Diophantine and tropical geometry

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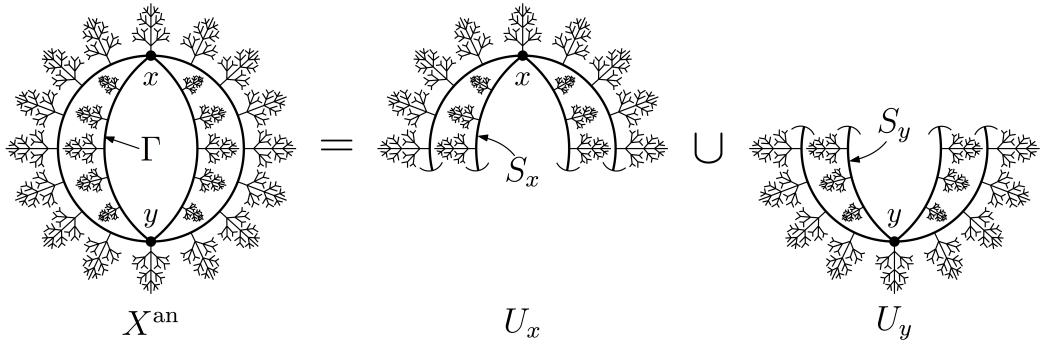
joint with Eric Katz (Waterloo) and Joe Rabinoff (Georgia Tech)

Slides available at `http://www.mathcs.emory.edu/~dzb/slides/`

Joint math meetings, Atlanta, GA
January 7, 2017



Basic Problem: given a system of diophantine equations,

Qualitative:

- Does there **exist** a solution?
- Do there exist **infinitely many** solutions?
- Does the set of solutions have some **extra structure** (e.g., geometric structure, group structure).

Quantitative

- How **many** solutions are there?
- How **large** is the **smallest** solution?
- How can we explicitly **find** all solutions? (With proof?)

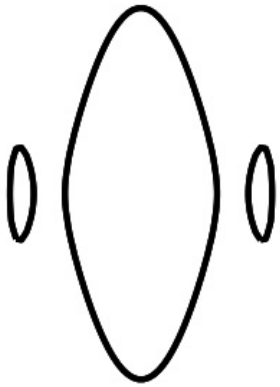
Implicit question

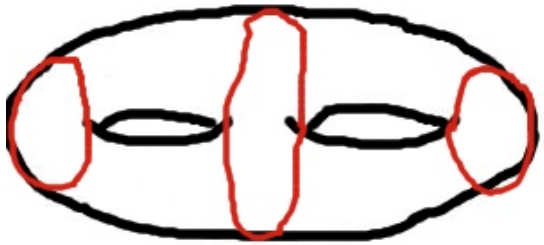
- Why do equations **have** (or fail to have) solutions?
- Why do some have **many** and some have **none**?
- What **underlying mathematical structures** control this?

Mordell Conjecture

Example

$$y^2 = (x^2 - 1)(x^2 - 2)(x^2 - 3)$$





This is a cross section of a two holed torus. The **genus** is the number of holes.

Conjecture (Mordell); Theorem (Faltings, Bombieri, Vojta)

A curve of genus $g \geq 2$ has only finitely many rational solutions.

Uniformity

Problem

1. *Given X, compute $X(\mathbb{Q})$ exactly.*
2. *Compute bounds on $\#X(\mathbb{Q})$.*

Conjecture (Uniformity)

There exists a constant $N(g)$ such that every smooth curve of genus g over $\mathbb{Q}$ has at most $N(g)$ rational points.

Theorem (Caporaso, Harris, Mazur)

Lang's conjecture $\Rightarrow$ uniformity.

Uniformity numerics

g	2	3	4	5	10	45	g
$B_g(\mathbb{Q})$	642	112	126	132	192	781	$16(g+1)$

Remark

Elkies studied K3 surfaces of the form

$$y^2 = S(t, u, v)$$

with lots of rational lines, such that S restricted to such a line is a perfect square.

Coleman's bound

Theorem (Coleman)

Let X be a curve of genus g and let $r = \operatorname{rank}_{\mathbb{Z}} \operatorname{Jac}_X(\mathbb{Q})$. Suppose $p > 2g$ is a prime of good reduction. Suppose $r < g$. Then

$$\#X(\mathbb{Q}) \leq \#X(\mathbb{F}_p) + 2g - 2.$$

Remark

This can be used to provably compute $X(\mathbb{Q})$.

Example (Gordon, Grant)

$$y^2 = x(x-1)(x-2)(x-5)(x-6)$$

Analysis

1. $\operatorname{rank}_{\mathbb{Z}} \operatorname{Jac}_X(\mathbb{Q}) = 1$, $g = 2$
2. $X(\mathbb{Q})$ contains

$$\{\infty, (0,0), (1,0), (2,0), (5,0), (6,0), (3, \pm 6), (10, \pm 120).\}$$

3. $\#\mathscr{X}_5^{\mathrm{sm}}(\mathbb{F}_7) = 8$
4. $10 \leq \#X(\mathbb{Q}) \leq \#X(\mathbb{F}_7) + 2g - 2 = 10$

This determines $X(\mathbb{Q})$.

Coleman's bound

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Remark

1. A modified statement holds for $p \leq 2g$ or for $K \neq \mathbb{Q}$.
2. Note: this does not prove uniformity (since the first good p might be large).

Tools

p-adic integration and Riemann–Roch

Chabauty's method

(*p*-**adic integration**) There exists $V \subset H^0(X_{\mathbb{Q}_p}, \Omega^1_X)$ with $\dim_{\mathbb{Q}_p} V \geq g - r$ such that,

$$\int_P^Q \omega = 0 \qquad \forall P, Q \in X(\mathbb{Q}), \omega \in V$$

(*p*-**adic Rolle's (Coleman), via Newton Polygons**)
Number of zeroes in a residue disc D_P is $\leq 1 + n_P$, where

$$n_P = \#\left(\operatorname{div}\omega \cap D_P\right)$$

(**Riemann-Roch**) $\sum n_P = 2g - 2$.

(**Coleman's bound**) $\sum_{P \in X(\mathbb{F}_p)}(1 + n_P) = \#X(\mathbb{F}_p) + 2g - 2$.

Example (from McCallum-Poonen's survey paper)

Example

$$X\colon y^2 = x^6 + 8x^5 + 22x^4 + 22x^3 + 5x^2 + 6x + 1$$

1. Points reducing to $\widetilde{Q} = (0,1)$ are given by

$$\begin{aligned} x &= p \cdot t, \text{ where } t \in \mathbb{Z}_p \\ y &= \sqrt{x^6 + 8x^5 + 22x^4 + 22x^3 + 5x^2 + 6x + 1} = 1 + x^2 + \cdots \end{aligned}$$

2. $$\int_{(0,1)}^{P_t} \frac{x dx}{y} = \int_0^t (x - x^3 + \cdots) dx$$

Chabauty's method

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Bad reduction bound

Theorem (Lorenzini-Tucker, McCallum-Poonen)

Let X be a curve of genus g and let $r = \operatorname{rank}_{\mathbb{Z}} \operatorname{Jac}_X(\mathbb{Q})$. Suppose $p > 2g$ is a prime. Suppose $r < g$.

Let $\mathscr{X}$ be a regular proper model of X. Then

$$\#X(\mathbb{Q}) \leq \#\mathscr{X}^{sm}(\mathbb{F}_p) + 2g - 2.$$

Remark (Still doesn't prove uniformity)

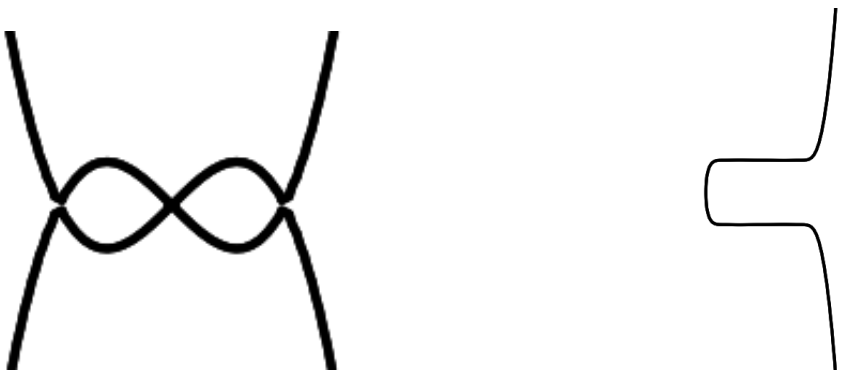
$\#\mathscr{X}^{\mathrm{sm}}(\mathbb{F}_p)$ can contain an n-gon, for n arbitrarily large.

Tools

p-adic integration and arithmetic Riemann–Roch ($\mathcal{K} \cdot \mathscr{X}_p = 2g - 2$)

Models – semistable example

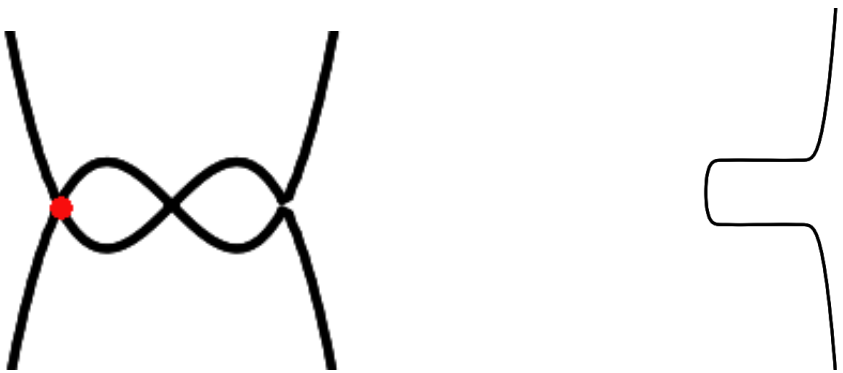
$$\begin{aligned} y^2 &= (x(x-1)(x-2))^3 - 5 \\ &= (x(x-1)(x-2))^3 \mod 5. \end{aligned}$$



Note: no point can reduce to $(0,0)$. Local equation looks like $xy = 5$

Models – semistable example (not regular)

$$\begin{aligned} y^2 &= (x(x-1)(x-2))^3 - 5^4 \\ &= (x(x-1)(x-2))^3 \mod 5 \end{aligned}$$



Now: $(0, 5^2)$ reduces to $(0, 0)$. Local equation looks like $xy = 5^4$

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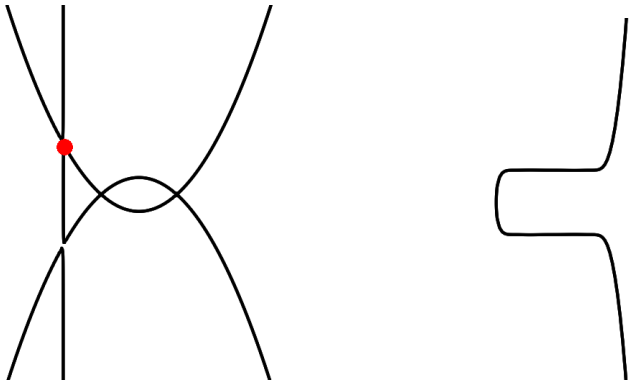

Blow up. Local equation looks like $xy = 5^3$

Models – semistable example (regular at (0,0))

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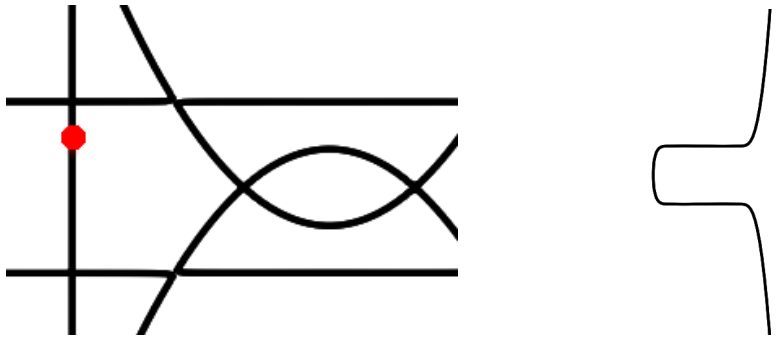

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Tools

p-adic integration and arithmetic Riemann–Roch ($\mathcal{K} \cdot \mathscr{X}_p = 2g - 2$)

Stoll's hyperelliptic uniformity theorem

Theorem (Stoll)

Let X be a hyperelliptic curve of genus g and let $r = \operatorname{rank}_{\mathbb{Z}} \operatorname{Jac}_X(\mathbb{Q})$. Suppose $r < g - 2$.

Then

$$\#X(\mathbb{Q}) \leq 8(r+4)(g-1) + \max\{1, 4r\} \cdot g$$

Tools

p-adic integration on annuli

comparison of different analytic continuations of p-adic integration

p-adic Rolle's on hyperelliptic annuli

Analytic continuation of integrals

(**Residue Discs.**)

$$P \in \mathcal{X}^{\mathrm{sm}}(\mathbb{F}_p),\ t\colon D_P \cong p\mathbb{Z}_p,\ \omega|_{D_P} = f(t)dt$$

(**Integrals on a disc.**)

$$Q, R \in D_P,\ \int_Q^R \omega := \int_{t(Q)}^{t(R)} f(t)dt.$$

(**Integrals between discs.**)

$$Q \in D_{P_1},\ R \in D_{P_2},\ \int_Q^R \omega := ?$$

Analytic continuation of integrals via Abelian varieties

(**Integrals between discs.**)

$$Q \in D_{P_1},\ R \in D_{P_2},\ \int_Q^R \omega := ?$$

(**Albanese map.**)

$$\iota\colon X \hookrightarrow \mathrm{Jac}_X,\ Q \mapsto [Q - \infty]$$

(**Abelian integrals via functorality and additivity.**)

$$\int_Q^R \iota^*\omega = \int_{\iota(Q)}^{\iota(R)} \omega = \int_{[Q-\infty]}^{[R-\infty]} \omega = \int_0^{[R-Q]} \omega = \frac{1}{n}\int_0^{n[R-Q]} \omega$$

Analytic continuation of integrals via Frobenius

(**Integrals between discs.**)

$$Q \in D_{P_1},\ R \in D_{P_2},\ \int_Q^R \omega := ?$$

(**Abelian integrals via functorality and Frobenius.**)

$$\int_Q^R \omega = \int_Q^{\phi(Q)} \omega + \int_{\phi(Q)}^{\phi(R)} \omega + \int_{\phi(R)}^R \omega$$

(**Very clever trick (Coleman)**)

$$\int_{\phi(Q)}^{\phi(R)} \omega_i = \int_Q^R \phi^* \omega_i = df_i + \sum_j \int_Q^R a_{ij}\omega_j$$

Comparison of integrals

Facts

1. For X with good reduction, the **Abelian** and **Coleman** integrals agree.
2. ~~A mystery~~. The associated Berkovich curve is contractable.
3. For X with bad reduction they differ.

Theorem (Stoll; Katz-Rabinoff-Zureick–Brown)

There exist linear functions $a(\omega), c(\omega)$ such that

$$\oint_Q^R \omega - \int_Q^R \omega = a(\omega)\left(\log(t(R)) - \log(t(Q))\right) + c(\omega)\left(t(Q) - t(R)\right)$$

Why bother? Integration on Annuli (a trade off)

Assumption

Assume $\mathcal{X}/\mathbb{Z}_p$ is **stable**, but not regular.

(**Residue Discs.**)

$P \in \mathcal{X}^{\mathrm{sm}}(\mathbb{F}_p)$, $t\colon D_P \cong p\mathbb{Z}_p$, $\omega|_{D_P} = f(t)dt$

(**Residue Annuli.**)

$P \in \mathcal{X}^{\mathrm{sing}}(\mathbb{F}_p)$, $t\colon D_P \cong p\mathbb{Z}_p - p^r\mathbb{Z}_p, \omega|_{D_P} = f(t, t^{-1})dt$

(**Integrals on an annulus are multivalued.**)

$$\int_Q^R \omega := \int_{t(Q)}^{t(R)} f(t, t^{-1})dt = \cdots + a(\omega)\log t + \cdots$$

(**Cover the annulus with discs**)

Each analytic continuation implicitly chooses a branch of log.

Why bother? Integration on Annuli (a trade off)

(**Abelian integrals.**) Analytically continue via Albanese.

$$\oint_Q^R \omega = 0 \text{ if } R, Q \in X(\mathbb{Q}), \omega \in V$$

(**Berkovich-Coleman integrals.**) Analytically continue via Frobenius.

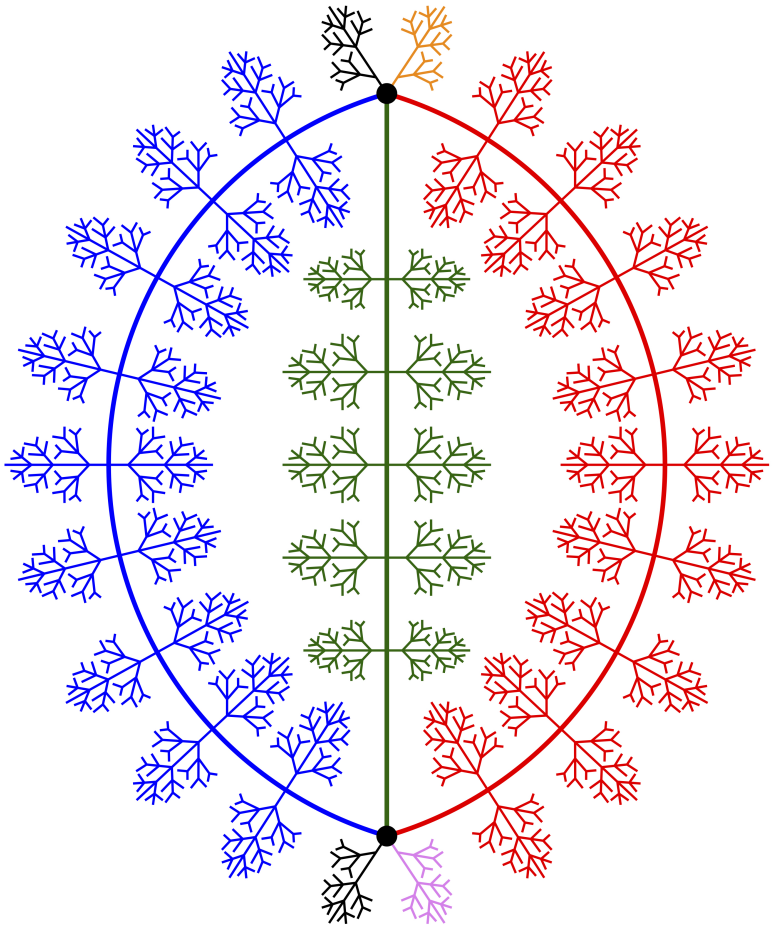
$$\int_Q^R \omega := \int_{t(Q)}^{t(R)} f(t, t^{-1}) dt = \cdots + a(\omega) \log_{\text{Col}} t + \cdots$$

(**Stoll's theorem.**)

$$\oint_Q^R \omega - \int_Q^R \omega = a(\omega) \left(\log_{\text{ab}}(r(R)) - \log_{\text{ab}}(t(Q))\right) + c(\omega) \left(t(Q) - t(R)\right)$$

Berkovich picture

X^{an}



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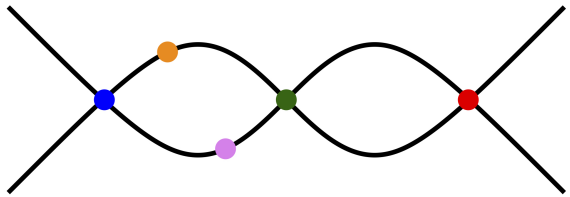


$\mathcal{X}_k$

Stoll's comparison theorem, tropical geometry edition

Theorem (Katz, Rabinoff, ZB)

The difference $\log_{Col} - \log_{ab}$ *is the unique homomorphism that takes the value*

$$\int_\gamma \omega$$

on $Trop(\gamma)$*, where* $Trop\colon G(\mathbb{K}) \to T(\mathbb{K})/T(\mathcal{O})$.

$$\begin{array}{ccccc} & & T & & \\ & & \downarrow & & \\ \Lambda & \longrightarrow & G & \longrightarrow & (\mathrm{Jac}_X)^{\mathrm{an}} \\ & & \downarrow & & \\ & & B & & \end{array}$$

T = torus, Λ = discrete, and B = Abelian w/ good reduction.

Main Theorem (partial uniformity for curves)

Theorem (Katz, Rabinoff, ZB)

Let X be any curve of genus g and let $r = \operatorname{rank}_{\mathbb{Z}} \operatorname{Jac}_X(\mathbb{Q})$. Suppose $r \leq g - 2$. Then

$$\#X(\mathbb{Q}) \leq 84g^2 - 98g + 28$$

Tools

p-adic integration on annuli

comparison of different analytic continuations of p-adic integration

Non-Archimedean (Berkovich) structure of a curve [BPR]

Combinatorial restraints coming from the Tropical canonical bundle

p-adic Rolle's on annuli for arbitrary curves

Comments

Corollary ((Partially) effective Manin-Mumford)

There is an effective constant $N(g)$ such that if $g(X) = g$, then

$$\#\left(X \cap \mathrm{Jac}_{X,tors}\right)(\mathbb{Q}) \leq N(g)$$

Corollary

There is an effective constant $N'(g)$ such that if $g(X) = g > 3$ and $X/\mathbb{Q}$ has totally degenerate, trivalent reduction mod 2, then

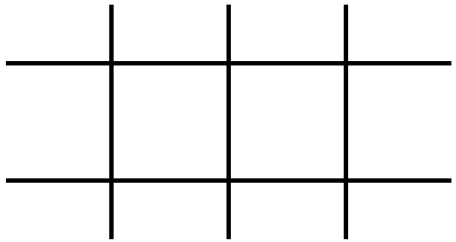
$$\#\left(X \cap \mathrm{Jac}_{X,tors}\right)(\mathbb{C}) \leq N'(g)$$

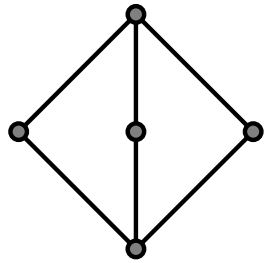
The second corollary is a big improvement

1. It requires working over a non-discretely valued field.
2. The bound only depends on the reduction type.
3. Integration over wide opens (c.f. Coleman) instead of discs and annuli.

Baker-Payne-Rabinoff and the slope formula

(**Dual graph Γ of $X_{\mathbb{F}_p}$**)





(**Contraction Theorem**) $\tau\colon X^{\mathrm{an}} \to \Gamma$.

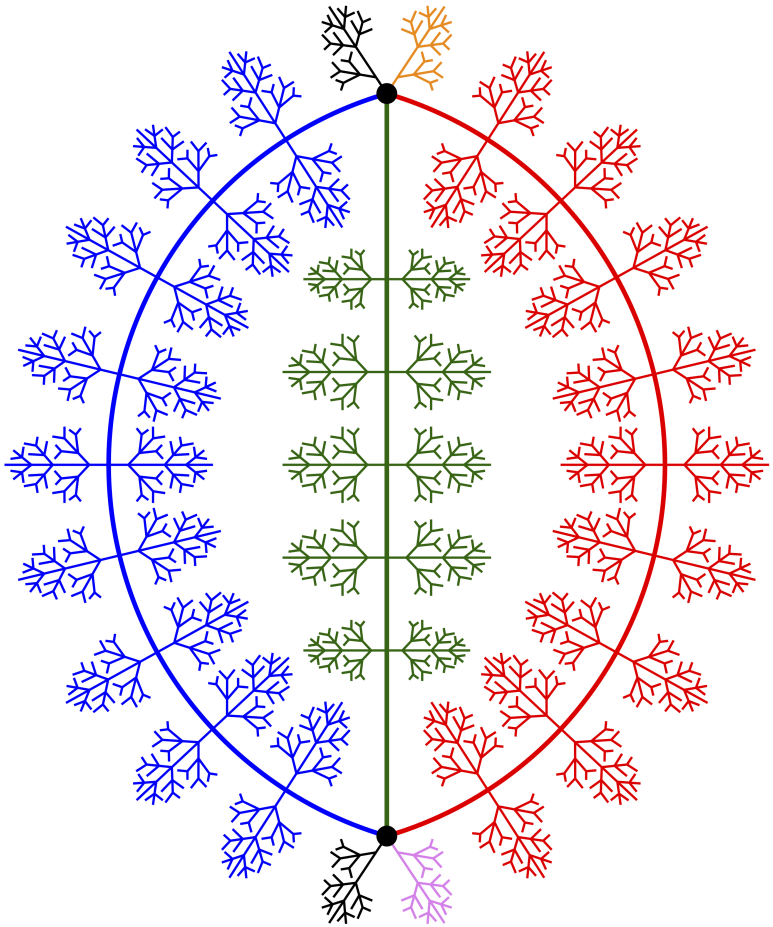
(**Combinatorial harmonic analysis/potential theory**)

f	a meromorphic function on X^{an}
$F := (-\log\lvert f\rvert)\big\rvert_\Gamma$	associated tropical, piecewise linear function
$\operatorname{div} F$	combinatorial record of the slopes of F

(**Slope formula**) $\tau_* \operatorname{div} f = \operatorname{div} F$

Berkovich picture

X^{an}



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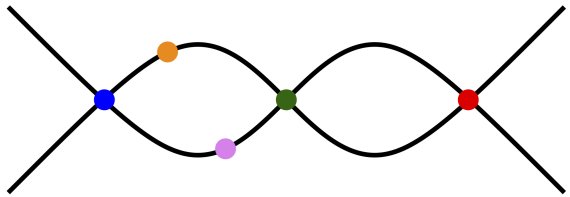

$\mathcal{X}_k$