Beyond Fermat's Last Theorem

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Slides available at `http://www.mathcs.emory.edu/~dzb/slides/`

2018 Joint Mathematics Meetings
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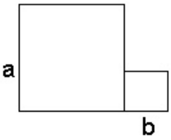
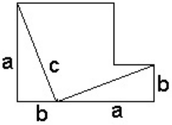
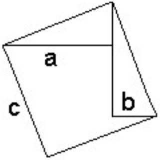
$$\boxed{a^2 + b^2 = c^2}$$







Basic Problem (Solving Diophantine Equations)

Setup

Let $f_1, \ldots, f_m \in \mathbb{Z}[x_1, ..., x_n]$ be polynomials.

Let R be a ring (e.g., $R = \mathbb{Z}$, $\mathbb{Q}$).

Problem

Describe the set

$$\left\{(a_1, \ldots, a_n) \in R^n : \forall i, f_i(a_1, \ldots, a_n) = 0\right\}.$$

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Fact

Solving diophantine equations is hard.

Hilbert's Tenth Problem

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There does not exist an algorithm solving the following problem:

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This is also *known* for many rings (e.g., $R = \mathbb{C}, \mathbb{R}, \mathbb{F}_q, \mathbb{Q}_p, \mathbb{C}(t)$).

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This is also *known* for many rings (e.g., $R = \mathbb{C}, \mathbb{R}, \mathbb{F}_q, \mathbb{Q}_p, \mathbb{C}(t)$).
This is *still open* for many other rings (e.g., $R = \mathbb{Q}$).

Fermat's Last Theorem

Theorem (Wiles et. al)

The only solutions to the equation

$$x^n + y^n = z^n, n \geq 3$$

are multiples of the triples

$$(0,0,0), \quad (\pm 1, \mp 1, 0), \quad \pm(1,0,1), \quad (0, \pm 1, \pm 1).$$

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GAP

Basic Problem: $f_1, \ldots, f_m \in \mathbb{Z}[x_1, ..., x_n]$

Qualitative:

- Does there exist a solution?
- Do there exist infinitely many solutions?
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Implicit question

- Why do equations have (or fail to have) solutions?
- Why do some have many and some have none?
- What underlying mathematical structures control this?

The Mordell Conjecture

Example

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For $n \geq 5$, the equation

$$y^2 = f(x)$$

has only finitely many solutions if $f(x)$ is squarefree, with degree > 4.

Fermat Curves

Question

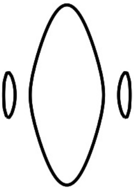
Why is Fermat's last theorem believable?

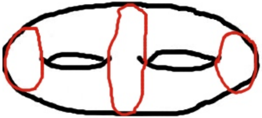
1. $x^n + y^n - z^n = 0$ looks like a surface (3 variables)
2. $x^n + y^n - 1 = 0$ looks like a curve (2 variables)

Mordell Conjecture

Example

$$y^2 = (x^2 - 1)(x^2 - 2)(x^2 - 3)$$





This is a cross section of a two holed torus. The **genus** is the number of holes.

Conjecture (Mordell)

A curve of genus $g \geq 2$ has only finitely many rational solutions.

Fermat Curves

Question

Why is Fermat's last theorem believable?

1. $x^n + y^n - 1 = 0$ is a curve of genus $(n-1)(n-2)/2$.
2. Mordell implies that for **fixed** $n > 3$, the nth Fermat equation has only finitely many solutions.

Fermat Curves

Question

What if $n = 3$?

1. $x^3 + y^3 - 1 = 0$ is a curve of genus $(3-1)(3-2)/2 = 1$.
2. We were lucky; $Ax^3 + By^3 = Cz^3$ can have infinitely many solutions.

Fermat Surfaces

Conjecture

The only solutions to the equation

$$x^n + y^n = z^n + w^n, n \geq 5$$

satisfy $xyzw = 0$ or lie on the lines 'lines' $x = \pm y$, $z = \pm w$ (and permutations).

Fermat-like equations

Theorem (Poonen, Schaefer, Stoll)

The coprime integer solutions to $x^2 + y^3 = z^7$ *are the* 16 *triples*

$$(\pm 1, -1, 0), \quad (\pm 1, 0, 1), \quad \pm(0, 1, 1),$$

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$$(\pm 21063928, -76271, 17).$$

Generalized Fermat Equations

Problem

What are the solutions to the equation $x^a + y^b = z^c$?

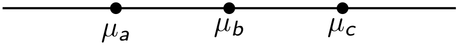
Generalized Fermat Equations

Problem

What are the solutions to the equation $x^a + y^b = z^c$*?*

Theorem (Darmon and Granville)

Fix $a, b, c \geq 2$*. Then the equation* $x^a + y^b = z^c$ *has only finitely many coprime integer solutions iff* $\chi = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1 \leq 0$.



Known Solutions to $x^a + y^b = z^c$

The 'known' solutions with

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$$

are the following:

$$1^p + 2^3 = 3^2$$

$$2^5 + 7^2 = 3^4,\ 7^3 + 13^2 = 2^9,\ 2^7 + 17^3 = 71^2,\ 3^5 + 11^4 = 122^2$$

$$17^7 + 76271^3 = 21063928^2,\ 1414^3 + 2213459^2 = 65^7$$

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Problem (Beal's conjecture)

These are all solutions with $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1 < 0$.

Generalized Fermat Equations – Known Solutions

Conjecture (Beal, Granville, Tijdeman-Zagier)

This is a complete list of coprime non-zero solutions such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1 < 0$.

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...or even for a counterexample.

Examples of Generalized Fermat Equations

Theorem (Poonen, Schaefer, Stoll)

The coprime integer solutions to $x^2 + y^3 = z^7$ are the 16 triples

$$(\pm 1, -1, 0), \quad (\pm 1, 0, 1), \quad \pm(0, 1, 1), \quad (\pm 3, -2, 1),$$
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$$\frac{1}{2} + \frac{1}{3} + \frac{1}{6} - 1 = 0$$

Examples of Generalized Fermat Equations

Theorem (Darmon, Merel)

Any pairwise coprime solution to the equation

$$x^n + y^n = z^2, n > 4$$

satisfies $xyz = 0$.

$$\frac{1}{n} + \frac{1}{n} + \frac{1}{2} - 1 = \frac{2}{n} - \frac{1}{2} < 0$$

Examples of Generalized Fermat Equations

Theorem (Klein, Zagier, Beukers, Edwards, others)

The equation

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$$(T/2)^2 + H^3 + (f/12^3)^5$$

1. $f = st(t^{10} - 11t^5s^5 - s^{10})$,
2. $H =$ Hessian of f,
3. $T =$ a degree 3 covariant of the dodecahedron.

(p, q, r) such that $\chi < 0$ and the solutions to $x^p + y^q = z^r$ have been determined.

$\{n, n, n\}$	Wiles,Taylor-Wiles, building on work of many others
$\{2, n, n\}$	Darmon-Merel, others for small n
$\{3, n, n\}$	Darmon-Merel, others for small n
$\{5, 2n, 2n\}$	Bennett
$(2, 4, n)$	Ellenberg, Bruin, Ghioca $n \geq 4$
$(2, n, 4)$	Bennett-Skinner; $n \geq 4$
$\{2, 3, n\}$	Poonen-Shaefer-Stoll, Bruin. $6 \leq n \leq 9$
$\{2, 2\ell, 3\}$	Chen, Dahmen, Siksek; primes $7 < \ell < 1000$ with $\ell \neq 31$
$\{3, 3, n\}$	Bruin; $n = 4, 5$
$\{3, 3, \ell\}$	Kraus; primes $17 \leq \ell \leq 10000$
$(2, 2n, 5)$	Chen $n \geq 3^*$
$(4, 2n, 3)$	Bennett-Chen $n \geq 3$
$(6, 2n, 2)$	Bennett-Chen $n \geq 3$
$(2, 6, n)$	Bennett-Chen $n \geq 3$

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$(2,3,10)$	**ZB**

Faltings' theorem / Mordell's conjecture

Theorem (Faltings, Vojta, Bombieri)

Let X be a smooth curve over $\mathbb{Q}$ with genus at least 2. Then $X(\mathbb{Q})$ is finite.

Example

For $g \geq 2$, $y^2 = x^{2g+1} + 1$ has only finitely many solutions with $x, y \in \mathbb{Q}$.

Uniformity

Problem

1. *Given X, compute $X(\mathbb{Q})$ exactly.*
2. *Compute bounds on $\#X(\mathbb{Q})$.*

Conjecture (Uniformity)

There exists a constant $N(g)$ such that every smooth curve of genus g over $\mathbb{Q}$ has at most $N(g)$ rational points.

Theorem (Caporaso, Harris, Mazur)

Lang's conjecture $\Rightarrow$ uniformity.

Uniformity numerics

g	2	3	4	5	10	45	g
$B_g(\mathbb{Q})$	642	112	126	132	192	781	$16(g+1)$

Remark

Elkies studied K3 surfaces of the form

$$y^2 = S(t, u, v)$$

with lots of rational lines, such that S restricted to such a line is a perfect square.

Coleman's bound

Theorem (Coleman)

Let X be a curve of genus g and let $r = \operatorname{rank}_{\mathbb{Z}} \operatorname{Jac}_X(\mathbb{Q})$. Suppose $p > 2g$ is a prime of good reduction. Suppose $r < g$. Then

$$\#X(\mathbb{Q}) \leq \#X(\mathbb{F}_p) + 2g - 2.$$

Remark

1. A modified statement holds for $p \leq 2g$ or for $K \neq \mathbb{Q}$.
2. Note: this does not prove uniformity (since the first good p might be large).

Tools

p-adic integration and Riemann–Roch

Main Theorem (partial uniformity for curves)

Theorem (Katz, Rabinoff, ZB)

Let X be any curve of genus g and let $r = \operatorname{rank}_{\mathbb{Z}} \operatorname{Jac}_X(\mathbb{Q})$. Suppose $r < g - 2$. Then

$$\#X(\mathbb{Q}) \leq 84g^2 - 98g + 28$$

Tools

p-adic integration on annuli

comparison of different analytic continuations of p-adic integration

Non-Archimedean (Berkovich) structure of a curve [BPR]

Combinatorial restraints coming from the Tropical canonical bundle

Chabauty's method

(p-**adic integration**) There exists $V \subset H^0(X_{\mathbb{Q}_p}, \Omega^1_X)$ with $\dim_{\mathbb{Q}_p} V \geq g - r$ such that,

$$\int_P^Q \omega = 0 \qquad \forall P, Q \in X(\mathbb{Q}), \omega \in V$$

(**Coleman, via Newton Polygons**) Number of zeroes in a residue disc D_P is $\leq 1 + n_P$, where $n_P = \#\left(\operatorname{div}\omega \cap D_P\right)$

(**Riemann–Roch**) $\sum n_P = 2g - 2$.

(**Coleman's bound**) $\sum_{P \in X(\mathbb{F}_p)} (1 + n_P) = \#X(\mathbb{F}_p) + 2g - 2$.

Example (from McCallum–Poonen's survey paper)

Example

$$X\colon y^2 = x^6 + 8x^5 + 22x^4 + 22x^3 + 5x^2 + 6x + 1$$

1. Points reducing to $\widetilde{Q} = (0,1)$ are given by

$$\begin{aligned} x &= \ p \cdot t, \text{ where } t \in \mathbb{Z}_p \\ y &= \ \sqrt{x^6 + 8x^5 + 22x^4 + 22x^3 + 5x^2 + 6x + 1} = 1 + x^2 + \cdots \end{aligned}$$

2. $\displaystyle\int_{(0,1)}^{P_t} \frac{xdx}{y} = \int_0^t (x - x^3 + \cdots)dx$

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Comments

Corollary ((Partially) effective Manin-Mumford)

There is an effective constant $N(g)$ such that if $g(X) = g$, then

$$\#\left(X \cap \mathrm{Jac}_{X,tors}\right)(\mathbb{Q}) \leq N(g)$$

Corollary

There is an effective constant $N'(g)$ such that if $g(X) = g > 3$ and $X/\mathbb{Q}$ has totally degenerate, trivalent reduction mod 2, then

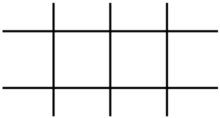
$$\#\left(X \cap \mathrm{Jac}_{X,tors}\right)(\mathbb{C}) \leq N'(g)$$

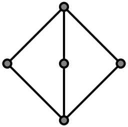
The second corollary is a big improvement

1. It requires working over a non-discretely valued field.
2. The bound only depends on the reduction type.
3. Integration over wide opens (c.f. Coleman) instead of discs and annuli.

Baker-Payne-Rabinoff and the slope formula

(**Dual graph** Γ **of** $X_{\mathbb{F}_p}$)





(**Contraction Theorem**) $\tau \colon X^{\mathrm{an}} \to \Gamma$.

(**Combinatorial harmonic analysis/potential theory**)

f	a meromorphic function on X^{an}
$F := (-\log\lvert f\rvert)\big\rvert_{\Gamma}$	associated tropical, piecewise linear function
$\operatorname{div} F$	combinatorial record of the slopes of F

(**Slope formula**) $\tau_* \operatorname{div} f = \operatorname{div} F$