

Progress on Mazur's program B

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Slides available at <http://www.mathcs.emory.edu/~dzb/slides/>

Rational points on irrational varieties

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Galois Representations

$$\mathbb{Q} \subset K$$

$$G_K := \text{Aut}(\overline{K}/K)$$

$$E[n](\overline{K}) \cong (\mathbb{Z}/n\mathbb{Z})^2$$

$$\rho_{E,n}: G_K \rightarrow \text{Aut } E[n] \cong \text{GL}_2(\mathbb{Z}/n\mathbb{Z})$$

$$\rho_{E,\ell^\infty}: G_K \rightarrow \text{GL}_2(\mathbb{Z}_\ell) = \varprojlim_n \text{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z})$$

$$\rho_E: G_K \rightarrow \text{GL}_2(\widehat{\mathbb{Z}}) = \varprojlim_n \text{GL}_2(\mathbb{Z}/n\mathbb{Z})$$

Serre's Open Image Theorem

Theorem (Serre, 1972)

Let E be an elliptic curve over K without CM. The image

$$\rho_E(G_K) \subset \mathrm{GL}_2(\hat{\mathbb{Z}})$$

of ρ_E is open.

Note:

$$\mathrm{GL}_2(\hat{\mathbb{Z}}) \cong \prod_p \mathrm{GL}_2(\mathbb{Z}_p)$$

Image of Galois

$$\rho_{E,n}: G_{\mathbb{Q}} \twoheadrightarrow H(n) \hookrightarrow \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$$

$$\left\{ \begin{array}{c} \overline{\mathbb{Q}} \\ \downarrow \\ \overline{\mathbb{Q}}^{\ker \rho_{E,n}} = \mathbb{Q}(E[n]) \\ \downarrow \\ \mathbb{Q} \end{array} \right\} H(n)$$

Problem (Mazur's "program B")

Classify all possibilities for $H(n)$.

Mazur's Program B

As presented at Modular functions in one variable V in Bonn

Theorem 1 also fits into a general program:

B. Given a number field K and a subgroup H of $GL_2 \hat{\mathbb{Z}} = \prod_p GL_2 \mathbb{Z}_p$ classify
all elliptic curves E/K whose associated Galois representation on torsion points
maps $\text{Gal}(\bar{K}/K)$ into $H \subset GL_2 \hat{\mathbb{Z}}$.

Mazur - Rational points on modular curves (1977)

Example - torsion on an elliptic curve

If E has a K -rational **torsion point** $P \in E(K)[n]$ (of exact order n) then:

$$H(n) \subset \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$$

since for $\sigma \in G_K$ and $Q \in E(\overline{K})[n]$ such that $E(\overline{K})[n] \cong \langle P, Q \rangle$,

$$\sigma(P) = P$$

$$\sigma(Q) = a_\sigma P + b_\sigma Q$$

Example - Isogenies

If E has a K -rational, **cyclic isogeny** $\phi: E \rightarrow E'$ with $\ker \phi = \langle P \rangle$ then:

$$H(n) \subset \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

since for $\sigma \in G_K$ and $Q \in E(\overline{K})[n]$ such that $E(\overline{K})[n] \cong \langle P, Q \rangle$,

$$\sigma(P) = a_\sigma P$$

$$\sigma(Q) = b_\sigma P + c_\sigma Q$$

Example - other maximal subgroups

Normalizer of a split Cartan:

$$N_{\text{sp}} = \left\langle \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle$$

$H(n) \subset N_{\text{sp}}$ and $H(n) \not\subset C_{\text{sp}}$ iff

- there exists an unordered pair $\{\phi_1, \phi_2\}$ of cyclic isogenies,
- whose kernels intersect trivially,
- neither of which is defined over K ,
- but which are both defined over some quadratic extension of K ,
- and which are Galois conjugate.

Example - other maximal subgroups

Normalizer of a non-split Cartan:

$$C_{\text{ns}} = \text{im} \left(\mathbb{F}_{p^2}^* \rightarrow \text{GL}_2(\mathbb{F}_p) \right) \subset N_{\text{ns}}$$

$H(n) \subset N_{\text{ns}}$ and $H(n) \not\subset C_{\text{ns}}$ iff

E admits a “necklace” (Rebolledo, Wuthrich)

Modular curves

Definition

- $X(N)(K) := \{(E/K, P, Q) : E[N] = \langle P, Q \rangle\} \cup \{\text{cusps}\}$
- $X(N)(K) \ni (E/K, P, Q) \Leftrightarrow \rho_{E,N}(G_K) = \{I\}$

Definition

$\Gamma(N) \subset H \subset \text{GL}_2(\widehat{\mathbb{Z}})$ (finite index)

- $X_H := X(N)/H$
- $X_H(K) \ni (E/K, \iota) \Leftrightarrow H(N) \subset H \pmod{N}$

Stacky disclaimer

This is only true up to twist; there are some subtleties if

- 1 $j(E) \in \{0, 12^3\}$ (plus some minor group theoretic conditions), or
- 2 if $-I \in H$.

Rational Points on modular curves

Mazur's program B

Compute $X_H(\mathbb{Q})$ for all H .

Remark

- Sometimes $X_H \cong \mathbb{P}^1$ or elliptic with rank $X_H(\mathbb{Q}) > 0$.
- Some X_H have *sporadic* points.
- Can compute $g(X_H)$ group theoretically (via Riemann–Hurwitz).
- Can compute $\#X_H(\mathbb{F}_q)$ via moduli and enumeration [Sutherland].

Fact

$$g(X_H), \gamma(X_H) \rightarrow \infty \text{ as } [\mathrm{GL}_2(\widehat{\mathbb{Z}}) : H] \rightarrow \infty.$$

Sample subgroup (Serre)

$$\begin{array}{ccccc} \ker \phi_2 & \subset & H(8) & \subset & \mathrm{GL}_2(\mathbb{Z}/8\mathbb{Z}) & \dim_{\mathbb{F}_2} \ker \phi_2 = 3 \\ & & \downarrow \phi_2 & & \downarrow & \\ I + 2M_2(\mathbb{Z}/2\mathbb{Z}) & \subset & H(4) & = & \mathrm{GL}_2(\mathbb{Z}/4\mathbb{Z}) & \dim_{\mathbb{F}_2} \ker \phi_1 = 4 \\ & & \downarrow \phi_1 & & \downarrow & \\ & & H(2) & = & \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z}) & \end{array}$$

$$\chi: \mathrm{GL}_2(\mathbb{Z}/8\mathbb{Z}) \rightarrow \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/8\mathbb{Z})^* \rightarrow \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/8\mathbb{Z})^* \cong \mathbb{F}_2^3.$$

$$\chi = \mathrm{sgn} \times \det$$

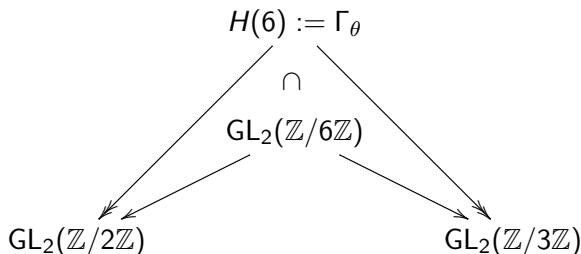
$$H(8) := \chi^{-1}(G), \quad G \subset \mathbb{F}_2^3.$$

A typical subgroup

$\ker \phi_4$	\subset	$H(32)$	\subset	$GL_2(\mathbb{Z}/32\mathbb{Z})$	$\dim_{\mathbb{F}_2} \ker \phi_4 = 4$
		$\downarrow \phi_4$		\downarrow	
$\ker \phi_3$	\subset	$H(16)$	\subset	$GL_2(\mathbb{Z}/16\mathbb{Z})$	$\dim_{\mathbb{F}_2} \ker \phi_3 = 3$
		$\downarrow \phi_3$		\downarrow	
$\ker \phi_2$	\subset	$H(8)$	\subset	$GL_2(\mathbb{Z}/8\mathbb{Z})$	$\dim_{\mathbb{F}_2} \ker \phi_2 = 2$
		$\downarrow \phi_2$		\downarrow	
$\ker \phi_1$	\subset	$H(4)$	\subset	$GL_2(\mathbb{Z}/4\mathbb{Z})$	$\dim_{\mathbb{F}_2} \ker \phi_1 = 3$
		$\downarrow \phi_1$		\downarrow	
		$H(2)$	$=$	$GL_2(\mathbb{Z}/2\mathbb{Z})$	

Non-abelian entanglements

There exists a surjection $\theta: \mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z}) \rightarrow \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$.



$$\mathrm{im} \rho_{E,6} \subset H(6) \Leftrightarrow j(E) = 2^{10}3^3t^3(1-4t^3) \Rightarrow K(E[2]) \subset K(E[3]).$$
$$X_H \cong \mathbb{P}^1 \xrightarrow{j} X(1).$$

Classification of Images - Mazur's Theorem

Theorem

Let E be an elliptic curve over \mathbb{Q} . Then for $\ell > 11$, $E(\mathbb{Q})[\ell] = \{0\}$.

In other words, for $\ell > 11$, $H(\ell)$ is not contained in a subgroup conjugate to

$$\begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}.$$

Classification of Images - Mazur; Bilu, Parent, Rebolledo

Theorem (Mazur)

Let E be an elliptic curve over \mathbb{Q} without CM. Then for $\ell > 37$, $H(\ell)$ is not contained in a subgroup conjugate to

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$$

Theorem (Bilu, Parent, Rebolledo)

Let E be an elliptic curve over \mathbb{Q} without CM. Then for $\ell > 13$, $H(\ell)$ is not contained in a subgroup conjugate to

$$\left\langle \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle.$$

Main conjecture

Conjecture (Serre)

Let E be an elliptic curve over \mathbb{Q} without CM. Then for $\ell > 37$, $\rho_{E,\ell}$ is surjective.

In other words, conjecturally, $H(\ell) = \mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z})$ for $\ell > 37$.

“Vertical” image conjecture

Conjecture

There exists a constant N such that for every E/\mathbb{Q} without CM

$$\left[\mathrm{GL}_2(\hat{\mathbb{Z}}) : \rho_E(G_{\mathbb{Q}}) \right] \leq N.$$

Remark

This follows from the “ $\ell > 37$ ” conjecture.

Problem

Assume the “ $\ell > 37$ ” conjecture and compute N .

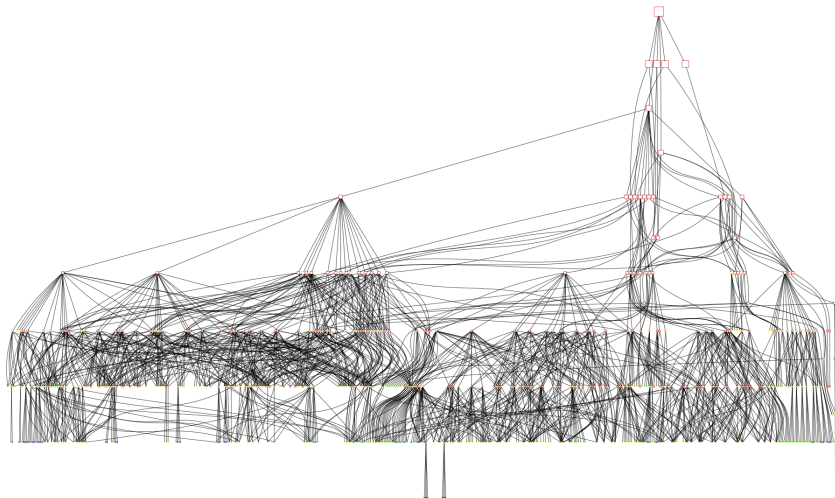
Main Theorem

Rouse, ZB (2-adic)

The index of $\rho_{E,2^\infty}(G_{\mathbb{Q}})$ divides 64 or 96; all such indices occur.

- 1 All indices dividing 96 occur infinitely often; 64 occurs only twice.
- 2 The 2-adic image is determined by the mod 32 image.
- 3 1208 different images can occur for non-CM elliptic curves.
- 4 There are 8 “sporadic” subgroups.

Subgroups of $GL_2(\mathbb{Z}_2)$



Index, # of isogeny classes

1 , 727995

2 , 7281

3 , 175042

4 , 1769

6 , 57500

8 , 577

12 , 29900

16 , 235

24 , 5482

32 , 20

48 , 1544

64 , 0 (two examples)

96 , 241 (first example - $X_0(15)$)

CM , 1613

Index, # of isogeny classes

64 , 0

$$j = -3 \cdot 2^{18} \cdot 5^3 \cdot 13^3 \cdot 41^3 \cdot 107^3 \cdot 17^{-16}$$

$$j = -2^{21} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13^3 \cdot 23^3 \cdot 41^3 \cdot 179^3 \cdot 409^3 \cdot 79^{-16}$$

Rational points on $X_{ns}^+(16)$ (Heegner, Baran)

Applications

Theorem (R. Jones, Rouse, ZB)

- 1 **Arithmetic dynamics:** let $P \in E(\mathbb{Q})$.
- 2 How often is the order of $\tilde{P} \in E(\mathbb{F}_p)$ odd?
- 3 Answer depends on $\rho_{E,2^\infty}(G_{\mathbb{Q}})$.
- 4 Examples: 11/21 (generic), 121/168 (maximal), 1/28 (minimal)

Theorem (Various authors)

Computation of $S_{\mathbb{Q}}(d)$ for particular d .

Theorem (Daniels, Lozano-Robledo, Najman, Sutherland)

Classification of $E(\mathbb{Q}(3^\infty))_{\text{tors}}$

Theorem (Gonzalez-Jimenez, Lozano-Robledo)

Classify E/\mathbb{Q} with $\rho_{E,N}(G_{\mathbb{Q}})$ abelian.

More applications

Theorem (Sporadic points)

Najman's example $X_1(21)^{(3)}(\mathbb{Q})$; "easy production" of other examples.

Theorem (Jack Thorne)

Elliptic curves over \mathbb{Q}_∞ are modular.

(One step is to show $X_0(15)(\mathbb{Q}_\infty) = X_0(15)(\mathbb{Q}) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.)

Recent theorems

Zywina (mod ℓ)

Classifies $\rho_{E,\ell}(G_{\mathbb{Q}})$ (modulo some conjectures).

Zywina (indices occurring infinitely often; modulo conjectures)

The **index** of $\rho_{E,N}(G_{\mathbb{Q}})$ divides 220, 336, 360, 504, 864, 1152, 1200, 1296 or 1536.

Sutherland–Zywina

Parametrizations in all **prime power** levels, $g = 0$ and $g = 1, r > 0$ cases.

Brau–N. Jones, N. Jones–McMurdy (in progress)

Equations for X_H for entanglement groups H .

Morrow; Camacho–Li–Morrow–Petok–ZB (composite level)

Classifies $\rho_{E, \ell_1^n \cdot \ell_2^m}(G_{\mathbb{Q}})$ (partially).

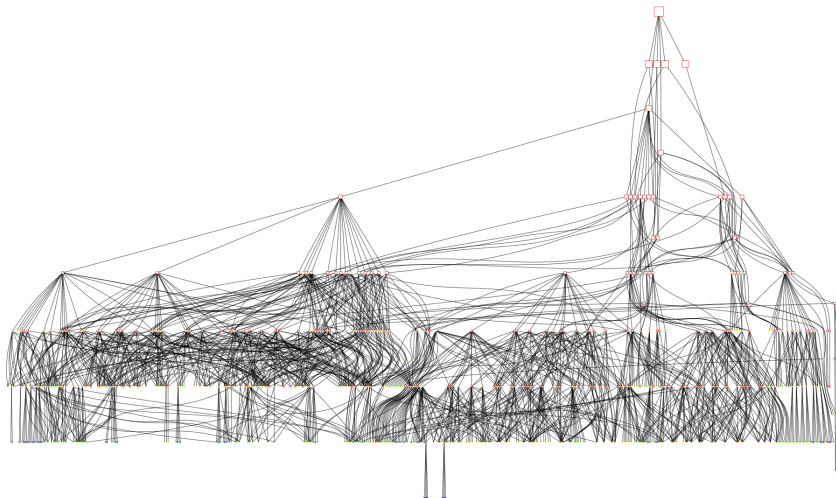
Derickx–Rouse–Sutherland–ZB for other prime powers (in progress)

Partial progress; e.g. for $N = 3^n$.

Proof template

- 1 Compute all arithmetically minimal $H \subset \mathrm{GL}_2(\mathbb{Z}_2)$
- 2 Compute equations for each X_H
- 3 Find (with proof) all rational points on each X_H .

Subgroups of $GL_2(\mathbb{Z}_2)$



Finding Equations – Basic idea

- 1 The canonical map $C \hookrightarrow \mathbb{P}^{g-1}$ is given by $P \mapsto [\omega_1(P) : \cdots : \omega_g(P)]$.
- 2 For a general curve, this is an embedding, and the relations are quadratic.
- 3 For a modular curve,

$$M_k(H) \cong H^0(X_H, \Omega^1(\Delta)^{\otimes k/2})$$

given by

$$f(z) \mapsto f(z) dz^{\otimes k/2}.$$

Equations – Example: $X_1(17) \subset \mathbb{P}^4$

$$q - 11q^5 + 10q^7 + O(q^8)$$

$$q^2 - 7q^5 + 6q^7 + O(q^8)$$

$$q^3 - 4q^5 + 2q^7 + O(q^8)$$

$$q^4 - 2q^5 + O(q^8)$$

$$q^6 - 3q^7 + O(q^8)$$

$$xu + 2xv - yz + yu - 3yv + z^2 - 4zu + 2u^2 + v^2 = 0$$

$$xu + xv - yz + yu - 2yv + z^2 - 3zu + 2uv = 0$$

$$2xz - 3xu + xv - 2y^2 + 3yz + 7yu - 4yv - 5z^2 - 3zu + 4zv = 0$$

Equations – general

- ① $H' \subset H$ of index 2, $X_{H'} \rightarrow X_H$ degree 2.
- ② Given equations for X_H , compute equations for $X_{H'}$.
- ③ Compute a new modular form on H' , compute (quadratic) relations between this and modular forms on H .
- ④ **Main technique** – if $X_{H'}$ has “new cusps”, then write down Eisenstein series which vanish at “one new cusp, not others”.

Rational points rundown, $\ell = 2$

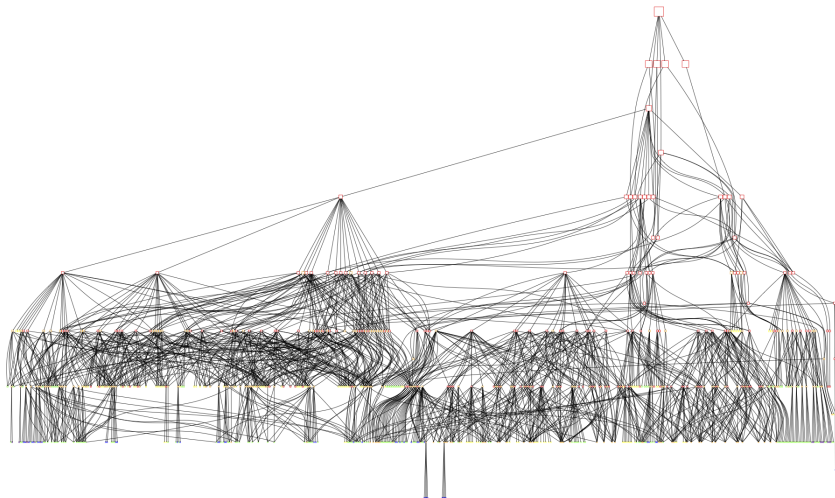
318 curves (excluding pointless conics)

Genus	0	1	2	3	5	7
Number	175	52	56	18	20	4
Rank of Jacobian						
0		25	46	–	–	??
1		27	3	9	10	??
2			7	–	–	??
3				9	–	??
4					–	??
5					10	??

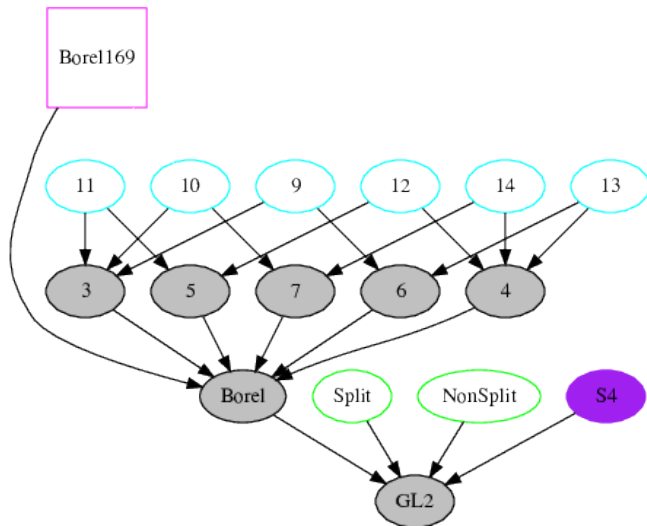
More 2-adic facts

- ① There are 8 “sporadic” subgroups
 - ① Only one genus 2 curve has a sporadic point
 - ② Six genus 3 curves each have a single sporadic point
 - ③ The genus 1, 5, and 7 curves have no sporadic points
- ② Many accidental isomorphisms of $X_H \cong X_{H'}$.
- ③ There is one H such that $g(X_H) = 1$ and $X_H \in X_H(\mathbb{Q})$.

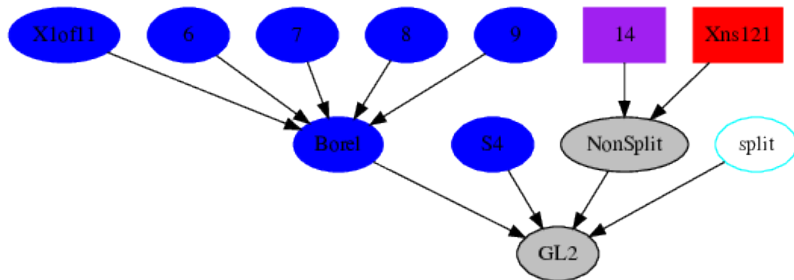
Subgroups of $GL_2(\mathbb{Z}_2)$



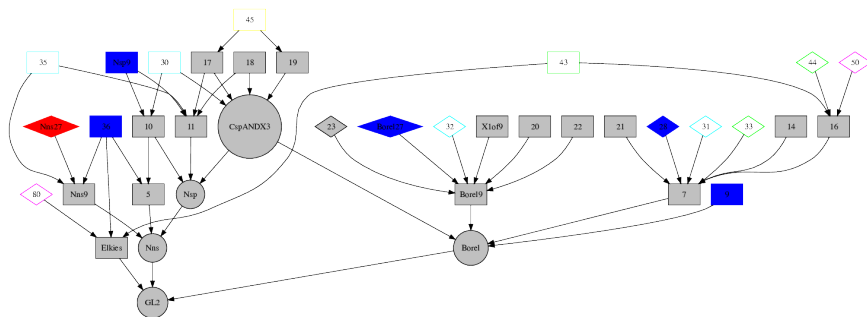
Subgroups of $GL_2(\mathbb{Z}_{13})$



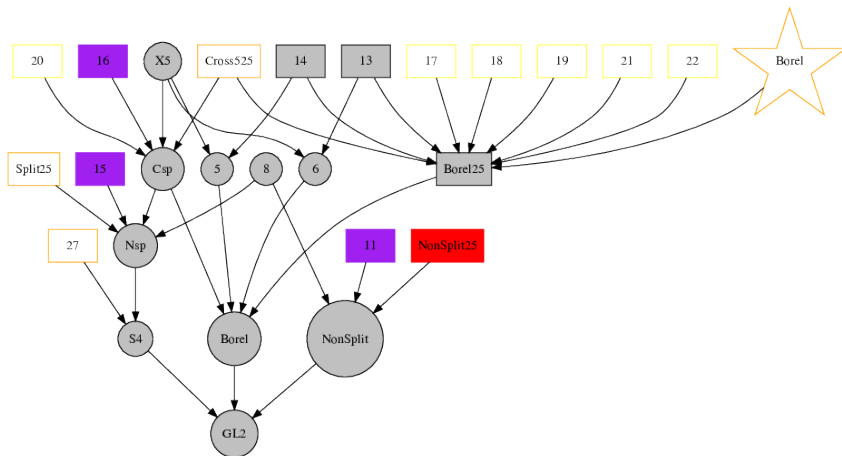
Subgroups of $GL_2(\mathbb{Z}_{11})$



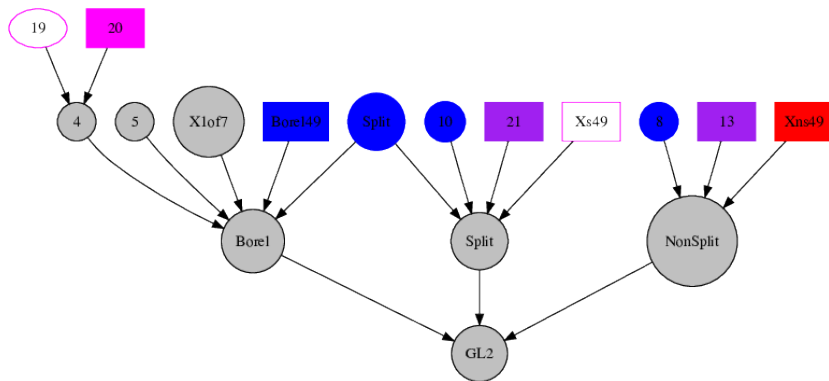
Subgroups of $GL_2(\mathbb{Z}_3)$



Subgroups of $GL_2(\mathbb{Z}_5)$



Subgroups of $GL_2(\mathbb{Z}_7)$



Rational Points: summary of remaining work.

3	$g = 12$
5	$g = 2, 4, 14$
7	$g = 9, 12, 69$
11	$g = 41, 511$
13	$X_{S_4}(13)$ (genus 3)

Rational Points: summary of remaining work – more info.

The Untouchables	$X_{ns}^+(27), X_{ns}^+(25), X_{ns}^+(49), X_{ns}^+(121)$ $g = 12, 14, 69, 511$
Also probably untouchable ($r \geq g$)	X_{13}, X_{21}, X_{14} $g = 9, 9, 41$ level 7, 7, 11
Cautiously optimistic ($r \geq g$)	$X_{11}, X_{15}, X_{16}, X_{S_4}$ $g = 2, 2, 4, 3$ level 5, 5, 5, 13
Optimistic ($r = 3 < g$)	$g = 12$, level 7

Explicit methods: highlight reel

- Local methods
- Chabauty
- Elliptic Chabauty
- Mordell–Weil sieve
- étale descent
- Pryms
- **Equationless descent via group theory.**
- **New techniques for computing** Aut C .

Thank you!