Beyond Fermat's Last Theorem

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Slides available at `http://www.mathcs.emory.edu/~dzb/slides/`

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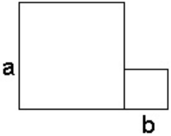
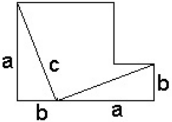
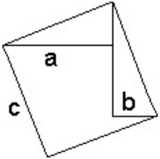
$$\boxed{a^2 + b^2 = c^2}$$







Basic Problem (Solving Diophantine Equations)

Setup

Let $f_1, \ldots, f_m \in \mathbb{Z}[x_1, ..., x_n]$ be polynomials.

Let R be a ring (e.g., $R = \mathbb{Z}$, $\mathbb{Q}$).

Problem

Describe the set

$$\left\{(a_1, \ldots, a_n) \in R^n : \forall i, f_i(a_1, \ldots, a_n) = 0\right\}.$$

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Fact

Solving diophantine equations is hard.

Hilbert's Tenth Problem

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Theorem (Davis-Putnam-Robinson 1961, Matijasevič 1970)

There does not exist an algorithm solving the following problem:

input: $f_1, \ldots, f_m \in \mathbb{Z}[x_1, ..., x_n]$;

output: YES / NO *according to whether the set*

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This is also *known* for many rings (e.g., $R = \mathbb{C}, \mathbb{R}, \mathbb{F}_q, \mathbb{Q}_p, \mathbb{C}(t)$).

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> *is non-empty.*

This is also *known* for many rings (e.g., $R = \mathbb{C}, \mathbb{R}, \mathbb{F}_q, \mathbb{Q}_p, \mathbb{C}(t)$).
This is *still open* for many other rings (e.g., $R = \mathbb{Q}$).

Fermat's Last Theorem

Theorem (Wiles et. al)

The only solutions to the equation

$$x^n + y^n = z^n, n \geq 3$$

are multiples of the triples

$$(0,0,0), \quad (\pm 1, \mp 1, 0), \quad \pm(1,0,1), \quad (0, \pm 1, \pm 1).$$

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Qualitative:

- Does there exist a solution?
- Do there exist infinitely many solutions?
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Implicit question

- Why do equations have (or fail to have) solutions?
- Why do some have many and some have none?
- What underlying mathematical structures control this?

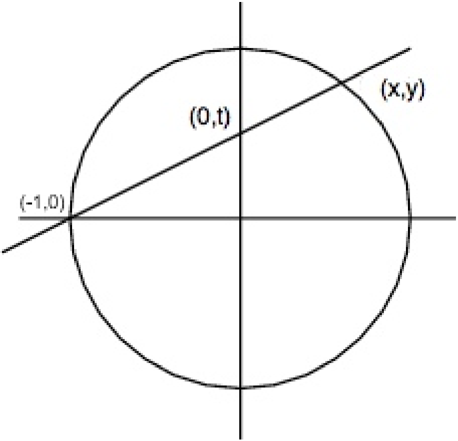
Example: Pythagorean triples

Lemma

The equation

$$x^2 + y^2 = z^2$$

has infinitely many non-zero coprime solutions.



$$\text{Slope} = \quad t = \frac{y}{x+1}$$

$$x = \frac{1-t^2}{1+t^2}$$

$$y = \frac{2t}{1+t^2}$$

Pythagorean triples

Lemma

The solutions to

$$a^2 + b^2 = c^2$$

are all multiples of the triples

$$\boxed{a = 1 - t^2} \; \boxed{b = 2t} \; \boxed{c = 1 + t^2}$$

The Mordell Conjecture

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Theorem (Faltings)

For $n \geq 5$, the equation

$$y^2 = f(x)$$

has only finitely many solutions if $f(x)$ is squarefree, with degree > 4.

Fermat Curves

Question

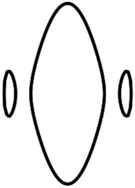
Why is Fermat's last theorem believable?

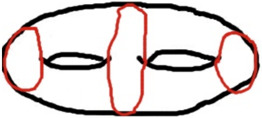
1. $x^n + y^n - z^n = 0$ looks like a surface (3 variables)
2. $x^n + y^n - 1 = 0$ looks like a curve (2 variables)

Mordell Conjecture

Example

$$y^2 = (x^2 - 1)(x^2 - 2)(x^2 - 3)$$





This is a cross section of a two holed torus. The **genus** is the number of holes.

Conjecture (Mordell)

A curve of genus $g \geq 2$ has only finitely many rational solutions.

Fermat Curves

Question

Why is Fermat's last theorem believable?

1. $x^n + y^n - 1 = 0$ is a curve of genus $(n-1)(n-2)/2$.
2. Mordell implies that for **fixed** $n > 3$, the nth Fermat equation has only finitely many solutions.

Fermat Curves

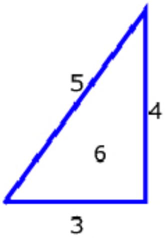
Question

What if $n = 3$?

1. $x^3 + y^3 - 1 = 0$ is a curve of genus $(3-1)(3-2)/2 = 1$.
2. We were lucky; $Ax^3 + By^3 = Cz^3$ can have infinitely many solutions.

Congruent number problem

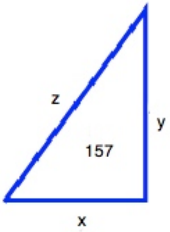
$$\boxed{x^2 + y^2 = z^2,\ xy = 2 \cdot 6}$$



$$3^2 + 4^2 = 5^2, \quad 3 \cdot 4 = 2 \cdot 6$$

Congruent number problem

$$\boxed{x^2 + y^2 = z^2,\ xy = 2 \cdot 157}$$



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The pair of equations

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$$x = \frac{157841 \cdot 4947203 \cdot 52677109576}{2 \cdot 3^2 \cdot 5 \cdot 13 \cdot 17 \cdot 37 \cdot 101 \cdot 17401 \cdot 46997 \cdot 356441}$$

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"Next" soluton has **176 digits**!

Back of the envelope calculation (as of 2011)

$$\boxed{x^2 + y^2 = z^2,\ xy = 2 \cdot 157}$$

- Num, den$(x, y, z) \leq 10 \sim 10^6$ many, **1 min** on Emory's computers.

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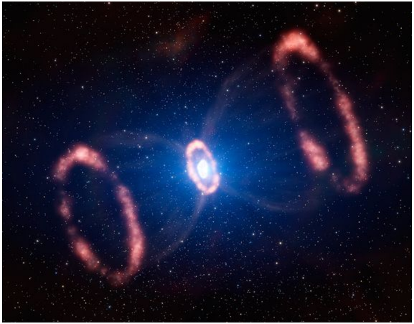
- Expected time until 'heat death' of universe – $\mathbf{10^{100}}$ **years**.



Fermat Surfaces

Conjecture

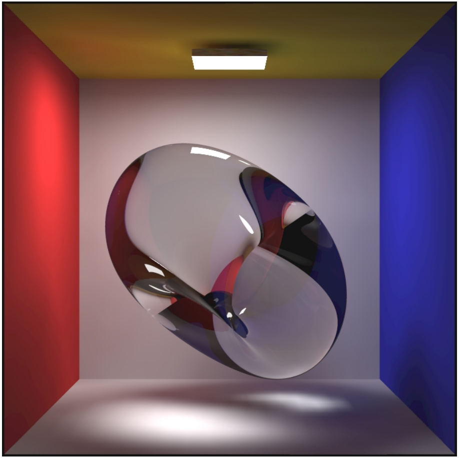
The only solutions to the equation

$$x^n + y^n = z^n + w^n, n \geq 5$$

satisfy $xyzw = 0$ or lie on the lines 'lines' $x = \pm y$, $z = \pm w$ (and permutations).

The Swinnerton-Dyer K3 surface

$$\boxed{x^4 + 2y^4 = 1 + 4z^4}$$



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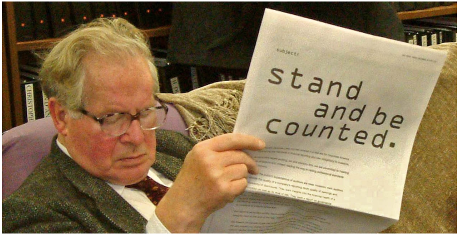

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The Swinnerton-Dyer K3 surface

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- Two 'obvious' solutions – $(\pm 1 : 0 : 0)$.
- The next smallest solutions are $\left(\pm\frac{1484801}{1169407}, \pm\frac{1203120}{1169407}, \pm\frac{1157520}{1169407}\right)$.

Problem

Find another solution.

Remark

1. $\mathbf{10^{16}}$ **years** to find via brute force.
2. Age of the universe – **13.75** $\pm$ **.11 billion years** (roughly $\mathbf{10^{10}}$).

Fermat-like equations

Theorem (Poonen, Schaefer, Stoll)

The coprime integer solutions to $x^2 + y^3 = z^7$ *are the* 16 *triples*

$$(\pm 1, -1, 0), \quad (\pm 1, 0, 1), \quad \pm(0, 1, 1),$$

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$$(\pm 21063928, -76271, 17).$$

Generalized Fermat Equations

Problem

What are the solutions to the equation $x^a + y^b = z^c$*?*

Generalized Fermat Equations

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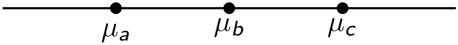
What are the solutions to the equation $x^a + y^b = z^c$?

Theorem (Darmon and Granville)

Fix $a, b, c \geq 2$. Then the equation $x^a + y^b = z^c$ has only finitely many coprime integer solutions iff $\chi = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1 \leq 0$.



Known Solutions to $x^a + y^b = z^c$

The 'known' solutions with

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$$

are the following:

$$1^p + 2^3 = 3^2$$

$$2^5 + 7^2 = 3^4,\ 7^3 + 13^2 = 2^9,\ 2^7 + 17^3 = 71^2,\ 3^5 + 11^4 = 122^2$$

$$17^7 + 76271^3 = 21063928^2,\ 1414^3 + 2213459^2 = 65^7$$

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Problem (Beal's conjecture)

These are all solutions with $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1 < 0$.

Generalized Fermat Equations – Known Solutions

Conjecture (Beal, Granville, Tijdeman-Zagier)

This is a complete list of coprime non-zero solutions such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1 < 0$.

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...or even for a counterexample.

Examples of Generalized Fermat Equations

Theorem (Poonen, Schaefer, Stoll)

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$$\frac{1}{2} + \frac{1}{3} + \frac{1}{6} - 1 = 0$$

Examples of Generalized Fermat Equations

Theorem (Darmon, Merel)

Any pairwise coprime solution to the equation

$$x^n + y^n = z^2, n > 4$$

satisfies $xyz = 0$.

$$\frac{1}{n} + \frac{1}{n} + \frac{1}{2} - 1 = \frac{2}{n} - \frac{1}{2} < 0$$

Other applications of the modular method

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Theorem (Bugeaud, Mignotte, Siksek 2006)

The only Fibonacci numbers that are perfect powers are

$$F_0 = 0,\ F_1 = F_2 = 1,\ F_6 = 8,\ F_{12} = 144.$$

Examples of Generalized Fermat Equations

Theorem (Klein, Zagier, Beukers, Edwards, others)

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$$(T/2)^2 + H^3 + (f/12^3)^5$$

1. $f = st(t^{10} - 11t^5s^5 - s^{10})$,
2. $H =$ Hessian of f,
3. $T =$ a degree 3 covariant of the dodecahedron.

(p, q, r) such that $\chi < 0$ and the solutions to $x^p + y^q = z^r$ have been determined.

$\{n, n, n\}$	Wiles,Taylor-Wiles, building on work of many others
$\{2, n, n\}$	Darmon-Merel, others for small n
$\{3, n, n\}$	Darmon-Merel, others for small n
$\{5, 2n, 2n\}$	Bennett
$(2, 4, n)$	Ellenberg, Bruin, Ghioca $n \geq 4$
$(2, n, 4)$	Bennett-Skinner; $n \geq 4$
$\{2, 3, n\}$	Poonen-Shaefer-Stoll, Bruin. $6 \leq n \leq 9$
$\{2, 2\ell, 3\}$	Chen, Dahmen, Siksek; primes $7 < \ell < 1000$ with $\ell \neq 31$
$\{3, 3, n\}$	Bruin; $n = 4, 5$
$\{3, 3, \ell\}$	Kraus; primes $17 \leq \ell \leq 10000$
$(2, 2n, 5)$	Chen $n \geq 3^*$
$(4, 2n, 3)$	Bennett-Chen $n \geq 3$
$(6, 2n, 2)$	Bennett-Chen $n \geq 3$
$(2, 6, n)$	Bennett-Chen $n \geq 3$

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$(2, 3, 10)$	**ZB**

Faltings' theorem / Mordell's conjecture

Theorem (Faltings, Vojta, Bombieri)

Let X be a smooth curve over $\mathbb{Q}$ with genus at least 2. Then $X(\mathbb{Q})$ is finite.

Example

For $g \geq 2$, $y^2 = x^{2g+1} + 1$ has only finitely many solutions with $x, y \in \mathbb{Q}$.

Uniformity

Problem

1. *Given X, compute $X(\mathbb{Q})$ exactly.*
2. *Compute bounds on $\#X(\mathbb{Q})$.*

Conjecture (Uniformity)

There exists a constant $N(g)$ such that every smooth curve of genus g over $\mathbb{Q}$ has at most $N(g)$ rational points.

Theorem (Caporaso, Harris, Mazur)

Lang's conjecture $\Rightarrow$ uniformity.

Uniformity numerics

g	2	3	4	5	10	45	g
$B_g(\mathbb{Q})$	642	112	126	132	192	781	$16(g+1)$

Remark

Elkies studied K3 surfaces of the form

$$y^2 = S(t, u, v)$$

with lots of rational lines, such that S restricted to such a line is a perfect square.

Main Theorem (partial uniformity for curves)

Theorem (Katz, Rabinoff, ZB)

Let X be any curve of genus g and let $r = \operatorname{rank}_{\mathbb{Z}} \operatorname{Jac}_X(\mathbb{Q})$. Suppose $r < g - 2$. Then

$$\#X(\mathbb{Q}) \leq 84g^2 - 98g + 28$$

Tools

p-adic integration on annuli

comparison of different analytic continuations of p-adic integration

Non-Archimedean (Berkovich) structure of a curve [BPR]

Combinatorial restraints coming from the Tropical canonical bundle

Berkovich picture