
MATH 220, Mathematical Reasoning and Proof
MWF 1 - 1:50

All assignments
Last updated: September 18, 2023
Gradescope code: 7DVWGG

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Assignment 1

Topics: Introduction to the course. Mathematical reasoning. Logic.

Reading: 1.1, 1.2, 1.5, 2.1, 2.2

Suggested problems (do not hand in; these are just for extra practice): [Handout 1](#)

Assignment 1, due Friday, Sep 15, 12:55pm via Gradescope:

1. Write the negation of each of the following statements.
 - (a) All triangles are isosceles.
 - (b) Every door in the building was locked.
 - (c) Some even numbers are multiples of three.
 - (d) Every real number is less than 100.
 - (e) Every integer is positive or negative.
 - (f) If f is a polynomial function, then f is continuous at 0.
 - (g) If $x^2 > 0$, then $x > 0$.
 - (h) There exists a $y \in \mathbf{R}$ such that $xy = 1$.
 - (i) $(2 > 1)$ and $(\forall x, x^2 > 0)$
 - (j) $\forall \epsilon > 0, \exists \delta > 0$ such that if $|x| < \delta$, then $|f(x)| < \epsilon$.
2. Write the converse, contrapositive, and negation of each of the following implications.
 - (a) If a quadrilateral is a rectangle, then it has two pairs of parallel sides.
 - (b) $(P \wedge \neg Q) \Rightarrow R$
 - (c) $P \Rightarrow (R \Rightarrow \forall x, Q(x))$
3. Let P and Q be statements. Write the truth table for
 - (a) $(\neg P) \vee Q$
 - (b) $(P \wedge (\neg Q)) \Rightarrow Q$
4. Are the statements $(P \vee Q) \wedge R$ and $P \vee (Q \wedge R)$ equivalent? If so, give a proof. If not, explain why by giving a counterexample.

(Two statement forms are equivalent if they have the same truth tables, and here, a counterexample simply means some choice of truth values for P, Q , and R such that the two statement forms give different outputs.)
5. Let P and Q be statements.
 - (a) Prove that $\neg(P \Rightarrow Q)$ is equivalent to $P \wedge \neg Q$.
 - (b) Prove that $\neg(P \Rightarrow Q)$ is *not* equivalent to $\neg P \wedge Q$.

(c) Give an example of statements P and Q such that $\neg P \Rightarrow \neg Q$ is true and $\neg(P \Rightarrow Q)$ is false.

6. Suppose that n is an even integer, and let m be any integer. Prove that nm is even.
7. Suppose that n is an odd integer. Prove that n^2 is an odd integer. (Hint: an integer n is odd if and only if there exists an integer k such that $n = 2k + 1$.)
8. Prove that if n^2 is even, then n is even. (Hint: ~~page 67~~ contrapositive.)

Assignment 2

Topics: “Direct” proofs, proof by cases, and divisibility problems.

Reading:

- 3.3, from Definition 3.3.6
- 6.4, just Definition 6.4.1
- see Index to find definitions like prime, etc

Suggested problems (do not hand in; these are just for extra practice)

1. [Handout 2](#)

Assignment, due Friday, Sep 22, 12:55pm via Gradescope:

1. Suppose that $a \mid b$. Prove that for all $n \in \mathbb{Z}_{>0}$, $a^n \mid b^n$.
2. Suppose that there exists an integer $n \in \mathbb{Z}_{>0}$ such that $a \mid b^n$. Is it true that $a \mid b$? Prove or disprove your answer. (For a disproof, please give a counterexample that demonstrates that the statement is false.)
3. Prove that for all $a \in \mathbb{Z}$ and for $n \in \mathbb{Z}_{\geq 0}$, $a - 1$ divides $a^n - 1$.
4. Prove that for all integers n , n and $n + 1$ have no common divisors other than ± 1 .
5. Prove that if x is an integer, then $x^2 + 2$ is not divisible by 4. (Hint: there are two cases: x is even, x is odd. Also, feel free to use basic facts about even or odd, e.g., “odd + odd = even”, without additional proof.)
6. Prove that the product of three consecutive integers is divisible by 6. (It suffices to prove that it is divisible by 2 and 3 separately.)
7. Show that for all integers a and b ,
$$a^2b^2(a^2 - b^2)$$
is divisible by 12. (It suffices to prove that it is divisible by 4 and 3 separately.)
8. Find all positive integers n such that $n^2 - 1$ is prime. Prove that your answer is correct.

Assignment 3

Topics: Proof by contradiction. Unsolvability of equations. Irrationality.

Reading: 3.2

Suggested problems (do not hand in; these are just for extra practice)

1. [Handout 3](#)

Assignment, due Friday, Sep 29, 12:55pm via Gradescope TODO:

1. Prove that there do not exist integers a , and b such that $21a + 30b = 1$.
2. Prove that $2^{1/3}$ is irrational.
3. Suppose that x is a real number such that $0 \leq x \leq \pi/2$. Prove that $\sin x + \cos x \geq 1$. (Hint: at some point in your proof, use that $(\sin x)^2 + (\cos x)^2 = 1$.)
4. Prove that there are no positive integer solutions to the equation $x^2 - y^2 = 10$.
5. Let a, b, c be integers satisfying $a^2 + b^2 = c^2$. Show that abc must be even. (Harder problem, just for fun: show that a or b must be even.)
6. Suppose that a and n are integers that are both at least 2. Prove that if $a^n - 1$ is prime, then $a = 2$ and n is a prime. (Primes of the form $2^n - 1$ are called Mersenne primes.)
7. Suppose that $a, b \in \mathbb{Z}$. Prove that $a^2 - 4b \neq 2$.
8. Prove that $\log_{10} 7$ is irrational

Assignment 4

Topics: Induction.

Reading: Chapter 6

Fun Video (optional): Vi Hart; “Doodling in Math: Spirals, Fibonacci, and Being a Plant”

<https://www.youtube.com/watch?v=ahXIMUkSXX0>

Suggested problems (do not hand in; these are just for extra practice)

1. [Handout 4](#)

2. [Handout 5](#)

Assignment, due Friday, Oct 06, 12:55pm via Gradescope TODO:

1. Prove that for every positive integer n ,

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

2. Let a_n be defined recursively by $a_1 = 1$ and $a_n = \sqrt{1 + a_{n-1}}$. Prove that for all positive integers n , $a_n < 2$.

3. Prove by induction that if b_1, b_2, \dots, b_n are even integers, then $b_1 + b_2 + \cdots + b_n$ is even.

4. Let $F_1, F_2, F_3, \dots = 1, 1, 2, 3, 5, 8, \dots$ be the Fibonacci sequence. Prove that $F_1^2 + \cdots + F_n^2 = F_n F_{n+1}$.

5. Prove that $n! > 2^n$ for all $n \geq 4$.

6. *Bernoulli's inequality*: let $\beta \in \mathbb{R}$ be a real number such that $\alpha > -1$ and $\alpha \neq 0$. Prove that for all integers $n \geq 2$, $(1 + \beta)^n > 1 + n\beta$.

7. Prove that for all integers $n \geq 1$,

$$1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{n}.$$

8. Prove (using induction) that for all integers $n \geq 1$, $2^{2n} - 1$ is divisible by 3.

Assignment 5

Topics: Basics of set theory. Basic operations. Proofs with sets.

Reading:

1. 1.3, 1.4, 2.3

Suggested problems (do not hand in; these are just for extra practice): [Handout 6](#)

Assignment 6, due Friday, Oct 20, 12:55pm via Gradescope:

1. Let $A = \{n \in \mathbb{Z} \mid n \text{ is a multiple of } 4\}$ and $B = \{n \in \mathbb{Z} \mid n^2 \text{ is a multiple of } 4\}$
 - (a) Prove or disprove: $A \subseteq B$.
 - (b) Prove or disprove: $B \subseteq A$.
2. Prove that $A \cup (A \cap B) = A$.
3. Let A, B and C be sets.
 - (a) Prove that $(A \subseteq C) \wedge (B \subseteq C) \Rightarrow A \cup B \subseteq C$.
 - (b) State the contrapositive of part (a).
 - (c) State the converse of part (a). Prove or disprove it.
4. Let n and m be integers. Prove that if $n\mathbb{Z} \subseteq m\mathbb{Z}$ then m divides n .

Midterm

Topics: Friday, October 13 will be an in class Exam.

Content: The questions will all be either

1. homework problems,
2. suggested problems,
3. problems we worked in class, or
4. minor variations of one of these.

A typical exam will have one or two questions from each week of the course. You can expect problems like the following:

- Negations
- Give definitions
- Contrapositive
- Contradiction
- Induction