Gauss composition and integral arithmetic invariant theory

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Sums of Squares

Recall (p prime)

$$p = x^2 + y^2 \text{ if and only if } p = 1 \bmod 4 \text{ or } p = 2.$$

For products

$$(x^2 + y^2)(z^2 + w^2) = (xz + yw)^2 + (xw - yz)^2$$

Sums of Squares

Recall (p prime)

$p = x^2 + dy^2$ if and only if **[more complicated condition]**.

Example

$p = x^2 + 2y^2$ for some $x, y \in \mathbb{Z}$ if and only if $p = 2$ or $p = 1, 3 \mod 8$.

Example

$p = x^2 + 3y^2$ for some $x, y \in \mathbb{Z}$ if and only if $p = 3$ or $p = 1 \mod 3$.

Sums of Squares

Recall (p prime)

$p = x^2 + dy^2$ if and only if **[more complicated condition]**.

For products

$$(x^2 + dy^2)(z^2 + dw^2) = (xz + dyw)^2 + d(xw - yz)^2$$

Integers represented by a quadratic form

General quadratic forms (initiated by Lagrange)

$$Q(x, y) \in \mathbb{Z}[x, y]_2$$

Recall (p prime)

$p = Q(x, y)$ for some $x, y \in \mathbb{Z}$ if and only if **[more complicated condition]**.

Composition law?

$$Q(x, y)Q(z, w) = Q(a, b)$$

Sums of Squares (Euler's conjecture)

Example

$p = x^2 + 14y^2$ for some $x, y \in \mathbb{Z}$ if and only if $\left(\frac{-14}{p}\right) = -1$ and $(z^2+1)^2 = 8$ has a solution mod p.

Example

$p = 2x^2 + 7y^2$ for some $x, y \in \mathbb{Z}$ if and only if $\left(\frac{-14}{p}\right) = -1$ and $(z^2+1)^2 - 8$ factors into two irreducible quadratics mod p.

Integers represented by a quadratic form (equivalence)

Equivalence of forms

1. $Q(x, y) \in \mathbb{Z}[x, y]_2$
2. $M \in \mathrm{SL}_2(\mathbb{Z})$, $Q^M(x, y) := Q(ax + by, cx + dy)$
3. $n \in \mathbb{Z}$ is represented by Q iff it is represented by Q^M.
4. **Reduced forms:** $|b| \leq a \leq c$ and $b \geq 0$ if $a = c$ or $a = |b|$.

Example

$$29x^2 + 82xy + 58y^2 \sim x^2 + y^2.$$

Gauss composition

Theorem (Gauss composition)

The reduced, non-degenerate positive definite forms of discriminant $-D$ form a finite abelian group, isomorphic to the class group of $\mathbb{Q}(\sqrt{-D})$.

Example ($D = -56$)

$$x^2 + 14y^2,\ 2x^2 + 7y^2,\ 3x^2 \pm 2xy + 5y^2$$

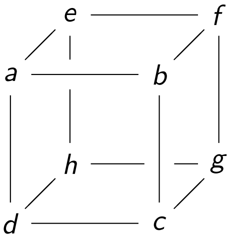
Remark

1. Gauss's proof was long and complicated; difficult to compute with.
2. Later reformulated by Dirichlet.
3. Much later reformulated by Bhargava.

Bhargava cubes

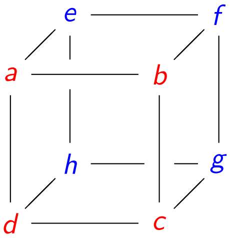
$$
\begin{tikzcd}
& e \arrow[rr, dash] \arrow[dl, dash] \arrow[dd, dash] & & f \arrow[dl, dash] \arrow[dd, dash] \\
a \arrow[rr, dash] \arrow[dd, dash] & & b \arrow[dd, dash] & \\
& h \arrow[rr, dash] \arrow[dl, dash] & & g \arrow[dl, dash] \\
d \arrow[rr, dash] & & c &
\end{tikzcd}
$$

Bhargava cubes



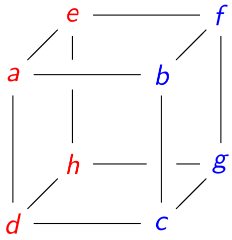
1. $a, b, d, c, e, f, h, g \in \mathbb{Z}$,
2. Cube is really an element of $\mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2$, with a natural $\mathrm{SL}_2(\mathbb{Z})^3$ action

Gauss composition via Bhargava cubes



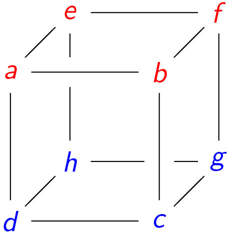
$$Q_1(x,y) := -\mathrm{Det}\left(\begin{pmatrix} a & b \\ d & c \end{pmatrix} x - \begin{pmatrix} e & f \\ h & g \end{pmatrix} y\right)$$

Gauss composition via Bhargava cubes



$$Q_i(x, y) := -\mathrm{Det}\,(M_i x - N_i y)$$

Gauss composition via Bhargava cubes



$$Q_i(x, y) := -\mathrm{Det}\,(M_i x - N_i y)$$

Bhargava's theorem

$$
\begin{tikzcd}
& e \arrow[rr, dash] \arrow[dl, dash] \arrow[dd, dash] & & f \arrow[dl, dash] \arrow[dd, dash] \\
a \arrow[rr, dash] \arrow[dd, dash] & & b \arrow[dd, dash] & \\
& h \arrow[rr, dash] \arrow[dl, dash] & & g \arrow[dl, dash] \\
d \arrow[rr, dash] & & c &
\end{tikzcd}
$$

$$Q_i(x, y) := -\mathrm{Det}\,(M_i x - N_i y)$$

Theorem (Bhargava)

$$Q_1(x, y) + Q_2(x, y) + Q_3(x, y) = 0$$

Lots of parameterizations

Example

binary cubic forms	$\leftrightarrow$	cubic fields
pairs (ternary, quadratic) forms	$\leftrightarrow$	quartic fields
quadruples of quinary alternating bilinear forms	$\leftrightarrow$	quintic fields
binary quartic forms	$\leftrightarrow$	2-Selmer elements of Elliptic curves

Remark

1. 14 more (Bhargava)
2. many more (Bhargava-Ho)

Representation theoretic framework

Space of forms

1. The space V of binary quadratic forms is 3-dimensional vector space (resp. R-module).
2. $V = \operatorname{Sym}^2 \mathbb{C}^2$

Representations

$\mathrm{SL}_2(\mathbb{C}) \circlearrowleft \operatorname{Sym}^2 \mathbb{C}^2$

$\mathrm{SL}_2(\mathbb{R}) \circlearrowleft \operatorname{Sym}^2 \mathbb{R}^2$

$\mathrm{SL}_2(\mathbb{Z}) \circlearrowleft \operatorname{Sym}^2 \mathbb{Z}^2$ etc..

Invariants

1. $\mathbb{C}$-**Invariants**: two non-zero forms f, g are $\mathbb{C}$ equivalent iff $\Delta(f) = \Delta(g)$.
2. $\mathbb{Z}$-**Invariants**: $\Delta(f) = \Delta(g) \not\Rightarrow \mathbb{Z}$ equivalence.

Representation theoretic framework

Invariants

1. $\mathbb{C}$-**Invariants**: two non-zero forms f, g are $\mathbb{C}$ equivalent iff $\Delta(f) = \Delta(g)$.
2. $\mathbb{Z}$-**Invariants**: $\Delta(f) = \Delta(g) \not\Rightarrow \mathbb{Z}$ equivalence.

Example ($D = -14 \cdot 4$)

$x^2 + 14y^2$ is not equivalent to $2x^2 + 7y^2$.

Fundamental object of study

1. $\mathrm{SL}_2(\mathbb{Z})$-orbits of an $\mathrm{SL}_2(\overline{\mathbb{Q}})$-orbit

General representation theoretic framework

Framework

1. $V =$ free R module
2. $G \circlearrowleft V$
3. $R \to R'$ ring extension
4. $v \in V(R)$

Goal

Understand the $G(R)$-orbits of the $G(R')$-orbit of v

Arithmetic invariant theory

"Is every group a cohomology group

$H^1_{\text{ét}}(\operatorname{Spec}\mathbb{Z}, \operatorname{Res}_{\mathcal{O}/\mathbb{Z}}\mathbb{G}_m)$

Arithmetic invariant theory

"Is every group a cohomology group or a Manjul shaped asteroid that fell from the sky?" – Jordan Ellenberg

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Arithmetic invariant theory

"Is every group a cohomology group or a Manjul shaped asteroid that fell from the sky?" – *Jordan Ellenberg*

$$H^1_{\text{ét}}(\operatorname{Spec}\mathbb{Z}, \operatorname{Res}_{\mathcal{O}/\mathbb{Z}}\mathbb{G}_m)$$

Bhargava–Gross–Wang

Setup

1. $f, g \in V(\mathbb{Q})$
2. $M \in G(\overline{\mathbb{Q}})$ s.t. $g = M \cdot f$
3. $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$
4. Then $g = M^\sigma \cdot f$, so $f = M^{-1}M^\sigma \cdot f$, i.e. $M^{-1}M^\sigma \in \mathrm{Stab}_f$

Cohomological framework

The map

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Stab}_f;\ \sigma \mapsto M^{-1}M^\sigma$$

is a **cocycle**, and gives an element of $H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Stab}_f)$.

Integral arithmetic invariant theory

Remark

1. AIT only works for fields; can't recover Gauss composition
2. Analogue of Galois cohomology is étale cohomology.

Integral arithmetic invariant theory – setup

Setup

1. S any base (e.g. $\mathbb{Z}$);
2. G/S any group scheme (not necessarily smooth, or even flat);
3. X (usually a vector space);
4. $G \circlearrowleft X$ an action.

Example ("Gauss")

$G = \mathrm{SL}_{2,\mathbb{Z}}$, acting on $X = \mathrm{Sym}^2 \mathbb{A}^2_{\mathbb{Z}}$

Main Theorem

Theorem (Giraud; Geraschenko-ZB)

Let $v \in X(S)$. Then there is a functorial long exact sequence (of groups and pointed sets)

$$0 \to Stab_v(S) \to G(S) \xrightarrow{g \mapsto g \cdot v} Orbit_v(S) \to H^1(S, Stab_v) \to H^1(S, G).$$

If $Stab_v$ is commutative, then

$$Orbit_v(S)/G(S) \cong \ker\left(H^1(S, Stab_v) \to H^1(S, G)\right)$$

is a group.

Remark

The image $\mathrm{Orbit}_v(S)/G(S)$ of $X(S)$ is the set of $G(S)$ equivalence classes of $v' \in \mathrm{Orbit}_v(S)$ in the same local orbit as v.

Example: Gauss composition revisited

Example ("Gauss")

$G = \mathrm{SL}_{2,\mathbb{Z}}$ acts on $X = \mathrm{Sym}^2 \mathbb{A}^2_{\mathbb{Z}}$; Stab_v is a non-split torus (thus *commutative*).

Let $f \in X(\mathbb{Z})$ be a *primitive* (non-zero mod all p) integral quadratic form.

$$0 \to \mathrm{Stab}_v(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}) \xrightarrow{g \mapsto g \cdot f} \mathrm{Orbit}_f(\mathbb{Z}) \to H^1(\mathbb{Z}, \mathrm{Stab}_v) \to H^1(\mathbb{Z}, \mathrm{SL}_2).$$

Remark

1. $H^1(\mathbb{Z}, \mathrm{SL}_2) = 0$ (this is Hilbert's theorem 90).
2. $\mathrm{Orbit}_f(\mathbb{Z}) / \mathrm{SL}_2(\mathbb{Z}) =$ integral equivalence classes of primitive forms with the same discriminantn.
3. $H^1(\mathbb{Z}, \mathrm{Stab}_v) \cong \mathrm{Orbit}_f(\mathbb{Z}) / \mathrm{SL}_2(\mathbb{Z})$.
4. $H^1(\mathbb{Z}, \mathrm{Stab}_v) \cong \mathrm{Pic}\, \mathbb{Z}[(\Delta_f + \sqrt{\Delta_f})/2] = \mathrm{Cl}\, \mathbb{Q}[\sqrt{\Delta_f}]$.

Example: Gauss composition (non-primitive)

Remark

1. If $f \in \mathbb{Z}^2$ is *not* primitive, then Stab_f is not *flat* over $\operatorname{Spec}\mathbb{Z}$.
2. (Easiest way to not be flat: $\dim \mathrm{Stab}_{f,\mathbb{F}_p}$ is not constant.)
3. Our machinery does not care; and recovers Gauss composition for non-primitive forms.

Still to come

$$\boxed{\text{More applications wanted.}}$$

1. We're currently iterating through the known literature, deriving paramaterizations where possible.
2. E.g. Delone–Faddeev (ternary cubic forms vs cubic rings): stabilizer is a finite flat group scheme.
3. Future predictive power, especially of degenerate objects/orbits.