MATH 220 HANDOUT 2 - DIVISIBILITY

(1) Show that if $d \neq 0$ and $d \mid a$, then $d \mid (-a)$ and $-d \mid a$.
(2) Show that if $a \mid b$ and $b \mid a$, then $a = b$ or $a = -b$.
(3) Suppose that n is an integer such that $5 \mid (n+2)$. Which of the following are divisible by 5?
  (a) $n^2 - 4$
  (b) $n^2 + 8n + 7$
  (c) $n^4 - 1$
  (d) $n^2 - 2n$
(4) Prove that the square of any integer of the form $5k + 1$ for $k \in \mathbf{Z}$ is of the form $5k' + 1$ for some $k' \in \mathbf{Z}$.
(5) Show that if $ac \mid bc$ and $c \neq 0$, then $a \mid b$.

(6) (a) Prove that the product of three consecutive integers is divisible by 6.
  (b) Prove that the product of four consecutive integers is divisible by 24.
  (c) Prove that the product of n consecutive integers is divisible by $n(n-1)$.
  (d) (Challenge problem) Prove that the product of n consecutive integers is divisible by $n!$.
(7) Find all integers $n \geq 1$ so that $n^3 - 1$ is prime. Hint: $n^3 - 1 = (n^2 + n + 1)(n - 1)$.
(8) Show that for all integers a and b,
$$a^2 b^2 (a^2 - b^2)$$
is divisible by 12.
(9) Suppose that a is an integer greater than 1 and that n is a positive integer. Prove that if $a^n + 1$ is prime, then a is even and n is a power of 2. Primes of the form $2^{2^k} + 1$ are called Fermat primes.
(10) Suppose that a and n are integers that are both at least 2. Prove that if $a^n - 1$ is prime, then $a = 2$ and n is a prime. (Primes of the form $2^n - 1$ are called Mersenne primes.)
(11) Let n be an integer greater than 1. Prove that if one of the numbers $2^n - 1, 2^n + 1$ is prime, then the other is composite.
(12) Show that every integer of the form $4 \cdot 14^k + 1$, $k \geq 1$ is composite. Hint: show that there is a factor of 3 when k is odd and a factor of 5 when k is even.

(13) Can you find an integer $n > 1$ such that the sum
$$1 + \frac{1}{2} + \frac{1}{3} + + \cdots + \frac{1}{n}$$
is an integer?