Families of Abelian Varieties with Big Monodromy

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Background - Galois Representations

$$\rho_{A,n}\colon \quad G_K \to \operatorname{Aut} A[n] \quad \cong \quad \mathrm{GL}_{2g}(\mathbb{Z}/n\mathbb{Z})$$

$$\rho_{A,\ell^\infty}\colon \quad G_K \to \mathrm{GL}_{2g}(\mathbb{Z}_\ell) \quad = \quad \varprojlim_n \mathrm{GL}_{2g}\left(\mathbb{Z}/\ell^n\mathbb{Z}\right)$$

$$\rho_{A}\colon \quad G_K \to \mathrm{GL}_{2g}(\widehat{\mathbb{Z}}) \quad = \quad \varprojlim_n \mathrm{GL}_{2g}\left(\mathbb{Z}/n\mathbb{Z}\right)$$

Background - Galois Representations

$$\rho_{A,n} \colon G_K \twoheadrightarrow G_n \hookrightarrow \mathrm{GSp}_{2g}(\mathbb{Z}/n\mathbb{Z})$$

$$G_n \cong \mathrm{Gal}(K(A[n])/K)$$

Example - torsion on an ellitpic curve

If E has a K-rational torsion point $P \in E(K)[n]$ (of exact order n), then the image is constrained:

$$G_n \subset \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$$

since for $\sigma \in G_K$ and $Q \in E(\overline{K})[n]$ such that $E(\overline{K})[n] \cong \langle P, Q \rangle$,

$$\begin{aligned} \sigma(P) &= \quad P \\ \sigma(Q) &= \quad a_\sigma P \quad + \quad b_\sigma Q \end{aligned}$$

Monodromy of a family

1. $U \subset \mathbb{P}^N_K$ (non-empty open)
2. $\eta \in U$ (generic point)
3. $\mathscr{A} \to U$ (family of principally polarized abelian varieties)
4. $\rho_{\mathscr{A}_\eta} \colon G_{K(U)} \to \mathrm{GSp}_{2g}(\widehat{\mathbb{Z}})$

Definition

The **monodromy** of $\mathscr{A} \to U$ is the image H_η of $\rho_{\mathscr{A}_\eta}$. We say that $\mathscr{A} \to U$ has **big monodromy** if H_η is an open subgroup of $\mathrm{GSp}_{2g}(\widehat{\mathbb{Z}})$.

Monodromy of a family over a stack

1. U is now a stack.

Definition

The **monodromy** of $\mathscr{A} \to U$ is the image H of $\rho_{\mathscr{A}}$. We say that $\mathscr{A} \to U$ has **big monodromy** if H is an open subgroup of $\mathrm{GSp}_{2g}(\widehat{\mathbb{Z}})$.

1. $\operatorname{Spec} \Omega \xrightarrow{\eta} U$ (geometric generic point)
2. $\pi_{1,\mathrm{et}}(U)$

1. $\mathscr{A} \to U$ (family of principally polarized abelian varieties)
2. $\rho_{\mathscr{A}} \colon \pi_{1,\mathrm{et}}(U) \to \mathrm{GSp}_{2g}(\widehat{\mathbb{Z}})$

(Example) standard family of elliptic curves

$$E\colon y^2 = x^3 + ax + b$$

$$U = \mathbb{A}^2_K - \Delta$$

$$H = \left\{A \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) : \det(A) \in \chi_K(\mathrm{Gal}(\overline{K}/K))\right\}$$

(Example) elliptic curves with full two torsion

$$E\colon y^2 = x(x-a)(x-b)$$

$$U = \mathbb{A}^2_{\mathbb{Q}} - \Delta$$

$$H = \left\{ A \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) : A \equiv I \pmod 2 \right\}$$

Exotic example from Zywina's HIT paper

$$E\colon y^2 + xy = x^3 - \frac{36}{j-1728}x - \frac{1}{j-1728}$$

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Exotic example from Zywina's HIT paper

$$E\colon y^2 + xy = x^3 - \frac{36}{j - 1728}x - \frac{1}{j - 1728} \text{ over } U \subset \mathbb{A}^1_K$$

$$j = \frac{(T^{16} + 256T^8 + 4096)^3}{T^{32}(T^8 + 16)}$$

$$[\mathrm{GL}_2(\widehat{\mathbb{Z}}) : H] = 1536$$

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$$[\mathrm{GL}_2(\widehat{\mathbb{Z}}) : H] = 1536$$

H is the subgroup of matricies preserving $h(z) = \eta(z)^4/\eta(4z)$.

(Example) Hyperelliptic

$$E\colon y^2 = x^{2g+2} + a_{2g+1}x^{2g+1} + \ldots + a_0$$

$$\text{over } U \subset \mathbb{A}^{2g+2}$$

$$H = \left\{ A \in \mathrm{GSp}_{2g}(\widehat{\mathbb{Z}}) : A \pmod 2 \in S_{2g+2} \right\}$$

Main Theorem

Theorem (ZB-Zywina)

Let U be a non-empty open subset of $\mathbb{P}^N_K$ and let $\mathscr{A} \to U$ be a family of principally polarized abelian varieties. Let η be the generic point of U and suppose moreover that $\mathscr{A}_\eta/K(\eta)$ has big monodromy. Let H_η be the image of $\rho_{\mathscr{A}_\eta}$.

Let

$$B_K(N) = \{u \in U(K) : \ h(u) \leq N\}.$$

Then a **random fiber has maximal monodromy**, *i.e. (if $K \neq \mathbb{Q}$)*

$$\lim_{N \to \infty} \frac{|\{u \in B_K(N) : \rho_{\mathcal{A}_u}(G_K) = H_\eta\}|}{|B_K(N)|} = 1.$$

Corollary - Variant of Inverse Galois Problem

Corollary

For every $g > 2$, there exists an abelian variety $A/\mathbb{Q}$ such that

$$\mathrm{Gal}(\mathbb{Q}(A_{\mathrm{tors}})/\mathbb{Q}) \cong \mathrm{GSp}_{2g}(\widehat{\mathbb{Z}}),$$

i.e, for every n,

$$\mathrm{Gal}(\mathbb{Q}(A[n])/\mathbb{Q}) \cong \mathrm{GSp}_{2g}(\mathbb{Z}/n\mathbb{Z}).$$

Monodromy of trigonal curves

Theorem (ZB, Zywina)

For every $g > 2$

1. *the stack* T_g *of trigonal curves has monodromy* $\mathrm{GSp}_{2g}(\widehat{\mathbb{Z}})$*, and*
2. *there is a family of trigonal curves over a nonempty rational base* $U \subset \mathbb{P}^N_{\mathbb{Q}}$ *with monodromy* $\mathrm{GSp}_{2g}(\widehat{\mathbb{Z}})$

Monodromy of families of Pryms

Question

For every g, does there exists a family $\mathcal{A} \to U$ of PP abelian varieties of dimension g, U rational, which **are not generically isogenous to Jacobians**, with monodromy $\mathrm{GSp}_{2g}(\widehat{\mathbb{Z}})$?

1. One can (probably) take $\mathcal{A} \to U$ to be a family of Prym varieties associated to **tetragonal curves**, or
2. (Tsimerman) one can take $\mathcal{A} \to U$ to be a family of Prym varieties associated to **bielliptic curves**.

Sketch of trigonal proof

Theorem

For every g the stack T_g of trigonal curves has monodromy $\mathrm{GSp}_{2g}(\widehat{\mathbb{Z}})$.

Proof.

1. $\mathcal{M}_{g,d-1} \subset \overline{\mathcal{M}_{g,d}}$ (suffices for $\ell > 2$)
2. $\mathcal{M}_{g-2} \subset \overline{\mathcal{M}_g}$
3. the mod 2 monodromy thus contains subgroups isomorphic to
    1. S_{2g+2}
    2. $\mathrm{Sp}_{2(g-2)+2}(\mathbb{Z}/2\mathbb{Z})$

□

(Example) Hyperelliptic

$$E\colon y^2 = x^{2g+2} + a_{2g+1}x^{2g+1} + \ldots + a_0$$

$$\text{over } U \subset \mathbb{A}^{2g+2}$$

$$H = \left\{ A \in \mathrm{GSp}_{2g}(\widehat{\mathbb{Z}}) : A \pmod 2 \in S_{2g+2} \right\}$$

Hyperelliptic example continued

Theorem

1. *(Yu) unpublished*
2. *(Achter, Pries) the stack of hyperelliptic curves has maximal monodromy*
3. *(Hall) any 1-paramater family* $y^2 = (t - x)f(t)$ *over* $K(x)$ *has full monodromy*

Hyperelliptic example proof

Corollary

$$E\colon y^2 = x^{2g+2} + a_{2g+1}x^{2g+1} + \ldots + a_0$$

has monodromy $\left\{A \in \mathrm{GSp}_{2g}(\widehat{\mathbb{Z}}) : A \pmod 2 \in S_{2g+2}\right\}$.

Proof.

1. $U =$ space of distinct unordered $2g+2$-tuples of points on $\mathbb{P}^1$
2. $U \twoheadrightarrow \mathcal{H}_{g,2}$
3. $\mathcal{H}_{g,2} \cong [U/\operatorname{Aut}\mathbb{P}^1]$
4. fibers are irreducible, thus

$$\pi_{1,\mathrm{et}}(U) \twoheadrightarrow \pi_{1,\mathrm{et}}(\mathcal{H}_{g,2})$$

is surjective.

□

Sketch of trigonal proof

Theorem (ZB, Zywina)

For every $g > 2$ there is a family of trigonal curves over a nonempty rational base $U \subset \mathbb{P}^N_{\mathbb{Q}}$ with monodromy $\mathrm{GSp}_{2g}(\widehat{\mathbb{Z}})$

Proof.

1. **Main issue**:

$$f_3(x)y^3 + f_2(x)y^2 + f_1(x)y + f_0(x) = 0$$

2. The stack $\mathcal{T}_g$ is unirational, need to make this explicit
3. (Bolognesi, Vistoli) $\mathcal{T}_g \cong [U/G]$ where U is rational and G is a connected algebraic group.
4. **Maroni-invariant** (normal form for trigonal curves).

□

Sketch of trigonal proof - Maroni Invariant

Maroni-invariant

1. The image of the canonical map lands in a scroll

$$C \hookrightarrow \mathbb{F}_n \hookrightarrow \mathbb{P}^{g-1}$$

$$\mathbb{F}_n \cong \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-n))$$
$$\mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$$
$$\mathbb{F}_1 \cong \mathrm{Bl}_P \mathbb{P}^2$$

2. n has the same parity as g
3. generically $n = 0$ or 1
4. e.g., if g even we can take $U =$ space of bihomogenous polynomials of bi-degree $(3, d)$
5. $[U/G] \cong \mathcal{T}_g^0 \subset \mathcal{T}_g$.

Pryms

$C \to D \rightsquigarrow \ker_0(J_C \to J_D)$, generally not a Jacobian

Monodromy of families of Pryms, bielliptic target

Example (Tsimerman)

The space of (ramified) double covers of a fixed elliptic curve is rational, so the space of Pryms is also rational, with base isomorphic to a projective space over $X_1(2)$. The associated family of Prym's has big monodromy.