Beyond Fermat's Last Theorem

David Zureick-Brown

Emory University
Department of Mathematics and Computer Science

February 21, 2012

Slides available at `http://www.mathcs.emory.edu/~dzb/slides/`

Basic Problem

Problem (Solving Diophantine Equations)

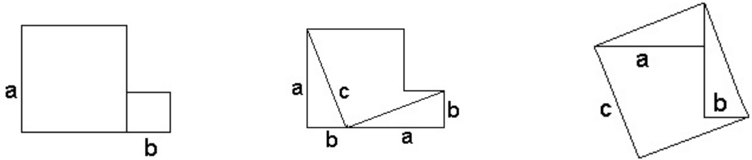
Let $f(x_1, \ldots, x_n)$ be a polynomial and find all solutions to the equation

$$f(a_1, \ldots, a_n) = 0,$$

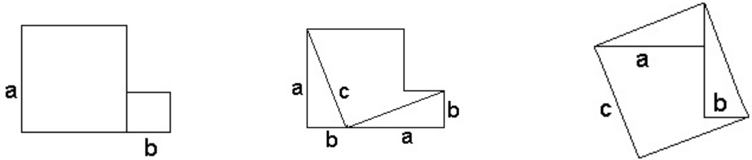
where $a_1, \ldots, a_n$ are **rational** *numbers.*

Pythagorean Triples

$$\boxed{a^2 + b^2 = c^2}$$



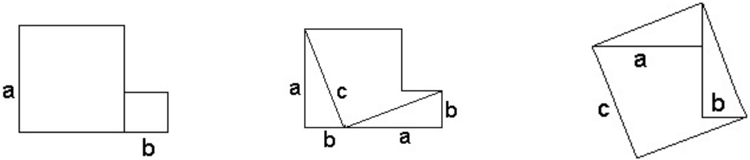
$$\boxed{a^2 + b^2 = c^2}$$



- $(3, 4, 5) : 3^2 + 4^2 - 5^2 = 9 + 16 - 25 = 0.$

Pythagorean Triples

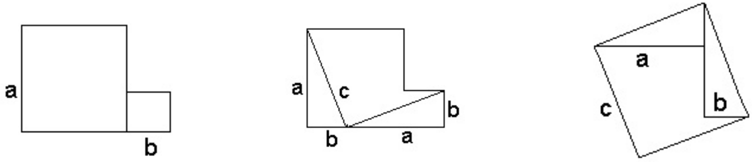
$$\boxed{a^2 + b^2 = c^2}$$



- $(3, 4, 5) : 3^2 + 4^2 - 5^2 = 9 + 16 - 25 = 0.$
- $(5, 12, 13) : 5^2 + 12^2 - 13^2 = 25 + 144 - 169 = 0.$

Pythagorean Triples

$$\boxed{a^2 + b^2 = c^2}$$



- $(3,4,5) : 3^2 + 4^2 - 5^2 = 9 + 16 - 25 = 0.$
- $(5,12,13) : 5^2 + 12^2 - 13^2 = 25 + 144 - 169 = 0.$
- $(1,2,\sqrt{5}) : 1^2 + 2^2 - \sqrt{5}^2 = 1 + 4 - 5 = 0.$

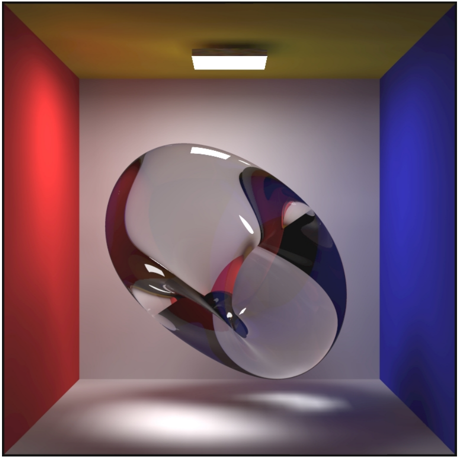
Basic problem

Fact

Solving Diophantine equations is **hard***!*

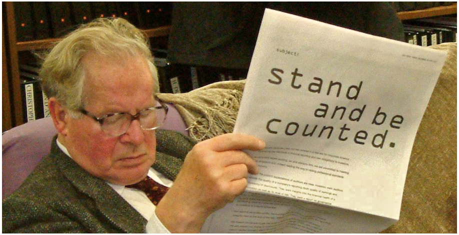
The Swinnerton-Dyer K3 surface

$$\boxed{x^4 + 2y^4 = 1 + 4z^4}$$



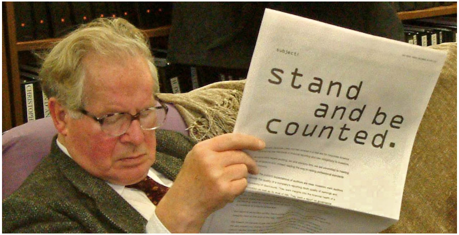
The Swinnerton-Dyer K3 surface

$$\boxed{x^4 + 2y^4 = 1 + 4z^4}$$



The Swinnerton-Dyer K3 surface

$$\boxed{x^4 + 2y^4 = 1 + 4z^4}$$



Two 'obvious' solutions – $(\pm 1 : 0 : 0)$.

The Swinnerton-Dyer K3 surface

$$\boxed{x^4 + 2y^4 = 1 + 4z^4}$$

- Two ‘obvious’ solutions – $(\pm 1 : 0 : 0)$.

The Swinnerton-Dyer K3 surface

$$\boxed{x^4 + 2y^4 = 1 + 4z^4}$$

- Two 'obvious' solutions – $(\pm 1 : 0 : 0)$.
- The next smallest solutions are $(\pm\frac{1484801}{1169407}, \pm\frac{1203120}{1169407}, \pm\frac{1157520}{1169407})$.

The Swinnerton-Dyer K3 surface

$$\boxed{x^4 + 2y^4 = 1 + 4z^4}$$

- Two 'obvious' solutions – $(\pm 1 : 0 : 0)$.
- The next smallest solutions are $(\pm\frac{1484801}{1169407}, \pm\frac{1203120}{1169407}, \pm\frac{1157520}{1169407})$.

Problem

Find another solution.

The Swinnerton-Dyer K3 surface

$$\boxed{x^4 + 2y^4 = 1 + 4z^4}$$

- Two 'obvious' solutions – $(\pm 1 : 0 : 0)$.
- The next smallest solutions are $(\pm\frac{1484801}{1169407}, \pm\frac{1203120}{1169407}, \pm\frac{1157520}{1169407})$.

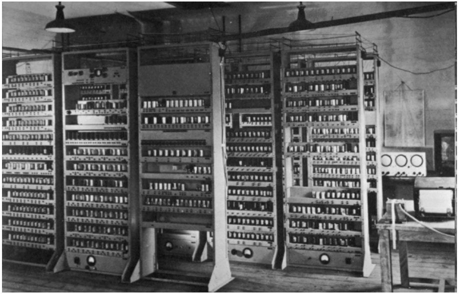
Problem

Find another solution.

(Mathematicians aren't sure if there are any more!)

Back of the envelope calculation

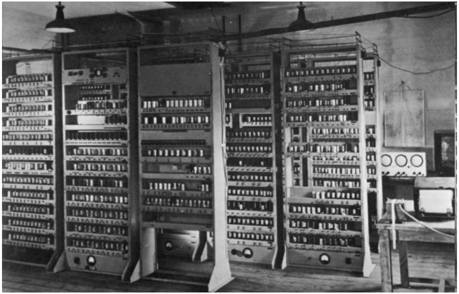
$$\boxed{x^4 + 2y^4 = 1 + 4z^4}$$



- Num, den$(x, y, z) \leq 10 \sim 10^6$ many, **1 min** on Emory's computers.

Back of the envelope calculation

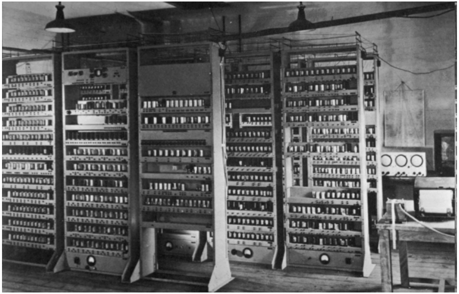
$$\boxed{x^4 + 2y^4 = 1 + 4z^4}$$



- Num, den$(x, y, z) \leq 10 \sim 10^6$ many, **1 min** on Emory's computers.
- Num, den$(x, y, z) \leq 1500000 \sim 10^{37}$ many, $\mathbf{10^{31}}$ **mins** $=$ $\mathbf{10^{25}}$ **years**.

Back of the envelope calculation

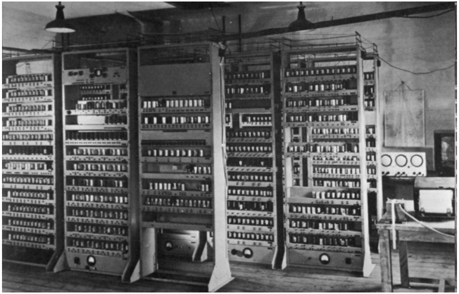
$$\boxed{x^4 + 2y^4 = 1 + 4z^4}$$



- Num, den$(x, y, z) \leq 10 \sim 10^6$ many, **1 min** on Emory's computers.
- Num, den$(x, y, z) \leq 1500000 \sim 10^{37}$ many, $\mathbf{10^{31}}$ **mins** $=$ $\mathbf{10^{25}}$ **years**.
- 10^9 many computers in the world – so $\mathbf{10^{16}}$ **years**

Back of the envelope calculation

$$\boxed{x^4 + 2y^4 = 1 + 4z^4}$$



- Num, den$(x, y, z) \leq 10 \sim 10^6$ many, **1 min** on Emory's computers.
- Num, den$(x, y, z) \leq 1500000 \sim 10^{37}$ many, $\mathbf{10^{31}}$ **mins** $=$ $\mathbf{10^{25}}$ **years**.
- 10^9 many computers in the world – so $\mathbf{10^{16}}$ **years**
- Age of the universe – **13.75** $\pm$ **.11 billion years** (roughly $\mathbf{10^{10}}$).

Congruent number problem

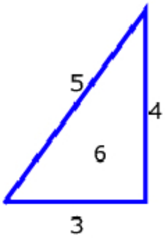
The pair of equations

$$\boxed{x^2 + y^2 = z^2,\ xy = 2 \cdot 157}$$

has **infinitely many** solutions. **How large** Is the smallest solution? How many **digits** does the smallest solution have?

Congruent number problem

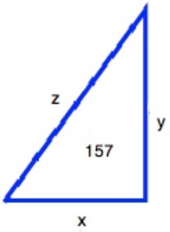
$$\boxed{x^2 + y^2 = z^2,\ xy = 2 \cdot 6}$$



$$3^2 + 4^2 = 5^2,\ 3 \cdot 4 = 2 \cdot 6$$

Congruent number problem

$$\boxed{x^2 + y^2 = z^2,\ xy = 2 \cdot 157}$$



Congruent number problem

$$\boxed{x^2 + y^2 = z^2,\ xy = 2 \cdot 157}$$

has **infinitely many** solutions. **How large** Is the smallest solution? How many **digits** does the smallest solution have?

Congruent number problem

$$\boxed{x^2 + y^2 = z^2,\ xy = 2 \cdot 157}$$

has **infinitely many** solutions. **How large** Is the smallest solution? How many **digits** does the smallest solution have?

$$x = \frac{157841 \cdot 4947203 \cdot 52677109576}{2 \cdot 3^2 \cdot 5 \cdot 13 \cdot 17 \cdot 37 \cdot 101 \cdot 17401 \cdot 46997 \cdot 356441}$$

$$y = \frac{2 \cdot 3^2 \cdot 5 \cdot 13 \cdot 17 \cdot 37 \cdot 101 \cdot 157 \cdot 17401 \cdot 46997 \cdot 356441}{157841 \cdot 4947203 \cdot 52677109576}$$

$$z = \frac{20085078913 \cdot 1185369214457 \cdot 942545825502442041907480}{2 \cdot 3^2 \cdot 5 \cdot 13 \cdot 17 \cdot 37 \cdot 101 \cdot 17401 \cdot 46997 \cdot 356441 \cdot 157841 \cdot 4947203 \cdot 52677109576}$$

Congruent number problem

$$\boxed{x^2 + y^2 = z^2,\ xy = 2 \cdot 157}$$

has **infinitely many** solutions. **How large** Is the smallest solution? How many **digits** does the smallest solution have?

$$x = \frac{157841 \cdot 4947203 \cdot 52677109576}{2 \cdot 3^2 \cdot 5 \cdot 13 \cdot 17 \cdot 37 \cdot 101 \cdot 17401 \cdot 46997 \cdot 356441}$$

$$y = \frac{2 \cdot 3^2 \cdot 5 \cdot 13 \cdot 17 \cdot 37 \cdot 101 \cdot 157 \cdot 17401 \cdot 46997 \cdot 356441}{157841 \cdot 4947203 \cdot 52677109576}$$

$$z = \frac{20085078913 \cdot 1185369214457 \cdot 942545825502442041907480}{2 \cdot 3^2 \cdot 5 \cdot 13 \cdot 17 \cdot 37 \cdot 101 \cdot 17401 \cdot 46997 \cdot 356441 \cdot 157841 \cdot 4947203 \cdot 52677109576}$$

The denominator of z has **44 digits**!

Congruent number problem

$$\boxed{x^2 + y^2 = z^2,\ xy = 2 \cdot 157}$$

has **infinitely many** solutions. **How large** Is the smallest solution? How many **digits** does the smallest solution have?

$$x = \frac{157841 \cdot 4947203 \cdot 52677109576}{2 \cdot 3^2 \cdot 5 \cdot 13 \cdot 17 \cdot 37 \cdot 101 \cdot 17401 \cdot 46997 \cdot 356441}$$

$$y = \frac{2 \cdot 3^2 \cdot 5 \cdot 13 \cdot 17 \cdot 37 \cdot 101 \cdot 157 \cdot 17401 \cdot 46997 \cdot 356441}{157841 \cdot 4947203 \cdot 52677109576}$$

$$z = \frac{20085078913 \cdot 1185369214457 \cdot 942545825502442041907480}{2 \cdot 3^2 \cdot 5 \cdot 13 \cdot 17 \cdot 37 \cdot 101 \cdot 17401 \cdot 46997 \cdot 356441 \cdot 157841 \cdot 4947203 \cdot 52677109576}$$

The denominator of z has **44 digits**!
How did anyone ever find this solution?

Congruent number problem

$$\boxed{x^2 + y^2 = z^2,\ xy = 2 \cdot 157}$$

has **infinitely many** solutions. **How large** Is the smallest solution? How many **digits** does the smallest solution have?

$$x = \frac{157841 \cdot 4947203 \cdot 52677109576}{2 \cdot 3^2 \cdot 5 \cdot 13 \cdot 17 \cdot 37 \cdot 101 \cdot 17401 \cdot 46997 \cdot 356441}$$

$$y = \frac{2 \cdot 3^2 \cdot 5 \cdot 13 \cdot 17 \cdot 37 \cdot 101 \cdot 157 \cdot 17401 \cdot 46997 \cdot 356441}{157841 \cdot 4947203 \cdot 52677109576}$$

$$z = \frac{20085078913 \cdot 1185369214457 \cdot 942545825502442041907480}{2 \cdot 3^2 \cdot 5 \cdot 13 \cdot 17 \cdot 37 \cdot 101 \cdot 17401 \cdot 46997 \cdot 356441 \cdot 157841 \cdot 4947203 \cdot 52677109576}$$

The denominator of z has **44 digits**!
How did anyone ever find this solution?
"Next" soluton has **176 digits**!

Back of the envelope calculation

$$\boxed{x^2 + y^2 = z^2,\ xy = 2 \cdot 157}$$

- Num, den$(x, y, z) \leq 10 \sim 10^6$ many, **1 min** on Emory's computers.

Back of the envelope calculation

$$\boxed{x^2 + y^2 = z^2,\ xy = 2 \cdot 157}$$

- Num, den$(x, y, z) \leq 10 \sim 10^6$ many, **1 min** on Emory's computers.
- Num, den$(x, y, z) \leq 10^{44} \sim 10^{264}$ many, $\mathbf{10^{258}}$ **mins** $=$ $\mathbf{10^{252}}$ **years**.

Back of the envelope calculation

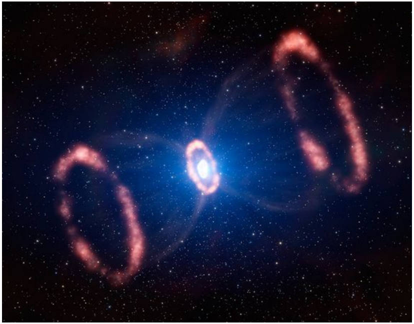
$$\boxed{x^2 + y^2 = z^2,\ xy = 2 \cdot 157}$$

- Num, den$(x, y, z) \leq 10 \sim 10^6$ many, **1 min** on Emory's computers.
- Num, den$(x, y, z) \leq 10^{44} \sim 10^{264}$ many, $\mathbf{10^{258}}$ **mins** $=$ $\mathbf{10^{252}}$ **years**.
- 10^9 many computers in the world – so $\mathbf{10^{243}}$ **years**

Back of the envelope calculation

$$\boxed{x^2 + y^2 = z^2,\ xy = 2 \cdot 157}$$

- Num, den$(x, y, z) \leq 10 \sim 10^6$ many, **1 min** on Emory's computers.
- Num, den$(x, y, z) \leq 10^{44} \sim 10^{264}$ many, $\mathbf{10^{258}}$ **mins** $=$ $\mathbf{10^{252}}$ **years**.
- 10^9 many computers in the world – so $\mathbf{10^{243}}$ **years**
- Expected time of 'heat death' of universe – $\mathbf{10^{100}}$ **years**.



Pythagorean triples

Lemma

The equation

$$a^2 + b^2 = c^2$$

has infinitely many non-zero coprime solutions.

Pythagorean triples

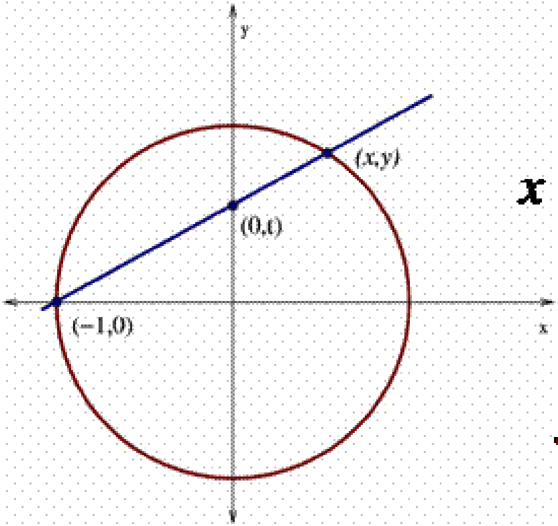
Lemma

The equation

$$a^2 + b^2 = c^2$$

has infinitely many non-zero coprime solutions.

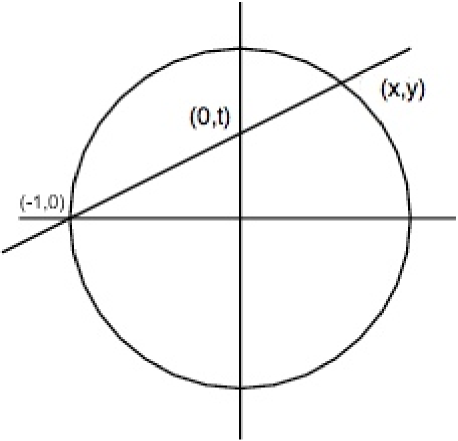
Lets work this out in detail.



$$x = \frac{a}{c} \qquad y = \frac{b}{c}$$

$$x^2 + y^2 = 1$$

Pythagorean triples



$$\text{Slope} = \quad t = \frac{y}{x+1}$$
$$x = \frac{1-t^2}{1+t^2}$$
$$y = \frac{2t}{1+t^2}$$

Pythagorean triples

Lemma

The solutions to

$$a^2 + b^2 = c^2$$

are all multiples of the triples

$$\boxed{a = 1 - t^2} \; \boxed{b = 2t} \; \boxed{c = 1 + t^2}$$

Fermat's Last Theorem

Theorem (Wiles et. al)

The only solutions to the equation

$$x^n + y^n = z^n, n \geq 3$$

are multiples of the triples

$$(0,0,0), \quad (\pm 1, \mp 1, 0), \quad \pm(1,0,1), \quad (0, \pm 1, \pm 1).$$

Fermat's Last Theorem

Theorem (Wiles et. al)

The only solutions to the equation

$$x^n + y^n = z^n, n \geq 3$$

are multiples of the triples

$$(0,0,0), \quad (\pm 1, \mp 1, 0), \quad \pm(1,0,1), \quad (0, \pm 1, \pm 1).$$

This took 300 years to prove!

Fermat's Last Theorem

Theorem (Wiles et. al)

The only solutions to the equation

$$x^n + y^n = z^n, n \geq 3$$

are multiples of the triples

$$(0,0,0), \quad (\pm 1, \mp 1, 0), \quad \pm(1,0,1), \quad (0, \pm 1, \pm 1).$$

This took 300 years to prove!

Basic Problem: $f_1, \ldots, f_m \in \mathbb{Z}[x_1, ..., x_n]$

- **Qualitative**:
  - Does there exist a solution?
  - Do there exist infinitely many solutions?
  - Does the set of solutions have some extra structure (e.g., geometric structure, group structure).

Basic Problem: $f_1, \ldots, f_m \in \mathbb{Z}[x_1, ..., x_n]$

- **Qualitative**:
  - Does there exist a solution?
  - Do there exist infinitely many solutions?
  - Does the set of solutions have some extra structure (e.g., geometric structure, group structure).
- **Quantitative**
  - How many solutions are there?
  - How large is the smallest solution?
  - How can we expliclty find all solutions (and prove that we have them all)?

Basic Problem: $f_1, \ldots, f_m \in \mathbb{Z}[x_1, ..., x_n]$

- **Qualitative**:
  - Does there exist a solution?
  - Do there exist infinitely many solutions?
  - Does the set of solutions have some extra structure (e.g., geometric structure, group structure).
- **Quantitative**
  - How many solutions are there?
  - How large is the smallest solution?
  - How can we expliclty find all solutions (and prove that we have them all)?
- **Implicit question**
  - Why do equations have (or fail to have) solutions?
  - Why do some have many and some have none?
  - What underlying mathematical structures control this?

Hilbert's Tenth Problem

Fact

Solving diophantine equations is hard.

Hilbert's Tenth Problem

Fact

Solving diophantine equations is hard.

The ring $R = \mathbb{Z}$ is especially hard.

Hilbert's Tenth Problem

Fact

Solving diophantine equations is hard.

The ring $R = \mathbb{Z}$ is especially hard.

Theorem (Davis-Putnam-Robinson 1961, Matijasevič 1970)

There does not exist an algorithm solving the following problem:

> **input**: $f_1, \ldots, f_m \in \mathbb{Z}[x_1, ..., x_n]$;
>
> **output**: YES / NO *according to whether there exists integers* $a_1, \ldots, a_n \in \mathbb{Z}$ *such that for each* i, $f_i(a_1, \ldots, a_n) = 0$.

Hilbert's Tenth Problem

Fact

Solving diophantine equations is hard.

The ring $R = \mathbb{Z}$ is especially hard.

Theorem (Davis-Putnam-Robinson 1961, Matijasevič 1970)

There does not exist an algorithm solving the following problem:

input: $f_1, \ldots, f_m \in \mathbb{Z}[x_1, ..., x_n]$;

output: YES / NO *according to whether there exists integers* $a_1, \ldots, a_n \in \mathbb{Z}$ *such that for each* i, $f_i(a_1, \ldots, a_n) = 0$.

The case $R = \mathbb{Q}$ is *still unknown*!

The Mordell Conjecture

Example

The equation $y^2 = x^2 + 1$ has infinitely many solutions.

The Mordell Conjecture

Example

The equation $y^2 = x^2 + 1$ has infinitely many solutions.

Theorem (Faltings)

For $n \geq 3$, the equation

$$y^2 = x^n + 1$$

has only finitely many solutions.

Fermat-like equations

Theorem (Poonen, Schaefer, Stoll)

The coprime integer solutions to $x^2 + y^3 = z^7$ *are the* 16 *triples*

$$(\pm 1, -1, 0), \quad (\pm 1, 0, 1), \quad \pm(0, 1, 1),$$

Fermat-like equations

Theorem (Poonen, Schaefer, Stoll)

The coprime integer solutions to $x^2 + y^3 = z^7$ *are the* 16 *triples*

$$(\pm 1, -1, 0), \quad (\pm 1, 0, 1), \quad \pm(0, 1, 1), \quad (\pm 3, -2, 1),$$

Fermat-like equations

Theorem (Poonen, Schaefer, Stoll)

The coprime integer solutions to $x^2 + y^3 = z^7$ *are the* 16 *triples*

$$(\pm 1, -1, 0), \quad (\pm 1, 0, 1), \quad \pm(0, 1, 1), \quad (\pm 3, -2, 1),$$
$$(\pm 71, -17, 2),$$

Fermat-like equations

Theorem (Poonen, Schaefer, Stoll)

The coprime integer solutions to $x^2 + y^3 = z^7$ *are the* 16 *triples*

$$(\pm 1, -1, 0), \quad (\pm 1, 0, 1), \quad \pm(0, 1, 1), \quad (\pm 3, -2, 1),$$
$$(\pm 71, -17, 2), (\pm 2213459, 1414, 65), \quad (\pm 15312283, 9262, 113),$$
$$(\pm 21063928, -76271, 17)\,.$$

Generalized Fermat Equations

Problem

What are the solutions to the equation $x^a + y^b = z^c$*?*

Generalized Fermat Equations

Problem

What are the solutions to the equation $x^a + y^b = z^c$?

Theorem (Darmon and Granville)

Fix $a, b, c \geq 2$. Then the equation $x^a + y^b = z^c$ has only finitely many coprime integer solutions iff $\chi = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1 \leq 0$.

Examples of Generalized Fermat Equations

Theorem (Poonen, Schaefer, Stoll)

The coprime integer solutions to $x^2 + y^3 = z^7$ are the 16 triples

$$(\pm 1, -1, 0), \quad (\pm 1, 0, 1), \quad \pm(0, 1, 1), \quad (\pm 3, -2, 1),$$
$$(\pm 71, -17, 2), (\pm 2213459, 1414, 65), \quad (\pm 15312283, 9262, 113),$$
$$(\pm 21063928, -76271, 17).$$

Examples of Generalized Fermat Equations

Theorem (Poonen, Schaefer, Stoll)

The coprime integer solutions to $x^2 + y^3 = z^7$ are the 16 triples

$$(\pm 1, -1, 0), \quad (\pm 1, 0, 1), \quad \pm(0, 1, 1), \quad (\pm 3, -2, 1),$$
$$(\pm 71, -17, 2), (\pm 2213459, 1414, 65), \quad (\pm 15312283, 9262, 113),$$
$$(\pm 21063928, -76271, 17).$$

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{7} - 1 = -\frac{1}{42} < 0$$

Examples of Generalized Fermat Equations

Theorem (Poonen, Schaefer, Stoll)

The coprime integer solutions to $x^2 + y^3 = z^7$ are the 16 triples

$$(\pm 1, -1, 0), \quad (\pm 1, 0, 1), \quad \pm(0, 1, 1), \quad (\pm 3, -2, 1),$$
$$(\pm 71, -17, 2), (\pm 2213459, 1414, 65), \quad (\pm 15312283, 9262, 113),$$
$$(\pm 21063928, -76271, 17).$$

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{7} - 1 = -\frac{1}{42} < 0$$

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{6} - 1 = 0$$

Examples of Generalized Fermat Equations

Question

What are the solutions to solutions to $x^3 + y^5 = z^7$?

$$(\pm 1, -1, 0), \quad (\pm 1, 0, 1), \quad \pm(0, 1, 1)$$

Examples of Generalized Fermat Equations

Question

What are the solutions to solutions to $x^3 + y^5 = z^7$?

$$(\pm 1, -1, 0), \quad (\pm 1, 0, 1), \quad \pm(0, 1, 1)$$

$$\frac{1}{3} + \frac{1}{5} + \frac{1}{7} - 1 = -\frac{34}{105} < 0$$

Examples of Generalized Fermat Equations

Question

What are the solutions to solutions to $x^3 + y^5 = z^7$?

$$(\pm 1, -1, 0), \quad (\pm 1, 0, 1), \quad \pm(0, 1, 1), \ ???$$

$$\frac{1}{3} + \frac{1}{5} + \frac{1}{7} - 1 = -\frac{34}{105} < 0$$

Examples of Generalized Fermat Equations

Theorem (Bennett, Skinner)

Any pairwise coprime solution to the equation

$$x^n + y^n = z^2, n > 4$$

satisfies $xyz = 0$.

Examples of Generalized Fermat Equations

Theorem (Bennett, Skinner)

Any pairwise coprime solution to the equation

$$x^n + y^n = z^2, n > 4$$

satisfies $xyz = 0$.

$$\frac{1}{n} + \frac{1}{n} + \frac{1}{2} - 1 = \frac{2}{n} - \frac{1}{2} < 0$$

Examples of Generalized Fermat Equations

Exercise

The coprime integer solutions to $x^2 - y^2 = z^n$ are ...

Examples of Generalized Fermat Equations

Exercise

The coprime integer solutions to $x^2 - y^2 = z^n$ are ...

$$\frac{1}{2} + \frac{1}{2} + \frac{1}{n} - 1 = \frac{1}{n} > 0$$

Examples of Generalized Fermat Equations

Exercise

The coprime integer solutions to $x^2 - y^2 = z^n$ are ...

$\boxed{x = \frac{1}{2}(s^n + t^n)}$ $\boxed{y = \frac{1}{2}(s^n - t^n)}$ $\boxed{z = st}$

$$\frac{1}{2} + \frac{1}{2} + \frac{1}{n} - 1 = \frac{1}{n} > 0$$

Examples of Generalized Fermat Equations

Theorem (Johnny Edwards)

The coprime integer solutions to $x^2 + y^3 = z^5$ *are*

Examples of Generalized Fermat Equations

Theorem (Johnny Edwards)

The coprime integer solutions to $x^2 + y^3 = z^5$ are

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} - 1 = \frac{1}{30} > 0$$

Examples of Generalized Fermat Equations

Theorem (Johnny Edwards)

The coprime integer solutions to $x^2 + y^3 = z^5$ are

$$\boxed{x = f_i(s,t)} \quad \boxed{y = g_i(s,t)} \quad \boxed{z = H_i(s,t)}$$

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} - 1 = \frac{1}{30} > 0$$

Examples of Generalized Fermat Equations

Theorem (Johnny Edwards)

The coprime integer solutions to $x^2 + y^3 = z^5$ are

$$\boxed{x = f_i(s,t)} \quad \boxed{y = g_i(s,t)} \quad \boxed{z = H_i(s,t)}$$

$$\deg f_i = 30, \ \deg g_i = 20, \ \deg H_i = 12$$

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} - 1 = \frac{1}{30} > 0$$

Examples of Generalized Fermat Equations

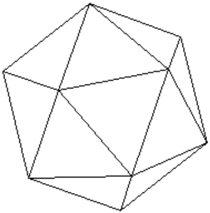
Theorem (Johnny Edwards)

The coprime integer solutions to $x^2 + y^3 = z^5$ *are*

$$\boxed{x = f_i(s,t)} \quad \boxed{y = g_i(s,t)} \quad \boxed{z = H_i(s,t)}$$

$$\deg f_i = 30, \ \deg g_i = 20, \ \deg H_i = 12$$

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} - 1 = \frac{1}{30} > 0$$



Known Solutions to $x^a + y^b = z^c$

The 'known' solutions with

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$$

are the following:

$$1^p + 2^3 = 3^2$$

$$2^5 + 7^2 = 3^4, 7^3 + 13^2 = 2^9, 2^7 + 17^3 = 71^2, 3^5 + 11^4 = 122^2$$

$$17^7 + 76271^3 = 21063928^2, 1414^3 + 2213459^2 = 65^7, 9262^3 153122832 = 113^7$$

$$43^8 + 96222^3 = 30042907^2, 33^8 + 1549034^2 = 15613^3$$

Known Solutions to $x^a + y^b = z^c$

The 'known' solutions with

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$$

are the following:

$$1^p + 2^3 = 3^2$$

$$2^5 + 7^2 = 3^4, 7^3 + 13^2 = 2^9, 2^7 + 17^3 = 71^2, 3^5 + 11^4 = 122^2$$

$$17^7 + 76271^3 = 21063928^2, 1414^3 + 2213459^2 = 65^7, 9262^3 153122832 = 113^7$$

$$43^8 + 96222^3 = 30042907^2, 33^8 + 1549034^2 = 15613^3$$

Problem (Beal's conjecture)

These are all solutions with $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1 < 0.$

Known Solutions to $x^a + y^b = z^c$

Problem (Beal's conjecture)

These are all solutions with $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1 < 0$.

Big money for solution:

Known Solutions to $x^a + y^b = z^c$

Problem (Beal's conjecture)

These are all solutions with $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1 < 0$.

Big money for solution: \$100,000.

Thank you

Thank You!