

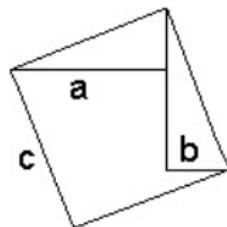
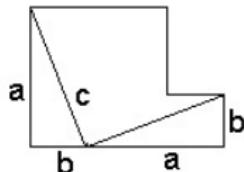
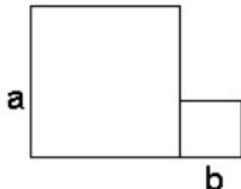
# Beyond Fermat's Last Theorem

David Zureick-Brown

Slides available at <http://dmzb.github.io/>

Amherst College Colloquium  
February 12, 2024

$$a^2 + b^2 = c^2$$



## Basic Problem (Solving Diophantine Equations)

Let  $f_1, \dots, f_m$  be polynomials with integer coefficients, e.g.,

$$x^2 + y^2 + 1$$

$$x^3 - y^2 - 2$$

$$2y^2 + 17x^4 - 1$$

Basic problem: solve polynomial equations

Describe the set

$$V(f_1, \dots, f_m) = \{(a_1, \dots, a_n) \in \mathbb{Z}^n : \forall i, f_i(a_1, \dots, a_n) = 0\},$$

i.e., the set of integer solutions to those polynomials

Fact

*Solving Diophantine equations is hard.*

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# Hilbert's Tenth Problem

Theorem (Davis–Putnam–Robinson 1961, Matijasevič 1970)

*There does not exist an algorithm solving the following problem:*

**input:** integer polynomials  $f_1, \dots, f_m$  in variables  $x_1, \dots, x_n$ ;

**output:** YES / NO according to whether the set of solutions

$$\{(a_1, \dots, a_n) \in \mathbb{Z}^n : \forall i, f_i(a_1, \dots, a_n) = 0\}$$

*is non-empty.*

This is *known* to be true for many other cases (e.g.,  $\mathbb{C}, \mathbb{R}, \mathbb{F}_q, \mathbb{Q}_p, \mathbb{C}(t)$ ).

This is *still unknown* in many other cases (e.g.,  $\mathbb{Q}$ ).

# Fermat's Last Theorem - A Marvelous Proof

## Theorem (Wiles; Taylor)

*For primes  $p \geq 3$  the only integer solutions to the equation*

$$x^p + y^p = z^p$$

*are integer multiples of the triples*

$$(0, 0, 0), \quad (\pm 1, \mp 1, 0), \quad \pm(1, 0, 1), \quad \pm(0, 1, 1).$$

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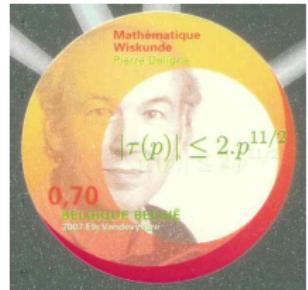
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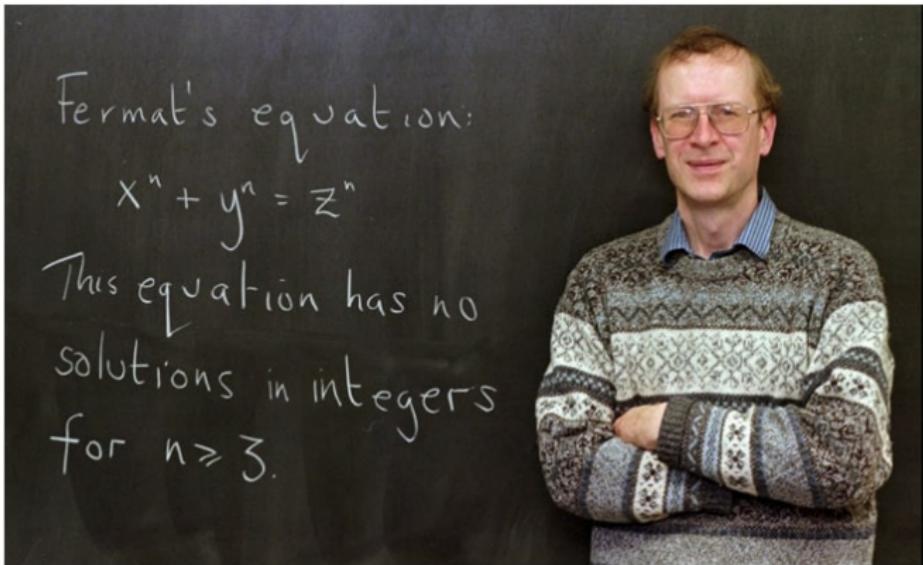
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# Fermat's Last Theorem - aftermath

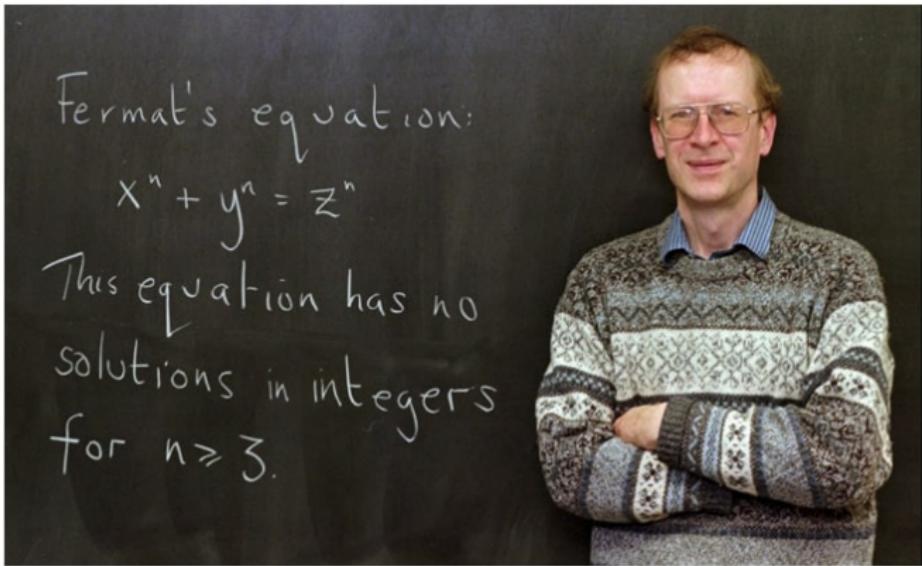


Fermat's equation:

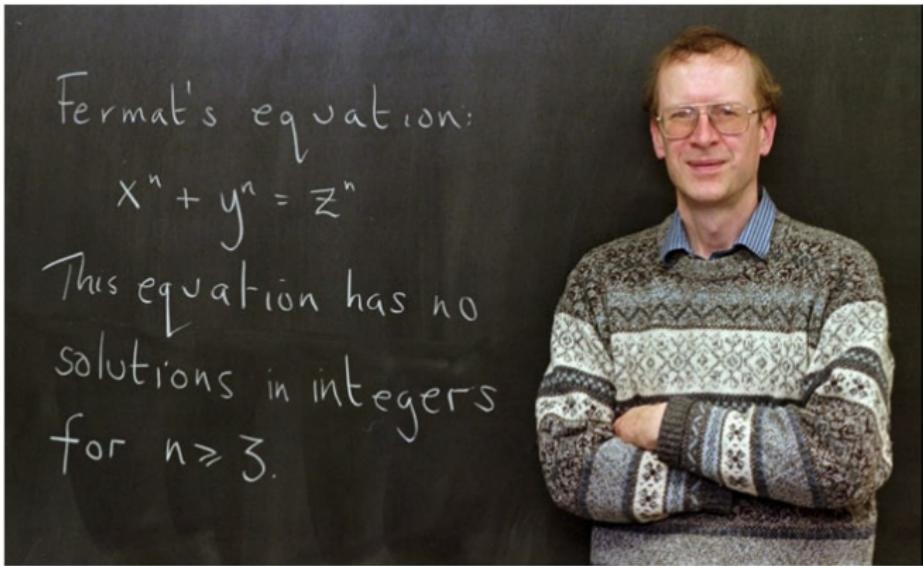
$$x^n + y^n = z^n$$

This equation has no  
solutions in integers  
for  $n \geq 3$ .

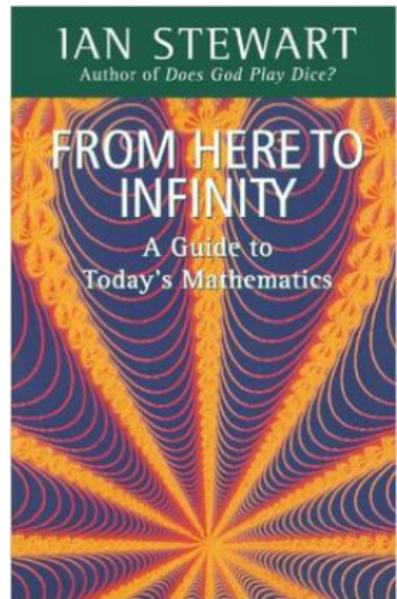
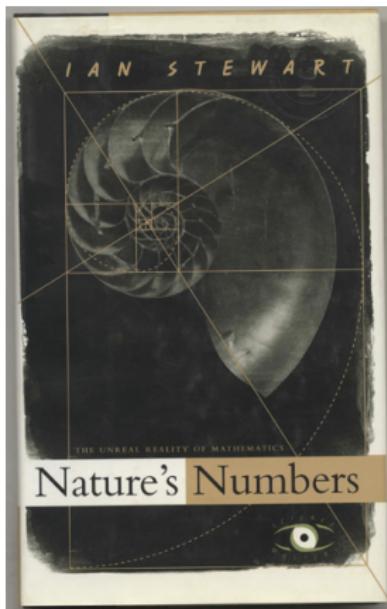
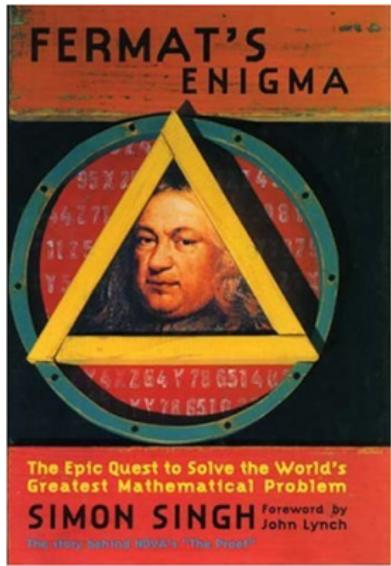
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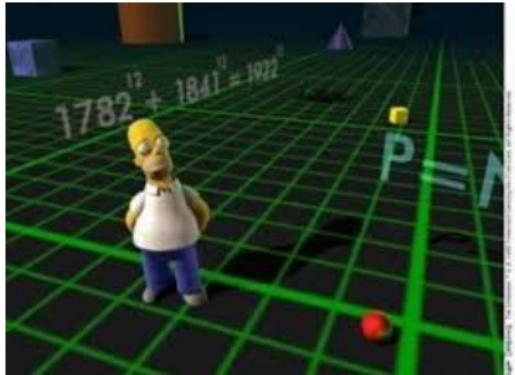
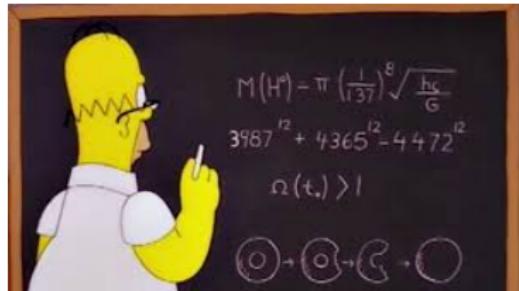
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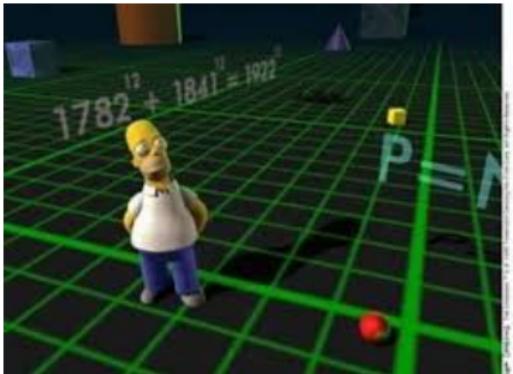
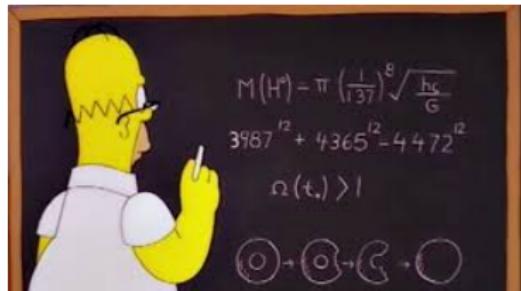
# Books



# Fermat trolling



# Fermat trolling



See <https://youtu.be/ReOQ300AcSU?si=--fAdsdPttt4HR3N>

**Basic Problem:**  $f_1, \dots, f_m \in \mathbb{Z}[x_1, \dots, x_n]$

**Qualitative:**

- ▶ Does there **exist** a solution?
- ▶ Do there exist **infinitely many** solutions?
- ▶ Does the set of solutions have some **extra structure** (e.g., geometric structure, group structure).

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### **Implicit question**

- ▶ Why do equations **have** (or fail to have) solutions?
- ▶ Why do some have **many** and some have **none**?
- ▶ What **underlying mathematical structures** control this?

## Example: Pythagorean triples

$$3^2 + 4^2 = 5^2$$

$$5^2 + 12^2 = 13^2$$

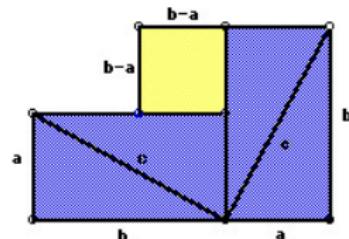
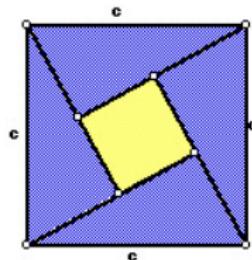
$$7^2 + 24^2 = 25^2$$

### Lemma

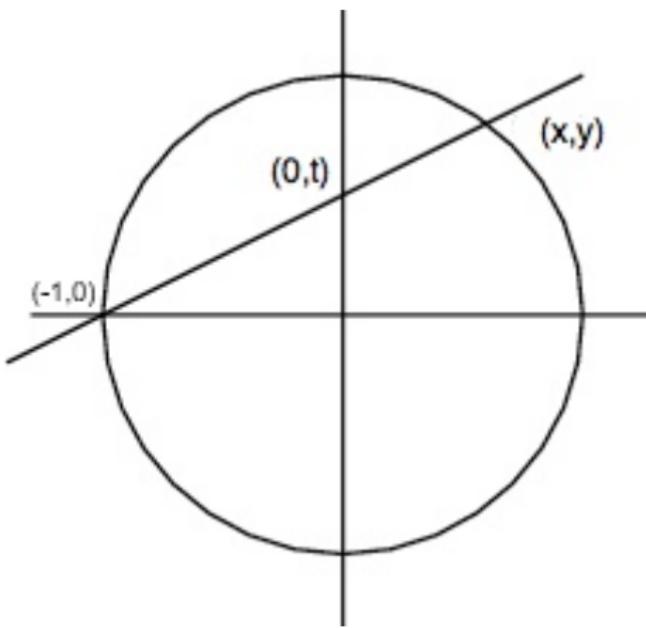
*The equation*

$$x^2 + y^2 = z^2$$

*has infinitely many non-zero coprime solutions.*



## Pythagorean triples



$$\text{Slope} = t = \frac{y}{x+1}$$

$$x = \frac{1-t^2}{1+t^2}$$

$$y = \frac{2t}{1+t^2}$$

# Pythagorean triples

## Lemma

*The solutions to*

$$a^2 + b^2 = c^2$$

*(with  $c \neq 0$ ) are all multiples of the triples*

$$a = 1 - t^2$$

$$b = 2t$$

$$c = 1 + t^2$$

# The Mordell Conjecture

## Example

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For  $n \geq 5$ , the equation

$$y^2 = f(x)$$

has only finitely many solutions if  $f(x)$  is squarefree, with degree  $> 4$ .

# Fermat Curves

## Question

Why is Fermat's last theorem believable?

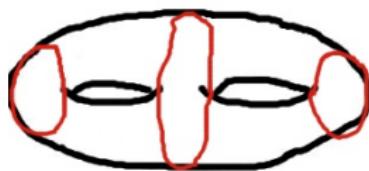
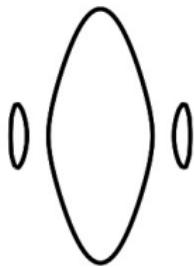
- ①  $x^n + y^n - z^n = 0$  looks like a surface (3 variables)
- ②  $x^n + y^n - 1 = 0$  looks like a curve (2 variables)

# Mordell Conjecture

## Example

$$y^2 = -(x^2 - 1)(x^2 - 2)(x^2 - 3)$$

This is a cross section of a two holed torus.



The **genus** is the number of holes.

Conjecture (Mordell, 1922)

*A curve of genus  $g \geq 2$  has only finitely many rational solutions.*

# Fermat Curves

## Question

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- ①  $x^n + y^n - z^n = 0$  looks like a surface (3 variables)
- ②  $x^n + y^n - 1 = 0$  looks like a curve (2 variables)
- ③ and has genus

$$(n - 1)(n - 2)/2$$

which is  $\geq 2$  iff  $n \geq 4$ .

# Fermat Curves

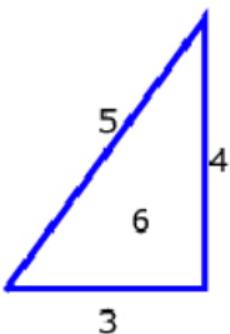
## Question

What if  $n = 3$ ?

- ①  $x^3 + y^3 - 1 = 0$  is a curve of genus  $(3 - 1)(3 - 2)/2 = 1$ .
- ② We were lucky;  $Ax^3 + By^3 = Cz^3$  can have infinitely many solutions.

## Congruent number problem

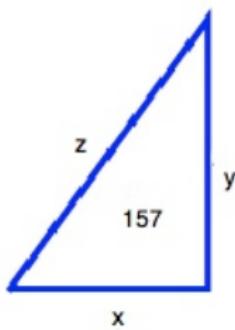
$$x^2 + y^2 = z^2, xy = 2 \cdot 6$$



$$3^2 + 4^2 = 5^2, \quad 3 \cdot 4 = 2 \cdot 6$$

# Congruent number problem

$$x^2 + y^2 = z^2, xy = 2 \cdot 157$$



## Assume the Birch–Swinnerton-Dyer conjectures

If you assume \$1,000,000 worth of conjectures, then the equations

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The denominator of  $z$  has **44 digits**!

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“Next” soluton has **176 digits**!

## Back of the envelope calculation (as of 2011)

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- Num, den( $x, y, z$ )  $\leq 10 \sim 10^6$  many, **1 min** on Emory's computers.

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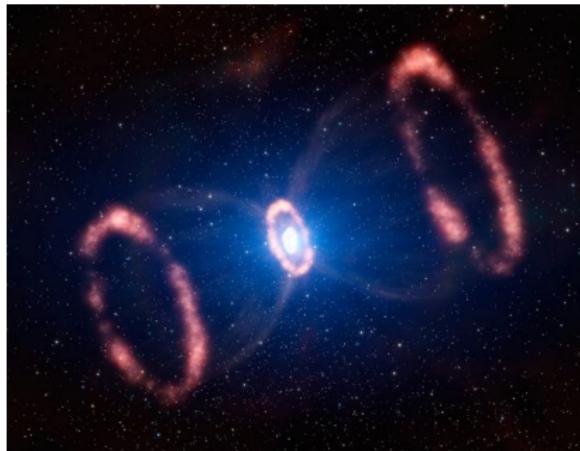
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- Expected time until ‘heat death’ of universe –  **$10^{100}$  years**.



# Fermat Surfaces

## Conjecture

*The only solutions to the equation*

$$x^n + y^n = z^n + w^n, n \geq 5$$

*satisfy  $xyzw = 0$  or lie on the lines ‘lines’  $x = z$ ,  $y = w$  (and permutations).*

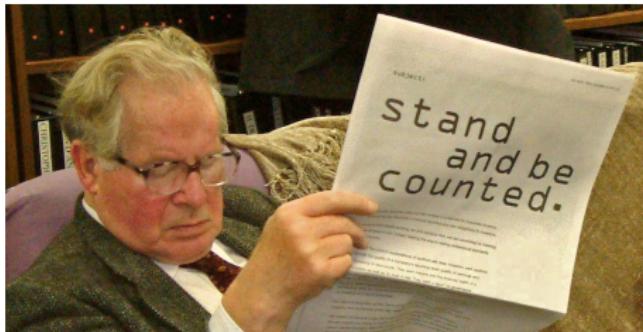
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- Two ‘obvious’ solutions –  $(\pm 1 : 0 : 0)$ .
- The next smallest solutions are  $(\pm \frac{1484801}{1169407}, \pm \frac{1203120}{1169407}, \pm \frac{1157520}{1169407})$ .

## Problem

*Find another solution. (Probably impossible.)*

## Back of envelope calculation

- ➊  **$10^{16}$  years** to find via brute force.
- ➋ Age of the universe –  **$13.75 \pm .11$  billion years** (roughly  $10^{10}$ ).

## Sums of cubes

$$1 = 1^3 + 0^3 + 0^3$$

$$2 = 1^3 + 1^3 + 0^3$$

$$3 = 1^3 + 1^3 + 1^3$$

$$3 = 4^3 + 4^3 + (-5)^3$$

$$4 \neq x^3 + y^3 + z^3$$

$$5 \neq x^3 + y^3 + z^3$$

$$6 = 1^3 + 1^3 + 2^3$$

Conjecture (Heath-Brown)

*The equation*

$$x^3 + y^3 + z^3 = n$$

*has an integer solution if and only if  $n$  is not 4 or 5 mod 9.*

# Solved by Booker–Sutherland

$$32 \neq x^3 + y^3 + z^3$$

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$$3 = 569936821221962380720^3 + (-569936821113563493509)^3 + (-472715493453327032)^3$$



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$$42 = (-80538738812075974)^3 + 80435758145817515^3 + 12602123297335631^3$$

$$3 = 569936821221962380720^3 + (-569936821113563493509)^3 + (-472715493453327032)^3$$

$$114 = x^3 + y^3 + z^3?$$



<https://www.quantamagazine.org/why-the-sum-of-three-cubes-is-a-hard-math-problem-20191105/>

## “Generalized” Fermat equations

Theorem (Poonen, Schaefer, Stoll)

*The coprime integer solutions to  $x^2 + y^3 = z^7$  are the 16 triples*

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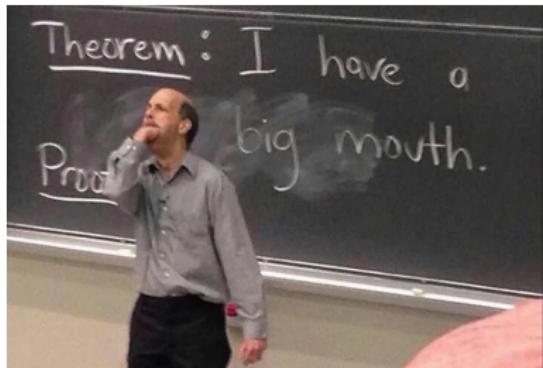
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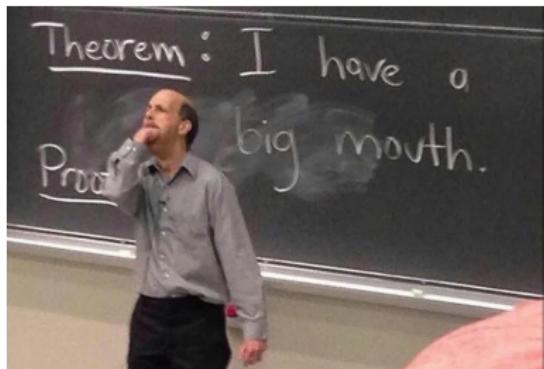


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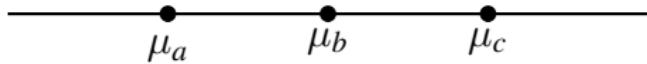
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## Theorem (Darmon and Granville)

*Fix  $a, b, c \geq 2$ . Then the equation  $x^a + y^b = z^c$  has only finitely many coprime integer solutions iff  $\chi = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1 \leq 0$ .*



Known Solutions to  $x^a + y^b = z^c$  with  $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$

$$1^p + 2^3 = 3^2, \quad 2^5 + 7^2 = 3^4$$

$$7^3 + 13^2 = 2^9, \quad 2^7 + 17^3 = 71^2$$

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Problem (Beal's conjecture)

*These are all solutions with  $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1 < 0$ .*

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Conjecture (Beal, Granville, Tijdeman–Zagier)

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Theorem (Darmon, Merel)

*Any pairwise coprime solution to the equation*

$$x^n + y^n = z^2, n > 4$$

*satisfies*  $xyz = 0$ .

$$\frac{1}{n} + \frac{1}{n} + \frac{1}{2} - 1 = \frac{2}{n} - \frac{1}{2} < \frac{2}{4} - \frac{1}{2} = 0$$

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1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...

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Theorem (Silliman–Vogt; 2013 REU)

*0 and 1 are the only perfect powers in the Lucas sequence*

$$L_1 = 0, L_2 = 1, \quad L_n = 3L_{n-1} - 2L_{n-2}.$$

$$0, \textcolor{red}{1}, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, 4095, 8191, \dots, 2^n - 1, \dots$$

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$$(T/2)^2 + H^3 + (f/12^3)^5$$

- ①  $f = st(t^{10} - 11t^5s^5 - s^{10})$ ,
- ②  $H = \text{Hessian of } f$ ,
- ③  $T = \text{a degree 3 covariant of the dodecahedron}$ .

$(a, b, c)$  such that  $\chi < 0$  and the solutions to  $x^a + y^b = z^c$  have been determined.

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$\{2, n, n\}$	Darmon–Merel, others for small $n$
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$(2, n, 4)$	Bennett–Skinner; $n \geq 4$
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$(2, 3, 10)$	<b>ZB</b>