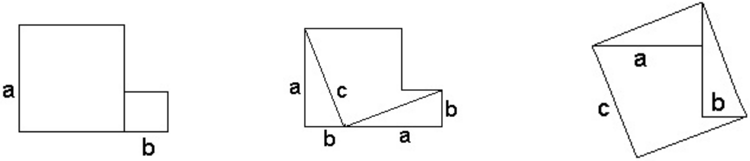
# Beyond Fermat's Last Theorem

David Zureick-Brown

Emory University

Georgia state math and stat club

$$\boxed{a^2 + b^2 = c^2}$$



# Basic Problem (Solving Diophantine Equations)

## Analysis

Let $f_1, \ldots, f_m \in \mathbb{Z}[x_1, ..., x_n]$ be polynomials.

## Problem

*Describe the set*

$$\left\{(a_1, \ldots, a_n) \in \mathbb{Z}^n : \forall i, f_i(a_1, \ldots, a_n) = 0\right\}.$$

## Example

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$$x_1^2 + x_2^2 = -1 \text{ has no integer solutions}$$

## Fact

*Solving diophantine equations is hard.*

# Hilbert's Tenth Problem

In fact, it is provably hard.

**Theorem (Davis-Putnam-Robinson 1961, Matijasevič 1970)**

*There does not exist an algorithm solving the following problem:*

**input**: $f_1, \ldots, f_m \in \mathbb{Z}[x_1, ..., x_n]$;

**output**: YES / NO *according to whether the set*

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*is non-empty.*

This is *still open* for many other rings (e.g., $R = \mathbb{Q}$).

**Theorem (Wiles et. al)**

*The only solutions to the equation*

$$x^n + y^n = z^n, n \geq 3$$

*are multiples of the triples*

$$(0,0,0), \quad (\pm 1, \mp 1, 0), \quad \pm(1,0,1), \quad (0, \pm 1, \pm 1).$$

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**Qualitative**:

- Does there exist a solution?
- Do there exist infinitely many solutions?
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**Implicit question**

- Why do equations have (or fail to have) solutions?
- Why do some have many and some have none?
- What underlying mathematical structures control this?

# Example: Pythagorean triples

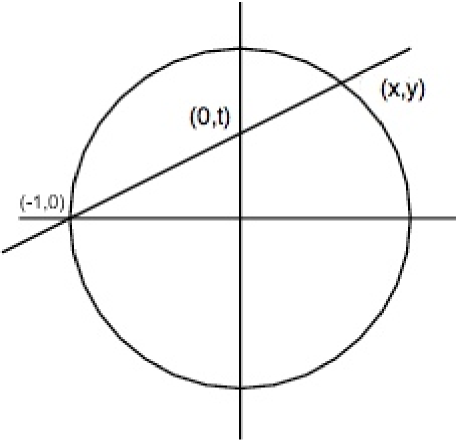
**Lemma**

*The equation*

$$x^2 + y^2 = z^2$$

*has infinitely many non-zero coprime solutions.*

# Pythagorean triples




$$\text{Slope} = \quad t = \frac{y}{x+1}$$

$$x = \frac{1-t^2}{1+t^2}$$

$$y = \frac{2t}{1+t^2}$$

# Pythagorean triples

## Lemma

*The solutions to*

$$a^2 + b^2 = c^2$$

*are all multiples of the triples*

$$\boxed{a = 1 - t^2} \; \boxed{b = 2t} \; \boxed{c = 1 + t^2}$$

# The Mordell Conjecture

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*For $n \geq 5$, the equation*

$$y^2 = f(x)$$

*has only finitely many solutions if $f(x)$ is squarefree, with degree $> 4$.*

# Fermat Curves

**Question**

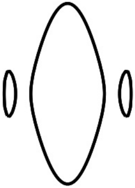
Why is Fermat's last theorem believable?

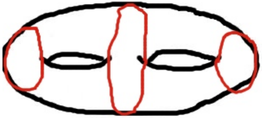
1. $x^n + y^n - z^n = 0$ looks like a surface (3 variables)
2. $x^n + y^n - 1 = 0$ looks like a curve (2 variables)

## Mordell Conjecture

### Example

$$y^2 = (x^2 - 1)(x^2 - 2)(x^2 - 3)$$





This is a cross section of a two holed torus. The **genus** is the number of holes.

### Conjecture (Mordell)

A curve of genus $g \geq 2$ has only finitely many rational solutions.

# Fermat Curves

**Question**

Why is Fermat's last theorem believable?

1. $x^n + y^n - 1 = 0$ is a curve of genus $(n-1)(n-2)/2$.
2. Mordell implies that for **fixed** $n > 3$, the $n$th Fermat equation has only finitely many solutions.

# Fermat Curves

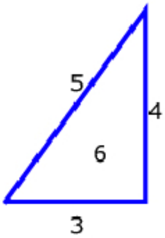
**Question**

What if $n = 3$?

1. $x^3 + y^3 - 1 = 0$ is a curve of genus $(3-1)(3-2)/2 = 1$.
2. We were lucky; $Ax^3 + By^3 = Cz^3$ can have infinitely many solutions.

# Congruent number problem

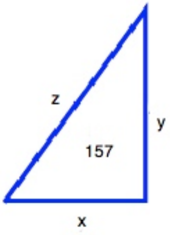
$$\boxed{x^2 + y^2 = z^2,\ xy = 2 \cdot 6}$$




$$3^2 + 4^2 = 5^2, \quad 3 \cdot 4 = 2 \cdot 6$$

# Congruent number problem

$$\boxed{x^2 + y^2 = z^2,\ xy = 2 \cdot 157}$$



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The pair of equations

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$$x = \frac{157841 \cdot 4947203 \cdot 52677109576}{2 \cdot 3^2 \cdot 5 \cdot 13 \cdot 17 \cdot 37 \cdot 101 \cdot 17401 \cdot 46997 \cdot 356441}$$

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The denominator of $z$ has **44 digits**!

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"Next" soluton has **176 digits**!

# Back of the envelope calculation

$$\boxed{x^2 + y^2 = z^2,\ xy = 2 \cdot 157}$$

- Num, den$(x, y, z) \leq 10 \sim 10^6$ many, **1 min** on Emory's computers.

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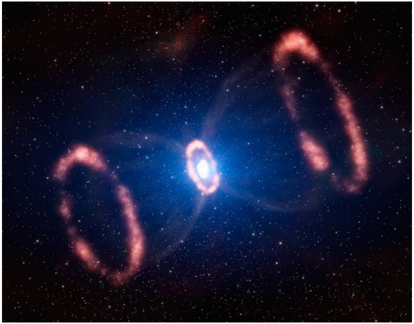
- $10^9$ many computers in the world – so $\mathbf{10^{243}}$ **years**
- Expected time of 'heat death' of universe – $\mathbf{10^{100}}$ **years**.



# Fermat Surfaces

## Conjecture

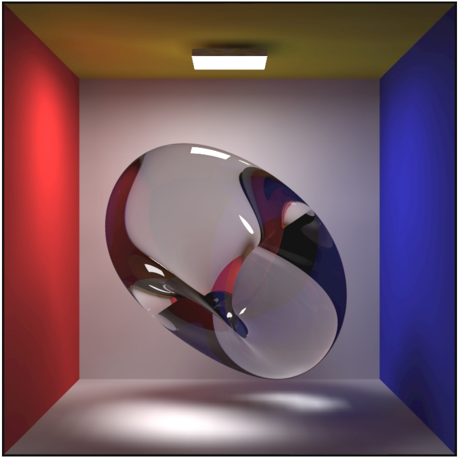
The only solutions to the equation

$$x^n + y^n = z^n + w^n, n \geq 5$$

satisfy $xyzw = 0$ or lie on the lines 'lines' $x = \pm y$, $z = \pm w$ (and permutations).

# The Swinnerton-Dyer K3 surface

$$\boxed{x^4 + 2y^4 = 1 + 4z^4}$$



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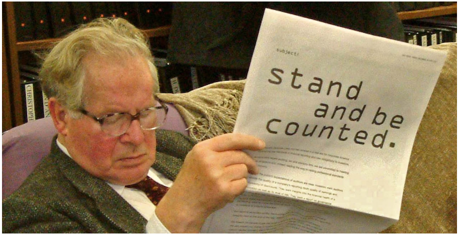


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# The Swinnerton-Dyer K3 surface

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- Two 'obvious' solutions – $(\pm 1 : 0 : 0)$.
- The next smallest solutions are $(\pm \frac{1484801}{1169407}, \pm \frac{1203120}{1169407}, \pm \frac{1157520}{1169407})$.

## Problem

*Find another solution.*

1. $\mathbf{10^{16}}$ **years** to find via brute force.
2. Age of the universe – $\mathbf{13.75} \pm \mathbf{.11}$ **billion years** (roughly $\mathbf{10^{10}}$).

# Fermat-like equations

**Theorem (Poonen, Schaefer, Stoll)**

*The coprime integer solutions to* $x^2 + y^3 = z^7$ *are the* 16 *triples*

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**Problem**

*What are the solutions to the equation* $x^a + y^b = z^c$*?*

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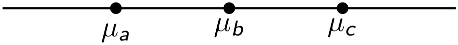
## Theorem (Darmon and Granville)

*Fix $a, b, c \geq 2$. Then the equation $x^a + y^b = z^c$ has only finitely many coprime integer solutions iff $\chi = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1 \leq 0$.*



# Known Solutions to $x^a + y^b = z^c$

The 'known' solutions with

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$$

are the following:

$$1^p + 2^3 = 3^2$$

$$2^5 + 7^2 = 3^4,\ 7^3 + 13^2 = 2^9,\ 2^7 + 17^3 = 71^2,\ 3^5 + 11^4 = 122^2$$

$$17^7 + 76271^3 = 21063928^2,\ 1414^3 + 2213459^2 = 65^7$$

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**Problem (Beal's conjecture)**

*These are all solutions with* $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1 < 0$.

# Generalized Fermat Equations – Known Solutions

**Conjecture (Beal, Granville, Tijdeman-Zagier)**

This is a complete list of coprime non-zero solutions such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1 < 0$.

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...or even for a counterexample.

# Examples of Generalized Fermat Equations

**Theorem (Poonen, Schaefer, Stoll)**

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$$\frac{1}{2} + \frac{1}{3} + \frac{1}{6} - 1 = 0$$

# Examples of Generalized Fermat Equations

**Theorem (Darmon, Merel)**

*Any pairwise coprime solution to the equation*

$$x^n + y^n = z^2, n > 4$$

*satisfies* $xyz = 0$.

$$\frac{1}{n} + \frac{1}{n} + \frac{1}{2} - 1 = \frac{2}{n} - \frac{1}{2} < 0$$

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**Theorem (Bugeaud, Mignotte, Siksek 2006)**

*The only Fibonacci numbers that are perfect powers are*

$$F_0 = 0,\ F_1 = F_2 = 1,\ F_6 = 8,\ F_{12} = 144.$$

# Examples of Generalized Fermat Equations

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$$(T/2)^2 + H^3 + (f/12^3)^5$$

1. $f = st(t^{10} - 11t^5s^5 - s^{10})$,
2. $H =$ Hessian of $f$,
3. $T =$ a degree 3 covariant of the dodecahedron.

# $(p, q, r)$ such that $\chi < 0$ and the solutions to $x^p + y^q = z^r$ have been determined.

| | |
|---|---|
| $\{n, n, n\}$ | Wiles,Taylor-Wiles, building on work of many others |
| $\{2, n, n\}$ | Darmon-Merel, others for small $n$ |
| $\{3, n, n\}$ | Darmon-Merel, others for small $n$ |
| $\{5, 2n, 2n\}$ | Bennett |
| $(2, 4, n)$ | Ellenberg, Bruin, Ghioca $n \geq 4$ |
| $(2, n, 4)$ | Bennett-Skinner; $n \geq 4$ |
| $\{2, 3, n\}$ | Poonen-Shaefer-Stoll, Bruin. $6 \leq n \leq 9$ |
| $\{2, 2\ell, 3\}$ | Chen, Dahmen, Siksek; primes $7 < \ell < 1000$ with $\ell \neq 31$ |
| $\{3, 3, n\}$ | Bruin; $n = 4, 5$ |
| $\{3, 3, \ell\}$ | Kraus; primes $17 \leq \ell \leq 10000$ |
| $(2, 2n, 5)$ | Chen $n \geq 3^*$ |
| $(4, 2n, 3)$ | Bennett-Chen $n \geq 3$ |
| $(6, 2n, 2)$ | Bennett-Chen $n \geq 3$ |
| $(2, 6, n)$ | Bennett-Chen $n \geq 3$ |

## $(p, q, r)$ such that $\chi < 0$ and the solutions to $x^p + y^q = z^r$ have been determined.

| | |
|---|---|
| $\{n, n, n\}$ | Wiles,Taylor-Wiles, building on work of many others |
| $\{2, n, n\}$ | Darmon-Merel, others for small $n$ |
| $\{3, n, n\}$ | Darmon-Merel, others for small $n$ |
| $\{5, 2n, 2n\}$ | Bennett |
| $(2, 4, n)$ | Ellenberg, Bruin, Ghioca $n \geq 4$ |
| $(2, n, 4)$ | Bennett-Skinner; $n \geq 4$ |
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| $(2, 2n, 5)$ | Chen $n \geq 3^*$ |
| $(4, 2n, 3)$ | Bennett-Chen $n \geq 3$ |
| $(6, 2n, 2)$ | Bennett-Chen $n \geq 3$ |
| $(2, 6, n)$ | Bennett-Chen $n \geq 3$ |
| $(2, 3, 10)$ | **ZB** |

# Faltings' theorem / Mordell's conjecture

**Theorem (Faltings, Vojta, Bombieri)**

*Let $X$ be a smooth curve over $\mathbb{Q}$ with genus at least 2. Then $X(\mathbb{Q})$ is finite.*

**Example**

For $g \geq 2$, $y^2 = x^{2g+1} + 1$ has only finitely many solutions with $x, y \in \mathbb{Q}$.

# Uniformity

## Problem

1. *Given $X$, compute $X(\mathbb{Q})$ exactly.*
2. *Compute bounds on $\#X(\mathbb{Q})$.*

## Conjecture (Uniformity)

There exists a constant $N(g)$ such that every smooth curve of genus $g$ over $\mathbb{Q}$ has at most $N(g)$ rational points.

## Theorem (Caporaso, Harris, Mazur)

*Lang's conjecture $\Rightarrow$ uniformity.*

# Uniformity numerics

| $g$ | 2 | 3 | 4 | 5 | 10 | 45 | $g$ |
|---|---|---|---|---|---|---|---|
| $B_g(\mathbb{Q})$ | 642 | 112 | 126 | 132 | 192 | 781 | $16(g+1)$ |

Elkies studied K3 surfaces of the form

$$y^2 = S(t, u, v)$$

with lots of rational lines, such that S restricted to such a line is a perfect square.

# Main Theorem (partial uniformity for curves)

**Theorem (Katz, Rabinoff, ZB)**

*Let $X$ be any curve of genus $g$ and let $r = \operatorname{rank}_{\mathbb{Z}} \operatorname{Jac}_X(\mathbb{Q})$. Suppose $r \leq g - 2$. Then*

$$\#X(\mathbb{Q}) \leq 84g^2 - 123g + 48$$