ICERM: ALGEBRAIC POINTS ON CURVES PROBLEM SESSION

Last update: June 24, 2025

Problem 1. (Proposed by: Bianca Viray) Let $f\colon X \to Y$ be a morphism of nice curves. What is the relation between the density degree sets $\delta(X/K)$ and $\delta(Y/K)$ of X and Y?

Problem 2. (Proposed by: Lea Beneish) Let X be a surface and $d \in \delta(X/K)$, does there exist a curve $C \subset X$ such that $d \in \delta(C/K)$? Comment (Bianca): think about surfaces with infinitely many rational points, but no rational or elliptic curves.

Problem 3. (Proposed by: Nathan Chen) Let $f\colon X \to Y$ be a morphism from a geometrically ruled surface to a curve. What is the relation between the density degree sets $\delta(X/K)$ and $\delta(Y/K)$ of X and Y?

Problem 4. (Proposed by: Maarten Derickx) Make de Franchis' theorem (finiteness of the number of maps from a curve C to curves of genus at least 2) practically computable. (Poonen + others have an effectivity result that is not practical.)

Problem 5. (Proposed by: David Zywina) What are the possible indices for Serre's open image theorem for elliptic curves over the rationals? Assuming Serre's uniformity problem, this reduces to finding all the rational points on a list of 300 or so modular curves.

Groups can be found at: `https://github.com/davidzywina/Modular/tree/main/agreeable_groups`

Problem 6. (Proposed by: Bianca Viray) Let L/K be quadratic and $C = \mathbb{P}^1$ or an elliptic curve over K. Describe the locus in Sym^{2d}_C of closed points whose residue field contains L. What does this imply about the proportion or asymptotics of such points?

Problem 7. (Proposed by: Ari Shnidman) What is the largest known order of $J(\mathbb{Q})_{\mathrm{tors}}$ for J the Jacobian of a genus 2 curve? (Sutherland: there is a list of all cases that Sutherland knows on his webpage, and there are some missing cases.) Some known: simple (80), non simple (128). Might be helpful to actually paramaterize some of these moduli spaces; maybe some of them are unirational, but maybe not rational?

Problem 8. (Proposed by: Jef Laga) Does the $\mathrm{Gal}(\overline{k}/k)$ set $C(\overline{k})$ determine C up to twist? Are there any concrete statements that are consequences of this that can be tested?

Problem 9. (Proposed by: Álvaro Lozano-Robledo) Let

$$\pi_X\colon \mathrm{SL}(2, \mathbb{Z}[x]) \to \mathrm{SL}(2, \mathbb{Z})$$

given by $X \mapsto 0$ and let $\Gamma(X)$ be the kernel. Let $N \geq 2$ and $\widetilde{\Gamma}(N) = \Gamma(X)|_{x=N} \subset \Gamma(N)$. Are these equal? Yes for $N = 2, 3, 4, 5, 6, 7$. To understand the problem, take $N = 2$ and find a matrix that specializes to $-I$.

Problem 10. (Proposed by: Robert Lemke Oliver) Fix a curve C and a degree d. What groups (or rather, permutation groups) occur as the Galois groups of degree d points on C? Rather, which permutation groups occur, and when they occur infinitely often, with what density do they occur?

Problem 11. (Proposed by: Ari Shnidman) What are the possible Galois structures on the set of 4^6 fourth roots of the canonical divisor on a plane quartic? (This is related to Ceresa cycles.)

Problem 12. (Proposed by: Nathan Chen) How does $\delta(C, K)$ vary in algebraic families? Can one say something about the proportion where $\delta(C, K)$ is constant? (Vogt) Over $\overline{k}$ this is known by work of Abramovich.

Problem 13. (Proposed by: Isabel Vogt) Fix $d < \lfloor (g+3)/2 \rfloor$ (i.e., less than the expected gonality of a genus g curve). What does the locus of curves in M_g with potentially dense degree d points? E.g., what is the codimension of the closure of this locus?

Problem 14. (Proposed by: Santiago Arango-Piñeros) Let p be an odd prime. By Fermat's last theorem, we know the degree one points on the Fermat curves $F_p \colon x^p + y^p = z^p \subset \mathbb{P}^2_{\mathbb{Q}}$. What can we say about points of higher degree on F_p? By work of Debarre and Klassen, we know that there are finitely many points of degree $\leq p-2$ on F_p. Furthermore, Klassen and Tzermias conjecture that every point of degree $\leq p-2$ on F_p also lies on the line $x+y = z$.

[1] Debarre–Klassen, "Points of low degree on smooth plane curves"

[2] Klassen–Tzermias, "Algebraic points of low degree on the Fermat quintic"

[3] Tzermias, "Parametrization of low-degree points on a Fermat curve"

Problem 15. (Proposed by: Kenji Terao) Given an isolated point of $C(k)$, when does it remain isolated over an extension k'?

Revised question: Let $f \colon X \to Z$, $g \colon Y \to Z$ be morphisms of nice curves. What is the relation between the density degree set $\delta(X \times_Z Y)$ and those of X and Y? What about the set of isolated points on $X \times_Z Y$?

Problem 16. (Proposed by: James Rawson) If there is an embedding $J_C \subset W^d \subset J_{C'}$, what can we say about the degrees of correspondences between C and C'?

Problem 17. (Proposed by: Drew Sutherland) Identify the isolated points among the known points on modular curves in the LMFDB (and add this to the database). Even just generating a larger dataset of isolated points would be useful.

Problem 18. (Proposed by: Ari Shnidman) Can we produce an example of a non-Ueno isolated point? (Viray) There are examples, but it would be interesting to find some in the database.

Problem 19. (Proposed by: Drew Sutherland) Improve the gonality bounds on particular curves in the LMFDB. Then turn your explanation into an algorithm and run it on the LMFDB.

Problem 20. (Proposed by: Robert Lemke Oliver) Let $\phi\colon X \to \mathbb{P}^1$ be a degree d map of curves over a number field k. Let $G \leq S_d$ be the generic Galois group, viewed as a permutation group acting on the d preimages of $\phi^{-1}(a)$ for $a \in \mathbb{P}^1$. What bounds may be placed on the number of points of $\mathbb{P}^1(k)$ of bounded height for which the Galois group is a proper subgroup of G?

Problem 21. (Proposed by: Robert Lemke Oliver) Masser and Vaaler have shown that the number of degree d points on $\mathbb{P}^1/\mathbb{Q}$ of height at most H is $\sim cH^{d(d+1)}$ for a suitable constant c. For a transitive subgroup $G \leq S_d$, how many of these points have residue fields with Galois group G? (Here, as is common in arithmetic statistics, we consider the Galois group of a non-Galois extension $K/\mathbb{Q}$ to be the image of the permutation action of the absolute Galois group on the d embeddings $K \hookrightarrow \overline{\mathbb{Q}}$.)

Problem 22. (Proposed by: Robert Lemke Oliver) Does there exist an elliptic curve $E/\mathbb{Q}$, a prime $p \geq 23$, and a cyclic degree p extension $K/\mathbb{Q}$ such that E gains rank over K? I have a heuristic in the spirit of David–Fearnley–Kisilevsky, Mazur–Rubin, and (to a lesser extent) Park–Poonen–Voight–Wood that there should be only finitely many such triples (E, p, K), but I am intentionally stating this in a more provocative manner with the aim of providing more falsifiable bounded rank type heuristics.

Problem 23. (Proposed by: Maarten Derickx) How can one determine whether a given curve is Debarre–Fahloui? And can this be made explicit enough so that we decide for all genus 2 curves in the LMFDB whether they are Debarre–Fahloui or not?

Problem 24. (Proposed by: Niven Achenjang) Are there geometric characterizations of when the density degree set of a curve C/k is *not* a semigroup?

For example, do all such C have to sit in a product $E \times \mathbb{P}^1$ realizing a particular type of class in CH^1? (This isn't an actual guess, but hopefully it makes the question slightly less vague.)