Rigid Cohomology for Algebraic Stacks

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Slides available at `http://www.math.wisc.edu/~brownda/slides/`

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- Construct variants (e.g., cohomology supported in a closed subscheme);
- Weil formalism (e.g., excision, Gysin, trace formulas).

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- Lafforgue constructs a 'compactified moduli stack of shtukas' $\mathcal{X}$ (actually a compactification of a stratification of a moduli stack of shtukas).
- The ℓ-adic étale cohomology of étale sheaves on $\mathcal{X}$ realize a Langlands correspondence between certain Galois and automorphic representations.

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- $\ell = p$ is bad for étale cohomology.
- $\mathcal{X}$ is a singular, separated Artin stack, so crystalline cohomology won't work.
- Generalizing rigid cohomology to Artin stacks would extend Lafforgue's work to the $\ell = p$ case.

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- Logarithmic rigid cohomology and crystalline fundamental groups;
- Arithmetic Statistics – Cohen-Lenstra heuristics for p-divisible groups.

Rigid Cohomology Setup (for Schemes)

$$\begin{tikzcd}]X[\arrow[r, phantom, "\overset{i}{\subset}"] \arrow[d, "sp"'] & \mathbb{P}^{n,\mathrm{an}}_{\mathbb{Q}_p} \arrow[d, "sp"] \\ X \arrow[r, hook, "j"] & \mathbb{P}^n_{\mathbb{F}_p} \end{tikzcd}$$

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(OK to replace $\mathbb{P}^n$ with a formal scheme which is smooth and proper over $\operatorname{Spf} \mathbb{Z}_p$.)

Example: $X = \mathbb{A}^1_{\mathbb{F}_p}$

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$$d(f(x)) := f'(x)dx$$

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$$H^i_{\mathrm{rig}}(\mathbb{A}^1_{\mathbb{F}_p}) = \begin{cases} \mathbb{Q}_p, & \text{if } i = 0 \\ 0, & \text{if } i \geq 1 \end{cases}$$

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- Functorality is another theorem (Berthelot).
- Hard to prove results about relative rigid cohomolgy (e.g., coherence is still an open problem).
- How to define for a scheme which isn't quasi-projective?

Le Stum's overconvergent site: $\mathrm{AN}^\dagger(X)$

- Objects: $(X', V') =$

$$
\begin{tikzcd}
X' \arrow[r, hook] \arrow[d] & P' \arrow[dd] & P'_{\mathbb{Q}_p} \arrow[l, "sp"'] \arrow[dd] & V' \arrow[l, "\lambda"'] \\
X \arrow[d] & & & \\
\operatorname{Spec} \mathbb{F}_p \arrow[r, hook] & \operatorname{Spf} \mathbb{Z}_p & \mathcal{M}(\mathbb{Q}_p) \arrow[l, "sp"'] &
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\arrow[from=1-2, to=3-2, crossing over]
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- A morphism $(X', V') \to (X'', V'')$ is a triple of morphisms compatible with the diagram
- $\{(X, V_i) \to (X', V')\}$ is a covering if $V = \cup V_i$ is an open covering of topological spaces.

Strict Neighborhoods

- The **tube** of $(X' \hookrightarrow P', P'_{\mathbb{Q}_p} \xleftarrow{\lambda} V')$ is $\lambda^{-1}(]X'[_{P'_{\mathbb{Q}_p}})$.

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- A morphism $(X', V') \to (X'', V'')$ induces a morphism $]X'[_{V'} \to]X''[_{V''}$.
- We declare $(X', W') \to (X', V')$ to be an isomorphism if the induced map on tubes $]X'[_{W'} \to]X'[_{V'}$ is an isomorphism.

Structure sheaf $\mathcal{O}_X^\dagger$

- $i\colon]X'[_{V'} \hookrightarrow V'$.
- $\mathcal{O}_X^\dagger(X', V') := \Gamma(]X'[_{V'}, i^{-1}\mathcal{O}_{V'})$.

Le Stum's main theorem

Theorem

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(ii) $\mathrm{Coh}\,\mathcal{O}_X^\dagger \cong \mathrm{Isoc}^\dagger X$

Rigid Cohomology for Stacks

Let $\mathcal{X}$ be a stack and define $\mathrm{AN}^\dagger(\mathcal{X})$ and $\mathcal{O}_X^\dagger$ the same way.

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- A morphism $(X, V) \to (X', V')$ is a triple of morphisms compatible with the diagram.
- $\{(X_i, V_i) \to (X, V)\}$ is a covering if $V = \cup V_i$ is an open covering of topological spaces.

Rigid Cohomology for Stacks

We define

- $\mathcal{O}^{\dagger}_{\mathcal{X}}$ as before: $(X, V) \mapsto \Gamma(i^{-1}\mathcal{O}_V)$;
- $\mathrm{Isoc}^{\dagger}(\mathcal{X}) := \mathrm{Coh}\,\mathcal{O}^{\dagger}_{\mathcal{X}}$;
- $H^i_{\mathrm{rig}}(\mathcal{X}) := H^i(\mathcal{X}_{\mathrm{AN}^{\dagger}}, \mathcal{O}^{\dagger}_{\mathcal{X}})$.

Sample Theorems: Finiteness and agreement

Theorem (B.)

(i) $H^i_{\mathrm{rig}}(\mathcal{X})$ *is finite dimensional.*

(ii) $H^i_{\mathrm{rig}}(\mathcal{X}) \otimes \mathbb{C}$ *agrees with the étale cohomology of* $\mathcal{X}$*.*

(iii) $H^i_{\mathrm{rig}}(\mathcal{X})$ *agrees with the crystalline cohomology of* $\mathcal{X}$ *when* $\mathcal{X}$ *is smooth and proper.*

Cohomology supported in a closed substack

- Given a closed substack $\mathcal{Z} \subset \mathcal{X}$ with open complement $\mathcal{U}$, I can define functors $H^i_{\text{rig},\mathcal{Z}}(\mathcal{X})$.
- Idea: use the very general notions of open and closed immersion of topoi (of SGA4).

Cohomology supported in a closed subscheme: Theorems

Theorem (B.)

(i) *$H^i_{\mathrm{rig},\mathcal{Z}}(\mathcal{X})$ agrees with Berthelot's construction when $\mathcal{X}$ is a scheme.*

(ii) *(Excision) There is a long exact sequence*

$$\cdots H^i_{\mathrm{rig},\mathcal{Z}}(\mathcal{X}) \to H^i_{\mathrm{rig}}(\mathcal{X}) \to H^i(\mathcal{U}) \to H^{i+1}_{\mathrm{rig},\mathcal{Z}}(\mathcal{X}) \cdots$$

(iii) *(Gysin) When $(\mathcal{X}, \mathcal{Z})$ is a smooth pair, there is an isomorphism*

$$H^i_{\mathrm{rig},\mathcal{Z}}(\mathcal{X}) \cong H^{i-2d}(\mathcal{Z})$$

Main tool: cohomological descent

Theorem (B.)

Cohomological descent holds on the overconvergent site with respect to smooth, flat, and étale hypercovers.

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Special case: let $X \to \mathcal{X}$ be a smooth cover. Then there is a spectral sequence

$$H^i_{\mathrm{rig}}(X_j) \Rightarrow H^{i+j}_{\mathrm{rig}}(\mathcal{X})$$

where X_j is the $j+1$ fold fiber product $X \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} X$.

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Previous proof for rigid cohomology is ~ 200 pages; mine is ~ 10.

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Thank you!