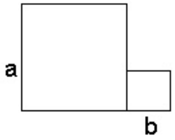
Beyond Fermat's Last Theorem

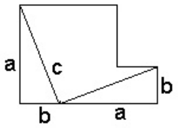
David Zureick-Brown

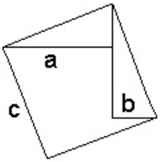
Emory University

Wake Forest University, Colloquium

$$\boxed{a^2 + b^2 = c^2}$$







Basic Problem (Solving Diophantine Equations)

Let $f_1, \ldots, f_m \in \mathbb{Z}[x_1, ..., x_n]$ be polynomials and let R be a ring (e.g., $R = \mathbb{Z}, \mathbb{Q}$).

Problem

Describe the set

$$\left\{(a_1, \ldots, a_n) \in R^n : \forall i, f_i(a_1, \ldots, a_n) = 0\right\}.$$

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Fact

Solving diophantine equations is hard.

Hilbert's Tenth Problem

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Theorem (Davis-Putnam-Robinson 1961, Matijasevič 1970)

There does not exist an algorithm solving the following problem:

input: $f_1, \ldots, f_m \in \mathbb{Z}[x_1, ..., x_n]$;

output: YES / NO *according to whether the set*

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This is *still open* for many other rings (e.g., $R = \mathbb{Q}$).

Fermat's Last Theorem

Theorem (Wiles et. al)

The only solutions to the equation

$$x^n + y^n = z^n, n \geq 3$$

are multiples of the triples

$$(0,0,0), \quad (\pm 1, \mp 1, 0), \quad \pm(1,0,1), \quad (0, \pm 1, \pm 1).$$

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Template for the proof of FLT

Step 1: Assume there is a counterexample $a^p + b^p = c^p$.

Step 2: (Frey) Build an elliptic curve with strange properties:

$$E_{(a,b,c)} \colon y^2 = x(x - a^p)(x + b^p)$$
$$j = \frac{2^8 \left(c^{2p} - a^p b^p\right)^3}{(abc)^{2p}}$$
$$\Delta = 2^{-8}(abc)^{2p}.$$

Step 3: (Ribet) Show that the Frey curve $E_{(a,b,c)}$ is not modular.

Step 4: Prove that every elliptic curve over $\mathbb{Q}$ is modular.

Modularity is now a theorem

Theorem (Wiles 1995; Breuil-Conrad-Diamond-Taylor 2002)

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Basic Problem: $f_1, \ldots, f_m \in \mathbb{Z}[x_1, ..., x_n]$

- **Qualitative**:
  - Does there exist a solution?
  - Do there exist infinitely many solutions?
  - Does the set of solutions have some extra structure (e.g., geometric structure, group structure).

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- **Implicit question**
  - Why do equations have (or fail to have) solutions?
  - Why do some have many and some have none?
  - What underlying mathematical structures control this?

Example: Pythagorean triples

Lemma

The equation

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has infinitely many non-zero coprime solutions.

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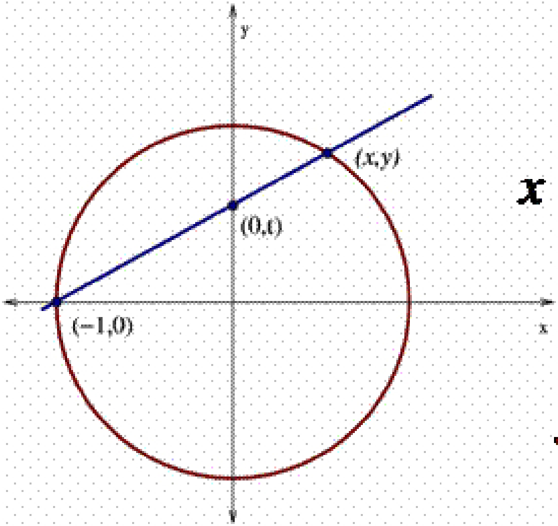
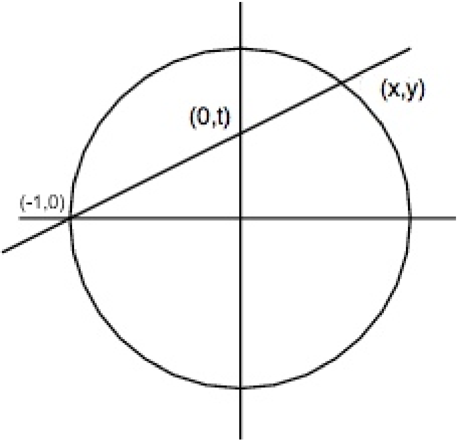
Lets work this out in detail.



$$x = \frac{a}{c} \qquad y = \frac{b}{c}$$

$$x^2 + y^2 = 1$$

Pythagorean triples



$$\text{Slope} = \quad t = \frac{y}{x+1}$$

$$x = \frac{1-t^2}{1+t^2}$$

$$y = \frac{2t}{1+t^2}$$

Pythagorean triples

Lemma

The solutions to

$$a^2 + b^2 = c^2$$

are all multiples of the triples

$$\boxed{a = 1 - t^2} \quad \boxed{b = 2t} \quad \boxed{c = 1 + t^2}$$

The Mordell Conjecture

Example

The equation $y^2 - x^2 = 1$ has infinitely many solutions.

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Theorem (Faltings)

For $n \geq 5$, the equation

$$y^2 - x^n = 1$$

has only finitely many solutions.

Fermat Curves

Question

Why is Fermat's last theorem believable?

1. $x^n + y^n - z^n = 0$ looks like a surface (3 variables)
2. $x^n + y^n - 1 = 0$ looks like a curve (2 variables)

Mordell Conjecture

Consider $y^2 = (x^2 - 1)(x^2 - 2)(x^2 - 3)$.

This is a cross section of a torus. The **genus** is the number of holes.

Conjecture (Mordell)

A curve of genus $g \geq 2$ has only finitely many rational solutions.

Fermat Curves

Question

Why is Fermat's last theorem believable?

1. $x^n + y^n - 1 = 0$ is a curve of genus $(n-1)(n-2)/2$.
2. Mordell implies that for **fixed** $n > 3$, the nth Fermat equation has only finitely many solutions.

Fermat Curves

Question

What if $n = 3$?

1. $x^3 + y^3 - 1 = 0$ is a curve of genus $(3-1)(3-2)/2 = 1$.
2. We were lucky; $Ax^3 + By^3 = Cz^3$ can have infinitely many solutions.

Congruent number problem

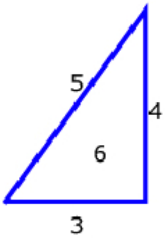
The pair of equations

$$\boxed{x^2 + y^2 = z^2,\ xy = 2 \cdot 157}$$

has **infinitely many** solutions. **How large** Is the smallest solution? How many **digits** does the smallest solution have?

Congruent number problem

$$\boxed{x^2 + y^2 = z^2,\ xy = 2 \cdot 6}$$



$$3^2 + 4^2 = 5^2,\ 3 \cdot 4 = 2 \cdot 6$$

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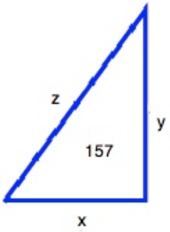

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"Next" soluton has **176 digits**!

Back of the envelope calculation

$$\boxed{x^2 + y^2 = z^2,\ xy = 2 \cdot 157}$$

- Num, den$(x, y, z) \leq 10 \sim 10^6$ many, **1 min** on Emory's computers.

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- Num, den$(x, y, z) \leq 10 \sim 10^6$ many, **1 min** on Emory's computers.
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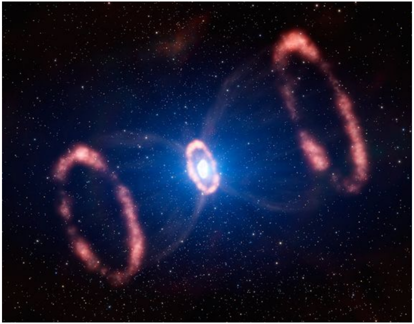
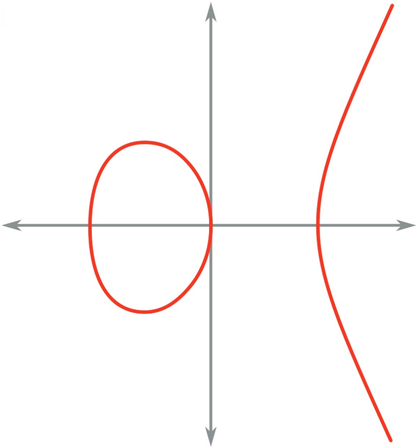
- Expected time of 'heat death' of universe – $\mathbf{10^{100}}$ **years**.

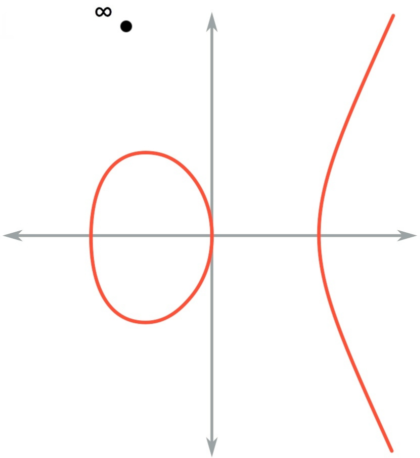


Elliptic Curves



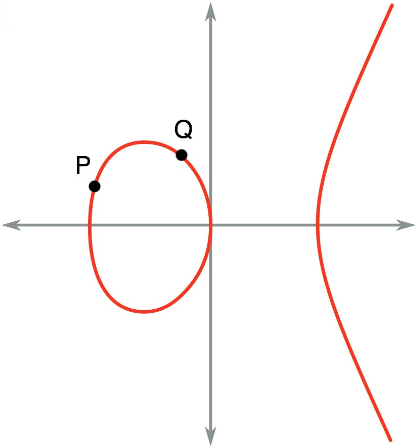
$$E : y^2 = x^3 + ax + b$$

Elliptic Curves - point at infinity



$$E\colon zy^2 = x^3 + axz^2 + bz^3$$
$$\infty = [0 : 1 : 0]$$

Elliptic Curves – addition



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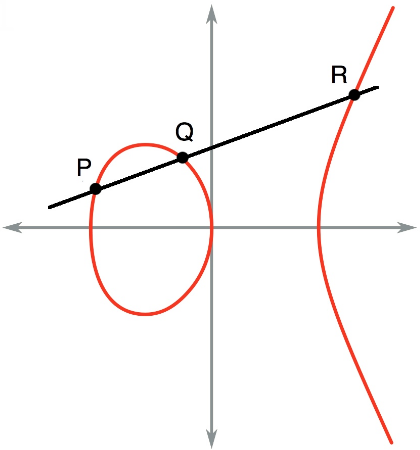

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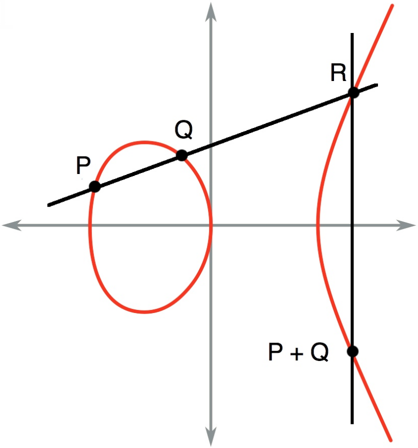

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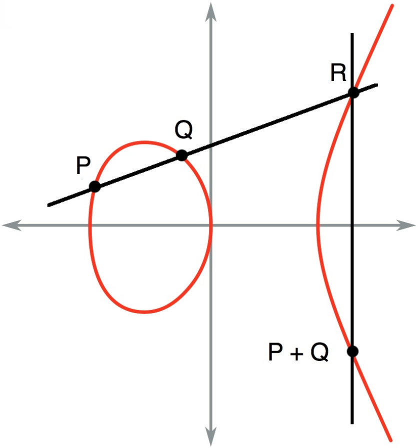
$$P + Q = (x_2, -y_2) \in \mathbb{Q}^2$$



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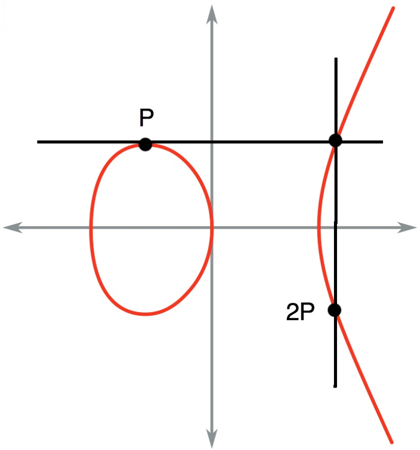
$$E(\mathbb{Q}) \times E(\mathbb{Q}) \to E(\mathbb{Q})$$

$$(P\ ,\ Q) \quad \mapsto P + Q$$

Elliptic Curves - duplication



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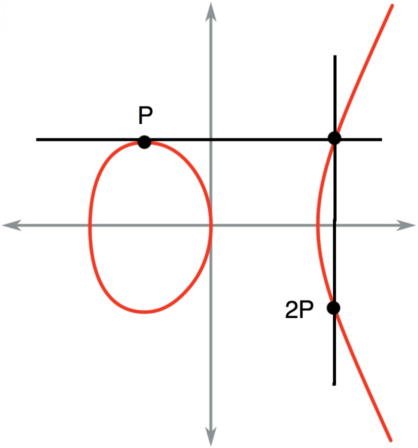

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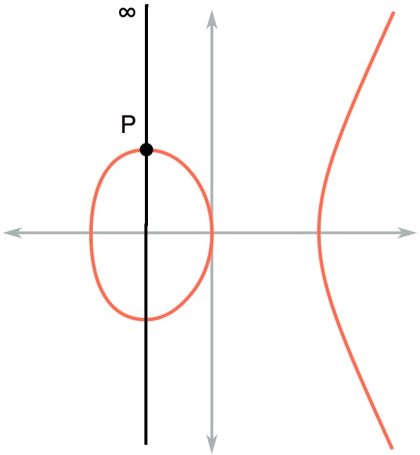
$$2P = (x_3, y_3) \in \mathbb{Q}^2$$

Elliptic Curves – identity



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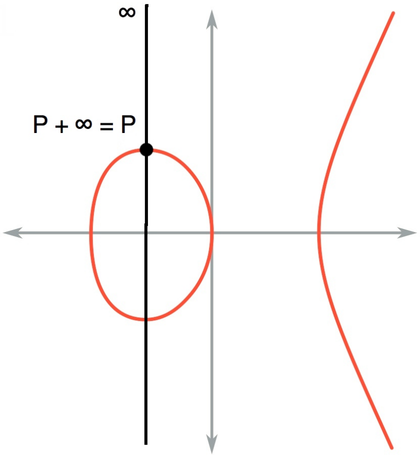

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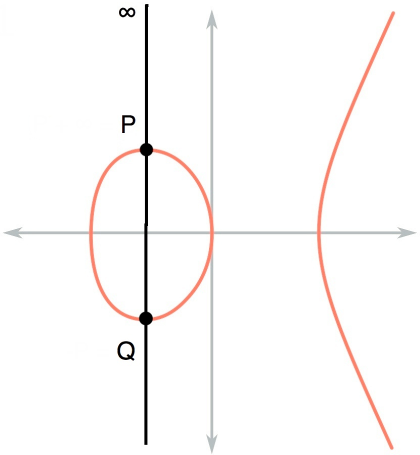
Elliptic Curves – inverses



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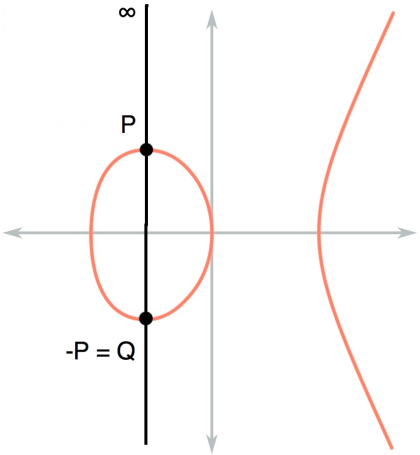

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Elliptic Curves – torsion subgroup

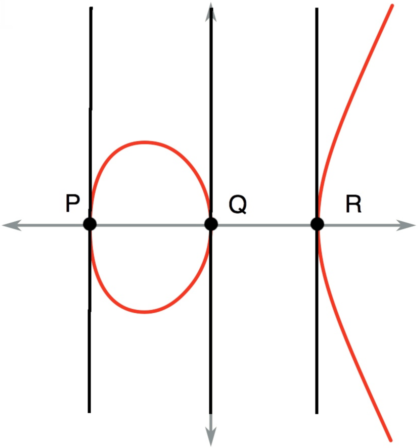
Let $n \in \mathbb{Z}$ be an integer.

Definition

The *n-torsion subgroup* $E[n]$ of E is defined to be

$$\ker\left(E \xrightarrow{[n]} E\right) := \{P \in E : nP := P + \ldots + P = \infty\}.$$

Elliptic Curves – two torsion



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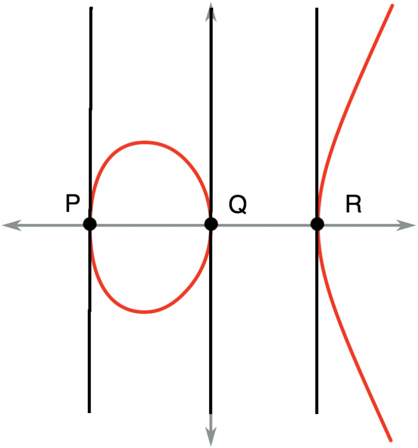


$$E\colon y^2 = x^3 + ax + b$$
$$2P = 2Q = 2R = \infty$$

Fermat Surfaces

Conjecture

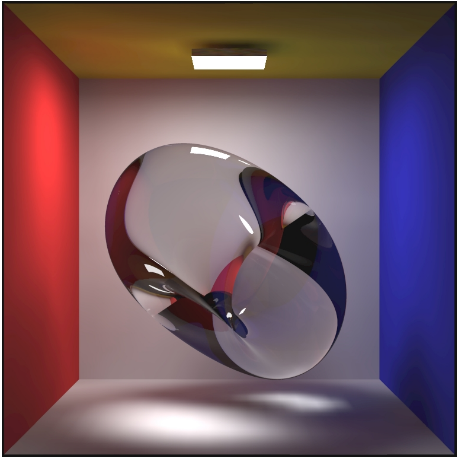
The only solutions to the equation

$$x^n + y^n = z^n + w^n, n \geq 5$$

satisfy $xyzw = 0$ or lie on the lines 'lines' $x = \pm y$, $z = \pm w$ (and permutations).

The Swinnerton-Dyer K3 surface

$$\boxed{x^4 + 2y^4 = 1 + 4z^4}$$



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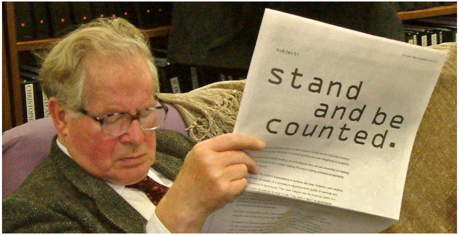

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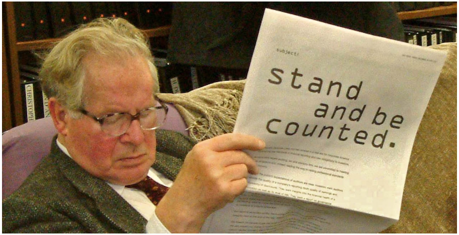

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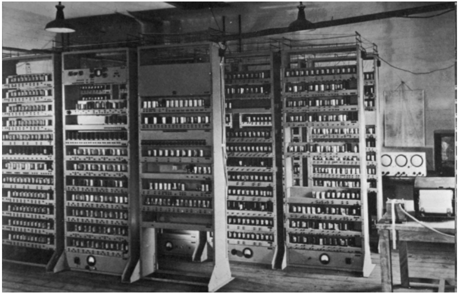
(Mathematicians aren't sure if there are any more!)

Back of the envelope calculation

$$\boxed{x^4 + 2y^4 = 1 + 4z^4}$$



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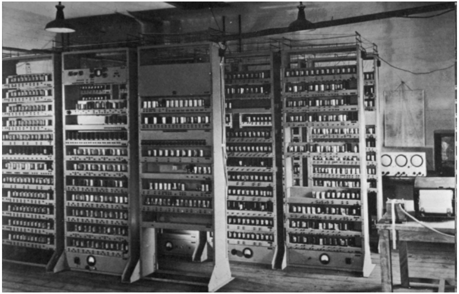

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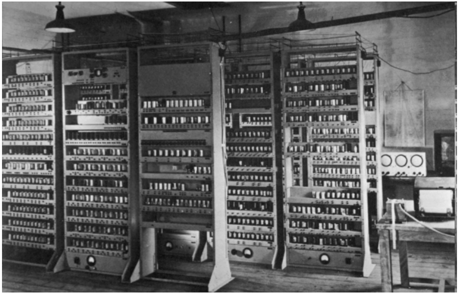

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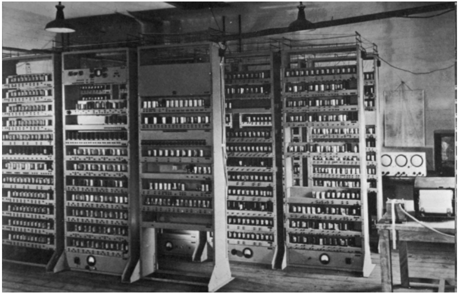

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- Age of the universe – **13.75** $\pm$ **.11 billion years** (roughly $\mathbf{10^{10}}$).

Fermat-like equations

Theorem (Poonen, Schaefer, Stoll)

The coprime integer solutions to $x^2 + y^3 = z^7$ *are the* 16 *triples*

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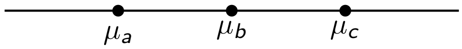
Theorem (Darmon and Granville)

Fix $a, b, c \geq 2$*. Then the equation* $x^a + y^b = z^c$ *has only finitely many coprime integer solutions iff* $\chi = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1 \leq 0$*.*



Known Solutions to $x^a + y^b = z^c$

The 'known' solutions with

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$$

are the following:

$$1^p + 2^3 = 3^2$$

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Problem (Beal’s conjecture)

These are all solutions with $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1 < 0$.

Generalized Fermat Equations – Known Solutions

Conjecture (Beal, Granville, Tijdeman-Zagier)

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...or even for a counterexample.

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Theorem (Bugeaud, Mignotte, Siksek 2006)

The only Fibonacci numbers that are perfect powers are

$$F_0 = 0,\ F_1 = F_2 = 1,\ F_6 = 8,\ F_{12} = 144.$$

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$$(T/2)^2 + H^3 + (f/12^3)^5$$

1. $f = st(t^{10} - 11t^5s^5 - s^{10})$,
2. $H =$ Hessian of f,
3. $T =$ a degree 3 covariant of the dodecahedron.

(p, q, r) such that $\chi < 0$ and the solutions to $x^p + y^q = z^r$ have been determined.

$\{n, n, n\}$	Wiles, Taylor-Wiles, building on work of many others
$\{2, n, n\}$	Darmon-Merel, others for small n
$\{3, n, n\}$	Darmon-Merel, others for small n
$\{5, 2n, 2n\}$	Bennett
$(2, 4, n)$	Ellenberg, Bruin, Ghioca $n \geq 4$
$(2, n, 4)$	Bennett-Skinner; $n \geq 4$
$\{2, 3, n\}$	Poonen-Shaefer-Stoll, Bruin. $6 \leq n \leq 9$
$\{2, 2\ell, 3\}$	Chen, Dahmen, Siksek; primes $7 < \ell < 1000$ with $\ell \neq 31$
$\{3, 3, n\}$	Bruin; $n = 4, 5$
$\{3, 3, \ell\}$	Kraus; primes $17 \leq \ell \leq 10000$
$(2, 2n, 5)$	Chen $n \geq 3^*$
$(4, 2n, 3)$	Bennett-Chen $n \geq 3$
$(6, 2n, 2)$	Bennett-Chen $n \geq 3$
$(2, 6, n)$	Bennett-Chen $n \geq 3$

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Theorem (ZB., 2011)

The only coprime integer solutions to the equation

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It is the first generalized Fermat equation of the form $x^2 + y^3 = z^n$ conjectured to have only trivial solutions.

$(3^2 + (-2)^3 = 1^n$ is considered to be trivial.)

The Mazur-Ellenberg threefold

$$V\colon\ x_0^4 y_0^5 + x_1^4 y_1^5 + x_2^4 y_2^5 = 0 \subset \mathbb{P}^2 \times \mathbb{P}^2.$$

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$$p^4 = (p)^4(1)^5$$

$$p^5 = (1)^4(p)^5$$

$$p^{12} = (p^3)^4(1)^5$$

$$p^{13} = (p^2)^4(p)^5$$

$$p^{20} = (p^5)^4(1)^5 = (1)^4(p^4)^5$$

Theorem

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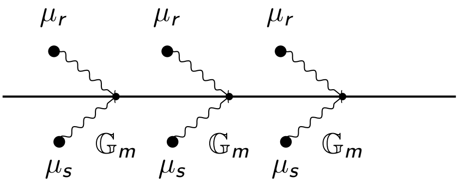
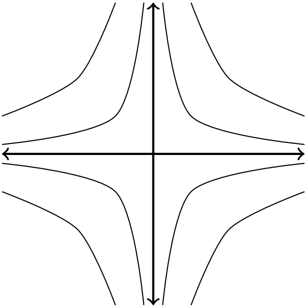
Theorem (ZB)

$V - Z$ is log general type; in particular, the Lang-Vojta conjecture implies Fermat's last theorem and most cases of the generalized Fermat conjecture.



Theorem

$$[\mathbb{C}^2/\mathbb{C}^*], \ \ t \cdot (x, y) = (t^r x, t^{r-1} y)$$



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