MATH 220, Mathematical Reasoning and Proof
MWF 1 - 1:50

All assignments
Last updated: December 7, 2023
Gradescope code: 7DVWGG

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Assignment 1

Topics: Introduction to the course. Mathematical reasoning. Logic.

Reading: 1.1, 1.2, 1.5, 2.1, 2.2

Suggested problems (do not hand in; these are just for extra practice): Handout 1

Assignment 1, due Friday, Sep 15, 12:55pm via Gradescope:

1. Write the negation of each of the following statements.
   (a) All triangles are isosceles.
   (b) Every door in the building was locked.
   (c) Some even numbers are multiples of three.
   (d) Every real number is less than 100.
   (e) Every integer is positive or negative.
   (f) If f is a polynomial function, then f is continuous at 0.
   (g) If $x^2 > 0$, then $x > 0$.
   (h) There exists a $y \in \mathbf{R}$ such that $xy = 1$.
   (i) $(2 > 1)$ and $(\forall x, x^2 > 0)$
   (j) $\forall \epsilon > 0$, $\exists \delta > 0$ such that if $|x| < \delta$, then $|f(x)| < \epsilon$.

2. Write the converse, contrapositive, and negation of each of the following implications.
   (a) If a quadrilateral is a rectangle, then it has two pairs of parallel sides.
   (b) $(P \wedge \neg Q) \Rightarrow R$
   (c) $P \Rightarrow (R \Rightarrow \forall x, Q(x))$

3. Let P and Q be statements. Write the truth table for
   (a) $(\neg P) \vee Q$
   (b) $(P \wedge (\neg Q)) \Rightarrow Q$

4. Are the statements $(P \vee Q) \wedge R$ and $P \vee (Q \wedge R)$ equivalent? If so, give a proof. If not, explain why by giving a counterexample.

   (Two statement forms are equivalent if they have the same truth tables, and here, a counterexample simply means some choice of truth values for P, Q, and R such that the two statement forms give different outputs.)

5. Let P and Q be statements.
   (a) Prove that $\neg(P \Rightarrow Q)$ is equivalent to $P \wedge \neg Q$.
   (b) Prove that $\neg(P \Rightarrow Q)$ is *not* equivalent to $\neg P \wedge Q$.

(c) Give an example of statements P and Q such that $\neg P \Rightarrow \neg Q$ is true and $\neg(P \Rightarrow Q)$ is false.

6. Suppose that n is an even integer, and let m be any integer. Prove that nm is even.

7. Suppose that n is an odd integer. Prove that n^2 is an odd integer. (Hint: an integer n is odd if and only if there exists an integer k such that $n = 2k + 1$.)

8. Prove that if n^2 is even, then n is even. (Hint: ~~page 67~~ contrapositive.)

Assignment 2

Topics: "Direct" proofs, proof by cases, and divisibility problems.

Reading:

- 3.3, from Definition 3.3.6
- 6.4, just Definition 6.4.1
- see Index to find definitions like prime, etc

Suggested problems (do not hand in; these are just for extra practice)

1. Handout 2

Assignment, due Friday, Sep 22, 12:55pm via Gradescope:

1. Suppose that $a \mid b$. Prove that for all $n \in \mathbb{Z}_{>0}$, $a^n \mid b^n$.
2. Suppose that there exists an integer $n \in \mathbb{Z}_{>0}$ such that $a \mid b^n$. Is it true that $a \mid b$? Prove or disprove your answer. (For a disproof, please give a counterexample that demonstrates that the statement is false.)
3. Prove that for all $a \in \mathbb{Z}$ and for $n \in \mathbb{Z}_{\geq 0}$, $a - 1$ divides $a^n - 1$.
4. Prove that for all integers n, n and $n + 1$ have no common divisors other than ± 1.
5. Prove that if x is an integer, then $x^2 + 2$ is not divisible by 4. (Hint: there are two cases: x is even, x is odd. Also, feel free to use basic facts about even or odd, e.g., "odd + odd = even", without additional proof.)
6. Prove that the product of three consecutive integers is divisible by 6. (It suffices to prove that it is divisible by 2 and 3 separately.)
7. Show that for all integers a and b,
$$a^2b^2(a^2 - b^2)$$
is divisible by 12. (It suffices to prove that it is divisible by 4 and 3 separately.)
8. Find all positive integers n such that $n^2 - 1$ is prime. Prove that your answer is correct.

Assignment 3

Topics: Proof by contradiction. Unsolvability of equations. Irrationality.

Reading: 3.2
Suggested problems (do not hand in; these are just for extra practice)

1. Handout 3

Assignment, due Friday, Sep 29, 12:55pm via Gradescope:

1. Prove that there do not exist integers a, and b such that $21a + 30b = 1$.
2. Prove that $2^{1/3}$ is irrational.
3. Suppose that x is a real number such that $0 \leq x \leq \pi/2$. Prove that $\sin x + \cos x \geq 1$. (Hint: at some point in your proof, use that $(\sin x)^2 + (\cos x)^2 = 1$.)
4. Prove that there are no positive integer solutions to the equation $x^2 - y^2 = 10$.
5. Let a, b, c be integers satisfying $a^2 + b^2 = c^2$. Show that abc must be even.
6. Suppose that a and n are integers that are both at least 2. Prove that if $a^n - 1$ is prime, then $a = 2$ and n is a prime. (Primes of the form $2^n - 1$ are called Mersenne primes.)
7. Suppose that $a, b \in \mathbb{Z}$. Prove that $a^2 - 4b \neq 2$.
8. Prove that $\log_{10} 7$ is irrational

Assignment 4

Topics: Induction.

Reading: Chapter 6

Fun Video (optional): Vi Hart; "Doodling in Math: Spirals, Fibonacci, and Being a Plant"
https://www.youtube.com/watch?v=ahXIMUkSXX0

Suggested problems (do not hand in; these are just for extra practice)

1. Handout 4
2. Handout 5

Assignment, due Friday, Oct 06, 12:55pm via Gradescope:

1. Prove that for every positive integer n,
$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$
2. Let a_n be defined recursively by $a_1 = 1$ and $a_n = \sqrt{1 + a_{n-1}}$. Prove that for all positive integers n, $a_n < 2$.
3. Prove by induction that if $b_1, b_2, \ldots, b_n$ are even integers, then $b_1 + b_2 + \cdots + b_n$ is even.
4. Let $F_1, F_2, F_3, \ldots = 1, 1, 2, 3, 5, 8, \ldots$ be the Fibonacci sequence. Prove that $F_1^2 + \cdots + F_n^2 = F_n F_{n+1}$.
5. Prove that $n! > 2^n$ for all $n \geq 4$.
6. *Bernoulli's inequality*: let $\beta \in \mathbb{R}$ be a real number such that $\beta > -1$ and $\beta \neq 0$. Prove that for all integers $n \geq 2$, $(1+\beta)^n > 1 + n\beta$.
7. Prove that for all integers $n \geq 1$,
$$1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{n}.$$
8. Prove (using induction) that for all integers $n \geq 1$, $2^{2n} - 1$ is divisible by 3.

Assignment 5

Topics: Basics of set theory. Basic operations. Proofs with sets.

Reading: 1.3, 1.4, 2.3
Suggested problems (do not hand in; these are just for extra practice): Handout 6

Assignment 5, due Friday, Oct 13, 12:55pm via Gradescope:

1. Let $A = \{n \in \mathbb{Z} | n \text{ is a multiple of } 4\}$ and $B = \{n \in \mathbb{Z} | n^2 \text{ is a multiple of } 4\}$
   (a) Prove or disprove: $A \subseteq B$.
   (b) Prove or disprove: $B \subseteq A$.
2. Prove that $A \cup (A \cap B) = A$.
3. Let A, B and C be sets.
   (a) Prove that $(A \subseteq C) \wedge (B \subseteq C) \Rightarrow A \cup B \subseteq C$.
   (b) State the contrapositive of part (a).
   (c) State the converse of part (a). Prove or disprove it.
4. Let n and m be integers. Prove that if $n\mathbb{Z} \subseteq m\mathbb{Z}$ then m divides n.

Midterm study guide

Topics: Friday, October 13 will be an in class Exam.

Content: The questions will all be either

1. homework problems,
2. suggested problems,
3. problems we worked in class, or
4. minor variations of one of these.

Problems with very long proofs or that involved some unusual trick will not be on the exam.

You are allowed to use any previous problem from class or from the homework (e.g., “additivity of divisibility” or “the 2 out of 3 rule”) on the exam without reproving it, unless otherwise noted on the exam. (E.g., if I ask you to prove “additivity of divisibility” on the exam, you will need to prove this using only the definition of divisibility, and I will remind you of this in the statement of the problem.)

A typical exam will have one or two questions from each week of the course. You can expect problems about following:

- Negations
- Give definitions (e.g., divides, rational, subset)
- Direct proofs
- Proof by contrapositive
- Divisibility problems
- Contradiction
- Induction
- Proofs with sets.

There will be a negation problem, at least one definition, and around 4-5 problems involving proofs (possibly including “prove or disprove” problems).

For sets: I will ask one problem, verbatim, from the homework or from class.

For definitions, I want a definition, in prose (complete sentences), and I want “just” the definition, and not any additional facts about the definition. (E.g., if you give the definition of rational, do not include that a rational number can be written in reduced form; that is a fact about rational numbers not part of the definition of rational.)

Assignment 6

Topics: More proofs with sets. DeMorgan's laws. Cartesian Products. Power sets

Suggested problems (do not hand in; these are just for extra practice)

1. Handout 7

Assignment 6, due Friday, Oct 20, 12:55pm via Gradescope:

1. Recall that $(a, b) = \{x : x \in \mathbb{R} \,|\, a < x < b\}$. Prove or disprove each of the following:
   (a) $(-1, 1) \subseteq (-2, 2)$.
   (b) $(-1, 2) \subseteq (-2, 1)$.
2. Let A, B be sets. Prove each of the following:
   (a) $A \cap B \subseteq A$;
   (b) $A \cap \emptyset = \emptyset$;
3. Prove that if $A \not\subseteq C$ then $A \not\subseteq B$ or $B \not\subseteq C$.
4. Let A, B, and C be sets. Prove or disprove the following. (For a disproof, please give an explicit counterexample; i.e., give an example of sets A, B and C demonstrating that the statement is false.)
   (a) If $A \not\subseteq B$ and $B \not\subseteq C$, then $A \not\subseteq C$.
   (b) If $A \subseteq B$ and $B \not\subseteq C$, then $A \not\subseteq C$.
5. Let A, B, C be sets. Prove each of the following:
   (a) Suppose that $B \subseteq C$. Prove that $A - C \subseteq A - B$.
   (b) $A \subseteq B$ if and only if $A \cap B = A$.
6. Let A, B and C be sets. Prove or disprove each of the following. (For a disproof, please give an explicit counterexample; i.e., give an example of sets A, B and C demonstrating that the statement is false.)
   (a) $(A \cap B) \cup C = A \cap (B \cup C)$.
   (b) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.
7. Let A and B be sets. Prove that $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$.

8. Let $A = \{0, 1, 2\}$. Which of the following statements are true? (No justification is needed.)

(a) $\{0\} \subseteq P(A)$;

(b) $\{1, 2\} \in P(A)$;

(c) $\{1, \{1\}\} \subseteq P(A)$.

(d) $\{\{0, 1\}, \{1\}\} \subseteq P(A)$;

(e) $\emptyset \in P(A)$;

(f) $\emptyset \subseteq P(A)$;

(g) $\{\emptyset\} \in P(A)$.

(h) $\{\emptyset\} \subseteq P(A)$;

Assignment 7

Topics: Introduction to functions; images and surjectivity

Reading: Chapter 5, especially 5.1 and 5.5

Suggested problems (do not hand in; these are just for extra practice) Handout 9

Assignment 7, due Friday, Oct 27, 12:55pm via Gradescope:

1. Let A and B be sets. Prove that $(A \cup B) \cap \overline{A} = B - A$.
2. Let A and B be sets. Prove that if $A \subseteq B$, then $P(A) \subseteq P(B)$. State the converse of this and prove or disprove it.
3. Let $f\colon \mathbf{R} \to \mathbf{R}$ be the function defined by $f(x) = 6x + 5$.
   (a) Prove that $f(\mathbf{R}) = \mathbf{R}$.
   (b) Compute $f([1, 4])$. Prove your answer.
4. Let $f\colon \mathbf{R} \to \mathbf{R}$ be the function defined by $x^4 + x^2$.
   (a) Compute the image of f. Prove that your answer is correct.
   (b) Compute $f([-1, 2])$. Prove that your answer is correct.
5. Let $g\colon \mathbb{R} \to \mathbb{Z}$ be the **ceiling function** $g(x) = \lceil x \rceil$, defined to be the smallest integer greater than or equal to x (i.e., "round x up to the nearest integer"; so $g(1.3) = 2$, and $g(3) = 3$.
   Compute the image of g. Prove that your answer is correct.
6. Consider the function $\sin\colon \mathbf{R} \to \mathbf{R}$.
   (a) Compute the image of sin. Prove that your answer is correct.
   (b) Compute $\sin([0, \pi/4])$. Prove that your answer is correct.
7. Let A and B be sets and let X and Y be subsets of A. Let $f\colon A \to B$ be a function. Prove or disprove each of the following. When giving a disproof, please give a counterexample.
   (a) $f(X \cup Y) \subseteq f(X) \cup f(Y)$.
   (b) $f(X \cup Y) \supseteq f(X) \cup f(Y)$.
8. Let A and B be sets and let X and Y be subsets of A. Let $f\colon A \to B$ be a function. Prove or disprove each of the following. When giving a disproof, please give an counterexample.
   (a) $f(X) - f(Y) \subseteq f(X - Y)$.
   (b) $f(X) - f(Y) \supseteq f(X - Y)$.

Assignment 8

Topics: Inverse Image (or "Preimage").

Reading: Chapter 5, especially 5.1 and 5.5

Suggested problems (do not hand in; these are just for extra practice) Handout 10

Assignment 8, due Friday, Nov 03, 12:55pm via Gradescope:

(REMINDER: you are allowed to use the results of previous problem as part of the proof of later problems.)

1. Let $f\colon \mathbf{R} \to \mathbf{R}$ be the function defined by $f(x) = 3x + 1$.
   (a) Compute $f^{-1}(\{1, 5, 8\})$ (do not give a proof).
   (b) Compute $f^{-1}(W)$, where $W = (4, \infty)$, and give a proof that your answer is correct.
   (c) Compute $f^{-1}(\mathbf{E})$, where $\mathbf{E}$ is the set of even integers, and give a proof that your answer is correct.

2. Let $f\colon \mathbf{Z} \to \mathbf{Z}$ be the function defined by $f(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ 2n+4, & \text{if } n \text{ is odd.} \end{cases}$

   Compute $f^{-1}(\mathbf{E})$. Prove that your answer is correct. (Reminder: $\mathbf{E}$ is the set of even integers.)

3. Let A and B be sets and let X and Y be subsets of B. Let $f\colon A \to B$ be a function. Prove or disprove the following. (For a disproof, please give an explicit counterexample.)
   (a) $f^{-1}(X \cap Y) \subseteq f^{-1}(X) \cap f^{-1}(Y)$.
   (b) $f^{-1}(X \cap Y) \supseteq f^{-1}(X) \cap f^{-1}(Y)$.

4. Let A and B be sets and let X be a subset of B. Let $f\colon A \to B$ be a function. Prove or disprove the following. (For a disproof, please give an explicit counterexample.)
   (a) $X \subseteq f(f^{-1}(X))$.
   (b) $X \supseteq f(f^{-1}(X))$.

5. Let A and B be sets. Let $S \subseteq A$ and let $T \subseteq B$. Let $f\colon A \to B$ be a function. Prove or disprove the following. (For a disproof, please give an explicit counterexample.)
   (a) $f(S) \subseteq T \Rightarrow S \subseteq f^{-1}(T)$.
   (b) $S \subseteq f^{-1}(T) \Rightarrow f(S) \subseteq T$.

In each of the following problems, let $f\colon A \to A$ be a function (note that the domain and codomain are the same) and suppose that $C \subseteq A$.

6. Prove or disprove the following. (For a disproof, please give an explicit counterexample).

   (a) $f^{-1}(C) \subseteq C$;

   (b) $C \subseteq f^{-1}(C)$;

   (c) $f(C) \subseteq C$;

   (d) $C \subseteq f(C)$;

7. Prove or disprove the following (for a disproof, please give an explicit counterexample): $C \subseteq f^{-1}(C) \iff f(C) \subseteq C$.

8. Prove or disprove the following (for a disproof, please give an explicit counterexample): $f^{-1}(C) \subseteq C \iff C \subseteq f(C)$.

Assignment 9

Topics: Injectivity.

Reading: Section 5.2

Suggested problems (do not hand in; these are just for extra practice): Handout 11

Assignment 9, due Friday, Nov 10, 12:55pm via Gradescope:

1. Let A and B be sets and let X and Y be subsets of B. Let $f\colon A \to B$ be a function. Prove or disprove the following. (For a disproof, please give an explicit counterexample.)
   (a) $f^{-1}(X - Y) \subseteq f^{-1}(X) - f^{-1}(Y)$.
   (b) $f^{-1}(X - Y) \supseteq f^{-1}(X) - f^{-1}(Y)$.
2. Let $f\colon A \to B$ be a function. Which of the followings statements are equivalent to the statement 'f is injective'? (No proof necessary.)
   (a) $f(a) = f(b)$ if $a = b$;
   (b) $f(a) = f(b)$ and $a = b$ for all $a, b \in A$;
   (c) If a and b are in A and $f(a) = f(b)$, then $a = b$;
   (d) If a and b are in A and $a = b$, then $f(a) = f(b)$;
   (e) If a and b are in A and $f(a) \neq f(b)$, then $a \neq b$;
   (f) If a and b are in A and $a \neq b$, then $f(a) \neq f(b)$.
3. Prove that the following functions are not injective.
   (a) $f\colon \mathbf{R} \to \mathbf{R}$, $f(x) = x^4 + x^2$;
   (b) $f\colon \mathbf{R} \to \mathbf{R}$, $f(x) = x^3 + x^2$;
   (c) $f\colon P(\mathbf{Z}) \to P(\mathbf{Z})$; $f(S) = S \cap \{1, 2\}$.
4. Prove that the following functions are injective.
   (a) $f\colon \mathbf{R}^2 \to \mathbf{R}^3$; $f(x, y) = (x + y, x - y, x^2 + y^2)$.
   (b) $f\colon \mathbf{R} \to \mathbf{R}$; $f(x) = e^{x+1}$.
   (c) $f\colon \mathbf{R} \to \mathbf{R}$, $f(x) = \begin{cases} -x - 1, & \text{if } x > 0 \\ x^2, & \text{if } x \leq 0. \end{cases}$
5. Let $f\colon \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4\}$ be a functon. Can f be injective? Explain your answer.
6. We say that a function $f\colon [a, b] \to \mathbf{R}$ is **decreasing** if for all $x_1, x_2 \in [a, b]$, if $x_1 < x_2$, then $f(x_1) > f(x_2)$.
   (a) Negate the definition of decreasing.
   (b) Prove that a decreasing function is injective.

7. Let A and B be sets and let X and Y be subsets of A. Let $f\colon A \to B$ be an injective function. Prove that $f(X \cap Y) = f(X) \cap f(Y)$.

8. Let A and B be sets and let W be a subset of B. Let $f\colon A \to B$ be a surjective function. Prove that $W \subseteq f(f^{-1}(W))$.

Assignment 10

Topics: Composition of functions.

Reading: Section 5.1

Suggested problems (do not hand in; these are just for extra practice): Handout 12

Assignment 10, due Friday, Nov 17, 12:55pm via Gradescope:

For problems 1, 2, and 3, let A, B and C be sets and let $f\colon A \to B$ and $g\colon B \to C$ be functions.

1. Prove or disprove: If $g \circ f$ is an injection, then g is an injection.
2. Prove or disprove: If $g \circ f$ is a surjection, then f is a surjection.
3. Prove or disprove: If $g \circ f$ is a surjection, then g is a surjection.
4. Let A and B be sets and let $f\colon A \to B$ and $g\colon B \to A$ be functions. Prove that if $g \circ f$ and $f \circ g$ are bijective, then so are f and g.
5. Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be functions. Suppose that f and g are both decreasing. Prove that $g \circ f$ is increasing.

Midterm (November 17)

Content: The questions will all be either

1. homework problems,
2. suggested problems,
3. problems we worked in class, or
4. minor variations of one of these.

Problems with very long proofs or that involved some unusual trick will not be on the exam.

You are allowed to use any previous problem from class or from the homework (e.g., "$A \subseteq A \cup B$") on the exam without reproving it, unless otherwise noted on the exam. (E.g., if I ask you to prove "$A \subseteq A \cup B$" on the exam, you will need to prove this using only the definition of subset and union, and I will remind you of this in the statement of the problem.)

A typical exam will have one or two questions from each week of the course. You can expect problems about following:

- Give definitions (e.g., subset, union, intersection, preimage, image)
- Proofs about sets (including power sets)
- Proofs about functions, images, preimages

There will be at least one definition, and around 4-5 problems involving proofs (possibly including "prove or disprove" problems).

For definitions, I want a definition, in prose (complete sentences), and I want "just" the definition, and not any additional facts about the definition. (E.g., if you give the definition of rational, do not include that a rational number can be written in reduced form; that is a fact about rational numbers not part of the definition of rational.)

There will be one problem verbatim from the second page of Handout 12 (on compositions).

Assignment 11

Thanksgiving break is Monday, November 20 through Friday, November 24;
There will no class those days.

Topics: Inverse functions.

Reading: Section 5.3

Suggested problems (do not hand in; these are just for extra practice) Handout 13

Assignment 11, due Friday, Dec 01, 12:55pm via Gradescope:

1. Define $f\colon \mathbf{R}-\{1\} \to \mathbf{R}-\{1\}$ by $f(x) = \frac{x+1}{x-1}$. Prove that f is a bijection. Find a formula for the inverse $f^{-1}(x)$, and prove that it is correct.

2. Let $f\colon \mathbf{R} \to \mathbf{R}$ be the function $f(x) = x^3 + x$. Prove that f is invertible without finding a formula for f^{-1}.

3. Let A, B and C be sets and let $f\colon A \to B$ and $g\colon B \to C$ be functions. Prove that if f and g are invertible, then so is $g \circ f$, and prove that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

4. Let A and B be sets and let $f\colon A \to B$ be a function. Suppose that f has a *left inverse* g; that is, suppose that there exists a function $g\colon B \to A$ such that $g \circ f = id_A$. Prove that f is injective.

5. Let $f\colon A \to B$ be a bijection. Let $g\colon B \to A$ be the inverse of f. Prove that g is also a bijection.

6. Let A be a set and let $f\colon A \to A$ be a function. Prove that $f \circ f = id_A$ if and only if "f is invertible and $f = f^{-1}$".

7. Let $f, g\colon A \to B$ be two functions, and let $h\colon B \to C$ be a function.

   (a) Prove that if h is injective and $h \circ f = h \circ g$ then $f = g$.

   (b) Give an example where h is not injective and $h \circ f = h \circ g$ but $f \neq g$.

8. Let $f\colon A \to B$ be a function, and let $g, h\colon B \to C$ be two functions.

   (a) Prove that if f is surjective and $g \circ f = h \circ f$ then $g = h$.

   (b) Give an example where f is not surjective and $g \circ f = h \circ f$ but $g \neq h$.

Assignment 12

Topics: Relations.

Reading: Chapter 4.

Suggested problems (do not hand in; these are just for extra practice): Handout 14

Assignment 12, due Friday, Dec 08, via Gradescope:

1. Define a relation R on the set $\mathbb{R}$ as follows: we say that $x \sim y$ if $|x| = |y|$. (Recall that $|x|$ is the absolute value of x) Determine whether this relation is reflexive, symmetric, transitive (and in each case give a proof or disproof).

2. Let $A = \{1, 2, 3\}$ and define a relation on A by $a \sim b$ if $a + b \neq 3$. Determine whether this relation is reflexive, symmetric, transitive (and in each case give a proof or disproof).

3. Define a relation on $\mathbf{Z}$ given by $a \sim b$ if $a - b$ is divisible by 3.

    (a) Prove that this is an equivalence relation.

    (b) What integers are in the equivalence class of 18? (No proof necessary.)

    (c) What integers are in the equivalence class of 31? (No proof necessary.)

    (d) How many distinct equivalence classes are there? What are they? (No proof necessary.)

4. Define a relation on $\mathbf{Z}$ given by $a \sim b$ if $a^2 - b^2$ is divisible by 4.

    (a) Prove that this is an equivalence relation.

    (b) How many distinct equivalence classes are there? What are they? (No proof necessary.)

5. Let A be a set, and let $P(A)$ be the power set of A. Assume that A is not the empty set. Define a relation on $P(A)$ by $X \sim Y$ if $X \subseteq Y$. Is this relation reflexive, symmetric, and/or transitive? In each case give a proof, or disprove with a counterexample. (For a counterexample, give an example of A, X, and Y that disproves the statement.)

6. Define a relation R on the set $\mathbb{R}$ as follows: we say that $x \sim y$ if $x - y \in \mathbb{Q}$. Determine whether this relation is reflexive, symmetric, transitive (and in each case give a proof or disproof).

7. Define a relation R on the set $\mathbb{Z}$ as follows: we say that $x \sim y$ if there exists a non-negative integer n such that $x = 2^n y$. Determine whether this relation is reflexive, symmetric, transitive (and in each case give a proof or disproof).

8. Define a relation R on the set $\mathbb{Z} \times (\mathbb{Z} - \{0\})$ as follows: we say that $(a, b) \sim (c, d)$ if $ad = bc$.

    (a) Prove that this is an equivalence relation.

    (b) What pairs (c, d) are in the equivalence class of $(1, 1)$? (No proof necessary.)

    (c) What pairs (c, d) are in the equivalence class of $(1, 2)$? (No proof necessary.)

    (d) How many distinct equivalence classes are there? What are they? (No proof necessary.)

Assignment 13

Topics: Binary Operations and Bijections/Countability

Reading: Chapter 4; Chapter 8

~~**Assignment**~~ **Topic 13, not to be handed in**: Consider the final two lectures "bonus content": there are a few topics that are very helpful to know in upper level courses, but which we were only able to briefly cover in class. I will not assign any graded work on these topics, and will not test you on these topics on the exam. If I were to assign some problems, I would assign the ones below (and some additional problems).

1. Let S be one of the following sets. Give an example of a bijection from S to $\mathbb{Z}$. (No proof is necessary.)
   (a) $S = n\mathbb{Z}$ (for some fixed positive integer n).
   (b) $S = \mathbb{Z} \times \{0\}$.
   (c) $S = \mathbb{Z} \times \{-1, 1\}$.
   (d) $S = \sin^{-1}(\{0\})$.
2. Give an example of a bijection from $\mathbb{N}$ to ..
3. Let $g\colon \mathrm{Fun}(S, \{0,1\}) \to P(S)$ be the function $f \mapsto f^{-1}(\{0\})$. Prove that f is a bijection.
4. As in class, let $P_{\mathrm{BD}}(S)$ be the set of all finite subsets of S, i.e.,
$$P_{\mathrm{BD}}(S) = \{A \subset S \mid |A| < \infty\}$$
   and for $i \in \mathbb{Z}_{>0}$ let
$$P_i(S) = \{A \subset S \mid |A| \leq i\}.$$
   (a) Prove that
$$P_{\mathrm{BD}}(S) = \bigcup_{i \in \mathbb{Z}_{>0}} P_i(S).$$
   (b) Prove that the map
$$S^i \to P_i(S)$$
   given by
$$(a_1, \ldots, a_i) \mapsto \{a_1, \ldots, a_i\}$$
   is a surjection.
   (c) Prove that $P_{\mathrm{BD}}(\mathbb{Z})$ is countable.
5. Prove that $|S| \neq |\mathrm{Fun}(S, \{0,1\})|$ by the same technique as the proof from class that $|S| \neq |P(S)|$. (In other words, proceed by contradiction, assuming that there is a bijection $f\colon S \to \mathrm{Fun}(S, \{0,1\})$ and then construct some $g \in \mathrm{Fun}(S, \{0,1\})$ which is not in the image of f.)

Final Exam

The **Final exam** is **December 18**, 2-5pm, in SMUD 204.

The **last day of class** is Wednesday, December 13.

There will be **office hours** before the exam. I will send out a survey to find a time that works for everyone who is planning to attend.

The final exam will be comprehensive.

The exam will be, roughly 8-10 questions, with multiple parts. Some questions will be "prove or disprove". For disproofs, please write out a counterexample as your disproof.

A typical exam will have one or two questions from each week of the course. You can expect a subset of the following:

- Negations
- Give definitions (e.g., divides, rational, subset)
- Direct proofs
- Proof by contrapositive
- Divisibility problems
- Contradiction
- Induction
- Proofs with sets.
- Images
- preimages
- Injectivity
- Surjectivity
- Compositions
- Invertibility
- Relations
- ~~Countability~~
- Problems from handouts 9-14