

# Rigid Cohomology for Algebraic Stacks

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# Problem (posed by Kiran Kedlaya):

Rigid Cohomology  
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Stacks

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Develop a theory of **Rigid Cohomology for Algebraic Stacks**

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Develop a theory of **Rigid Cohomology for Algebraic Stacks**; i.e.,

- ▶ (Coefficients) – define some notion of **overconvergent isocrystal**;

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- ▶ Construct **variants** (e.g., cohomology supported in a closed subscheme);
- ▶ **Weil formalism** (e.g., excision, Gysin, trace formulas).

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**Geometric Langlands** for  $\mathrm{GL}_n(\mathbb{F}_p(C))$ :

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**Geometric Langlands** for  $\mathrm{GL}_n(\mathbb{F}_p(C))$ :

- ▶ Lafforgue constructs a ‘compactified moduli stack of shtukas’  $\mathcal{X}$  (actually a compactification of a stratification of a moduli stack of shtukas).



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**Geometric Langlands** for  $GL_n(\mathbb{F}_p(C))$ :

- ▶ Lafforgue constructs a ‘compactified moduli stack of shtukas’  $\mathcal{X}$  (actually a compactification of a stratification of a moduli stack of shtukas).
- ▶ The  $\ell$ -adic étale cohomology of étale sheaves on  $\mathcal{X}$  **realize a Langlands correspondence** between certain **Galois** and **automorphic** representations.

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- ▶  $\ell = p$  is bad for étale cohomology.

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**Geometric Langlands** for  $\mathrm{GL}_n(\mathbb{F}_p(C))$ :

- ▶  $\ell = p$  is bad for étale cohomology.
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## Geometric Langlands for $\mathrm{GL}_n(\mathbb{F}_p(C))$ :

- ▶  $\ell = p$  is bad for étale cohomology.
- ▶  $\mathcal{X}$  is a singular, separated Artin stack, so crystalline cohomology won't work.
- ▶ Generalizing rigid cohomology to Artin stacks would extend Lafforgue's work to the  $\ell = p$  case.

## Applications:

- ▶ Geometric Langlands for  $GL_n(\mathbb{F}_p(C))$ ;
- ▶ Logarithmic rigid cohomology and crystalline fundamental groups;

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- ▶ Geometric Langlands for  $GL_n(\mathbb{F}_p(C))$ ;
- ▶ Logarithmic rigid cohomology and crystalline fundamental groups;
- ▶ Arithmetic Statistics – Cohen-Lenstra heuristics for  $p$ -divisible groups.

# Rigid Cohomology Setup (for Schemes)

$$\begin{array}{ccc}
 ]X[ & \overset{i}{\hookrightarrow} & \mathbb{P}_{\mathbb{Q}_p}^{n,\text{an}} \\
 \downarrow sp & & \downarrow sp \\
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$$H_{\text{rig}}^i(X) := H^i \left( ]X[, i^{-1} \Omega_{\mathbb{P}_{\mathbb{Q}_p}^{n,\text{an}}}^\bullet \right)$$

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(OK to replace  $\mathbb{P}^n$  with a formal scheme which is smooth and proper over  $\text{Spf } \mathbb{Z}_p$ .)

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Example:  $X = \mathbb{A}_{\mathbb{F}_p}^1$

$$\begin{array}{ccc}
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$$\Gamma \left( i^{-1} \Omega_{(\mathbb{P}_{\mathbb{Q}_p}^1)^{\text{an}}}^\bullet \right) = 0 \rightarrow \mathbb{Q}_p\{x\}^\dagger \xrightarrow{d} \mathbb{Q}_p\{x\}^\dagger dx \rightarrow 0$$

Example:  $X = \mathbb{A}_{\mathbb{F}_p}^1$

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$$\mathbb{Q}_p\{x\}^\dagger \subset \mathbb{Q}_p[|x|],$$

$$d(f(x)) := f'(x)dx$$

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$$H_{\text{rig}}^i(\mathbb{A}_{\mathbb{F}_p}^1) = \begin{cases} \mathbb{Q}_p, & \text{if } i = 0 \\ 0, & \text{if } i \geq 1 \end{cases}$$



# Problems with Berthelot's construction

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- **Independence** of choices is a theorem (Berthelot).

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- ▶ **Independence** of choices is a theorem (Berthelot).
- ▶ **Functorality** is another theorem (Berthelot).

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- ▶ **Independence** of choices is a theorem (Berthelot).
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- ▶ Hard to prove results about **relative** rigid cohomology (e.g., coherence is still an open problem).

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- ▶ **Independence** of choices is a theorem (Berthelot).
- ▶ **Functorality** is another theorem (Berthelot).
- ▶ Hard to prove results about **relative** rigid cohomology (e.g., coherence is still an open problem).
- ▶ How to **define** for a scheme which isn't quasi-projective?

# Le Stum's overconvergent site: $\text{AN}^\dagger(X)$

► **Objects:**  $(X', V') =$

$$\begin{array}{ccccccc}
 X' & \hookrightarrow & P' & \xleftarrow{sp} & P'_{\mathbb{Q}_p} & \xleftarrow{\lambda} & V' \\
 \downarrow & & \downarrow & & \downarrow & & \\
 X & & & & & & \\
 \downarrow & & \downarrow & & \downarrow & & \\
 \text{Spec } \mathbb{F}_p & \hookrightarrow & \text{Spf } \mathbb{Z}_p & \xleftarrow{sp} & \mathcal{M}(\mathbb{Q}_p) & & 
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- A **morphism**  $(X', V') \rightarrow (X'', V'')$  is a triple of morphisms compatible with the diagram.

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- A **morphism**  $(X', V') \rightarrow (X'', V'')$  is a triple of morphisms compatible with the diagram.
- $\{(X, V_i) \rightarrow (X', V')\}$  is a **covering** if  $V = \bigcup V_i$  is an open covering of topological spaces.

# Strict Neighborhoods

- The **tube** of  $(X' \hookrightarrow P', P'_{\mathbb{Q}_p} \xleftarrow{\lambda} V')$  is  $\lambda^{-1}(]X'[_{P'_{\mathbb{Q}_p}})$ .

$$\begin{array}{ccc}
 ]X'[_{V'} \hookrightarrow & & V' \\
 \downarrow & & \downarrow \lambda \\
 ]X'[_{P'_{\mathbb{Q}_p}} \hookrightarrow & & P'_{\mathbb{Q}_p} \\
 \downarrow & & \downarrow sp \\
 X' \hookrightarrow & & P'
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 \downarrow & & \downarrow sp \\
 X' \hookrightarrow P' & & 
 \end{array}$$

- A morphism  $(X', V') \rightarrow (X'', V'')$  induces a morphism  $]X'[_{V'} \rightarrow ]X''[_{V''}$ .
- We declare  $(X', W') \rightarrow (X', V')$  to be an isomorphism if the induced map on tubes  $]X'[_{W'} \rightarrow ]X'[_{V'}$  is an isomorphism.

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# Structure sheaf $\mathcal{O}_X^\dagger$

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- ▶  $i: ]X'[_{V'} \hookrightarrow V'$ .
- ▶  $\mathcal{O}_X^\dagger(X', V') := \Gamma(]X'[_{V'}, i^{-1}\mathcal{O}_{V'})$ .

# Le Stum's main theorem

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## Theorem

$$(i) \quad H^i(\mathrm{AN}^\dagger X, \mathcal{O}_X^\dagger) \cong H_{\mathrm{rig}}^i(X).$$

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## Theorem

- (i)  $H^i(\mathrm{AN}^\dagger X, \mathcal{O}_X^\dagger) \cong H_{\mathrm{rig}}^i(X).$
- (ii)  $\mathrm{Coh} \mathcal{O}_X^\dagger \cong \mathrm{Isoc}^\dagger X$

# Rigid Cohomology for Stacks

Let  $\mathcal{X}$  be a stack and define  $\mathrm{AN}^\dagger(\mathcal{X})$  and  $\mathcal{O}_X^\dagger$  the same way.

- **Objects:**  $(X, V) =$

$$\begin{array}{ccccccc}
 X & \hookrightarrow & P & \xleftarrow{sp} & P_{\mathbb{Q}_p} & \xleftarrow{\lambda} & V \\
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## Define:

- ▶  $\mathcal{O}_{\mathcal{X}}^{\dagger}$  as before:  $(X, V) \mapsto \Gamma(i^{-1}\mathcal{O}_V)$ ;
- ▶  $\text{Isoc}^{\dagger}(\mathcal{X}) := \text{Coh } \mathcal{O}_{\mathcal{X}}^{\dagger}$ ;
- ▶  $H_{\text{rig}}^i(\mathcal{X}) := H^i(\mathcal{X}_{\text{AN}^{\dagger}}, \mathcal{O}_{\mathcal{X}}^{\dagger})$ .

# Sample Theorems: Finiteness and agreement

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## Theorem (ZB.)

- (i)  $H_{\text{rig}}^i(\mathcal{X})$  is **finite dimensional**.
- (ii)  $H_{\text{rig}}^i(\mathcal{X}) \otimes \mathbb{C}$  **agrees with the étale cohomology of  $\mathcal{X}$** .
- (iii)  $H_{\text{rig}}^i(\mathcal{X})$  **agrees with the crystalline cohomology of  $\mathcal{X}$  when  $\mathcal{X}$  is smooth and proper**.



# Cohomology supported in a closed substack

Rigid Cohomology  
for Algebraic  
Stacks

David  
Zureick-Brown

Basic Problem

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- ▶ Given a closed substack  $\mathcal{Z} \subset \mathcal{X}$  with open complement  $\mathcal{U}$ , I can define functors  $H_{\text{rig}, \mathcal{Z}}^i(\mathcal{X})$ .
- ▶ **Idea:** use the very general notions of open and closed immersion of **topoi** (of SGA4).

# Cohomology supported in a closed substack: Theorems

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## Theorem (ZB.)

- (i)  $H_{\text{rig}, \mathcal{Z}}^i(\mathcal{X})$  **agrees** with Berthelot's construction when  $\mathcal{X}$  is a scheme.
- (ii) (Excision) There is a **long exact sequence**  
$$\cdots H_{\text{rig}, \mathcal{Z}}^i(\mathcal{X}) \rightarrow H_{\text{rig}}^i(\mathcal{X}) \rightarrow H^i(\mathcal{U}) \rightarrow H_{\text{rig}, \mathcal{Z}}^{i+1}(\mathcal{X}) \cdots$$
- (iii) (Gysin) When  $(\mathcal{X}, \mathcal{Z})$  is a smooth pair, there is an isomorphism

$$H_{\text{rig}, \mathcal{Z}}^i(\mathcal{X}) \cong H^{i-2d}(\mathcal{Z})$$

# Main tool: cohomological descent

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## Theorem (ZB.)

*Cohomological descent holds on the overconvergent site with respect to smooth, flat, and étale hypercovers.*

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## Theorem (ZB.)

*Cohomological descent holds on the overconvergent site with respect to smooth, flat, and étale hypercovers.*

Special case: let  $X \rightarrow \mathcal{X}$  be a smooth cover. Then there is a spectral sequence

$$H_{\text{rig}}^i(X_j) \Rightarrow H_{\text{rig}}^{i+j}(\mathcal{X})$$

where  $X_j$  is the  $j + 1$  fold fiber product  $X \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} X$ .

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Previous proof for rigid cohomology is  $\sim 200$  pages; mine is  $\sim 20$ .

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# Rigid Cohomology for Algebraic Stacks

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Thank you!