

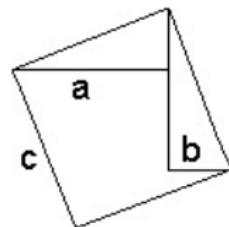
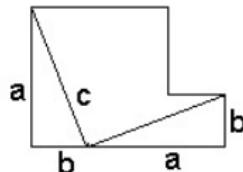
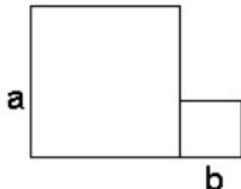
Beyond Fermat's Last Theorem

David Zureick-Brown

Slides available at <http://dmzb.github.io/>

Colby College Colloquium
April 15, 2024

$$a^2 + b^2 = c^2$$



Basic Problem (Solving Diophantine Equations)

Let f_1, \dots, f_m be polynomials with integer coefficients, e.g.,

$$x^2 + y^2 + 1$$

$$x^3 - y^2 - 2$$

$$2y^2 + 17x^4 - 1$$

Basic problem: solve polynomial equations

Describe the set

$$V(f_1, \dots, f_m) = \{(a_1, \dots, a_n) \in \mathbb{Z}^n : \forall i, f_i(a_1, \dots, a_n) = 0\},$$

i.e., the set of integer solutions to those polynomials

Fact

Solving Diophantine equations is hard.

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Hilbert's Tenth Problem

Theorem (Davis–Putnam–Robinson 1961, Matijasevič 1970)

There does not exist an algorithm solving the following problem:

input: integer polynomials f_1, \dots, f_m in variables x_1, \dots, x_n ;

output: YES / NO according to whether the set of solutions

$$\{(a_1, \dots, a_n) \in \mathbb{Z}^n : \forall i, f_i(a_1, \dots, a_n) = 0\}$$

is non-empty.

This is *known* to be true for many other cases (e.g., $\mathbb{C}, \mathbb{R}, \mathbb{F}_q, \mathbb{Q}_p, \mathbb{C}(t)$).

This is *still unknown* in many other cases (e.g., \mathbb{Q}).

Fermat's Last Theorem - A Marvelous Proof

Theorem (Wiles; Taylor)

For primes $p \geq 3$ the only integer solutions to the equation

$$x^p + y^p = z^p$$

are integer multiples of the triples

$$(0, 0, 0), \quad (\pm 1, \mp 1, 0), \quad \pm(1, 0, 1), \quad \pm(0, 1, 1).$$

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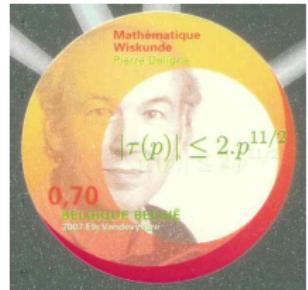
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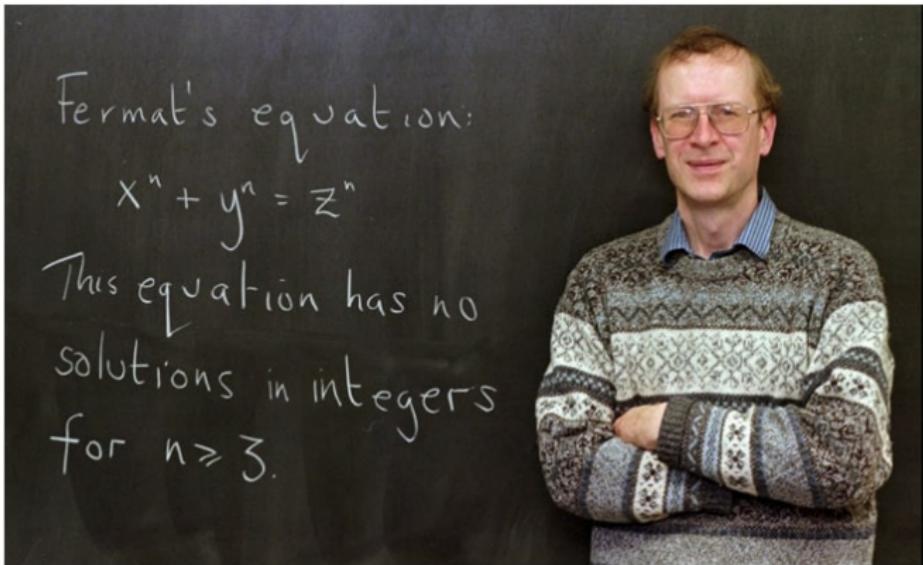
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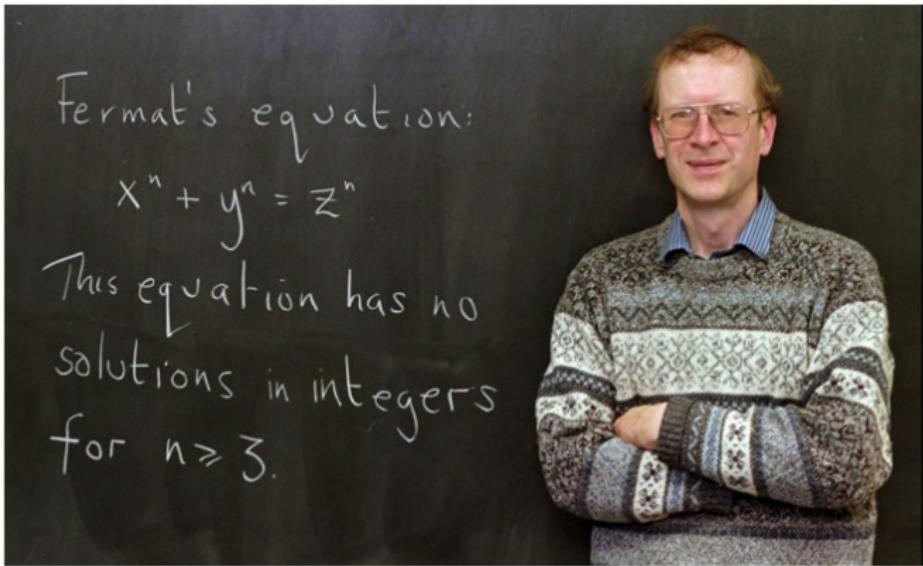
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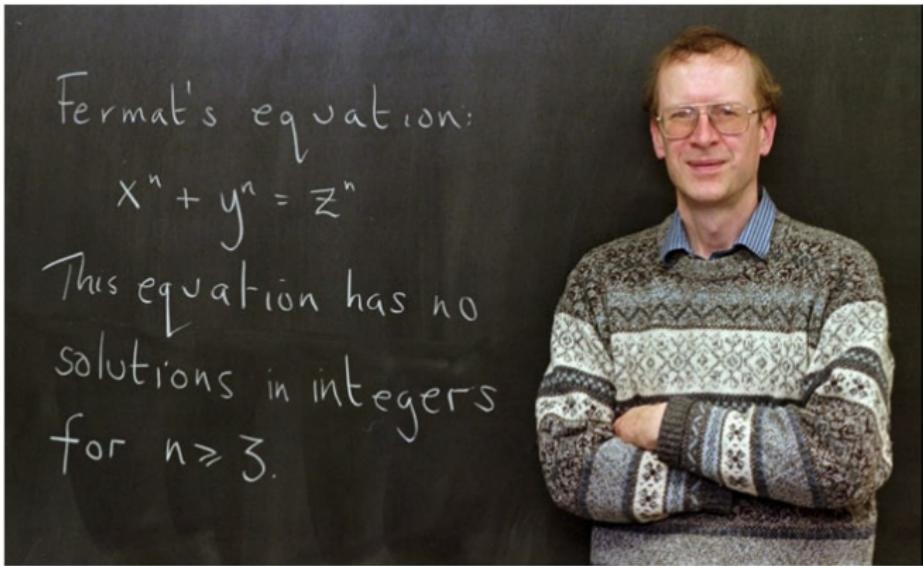
Fermat's Last Theorem - aftermath



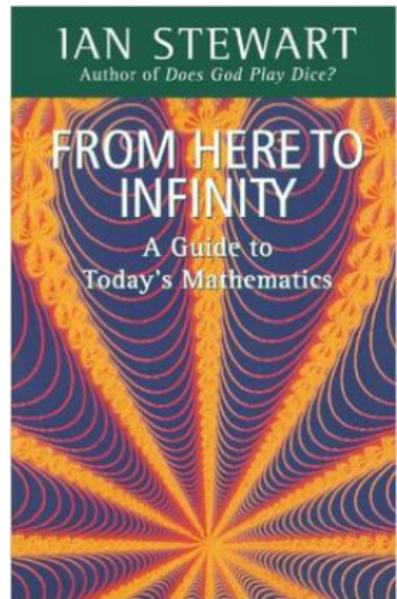
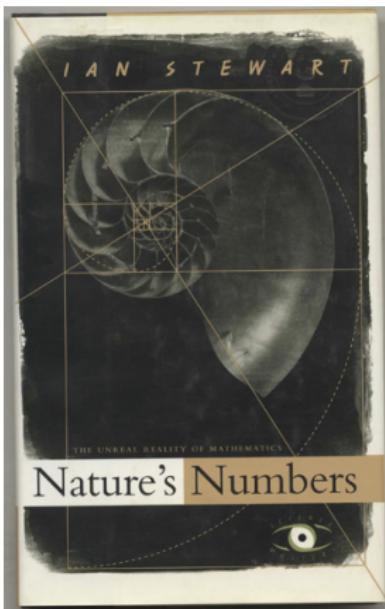
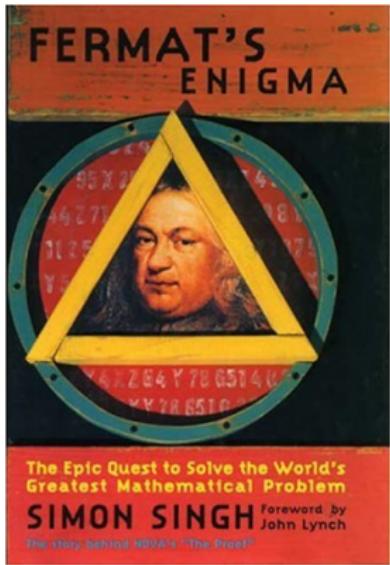
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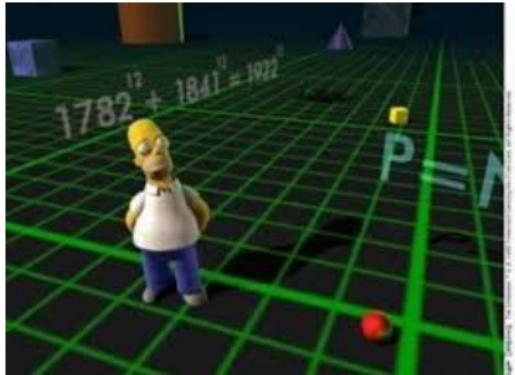
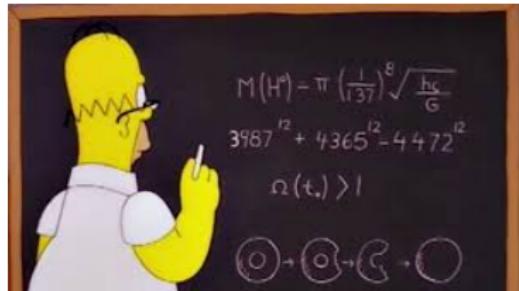
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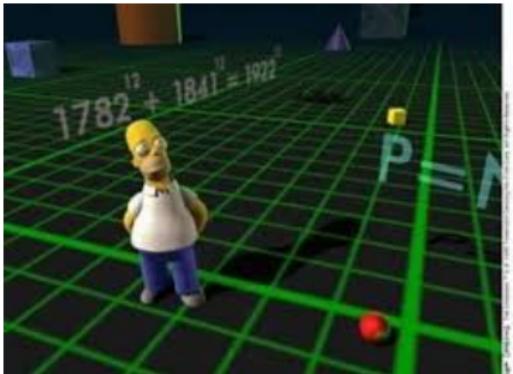
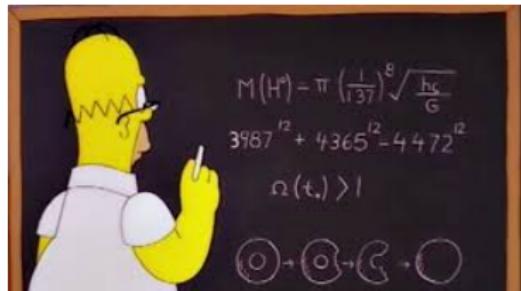
Books



Fermat trolling



Fermat trolling



See <https://youtu.be/ReOQ300AcSU?si=--fAdsdPttt4HR3N>

Basic Problem: $f_1, \dots, f_m \in \mathbb{Z}[x_1, \dots, x_n]$

Qualitative:

- ▶ Does there **exist** a solution?
- ▶ Do there exist **infinitely many** solutions?
- ▶ Does the set of solutions have some **extra structure** (e.g., geometric structure, group structure).

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- ▶ How **large** is the **smallest** solution?
- ▶ How can we explicitly **find** all solutions? (With proof?)

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Implicit question

- ▶ Why do equations **have** (or fail to have) solutions?
- ▶ Why do some have **many** and some have **none**?
- ▶ What **underlying mathematical structures** control this?

Example: Pythagorean triples

$$3^2 + 4^2 = 5^2$$

$$5^2 + 12^2 = 13^2$$

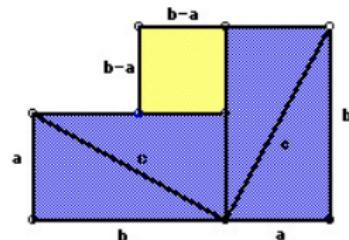
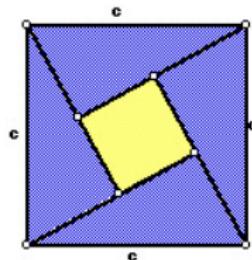
$$7^2 + 24^2 = 25^2$$

Lemma

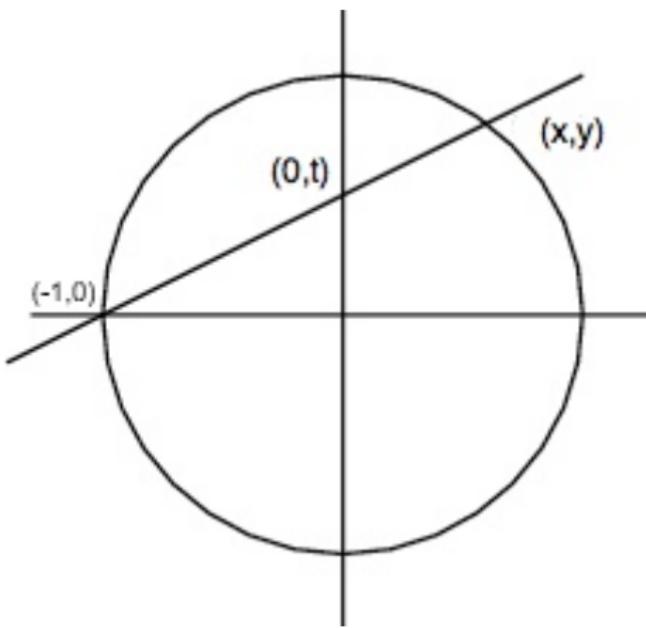
The equation

$$x^2 + y^2 = z^2$$

has infinitely many non-zero coprime solutions.



Pythagorean triples



$$\text{Slope} = t = \frac{y}{x+1}$$

$$x = \frac{1-t^2}{1+t^2}$$

$$y = \frac{2t}{1+t^2}$$

Pythagorean triples

Lemma

The solutions to

$$a^2 + b^2 = c^2$$

(with $c \neq 0$) are all multiples of the triples

$$a = 1 - t^2$$

$$b = 2t$$

$$c = 1 + t^2$$

The Mordell Conjecture

Example

The equation $y^2 + x^2 = 1$ has infinitely many solutions.

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Theorem (Faltings)

For $n \geq 5$, the equation

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Theorem (Faltings)

For $n \geq 5$, the equation

$$y^2 = f(x)$$

has only finitely many solutions if $f(x)$ is squarefree, with degree > 4 .

Fermat Curves

Question

Why is Fermat's last theorem believable?

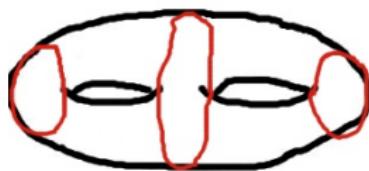
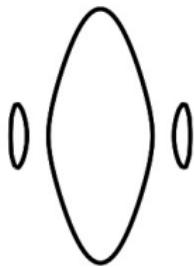
- ① $x^n + y^n - z^n = 0$ looks like a surface (3 variables)
- ② $x^n + y^n - 1 = 0$ looks like a curve (2 variables)

Mordell Conjecture

Example

$$y^2 = -(x^2 - 1)(x^2 - 2)(x^2 - 3)$$

This is a cross section of a two holed torus.



The **genus** is the number of holes.

Conjecture (Mordell, 1922)

A curve of genus $g \geq 2$ has only finitely many rational solutions.

Fermat Curves

Question

Why is Fermat's last theorem believable?

- ① $x^n + y^n - z^n = 0$ looks like a surface (3 variables)
- ② $x^n + y^n - 1 = 0$ looks like a curve (2 variables)
- ③ and has genus

$$(n - 1)(n - 2)/2$$

which is ≥ 2 iff $n \geq 4$.

Fermat Curves

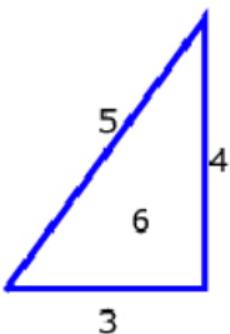
Question

What if $n = 3$?

- ① $x^3 + y^3 - 1 = 0$ is a curve of genus $(3 - 1)(3 - 2)/2 = 1$.
- ② We were lucky; $Ax^3 + By^3 = Cz^3$ can have infinitely many solutions.

Congruent number problem

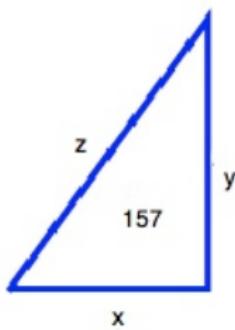
$$x^2 + y^2 = z^2, xy = 2 \cdot 6$$



$$3^2 + 4^2 = 5^2, \quad 3 \cdot 4 = 2 \cdot 6$$

Congruent number problem

$$x^2 + y^2 = z^2, xy = 2 \cdot 157$$



Assume the Birch–Swinnerton-Dyer conjectures

If you assume \$1,000,000 worth of conjectures, then the equations

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The denominator of z has **44 digits**!

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“Next” soluton has **176 digits!**

Back of the envelope calculation (as of 2011)

$$x^2 + y^2 = z^2, xy = 2 \cdot 157$$

- Num, den(x, y, z) $\leq 10 \sim 10^6$ many, **1 min** on Emory's computers.

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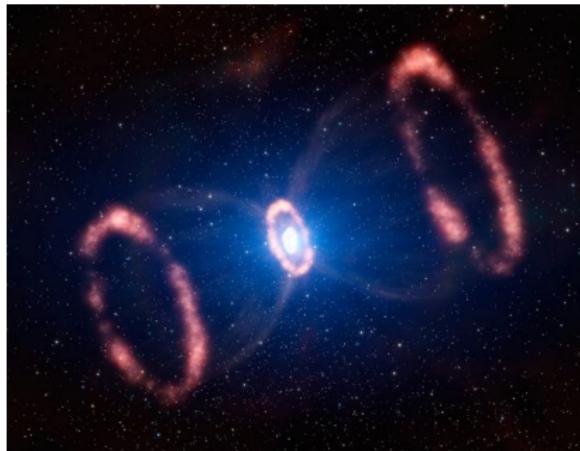
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- 10^9 many computers in the world – so **10^{243} years**
- Expected time until 'heat death' of universe – **10^{100} years**.



Fermat Surfaces

Conjecture

The only solutions to the equation

$$x^n + y^n = z^n + w^n, n \geq 5$$

satisfy $xyzw = 0$ or lie on the lines ‘lines’ $x = z$, $y = w$ (and permutations).

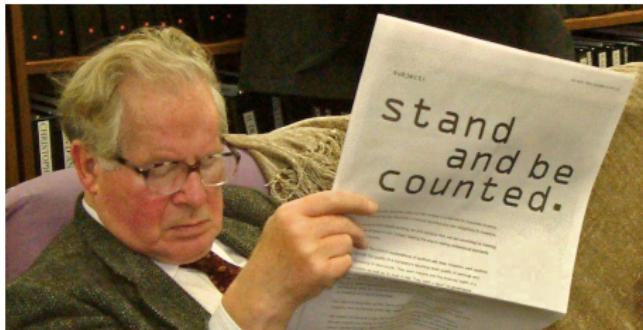
The Swinnerton-Dyer K3 surface

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The Swinnerton-Dyer K3 surface

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Two ‘obvious’ solutions – $(\pm 1 : 0 : 0)$.

The Swinnerton-Dyer K3 surface

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- Two ‘obvious’ solutions – $(\pm 1 : 0 : 0)$.
- The next smallest solutions are $(\pm \frac{1484801}{1169407}, \pm \frac{1203120}{1169407}, \pm \frac{1157520}{1169407})$.

Problem

Find another solution. (Probably impossible.)

Back of envelope calculation

- ➊ **10^{16} years** to find via brute force.
- ➋ Age of the universe – **$13.75 \pm .11$ billion years** (roughly 10^{10}).

Sums of cubes

$$1 = 1^3 + 0^3 + 0^3$$

$$2 = 1^3 + 1^3 + 0^3$$

$$3 = 1^3 + 1^3 + 1^3$$

$$3 = 4^3 + 4^3 + (-5)^3$$

$$4 \neq x^3 + y^3 + z^3$$

$$5 \neq x^3 + y^3 + z^3$$

$$6 = 1^3 + 1^3 + 2^3$$

Conjecture (Heath-Brown)

The equation

$$x^3 + y^3 + z^3 = n$$

has an integer solution if and only if n is not 4 or 5 mod 9.

Solved by Booker–Sutherland

$$32 \neq x^3 + y^3 + z^3$$

$$33 =$$

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<https://www.quantamagazine.org/why-the-sum-of-three-cubes-is-a-hard-math-problem-20191105/>

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$$114 = x^3 + y^3 + z^3?$$



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“Generalized” Fermat equations

Theorem (Poonen, Schaefer, Stoll)

The coprime integer solutions to $x^2 + y^3 = z^7$ are the 16 triples

$$(\pm 1, -1, 0), \quad (\pm 1, 0, 1), \quad \pm(0, 1, 1),$$

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The coprime integer solutions to $x^2 + y^3 = z^7$ are the 16 triples

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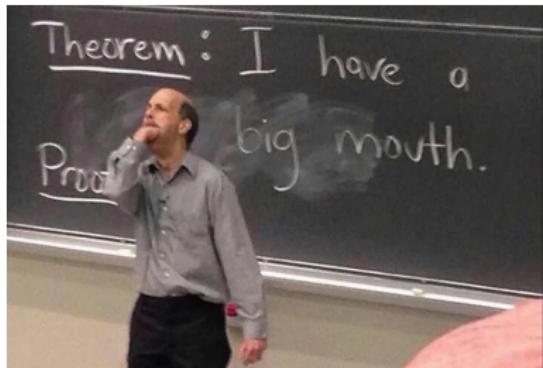
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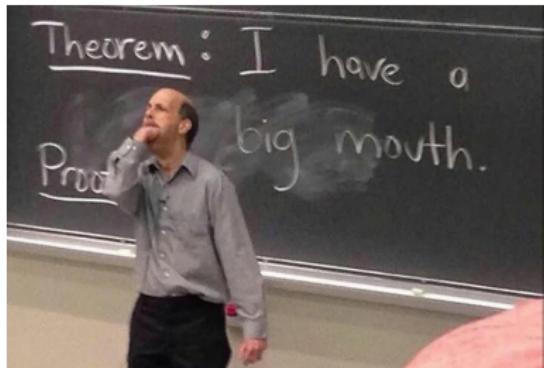


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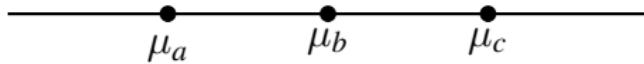
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Theorem (Darmon and Granville)

Fix $a, b, c \geq 2$. Then the equation $x^a + y^b = z^c$ has only finitely many coprime integer solutions iff $\chi = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1 \leq 0$.



Known Solutions to $x^a + y^b = z^c$ with $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$

$$1^p + 2^3 = 3^2, \quad 2^5 + 7^2 = 3^4$$

$$7^3 + 13^2 = 2^9, \quad 2^7 + 17^3 = 71^2$$

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Problem (Beal's conjecture)

These are all solutions with $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1 < 0$.

Generalized Fermat Equations – Known Solutions

Conjecture (Beal, Granville, Tijdeman–Zagier)

This is a complete list of coprime non-zero solutions such that

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Theorem (Darmon, Merel)

Any pairwise coprime solution to the equation

$$x^n + y^n = z^2, n > 4$$

satisfies $xyz = 0$.

$$\frac{1}{n} + \frac{1}{n} + \frac{1}{2} - 1 = \frac{2}{n} - \frac{1}{2} < \frac{2}{4} - \frac{1}{2} = 0$$

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The only Fibonacci numbers that are perfect powers are

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1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...

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Theorem (Silliman–Vogt; 2013 REU)

0 and 1 are the only perfect powers in the Lucas sequence

$$L_1 = 0, L_2 = 1, \quad L_n = 3L_{n-1} - 2L_{n-2}.$$

$$0, \textcolor{red}{1}, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, 4095, 8191, \dots, 2^n - 1, \dots$$

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$$(T/2)^2 + H^3 + (f/12^3)^5$$

- ① $f = st(t^{10} - 11t^5s^5 - s^{10})$,
- ② $H = \text{Hessian of } f$,
- ③ $T = \text{a degree 3 covariant of the dodecahedron}$.

(a, b, c) such that $\chi < 0$ and the solutions to $x^a + y^b = z^c$ have been determined.

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$\{2, n, n\}$	Darmon–Merel, others for small n
$\{3, n, n\}$	Darmon–Merel, others for small n
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$\{2, 2\ell, 3\}$	Chen, Dahmen, Siksek; primes $7 < \ell < 1000$ with $\ell \neq 31$
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$(2, 3, 10)$	ZB