ℓ-adic images of Galois for elliptic curves over $\mathbb{Q}$

David Zureick-Brown

Emory University

arXiv:2160.11141

with Jeremy Rouse and Andrew V. Sutherland
and an appendix with John Voight

Simons Collaboration on Arithmetic Geometry, Number Theory, and Computation Annual Meeting

January 12, 2023

Slides available at `http://www.math.emory.edu/~dzb/slides/`

Galois Representations

$$\begin{aligned} \mathbb{Q} &\subset K \subset \overline{\mathbb{Q}} \\ G_K &:= \mathrm{Aut}(\overline{K}/K) \\ E[n](\overline{K}) &\cong (\mathbb{Z}/n\mathbb{Z})^2 \end{aligned}$$

$$\begin{aligned} \rho_{E,n}\colon \quad G_K \to \mathrm{Aut}\, E[n] &\cong \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z}) \\ \rho_{E,\ell^\infty}\colon \quad G_K \to \mathrm{GL}_2(\mathbb{Z}_\ell) &= \varprojlim_n \mathrm{GL}_2\left(\mathbb{Z}/\ell^n\mathbb{Z}\right) \\ \rho_E\colon \quad G_K \to \mathrm{GL}_2(\widehat{\mathbb{Z}}) &= \varprojlim_n \mathrm{GL}_2\left(\mathbb{Z}/n\mathbb{Z}\right) \end{aligned}$$

Serre's Open Image Theorem

Theorem (Serre, 1972)

Let E be an elliptic curve over K without CM. The image

$$\rho_E(G_K) \subset \mathrm{GL}_2(\widehat{\mathbb{Z}})$$

of ρ_E is open.

Note:

$$\mathrm{GL}_2(\widehat{\mathbb{Z}}) \cong \prod_{\ell} \mathrm{GL}_2(\mathbb{Z}_\ell)$$

Thus ρ_{E,ℓ^∞} is surjective for all but finitely many ℓ.

Image of Galois

$$\rho_{E,n}\colon G_{\mathbb{Q}} \twoheadrightarrow H(n) \hookrightarrow \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$$

$$G_{\mathbb{Q}}\left\{\begin{array}{l}\overline{\mathbb{Q}}\\ \;| \\ \left.\begin{array}{l}\overline{\mathbb{Q}}^{\ker\rho_{E,n}} = \mathbb{Q}(E[n])\\ \;|\\ \mathbb{Q}\end{array}\right\}H(n)\end{array}\right.$$

Problem (Mazur's "program B")

Classify all possibilities for $H(n)$.

Mazur's Program B

As presented at Modular functions in one variable V in Bonn

Theorem 1 also fits into a general program:

B. Given a number field K and a subgroup H of $GL_2\hat{\mathbb{Z}} = \prod_p GL_2\mathbb{Z}_p$ classify all elliptic curves $E_{/K}$ whose associated Galois representation on torsion points maps $\mathrm{Gal}(\overline{K}/K)$ into $H \subset GL_2\hat{\mathbb{Z}}$.

Mazur - Rational points on modular curves (1977)

Example - torsion on an elliptic curve

If E has a K-rational **torsion point** $P \in E(K)[n]$ (of exact order n) then:

$$H(n) \subset \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$$

since for $\sigma \in G_K$ and $Q \in E(\overline{K})[n]$ such that $E(\overline{K})[n] \cong \langle P, Q \rangle$,

$$\begin{aligned} \sigma(P) &= \quad P \\ \sigma(Q) &= \quad a_\sigma P \quad + \quad b_\sigma Q \end{aligned}$$

Example - Isogenies

If E has a K-rational, **cyclic isogeny** $\phi\colon E \to E'$ with $\ker\phi = \langle P \rangle$ then:

$$H(n) \subset \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

since for $\sigma \in G_K$ and $Q \in E(\overline{K})[n]$ such that $E(\overline{K})[n] \cong \langle P, Q \rangle$,

$$\begin{aligned} \sigma(P) &= a_\sigma P \\ \sigma(Q) &= b_\sigma P + c_\sigma Q \end{aligned}$$

Example - other maximal subgroups

Normalizer of a split Cartan:

$$N_{\mathrm{sp}} = \left\langle \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle$$

$H(n) \subset N_{\mathrm{sp}}$ and $H(n) \not\subset C_{\mathrm{sp}}$ iff

- there exists an unordered pair $\{\phi_1, \phi_2\}$ of cyclic isogenies,
- whose kernels intersect trivially,
- neither of which is defined over K,
- but which are both defined over some quadratic extension of K,
- and which are Galois conjugate.

Example - other maximal subgroups

$\mathbb{F}_{p^2}^*$ acts on $\mathbb{F}_{p^2} \cong \mathbb{F}_p \times \mathbb{F}_p$

Normalizer of a non-split Cartan:

$$C_{\text{ns}} = \operatorname{im}\left(\mathbb{F}_{p^2}^* \to \mathrm{GL}_2(\mathbb{F}_p)\right) \subset N_{\text{ns}}$$

$H(n) \subset N_{\text{ns}}$ and $H(n) \not\subset C_{\text{ns}}$ iff

E admits a "necklace" (Rebolledo, Wuthrich)

Image of Galois

$$\rho_{E,n} \colon G_{\mathbb{Q}} \twoheadrightarrow H(n) \hookrightarrow \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$$

$$G_{\mathbb{Q}}\left\{\begin{array}{l}\overline{\mathbb{Q}}\\ \Big|\\ \left.\begin{array}{l}\overline{\mathbb{Q}}^{\ker\rho_{E,n}} = \mathbb{Q}(E[n])\\ \Big| \\ \mathbb{Q}\end{array}\right\}H(n)\end{array}\right.$$

Problem (Mazur's "program B")

Classify all possibilities for $H(n)$.

Modular curves

Definition

- $X(N)(K) := \{(E/K, P, Q) : E[N] = \langle P, Q\rangle\} \cup \{\text{cusps}\}$
- $X(N)(K) \ \ni \ (E/K, P, Q) \Leftrightarrow \rho_{E,N}(G_K) = \{I\}$

Let $\Gamma(N) \subset H \subset \mathrm{GL}_2(\widehat{\mathbb{Z}})$. The minimal such N is the **level** of H.

Definition

$X_H := X(N)/H(N)$ (where $H(N)$ is the image of H in $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$)

$X_H(K) \ni (E/K, \iota) \Leftrightarrow \rho_{E,N}(G_K) \subset H(N)$

Stacky disclaimer

This is only true up to twist; there are some subtleties if

1. $j(E) \in \{0, 12^3\}$ (plus some minor group theoretic conditions), or
2. if $-I \in H$.

Rational Points on modular curves

Mazur's program B

Compute $X_H(\mathbb{Q})$ for all H.

Remark

- *Sometimes* $X_H \cong \mathbb{P}^1$ *or elliptic with rank* $X_H(\mathbb{Q}) > 0$.
- *Some* X_H *have exceptional points (i.e, non-cuspidal non-CM points).*
- *Can compute* $g(X_H)$ *group theoretically (via Riemann–Hurwitz).*

Fact

$$g(X_H), \gamma(X_H) \to \infty \text{ as } \left[\mathrm{GL}_2(\widehat{\mathbb{Z}}) : H\right] \to \infty.$$

(Serre) Sample subgroup $H \subset \mathrm{GL}_2(\widehat{\mathbb{Z}})$

$$
\begin{tikzcd}[column sep=small]
\ker\phi_2 & \subset & H(8) \arrow[d, "\phi_2"] & \subset & \mathrm{GL}_2(\mathbb{Z}/8\mathbb{Z}) \arrow[d] \\
I + 2M_2(\mathbb{Z}/2\mathbb{Z}) & \subset & H(4) \arrow[d, "\phi_1"] & = & \mathrm{GL}_2(\mathbb{Z}/4\mathbb{Z}) \arrow[d] \\
 & & H(2) & = & \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})
\end{tikzcd}
$$

$\dim_{\mathbb{F}_2} \ker\phi_2 = 3$

$\dim_{\mathbb{F}_2} \ker\phi_1 = 4$

$\chi\colon \mathrm{GL}_2(\mathbb{Z}/8\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/8\mathbb{Z})^* \to \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/8\mathbb{Z})^* \cong \mathbb{F}_2^3.$

$\chi = \mathrm{sgn} \times \det$

$H(8) := \chi^{-1}(G)$, $G \subset \mathbb{F}_2^3$.

A typical subgroup $H \subset \mathrm{GL}_2(\widehat{\mathbb{Z}})$

$$\begin{tikzcd}
\ker \phi_4 & \subset & H(32) \arrow[d, "\phi_4"] & \subset & \mathrm{GL}_2(\mathbb{Z}/32\mathbb{Z}) \arrow[d] & & {\color{green}\dim_{\mathbb{F}_2} \ker \phi_4 = 4} \\
\ker \phi_3 & \subset & H(16) \arrow[d, "\phi_3"] & \subset & \mathrm{GL}_2(\mathbb{Z}/16\mathbb{Z}) \arrow[d] & & {\color{red}\dim_{\mathbb{F}_2} \ker \phi_3 = 3} \\
\ker \phi_2 & \subset & H(8) \arrow[d, "\phi_2"] & \subset & \mathrm{GL}_2(\mathbb{Z}/8\mathbb{Z}) \arrow[d] & & {\color{red}\dim_{\mathbb{F}_2} \ker \phi_2 = 2} \\
\ker \phi_1 & \subset & H(4) \arrow[d, "\phi_1"] & \subset & \mathrm{GL}_2(\mathbb{Z}/4\mathbb{Z}) \arrow[d] & & {\color{red}\dim_{\mathbb{F}_2} \ker \phi_1 = 3} \\
 & & H(2) & = & \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z}) & &
\end{tikzcd}$$

Non-abelian entanglements

There exists a surjection $\theta\colon \mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$.

$$
\begin{tikzcd}
 & H(6) := \Gamma_\theta \arrow[ddl, two heads] \arrow[ddr, two heads] \arrow[d, phantom, "\cap"] & \\
 & \mathrm{GL}_2(\mathbb{Z}/6\mathbb{Z}) \arrow[dl, two heads] \arrow[dr, two heads] & \\
\mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z}) & & \mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z})
\end{tikzcd}
$$

Brau–Jones

$$\operatorname{im} \rho_{E,6} \subset H(6) \Leftrightarrow j(E) = 2^{10}3^3t^3(1-4t^3) \Rightarrow K(E[2]) \subset K(E[3])$$

$$X_H \cong \mathbb{P}^1 \xrightarrow{j} X(1)$$

Main conjecture

Conjecture (Serre)

Let E be an elliptic curve over $\mathbb{Q}$ without CM. Then for $\ell > 37$, $\rho_{E,\ell}$ is surjective.

In other words, conjecturally, $\rho_{E,\ell^\infty} = \mathrm{GL}_2(\mathbb{Z}_\ell)$ for $\ell > 37$.

“Vertical” image conjecture

Conjecture

There exists a constant N such that for every $E/\mathbb{Q}$ without CM

$$\left[\mathrm{GL}_2(\widehat{\mathbb{Z}}) : \rho_E(G_{\mathbb{Q}})\right] \leq N.$$

Remark

This follows from the “$\ell > 37$” conjecture.

Problem

Assume the “$\ell > 37$” conjecture and compute N.

Subgroups of $\mathrm{GL}_2(\widehat{\mathbb{Z}})$

To identify open subgroups $H \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}})$ (up to conjugacy) we assign them unique labels.

Definition

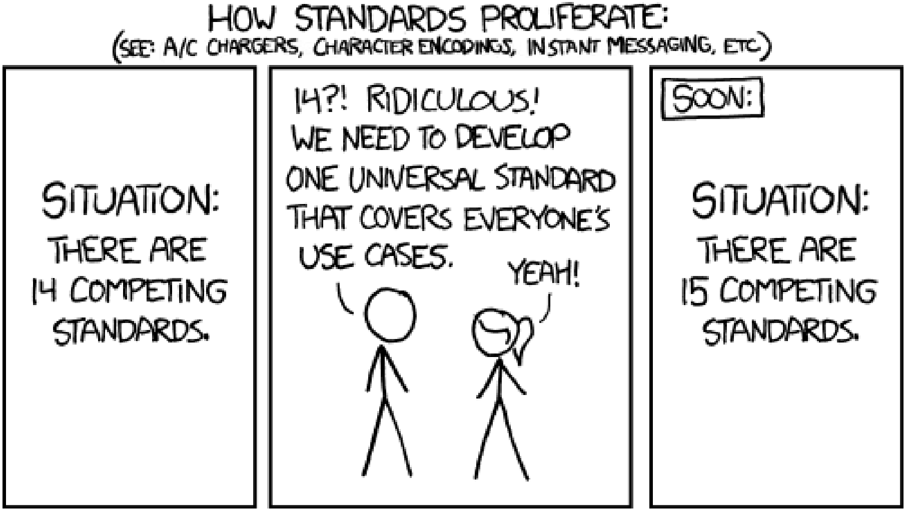
When $\det(H) = \widehat{\mathbb{Z}}^\times$ these labels have the form `N.i.g.n`, where N is the level, i is the index, g is the genus, and n is a tiebreaker given by ordering the subgroups of $\mathrm{GL}_2(N)$.

Example

- The Borel subgroup $B(13)$ has label `13.14.0.1`.
- The normalizer of the split Cartan $N_{\mathrm{sp}}(13)$ has label `13.91.3.1`.
- The normalizer of the nonsplit Cartan $N_{\mathrm{ns}}(13)$ has label `13.78.3.1`.
- The maximal S_4 exceptional group $S_4(13)$ has label `13.91.3.2`.

When $N = \ell^e$ we can also view these as labels of subgroups of $\mathrm{GL}_2(\mathbb{Z}_\ell)$.

Obligatory XKCD cartoon



Main Theorem

Definition

A point $P \in X_H(K)$ is exceptional if $X_H(K)$ is finite and P corresponds to a non-CM E/K.

Theorem (Rouse–Sutherland–ZB 2021)

Let ℓ be a prime, let $E/\mathbb{Q}$ be a non-CM elliptic curve, and let $H = \rho_{E,\ell^\infty}(G_\mathbb{Q})$.
Exactly one of the following is true:

1. *$X_H(\mathbb{Q})$ is infinite and H is listed in (Sutherland–Zywina 2017);*
2. *X_H has a rational exceptional point listed in Table 1;*
3. *$H \leq N_{\mathrm{ns}}(3^3), N_{\mathrm{ns}}(5^2), N_{\mathrm{ns}}(7^2), N_{\mathrm{ns}}(11^2)$, or $N_{\mathrm{ns}}(\ell)$ for some $\ell > 13$;*
4. *H is a subgroup of `49.179.9.1` or `49.196.9.1`.*

We conjecture that cases (3) and (4) never occur.
If they do, the exceptional points have very large heights (e.g. $10^{10^{200}}$ for $X_{\mathrm{ns}}^{+}(11^2)(\mathbb{Q})$).

label	level	notes	j-invariants/models of exceptional points
16.64.2.1	2^4	$N_{ns}(16)$	$-2^{18}\cdot 3\cdot 5^3\cdot 13^3\cdot 41^3\cdot 107^3/17^{16}$, $-2^{21}\cdot 3^3\cdot 5^3\cdot 7\cdot 13^3\cdot 23^3\cdot 41^3\cdot 179^3\cdot 409^3/79^{16}$
16.96.3.335	2^4	$H(4)\subsetneq N_{sp}(4)$	$257^3/2^8$
16.96.3.343	2^4	$H(4)\subsetneq N_{sp}(4)$	$17^3\cdot 241^3/2^4$
16.96.3.346	2^4	$H(4)\subsetneq N_{sp}(4)$	$2^4\cdot 17^3$
16.96.3.338	2^4	$H(4)\subsetneq N_{sp}(4)$	2^{11}
32.96.3.230	2^5	$H(4)\subsetneq N_{sp}(4)$	$-3^3\cdot 5^3\cdot 47^3\cdot 1217^3/(2^8\cdot 31^8)$
32.96.3.82	2^5	$H(8)\subsetneq N_{sp}(8)$	$3^3\cdot 5^6\cdot 13^3\cdot 23^3\cdot 41^3/(2^{16}\cdot 31^4)$
25.50.2.1	5^2	$H(5)=N_{ns}(5)$	$2^4\cdot 3^2\cdot 5^7\cdot 23^3$
25.75.2.1	5^2	$H(5)=N_{sp}(5)$	$2^{12}\cdot 3^3\cdot 5^7\cdot 29^3/7^5$
7.56.1.2	7	$\subsetneq N_{ns}(7)$	$3^3\cdot 5\cdot 7^5/2^7$
7.112.1.2	7	$-I\notin H$	$y^2+xy+y=x^3-x^2-2680x-50053$, $y^2+xy+y=x^3-x^2-131305x+17430697$
11.60.1.3	11	$\subsetneq B(11)$	$-11\cdot 131^3$
11.120.1.8	11	$-I\notin H$	$y^2+xy+y=x^3+x^2-30x-76$
11.120.1.9	11	$-I\notin H$	$y^2+xy=x^3+x^2-2x-7$
11.60.1.4	11	$\subsetneq B(11)$	-11^2
11.120.1.3	11	$-I\notin H$	$y^2+xy=x^3+x^2-3632x+82757$
11.120.1.4	11	$-I\notin H$	$y^2+xy+y=x^3+x^2-305x+7888$
13.91.3.2	13	$S_4(13)$	$2^4\cdot 5\cdot 13^4\cdot 17^3/3^{13}$, $-2^{12}\cdot 5^3\cdot 11\cdot 13^4/3^{13}$, $2^{18}\cdot 3^3\cdot 13^4\cdot 127^3\cdot 139^3\cdot 157^3\cdot 283^3\cdot 929/(5^{13}\cdot 61^{13})$
17.72.1.2	17	$\subsetneq B(17)$	$-17\cdot 373^3/2^{17}$
17.72.1.4	17	$\subsetneq B(17)$	$-17^2\cdot 101^3/2$
37.114.4.1	37	$\subsetneq B(37)$	$-7\cdot 11^3$
37.114.4.2	37	$\subsetneq B(37)$	$-7\cdot 137^3\cdot 2083^3$

Table 1. All known exceptional groups, j-invariants, and points of prime power level.

UNSOLVED mysteries

label	level	group	genus
`27.243.12.1`	3^3	$N_{\mathrm{ns}}(3^3)$	12
`25.250.14.1`	5^2	$N_{\mathrm{ns}}(5^2)$	14
`49.1029.69.1`	7^2	$N_{\mathrm{ns}}(7^2)$	69
`49.147.9.1`	7^2	$\left\langle \left(\begin{smallmatrix} 16 & 6 \\ 20 & 45 \end{smallmatrix}\right), \left(\begin{smallmatrix} 20 & 17 \\ 40 & 36 \end{smallmatrix}\right) \right\rangle$	9
`49.196.9.1`	7^2	$\left\langle \left(\begin{smallmatrix} 42 & 3 \\ 16 & 31 \end{smallmatrix}\right), \left(\begin{smallmatrix} 16 & 23 \\ 8 & 47 \end{smallmatrix}\right) \right\rangle$	9
`121.6655.511.1`	11^2	$N_{\mathrm{ns}}(11^2)$	511

Arithmetically maximal groups of level ℓ^n with $\ell \leq 13$ for which $X_H(\mathbb{Q})$ is unknown; each has rank $=$ genus, rational CM points, no rational cusps, and no known exceptional points.

Summary of ℓ-adic images of Galois for non-CM $E/\mathbb{Q}$.

ℓ	2	3^*	5^*	7^*	11^*	13	17^*	37^*	other*
subgroups	1208	47	25	17	8	12	3	3	1
exceptional subgroups	7	0	2	2	6	1	2	2	0
unexceptional subgroups	1201	47	23	15	2	11	1	1	1
max level	32	27	25	7	11	13	17	37	1
max index	96	72	120	112	120	91	72	114	1
max genus	3	0	2	1	1	3	1	4	0

Summary of the $H \leq \mathrm{GL}_2(\mathbb{Z}_\ell)$ which occur as $\rho_{E,\ell^\infty}(G_\mathbb{Q})$ for some non-CM elliptic curve $E/\mathbb{Q}$. Starred primes depend on the conjecture that cases (3) and (4) of our theorem do not occur.

In particular, we conjecture that there are $1207, 46, 24, 16, 7, 11, 2, 2$ proper subgroups of $\mathrm{GL}_2(\mathbb{Z}_\ell)$ that arise as $\rho_{E,\ell^\infty}(G_\mathbb{Q})$ for non-CM $E/\mathbb{Q}$ for $\ell = 2, 3, 5, 7, 11, 13, 17, 37$ and none for any other ℓ.

Arithmetically maximal groups

Definition

We say that an open subgroup $H \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}})$ is arithmetically maximal if

1. $\det(H) = \mathbb{Z}^\times$ (necessary for $\mathbb{Q}$-points),
2. a conjugate of $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}1 & 1\\ 0 & -1\end{smallmatrix}\right)$ lies in H (necessary for $\mathbb{R}$-points),
3. $j(X_H(\mathbb{Q}))$ is finite but $j(X_{H'}(\mathbb{Q}))$ is infinite for $H \subsetneq H' \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}})$.

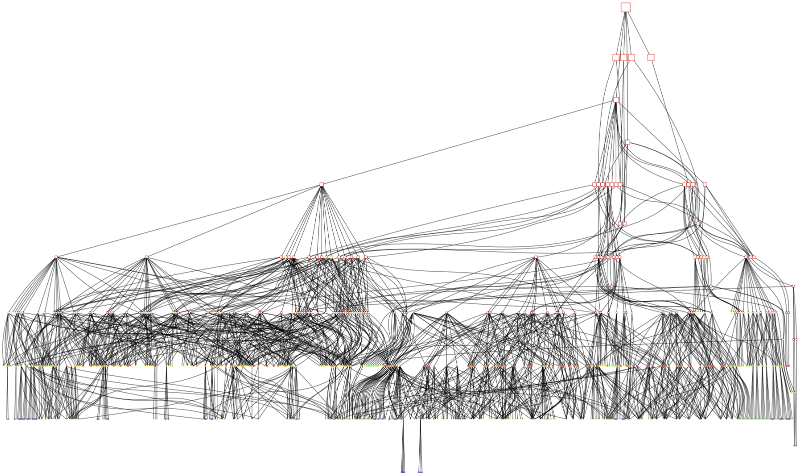
Arithmetically maximal groups H arise as maximal subgroups of an H' with $X_{H'}(\mathbb{Q})$ infinite.

Theorem (Sutherland–Zywina 2017)

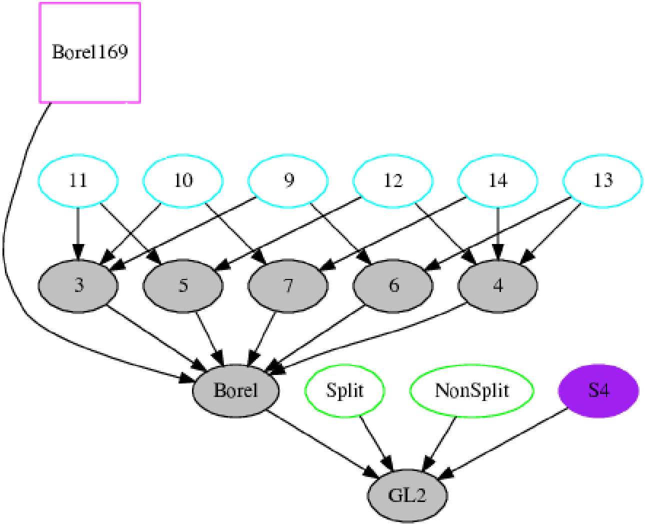
For $\ell = 2, 3, 5, 7, 11, 13$ there are $1208, 47, 23, 15, 2, 11$ subgroups $H \leq \mathrm{GL}_2(\widehat{\mathbb{Z}})$ of ℓ-power level with $X_H(\mathbb{Q})$ infinite, and only $H = \mathrm{GL}_2(\widehat{\mathbb{Z}})$ for $\ell > 13$.

This allows us to compute explicit upper bounds on the level and index of arithmetically maximal subgroup of prime power level ℓ and we can then exhaustively enumerate them.

Subgroups of $\mathrm{GL}_2(\mathbb{Z}_2)$



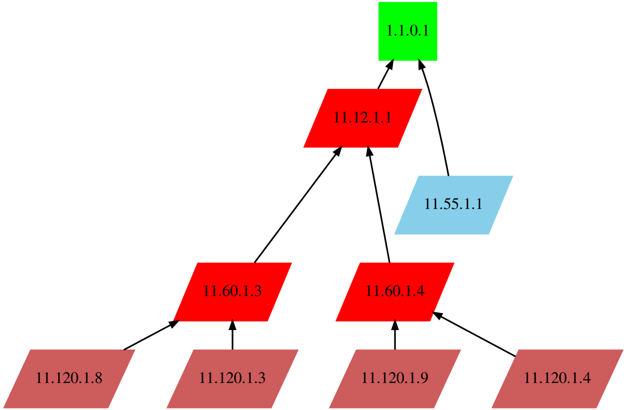
Subgroups of $\mathrm{GL}_2(\mathbb{Z}_{13})$



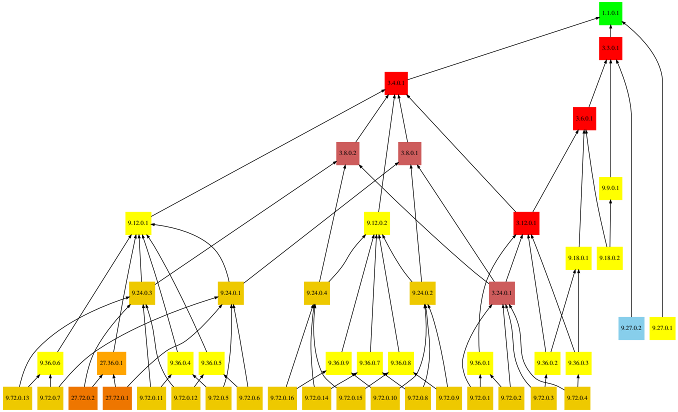
Steps of the proof

1. Compute the set $\mathcal{S}$ of **arithmetically maximal** subgroups of ℓ-power level for $\ell \leq 37$ (for all $\ell > 37$ we already know $N_{\mathrm{ns}}(\ell)$ is the only possible exceptional group).
2. For $H \in \mathcal{S}$ check for **local obstructions** and compute the **isogeny decomposition** of the Jacobian of X_H and the analytic ranks of all its simple factors.
3. For $H \in \mathcal{S}$ **compute equations** for X_H and $j_H \colon X_H \to X(1)$ (if needed). In several cases we can prove $X_H(\mathbb{Q})$ is empty without a model for X_H.
4. For $H \in \mathcal{S}$ with $-I \in H$ **determine the rational points** in $X_H(\mathbb{Q})$ (if possible). In several cases we are able to exploit recent progress by others ($\ell = 13$ for example).
5. For $H \in \mathcal{S}$ with $-I \notin H$ **compute equations** for the universal curve $\mathcal{E} \to U$, where $U \subseteq X_H$ is the locus with $j(P) \neq 0, 1728, \infty$.

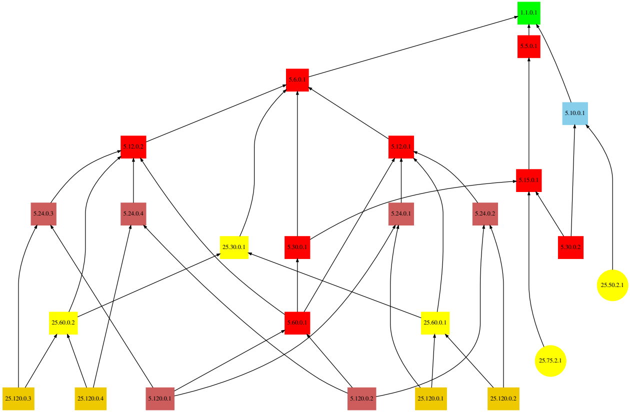
Subgroups of $\mathrm{GL}_2(\mathbb{Z}_{11})$



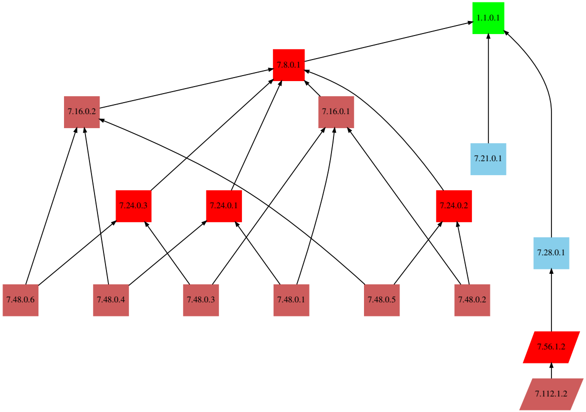
Subgroups of $\mathrm{GL}_2(\mathbb{Z}_3)$



Subgroups of $\mathrm{GL}_2(\mathbb{Z}_5)$



Subgroups of $\mathrm{GL}_2(\mathbb{Z}_7)$



Finding Equations for X_H – Basic idea

1. The canoncial map $C \hookrightarrow \mathbb{P}^{g-1}$ is given by $P \mapsto [\omega_1(P) : \cdots : \omega_g(P)]$.
2. For a general curve, this is an embedding, and the relations are quadratic.
3. For a modular curve,
$$M_k(H) \cong H^0(X_H, \Omega^1(\Delta)^{\otimes k/2})$$
given by
$$f(z) \mapsto f(z)\, dz^{\otimes k/2}.$$

Equations – Example: $X_1(17) \subset \mathbb{P}^4$

Cusp forms

$$
\begin{aligned}
q - 11q^5 + 10q^7 + O(q^8) \\
q^2 - 7q^5 + 6q^7 + O(q^8) \\
q^3 - 4q^5 + 2q^7 + O(q^8) \\
q^4 - 2q^5 + O(q^8) \\
q^6 - 3q^7 + O(q^8)
\end{aligned}
$$

$$
\begin{aligned}
xu + 2xv - yz + yu - 3yv + z^2 - 4zu + 2u^2 + v^2 = 0 \\
xu + xv - yz + yu - 2yv + z^2 - 3zu + 2uv = 0 \\
2xz - 3xu + xv - 2y^2 + 3yz + 7yu - 4yv - 5z^2 - 3zu + 4zv = 0
\end{aligned}
$$

Computing models of modular curves

- We introduce a variety of improvements and tricks to compute models of various X_H.
- See Rouse's VaNTAGe talk for more details and interesting examples.
- To compute $j_H \colon X_H \to X(1)$ we represent E_4 and E_6 as ratios of elements of the canonical ring.
- We show that E_4 is a rational function of an element of weight k and weight $k-4$ if

$$k \geq \frac{2e_\infty + e_2 + e_3 + 5g - 4}{2(g-1)}$$

- We used this method to compute canonical models for many curves of large genus.
- See Assaf's recent paper and Zywina's BIRS talk for other efficient approaches.

Explicit methods: highlight reel

- Local methods
- Chabauty
- Elliptic Chabauty
- Mordell–Weil sieve
- étale descent
- Pryms
- *Equationless étale descent via group theory*
- *New techniques for computing* Aut C

- **"Equationless" local methods** and **Mordell–Weil sieve**
- **Greenberg Transforms** (and big computations)
- **Novel variants of existing techniques**
- **Modularity of isogeny factors of** J_H (w/ Voight)

Computing $X_H(\mathbb{F}_p)$ "via moduli"

Idea: one can compute $\#X_1(N)(\mathbb{F}_p)$ by enumerating elliptic curves over $\mathbb{F}_p$, then computing their N torsion subgroups.

Deligne–Rapoport 1973

The modular curves X_H and Y_H are coarse spaces for the stacks $\mathcal{M}_H$ and $\mathcal{M}_H^0$ that parameterize elliptic curves E with H-level structure, by which we mean an equivalence class $[\iota]_H$ of isomorphisms $\iota\colon E[N] \to \mathbb{Z}(N)^2$, where $\iota \sim \iota'$ if $\iota = h \circ \iota'$ for some $h \in H$.

- $Y_H(\bar{k}) = \{(j(E), \alpha) : \alpha = Hg\mathcal{A}_E\}$ with $\mathcal{A}_E := \{\varphi_N : \varphi \in \mathrm{Aut}(E_{\bar{k}})\}$, and $Y_H(k) = Y_H(\bar{k})^{G_k}$.
- $X_H^\infty(k) = \{\alpha \in H\backslash \mathrm{GL}_2(N)/U(N) : \alpha^{\chi_N(G_K)} = \alpha\}$ where $U(N) := \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), -1 \rangle$.
- For $k = \mathbb{F}_q$, to compute $\#X_H(k) = \#Y_H(k) + \#X_H^\infty(k)$ count double cosets fixed by G_k.

- See Drew's Slides for a nice summary of the implementation.

Arithmetically maximal modular curves with $\#X_H(\mathbb{F}_p) = 0$

label	level	generators	p	rank	genus
16.48.2.17	2^4	$\begin{pmatrix} 11 & 9 \\ 4 & 13 \end{pmatrix}, \begin{pmatrix} 13 & 5 \\ 4 & 11 \end{pmatrix}, \begin{pmatrix} 1 & 9 \\ 12 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 9 \\ 0 & 5 \end{pmatrix}$	$3, 11$	0	2
27.108.4.5	3^3	$\begin{pmatrix} 4 & 25 \\ 6 & 14 \end{pmatrix}, \begin{pmatrix} 8 & 0 \\ 3 & 1 \end{pmatrix}$	$7, 31$	0	4
25.150.4.2	5^2	$\begin{pmatrix} 7 & 20 \\ 20 & 7 \end{pmatrix}, \begin{pmatrix} 22 & 2 \\ 13 & 22 \end{pmatrix}$	2	0	4
25.150.4.7	5^2	$\begin{pmatrix} 24 & 24 \\ 0 & 18 \end{pmatrix}, \begin{pmatrix} 2 & 5 \\ 0 & 23 \end{pmatrix}$	$3, 23$	4	4
25.150.4.8	5^2	$\begin{pmatrix} 8 & 4 \\ 0 & 23 \end{pmatrix}, \begin{pmatrix} 16 & 7 \\ 0 & 8 \end{pmatrix}$	2	0	4
25.150.4.9	5^2	$\begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix}, \begin{pmatrix} 3 & 18 \\ 0 & 14 \end{pmatrix}$	2	0	4
49.168.12.1	7^2	$\begin{pmatrix} 39 & 6 \\ 36 & 24 \end{pmatrix}, \begin{pmatrix} 11 & 9 \\ 24 & 2 \end{pmatrix}$	2	3	12
13.84.2.2	13	$\begin{pmatrix} 3 & 7 \\ 0 & 8 \end{pmatrix}, \begin{pmatrix} 12 & 4 \\ 0 & 12 \end{pmatrix}$	2	0	2
13.84.2.3	13	$\begin{pmatrix} 9 & 2 \\ 0 & 7 \end{pmatrix}, \begin{pmatrix} 4 & 4 \\ 0 & 7 \end{pmatrix}$	3	0	2
13.84.2.4	13	$\begin{pmatrix} 8 & 12 \\ 0 & 10 \end{pmatrix}, \begin{pmatrix} 8 & 3 \\ 0 & 9 \end{pmatrix}$	2	0	2
13.84.2.6	13	$\begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 11 & 3 \\ 0 & 10 \end{pmatrix}$	3	0	2

Arithmetically maximal H of ℓ-power level for which $X_H(\mathbb{F}_p)$ is empty for some $p \neq \ell \leq 37$.

Decomposing the Jacobian of X_H

Let H be an open subgroup of $\mathrm{GL}_2(\widehat{\mathbb{Z}})$ of level N and let J_H denote the Jacobian of X_H.

Theorem (Rouse–Sutherland–Voight–ZB 2021)

Each simple factor A of J_H is isogenous to A_f for a weight-2 eigenform f on $\Gamma_0(N^2) \cap \Gamma_1(N)$.

Corollary (Kolyvagin's theorem)

If A is an isogeny factor of X_H, and if the analytic rank of A is zero, then $A(\mathbb{Q})$ is finite.

Corollary (Decomposition)

We can decompose J_H up to isogeny using linear algebra and point-counting.

Mordell–Weil sieve

Let X be a curve and A be an abelian variety.

$$\begin{array}{c} X(\mathbb{Q}) \\ \big\downarrow \\ X(\mathbb{F}_p) \end{array}$$

- If $X(\mathbb{F}_p)$ is empty for some p then $X(\mathbb{Q})$ is empty.

Mordell–Weil sieve

Let X be a curve and A be an abelian variety.

$$\begin{tikzcd} X(\mathbb{Q}) \arrow[r] \arrow[d] & A(\mathbb{Q}) \arrow[d, "\beta"] \\ X(\mathbb{F}_p) \arrow[r, "\pi"] & A(\mathbb{F}_p). \end{tikzcd}$$

- If $X(\mathbb{F}_p)$ is empty for some p then $X(\mathbb{Q})$ is empty.
- If $\operatorname{im}\pi \cap \operatorname{im}\beta$ is empty then $X(\mathbb{Q})$ is empty.

Mordell–Weil sieve

Let X be a curve and A be an abelian variety.

$$
\begin{tikzcd}
X(\mathbb{Q}) \arrow[r] \arrow[d] & A(\mathbb{Q}) \arrow[d, "\beta"] \\
\prod_{p\in S} X(\mathbb{F}_p) \arrow[r, "\pi_S"] & \prod_{p\in S} A(\mathbb{F}_p).
\end{tikzcd}
$$

- If $X(\mathbb{F}_p)$ is empty for some p then $X(\mathbb{Q})$ is empty.
- If $\operatorname{im}\pi_S \cap \operatorname{im}\beta$ is empty then $X(\mathbb{Q})$ is empty.
- This is explicit and is implemented in Magma.

An equationless Mordell–Weil sieve for the group `121.605.41.1`

The curve X_H has local points everywhere, and analytic rank = genus = 41.

$H(11) \subset N_{\text{ns}}(11)$, so X_H maps to $X_{\text{ns}}^+(11)$, which is an elliptic curve of rank 1.

$$\begin{tikzcd}
X_H(\mathbb{Q}) \arrow[rr] \arrow[d] & & X_{\text{ns}}^+(11)(\mathbb{Q}) \arrow[r, equal] \arrow[d, "\beta"] & \langle R \rangle \cong \mathbb{Z} \\
\prod_{p\in S} X_H(\mathbb{F}_p) \arrow[rr, "\pi_S"] & & \prod_{p\in S} X_{\text{ns}}^+(11)(\mathbb{F}_p) &
\end{tikzcd}$$

- For $p = 13$ the image of any point in $Y_H(\mathbb{Q})$ maps to nR with $n \equiv 1, 5 \bmod 7$.
- For $p = 307$ any point in $Y_H(\mathbb{Q})$ maps to nR with $n \equiv 2, 3, 4, 7, 10, 13 \bmod 14$.
- Therefore $Y_H(\mathbb{Q}) = \emptyset$ (and in fact $X_H(\mathbb{Q}) = \emptyset$, there are no rational cusps).
- A point of $X_{\text{ns}}^+(11)(\mathbb{F}_p)$ corresponds to E with $\rho_{E,11}(G_{\mathbb{F}_p}) \subset N_{\text{ns}}(11)$ and lifts to a point of $X_H(\mathbb{F}_p)$ if and only if $\rho_{E,121}(G_{\mathbb{F}_p}) \subset H(121)$.

Gargantuan models of modular curves[1]

- We computed canonical models (over $\mathbb{Q}$) for `27.729.43.1` (resp. `25.625.36.1`).
- We use these models to prove that X_H has no $\mathbb{Q}_3$ (resp. $\mathbb{Q}_5$) as follows.
- These models have very bad reduction at $p = 3$ (resp. 5). (They're not even flat.)
- $X_H(\mathbb{F}_p) \neq \emptyset$ for all p, but $X_H(\mathbb{Z}/p^2\mathbb{Z}) = \emptyset$ for $p = 3$ (resp. 5).
- The "Greenberg transform" (i.e., the "Wittferential tangent space" of Buium) is adjoint to Witt vectors: $X_H^{(1)}(\mathbb{F}_p) = X_H(\mathbb{Z}/p^2\mathbb{Z})$.
- The fibers of the map $X_H^{(1)} \to X_H$ have no $\mathbb{F}_p$ points.

[1] We give thanks to Poonen and Zywina

Thank you!