

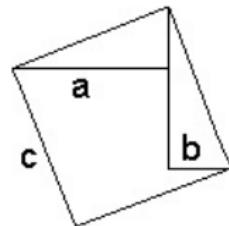
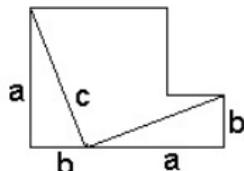
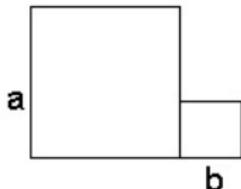
Beyond Fermat's Last Theorem

David Zureick-Brown

Emory University

UC Irvine math club

$$a^2 + b^2 = c^2$$



Basic Problem (Solving Diophantine Equations)

Analysis

Let $f_1, \dots, f_m \in \mathbb{Z}[x_1, \dots, x_n]$ be polynomials.

Problem

Describe the set

$$\{(a_1, \dots, a_n) \in \mathbb{Z}^n : \forall i, f_i(a_1, \dots, a_n) = 0\}.$$

Example

$x_1^2 + x_2^2 = -1$ has no integer solutions

Basic Problem (Solving Diophantine Equations)

Analysis

Let $f_1, \dots, f_m \in \mathbb{Z}[x_1, \dots, x_n]$ be polynomials.

Problem

Describe the set

$$\{(a_1, \dots, a_n) \in \mathbb{Z}^n : \forall i, f_i(a_1, \dots, a_n) = 0\}.$$

Example

$x_1^2 + x_2^2 = -1$ has no integer solutions

Fact

Solving diophantine equations is hard.

Hilbert's Tenth Problem

In fact, it is provably hard.

Theorem (Davis-Putnam-Robinson 1961, Matijasevič 1970)

There does not exist an algorithm solving the following problem:

input: $f_1, \dots, f_m \in \mathbb{Z}[x_1, \dots, x_n]$;

output: YES / NO according to whether the set

$$\{(a_1, \dots, a_n) \in \mathbb{Z}^n : \forall i, f_i(a_1, \dots, a_n) = 0\}$$

is non-empty.

Hilbert's Tenth Problem

In fact, it is provably hard.

Theorem (Davis-Putnam-Robinson 1961, Matijasevič 1970)

There does not exist an algorithm solving the following problem:

input: $f_1, \dots, f_m \in \mathbb{Z}[x_1, \dots, x_n]$;

output: YES / NO according to whether the set

$$\{(a_1, \dots, a_n) \in \mathbb{Z}^n : \forall i, f_i(a_1, \dots, a_n) = 0\}$$

is non-empty.

Hilbert's Tenth Problem

In fact, it is provably hard.

Theorem (Davis-Putnam-Robinson 1961, Matijasevič 1970)

There does not exist an algorithm solving the following problem:

input: $f_1, \dots, f_m \in \mathbb{Z}[x_1, \dots, x_n]$;

output: YES / NO according to whether the set

$$\{(a_1, \dots, a_n) \in \mathbb{Z}^n : \forall i, f_i(a_1, \dots, a_n) = 0\}$$

is non-empty.

This is *still open* for many other rings (e.g., $R = \mathbb{Q}$).

Fermat's Last Theorem

Theorem (Wiles et. al)

The only solutions to the equation

$$x^n + y^n = z^n, n \geq 3$$

are multiples of the triples

$$(0, 0, 0), \quad (\pm 1, \mp 1, 0), \quad \pm(1, 0, 1), \quad (0, \pm 1, \pm 1).$$



Fermat's Last Theorem

Theorem (Wiles et. al)

The only solutions to the equation

$$x^n + y^n = z^n, n \geq 3$$

are multiples of the triples

$$(0, 0, 0), \quad (\pm 1, \mp 1, 0), \quad \pm(1, 0, 1), \quad (0, \pm 1, \pm 1).$$

This took 300 years to prove!



Fermat's Last Theorem

Theorem (Wiles et. al)

The only solutions to the equation

$$x^n + y^n = z^n, n \geq 3$$

are multiples of the triples

$$(0, 0, 0), \quad (\pm 1, \mp 1, 0), \quad \pm(1, 0, 1), \quad (0, \pm 1, \pm 1).$$

This took 300 years to prove!



Basic Problem: $f_1, \dots, f_m \in \mathbb{Z}[x_1, \dots, x_n]$

Qualitative:

- Does there **exist** a solution?
- Do there exist **infinitely many** solutions?
- Does the set of solutions have some **extra structure** (e.g., geometric structure, group structure).

Basic Problem: $f_1, \dots, f_m \in \mathbb{Z}[x_1, \dots, x_n]$

Qualitative:

- Does there **exist** a solution?
- Do there exist **infinitely many** solutions?
- Does the set of solutions have some **extra structure** (e.g., geometric structure, group structure).

Quantitative

- How **many** solutions are there?
- How **large** is the **smallest** solution?
- How can we explicitly **find** all solutions? (With proof?)

Basic Problem: $f_1, \dots, f_m \in \mathbb{Z}[x_1, \dots, x_n]$

Qualitative:

- Does there **exist** a solution?
- Do there exist **infinitely many** solutions?
- Does the set of solutions have some **extra structure** (e.g., geometric structure, group structure).

Quantitative

- How **many** solutions are there?
- How **large** is the **smallest** solution?
- How can we explicitly **find** all solutions? (With proof?)

Implicit question

- Why do equations **have** (or fail to have) solutions?
- Why do some have **many** and some have **none**?
- What **underlying mathematical structures** control this?

Example: Pythagorean triples

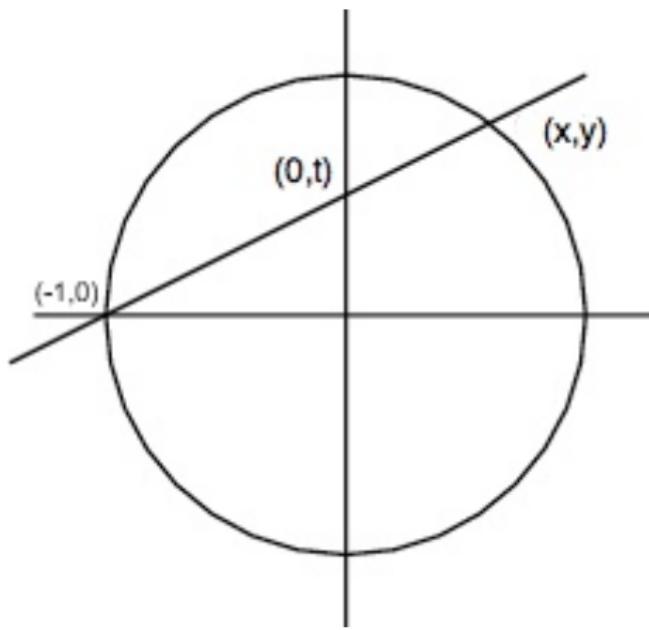
Lemma

The equation

$$x^2 + y^2 = z^2$$

has infinitely many non-zero coprime solutions.

Pythagorean triples



$$\text{Slope} = t = \frac{y}{x+1}$$

$$x = \frac{1-t^2}{1+t^2}$$

$$y = \frac{2t}{1+t^2}$$

Pythagorean triples

Lemma

The solutions to

$$a^2 + b^2 = c^2$$

are all multiples of the triples

$$a = 1 - t^2$$

$$b = 2t$$

$$c = 1 + t^2$$

The Mordell Conjecture

Example

The equation $y^2 + x^2 = 1$ has infinitely many solutions.

The Mordell Conjecture

Example

The equation $y^2 + x^2 = 1$ has infinitely many solutions.

Theorem (Faltings)

For $n \geq 5$, the equation

$$y^2 + x^n = 1$$

has only finitely many solutions.

The Mordell Conjecture

Example

The equation $y^2 + x^2 = 1$ has infinitely many solutions.

Theorem (Faltings)

For $n \geq 5$, the equation

$$y^2 + x^n = 1$$

has only finitely many solutions.

Theorem (Faltings)

For $n \geq 5$, the equation

$$y^2 = f(x)$$

has only finitely many solutions if $f(x)$ is squarefree, with degree > 4 .

Fermat Curves

Question

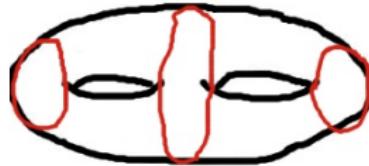
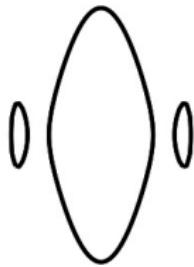
Why is Fermat's last theorem believable?

- ① $x^n + y^n - z^n = 0$ looks like a surface (3 variables)
- ② $x^n + y^n - 1 = 0$ looks like a curve (2 variables)

Mordell Conjecture

Example

$$y^2 = -(x^2 - 1)(x^2 - 2)(x^2 - 3)$$



This is a cross section of a two holed torus. The **genus** is the number of holes.

Conjecture (Mordell)

A curve of genus $g \geq 2$ has only finitely many rational solutions.

Fermat Curves

Question

Why is Fermat's last theorem believable?

- ① $x^n + y^n - 1 = 0$ is a curve of genus $(n - 1)(n - 2)/2$.
- ② Mordell implies that for **fixed** $n > 3$, the n th Fermat equation has only finitely many solutions.

Fermat Curves

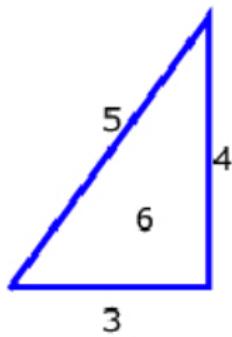
Question

What if $n = 3$?

- ① $x^3 + y^3 - 1 = 0$ is a curve of genus $(3 - 1)(3 - 2)/2 = 1$.
- ② We were lucky; $Ax^3 + By^3 = Cz^3$ can have infinitely many solutions.

Congruent number problem

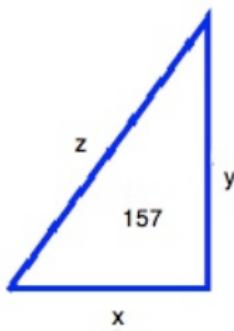
$$x^2 + y^2 = z^2, xy = 2 \cdot 6$$



$$3^2 + 4^2 = 5^2, \quad 3 \cdot 4 = 2 \cdot 6$$

Congruent number problem

$$x^2 + y^2 = z^2, xy = 2 \cdot 157$$



Congruent number problem

The pair of equations

$$x^2 + y^2 = z^2, \quad xy = 2 \cdot 157$$

has **infinitely many** solutions. **How large** is the smallest solution? How many **digits** does the smallest solution have?

Congruent number problem

$$x^2 + y^2 = z^2, xy = 2 \cdot 157$$

has **infinitely many** solutions. **How large** is the smallest solution? How many **digits** does the smallest solution have?

Congruent number problem

$$x^2 + y^2 = z^2, xy = 2 \cdot 157$$

has **infinitely many** solutions. **How large** is the smallest solution? How many **digits** does the smallest solution have?

$$x = \frac{157841 \cdot 4947203 \cdot 52677109576}{2 \cdot 3^2 \cdot 5 \cdot 13 \cdot 17 \cdot 37 \cdot 101 \cdot 17401 \cdot 46997 \cdot 356441}$$

$$y = \frac{2 \cdot 3^2 \cdot 5 \cdot 13 \cdot 17 \cdot 37 \cdot 101 \cdot 157 \cdot 17401 \cdot 46997 \cdot 356441}{157841 \cdot 4947203 \cdot 52677109576}$$

$$z = \frac{20085078913 \cdot 1185369214457 \cdot 942545825502442041907480}{2 \cdot 3^2 \cdot 5 \cdot 13 \cdot 17 \cdot 37 \cdot 101 \cdot 17401 \cdot 46997 \cdot 356441 \cdot 157841 \cdot 4947203 \cdot 52677109576}$$

Congruent number problem

$$x^2 + y^2 = z^2, xy = 2 \cdot 157$$

has **infinitely many** solutions. **How large** is the smallest solution? How many **digits** does the smallest solution have?

$$x = \frac{157841 \cdot 4947203 \cdot 52677109576}{2 \cdot 3^2 \cdot 5 \cdot 13 \cdot 17 \cdot 37 \cdot 101 \cdot 17401 \cdot 46997 \cdot 356441}$$

$$y = \frac{2 \cdot 3^2 \cdot 5 \cdot 13 \cdot 17 \cdot 37 \cdot 101 \cdot 157 \cdot 17401 \cdot 46997 \cdot 356441}{157841 \cdot 4947203 \cdot 52677109576}$$

$$z = \frac{20085078913 \cdot 1185369214457 \cdot 942545825502442041907480}{2 \cdot 3^2 \cdot 5 \cdot 13 \cdot 17 \cdot 37 \cdot 101 \cdot 17401 \cdot 46997 \cdot 356441 \cdot 157841 \cdot 4947203 \cdot 52677109576}$$

The denominator of z has **44 digits!**

Congruent number problem

$$x^2 + y^2 = z^2, xy = 2 \cdot 157$$

has **infinitely many** solutions. **How large** is the smallest solution? How many **digits** does the smallest solution have?

$$x = \frac{157841 \cdot 4947203 \cdot 52677109576}{2 \cdot 3^2 \cdot 5 \cdot 13 \cdot 17 \cdot 37 \cdot 101 \cdot 17401 \cdot 46997 \cdot 356441}$$

$$y = \frac{2 \cdot 3^2 \cdot 5 \cdot 13 \cdot 17 \cdot 37 \cdot 101 \cdot 157 \cdot 17401 \cdot 46997 \cdot 356441}{157841 \cdot 4947203 \cdot 52677109576}$$

$$z = \frac{20085078913 \cdot 1185369214457 \cdot 942545825502442041907480}{2 \cdot 3^2 \cdot 5 \cdot 13 \cdot 17 \cdot 37 \cdot 101 \cdot 17401 \cdot 46997 \cdot 356441 \cdot 157841 \cdot 4947203 \cdot 52677109576}$$

The denominator of z has **44 digits!**
How did anyone ever find this solution?

Congruent number problem

$$x^2 + y^2 = z^2, xy = 2 \cdot 157$$

has **infinitely many** solutions. **How large** is the smallest solution? How many **digits** does the smallest solution have?

$$x = \frac{157841 \cdot 4947203 \cdot 52677109576}{2 \cdot 3^2 \cdot 5 \cdot 13 \cdot 17 \cdot 37 \cdot 101 \cdot 17401 \cdot 46997 \cdot 356441}$$

$$y = \frac{2 \cdot 3^2 \cdot 5 \cdot 13 \cdot 17 \cdot 37 \cdot 101 \cdot 157 \cdot 17401 \cdot 46997 \cdot 356441}{157841 \cdot 4947203 \cdot 52677109576}$$

$$z = \frac{20085078913 \cdot 1185369214457 \cdot 942545825502442041907480}{2 \cdot 3^2 \cdot 5 \cdot 13 \cdot 17 \cdot 37 \cdot 101 \cdot 17401 \cdot 46997 \cdot 356441 \cdot 157841 \cdot 4947203 \cdot 52677109576}$$

The denominator of z has **44 digits**!
How did anyone ever find this solution?
“Next” solution has **176 digits**!

Back of the envelope calculation

$$x^2 + y^2 = z^2, xy = 2 \cdot 157$$

- Num, den(x, y, z) $\leq 10 \sim 10^6$ many, **1 min** on Emory's computers.

Back of the envelope calculation

$$x^2 + y^2 = z^2, xy = 2 \cdot 157$$

- Num, $\text{den}(x, y, z) \leq 10 \sim 10^6$ many, **1 min** on Emory's computers.
- Num, $\text{den}(x, y, z) \leq 10^{44} \sim 10^{264}$ many, **10^{258} mins = 10^{252} years**.

Back of the envelope calculation

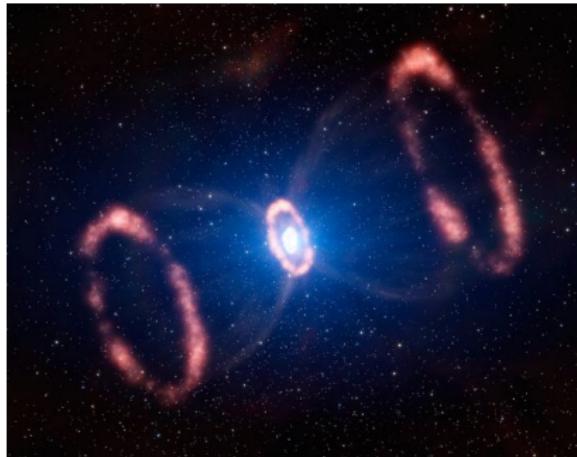
$$x^2 + y^2 = z^2, xy = 2 \cdot 157$$

- Num, $\text{den}(x, y, z) \leq 10 \sim 10^6$ many, **1 min** on Emory's computers.
- Num, $\text{den}(x, y, z) \leq 10^{44} \sim 10^{264}$ many, **10^{258} mins = 10^{252} years**.
- 10^9 many computers in the world – so **10^{243} years**

Back of the envelope calculation

$$x^2 + y^2 = z^2, xy = 2 \cdot 157$$

- Num, $\text{den}(x, y, z) \leq 10 \sim 10^6$ many, **1 min** on Emory's computers.
- Num, $\text{den}(x, y, z) \leq 10^{44} \sim 10^{264}$ many, **10^{258} mins = 10^{252} years**.
- 10^9 many computers in the world – so **10^{243} years**
- Expected time of 'heat death' of universe – **10^{100} years**.



Fermat Surfaces

Conjecture

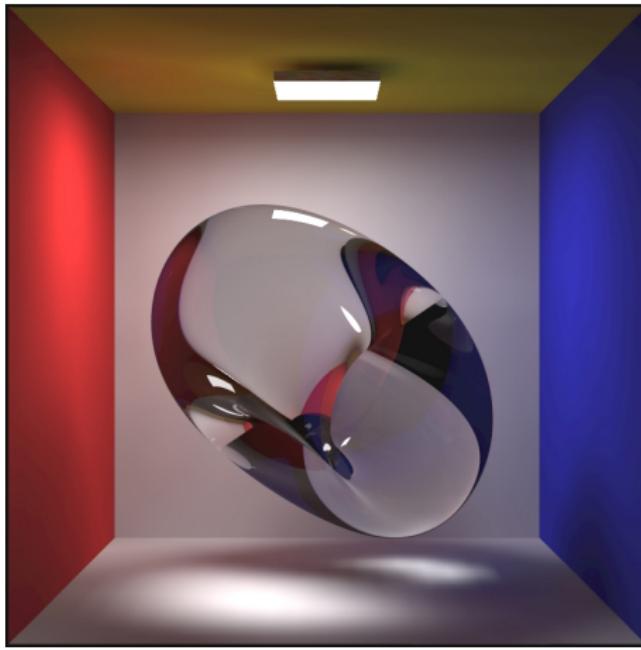
The only solutions to the equation

$$x^n + y^n = z^n + w^n, n \geq 5$$

satisfy $xyzw = 0$ or lie on the lines ‘lines’ $x = \pm y$, $z = \pm w$ (and permutations).

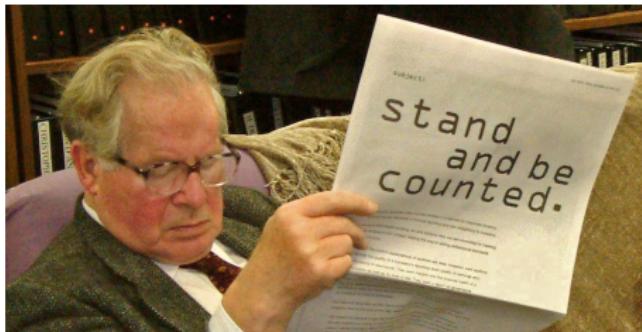
The Swinnerton-Dyer K3 surface

$$x^4 + 2y^4 = 1 + 4z^4$$



The Swinnerton-Dyer K3 surface

$$x^4 + 2y^4 = 1 + 4z^4$$



Two ‘obvious’ solutions – $(\pm 1 : 0 : 0)$.

The Swinnerton-Dyer K3 surface

$$x^4 + 2y^4 = 1 + 4z^4$$

- Two ‘obvious’ solutions – $(\pm 1 : 0 : 0)$.
- The next smallest solutions are $(\pm \frac{1484801}{1169407}, \pm \frac{1203120}{1169407}, \pm \frac{1157520}{1169407})$.

Problem

Find another solution.

- ➊ **10^{16} years** to find via brute force.
- ➋ Age of the universe – **$13.75 \pm .11$ billion years** (roughly **10^{10}**).

Fermat-like equations

Theorem (Poonen, Schaefer, Stoll)

The coprime integer solutions to $x^2 + y^3 = z^7$ are the 16 triples

$$(\pm 1, -1, 0), \quad (\pm 1, 0, 1), \quad \pm(0, 1, 1),$$

Fermat-like equations

Theorem (Poonen, Schaefer, Stoll)

The coprime integer solutions to $x^2 + y^3 = z^7$ are the 16 triples

$$(\pm 1, -1, 0), \quad (\pm 1, 0, 1), \quad \pm(0, 1, 1), \quad (\pm 3, -2, 1),$$

Fermat-like equations

Theorem (Poonen, Schaefer, Stoll)

The coprime integer solutions to $x^2 + y^3 = z^7$ are the 16 triples

$$(\pm 1, -1, 0), \quad (\pm 1, 0, 1), \quad \pm(0, 1, 1), \quad (\pm 3, -2, 1), \\ (\pm 71, -17, 2),$$

Fermat-like equations

Theorem (Poonen, Schaefer, Stoll)

The coprime integer solutions to $x^2 + y^3 = z^7$ are the 16 triples

$$\begin{aligned} & (\pm 1, -1, 0), \quad (\pm 1, 0, 1), \quad \pm(0, 1, 1), \quad (\pm 3, -2, 1), \\ & (\pm 71, -17, 2), (\pm 2213459, 1414, 65), \quad (\pm 15312283, 9262, 113), \\ & (\pm 21063928, -76271, 17). \end{aligned}$$

Generalized Fermat Equations

Problem

What are the solutions to the equation $x^a + y^b = z^c$?

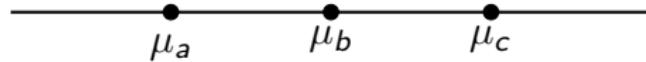
Generalized Fermat Equations

Problem

What are the solutions to the equation $x^a + y^b = z^c$?

Theorem (Darmon and Granville)

Fix $a, b, c \geq 2$. Then the equation $x^a + y^b = z^c$ has only finitely many coprime integer solutions iff $\chi = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1 \leq 0$.



Known Solutions to $x^a + y^b = z^c$

The ‘known’ solutions with

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$$

are the following:

$$1^p + 2^3 = 3^2$$

$$2^5 + 7^2 = 3^4, 7^3 + 13^2 = 2^9, 2^7 + 17^3 = 71^2, 3^5 + 11^4 = 122^2$$

$$17^7 + 76271^3 = 21063928^2, 1414^3 + 2213459^2 = 65^7$$

$$9262^3 + 153122832^2 = 113^7$$

$$43^8 + 96222^3 = 30042907^2, 33^8 + 1549034^2 = 15613^3$$

Known Solutions to $x^a + y^b = z^c$

The ‘known’ solutions with

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$$

are the following:

$$1^p + 2^3 = 3^2$$

$$2^5 + 7^2 = 3^4, 7^3 + 13^2 = 2^9, 2^7 + 17^3 = 71^2, 3^5 + 11^4 = 122^2$$

$$17^7 + 76271^3 = 21063928^2, 1414^3 + 2213459^2 = 65^7$$

$$9262^3 + 153122832^2 = 113^7$$

$$43^8 + 96222^3 = 30042907^2, 33^8 + 1549034^2 = 15613^3$$

Problem (Beal's conjecture)

These are all solutions with $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1 < 0$.

Generalized Fermat Equations – Known Solutions

Conjecture (Beal, Granville, Tijdeman-Zagier)

This is a complete list of coprime non-zero solutions such that

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1 < 0.$$

Generalized Fermat Equations – Known Solutions

Conjecture (Beal, Granville, Tijdeman-Zagier)

This is a complete list of coprime non-zero solutions such that

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1 < 0.$$

\$1,000,000 prize for proof of conjecture...

Generalized Fermat Equations – Known Solutions

Conjecture (Beal, Granville, Tijdeman-Zagier)

This is a complete list of coprime non-zero solutions such that

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1 < 0.$$

\$1,000,000 prize for proof of conjecture...

...or even for a counterexample.

Examples of Generalized Fermat Equations

Theorem (Poonen, Schaefer, Stoll)

The coprime integer solutions to $x^2 + y^3 = z^7$ are the 16 triples

$$\begin{aligned} & (\pm 1, -1, 0), \quad (\pm 1, 0, 1), \quad \pm(0, 1, 1), \quad (\pm 3, -2, 1), \\ & (\pm 71, -17, 2), (\pm 2213459, 1414, 65), \quad (\pm 15312283, 9262, 113), \\ & (\pm 21063928, -76271, 17). \end{aligned}$$

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{7} - 1 = -\frac{1}{42} < 0$$

Examples of Generalized Fermat Equations

Theorem (Poonen, Schaefer, Stoll)

The coprime integer solutions to $x^2 + y^3 = z^7$ are the 16 triples

$$\begin{aligned} & (\pm 1, -1, 0), \quad (\pm 1, 0, 1), \quad \pm(0, 1, 1), \quad (\pm 3, -2, 1), \\ & (\pm 71, -17, 2), (\pm 2213459, 1414, 65), \quad (\pm 15312283, 9262, 113), \\ & (\pm 21063928, -76271, 17). \end{aligned}$$

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{7} - 1 = -\frac{1}{42} < 0$$

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{6} - 1 = 0$$

Examples of Generalized Fermat Equations

Theorem (Darmon, Merel)

Any pairwise coprime solution to the equation

$$x^n + y^n = z^2, n > 4$$

satisfies $xyz = 0$.

$$\frac{1}{n} + \frac{1}{n} + \frac{1}{2} - 1 = \frac{2}{n} - \frac{1}{2} < 0$$

Other applications of the modular method

The ideas behind the proof of FLT now permeate the study of diophantine problems.

Other applications of the modular method

The ideas behind the proof of FLT now permeate the study of diophantine problems.

Theorem (Bugeaud, Mignotte, Siksek 2006)

The only Fibonacci numbers that are perfect powers are

$$F_0 = 0, F_1 = F_2 = 1, F_6 = 8, F_{12} = 144.$$

Examples of Generalized Fermat Equations

Theorem (Klein, Zagier, Beukers, Edwards, others)

The equation

$$x^2 + y^3 = z^5$$

Examples of Generalized Fermat Equations

Theorem (Klein, Zagier, Beukers, Edwards, others)

The equation

$$x^2 + y^3 = z^5$$

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} - 1 = \frac{1}{30} > 0$$

Examples of Generalized Fermat Equations

Theorem (Klein, Zagier, Beukers, Edwards, others)

The equation

$$x^2 + y^3 = z^5$$

has infinitely many coprime solutions

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} - 1 = \frac{1}{30} > 0$$

Examples of Generalized Fermat Equations

Theorem (Klein, Zagier, Beukers, Edwards, others)

The equation

$$x^2 + y^3 = z^5$$

has infinitely many coprime solutions

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} - 1 = \frac{1}{30} > 0$$

$$(T/2)^2 + H^3 + (f/12^3)^5$$

- ① $f = st(t^{10} - 11t^5s^5 - s^{10})$,
- ② $H = \text{Hessian of } f$,
- ③ $T = \text{a degree 3 covariant of the dodecahedron}$.

(p, q, r) such that $\chi < 0$ and the solutions to $x^p + y^q = z^r$ have been determined.

$\{n, n, n\}$	Wiles, Taylor-Wiles, building on work of many others
$\{2, n, n\}$	Darmon-Merel, others for small n
$\{3, n, n\}$	Darmon-Merel, others for small n
$\{5, 2n, 2n\}$	Bennett
$(2, 4, n)$	Ellenberg, Bruin, Ghioca $n \geq 4$
$(2, n, 4)$	Bennett-Skinner; $n \geq 4$
$\{2, 3, n\}$	Poonen-Shaefer-Stoll, Bruin. $6 \leq n \leq 9$
$\{2, 2\ell, 3\}$	Chen, Dahmen, Siksek; primes $7 < \ell < 1000$ with $\ell \neq 31$
$\{3, 3, n\}$	Bruin; $n = 4, 5$
$\{3, 3, \ell\}$	Kraus; primes $17 \leq \ell \leq 10000$
$(2, 2n, 5)$	Chen $n \geq 3^*$
$(4, 2n, 3)$	Bennett-Chen $n \geq 3$
$(6, 2n, 2)$	Bennett-Chen $n \geq 3$
$(2, 6, n)$	Bennett-Chen $n \geq 3$

(p, q, r) such that $\chi < 0$ and the solutions to $x^p + y^q = z^r$ have been determined.

$\{n, n, n\}$	Wiles, Taylor-Wiles, building on work of many others
$\{2, n, n\}$	Darmon-Merel, others for small n
$\{3, n, n\}$	Darmon-Merel, others for small n
$\{5, 2n, 2n\}$	Bennett
$(2, 4, n)$	Ellenberg, Bruin, Ghioca $n \geq 4$
$(2, n, 4)$	Bennett-Skinner; $n \geq 4$
$\{2, 3, n\}$	Poonen-Shaefer-Stoll, Bruin. $6 \leq n \leq 9$
$\{2, 2\ell, 3\}$	Chen, Dahmen, Siksek; primes $7 < \ell < 1000$ with $\ell \neq 31$
$\{3, 3, n\}$	Bruin; $n = 4, 5$
$\{3, 3, \ell\}$	Kraus; primes $17 \leq \ell \leq 10000$
$(2, 2n, 5)$	Chen $n \geq 3^*$
$(4, 2n, 3)$	Bennett-Chen $n \geq 3$
$(6, 2n, 2)$	Bennett-Chen $n \geq 3$
$(2, 6, n)$	Bennett-Chen $n \geq 3$
$(2, 3, 10)$	ZB

Faltings' theorem / Mordell's conjecture

Theorem (Faltings, Vojta, Bombieri)

Let X be a smooth curve over \mathbb{Q} with genus at least 2. Then $X(\mathbb{Q})$ is finite.

Example

For $g \geq 2$, $y^2 = x^{2g+1} + 1$ has only finitely many solutions with $x, y \in \mathbb{Q}$.

Uniformity

Problem

- ① Given X , compute $X(\mathbb{Q})$ exactly.
- ② Compute bounds on $\#X(\mathbb{Q})$.

Conjecture (Uniformity)

There exists a constant $N(g)$ such that every smooth curve of genus g over \mathbb{Q} has at most $N(g)$ rational points.

Theorem (Caporaso, Harris, Mazur)

Lang's conjecture \Rightarrow uniformity.

Uniformity numerics

g	2	3	4	5	10	45	g
$B_g(\mathbb{Q})$	642	112	126	132	192	781	$16(g + 1)$

Elkies studied K3 surfaces of the form

$$y^2 = S(t, u, v)$$

with lots of rational lines, such that S restricted to such a line is a perfect square.

Main Theorem (partial uniformity for curves)

Theorem (Katz, Rabinoff, ZB)

Let X be *any* curve of genus g and let $r = \text{rank}_{\mathbb{Z}} \text{Jac}_X(\mathbb{Q})$. Suppose $r \leq g - 2$. Then

$$\#X(\mathbb{Q}) \leq 84g^2 - 123g + 48$$