Math 271-02: Linear Algebra (Professor Zureick-Brown)

All assignments

Last updated: September 8, 2023

Gradescope code: 8EG7ND

Show all work for full credit!
Proofs should be written in full sentences whenever possible.

Learning goals for this problem set:

- Know and understand the definition of a vector space
- Understand the role of the phrases 'for all' and 'there exists' in the definition of a vector space
- Be able to prove that a given set with given addition and scalar multiplication operations is or is not a vector space
- Be able to prove facts that are true in any vector space
- Determine if a given quantified statement is true or false.

Suggested readings for this problem set: All readings are from Damiano and Little, *A Course in Linear Algebra*.

- (1.1) Vector spaces are the most important objects in linear algebra. (This is why we do them right at the beginning so you have plenty of time to get used to them.) You should know by heart the eight properties that make up the definition of a vector space. Pay particular attention to the phrases 'there exists', 'for all' and 'for each'. How do these phrases tell us how to approach the task of verifying that a given set and operations form a vector space?

Here are some additional readings if you would like review some concepts concerning proofs!: All readings are from Richard Hammack, *Book of Proof, 3rd ed.* (Note: page numbers refer to the page numbers on the book page not the page number in the PDF viewer!)

- Pages 34-45: Here you will read about what is and is not a mathematical statement. You will also see different types of mathematical statements including "conditional statements" (which are also known as "if-then" statements). In fact, if-then statements are some of the most common types of statements in mathematics and we will see them a lot this semester!

- Pages 53-57: This is all about quantifiers and statements that use them.

Book Problems from A Course in Linear Algebra by D. Damiano and J. Little:

- **§1.1:** 1, 2, 3 [properties 3,4,6 only for problem 3], 4 [properties 3,8 only for problem 4], 5, 6(a,c), 7(c), 8 [be careful with the presentation of your proof for 8!]

Work-out Problems: Let $\mathbb{Z} := \{\dots, -2, -1, 0, 1, 2, \dots\}$ be the set of integers and $\mathbb{N} := \{1, 2, 3, \dots\}$ be the set of natural numbers.

9. Determine if each of the following statements are true or false. Be sure to justify your answer.

   (a) $\exists x \in \mathbb{Z}$ s.t. $x^2 - x = 0$. (b) $\exists x \in \mathbb{N}$ s.t. $x < 0$. (c) $\forall x \in \mathbb{Z}$, $\sqrt{x^2} = x$.

10. Determine if each of the following statements are true or false. Be sure to justify your answer.

    (a) $\forall x \in \mathbb{N}, \exists\, y \in \mathbb{N}$ s.t. $x + y = 1$.

    (b) $\exists y \in \mathbb{N}, \forall\, x \in \mathbb{N}, x + y = 1$.

Extra Practice Problems from from A Course in Linear Algebra by D. Damiano and J. Little (not to turn in):

- **§1.1:** 3 [in full], 4 [in full], 6(b), 10, 11

IN PROGRESS

Assignment 2 *Due by 1:55pm eastern on Friday, Sep 22*

Learning goals for this problem set:

- Know, and be able to use, the definition of a subspace of a vector space
- Be able to show that a given subset either is or is not a subspace.
- Know and be able to use the definitions of linear combinations and the span of a set of vectors
- Be able to prove basic facts about subspaces, linear combinations and the span of a set of vectors
- Understand what is means to say that a subspace is generated (spanned) by a set of vectors

Suggested readings for this problem set: All readings are from Damiano and Little, *A Course in Linear Algebra*.

- §1.2: It's important to understand the different ways you can check that a subset is a subspace. While Theorem 1.2.8 is the quickest, the three conditions in the paragraph before that theorem give a better perspective on what a subspace really is. For example, if you are showing that something is *not* a subspace, these might be easier to think about.

- §1.3: Given a vector space V that contains some given set of vectors, we know that that V must contain the set of all linear combinations of the given vectors. This set is called the span of these vectors, and we say that this set generates (or spans) if the span equals all of V . Finding sets that generate V is a good way to start to understand the structure of V. To do this, we start by studying the properties of the span of a set of vectors.

Here are some additional readings if you would like review some concepts concerning proofs!: All readings are from Richard Hammack, *Book of Proof, 3rd ed.* (Note: page numbers refer to the page numbers on the book page not the page number in the PDF viewer!)

- Pages 113-117: This covers the basic meanings behind three important words used in mathematics: *Theorem*, *proof*, and *definition*.

- Pages 118-123: This is all about how to prove "if-then" type statements using a proof technique called *direct proof*.

Book Problems from A Course in Linear Algebra by D. Damiano and J. Little:

- **§1.2:** 1, 3(c,d,h,i), 5, 10, 11(*read the paragraph above 11!*)

  Note: For #1 in §1.2, we need to know that $c \cdot \vec{0} = \vec{0}$ for any $c \in \mathbb{R}$. You will prove this below!
- **§1.3:** 1(a,c,d), 3, 7

Work-out Problems:

9. Let V be a vector space and let $\vec{0}$ denote the zero vector in V. Using the axioms of a vector space, prove that for every $c \in \mathbb{R}$, $c \cdot \vec{0} = \vec{0}$.

10. Let S be a *subset* of a vector space V. Prove that $\text{span}(S) \subseteq V$.

Extra Practice Problems from from A Course in Linear Algebra by D. Damiano and J. Little (not to turn in):

- **§1.2:** 2, 6, 9, 12
- **§1.3:** 5, 8

IN PROGRESS
Assignment 3 *Due by 1:55pm eastern on Friday, Sep 29*

Learning goals for this problem set:

- Be able to prove basic facts about subspaces, linear combinations and the span of a set of vectors
- Understand the definitions related to linear dependence and linear independence
- Determine if a given set of vectors is linearly dependent or linearly independent
- Prove facts about linearly dependent and independent sets
- Take a given system of linear equations and write it in echelon form.
- Determine the set of solutions to a system of linear equations.

Suggested readings for this problem set: All readings are from Damiano and Little, *A Course in Linear Algebra*.

- §1.4: Once we find a set of vectors that generates a given vector space, we want to make sure that we have picked these vectors efficiently. That is to say, we will want to find a set of vectors that spans the given vector space that is as small as possible. It turns out that out spanning set is as small as possible if and only if that set is linearly independent. In order to see this, we first need to prove some basic fact about linearly dependent and independent sets.

- §1.5: Many questions about linear independence and the span of a set of vectors can be reduced down to solving systems of linear equations. Because of this, we will spend the section working out an algorithm to parametrize the solutions to a given systems of linear equations. This algorithm will play a fundamental role in the computations and theory that arise in the rest of the chapter.

Book Problems from A Course in Linear Algebra by D. Damiano and J. Little:

- **§1.3:** 6(a),
- **§1.4:** 1(e,f,h,i,j), 4(a), 5, 8, 9(a)
- **§1.5:** 1(b,d,e), 2(a,b)

Work-out Problems:

9. Let S and T be nonempty subsets of a vector space V. Suppose that $\text{span}(T) = V$ and $T \subseteq \text{span}(S)$. Prove that $\text{span}(S) = V$.

10. Let $S := \{\vec{v}_1, \ldots, \vec{v}_n\}$ be a *subset* of a vector space V and let $\alpha \in \mathbb{R}$ with $\alpha \neq 0$. Prove that S is linearly independent if and only if $S' := \{\alpha\vec{v}_1, \ldots, \alpha\vec{v}_n\}$ is linearly independent.

Extra Practice Problems from from A Course in Linear Algebra by D. Damiano and J. Little (not to turn in):

- **§1.4:** 2, 9(b), 10, 11

IN PROGRESS
Assignment 4 *Due by 1:55pm eastern on Friday, Oct 06*

Learning goals for this problem set:

- Take a given system of linear equations and write it in echelon form.
- Determine the set of solutions to a system of linear equations.
- Given a set of vectors, determine if a different vector is in the span of that set.

Suggested readings for this problem set: All readings are from Damiano and Little, *A Course in Linear Algebra*.

- §1.5: Many questions about linear independence and the span of a set of vectors can be reduced down to solving systems of linear equations. Because of this, we will spend the section working out an algorithm to parametrize the solutions to a given systems of linear equations. This algorithm will play a fundamental role in the computations and theory that arise in the rest of the chapter.

Book Problems from A Course in Linear Algebra by D. Damiano and J. Little:

- **§1.5:** 1(a,c), 2(c,d), 3(a,b,d), 5(a,b,c)

  For 1.5.3: be sure to explain how your calculations relate to the what the problem is asking!

Work-out Problems:

5. Let V be a vector space and suppose that S and S' are two *subsets* of V such that $S \subseteq S'$. Prove that $\text{span}(S) \subseteq \text{span}(S')$.

6. Let $\vec{v}$ be a vector in a vector space V and suppose that the set $\{\vec{v}\}$ is linearly independent. Prove that $\vec{v} \neq \vec{0}$.

7. Let $S := \{\vec{v}_1, \ldots, \vec{v}_n\}$ be a *subset* of a vector space V and let $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ be such that all $\alpha_i \neq 0$. Prove that S is linearly independent if and only if $S' := \{\alpha_1\vec{v}_1, \ldots, \alpha_n\vec{v}_n\}$ is linearly independent.

8. Let $T := \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ be a collection of vectors in a vector space V, and let $S := \{\vec{v}_1, \vec{v}_3\}$. Prove the following two statements:

   (a) If S is a linearly dependent set then T is a linearly dependent set.

   (b) If T is a linearly independent set then S is a linearly independent set.

   ***Problem 8 is a simplified version of a statement covered in lecture. You cannot use that to prove this!*

IN PROGRESS

Assignment 5 *Due by 1:55pm eastern on Friday, Oct 13*

Learning goals for this problem set:

- Be able to determine if a given set is a basis for a vector space.
- Be able to find a basis contained in a given spanning set.
- Be able to extend a given linearly independent set to a basis
- Be able to use bases to solve theoretical problems about vector spaces.
- Be able to use the fact that in a finite-dimensional vector space, any object can be written as a unique linear combination of the objects in a basis
- Be able to show that a given function either is or is not a linear transformation.

Suggested readings for this problem set: All readings are from Damiano and Little, *A Course in Linear Algebra*.

- §1.6: One of the key results of this section is (1.6.6). This is important both for theoretical reasons (it tells you that any vector space with a finite spanning set has a basis) and also for practical reasons (it gives us a method for producing a basis that contains a given linearly independent set). Look at the proof of this Theorem and make sure you can carry out this process. Also look back at (1.5.12) from the previous section, and #2 from the exercises of §1.5. These were all examples of finding a basis, even though that language was not used there.

- §2.1: You should know the definition of linear transformation and be able to decide if a given function is or is not linear. Proposition (2.1.14) and the Examples that come after it provide a really useful way to think about linear transformations, and how they relate to bases

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Book Problems from A Course in Linear Algebra by D. Damiano and J. Little:

- **§1.2:** 12 (only do Axioms A3 and A4)
- **§1.6:** 2(a,b,d), 3, 7(b), 9(a,b), 14(a,b,c)
- **§2.1:** 3(a,b,e,f,g), 4, 8

Work-out Problems:

10. What are all of the different kinds of subspaces of $\mathbb{R}^2$ and $\mathbb{R}^3$? What are they geometrically? (*Hint: Think about all of the possible dimensions of a subspace can have.*)

11. Let V be a finite dimensional vector space and set $n := \dim(V)$. Suppose that S is a linearly independent subset of V that contains n vectors. Prove that $\text{span}(S) = V$.

12. Let V be a finite dimensional vector space and set $n := \dim(V)$. Prove that no set containing fewer than n vectors can span V.

Learning goals for this problem set:

- Know how a linear transformation is determined by its values on the vectors in a basis.
- Calculate the coordinate vector of a given vector with respect to a given ordered basis.
- Calculate the matrix associated to a linear transformation with respect to given ordered bases.

Suggested readings for this problem set: All readings are from Damiano and Little, *A Course in Linear Algebra*.

- §2.1: You should know the definition of linear transformation and be able to decide if a given function is or is not linear. Proposition (2.1.14) and the Examples that come after it provide a really useful way to think about linear transformations, and how they relate to bases.
- §2.2: This section is about the relationship between linear transformations and matrices. The key skills are to be able to translate between objects in a vector space and their coordinate vectors, and between linear transformations and their matrices. Proposition (2.2.15) is the result that tells us how linear transformations are related to matrix multiplication. There is a lot of notation in this section, so work through it carefully and pay attention to all the different indices!

Book Problems from A Course in Linear Algebra by D. Damiano and J. Little:

- **§2.1:** 10, 11
- **§2.2:** 3(a,b), 6(a,b), 7(a,b,c), 8(a), 9(a)[1] , 11(a,b)

Work-out Problems (to turn in):

9. Let V be a vector space. We have regularly used the fact that vector spaces are closed under *finite* linear combinations of vectors but we have not rigorously proved, but we will now–yay! See the vector space axioms only tell us that $a\vec{x} + b\vec{y} \in V$ whenever $a, b \in \mathbb{R}$ and $\vec{x}, \vec{y} \in V$ and so we will now use induction to prove the following fact: *for every $n \in \mathbb{N}$, if $c_1, \dots, c_n \in \mathbb{R}$ and $\vec{v}_1, \dots, \vec{v}_n \in V$ then $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n \in V$.*

10. Let $\alpha := \{\vec{v}_1, \dots, \vec{v}_n\}$ be a basis for a finite dimensional vector space V and define a mapping $T\colon V \to \mathbb{R}^n$ by setting $T(\vec{v}) := [\vec{v}]_\alpha$ for every $\vec{v} \in V$. Prove that T is a linear map.

11. Let $\alpha = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ be the standard basis in $\mathbb{R}^3$ and let $\beta := \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$. Take my word for it that β is a basis for $\mathbb{R}^3$.

    (a) Compute the coordinate vectors $[(3, 5, 7)]_\alpha$ and $[(3, 5, 7)]_\beta$.

    (b) For an arbitrary vector $(x, y, z) \in \mathbb{R}^3$, compute the coordinate vector $[(x, y, z)]_\alpha$ and $[(x, y, z)]_\beta$.

[1] Compare $[I]_\alpha^\beta$ from this problem to the matrix $[I]_{\text{std}}^{\text{std}}$ that we computed in class. See how different choices of bases α and β can alter the matrix representation of a linear operator??

12. Let $T : P_2(\mathbb{R}) \to P_1(\mathbb{R})$ be the function given by

$$T(p(x)) = 3p'(x) + p(2), \quad \text{for all } p \in P_2(\mathbb{R}).$$

Let $\alpha := \{1, x, x^2\}$ and $\beta := \{1, x\}$ be the standard bases for $P_2(\mathbb{R})$ and $P_1(\mathbb{R})$, respectively. Also, let $\gamma := \{1, 1 - x, 1 + x + x^2\}$. Take my word for it that γ is a basis for $P_2(\mathbb{R})$.

(a) Calculate the coordinate vectors $[2x^2 - x - 3]_\alpha$ and $[2x^2 - x - 3]_\gamma$.

(b) Prove that T is a linear transformation.

(c) Calculate the matrices $[T]_\alpha^\beta$ and $[T]_\gamma^\beta$.

Extra Practice Problems from from A Course in Linear Algebra by D. Damiano and J. Little (not to turn in):

- **§2.1:** 12
- **§2.2:** 2.2: 12, 13

Learning goals for this problem set:

- Calculate the coordinate vector of a given vector with respect to a given ordered basis.
- Calculate the matrix associated to a linear transformation with respect to given ordered bases.
- Calculate the linear transformation associated to a matrix with respect to given ordered bases.
- Use coordinate representations of objects to make calculations for a linear transformation.

Suggested readings for this problem set: All readings are from Damiano and Little, *A Course in Linear Algebra*.

- §2.3: There are two important subspaces associated with every linear transformation, namely, the kernel and image. Understanding the properties of these subspaces reveal information about the transformation. In particular, computing bases for these two subspaces will shed light on the behavior of a given linear transformation. We will also see that, while these spaces seem to be unrelated to each other, their dimensions are linked through The Dimension Theorem (Theorem 2.3.17 in the book).

Book Problems from A Course in Linear Algebra by D. Damiano and J. Little:

- **§2.3:** 1(a,d,e,f), 3(b,c), 5, 7(a),

Work-out Problems (to turn in):

5. Let $\alpha := \{1 + x^2, x - 3x^2, 1 + x - 3x^2\}$. Take my word for it that α is a basis for $P_2(\mathbb{R})$. Find a vector $p \in P_2(\mathbb{R})$ such that
$$[p]_\alpha = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}.$$

6. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation given by $T((x, y)) := (2x - y, x + 3y, -y)$ for all $(x, y) \in \mathbb{R}^2$.
   (a) Find the matrix of T with respect to the standard basis $\alpha := \{\vec{e}_1, \vec{e}_2\}$ of $\mathbb{R}^2$ and the standard basis $\beta := \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ of $\mathbb{R}^2$. That is, find $[T]_\alpha^\beta$.
   (b) Let $\alpha' := \{\vec{e}_2, \vec{e}_1\}$ be the standard basis of $\mathbb{R}^2$, but in the *opposite order*. Find the matrix $[T]_{\alpha'}^\beta$.
   (c) Let $\beta' := \{\vec{e}_1, \vec{e}_3, \vec{e}_2\}$ be the standard basis of $\mathbb{R}^3$, but with the *order of the last two vectors swapped*. Find the matrix $[T]_\alpha^{\beta'}$.
   (d) Now let V and W be any finite-dimensional vector spaces, and let $T\colon V \to W$ be a linear transformation. Let α be an ordered basis of V, and let β be an ordered basis of W. Based on your answers to parts (a)-(c), write down a general rule for what happens to the matrix $[T]_\alpha^\beta$ when you (i) swap two of the vectors in the basis α, or (ii) swap two of the vectors in the basis β. (*You do not need to prove that your rule is correct.*)

7. Let $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ be the function given by
$$T(p(x)) = xp(x) - p'(x).$$

(a) Prove that T is a linear transformation.

(b) Calculate the matrix $[T]_\alpha^\beta$, where $\alpha := \{1, x, x^2\}$ and $\beta := \{1, x, x^2, x^3\}$, are the standard bases of $P_2(\mathbb{R})$ and $P_3(\mathbb{R})$, respectively.

(c) Calculate the coordinate vectors $[2x^2 - x - 3]_\alpha$ and $[T(2x^2 - x - 3)]_\beta$.

(d) Do matrix multiplication to verify that the equation

$$[T(2x^2 - x - 3)]_\beta = [T]_\alpha^\beta [2x^2 - x - 3]_\alpha$$

is true.

(e) Find a basis for the $\ker(T)$.

(f) Find a basis for the $\operatorname{im}(T)$.

8. Let $\alpha := \{1, 1 + x, 1 + x + x^2\} \subseteq P_2(\mathbb{R})$ and $\beta := \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ be the standard basis for $\mathbb{R}^3$. (You can take my word that α is a bases for $P_2(\mathbb{R})$.) Suppose that $T\colon P_2(\mathbb{R}) \to \mathbb{R}^3$ is a linear transformation such that

$$[T]_\alpha^\beta = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 2 & 3 & 1 \end{bmatrix}.$$

(a) Find the value of $T(1 + 2x + x^2)$.

(b) Find a basis for $\ker(T)$.

(c) Find a basis for $\operatorname{im}(T)$.

9. Let $\alpha := \{\vec{v}_1, \ldots, \vec{v}_n\}$ be a basis for a finite dimensional vector space V and define a mapping $T\colon V \to \mathbb{R}^n$ by setting $T(\vec{v}) := [\vec{v}]_\alpha$ for every $\vec{v} \in V$. Recall from your last homework assignment that T is a linear map.

(a) Let $\beta := \{\vec{e}_1, \ldots, \vec{e}_n\}$ be the standard basis in $\mathbb{R}^n$. Calculate the matrix $[T]_\alpha^\beta$.

(b) Find a basis for the $\ker(T)$.

(c) Find a basis for the $\operatorname{im}(T)$.

Extra Practice Problems from from A Course in Linear Algebra by D. Damiano and J. Little (not to turn in):

- **§2.3:** 1(b,e), 4, 8, 12

IN PROGRESS
Assignment 8 *Due by 1:55pm eastern on Friday, Nov 03*

Learning goals for this problem set:

- Decide if a given linear transformation is injective and/or surjective.
- Solve an inhomogeneous linear system by solving the associated homogeneous system.
- Prove basic facts about the composition of two linear transformations.

Suggested readings for this problem set: All readings are from Damiano and Little, *A Course in Linear Algebra*.

- §2.4: Our main application of the Rank-Nullity Theorem is to understand when a linear transformation can be injective and/or surjective. You should be able to prove the main results on this topic, and be able to apply these concepts in specific cases. This section also considers how to solve an equation of the form $T(\vec{x}) = \vec{b}$ where $\vec{b}$ is not the zero object.
- §2.5: The composition of two compatible linear transformations is again a linear transformation. Restricting to the case of finite-dimensional vector spaces, we are interested in computing the composition's matrix representation as a function of the matrix representations of the components. Motivated by this formula, we define matrix multiplication so that the product of two matrix representations give the matrix representation of the composition.

Book Problems from A Course in Linear Algebra by D. Damiano and J. Little:

- **§2.4:** 1(a,c), 2(b,c,d), 3(a,b), 4(d,e), 5(a,c), 6(a)

  Comments: For 2.4.2, the blank entries in the matrix should be treated as zeros.
- **§2.5:** 2 (only prove part (ii))

Work-out Problems (to turn in):

8. Suppose that $T\colon V \to W$ is a linear map where V is finite dimensional (no assumptions are made about the dimension of W). Prove that T is injective if and only if $\dim(\operatorname{im}(T)) = \dim(V)$.
9. Suppose that $T\colon V \to W$ is a linear map where V and W are finite dimensional. Prove that T is surjective if and only if $\dim(\operatorname{im}(T)) = \dim(W)$.

Extra Practice Problems from from A Course in Linear Algebra by D. Damiano and J. Little (not to turn in):

- **§2.4:** 1(b,d,e), 2(e,f), 3(c), 4(a,b,c,f), 10
- **§2.5:** 2 (part (iii))

IN PROGRESS
Assignment 9 *Due by 1:55pm eastern on Friday, Nov 10*

Learning goals for this problem set:

- Prove basic facts about the composition of two linear transformations.
- Compute the composition of two linear transformations.
- Compute the matrix representation of a composition of linear transformations.
- Be able to multiply matrices when compatible
- Show a particular function is invertible.
- Compute the inverse of a give linear transformation.
- Determine if two vector spaces are isomorphic.

Suggested readings for this problem set: All readings are from Damiano and Little, *A Course in Linear Algebra*.

- §2.5: The composition of two compatible linear transformations is again a linear transformation. Restricting to the case of finite-dimensional vector spaces, we are interested in computing the composition's matrix representation as a function of the matrix representations of the components. Motivated by this formula, we define matrix multiplication so that the product of two matrix representations give the matrix representation of the composition.
- §2.6: A natural question to ask is given a linear map, when is that map invertible? In this section, we will start to answer this questions as well as a few other. The main goals are to be able to determine when a linear map is invertible and when it is, compute the inverse.

Book Problems from A Course in Linear Algebra by D. Damiano and J. Little:

- **§2.5:** 5(b,c,d,e), 6(b), 7(b), 8(a,b), 12(a,b)

  Comments: For 2.5.5, if the matrix product is not possible then explain why.
- **§2.6:** 1(a,b), 2(a,c),

  Comments: For 2.6.2c, think about what is the "standard basis" for $M_{2\times 3}(\mathbb{R})$!

Work-out Problems (to turn in):

8. Recall the following result from class: If $S : U \to V$ and $T : V \to W$ are linear maps then $\ker(S) \subseteq \ker(TS)$ and $\operatorname{im}(TS) \subseteq \operatorname{im}(T)$.
   (a) Find examples of linear maps S_1, S_2, T_1, T_2 where $\ker(S_1)$ is a *proper* subset of $\ker(T_1S_1)$ and $\operatorname{im}(T_2S_2)$ is a *proper* subset of $\operatorname{im}(T_2)$. (You do not need all of your linear maps to be different!)
   (b) Let $S : U \to V$ be a surjective linear transformation and $T : V \to W$ be *any* linear map. Prove that $\operatorname{im}(TS) = \operatorname{im}(T)$.
9. Let $T : V \to W$ be an isomorphism and suppose that $\alpha := \{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis for V. Prove that $\{T(\vec{v}_1), \ldots, T(\vec{v}_n)\}$ is a basis for W.

10. Suppose $A, B, C \in M_{n\times n}(\mathbb{R})$. Prove that $A(B + C) = AB + AC$. (*Hint: Think of matrices as linear maps!*)

Extra Practice Problems from from A Course in Linear Algebra by D. Damiano and J. Little (not to turn in):

- **§2.5:** 10, 11, 12(b), 13, 17
- **§2.6:** The rest of 1, 2, and 3

Learning goals for this problem set:

- Compute the change-of-basis matrix for two different bases of the same vector space.
- Use the change-of-basis formula to compute the matrix representation of a linear transformation.
- Calculate the determinant of an 2×2 matrix using row operations as appropriate to simplify the calculation.

Suggested readings for this problem set: All readings are from Damiano and Little, *A Course in Linear Algebra*.

- §2.7: Many times we have been given a matrix representation of a linear transformation with respect to two fixed bases. It is reasonable to think that we might want to be able to use that given information to find the matrix representation of that same linear transformation with respect to two new, possibly more convenient bases. At the heart of this task is finding a way to convert coordinate vectors with respect to one basis into coordinate vectors with respect to a different basis.

Book Problems from A Course in Linear Algebra by D. Damiano and J. Little:

- **§2.7:** 1(a,b), 2, 3, 4(b,c)
- **§3.1:** 3, 4, 6

Work-out Problems (to turn in):

8. Let $A, B \in M_{n\times n}(\mathbb{R})$.
   (a) Prove that A is similar to itself.
   (b) Suppose that A is similar to B. Prove that B is similar to A.
   (*This means that order does not matter when talking about similar matrices and we can simply say "A and B are similar matrices".*)
   (c) Suppose that A is similar to B and B is similar to $C \in M_{n\times n}(\mathbb{R})$. Prove that A is similar to C.

IN PROGRESS

Assignment 11 *Due by 1:55pm eastern on Friday, Dec 01*

Learning goals for this problem set:

- Compute the change-of-basis matrix for two different bases of the same vector space.
- Use the change-of-basis formula to compute the matrix representation of a linear transformation.
- Calculate the determinant of an 2×2 matrix using row operations as appropriate to simplify the calculation.

Suggested readings for this problem set: All readings are from Damiano and Little, *A Course in Linear Algebra*.

- §2.7: Many times we have been given a matrix representation of a linear transformation with respect to two fixed bases. It is reasonable to think that we might want to be able to use that given information to find the matrix representation of that same linear transformation with respect to two new, possibly more convenient bases. At the heart of this task is finding a way to convert coordinate vectors with respect to one basis into coordinate vectors with respect to a different basis.

- §3.1-3.2: The book has a lot of information on determinants that you do not really need to know. It goes into a lot more depth than is necessary for this course. (But it is all interesting stuff, so I encourage you to read it!) For us, the important parts are the determinant of a 2×2-matrix (3.1.5) and of an n×n-matrix (3.2.7). It is also important to be able to use row operations to help simplify the calculation of a determinant, such as in (3.2.15).

Book Problems from A Course in Linear Algebra by D. Damiano and J. Little:

- **§2.7:** 2, 3, 4(b,c)
- **§3.1:** 10
- **§3.2:** 1(a,c), 2(b,c), 4(b,d), 6(a)

Work-out Problems (to turn in):

9. Let $A, B \in M_{n\times n}(\mathbb{R})$.

   (a) Prove that A is similar to itself.

   (b) Suppose that A is similar to B. Prove that B is similar to A.
   (*This means that order does not matter when talking about similar matrices and we can simply say "A and B are similar matrices".*)

   (c) Suppose that A is similar to B and B is similar to $C \in M_{n\times n}(\mathbb{R})$. Prove that A is similar to C.

10. Let $A, B \in M_{n\times n}(\mathbb{R})$ and suppose that there exists an invertible matrix $Q \in M_{n\times n}(\mathbb{R})$ such that $A = Q^{-1}BQ$. (So A and B are similar matrices.) Use induction to prove that $A^k = Q^{-1}B^kQ$ for all $k \in \mathbb{N}$.

11. Let $c \in \mathbb{R}$ and consider the following elementary matrices:

$$E_{13}(c) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c & 0 & 1 \end{bmatrix} \qquad E_2(c) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad G_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Prove that

(a) $\det(E_{13}(c)) = 1$

(b) $\det(E_2(c)) = c$

(c) $\det(G_{13}) = -1$

Note that you are just verifying the proposition from class regarding the determinants of elementary matrices! You cannot use that result for this problem!

12. Let $A := \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Verify the formula

$$\det(EA) = \det(E)\det(A)$$

for each of the following elementary matrices:

$$E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad E = \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix} \qquad E = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$$

IN PROGRESS

Assignment 12 — *Due by 1:55pm eastern on Friday, Dec 08*

Learning goals for this problem set:

- Use the determinant to decide if a matrix is, or is not, invertible.
- Calculate the determinant of an $n \times n$ matrix using row operations as appropriate to simplify the calculation.
- Prove basic facts using properties of determinants.
- Calculate the eigenvalues and eigenvectors of a linear map or $n \times n$ matrix.
- Calculate the characteristic polynomial of a linear map or $n \times n$ matrix.

Suggested readings for this problem set: All readings are from Damiano and Little, *A Course in Linear Algebra*.

- §3.3: The most important part of this is the multiplicative property for the determinant (3.3.7). The consequences of this include (3.3.8) which tells us that the determinant is something that can be assigned to any linear transformation $T : V \to V$.
- §4.1: This whole section is very important. Focus on the process of finding the eigenvalues and eigenvectors of a matrix, or of a linear transformation.

Book Problems from A Course in Linear Algebra by D. Damiano and J. Little:

- **§3.3:** 2(b), 6, 7(a), 10
- **§4.1:** 1(b,c), 2(b,c), 3(b,d)

  Comments: For 3.3.7a, check you answer by solving the system using the techniques we learned in §1.5 (row reduction!). Try using induction for 3.3.10!

Work-out Problems (to turn in):

8. Let $A, B \in M_{n\times n}(\mathbb{R})$.
   (a) Suppose that A and B are similar. Prove that $\det(A) = \det(B)$.
   (b) Prove that the converse to the statement in part a is false, in general. That is, find matrices A and B (of any size you wish) such that $\det(A) = \det(B)$ but A is not similar to B.
9. We saw in class that $\det(AB) = \det(A)\det(B)$. Provide an example of two matrices A and B to show that the equality $\det(A + B) = \det(A) + \det(B)$ is not true, in general.
10. Let $A \in M_{n\times n}(\mathbb{R})$. Use induction to prove that $\det\left(A^k\right) = [\det(A)]^k$ for every $k \in \mathbb{N}$.
11. Let $A \in M_{n\times n}(\mathbb{R})$ and suppose that $\lambda \in \mathbb{R}$ is an eigenvalue of A with eigenvector $\vec{x} \in \mathbb{R}^n$. Use induction to prove that for every $k \in \mathbb{N}$, $\lambda^k \in \mathbb{R}$ is an eigenvalue of A^k with eigenvector $\vec{x}$.
12. Prove that a matrix $A \in M_{n\times n}(\mathbb{R})$ is not invertible if and only if $\lambda = 0 \in \mathbb{R}$ is an eigenvalue of A.