# A positive proportion of hyperelliptic curves have no unexpected quadratic points


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# Standard families of hyperelliptic curves of genus $g \geq 2$

- Recall: A *hyperelliptic curve* over $\mathbb{Q}$ is a nice (i.e., smooth, projective, and geometrically integral) curve with a degree-2 map to $\mathbb{P}^1_{\mathbb{Q}}$

**Families with marked $\mathbb{Q}$-rational points**

- Monic odd degree: $y^2 = x^{2g+1} + c_{2g-1}x^{2g-1} + \cdots + c_0$, for $c_i \in \mathbb{Z}$
  - Have marked $\mathbb{Q}$-rational Weierstrass point at $\infty$
- Monic even degree: $y^2 = x^{2g+2} + c_{2g}x^{2g} + \cdots + c_0$, for $c_i \in \mathbb{Z}$
  - Have a pair of marked $\mathbb{Q}$-rational non-Weierstrass points $\{\infty, \tau(\infty)\}$

**Families without marked $\mathbb{Q}$-rational points**

- Non-monic even degree: $y^2 = cx^{2g+2} + c_{2g}x^{2g} + \cdots + c_0$, for $c_i \in \mathbb{Z}$ and fixed $c \in \mathbb{Z} \smallsetminus \{n^2 : n \in \mathbb{Z}\}$
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# Motivating questions

- Let $\mathscr{F}$ be a standard family of hyperelliptic curves of genus $g \geq 2$
- Given $C \in \mathscr{F}$ and $P \in C(\mathbb{Q})$, we call $P$ *expected* if $P$ is among the marked points of the family $\mathscr{F}$, and *unexpected* otherwise
- Falting's Theorem $\Longrightarrow \#C(\mathbb{Q}) < \infty$ for each $C \in \mathscr{F}$; i.e., the set of unexpected $\mathbb{Q}$-rational points on $C$ is finite

**Question**

When curves $C \in \mathscr{F}$ are ordered by height ($\approx$ the sizes of their coeff-icients), how often does $C$ have no unexpected $\mathbb{Q}$-rational points?

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# Earlier work on pointlessness of hyperelliptic curves

**Families without marked $\mathbb{Q}$-rational points**

**Theorem (Bhargava, 2013)**

*When even-degree hyperelliptic curves $C/\mathbb{Q}$ of genus $g$ are ordered by height, the proportion of curves $C$ such that $C(\mathbb{Q}) = \varnothing$*

- *is $> 0$ for every $g \geq 1$ (and is $> 50\%$ for every $g \geq 2$); and*
- *tends to $100\%$ as $g \to \infty$.*

**Theorem (Bhargava-Gross-Wang, 2017 [corrected in BSS, 2021])**

*Let $k$ be odd. When even-degree hyperelliptic curves $C/\mathbb{Q}$ of genus $g$ are ordered by height, the proportion of curves $C$ such that $(\mathrm{Sym}^k C)(\mathbb{Q}) = \varnothing$*

- *is $> 0$ for every $g \geq 1$; and*
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**Proof strategy:** Show that most curves have no locally soluble 2-covers

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**Families with marked $\mathbb{Q}$-rational points**

**Theorem (Poonen-Stoll, 2013)**

*When monic odd-degree hyperelliptic curves $C/\mathbb{Q}$ of genus $g$ are ordered by height, the proportion of curves $C$ such that $C(\mathbb{Q}) = \{\infty\}$*

- *is $\gg 16^{-g} > 0$ for every $g \geq 3$; and*
- *tends to 100% as $g \to \infty$.*

**Theorem (Shankar-Wang, 2018)**

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**Proof strategy:** "Selmer-group Chabauty"

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- *is $\gg 16^{-g} > 0$ for every $g \geq 9$; and*
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# Motivating questions (cont.)

What about points of even degree?

- Given $C \in \mathscr{F}$ and $P \in (\mathrm{Sym}^2 C)(\mathbb{Q})$, we call $P$ *expected* if $P$ is the preimage of $\mathbb{Q}$-rational point under the hyperelliptic map $C \to \mathbb{P}^1_{\mathbb{Q}}$, and *unexpected* otherwise
- Faltings proved that if $g \geq 4$, the set of unexpected points in $(\mathrm{Sym}^2 C)(\mathbb{Q})$ is finite

**Question**

When curves $C \in \mathscr{F}$ are ordered by height ($\approx$ the sizes of their coefficients), how often does $\mathrm{Sym}^2 C$ have no unexpected $\mathbb{Q}$-rational points? (I.e., how often does $C$ have no unexpected quadratic points?)

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**Monic odd hyperelliptic curves**

**Theorem (Gunther-Morrow, 2017)**

*Under a technical assumption, when monic odd-degree hyperelliptic curves $C/\mathbb{Q}$ of genus $g \geq 4$ are ordered by height, a positive proportion of curves $C$ are such that $\mathrm{Sym}^2 C$ has $\leq 24$ unexpected $\mathbb{Q}$-rational points.*

**Proof strategy**:

- Park (2016) developed Chabauty for symmetric powers of curves, under the hypothesis $\mathrm{rk}\, J(\mathbb{Q}) \leq 1$; combine with the result of Bhargava-Gross (2013) that $\mathrm{Avg}\,\#\,\mathrm{Sel}_2(\mathrm{J}(C)) \leq^* 3 \implies$ $\mathrm{Avg}\,\mathrm{rk}\, J(\mathbb{Q}) \leq 3/2 \implies$ a positive proportion of $C$ have $\mathrm{rk}\, J(\mathbb{Q}) \leq 1$
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# Main result

**Theorem (BLSS, work in progress, 2024)**

*Let $\mathscr{F}$ be any one of the standard families of hyperelliptic curves of genus $g$ over $\mathbb{Q}$. When ordered by height, the proportion of curves $C \in \mathscr{F}$ with the property that $\operatorname{Sym}^2 C$ has no unexpected $\mathbb{Q}$-rational points is $\gg 16^{-g} > 0$ for every $g \geq 4$.*

# Selmer-group Chabauty

- Let $C$ be monic odd hyperelliptic of genus $g \geq 4$. Let $J = \mathrm{J}(C)$ be the Jacobian, and let $X = \mathrm{im}(\mathrm{Sym}^2 C \to J)$
- For a prime $p$, let $\overline{J(\mathbb{Q})} \subset J(\mathbb{Q}_p)$ be the $p$-adic closure of $J(\mathbb{Q})$ in $J(\mathbb{Q}_p)$. Consider the following diagram:

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J(\mathbb{Q})/pJ(\mathbb{Q}) \arrow[r, two heads] \arrow[dr] & \overline{J(\mathbb{Q})}/p\overline{J(\mathbb{Q})} \arrow[r] & J(\mathbb{Q}_p)/pJ(\mathbb{Q}_p) \arrow[r, two heads, "\log \otimes \mathbb{F}_p"] & \mathbb{F}_p^g \arrow[r, dashed, "\mathbb{P}"] & \mathbb{P}^{g-1}(\mathbb{F}_p) \\
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\end{tikzcd}$$

- The map $\log \colon J(\mathbb{Q}_p) \to T_0 J \simeq \mathbb{Q}_p^g$ is a local diffeomorphism onto its image, which can be identified with $\mathbb{Z}_p^g$; $\ker \log = J(\mathbb{Q}_p)^{\mathrm{tors}}$
- The map $\rho$ is defined on $\mathbb{Z}_p^g \setminus \{0\}$, given by scaling to an element of $\mathbb{P}^{g-1}(\mathbb{Z}_p)$ and reducing mod $p$

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- The map $\log\colon J(\mathbb{Q}_p) \to T_0 J \simeq \mathbb{Q}_p^g$ is a local diffeomorphism onto its image, which can be identified with $\mathbb{Z}_p^g$; $\ker \log = J(\mathbb{Q}_p)^{\mathrm{tors}}$
- The map $\rho$ is defined on $\mathbb{Z}_p^g \setminus \{0\}$, given by scaling to an element of $\mathbb{P}^{g-1}(\mathbb{Z}_p)$ and reducing mod $p$

# Selmer-group Chabauty

- Let $C$ be monic odd hyperelliptic of genus $g \geq 4$. Let $J = \mathrm{J}(C)$ be the Jacobian, and let $X = \mathrm{im}(\mathrm{Sym}^2 C \to J)$
- For a prime $p$, let $\overline{J(\mathbb{Q})} \subset J(\mathbb{Q}_p)$ be the $p$-adic closure of $J(\mathbb{Q})$ in $J(\mathbb{Q}_p)$. Consider the following diagram:

$$\begin{tikzcd}
X(\mathbb{Q}) \arrow[rr, hook] \arrow[d, hook] & & X(\mathbb{Q}_p) \arrow[d, hook] & & \\
J(\mathbb{Q}) \arrow[r, hook] \arrow[d, two heads] & \overline{J(\mathbb{Q})} \arrow[r, hook] \arrow[d, two heads] & J(\mathbb{Q}_p) \arrow[r, two heads, "\log"] \arrow[d, two heads] & \mathbb{Z}_p^g \arrow[d, two heads] \arrow[dr, dashed, "\rho"] & \\
J(\mathbb{Q})/pJ(\mathbb{Q}) \arrow[r, two heads] \arrow[dr, hook] & \overline{J(\mathbb{Q})}/p\overline{J(\mathbb{Q})} \arrow[r] & J(\mathbb{Q}_p)/pJ(\mathbb{Q}_p) \arrow[r, two heads, "\log \otimes \mathbb{F}_p"] & \mathbb{F}_p^g \arrow[r, dashed, "\mathbb{P}"] & \mathbb{P}^{g-1}(\mathbb{F}_p) \\
& \mathrm{Sel}_p J \arrow[ur] \arrow[urr, "\sigma"'] & & &
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# Selmer-group Chabauty (cont.)

**Lemma**

*Suppose that*

1. *The composite map* $\sigma\colon \operatorname{Sel}_p J \to J(\mathbb{Q}_p)/pJ(\mathbb{Q}_p) \to \mathbb{F}_p^g$ *is injective;*
2. $\mathbb{P}\sigma(\operatorname{Sel}_p J) \cap \rho \log(X(\mathbb{Q}_p)) = \varnothing$

*Then we have* $X(\mathbb{Q}_p) \cap \overline{J(\mathbb{Q})} \subset J(\mathbb{Q}_p)[p']$.

**Lemma**

*The generic monic odd hyperelliptic curve $C$ over $\mathbb{Q}_p$ has the property that* $X(\overline{\mathbb{Q}}_p) \cap J(\overline{\mathbb{Q}}_p)_{\mathrm{tors}} \subset J(\overline{\mathbb{Q}}_p)[2]$.

- Thus, under conditions of the first lemma, $X(\mathbb{Q}) \subset X(\mathbb{Q}_p) \cap \overline{J(\mathbb{Q})} = 0$ $\implies \operatorname{Sym}^2 C$ has no unexpected quadratic points
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# Selmer-group Chabauty (cont.)

**Condition 1: Injectivity of $\sigma$**

- Bhargava-Gross (2013) proved two crucial facts about the statistical behavior of 2-Selmer groups of monic odd hyperelliptic Jacobians:
  - $\text{Avg}\,\#\,\text{Sel}_2(J) \leq 3$
  - Let $C$ vary in a sufficiently small subfamily defined by local conditions at 2 so that the group $J(\mathbb{Q}_2)/2J(\mathbb{Q}_2) \simeq \Gamma$ is constant. Then the non-identity elements of the $\text{Sel}_2 J$ equidistribute in $\Gamma$
- Thus, the composite map $\sigma\colon \text{Sel}_2 J \to J(\mathbb{Q}_2)/2J(\mathbb{Q}_2) \to \mathbb{F}_2^g$ is very typically injective
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# Selmer-group Chabauty (cont.)

**Condition 2:** $\mathbb{P}\sigma(\mathrm{Sel}_2 J) \cap \rho \log(X(\mathbb{Q}_2)) = \varnothing$

- Consider the subfamily of curves $C$ with special fiber at 2 given by $y^2 + y = x^{2g+1} + x + 1$ if $g \equiv 1 \pmod 3$ and $y^2 + y = x^{2g+1} + x^3 + 1$ otherwise.
  (This happens a fraction of $\gg 16^{-g}$ of the time)
- $C(\mathbb{F}_2) = \{\infty\}$ and $C(\mathbb{F}_4) = \{\infty, (0, \alpha), (0, \alpha + 1), (1, \alpha), (1, \alpha + 1)\}$
- Compute log explicitly in terms of power series on the residue disks lying above these points, and then use this explicit formula for log to compute a fixed subset $S \subset \mathbb{P}^{g-1}(\mathbb{F}_2)$ such that $\#S = 5$ and $\rho \log(X(\mathbb{Q}_2)) \subset S$ for every curve $C$ in the subfamily
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# 2-Selmer groups of even-degree hyperelliptic Jacobians

- Let $f(x,y) \in \mathbb{Z}[x,y]$ be a separable form of degree $n = 2g+2 \geq 4$; consider hyperelliptic curve $C_f : z^2 = f(x,y)$ with Jacobian $\mathrm{J}(C_f)$
- Objective: apply "parametrize-and-count strategy" to study the distribution of $\mathrm{Sel}_2(\mathrm{J}(C_f))$ as $f$ varies among:
  - Non-monic binary $n$-ic forms with fixed leading coefficient; or
  - Among the family of all binary $n$-ic forms

**Conjecture (Poonen and Rains, 2010)**

*Let $n \geq 6$ with $n \equiv 2 \pmod 4$. When binary $n$-ic forms $f$ are ordered by the max norm on their coefficients, we have* $\mathrm{Avg}\,\#\,\mathrm{Sel}_2(\mathrm{J}(C_f)) = 6$.

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# Primer on parametrize-and-count strategy

- **Step 1** (algebraic): Parametrize arithmetic objects of interest in terms of integral/rational orbits of a coregular representation $G \curvearrowright V$; if rational, check that these orbits have integral representatives
- E.g., let $V = \{\text{binary quartic forms}\}$ and $G = \mathrm{PGL}_2$; $\mathrm{PGL}_2 \curvearrowright V$, with ring of invariants $= \mathbb{Z}\langle I, J\rangle$
- **Step 2** (analytic): Use geometry-of-numbers methods and sieve techniques to count integral representatives

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# Parametrization of $\mathrm{Sel}_2(\mathrm{J}(C_f))$

- Warmup case: $C_f(\mathbb{Q}) \neq \varnothing$. Then pullback via isomorphism $\mathrm{Pic}^1(C_f) \simeq \mathrm{Pic}^0(C_f) = \mathrm{J}(C_f)$ induces

$$\{\text{loc. sol. 2-covers of } \mathrm{J}(C_f)\} \leftrightarrow \{\text{loc. sol. 2-covers of } \mathrm{Pic}^1(C_f)\}$$

**Theorem (Bhargava–Gross–Wang, 2017 (via Wood, 2010))**

$$\left\{\begin{array}{c} \textit{loc. sol. 2-cover} \\ \textit{of } \mathrm{Pic}^1(C_f) \end{array}\right\} \hookrightarrow \left\{\begin{array}{c} (A,B) \in \mathbb{Z}^2 \otimes_{\mathbb{Z}} \mathrm{Sym}_2 \mathbb{Z}^n \textit{ s.t.} \\ \det(xA + yB) = f(x,y) \end{array}\right\} \Big/ (\mathrm{SL}_n/\mu_2)(\mathbb{Z})$$

- Serious problem: If $C_f(\mathbb{Q}) = \varnothing$, there may *not* exist $(A,B) \in \mathbb{Z}^2 \otimes_{\mathbb{Z}} \mathrm{Sym}_2 \mathbb{Z}^n$ with $\det(xA + yB) = f(x,y)$!
- Solution: Create $\mathbb{Q}$-point by replacing $f$ with $f^{\mathrm{mon}} := f_0^{-1} \times f(x, f_0 y)$
- $C_{f^{\mathrm{mon}}}(\mathbb{Q}) \neq \varnothing$, twist of $C_f$ by $\mathbb{Q}(\sqrt{f_0})$
- $\mathrm{J}(C_f)[2] \simeq \mathrm{J}(C_{f^{\mathrm{mon}}})[2]$, induces $H^1(\mathbb{Q}, \mathrm{J}(C_f)[2]) \simeq H^1(\mathbb{Q}, \mathrm{J}(C_{f^{\mathrm{mon}}})[2])$, which identifies elts of $\mathrm{Sel}_2(\mathrm{J}(C_f))$ with certain 2-covers of $\mathrm{J}(C_{f^{\mathrm{mon}}})$

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- Let $R_f := H^0\big(\operatorname{Proj}\frac{\mathbb{Z}[x,y]}{(f(x,y))}\big)$, $K_f := \operatorname{Frac}(R_f)$, $D_f := (\operatorname{different}(R_f))^{-1}$

**Theorem (Bhargava, Shankar, and S., 2021)**

*Let $f \in \mathbb{Z}[x,y]$ be a binary form of even degree $n \geq 4$, and suppose $C_f$ loc. sol. if $n \equiv 0 \pmod 4$. Elements of $\mathrm{Sel}_2(\mathrm{J}(C_f))$ correspond to certain pairs $(I, \alpha)$, where:*

- *$I$ = ideal class of $R_f$; $\alpha \in K_f^\times / K_f^{\times 2}$*
- *$I^2 \subset \alpha \times D_f$ and $\mathrm{N}(I)^2 = \mathrm{N}(\alpha) \times \mathrm{N}(D_f)$*

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# Parametrization (cont'd.)

**Theorem (S., 2020)**

*Let $f \in \mathbb{Z}[x,y]$ be a binary n-ic form with leading coefficient $f(1,0) = f_0 \neq 0$. Then we have an injection:*

$$\left\{ \begin{array}{c} (I,\alpha) \text{ s.t. } I^2 \subset \alpha \times D_f, \\ \mathrm{N}(I)^2 = \mathrm{N}(\alpha) \times \mathrm{N}(D_f) \end{array} \right\} \hookrightarrow \left\{ \begin{array}{c} (A,B) \in \mathbb{Z}^2 \otimes_{\mathbb{Z}} \mathrm{Sym}^2 \mathbb{Z}^n \text{ s.t.} \\ \det(xA + yB) = f^{mon}(x,y) \end{array} \right\} \Big/ \mathrm{SL}_n^{\pm}(\mathbb{Z})$$

*The image is defined by congruence conditions mod $f_0^{n-1}$.*

- Combining yields parametrization of 2-Selmer elements:

$$\mathrm{Sel}_2(\mathrm{J}(C_f)) \hookrightarrow \left\{ \begin{array}{c} (A,B) \in \mathbb{Z}^2 \otimes_{\mathbb{Z}} \mathrm{Sym}^2 \mathbb{Z}^n \text{ s.t.} \\ \det(xA + yB) = f^{\mathrm{mon}}(x,y) \end{array} \right\} \Big/ (\mathrm{SL}_n/\mu_2)(\mathbb{Z})$$

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# Main result: fixed leading coefficient

- Construction seems leading-coefficient dependent, so natural to apply it to families of binary forms with fixed leading coefficient

**Theorem (Bhargava, Shankar, and S., 2022)**

*Let $n \geq 4$ be even. Consider binary $n$-ic forms $f$ with fixed nonzero leading coefficient such that $C_f$ is loc. sol. if $n \equiv 0 \pmod 4$. When such $f$ are ordered by "height," we have* $\operatorname{Avg} \# \operatorname{Sel}_2(J(C_f)) \leq^* 6$.

- Shows robustness of Poonen–Rains conjecture — average remains 6 even on thin families of curves with fixed leading coefficient
- Previously proven by Shankar and Wang (2018) for leading coefficient 1 (marked rational non-Weierstrass points at $\infty$)
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# Varying the leading coefficient

- Goal: Compute $\mathrm{Avg}\,\#\,\mathrm{Sel}_2(\mathrm{J}(C_f))$ over all $f$ (loc. sol. if $4 \mid n$)
- Naïve approach: Determine asymptotic count of Selmer elements for each fixed $f_0$, and then simply sum over all possible values of $f_0$
- Given $f_0 \in \mathbb{Z} \setminus \{0\}$, let $S_{f_0}(X) := \{f : \mathrm{H}^*(f) < X, f(1,0) = f_0\}$, where

$$\mathrm{H}^*(f) = \max_i \{|f_0^{i-1} f_i|^{1/i}\}$$

Then we have

$$\sum_{f \in S_{f_0}(X)} \#\,\mathrm{Sel}_2(\mathrm{J}(C_f)) \ll f_0^{-\frac{n(n-1)}{2}} X^{\frac{n(n+1)}{2}} + \text{error}$$

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# Varying the leading coefficient (cont'd.)

- Summing bound over $f_0 \asymp X$ and ignoring error term, we find that $\operatorname{Avg} \# \operatorname{Sel}_2(\mathrm{J}(C_f)) \ll$

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# Orbits of rk $\leq 1$ mod $f_0$

- To control error, need to understand image of parametrization better
- Recall that image *a priori* defined by congruence conditions mod $f_0^{n-1}$: $(A, B)$ arises if for each $i \in \{2, \ldots, n-1\}$ certain linear combinations of the $i \times i$ minors of $B$ vanish modulo $f_0^{i-1}$
- Helpful if image is defined mod $f_0$, rather than a higher power
- Say $B \in \mathrm{Sym}^2 \mathbb{Z}^n$ has rk $\leq 1$ mod $f_0$ if $B \propto (\text{linear form})^2 \pmod{f_0}$

**Theorem (Bhargava, Shankar, S., 2021)**

- *If $R_f = \mathcal{O}_{K_f}$ is the maximal order, or if first two coefficients of $f$ are coprime, then $B$ has rk $\leq 1$ mod $f_0$*
- *Let $n = 4$. If $(A, B)$ is not defined mod $f_0$, then $\exists$ $(\mathrm{SL}_4/\mu_2)(\mathbb{Q})$-translate $(A', B') \in \mathbb{Z}^2 \otimes_{\mathbb{Z}} \mathrm{Sym}_2 \mathbb{Z}^4$ such that $i \times i$ minors of $B'$ vanish modulo $f_0^{i-1}$ for each $i \in \{2, 3, 4\}$*

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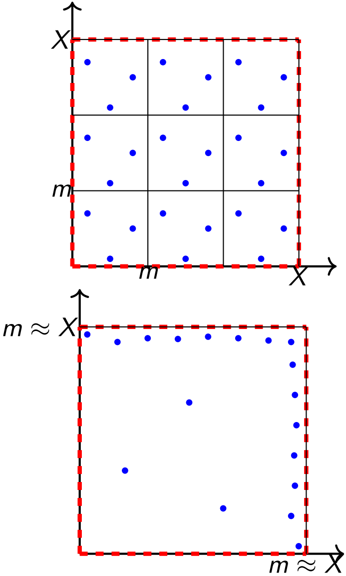


- Want to count lattice pts cut out by congruence conditions mod $m$ in box of sidelength $X$
- If $m/X$ is tiny, Davenport's lemma gives good estimate
- But *a priori*, orbits we want to count are defined by conditions mod $f_0$, and $f_0$ can be as big as $X$
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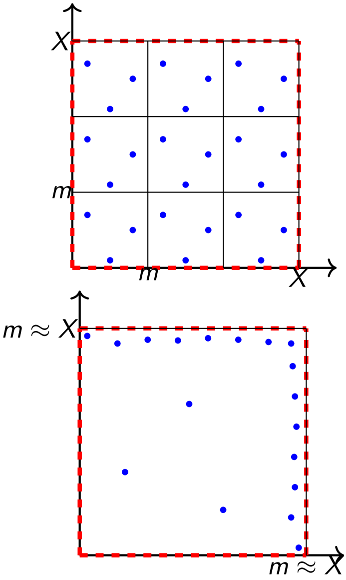


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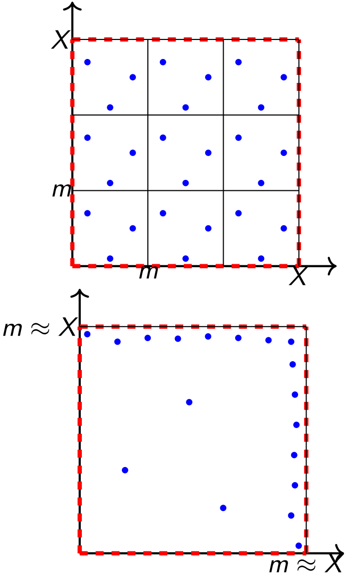


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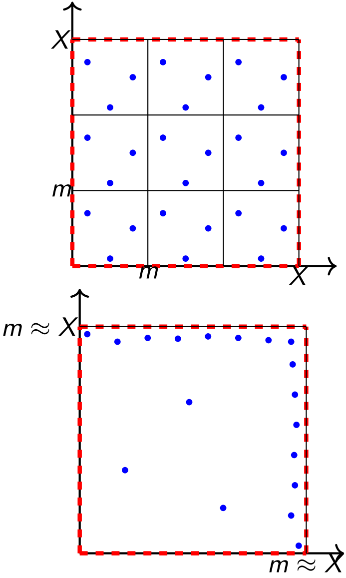


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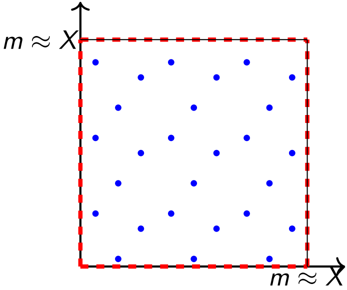


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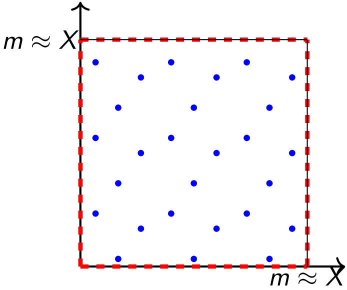


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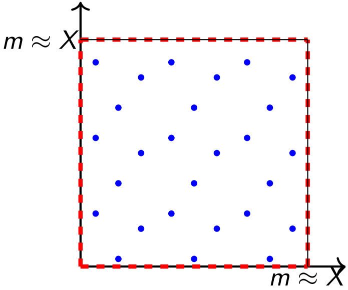


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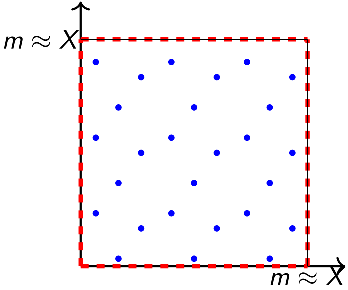


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**Theorem (Bhargava, Shankar, and S., 2021)**

*When binary quartic forms $f$ such that $C_f$ is loc. sol. are ordered by the max norm on their coefficients, we have* $\mathrm{Avg}\,\#\,\mathrm{Sel}_2(\mathrm{J}(C_f)) \leq^{\star} 6$.

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**Thank You!!**