

Math 355: Real Analysis
Instructor: David Zureick-Brown (“DZB”)

All assignments

Last updated: February 6, 2025

Gradescope code: DKNX3W

Show all work for full credit!

Proofs should be written in full sentences whenever possible.

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Gradescope Instructions for submitting work in Math 355

You will be using the online Gradescope program to submit your homework and exams. These instructions tell you how to sign up initially, and how to submit your written work.

Signing up for Gradescope the first time.

If you haven't used Gradescope for an **Amherst College** course before:

- Go to <http://www.gradescope.com>, click on "Sign up for free" (which may auto-scroll you to the bottom of the page), and select Sign up as [a] "Student".
- In the signup box:
 - Use the course entry code **DKNX3W**
 - Use your full name
 - Use your **Amherst College email** address. Or, if you are a Five-College student, use your email address from your own school.
 - Leave the "Student ID" entry blank.
- You will probably get an email asking to set a password for your account, so check your **amherst.edu** email inbox. (Or your email inbox through your own school, for Five-College students.)

Adding Math 355 to Gradescope.

If you **have** used Gradescope for an Amherst course before, and so you already have an account through your **amherst.edu** email, you still need to add Math 355, so:

- Go to <http://www.gradescope.com> and log in.
- Go to your Account Dashboard (click the Gradescope logo at upper left), and click "Add Course" at bottom right.
- Use the course code **DKNX3W**

(submission instructions on next page)

Submitting written work

First write it out on paper as you would normally. Then **scan it** to create a PDF. One method for scanning is the smartphone app **DropBox**. It makes nice clear scans, and it saves them directly into a folder so that you can have all your assignments in one place. **CamScanner** is another free scanning App, and there are others, too. **Gradescope** now has its own scanning app. You can also use a printer/scanner if you prefer.

Please be kind to our dear graders and make sure your submission is legible !

In particular, please leave some spacing between separate problems.

If you have a tablet computer, you may write your work there (instead of on paper) and save it as a PDF.

Some of you may know the math formatting package LaTeX and may want to use it in Math 355. That's fine, too; if so, you may write up your work in LaTeX and save the resulting PDF.

In short, any method is fine as long as it creates a legible PDF file and NOT a photo.

For example, if you use the DropBox app, then in your created *Math 355 Homework* Dropbox folder, you can select create (+) at the bottom of the screen and click the *Scan Document* option. Snap a shot of the first page of your homework, and then click [+] to snap shots of any subsequent pages. Do **not** use the *Take Photo* option.

After you have scanned/saved your work as a PDF, submit it on Gradescope as follows:

- Go to <http://www.gradescope.com> and log in.
- Select the course “Math 355, Spring 2025” and the appropriate assignment.
- Select “submit pdf” to submit your work in PDF format. Browse to find your PDF and upload.
- Now it is time to **tag** your problems. **This is an important step**, where you are telling Gradescope which problems are on which page(s).

For each problem, select the pages of your submission where your written solution appears.

I think the easiest thing to do is to click on the page of **your** homework upload where you wrote the given problem, and then click on the assigned problem listed. Repeat for each problem.

You must tag the problems or else you will not get credit for your work.

Gradescope will give you a warning when you go to submit your assignment if you have not selected the pages correctly. If you tag a problem incorrectly, you can fix it by clicking “More” and “Reselect Pages”.

- Click Save or Submit.

After your assignment is graded, you will be able to see your score on the written problems, along with comments, on Gradescope. You should receive an email notifying you when each homework set is graded.

Assignment 1: Introduction to the course; numbers

Due by 12:55pm, eastern, on Thursday, February 06

- **Suggested readings for this problem set:** 1.1, 1.2
- **Syllabus:** <https://dmzb.github.io/teaching/2025Spring355/syllabus-math-355-S25.pdf>
- **Gradescope instructions** (previous page)

All readings are from Abbott, *Understanding Analysis*.

Assignment: due Thursday, February 06, 12:55pm, via Gradescope (DKNX3W):

1. Let $A := \{2, \{2\}, \{2, \{2\}\}\}$. How many elements are in the set A ? For each of the following parts, answer true or false. Briefly explain our answer.
(a) $2 \in A$ (b) $2 \subseteq A$ (c) $\{2\} \subseteq A$ (d) $\{2\} \in A$
2. Consider the following sentence: If $a > 0$, then there exists a natural number n such that $\frac{1}{n} < a$.
(a) Write the negation.
(b) Write the contrapositive of the original sentence.
(c) Which is true, the original or the negation? Explain your answer intuitively. (Don't worry about giving a formal proof.)
3. Let $A := \{3, 5, 8\}$ and $B := \{3, 6, 10\}$.
(a) Is the following statement true or false? Justify your claim. $\exists b \in B$ s.t. $\forall a \in A, a - b < 0$.
(b) Negate the statement: $\exists b \in B$ s.t. $\forall a \in A, a - b < 0$.
(c) Is the following statement true or false? Justify your claim. $\forall b \in B, \exists a \in A$ s.t. $a - b < 0$.
4. Prove that there is no smallest strictly positive rational number.
5. Negate each of the following statements (do not worry about the validity of the statement):
(a) For all $x, y \in \mathbb{R}$ satisfying $x < y$, there exists an $n \in \mathbb{N}$ such that $x + 1/n < y$.
(b) For all $x, y \in \mathbb{R}$ satisfying $x < y$, there exists an $r \in \mathbb{Q}$ such that $x < r < y$.
(c) There exists a real number $x > 0$ such that $x < 1/n$ for all $n \in \mathbb{N}$.
(d) For all $M \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ satisfying $n \geq N$, one has $1/n < 1/M$.

6. Suppose $z \in \mathbb{R}$ and $z \geq 0$. Which of the following statements imply that $z = 0$? Justify your answer either by proof or counterexample.

- (a) $\forall \varepsilon \in (0, \infty), z < \varepsilon$.
- (b) $\forall \varepsilon \in (0, \infty), z \leq \varepsilon$.
- (c) $\forall \varepsilon \in (0, \infty), \exists \delta \in (0, \infty)$ s.t. $z \leq \varepsilon \cdot \delta$.
- (d) $\forall \varepsilon \in (0, \infty), \exists \delta \in (0, \infty)$ s.t. $z \cdot \delta \leq \varepsilon$.
- (e) $\forall \varepsilon \in (0, \infty), \exists \delta \in (0, \infty)$ s.t. $z \leq \delta < \varepsilon$.

7. **De Morgan's Laws.** Let A and B be subsets of \mathbb{R} .

- (a) If $x \in (A \cap B)^c$, explain why $x \in A^c \cup B^c$. This shows that $(A \cap B)^c \subseteq A^c \cup B^c$.
- (b) Prove the reverse inclusion $(A \cap B)^c \supseteq A^c \cup B^c$, and conclude that $(A \cap B)^c = A^c \cup B^c$.
- (c) Show $(A \cup B)^c = A^c \cap B^c$ by demonstrating inclusion both ways.
- (d) Show how induction can be used to conclude that

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$$

for any finite $n \in \mathbb{N}$.

8. Decide which of the following are true statements. Provide a short justification for those that are valid and a counterexample for those that are not:

- (a) Two real numbers satisfy $a < b$ if and only if $a < b + \varepsilon$ for every $\varepsilon > 0$.
- (b) Two real numbers satisfy $a < b$ if $a < b + \varepsilon$ for every $\varepsilon > 0$.
- (c) Two real numbers satisfy $a \leq b$ if and only if $a < b + \varepsilon$ for every $\varepsilon > 0$.

Bonus Problem:¹ Prove that there are two irrational numbers α and β such that α^β is rational.

¹Bonus problems are just for “fun”; they are not worth any points, so please do not submit them.

Assignment 2: Axiom of Completeness; sup, inf, min, max, real numbers, density

Suggested readings for this problem set: 1.3, 1.4

All readings are from Abbott, *Understanding Analysis*.

Assignment: due Thursday, February 13, 12:55pm, via Gradescope (DKNX3W):

1. Give an example of each of the following, or state that the request is impossible:
 - (a) A set B with $\inf B \geq \sup B$.
 - (b) A finite set that contains its infimum but not its supremum.
 - (c) A bounded subset of \mathbb{Q} that contains its supremum but not its infimum.
 - (d) Two sets A and B with $A \cap B = \emptyset$, $\sup A = \sup B$, $\sup A \notin A$, and $\sup B \notin B$.
 - (e) A sequence of nested open intervals $J_1 \supseteq J_2 \supseteq J_3 \supseteq \dots$ with $\bigcap_{n=1}^{\infty} J_n$ nonempty but containing only a finite number of elements.
2. Compute, without proofs, the suprema and infima (if they exist) of the following sets:
 - (a) $\{m/n : m, n \in \mathbb{N} \text{ with } m < n\}$.
 - (b) $\{(-1)^m/n : m, n \in \mathbb{N}\}$.
 - (c) $\{n/(3n+1) : n \in \mathbb{N}\}$.
 - (d) $\{m/(m+n) : m, n \in \mathbb{N}\}$.
3. Prove that if a is an upper bound for A , and if a is also an element of A , then it must be that $a = \sup A$.
4. Let $a < b$ be real numbers and consider the set $T = \mathbb{Q} \cap [a, b]$. Show $\sup T = b$.
5. Given sets A and B , define $A + B = \{a + b : a \in A \text{ and } b \in B\}$. Follow these steps to prove that if A and B are nonempty and bounded above, then $\sup(A + B) = \sup A + \sup B$.
 - (a) Let $s = \sup A$ and $t = \sup B$. Show $s + t$ is an upper bound for $A + B$.
 - (b) Now let u be an arbitrary upper bound for $A + B$, and temporarily fix $a \in A$. Show $t \leq u - a$.
 - (c) Finally, show $\sup(A + B) = s + t$.
 - (d) Construct another proof of this same fact using Lemma 1.3.8.
6. Decide if the following statements about suprema and infima are true or false. Give a short proof for those that are true. For any that are false, supply an example where the claim in question does not appear to hold.
 - (a) If A and B are nonempty, bounded, and satisfy $A \subseteq B$, then $\sup A \leq \sup B$.
 - (b) If $\sup A < \inf B$ for sets A and B , then there exists a $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$.
 - (c) If there exists a $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$, then $\sup A < \inf B$.

7. (a) Let A be nonempty and bounded below, and define $B = \{b \in \mathbb{R} : b \text{ is a lower bound for } A\}$. Show that $\sup B = \inf A$.
- (b) Use (a) to explain why there is no need to assert that greatest lower bounds exist as part of the Axiom of Completeness.
8. Let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and let $s \in \mathbb{R}$ have the property that for all $n \in \mathbb{N}$, $s + \frac{1}{n}$ is an upper bound for A and $s - \frac{1}{n}$ is not an upper bound for A . Show $s = \sup A$.



IN PROGRESS! Check back later for the final assignment.



Assignment 3: Completeness; countable and uncountable sets; Cantor diagonalization

Suggested readings for this problem set: 1.4, 1.5

All readings are from Abbott, *Understanding Analysis*.

Assignment: due Thursday, February 20, 12:55pm, via Gradescope (DKNX3W):

1. TBA
2. TBA
3. TBA
4. TBA
5. TBA
6. TBA
7. TBA
8. TBA



IN PROGRESS! Check back later for the final assignment.





IN PROGRESS! Check back later for the final assignment.



Assignment 4: Sequences; convergence, monotonicity; algebraic limit theorem (ALT and OLT)

Due by 12:55pm, eastern, on Thursday, February 27

Suggested readings for this problem set: 2.2, 2.3, 2.4

All readings are from Abbott, *Understanding Analysis*.

Assignment: due Thursday, February 27, 12:55pm, via Gradescope (DKNX3W):

1. TBA
2. TBA
3. TBA
4. TBA
5. TBA
6. TBA
7. TBA
8. TBA



IN PROGRESS! Check back later for the final assignment.





IN PROGRESS! Check back later for the final assignment.



Assignment 5: Monotone convergence; subsequences; Cauchy sequences; Bolzano–Weierstrass

Due by 12:55pm, eastern, on Thursday, March 06

Suggested readings for this problem set: 2.5, 2.6, 2.7

All readings are from Abbott, *Understanding Analysis*.

Assignment: due Thursday, March 06, 12:55pm, via Gradescope (DKNX3W):

1. TBA
2. TBA
3. TBA
4. TBA
5. TBA
6. TBA
7. TBA
8. TBA



IN PROGRESS! Check back later for the final assignment.





Midterm 1 study guide

In class exam, Thursday, March 11.

Content: The questions will all be either

1. homework problems,
2. suggested problems,
3. problems we worked in class, or
4. minor variations of one of these.

Problems with very long proofs or that involved some unusual trick will not be on the exam.

A typical exam will have a few questions from each week of the course and will cover **assignments 1-5**. The problems will be similar to the homework problems, and the proofs will use the same techniques. You can expect problems about following:

- Definitions.
- Continuity
- TBA

For definitions, I want a definition, in prose (complete sentences), and I want “just” the definition, and not any additional facts about the definition. (E.g., if you give the definition of differentiable, do not include any facts about differentiability, such as “a differentiable function is continuous”)

Additionally:

- This is a closed-book, 75 minute examination.
- No books, notes, calculators, cell phones, communication devices of any sort, or other aids are permitted.
- You are allowed to use Theorems, lemmas, etc from the book or from class, and previous homework problems as part of your solutions, and you are not required to reprove these during the exam. Try to reference them by name if possible (i.e., “by the Mean Value Theorem” or “a differentiable function is continuous”), but if you are unsure of the name of the theorem, it is ok to say “we proved in class that”. The exception to this is that if I ask you to prove e.g. the mean value theorem, then you must prove it and can’t reference that we proved it in class.
- I will be in the hallway, and will check in every 25 minutes for questions.
- You can always ask me (the instructor) if you have clarifying questions, but asking for hints or asking if a proof is correct or sufficient is not allowed.
- You may leave when you are finished, in which case please hand me the exam in the hallway.
- You may leave to use the bathroom.



- You may use the backs of pages for additional work space. If you use any additional scratch paper as part of your solution, please let me know after the exam.
- Be sure to justify your answer to each of the problems!
- For proofs, please make sure that your answers are in complete sentences!
- **Academic integrity:**
 - You are required to **turn off your phone** during the exam. If I observe that your phone is on, the penalty is that you fail the exam. (If there is an exceptional reason that your phone needs to be on during the exam, please discuss it with me).
 - **Submit only your own work.** Do not share your work, and do not look at other students' exams.
 - The penalty for cheating is at my discretion, and can range from failing the exam to failing the course.



IN PROGRESS! Check back later for the final assignment.



Assignment 6: Infinite series; convergence; absolute and conditional convergence

Due by 12:55pm, eastern, on Thursday, March 27

Suggested readings for this problem set: 2.7

All readings are from Abbott, *Understanding Analysis*.

Assignment: due Thursday, March 27, 12:55pm, via Gradescope (DKNX3W):

1. TBA
2. TBA
3. TBA
4. TBA
5. TBA
6. TBA
7. TBA
8. TBA



IN PROGRESS! Check back later for the final assignment.





IN PROGRESS! Check back later for the final assignment.



Assignment 7: Open, closed, and compact sets

Due by 12:55pm, eastern, on Thursday, April 03

Suggested readings for this problem set: 3.2, 3.3

All readings are from Abbott, *Understanding Analysis*.

Assignment: due Thursday, April 03, 12:55pm, via Gradescope (DKNX3W):

1. TBA
2. TBA
3. TBA
4. TBA
5. TBA
6. TBA
7. TBA
8. TBA



IN PROGRESS! Check back later for the final assignment.





IN PROGRESS! Check back later for the final assignment.



Assignment 8: Limits of functions; continuity

Due by 12:55pm, eastern, on Thursday, April 10

Suggested readings for this problem set: 4.2, 4.3

All readings are from Abbott, *Understanding Analysis*.

Assignment: due Thursday, April 10, 12:55pm, via Gradescope (DKNX3W):

1. TBA
2. TBA
3. TBA
4. TBA
5. TBA
6. TBA
7. TBA
8. TBA



IN PROGRESS! Check back later for the final assignment.





Midterm 2 study guide

In class exam, Thursday, April 17.

Content: The questions will all be either

1. homework problems,
2. suggested problems,
3. problems we worked in class, or
4. minor variations of one of these.

Problems with very long proofs or that involved some unusual trick will not be on the exam.

A typical exam will have a few questions from each week of the course and will cover **assignments 1-5**. The problems will be similar to the homework problems, and the proofs will use the same techniques. You can expect problems about following:

- Definitions.
- Integration
- TBA

For definitions, I want a definition, in prose (complete sentences), and I want “just” the definition, and not any additional facts about the definition. (E.g., if you give the definition of differentiable, do not include any facts about differentiability, such as “a differentiable function is continuous”)

Additionally:

- This is a closed-book, 75 minute examination.
- No books, notes, calculators, cell phones, communication devices of any sort, or other aids are permitted.
- You are allowed to use Theorems, lemmas, etc from the book or from class, and previous homework problems as part of your solutions, and you are not required to reprove these during the exam. Try to reference them by name if possible (i.e., “by the Mean Value Theorem” or “a differentiable function is continuous”), but if you are unsure of the name of the theorem, it is ok to say “we proved in class that”. The exception to this is that if I ask you to prove e.g. the mean value theorem, then you must prove it and can’t reference that we proved it in class.
- I will be in the hallway, and will check in every 25 minutes for questions.
- You can always ask me (the instructor) if you have clarifying questions, but asking for hints or asking if a proof is correct or sufficient is not allowed.
- You may leave when you are finished, in which case please hand me the exam in the hallway.
- You may leave to use the bathroom.



- You may use the backs of pages for additional work space. If you use any additional scratch paper as part of your solution, please let me know after the exam.
- Be sure to justify your answer to each of the problems!
- For proofs, please make sure that your answers are in complete sentences!
- **Academic integrity:**
 - You are required to **turn off your phone** during the exam. If I observe that your phone is on, the penalty is that you fail the exam. (If there is an exceptional reason that your phone needs to be on during the exam, please discuss it with me).
 - **Submit only your own work.** Do not share your work, and do not look at other students' exams.
 - The penalty for cheating is at my discretion, and can range from failing the exam to failing the course.



IN PROGRESS! Check back later for the final assignment.



Assignment 9: Uniform convergence, intermediate value theorem

Due by 12:55pm, eastern, on Thursday, April 17

Suggested readings for this problem set: 4.3, 4.4, 4.5

All readings are from Abbott, *Understanding Analysis*.

Assignment: due Thursday, April 17, 12:55pm, via Gradescope (DKNX3W):

1. TBA
2. TBA
3. TBA
4. TBA
5. TBA
6. TBA
7. TBA
8. TBA



IN PROGRESS! Check back later for the final assignment.





IN PROGRESS! Check back later for the final assignment.



Assignment 10: Differentiability, quotient rule, mean value theorem

Due by 12:55pm, eastern, on Tuesday, 10

Suggested readings for this problem set: 5.2, 5.3

All readings are from Abbott, *Understanding Analysis*.

Assignment: due Thursday, April 24, 12:55pm, via Gradescope (DKNX3W):

1. TBA
2. TBA
3. TBA
4. TBA
5. TBA
6. TBA
7. TBA
8. TBA



IN PROGRESS! Check back later for the final assignment.





IN PROGRESS! Check back later for the final assignment.



Assignment 11: Pointwise vs. uniform convergence; Uniform convergence and differentiation; continuity and differentiability of limits

Due by 12:55pm, eastern, on Tuesday, May 01

Suggested readings for this problem set: 6.2, 6.3

All readings are from Abbott, *Understanding Analysis*.

Assignment: due Thursday, May 01, 12:55pm, via Gradescope (DKNX3W):

1. TBA
2. TBA
3. TBA
4. TBA
5. TBA
6. TBA
7. TBA
8. TBA



IN PROGRESS! Check back later for the final assignment.





Final exam study guide

Final exam is a take home exam, released at **TBA** (evening) on **Wednesday, May 7 (probably)** and due **Wednesday, May 14 at 4pm (probably)**.

The exam will be available on gradescope, and should be submitted via Gradescope.

Big request: if you are writing your exam in latex or on a tablet, please start each problem on a new page. (My eyes are not very good and this helps a lot when I am grading ≥ 25 exams.) Please leave spacing between different problems, and please try to organize your answers into multiple paragraphs, with spacing.

The **last day of class** is Tuesday, May 6.

The *types* of problems will include a subset of

1. TBA

Problems with extremely long proofs or that involved some unusual trick will not be on the exam.

Since this is a take home exam, none of the problems will be identical to homework problems, but many problems will be minor variations of homework or of problems we worked in class.

A good way to prepare is to:

1. Know all of the definitions and terminology;
2. Know all of the statements of theorems, and examples of how we use the theorems;
3. Make a list of all of the different *proof techniques* from class and from the homework and review how those techniques are used in proofs and problems;
4. Practice doing problems “from scratch” and use your solutions as “hints” when you get stuck.

Additionally:

1. You are allowed to use the course textbook, lecture notes and any materials from the course website.
2. You are not allowed to use any other resources, including AI, other students, Google, or other books.
3. You are allowed to use Theorems, lemmas, etc from the book or from class, and previous homework problems as part of your solutions, and you are not required to reprove these during the exam. Please do cite them (e.g., “Proposition 1.3.4 from our book”) or refer to them by name, if they have a special name (e.g., “by the Mean Value Theorem”).
4. On the other hand, please do not use any theorems that are not from class or from our book.
5. Do not discuss the problems or their solutions with your classmates.



6. You can always ask me (the instructor) if you have clarifying questions, but asking for hints or asking if a proof is correct is not allowed.
7. You are welcome to use a calculator or any other program to help with computations (e.g., arithmetic).
8. Submit your exam via Gradescope.
9. Again (since this is a take-home exam): please **submit only your own work**. Do not share your work, and do not look at other students' exams. The penalty for cheating is at my discretion, and can range from failing the exam to failing the course.

The exam will be **comprehensive**. A typical exam will have a few questions from each week of the course, and there will be more emphasis on content not covered on the midterms. The exam to be around 8-9 problems (some with multiple parts).

Finally: **please take a look at the exam as early as possible** so that you can estimate the amount of time it will take you.

Hints

X.X. No hints yet.