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MATH 220, Mathematical Reasoning and Proof  
MWF 1 - 1:50

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**All assignments**

Last updated: September 22, 2023

Gradescope code: 7DVWGG

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# Assignment 1

**Topics:** Introduction to the course. Mathematical reasoning. Logic.

**Reading:** 1.1, 1.2, 1.5, 2.1, 2.2

**Suggested problems** (do not hand in; these are just for extra practice): [Handout 1](#)

**Assignment 1, due Friday, Sep 15, 12:55pm via Gradescope:**

1. Write the negation of each of the following statements.
  - (a) All triangles are isosceles.
  - (b) Every door in the building was locked.
  - (c) Some even numbers are multiples of three.
  - (d) Every real number is less than 100.
  - (e) Every integer is positive or negative.
  - (f) If  $f$  is a polynomial function, then  $f$  is continuous at 0.
  - (g) If  $x^2 > 0$ , then  $x > 0$ .
  - (h) There exists a  $y \in \mathbf{R}$  such that  $xy = 1$ .
  - (i)  $(2 > 1)$  and  $(\forall x, x^2 > 0)$
  - (j)  $\forall \epsilon > 0, \exists \delta > 0$  such that if  $|x| < \delta$ , then  $|f(x)| < \epsilon$ .
2. Write the converse, contrapositive, and negation of each of the following implications.
  - (a) If a quadrilateral is a rectangle, then it has two pairs of parallel sides.
  - (b)  $(P \wedge \neg Q) \Rightarrow R$
  - (c)  $P \Rightarrow (R \Rightarrow \forall x, Q(x))$
3. Let  $P$  and  $Q$  be statements. Write the truth table for
  - (a)  $(\neg P) \vee Q$
  - (b)  $(P \wedge (\neg Q)) \Rightarrow Q$
4. Are the statements  $(P \vee Q) \wedge R$  and  $P \vee (Q \wedge R)$  equivalent? If so, give a proof. If not, explain why by giving a counterexample.

(Two statement forms are equivalent if they have the same truth tables, and here, a counterexample simply means some choice of truth values for  $P, Q$ , and  $R$  such that the two statement forms give different outputs.)
5. Let  $P$  and  $Q$  be statements.
  - (a) Prove that  $\neg(P \Rightarrow Q)$  is equivalent to  $P \wedge \neg Q$ .
  - (b) Prove that  $\neg(P \Rightarrow Q)$  is *not* equivalent to  $\neg P \wedge Q$ .

(c) Give an example of statements  $P$  and  $Q$  such that  $\neg P \Rightarrow \neg Q$  is true and  $\neg(P \Rightarrow Q)$  is false.

6. Suppose that  $n$  is an even integer, and let  $m$  be any integer. Prove that  $nm$  is even.
7. Suppose that  $n$  is an odd integer. Prove that  $n^2$  is an odd integer. (Hint: an integer  $n$  is odd if and only if there exists an integer  $k$  such that  $n = 2k + 1$ .)
8. Prove that if  $n^2$  is even, then  $n$  is even. (Hint: ~~page 67~~ contrapositive.)

# Assignment 2

**Topics:** “Direct” proofs, proof by cases, and divisibility problems.

**Reading:**

- 3.3, from Definition 3.3.6
- 6.4, just Definition 6.4.1
- see Index to find definitions like prime, etc

**Suggested problems (do not hand in; these are just for extra practice)**

1. [Handout 2](#)

**Assignment, due Friday, Sep 22, 12:55pm via Gradescope:**

1. Suppose that  $a \mid b$ . Prove that for all  $n \in \mathbb{Z}_{>0}$ ,  $a^n \mid b^n$ .
2. Suppose that there exists an integer  $n \in \mathbb{Z}_{>0}$  such that  $a \mid b^n$ . Is it true that  $a \mid b$ ? Prove or disprove your answer. (For a disproof, please give a counterexample that demonstrates that the statement is false.)
3. Prove that for all  $a \in \mathbb{Z}$  and for  $n \in \mathbb{Z}_{\geq 0}$ ,  $a - 1$  divides  $a^n - 1$ .
4. Prove that for all integers  $n$ ,  $n$  and  $n + 1$  have no common divisors other than  $\pm 1$ .
5. Prove that if  $x$  is an integer, then  $x^2 + 2$  is not divisible by 4. (Hint: there are two cases:  $x$  is even,  $x$  is odd. Also, feel free to use basic facts about even or odd, e.g., “odd + odd = even”, without additional proof.)
6. Prove that the product of three consecutive integers is divisible by 6. (It suffices to prove that it is divisible by 2 and 3 separately.)
7. Show that for all integers  $a$  and  $b$ ,
$$a^2b^2(a^2 - b^2)$$
is divisible by 12. (It suffices to prove that it is divisible by 4 and 3 separately.)
8. Find all positive integers  $n$  such that  $n^2 - 1$  is prime. Prove that your answer is correct.

# Assignment 3

**Topics:** Proof by contradiction. Unsolvability of equations. Irrationality.

**Reading:** 3.2

**Suggested problems (do not hand in; these are just for extra practice)**

1. [Handout 3](#)

**Assignment, due Friday, Sep 29, 12:55pm via Gradescope:**

1. Prove that there do not exist integers  $a$ , and  $b$  such that  $21a + 30b = 1$ .
2. Prove that  $2^{1/3}$  is irrational.
3. Suppose that  $x$  is a real number such that  $0 \leq x \leq \pi/2$ . Prove that  $\sin x + \cos x \geq 1$ . (Hint: at some point in your proof, use that  $(\sin x)^2 + (\cos x)^2 = 1$ .)
4. Prove that there are no positive integer solutions to the equation  $x^2 - y^2 = 10$ .
5. Let  $a, b, c$  be integers satisfying  $a^2 + b^2 = c^2$ . Show that  $abc$  must be even. (Harder problem, just for fun: show that  $a$  or  $b$  must be even.)
6. Suppose that  $a$  and  $n$  are integers that are both at least 2. Prove that if  $a^n - 1$  is prime, then  $a = 2$  and  $n$  is a prime. (Primes of the form  $2^n - 1$  are called Mersenne primes.)
7. Suppose that  $a, b \in \mathbb{Z}$ . Prove that  $a^2 - 4b \neq 2$ .
8. Prove that  $\log_{10} 7$  is irrational

# Assignment 4

**Topics:** Induction.

**Reading:** Chapter 6

**Fun Video** (optional): Vi Hart; “Doodling in Math: Spirals, Fibonacci, and Being a Plant”

<https://www.youtube.com/watch?v=ahXIMUkSXX0>

**Suggested problems (do not hand in; these are just for extra practice)**

1. [Handout 4](#)

2. [Handout 5](#)

**Assignment, due Friday, Oct 06, 12:55pm via Gradescope:**

1. Prove that for every positive integer  $n$ ,

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

2. Let  $a_n$  be defined recursively by  $a_1 = 1$  and  $a_n = \sqrt{1 + a_{n-1}}$ . Prove that for all positive integers  $n$ ,  $a_n < 2$ .

3. Prove by induction that if  $b_1, b_2, \dots, b_n$  are even integers, then  $b_1 + b_2 + \cdots + b_n$  is even.

4. Let  $F_1, F_2, F_3, \dots = 1, 1, 2, 3, 5, 8, \dots$  be the Fibonacci sequence. Prove that  $F_1^2 + \cdots + F_n^2 = F_n F_{n+1}$ .

5. Prove that  $n! > 2^n$  for all  $n \geq 4$ .

6. *Bernoulli's inequality*: let  $\beta \in \mathbb{R}$  be a real number such that  $\beta > -1$  and  $\beta \neq 0$ . Prove that for all integers  $n \geq 2$ ,  $(1 + \beta)^n > 1 + n\beta$ .

7. Prove that for all integers  $n \geq 1$ ,

$$1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{n}.$$

8. Prove (using induction) that for all integers  $n \geq 1$ ,  $2^{2n} - 1$  is divisible by 3.

# Assignment 5

**Topics:** Basics of set theory. Basic operations. Proofs with sets.

**Reading:**

1. 1.3, 1.4, 2.3

**Suggested problems (do not hand in; these are just for extra practice):** [Handout 6](#)

**Assignment 6, due Friday, Oct 20, 12:55pm via Gradescope:**

1. Let  $A = \{n \in \mathbb{Z} \mid n \text{ is a multiple of } 4\}$  and  $B = \{n \in \mathbb{Z} \mid n^2 \text{ is a multiple of } 4\}$ 
  - (a) Prove or disprove:  $A \subseteq B$ .
  - (b) Prove or disprove:  $B \subseteq A$ .
2. Prove that  $A \cup (A \cap B) = A$ .
3. Let  $A, B$  and  $C$  be sets.
  - (a) Prove that  $(A \subseteq C) \wedge (B \subseteq C) \Rightarrow A \cup B \subseteq C$ .
  - (b) State the contrapositive of part (a).
  - (c) State the converse of part (a). Prove or disprove it.
4. Let  $n$  and  $m$  be integers. Prove that if  $n\mathbb{Z} \subseteq m\mathbb{Z}$  then  $m$  divides  $n$ .

# Midterm

**Topics:** Friday, October 13 will be an in class Exam.

**Content:** The questions will all be either

1. homework problems,
2. suggested problems,
3. problems we worked in class, or
4. minor variations of one of these.

A typical exam will have one or two questions from each week of the course. You can expect problems like the following:

- Negations
- Give definitions
- Contrapositive
- Contradiction
- Induction