

Sporadic torsion

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Theorem (Mazur, 1978)

Let E/\mathbb{Q} be an elliptic curve. Then $E(\mathbb{Q})_{tors}$ is isomorphic to one of the following groups.

$$\mathbb{Z}/N\mathbb{Z}, \quad \text{for } 1 \leq N \leq 10 \text{ or } N = 12,$$

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Then $Y_1(N)(\mathbb{Q}) \neq \emptyset$ and $Y_1(2, 2N)(\mathbb{Q}) \neq \emptyset$ iff N are as above.

Modular curves

Example ($N = 9$)

$E(K) \cong \mathbb{Z}/9\mathbb{Z}$ if and only if there exists $t \in K$ such that E is isomorphic to

$$y^2 + (t - rt + 1)xy + (rt - r^2t)y = x^3 + (rt - r^2t)x^2$$

where r is $t^2 - t + 1$. The torsion point is $(0, 0)$.

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$E(K) \cong \mathbb{Z}/11\mathbb{Z}$ correspond to $a, b \in K$ such that

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Rational Points on $X_1(N)$ and $X_1(2, 2N)$

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So, in a sense, the simplest thing that could happen does happen for these modular curves.

Higher Degree Torsion Points

Theorem (Merel, 1996)

For every integer $d \geq 1$, there is a constant $N(d)$ such that for all K/\mathbb{Q} of degree at most d and all E/K ,

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Problem

Fix $d \geq 1$. Classify all groups which can occur as $E(K)_{\text{tors}}$ for K/\mathbb{Q} of degree d . Which of these occur infinitely often?

The Quadratic Case

Theorem (Kamienny-Kenku-Momose, 1980's)

Let E be an elliptic curve over a quadratic number field K . Then $E(K)_{tors}$ is one of the following groups.

$$\mathbb{Z}/N\mathbb{Z}, \quad \text{for } 1 \leq N \leq 16 \text{ or } N = 18,$$

$$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2N\mathbb{Z}, \quad \text{for } 1 \leq N \leq 6,$$

$$\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3N\mathbb{Z}, \quad \text{for } 1 \leq N \leq 2, \text{ or}$$

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In particular, the corresponding curves $X_1(M, N)$ all have $g \leq 2$, which guarantees that they have infinitely many quadratic points.

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If $Y_1(M, N)(K) \neq \emptyset$, are all of the points coming from the existence of such divisors?

If not, we call these outliers **sporadic** points.

Sporadic Cubic Points

Theorem (Jeon-Kim-Schweizer, 2004)

Let E be an elliptic curve over a cubic number field K . Then the subgroups which arise as $E(K)_{tors}$ infinitely often are exactly the following.

$$\mathbb{Z}/N\mathbb{Z}, \quad \text{for } 1 \leq N \leq 20, N \neq 17, 19, \text{ or}$$

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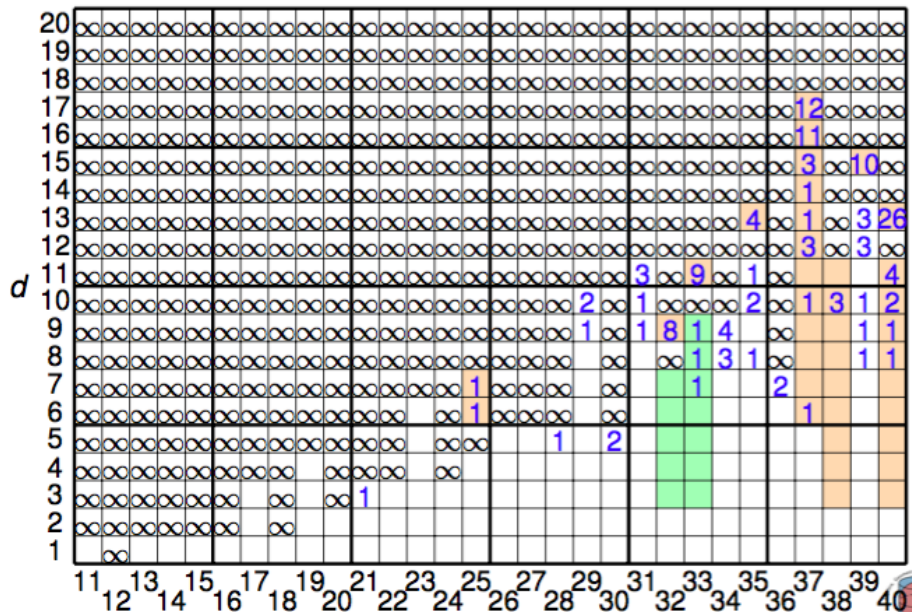
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Theorem (Najman, 2014)

There is an elliptic curve E/\mathbb{Q} whose torsion subgroup over a cubic field is $\mathbb{Z}/21\mathbb{Z}$.

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Classification of Cubic Torsion

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Remark

Parent showed that the largest prime that can divide $E(K)_{\text{tors}}$ in the cubic case is $p = 13$.

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For a finite set S of primes of good reduction, we have the following commutative diagram.

$$\begin{array}{ccc} X^{(d)}(\mathbb{Q}) & \longrightarrow & J(\mathbb{Q}) \\ \downarrow & & \downarrow \alpha \\ \prod_{p \in S} X(\mathbb{F}_q) & \xrightarrow{\beta} & \prod_{p \in S} J(\mathbb{F}_p) \end{array}$$

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We want to choose S so that, once we remove any known rational points, the images of α and β are disjoint.

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