Math 355: Real Analysis
Instructor: David Zureick-Brown ("DZB")

All assignments

Last updated: May 6, 2025

Gradescope code: DKNX3W

Show all work for full credit!
Proofs should be written in full sentences whenever possible.

Gradescope Instructions for submitting work in Math 355

You will be using the online Gradescope progam to submit your homework and exams. These instructions tell you how to sign up initially, and how to submit your written work.

Signing up for Gradescope the first time.

If you haven't used Gradescope for an **Amherst College** course before:

- Go to `http://www.gradescope.com`, click on "Sign up for free" (which may auto-scroll you to the bottom of the page), and select Sign up as [a] "Student".
- In the signup box:
  - Use the course entry code **DKNX3W**
  - Use your full name
  - Use your **Amherst College email** address. Or, if you are a Five-College student, use your email address from your own school.
  - Leave the "Student ID" entry blank.
- You will probably get an email asking to set a password for your account, so check your `amherst.edu` email inbox. (Or your email inbox through your own school, for Five-College students.)

Adding Math 355 to Gradescope.

If you **have** used Gradescope for an Amherst course before, and so you already have an account through your `amherst.edu` email, you still need to add Math 355, so:

- Go to `http://www.gradescope.com` and log in.
- Go to your Account Dashboard (click the Gradescope logo at upper left), and click "Add Course" at bottom right.
- Use the course code **DKNX3W**

(submission instructions on next page)

Submitting written work

First write it out on paper as you would normally. Then **scan it** to create a PDF. One method for scanning is the smartphone app **DropBox**. It makes nice clear scans, and it saves them directly into a folder so that you can have all your assignments in one place. **CamScanner** is another free scanning App, and there are others, too. **Gradescope** now has its own scanning app. You can also use a printer/scanner if you prefer.

Please be kind to our dear graders and make sure your submission is legible !

In particular, please leave some spacing between separate problems.

If you have a tablet computer, you may write your work there (instead of on paper) and save it as a PDF.

Some of you may know the math formatting package LaTeX and may want to use it in Math 355. That's fine, too; if so, you may write up your work in LaTeX and save the resulting PDF.

In short, any method is fine as long as it creates a legible **PDF** file and **NOT a photo**.

For example, if you use the DropBox app, then in your created *Math 355 Homework* Dropbox folder, you can select create (+) at the bottom of the screen and click the *Scan Document* option. Snap a shot of the first page of your homework, and then click [+] to snap shots of any subsequent pages. Do **not** use the *Take Photo* option.

After you have scanned/saved your work as a PDF, submit it on Gradescope as follows:

- Go to `http://www.gradescope.com` and log in.
- Select the course "Math 355, Spring 2025" and the appropriate assignment.
- Select "submit pdf" to submit your work in PDF format. Browse to find your PDF and upload.
- Now it is time to **tag** your problems. This is an **important step** , where you are telling Gradescope which problems are on which page(s).

  For each problem, select the pages of your submission where your written solution appears.

  I think the easiest thing to do is to click on the page of **your** homework upload where you wrote the given problem, and then click on the assigned problem listed. Repeat for each problem.

  You must tag the problems or else you will not get credit for your work.

  Gradescope will give you a warning when you go to submit your assignment if you have not selected the pages correctly. If you tag a problem incorrectly, you can fix it by clicking "More" and "Reselect Pages".
- Click Save or Submit.

After your assignment is graded, you will be able to see your score on the written problems, along with comments, on Gradescope. You should receive an email notifying you when each homework set is graded.

Assignment 1: Introduction to the course; numbers

Due by 12:55pm, eastern, on Thursday, February 06

- ***Suggested readings for this problem set:*** **1.1, 1.2**
- Syllabus: https://dmzb.github.io/teaching/2025Spring355/syllabus-math-355-S25.pdf
- Gradescope instructions (previous page)

All readings are from Abbott, *Understanding Analysis*.

Assignment: due Thursday, February 06, 12:55pm, via Gradescope (DKNX3W):

1. Let $A := \{2, \{2\}, \{2, \{2\}\}\}$. How many elements are in the set A? For each of the following parts, answer true or false. Briefly explain our answer.

   (a) $2 \in A$ (b) $2 \subseteq A$ (c) $\{2\} \subseteq A$ (d) $\{2\} \in A$

2. Consider the following sentence: If $a > 0$, then there exists a natural number n such that $\frac{1}{n} < a$.
   (a) Write the negation.
   (b) Write the contrapositive of the original sentence.
   (c) Which is true, the original or the negation? Explain your answer intuitively. (Don't worry about giving a formal proof.)

3. Let $A := \{3, 5, 8\}$ and $B := \{3, 6, 10\}$.
   (a) Is the following statement true or false? Justify your claim. $\exists\, b \in B$ s.t. $\forall\, a \in A$, $a - b < 0$.
   (b) Negate the statement: $\exists\, b \in B$ s.t. $\forall\, a \in A$, $a - b < 0$.
   (c) Is the following statement true or false? Justify your claim. $\forall\, b \in B$, $\exists\, a \in A$ s.t. $a - b < 0$.

4. Prove that there is no smallest strictly positive rational number.

5. Negate each of the following statements (do not worry about the validity of the statement):
   (a) For all $x, y \in \mathbb{R}$ satisfying $x < y$, there exists an $n \in \mathbb{N}$ such that $x + 1/n < y$.
   (b) For all $x, y \in \mathbb{R}$ satisfying $x < y$, there exists an $r \in \mathbb{Q}$ such that $x < r < y$.
   (c) There exists a real number $x > 0$ such that $x < 1/n$ for all $n \in \mathbb{N}$.
   (d) For all $M \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ satisfying $n \geq N$, one has $1/n < 1/M$.

6. Suppose $z \in \mathbb{R}$ and $z \geq 0$. Which of the following statements imply that $z = 0$? Justify your answer either by proof or counterexample.

   (a) $\forall\, \varepsilon \in (0, \infty)$, $z < \varepsilon$.

   (b) $\forall\, \varepsilon \in (0, \infty)$, $z \leq \varepsilon$.

   (c) $\forall\, \varepsilon \in (0, \infty)$, $\exists\, \delta \in (0, \infty)$ s.t. $z \leq \varepsilon \cdot \delta$.

   (d) $\forall\, \varepsilon \in (0, \infty)$, $\exists\, \delta \in (0, \infty)$ s.t. $z \cdot \delta \leq \varepsilon$.

   (e) $\forall\, \varepsilon \in (0, \infty)$, $\exists\, \delta \in (0, \infty)$ s.t. $z \leq \delta < \varepsilon$.

7. **De Morgan's Laws.** Let A and B be subsets of $\mathbb{R}$.

   (a) If $x \in (A \cap B)^c$, explain why $x \in A^c \cup B^c$. This shows that $(A \cap B)^c \subseteq A^c \cup B^c$.

   (b) Prove the reverse inclusion $(A \cap B)^c \supseteq A^c \cup B^c$, and conclude that $(A \cap B)^c = A^c \cup B^c$.

   (c) Show $(A \cup B)^c = A^c \cap B^c$ by demonstrating inclusion both ways.

   (d) Show how induction can be used to conclude that

   $$(A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$$

   for any finite $n \in \mathbb{N}$.

8. Decide which of the following are true statements. Provide a short justification for those that are valid and a counterexample for those that are not:

   (a) Two real numbers satisfy $a < b$ if and only if $a < b + \varepsilon$ for every $\varepsilon > 0$.

   (b) Two real numbers satisfy $a < b$ if $a < b + \varepsilon$ for every $\varepsilon > 0$.

   (c) Two real numbers satisfy $a \leq b$ if and only if $a < b + \varepsilon$ for every $\varepsilon > 0$.

Bonus Problem:[1] Prove that there are two irrational numbers α and β such that α^β is rational.

[1]Bonus problems are just for "fun"; they are not worth any points, so please do not submit them.

Assignment 2: Axiom of Completeness; sup, inf, min, max, real numbers, density

Suggested readings for this problem set: **1.3, 1.4**
All readings are from Abbott, *Understanding Analysis*.

Assignment: due Thursday, February 13, 12:55pm, via Gradescope (DKNX3W):

1. Give an example of each of the following, or state that the request is impossible:
   (a) A set B with $\inf B \geq \sup B$.
   (b) A finite set that contains its infimum but not its supremum.
   (c) A bounded subset of $\mathbb{Q}$ that contains its supremum but not its infimum.
   (d) Two sets A and B with $A \cap B = \emptyset$, $\sup A = \sup B$, $\sup A \notin A$, and $\sup B \notin B$.
   (e) A sequence of nested open intervals $J_1 \supseteq J_2 \supseteq J_3 \supseteq \ldots$ with $\bigcap_{n=1}^{\infty} J_n$ nonempty but containing only a finite number of elements.
2. Compute, without proofs, the suprema and infima (if they exist) of the following sets:
   (a) $\{m/n : m, n \in \mathbb{N} \text{ with } m < n\}$.
   (b) $\{(-1)^m/n : m, n \in \mathbb{N}\}$.
   (c) $\{n/(3n+1) : n \in \mathbb{N}\}$.
   (d) $\{m/(m+n) : m, n \in \mathbb{N}\}$.
3. Prove that if a is an upper bound for A, and if a is also an element of A, then it must be that $a = \sup A$.
4. Given sets A and B, define $A + B = \{a + b : a \in A \text{ and } b \in B\}$. Follow these steps to prove that if A and B are nonempty and bounded above, then $\sup(A + B) = \sup A + \sup B$.
   (a) Let $s = \sup A$ and $t = \sup B$. Show $s + t$ is an upper bound for $A + B$.
   (b) Now let u be an arbitrary upper bound for $A + B$, and temporarily fix $a \in A$. Show $t \leq u - a$.
   (c) Finally, show $\sup(A + B) = s + t$.
   (d) Construct another proof of this same fact using Lemma 1.3.8.
5. Decide if the following statements about suprema and infima are true or false. Give a short proof for those that are true. For any that are false, supply an example where the claim in question does not appear to hold.
   (a) If A and B are nonempty, bounded, and satisfy $A \subseteq B$, then $\sup A \leq \sup B$.
   (b) If $\sup A < \inf B$ for sets A and B, then there exists a $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$.
   (c) If there exists a $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$, then $\sup A < \inf B$.

6. (a) Let A be nonempty and bounded below, and define $B = \{b \in \mathbb{R} : b \text{ is a lower bound for } A\}$. Show that $\sup B = \inf A$.

   (b) Use (a) to explain why there is no need to assert that greatest lower bounds exist as part of the Axiom of Completeness.

7. Let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and let $s \in \mathbb{R}$ have the property that for all $n \in \mathbb{N}$, $s + \frac{1}{n}$ is an upper bound for A and $s - \frac{1}{n}$ is not an upper bound for A. Show $s = \sup A$.

Assignment 3: Completness; (un)countable; Cantor diagonalization; sequences, convergence

Suggested readings for this problem set: **1.4, 1.5, 2.2**
All readings are from Abbott, *Understanding Analysis*.

Assignment: due Thursday, February 20, 12:55pm, via Gradescope (DKNX3W):

1. Let $a < b$ be real numbers and consider the set $T = \mathbb{Q} \cap [a, b]$. Show $\sup T = b$.
2. Let $A = \{a, b, c\}$.
   (a) List the eight elements of $P(A)$. (Do not forget that $\emptyset$ is considered to be a subset of every set.)
   (b) If A is finite with n elements, show that $P(A)$ has 2^n elements.
   (c) Is the set of all functions from $\{0, 1\}$ to $\mathbb{N}$ countable or uncountable?
   (d) Is the set of all functions from $\mathbb{N}$ to $\{0, 1\}$ countable or uncountable?

   (Click here for a hint)
3. **Cantor's Theorem.** Prove that given any set A, there does not exist a function $f : A \to P(A)$ that is onto. Here is an outline for the proof.
   (a) First, suppose that there does exist such an f, and consider the set B
   $$B = \{a \in A : a \notin f(a)\}.$$
   Note that B is a subset of A. Follow the next two steps to show that B is not in the image of f. Suppose that B is in the image of f. Then there exists some element $a' \in A$ such that $f(a') = B$.
   (b) First, show (using the definition of B) that the case $a' \in B$ leads to a contradiction.
   (c) Finish the argument by showing that the case $a' \notin B$ is equally unacceptable.
4. Prove that the interval $[0, 1] \subset \mathbb{R}$ is uncountable without using decimal expansions by using the nested interval property (NIP).

   (Click here for a hint)
5. Verify, using the definition of convergence of a sequence, that the following sequences converge to the proposed limit:
   (a) $\lim \frac{2n+1}{5n+4} = \frac{2}{5}$.
   (b) $\lim \frac{2n^2}{n^3+3} = 0$.
   (c) $\lim \frac{\sin(n^2)}{\sqrt[3]{n}} = 0$.
6. Prove that if $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} a_n = b$, then $a = b$.

   (Click here for a hint)

7. Here are two useful definitions:

   (i) A sequence (a_n) is **eventually** in a set $A \subseteq \mathbb{R}$ if there exists an $N \in \mathbb{N}$ such that $a_n \in A$ for all $n \geq N$.

   (ii) A sequence (a_n) is **frequently** in a set $A \subseteq \mathbb{R}$ if, for every $N \in \mathbb{N}$, there exists an $n \geq N$ such that $a_n \in A$.

   (a) Is the sequence $(-1)^n$ eventually or frequently in the set $\{1\}$?

   (b) Which definition is stronger? Does *frequently* imply *eventually* or does *eventually* imply *frequently*?

   (c) Give an alternate rephrasing of Definition 2.2.3B using either *frequently* or *eventually*. Which is the term we want?

8. Use induction and the triangle inequality to prove the following fact:

$$\textit{For every } n \in \mathbb{N}, \textit{ if } c_1, \ldots c_n \in \mathbb{R}, \textit{ then } |c_1 + \cdots + c_n| \leq |c_1| + \cdots + |c_n|.$$

Assignment 4: Monotonicity; algebraic limit theorem (ALT and OLT)

Due by 12:55pm, eastern, on Thursday, February 27

Suggested readings for this problem set: **2.3, 2.4, 2.5**
All readings are from Abbott, *Understanding Analysis*.

Assignment: due Thursday, February 27, 12:55pm, via Gradescope (DKNX3W):

1. Let $x_n \geq 0$ for all $n \in \mathbb{N}$. Prove that if $(x_n) \to 0$, then $(\sqrt{x_n}) \to 0$.
2. Let $x_n \geq 0$ for all $n \in \mathbb{N}$. Prove that if $(x_n) \to x$, then $(\sqrt{x_n}) \to \sqrt{x}$.
3. **Squeeze Theorem.** Show that if $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, and if $\lim x_n = \lim z_n = l$, then $\lim y_n = l$ as well.

   Careful: if you are not using both $x_n \leq y_n$ **AND** $y_n \leq z_n$ then your proof very likely has a mistake.
4. Let $(a_n) \to 0$, and use the Algebraic Limit Theorem to compute the following limit (assuming the fractions are always defined). Be carefuly to justify each step by referencing the appropriate part of the Algebraic Limit Theorem.

$$\lim \frac{1 + 2a_n}{1 + 3a_n - 4a_n^2}$$

5. (a) Prove that if $(a_n)_{n=1}^{\infty}$ converges to $L \in \mathbb{R}$ then $(|a_n|)_{n=1}^{\infty} = (|a_1|, |a_2|, \dots)$ converges to $|L|$.

   (b) Prove that if $(|a_n|)_{n=1}^{\infty}$ converges to 0 then $(a_n)_{n=1}^{\infty}$ converges to 0.
   (*Note that if you combine the first two parts you get: A sequence $(a_n)_{n=1}^{\infty}$ converges to 0 if and only if the sequence $(|a_n|)_{n=1}^{\infty}$ converges to 0.*)

   (c) Give an example illustrating that $(a_n)_{n=1}^{\infty}$ need not converge if $(|a_n|)_{n=1}^{\infty}$ converges to a *nonzero* limit.
6. Determine if the following statement is true or false. Give a short proof if it is true, and provide a counterexample if it false. (Be careful, this is close to, but different from, the Order Limit Theorem!)

   Suppose that $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ converge to $L \in \mathbb{R}$ and $M \in \mathbb{R}$, respectively.
   If $a_n < b_n$ for every $n \in \mathbb{N}$, then $M < L$.
7. **Calculating Square Roots.** Let $x_1 = 2$, and define $x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$.

   (a) Show that $x_n^2 \geq 2$ for all n and use this to prove that $x_n - x_{n+1} \geq 0$. Conclude that $\lim x_n = \sqrt{2}$.

   (b) Modify the sequence (x_n) so that it converges to $\sqrt{c}$ for any given $c > 0$.

ONE MORE PROBLEM ON THE NEXT PAGE →

8. **Limit Superior.** Let (a_n) be a bounded sequence.

   (a) Prove that the sequence defined by $y_n = \sup\{a_k : k \geq n\}$ converges.

   (b) Define the limit superior of (a_n) as $\limsup a_n = \lim y_n$ and provide a reasonable definition for $\liminf a_n$. Explain why it always exists for any bounded sequence.

   (c) Prove that $\liminf a_n \leq \limsup a_n$ for every bounded sequence and give an example of a sequence where the inequality is strict.

   (d) Show that $\liminf a_n = \limsup a_n$ if and only if $\lim a_n$ exists. In this case, all three share the same value.

Assignment 5: Monotone convergence; subsequences; Cauchy sequences; Bolzano–Weierstrass

Due by 12:55pm, eastern, on Thursday, March 06

Suggested readings for this problem set: **2.5, 2.6, 2.7**
All readings are from Abbott, *Understanding Analysis*.

Assignment: due Thursday, March 06, 12:55pm, via Gradescope (DKNX3W):

1. Give an example of each of the following, or explain why the statement is impossible.
   (a) sequences (x_n) and (y_n), which both diverge, but whose sum $(x_n + y_n)$ converges;
   (b) sequences (x_n) and (y_n), where (x_n) converges, (y_n) diverges, and $(x_n + y_n)$ converges;
   (c) two sequences (a_n) and (b_n), where $(a_n b_n)$ and (a_n) converge but (b_n) does not.
   (d) A sequence that has a subsequence that is bounded but contains no subsequence that converges.
   (e) A sequence that does not contain 0 or 1 as a term but contains subsequences converging to each of these values.
   (f) A Cauchy sequence which contains a subsequence that has infinitely many 1's and infinitely many 0's.
   (g) A bounded and strictly increasing sequence which is not Cauchy.
   (h) A Cauchy sequence with an unbounded subsequence.
   (i) A divergent monotone sequence with a Cauchy subsequence.
   (j) An unbounded sequence containing a subsequence that is Cauchy.

2. Show that the following sequences are Cauchy, using only the definition of Cauchy.[2]
   (a) $a_n = 1/n$.
   (b) $a_n = 1/2^n$
   (c) $a_n = \sqrt{n+1} - \sqrt{n}$ (Click here for a hint)

3. Suppose that (x_n) and (y_n) are Cauchy sequences. Prove, without using the Cauchy Criterion, that the sequence $(z_n) = (x_n y_n)$ is also Cauchy.

4. Let (a_n) and (b_n) be Cauchy sequences. Decide whether each of the following sequences is a Cauchy sequence, justifying each conclusion.
   (a) $c_n = |a_n - b_n|$
   (b) $c_n = (-1)^n a_n$
   (c) $c_n = \lfloor a_n \rfloor$, where $\lfloor x \rfloor$ refers to the greatest integer less than or equal to x. (I.e., "round down".)

MORE PROBLEMS ON THE NEXT PAGE →

[2] I.e., do not use the Cauchy Criterion ("Cauchy if and only if convergent")

5. Let $A \subseteq \mathbb{R}$ be a nonempty set that is bounded above. Prove there exists a sequence $(a_n)_{n=1}^{\infty}$ of points in A that converges to $\sup(A) \in \mathbb{R}$. (Note that this is similar to, but different than the monotone convergence theorem. One approach to proving this is to modify the proof of the monotone convergence theorem.)

6. Assume (a_n) is a bounded sequence with the property that every convergent subsequence of (a_n) converges to the same limit $a \in \mathbb{R}$. Show that (a_n) must converge to a. (Note that we are not already assuming that the sequence (a_n) converges.)

This assignment only has 6 problems

Midterm 1 study guide

In class exam, Tuesday, March 11.

Content: The questions will all be either

1. Definitions,
2. homework problems,
3. problems we worked in class,
4. Theorems/Propositions/Lemmas that we proved in clasee,
5. True/False or "give an example" problems, or
6. minor variations of one of these.

Problems with very long proofs or that involved some unusual trick will not be on the exam. So for example, I am unlikey to ask you to prove the Bolzano–Weierstrauss theorem, but I am likely to ask you to prove that "a convergent sequence is bounded", or something of a similar length.

A typical exam will have a few questions from each week of the course and will cover **assignments 1-4**, which includes the following sections from our book.

Chapter 1, Chapter 2, sections 2.1-2.4.

The problems will be similar to the homework problems, and the proofs will use the same techniques. You can expect problems about following:

- Definitions
- Axiom of completness
- Sup and inf
- (Un)countability
- sequences
- convergences
- monotonicity
- Algebraic limit theorem
- Monotone convergence Theorem

The exam will **definitely include**

- One proof from the text that was also proved in class from among Theorems 1.4.1, 1.6.2, 2.3.2, 2.3.3 (ii), 2.4.2.
- One homework problem.

- Proof of convergence of a sequence using the definition.
- A problem that is not identical to any homework problem or theorem from class.
- True/false questions, or questions where you will need to come up with examples that illustrate important properties based on our definitions.

For definitions, please give **just** the definition, in prose (complete sentences, but use mathematical notation where appropriate), and not any additional facts about the definition. (E.g., if you give the definition of monotone, do not include any facts about monotone sequences, such as "a bounded monotone sequence converges".)
Additionally:

- This is a closed-book, closed-note, 75 minute examination.
- Books, outside notes, calculators, cell phones, communication devices of any sort, or other aids are permitted.
- You are allowed to use Theorems, lemmas, etc. from the book or from class, and previous homework problems as part of your solutions, and you are not required to reprove these during the exam. Try to reference them by name if possible (i.e., "by the Monotone Convergence Theorem" or "a differentiable function is continuous" or "Lemma 1.3.8"), but if you are unsure of the name or number of the theorem, it is ok to say, for example, "we proved in class that a convergent sequence is bounded". The exception to this is that if I ask you to prove e.g. that "a convergent sequence is bounded", then you must prove it and can't reference that we proved it in class.
- I will be in the hallway, and will check in every 25 minutes for questions.
- You can always ask me (the instructor) if you have clarifying questions, but asking for hints or asking if a proof is correct or sufficient is not allowed.
- You may leave when you are finished, in which case please hand me the exam in the hallway.
- You may leave to use the bathroom.
- You may use the backs of pages for additional work space. If you use any additional scratch paper as part of your solution, please include your name and problem number on the extra paper; give this to me with your exam, and let me know when you do
- Be sure to justify your answer to each of the problems!
- For proofs, please make sure that your answers are in complete sentences!
- My vision is not great. Please try to write legibly, and please try to leave some space between paragraphs.
- **Academic integrity**:
  - You are required to **turn off your phone** during the exam. If I observe that your phone is on, the penalty is that you fail the exam. (If there is an exceptional reason that your phone needs to be on during the exam, please discuss it with me).
  - **Submit only your own work**. Do not share your work, and do not look at other students' exams.
  - The penalty for cheating is at my discretion, and can range from failing the exam to failing the course.

Assignment 6: Infinite series; convergence; absolute and conditional convergence

Due by 12:55pm, eastern, on Thursday, March 27

Suggested readings for this problem set: **2.7**
All readings are from Abbott, *Understanding Analysis*.

Assignment: due Thursday, March 27, 12:55pm, via Gradescope (DKNX3W):

1. Determine if each of the following series converges absolutely, converges conditionally, or diverges. Prove your claims. You may use any of the results we proved or simply stated in class.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n + n}$.

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{3n + 1}$.

2. Suppose that the sequence $(a_n)_{n=1}^{\infty}$ converges to 0. Use *the definition* for convergent series to prove that $\sum_{n=1}^{\infty}(a_n - a_{n+1})$ converges. What is the value of its sum?

3. Determine if each of the following statements are true or false. Give a short proof if it is true, and provide a counterexample if it false (be sure to justify your example).

   (a) If both $\sum a_n$ and $\sum b_n$ diverge then $\sum(a_n + b_n)$ diverges.

   (b) If both $\sum a_n$ and $\sum b_n$ diverge then $\sum(a_n + b_n)$ converges.

   (c) If $\sum a_n$ converges and $\sum b_n$ diverges then $\sum(a_n + b_n)$ diverges.

4. Give an example of each or explain why the request is impossible referencing the proper theorem(s).

   (a) Two series $\sum x_n$ and $\sum y_n$ that both diverge but where $\sum x_n y_n$ converges.

   (b) A convergent series $\sum x_n$ and a bounded sequence (y_n) such that $\sum x_n y_n$ diverges.

   (c) Two sequences (x_n) and (y_n) where $\sum x_n$ and $\sum(x_n + y_n)$ both converge but $\sum y_n$ diverges.

   (d) A sequence (x_n) satisfying $0 \le x_n \le \frac{1}{n}$ where $\sum(-1)^n x_n$ diverges.

5. Prove the following.

   (a) Prove that if $a_n > 0$ and $\lim(na_n) = \ell$ with $\ell \neq 0$, then the series $\sum a_n$ diverges.

   (b) Assume $a_n > 0$ and $\lim(n^2 a_n)$ exists. Prove that $\sum a_n$ converges.

ONE MORE PROBLEM ON THE NEXT PAGE →

6. The following theorem is a great way to test *some* series for absolute convergence (does not always work!).

   Claim 1 (`Ratio Test`). *Suppose $\sum_{n=1}^{\infty} a_n$ is a series where $a_n \neq 0$ for all $n \in \mathbb{N}$ and assume that the* <u>*sequence*</u> *$(a_n)_{n=1}^{\infty}$ satisfies*

   $$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L \in [0, 1).$$

   Then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.

   Prove the Ratio Test using the following outline:

   (a) Fix a number $r \in (L, 1)$. Prove that there exists $N \in \mathbb{N}$ such that $|a_{n+1}| \leq r|a_n|$ for every $n \in \mathbb{N}$, $n \geq N$.

   (b) Use induction to prove that $|a_n| \leq r^{n-N}|a_N|$ for every $n \in \mathbb{N}$, $n \geq N$. (*Be careful about the base case!*)

   (c) Prove that $\sum\limits_{n=1}^{\infty} a_n$ converges absolutely using the Comparison Test (be careful about the starting point of the series and how that relates to the base case of your induction proof).

   (d) Modify the arguments in parts (a)-(c) to prove that if $L \in (1, \infty)$ then the series the series $\sum_{n=1}^{\infty} a_n$ diverges.

Assignment 7: Open, closed

Due by 12:55pm, eastern, on Thursday, April 03

Suggested readings for this problem set: **3.2, 3.3**
All readings are from Abbott, *Understanding Analysis*.

Assignment: due Thursday, April 03, 12:55pm, via Gradescope (DKNX3W):

1. Let
$$A = \left\{(-1)^n + \frac{2}{n} : n = 1, 2, 3, \ldots\right\} \quad \text{and} \quad B = \{x \in \mathbb{Q} : 0 < x < 1\}.$$
Answer the following questions for each set:
    (a) What are the limit points?
    (b) Is the set open? Closed?
    (c) Does the set contain any isolated points?
    (d) Find the closure of the set.
2. Decide whether the following sets are open, closed, or neither. If a set is not open, find a point in the set for which there is no ε-neighborhood contained in the set. If a set is not closed, find a limit point that is not contained in the set.
    (a) $\mathbb{Q}$.
    (b) $\mathbb{N}$.
    (c) $\{x \in \mathbb{R} : x \neq 0\}$.
    (d) $\left\{1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} : n \in \mathbb{N}\right\}$.
    (e) $\left\{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} : n \in \mathbb{N}\right\}$.
3. Let A be nonempty and bounded above so that $s = \sup A$ exists.
    (a) Prove that if $s \notin A$ then s is a limit point for A.
    (b) Provide an example illustrating that s may not be a limit point if $s \in A$. Justify your example.
    (c) Prove that if A is closed, then $s \in A$.
    (d) Can an open set contain its supremum? Explain your answer briefly.
4. Prove that a set $A \subseteq \mathbb{R}$ is closed if and only if every Cauchy Sequence contained in A has a limit that is also an element of A.
5. Let A and B be subsets of $\mathbb{R}$. Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
6. Is it true that $\overline{A \cap B} = \overline{A} \cap \overline{B}$? If so, prove it; if not, give a counterexample.
7. Suppose that $A \subseteq B$. Prove that $\overline{A} \subseteq \overline{B}$.
8. Prove that if $U \subseteq \mathbb{R}$ is an open set then U does not contain any isolated points.

Assignment 8: compact sets

Due by 12:55pm, eastern, on Thursday, April 10

Suggested readings for this problem set: **3.3, 4.2**
All readings are from Abbott, *Understanding Analysis*.

Assignment: due Thursday, April 10, 12:55pm, via Gradescope (DKNX3W):

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1. Decide whether the following sets are compact, and explain why or why not.

   (a) $\mathbb{Q} \cap [0, 1]$.

   (b) $\mathbb{N}$.

   (c) $\{x \in \mathbb{R} : x \neq 0\}$.

   (d) $\left\{1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} : n \in \mathbb{N}\right\}$.

   (e) $\left\{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} : n \in \mathbb{N}\right\}$.

2. Prove that an arbitrary[3] intersection of closed sets is closed without using the theorem that "a set is open iff its complement is closed". (I.e., just the definition of closed and Theorem 3.2.8.)

3. Decide whether the following statements are true or false. If the claim is valid, supply a short proof, and if the claim is false, provide a counterexample.

   (a) The intersection of any finite collection of compact sets is compact.

   (b) The intersection of any collection of compact sets is compact.

   (c) The union of any finite collection of compact sets is compact.

   (d) The union of any collection of compact sets is compact.

4. Show that if K is compact and nonempty, then $\sup K$ and $\inf K$ both exist and are elements of K.

5. Determine if the following statement is true or false:

   If $E \subseteq \mathbb{R}$ is a bounded infinite closed set, then E contains a rational number.

   Give a short proof if it is true, and provide a counterexample or disprove it if it false (be sure to justify your claims!).

MORE PROBLEMS ON THE NEXT PAGE →

[3]i.e., the intersection might include infinitely many sets

6. Compute each limit, and using just Definition 4.2.1, give a proof that your limit is correct.
   (a) $\lim_{x\to 2}(3x+4)$.
   (b) $\lim_{x\to 0} x^3$.
   (c) $\lim_{x\to 2} \frac{|x-2|}{x-2}$.
7. Decide if the following claims are true or false, and give short justifications for each conclusion.
   (a) If a particular δ has been constructed as a suitable response to a particular ε challenge, then any smaller positive δ will also suffice.
   (b) If $\lim_{x\to a} f(x) = L$ and a happens to be in the domain of f, then $L = f(a)$.
   (c) If $\lim_{x\to a} f(x) = L$, then $\lim_{x\to a} 3(f(x)-2)^2 = 3(L-2)^2$.
   (d) If $\lim_{x\to a} f(x) = 0$, then $\lim_{x\to a} f(x)g(x) = 0$ for any function g (with domain equal to the domain of f).
8. *Dirichlet's function* is the function $\mathcal{D}\colon \mathbb{R} \to \mathbb{R}$ which is defined by setting

$$\mathcal{D}(x) := \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases} \qquad \forall\, x \in \mathbb{R}.$$

   Let $c \in \mathbb{R}$. Prove that $\lim_{x\to c} \mathcal{D}(x)$ does not exist.

Midterm 2 study guide

In class exam, Thursday, April 17.

On *Tuesday, April 15* we will have an in class review for the exam.

Content: The questions will all be either

1. Definitions,
2. homework problems,
3. problems we worked in class,
4. Theorems/Propositions/Lemmas that we proved in clasee,
5. True/False or "give an example" problems, or
6. minor variations of one of these.

Problems with very long proofs or that involved some unusual trick will not be on the exam. So for example, I am unlikey to ask you to prove the Bolzano–Weierstrauss theorem, but I am likely to ask you to prove that "if $\sum a_n$ converges then $a_n \to 0$", or something of a similar length. A problem like $\overline{\overline{A}} = \overline{A}$ is an example of a problem that is a bit too long to ask on an exam.

A typical exam will have a few questions from each week of the course and will cover **assignments 5-8**, which includes the following sections from our book.

Chapter 2, sections 2.5-2.7; Chapter 3, sections 3.2, 3.3; Chapter 4, section 4.2

The problems will be similar to the homework problems and proofs from class, and the proofs will use the same techniques. You can expect problems about following:

- definitions,
- monotone convergence,
- subsequences,
- Cauchy sequences,
- the Bolzano–Weierstrauss theorem,
- series,
- absolute and conditional convergence,
- open and closed sets,
- compacts sets,
- limits of functions.

The exam will **definitely include**

- One proof from the text that was also proved in class from among Theorems 2.5.2, 2.6.3, 2.6.4, 3.2.3, 4.2.3.
- One problem using series tests to deduce convergence or divergence of a series (which may be an abstract series).
- One homework problem chosen from Homeworks 5-8.
- Proof of a functional limit or continuity using the $\epsilon - \delta$ definition.
- Examples involving compact sets, functional limits, or continuity.
- Definitions
- A problem that is not identical to any homework problem or theorem from class.
- True/false questions, or questions where you will need to come up with counterexamples, or examples that illustrate important properties based on our definitions.

For definitions, please give **just** the definition, in prose (complete sentences, but use mathematical notation where appropriate), and not any additional facts about the definition. (E.g., if you give the definition of monotone, do not include any facts about monotone sequences, such as "a bounded monotone sequence converges".)

Additionally:

- This is a closed-book, closed-note, 75 minute examination.
- Books, outside notes, calculators, cell phones, communication devices of any sort, or other aids are permitted.
- You are allowed to use Theorems, lemmas, etc. from the book or from class, and previous homework problems as part of your solutions, and you are not required to reprove these during the exam. Try to reference them by name if possible (i.e., "by the Monotone Convergence Theorem" or "a differentiable function is continuous" or "Lemma 1.3.8"), but if you are unsure of the name or number of the theorem, it is ok to say, for example, "we proved in class that a convergent sequence is bounded". The exception to this is that if I ask you to prove e.g. that "a convergent sequence is bounded", then you must prove it and can't reference that we proved it in class.
- I will be in the hallway, and will check in every 25 minutes for questions.
- You can always ask me (the instructor) if you have clarifying questions, but asking for hints or asking if a proof is correct or sufficient is not allowed.
- You may leave when you are finished, in which case please hand me the exam in the hallway.
- You may leave to use the bathroom.
- You may use the backs of pages for additional work space. If you use any additional scratch paper as part of your solution, please include your name and problem number on the extra paper; give this to me with your exam, and let me know when you do
- Be sure to justify your answer to each of the problems!
- For proofs, please make sure that your answers are in complete sentences!
- My vision is not great. Please try to write legibly, and please try to leave some space between paragraphs.
- **Academic integrity**:

- You are required to **turn off your phone** during the exam. If I observe that your phone is on, the penalty is that you fail the exam. (If there is an exceptional reason that your phone needs to be on during the exam, please discuss it with me).
- **Submit only your own work**. Do not share your work, and do not look at other students' exams.
- The penalty for cheating is at my discretion, and can range from failing the exam to failing the course.

Assignment 9: continuity; limits of functions, uniform convergence

Due by 12:55pm, eastern, on Thursday, April 17

Suggested readings for this problem set: **4.3, 6.2**
All readings are from Abbott, *Understanding Analysis*.

Assignment: due Thursday, April 17, 12:55pm, via Gradescope (DKNX3W):

1. Let $g\colon A \to \mathbb{R}$ and assume that f is a bounded function on A in the sense that there exists $M > 0$ satisfying $|f(x)| \leq M$ for all $x \in A$. Show that if $\lim_{x\to c} g(x) = 0$, then $\lim_{x\to c} g(x)f(x) = 0$ as well.

2. Let $g(x) = \sqrt[3]{x}$.

   (a) Prove that g is continuous at $c = 0$.

   (b) Prove that g is continuous at a point $c \neq 0$. (The identity $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ will be helpful.)

3. Prove that $\sin x$ is continuous. (Hint: one approach is to use the identity $\sin(a+h) = \sin a \cos h + \sin h \cos a$.)

4. Decide if the following claims are true or false, providing either a short proof or counterexample to justify each conclusion. Assume throughout that g is defined and continuous on all of $\mathbb{R}$.

   (a) If $g(x) \geq 0$ for all $x < 1$, then $g(1) \geq 0$ as well.

   (b) If $g(r) = 0$ for all $r \in \mathbb{Q}$, then $g(x) = 0$ for all $x \in \mathbb{R}$.

   (c) If $g(x_0) > 0$ for a single point $x_0 \in \mathbb{R}$, then $g(x)$ is in fact strictly positive for uncountably many points.

Assignment 10: Extreme, intermediate, and mean value theorems; differentiability

Due by 12:55pm, eastern, on Thursday, May 01

Suggested readings for this problem set: **4.4, 4.5, 5.2**
All readings are from Abbott, *Understanding Analysis*.

Assignment: due Thursday, May 01, 12:55pm, via Gradescope (DKNX3W):

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Let $A \subseteq \mathbb{R}$. Recall that the **image** of A is defined to be

$$f(A) = \{f(a) : a \in A\}$$

and the **preimage** of A is defined to be

$$f^{-1}(A) = \{a \in \mathbb{R} : f(a) \in A.\}$$

We proved in class that the continuous image of a compact set is compact (i.e., if A is compact, then $f(A)$ is compact). Here are some related problems.

1. Suppose that A is closed and that f is continuous. Prove that the preimage $f^{-1}(A)$ is closed.

2. Suppose that f is continuous. Decide if the following claims are true or false, providing either a short proof or counterexample to justify each conclusion.

   (a) If A is bounded, then $f(A)$ is bounded.
   (b) If A is bounded, then $f^{-1}(A)$ is bounded.
   (c) If A is compact, then $f^{-1}(A)$ is compact.
   (d) If A is closed , then $f(A)$ is closed.
   (e) If A is open, then $f(A)$ is open.
   (f) If A is open, then $f^{-1}(A)$ is open.

3. All of the statements from the previous problem can be false if f is not continuous. For each of the following, give an example of a function $f : \mathbb{R} \to \mathbb{R}$ and a set $A \subseteq \mathbb{R}$ such that

   (a) A is compact, but $f^{-1}(A)$ is not compact.
   (b) A is bounded, but $f(A)$ is not bounded.
   (c) A is closed , but $f(A)$ is not closed.
   (d) A is closed, but $f^{-1}(A)$ is not closed.
   (e) A is open, but $f(A)$ is not open.
   (f) A is open, but $f^{-1}(A)$ is not open.
   (g) A is compact, but $f(A)$ is not compact.

MORE PROBLEMS ON THE NEXT PAGE →

4. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) := |x|$ for all $x \in \mathbb{R}$. Prove that f is continuous at every $c \in \mathbb{R}$.

5. Finish proof I of the Intermediate Value Theorem using the Axiom of Completeness started previously (on page 138). (Hint: make sure that you are using the fact that f is continuous somewhere in your proof.)

6. Show how the Intermediate Value Theorem follows as a corollary to Theorem 4.5.2. (Be careful to show all of the relevant details. In particular, if you claim that a set is not connected, justify why it is not connected.)

7. Prove that if $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly increasing on (a, b). (Recall that *strictly increasing* means that if $x < y$ then $f(x) < f(y)$.)

8. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) := |x|$ for all $x \in \mathbb{R}$. Prove that f is not differentiable at 0.

Assignment 11: Pointwise vs. uniform convergence; Uniform convergence and differentiation; continuity and differentiability of limits

Due by 12:55pm, eastern, on Tuesday, May 01

Suggested readings for this problem set: **6.2, 6.3**
All readings are from Abbott, *Understanding Analysis*.

Assignment: due Thursday, May 01, 12:55pm, via Gradescope (DKNX3W):

There is no assignment 11. But there is some suggested reading (see above). Moreover, the final exam will include one problem where I ask about uniform convergence. The problem will be similar to the problems below.

1. Which of the following sequences of functions are uniformly convergent? If they converge, what function do they converge to? How would you prove or disprove this?
2. $f_n(x) = x^n$ on $\mathbb{R}$
3. $f_n(x) = x^n$ on $[0, 1]$
4. $f_n(x) = x^n$ on $[0, 1/2]$
5. $f_n(x) = x^n/n$ on $[0, 1]$
6. $f_n(x) = x^n/n$ on $\mathbb{R}$
7. $f_n(x) = x^n/n!$ on $\mathbb{R}$
8. $f_n(x) = \frac{x}{1+x^n}$ on $\mathbb{R}$
9. $f_n(x) = \frac{x}{1+x^n}$ on $[1, \infty)$
10. $f_n(x) = \sin(x/n)$ on $\mathbb{R}$
11. $f_n(x) = \frac{\sin(nx)}{\sqrt{x}}$ on $\mathbb{R}$
12. $f_n(x) = \frac{\cos(2^n x)}{2^n}$ on $\mathbb{R}$

Final exam study guide

Final exam is a take home exam, released (probably) the evening of **Thursday, May 8** and due **Wednesday, May 14 at 4pm**.

I will hold a final office hours **Wednesday, May 7, 1:30-3:30 pm**. There will also be fellow hours that week.

The exam will be available on gradescope, and should be submitted via Gradescope.

Big request: if you are writing your exam in latex or on a tablet, please start each problem on a new page. (My eyes are not very good and this helps a lot when I am grading ≥ 25 exams.) Please leave spacing between different problems, and please try to organize your answers into multiple paragraphs, with spacing.

The **last day of class** is Tuesday, May 6.

The exam will be comprehensive and cover sections 1.2-1.6, 2.2-2.7, 3.2-3.4, 4.2-4.5, 5.2-5.3, 6.2.

The exam will be extremely similar to the midterms.

Problems with extremely long proofs or that involved some unusual trick will not be on the exam.

Since this is a take home exam, none of the problems will be identical to homework problems, but many problems will be minor variations of homework or of problems we worked in class.

A good way to prepare is to:

1. Know all of the definitions and terminology;
2. Know all of the statements of theorems, and examples of how we use the theorems;
3. Make a list of all of the different *proof techniques* from class and from the homework and review how those techniques are used in proofs and problems;
4. Practice doing problems "from scratch" and use your solutions as "hints" when you get stuck.

Additionally:

1. You are allowed to use the course textbook, lecture notes and any materials from the course website.
2. You are not allowed to use any other resources, including AI, other students, Google, or other books.
3. You are allowed to use Theorems, lemmas, etc from the book or from class, and previous homework problems as part of your solutions, and you are not required to reprove these during the exam. Please do cite them (e.g., "Proposition 1.3.4 from our book") or refer to them by name, if they have a special name (e.g., "by the Mean Value Theorem").
4. On the other hand, please do not use any theorems that are not from class or from our book.
5. Do not discuss the problems or their solutions with your classmates.

6. You can always ask me (the instructor) if you have clarifying questions, but asking for hints or asking if a proof is correct is not allowed.

7. You are welcome to use a calulator or any other program to help with computations (e.g., arithmetic).

8. Submit your exam via Gradescope.

9. Again (since this is a take-home exam): please **submit only your own work**. Do not share your work, and do not look at other students' exams. The penalty for cheating is at my discretion, and can range from failing the exam to failing the course.

The exam will be **comprehensive**. A typical exam will have a few questions from each week of the course, and there will be more emphasis on content not covered on the midterms. The exam will be around 8-10 problems (some with multiple parts). The exam will include one problem asking you to prove that a particular sequence of functions is uniformly convergent.

Finally: **please take a look at the exam as early as possible** so that you can estimate the amount of time it will take you.

Hints

3.3. (b) induction. (c,d) given an example of a bijection to a set of known cardinality.

3.4. Assume that $[0, 1] = \{x_1, x_2, x_3, \dots\}$ and construct a sequence of nested intervals such that $x_n \notin I_n$ to obtain a contradiction.

3.6. To get started, assume $(a_n) \to a$ and also that $(a_n) \to b$. Now argue that $a = b$; one way to show this is to show that $|a - b| = 0$ by showing that $\forall \epsilon > 0, |a - b| < \epsilon$. (There are several ways to do this problem; this is just one suggestion.)

5.4. For (c), conjugate; i.e., use the identity

$$\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}}$$