 2006)

The only Fibonacci numbers that are perfect powers are

$$F_0 = 0,\ F_1 = F_2 = 1,\ F_6 = 8,\ F_{12} = 144.$$

More applications of the modular method

Theorem (Darmon, Merel 1997)

Any pairwise coprime integer solution to the equation

$$x^n + y^n = z^2, n \geq 4$$

satisfies $xyz = 0$.

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Step 5: (Easy) Find all triples (a, b, c) such that $E_{(a,b,c)}$ has CM.

Generalized Fermat Equations

Fix $p, q, r \in \mathbb{N}$ such that $\chi = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1 < 0$.

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Theorem (Darmon, Granville 1995)

The equation

$$x^p + y^q = z^r$$

has only finitely many coprime solutions with $xyz \neq 0$.

Examples of Generalized Fermat Equations

$$\chi = \frac{1}{2} + \frac{1}{3} + \frac{1}{7} - 1 = -\frac{1}{42} < 0$$

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Theorem (Poonen, Schaefer, Stoll 2008)

The coprime integer solutions to $x^2 + y^3 = z^7$ are the 16 triples

$$(\pm 1, -1, 0), \quad (\pm 1, 0, 1), \quad \pm(0, 1, 1), \quad (\pm 3, -2, 1),$$
$$(\pm 71, -17, 2), (\pm 2213459, 1414, 65), \quad (\pm 15312283, 9262, 113),$$
$$(\pm 21063928, -76271, 17).$$

Generalized Fermat Equations – Known Solutions

The 'known' solutions to the equation $x^p + y^q = z^r$ with $\chi = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1 < 0$ and $xyz \neq 0$ are the following:

The 'known' solutions to the equation $x^p + y^q = z^r$ with $\chi = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1 < 0$ and $xyz \neq 0$ are the following:

$$1^p + 2^3 = 3^2 \quad (-1)^{2p} + 2^3 = 3^2 \quad 2^5 + 7^2 = 3^4$$

$$7^3 + 13^2 = 2^9 \quad 2^7 + 17^3 = 71^2 \quad 3^5 + 11^4 = 122^2$$

$$17^7 + 76271^3 = 21063928^2 \quad 1414^3 + 2213459^2 = 65^7$$

$$9262^3 + 15312283^2 = 113^7 \quad 43^8 + 96222^3 = 30042907^2$$

$$33^8 + 1549034^2 = 15613^3$$

Generalized Fermat Equations – Known Solutions

Conjecture (Beal, Granville, Tijdeman-Zagier)

This is a complete list of coprime non-zero solutions such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1 < 0$.

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...or even for a counterexample.

(p, q, r) such that $\chi < 0$ and the solutions to $x^p + y^q = z^r$ have been determined.

$\{n, n, n\}$	Wiles, Taylor-Wiles, building on work of many others
$\{2, n, n\}$	Darmon-Merel, others for small n
$\{3, n, n\}$	Darmon-Merel, others for small n
$\{5, 2n, 2n\}$	Bennett
$(2, 4, n)$	Ellenberg, Bruin, Ghioca $n \geq 4$
$(2, n, 4)$	Bennett-Skinner; $n \geq 4$
$\{2, 3, n\}$	Poonen-Shaefer-Stoll, Bruin. $6 \leq n \leq 9$
$\{2, 2\ell, 3\}$	Chen, Dahmen, Siksek; primes $7 < \ell < 1000$ with $\ell \neq 31$
$\{3, 3, n\}$	Bruin; $n = 4, 5$
$\{3, 3, \ell\}$	Kraus; primes $17 \leq \ell \leq 10000$
$(2, 2n, 5)$	Chen $n \geq 3^*$
$(4, 2n, 3)$	Bennett-Chen $n \geq 3$
$(6, 2n, 2)$	Bennett-Chen $n \geq 3$
$(2, 6, n)$	Bennett-Chen $n \geq 3$

Main Theorem

$\chi = \frac{1}{2} + \frac{1}{3} + \frac{1}{10} - 1 = -\frac{1}{15}$ is maximal among unsolved Fermat equations.

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Theorem (B., 2011)

The only coprime integer solutions to the equation

$$x^2 + y^3 = z^{10}$$

are the 12 triples

$$(\pm 1, -1, 0), (\pm 1, 0, \pm 1), (0, 1, \pm 1), (\pm 3, -2, \pm 1).$$

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It is the first generalized Fermat equation of the form $x^2 + y^3 = z^n$ conjectured to have only trivial solutions.

$(3^2 + (-2)^3 = 1^n$ is considered to be trivial.)

Framework for solving $x^2 + y^3 = z^n$

Step 1: Assume there is a counterexample $a^2 + b^3 = c^n$.

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Step 2: Study the elliptic curve

$$\begin{aligned} E_{(a,b,c)}: y^2 &= x^3 + 3bx - 2a \\ \Delta &= -12^3 c^n \\ j &= 12^3 b^3 / c^n. \end{aligned}$$

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Step 5: For each such E, find all (a, b, c) such that $E \cong E_{(a,b,c)}$.

Framework for solving $x^2 + y^3 = z^n$

For large n, this template (conjecturally) works!

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Explicitly classify possibilities for $\rho_{E_{(a,b,c)},n}$.

- For large n, there are 13 possibilities for $\rho_{E_{(a,b,c)},n}$, which are 'independent of n'.

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Conjecture (Frey-Mazur)

Let $p > 23$ be a prime and E and E' be elliptic curves such that $\rho_{E,p} \cong \rho_{E',p}$. Then E is isogenous to E'.

Template breaks down for $x^2 + y^3 = z^{10}$

$$E_{(a,b,c)} \colon y^2 = x^3 + 3bx - 2a$$

Step 3:

Explicitly classify possibilities for $\rho_{E_{(a,b,c)},10}$.

Template breaks down for $x^2 + y^3 = z^{10}$

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Explicitly classify possibilities for $\rho_{E_{(a,b,c)},10}$.

- Known tools for classifying $\rho_{E_{(a,b,c)},10}$ fail.

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- Known tools for classifying $\rho_{E_{(a,b,c)},10}$ fail.
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  - $\rho_{E_{(a,b,c)},n}$ may be reducible for both $n = 2$ and 5.

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Definition

We say that $\rho \colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F}_\ell)$ is *reducible* if there is some subspace $W \subset \mathbb{F}_\ell^2$ such that for every $P \in W$, $P^{\rho(\sigma)} \in W$.

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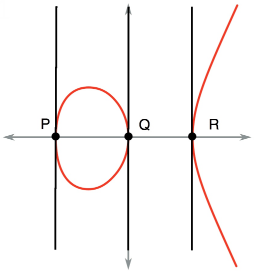
- Using one prime is not enough.
  - E.g., there are infinitely many elliptic curves over $\mathbb{Q}$ with trivial mod 2 representation ($E \colon y^2 = x(x-1)(x-\lambda)$).
- Multiprime approaches seem to be computationally infeasible.

Elliptic Curves – torsion



$$E[2](\mathbb{Q}) \cong \begin{cases} \{\infty\} & \text{if } f(x) \text{ has 0 rational roots} \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } f(x) \text{ has 1 rational roots} \\ (\mathbb{Z}/2\mathbb{Z})^2, & \text{if } f(x) \text{ has 3 rational roots} \end{cases}$$

Step 3: Classifying mod 2 Galois representations

Let E be given by the equation $y^2 = f(x) := x^3 + 3bx - 2a$

Fact

The splitting field of the polynomial $f(x)$ completely determines $\rho_{E,2}$.

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- The splitting field K of $x^3 + 3bx - 2a$ is unramified outside of $\{2, 3\}$ and of degree at most 6.
- (Hermite) There are only finitely many such fields.
- These days there are sophisticated algorithms for enumerating such K.

Step 3: Progress for $\ell = 2$

Lemma

There are elliptic curves $\{E_1, \ldots, E_n\}$ such that for every (a, b, c) such that $a^2 + b^3 = c^{10}$, there is an i such that

$$\rho_{E_{(a,b,c)},2} \cong \rho_{E_i,2}.$$

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- **Wanted**: a similar lemma for $\rho_{E_{(a,b,c)},5}$.

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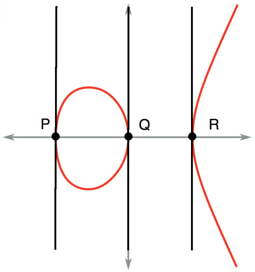
- **Wanted**: a similar lemma for $\rho_{E_{(a,b,c)},5}$.
- **Problem**: $\rho_{E_{(a,b,c)},5}$ may be reducible, thus modularity won't help!

Parameter spaces for Galois representations

Definition

$X_E(n)$ is the parameter space for pairs (E', ψ), where E' is an elliptic curve and $\psi\colon \rho_{E,n} \to \rho_{E',n}$ is a symplectic isomorphism of mod n Galois representations.

Elliptic Curves – torsion



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Example

Let E be an elliptic curve with $E(\mathbb{Q})[2] \cong (\mathbb{Z}/2\mathbb{Z})^2$ (so that $\rho_{E,2}$ is trivial). Then E is of the form

$$E\colon y^2 = x(x-1)(x-\lambda).$$

Other parameter spaces

Recall that $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F}_\ell)$ is *reducible* if there is some subspace $W \subset \mathbb{F}_\ell^2$ such that for every $P \in W$, $P^{\rho(\sigma)} \in W$.

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Definition

$X_0(p)$ is the parameter space for elliptic curves such that $\rho_{E,p}$ is reducible (more precisely – pairs $(E, W \subset E[p])$, where E is an elliptic curve and W is an invariant subgroup of size p).

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Definition

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Example ($X_0(5)$)

Let $E\colon y^2 = x^3 + 3bx - 2a$, and suppose $\rho_{E,5}$ is reducible. Then there exists a $t \in \mathbb{Z}$ such that

$$12^3 \frac{b^3}{a^2 + b^3} = \frac{(t^2 + 250t + 3125)^3}{t^5}.$$

Step 3: Intermediate Modular curves

Goal

Explicitly classify possibilities for $\rho_{E_{(a,b,c)},5}$.

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$$\begin{tikzcd}
X(10) \arrow[rr] \arrow[dd] & & X(5) \arrow[d] \\
 & & X_0(5) \arrow[d] \\
X(2) \arrow[r, "\pi"] & X_0(2) \arrow[r] & X(1)
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- $\pi\colon (E, \psi\colon \rho_{\text{triv}} \cong \rho_{E,2}) \mapsto (E, W)$.

Step 3: Intermediate Modular curves: $p = 2$

- $\mathrm{Aut}(X(2)/X(1)) \cong \mathrm{GL}_2(\mathbb{F}_2) \cong S_3$.

$$\begin{tikzcd} & X_0(2) \arrow[dr] & \\ X(2) \arrow[ur] \arrow[dr] & & X(1) \\ & X_\Delta \arrow[ur] & \end{tikzcd}$$

- $\mathrm{Aut}(X(2)/X(1)) \cong \mathrm{GL}_2(\mathbb{F}_2) \cong S_3$.
- $X_0(2)$ is the quotient of $X(2)$ by a transposition.

$$\begin{tikzcd} & X_0(2) \arrow[dr] & \\ X(2) \arrow[ur] \arrow[dr] & & X(1) \\ & X_\Delta \arrow[ur] & \end{tikzcd}$$

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- Define X_Δ to be the quotient of $X(2)$ by the normal subgroup A_3.

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Step 3: Intermediate Modular curves: $p = 2$

- Define X_Δ to be the quotient of $X(2)$ by the normal subgroup A_3.
- X_Δ classifies pairs (E, z) such that $z^2 = j(E) - 12^3 = c_6(E)^2/\Delta_E$.

$$\begin{tikzcd} & X_0(2) \arrow[dr] & \\ X(2) \arrow[ur] \arrow[dr] & & X(1) \\ & X_\Delta \arrow[ur] & \end{tikzcd}$$

Goal

Explicitly classify possibilities for $\rho_{E_{(a,b,c)},5}$.

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- X classifies **triples** (E, W, z) such that
  - $z^2 = j(E) - 12^2 = c_4(E)^2/\Delta_E$,
  - W is an invariant subspace of $E[5]$ of order 5.

$$E_{(a,b,c)}\colon\ y^2 = x^3 + 3bx - 2a$$
$$\Delta = -12^3 c^{10} = -3(2^3 \cdot 3 \cdot c^5)^2$$

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$$\begin{array}{ccc} X_{(-3)} & \longrightarrow & X_0(5) \\ \downarrow & & \downarrow \\ X_{(-3\Delta)} & \longrightarrow & X(1) \end{array}$$

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- $X_{(-3)}$ classifies **triples** (E, W, z) such that
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- Reducible $\rho_{E_{(a,b,c)},5}$ thus give rise to a point on $X_{(-3)}(\mathbb{Q})$.

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- Reducible $\rho_{E_{(a,b,c)},5}$ thus give rise to a point on $X_{(-3)}(\mathbb{Q})$.
- $X_{(-3)}$ turns out to be an elliptic curve, with $X_{(-3)}(\mathbb{Q}) \cong \mathbb{Z}/5\mathbb{Z}$.

Step 3: Classifying $\rho_{E_{(a,b,c)},\ell}$

Lemma

There are elliptic curves $\{E_1, \ldots, E_n\}$ *and* $\{E'_1, \ldots, E'_{n'}\}$ *such that for every* (a, b, c) *such that* $a^2 + b^3 = c^{10}$*, there exists an* i *and* j *such that*

$$\rho_{E_{(a,b,c)},2} \cong \rho_{E_i,2}$$

and

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Thus, $E_{(a,b,c)}$ gives rise to a point on $X_{E_i}(2)(\mathbb{Q})$ and a point on $X_{E'_j}(5)(\mathbb{Q})$.

Step 4: Classify elliptic curves with a given pair of Galois representations

Thus, $E_{(a,b,c)}$ gives rise to a point on $X_{E_i}(2)(\mathbb{Q})$ and a point on $X_{E'_j}(5)(\mathbb{Q})$.

Step 4:

For a fixed i, j, classify **all** elliptic curves E for which $\rho_{E,2} \cong \rho_{E_i,2}$ and $\rho_{E,5} \cong \rho_{E'_j,2}$.

Step 4: Elliptic Chabauty

$$
\begin{tikzcd}
X_{E_i,E'_j}(10) \arrow[rr] \arrow[dr, dashed, "/K_{E'_j}"'] \arrow[dd] & & X_{E'_j}(5) \arrow[d, "/K_{E'_j}"] \\
& X_{E_i} \arrow[r] \arrow[d] & X_0(5) \arrow[d] \\
X_{E_i}(2) \arrow[r] & X_{\Delta_{E_i}} \arrow[r] & X(1)
\end{tikzcd}
$$

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$$

- For every coprime (a, b, c) such that $a^2 + b^3 = c^{10}$, we can find some E_i, E'_j and a point on $P \in X_{E_i}(K_{E'_j})$ such that $j(P) \in X(1)(\mathbb{Q})$.

Step 4: Elliptic Chabauty

$$\begin{tikzcd}
X_{E_i,E_j'}(10) \arrow[rr] \arrow[dd] \arrow[dr, dashed, "/K_{E_j'}"'] & & X_{E_j'}(5) \arrow[d, "/K_{E_j'}"] \\
& X_{E_i} \arrow[r] \arrow[d] & X_0(5) \arrow[d] \\
X_{E_i}(2) \arrow[r] & X_{\Delta_{E_i}} \arrow[r] & X(1)
\end{tikzcd}$$

- For every coprime (a, b, c) such that $a^2 + b^3 = c^{10}$, we can find some E_i, E_j' and a point on $P \in X_{E_i}(K_{E_j'})$ such that $j(P) \in X(1)(\mathbb{Q})$.
- This latter set is finite, and in fact *computable* (via p-adic integration and other methods).

Conclusion: New ideas for $x^2 + y^3 = z^{10}$

The template fails for $x^2 + y^3 = z^{10}$.

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3) Classifying both the mod 2 and mod 5 at the same time leads to 'high genus parameter spaces'.
    - New idea: translate the work to low genus parameter spaces, but over larger number fields than $\mathbb{Q}$.