Counting points, counting fields, and heights on stacks.

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Slides available at `http://www.mathcs.emory.edu/~dzb/slides/`

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Batyrev–Manin–Malle

Let K be a number field or function field of a curve.

Let $X \hookrightarrow \mathbb{P}^N_K$ be a projective variety.

Conjecture (Batyrev–Manin)

There exists a nonempty open subscheme $U \subset X$ and constants a, b, c such that

$$N_U(B) \sim cB^a (\log B)^b.$$

Let $G \subset S_n$ be a transitive subgroup.

Conjecture (Malle)

There exists constants a, b, c such that

$$N_{G,K}(B) \sim cB^a (\log B)^b.$$

Bat–Man for stacks

1. Let $\mathscr{X}$ be a proper Artin stack with finite diagonal.
2. Let $V \in \operatorname{Vect} \mathscr{X}$ be a (Northcott) vector bundle.

Conjecture (Ellenberg–Satriano–ZB)

There exists a non-empty open substack $\mathscr{U} \subset \mathscr{X}$ and constants a, b, c such that

$$N_{\mathscr{U},V}(B) \sim cB^a (\log B)^b .$$

Why bother?

$$BG = [\operatorname{Spec} \mathbb{Z}/G]$$

$$BG(K) \leftrightarrow L \supset K \text{ with } \operatorname{Gal}(L/K) \cong G$$

Question

Is there an intrinsic notion of height on BG?

99 problems

There does not exist an embedding $\mathscr{X} \hookrightarrow \mathbb{P}^N$.

The coarse space of BG is a point.

1. $\operatorname{Vect} BG \cong \operatorname{Rep} G \Rightarrow$
2. $\operatorname{Pic} BG$ is torsion $\Rightarrow$
3. ht_V cannot be additive

99 problems

Let R be a DVR with fraction field K.

Problem

$\mathscr{X}(R) \to \mathscr{X}(K)$ is not surjective

$$\begin{tikzcd}
\operatorname{Spec} L \arrow[r] \arrow[d] & P \arrow[r] \arrow[d, "\text{ét}"] & \star \arrow[d] \\
\operatorname{Spec} K \arrow[r] & T \arrow[r] & BG
\end{tikzcd}$$

One must deal with non-tame Artin stacks.

Geometric heights

- Let $K = k(C)$, where C is a smooth proper curve over k.
- Let X be a proper variety over C.
- Let $L \in \operatorname{Pic} X$.
- Let $x \in X(K)$,
- with extension $\overline{x} \colon C \to X$

$$\begin{tikzcd} & X \arrow[d, "\pi"'] \\ \operatorname{Spec} k(C) \arrow[ur, "x"] \arrow[r] & C \arrow[u, bend right, "\overline{x}"'] \end{tikzcd}$$

$$\operatorname{ht}_L(x) := \deg \overline{x}^* L$$

(Also true for varieties over number fields if L is metrizied and deg is the Arakelov degree.)

Tuning stacks

Let $\mathscr{X}$ be a proper Artin stack with finite diagonal over C (either a smooth proper curve over a field or $\operatorname{Spec} \mathcal{O}_K$, with function field K).

Let $x \in \mathcal{X}(K)$.

Theorem

There exists a stack $\mathcal{C}$ an a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[rr, bend left, "x"] \arrow[dr] & \mathcal{C} \arrow[r, "\overline{x}"] \arrow[d, "\pi"'] & \mathscr{X} \arrow[dl, "p"] \\
& C &
\end{tikzcd}$$

such that π is a birational moduli space morphism.

We call such a $\mathcal{C}$ a **tuning stack** for x, and we call a terminal such $\mathcal{C}$ a "universal" tuning stack.

$B\mu_2$

$$
\begin{tikzcd}
\operatorname{Spec} k(H) \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & \star \arrow[d] \\
\operatorname{Spec} k(t) \arrow[r] \arrow[dr] & \mathcal{C} \arrow[r, "\overline{x}"] \arrow[d, "\pi"] & B\mu_2 \\
& \mathbb{P}^1 &
\end{tikzcd}
$$

$$\mathrm{ht}_L(x) = -(g+1)$$

BG redux

- Vect $BG \cong$ Rep G
- $p\colon \star \to BG$
- Let $V = p_*\mathcal{O}_\star$
- (corresponds to the regular representation of G)

Let $x \in BG(\mathbb{Q})$ be a rational point, corresponding to a G-extension $L \supset \mathbb{Q}$.

Proposition

$$\mathrm{ht}_V(x) = \frac{\Delta_L}{2}$$

BG redux: $H \subset G$

- Vect $BG \cong$ Rep G
- $p\colon BH \to BG$
- Let $V = p_*\mathcal{O}_{BH}$
- (corresponds to the permutation representation of G on G/H)

Let $x \in BG(\mathbb{Q})$ be a rational point, corresponding to a G-extension $L \supset \mathbb{Q}$.

Proposition

$$\mathrm{ht}_V(x) = \frac{\Delta_{L^H}}{2}$$

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Thanks

Thank you!