Diophantine and tropical geometry

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Slides available at `http://www.mathcs.emory.edu/~dzb/slides/`

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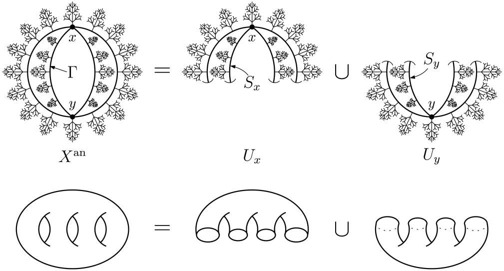


Example (from McCallum-Poonen's survey paper)

Example

$$X\colon y^2 = x^6 + 8x^5 + 22x^4 + 22x^3 + 5x^2 + 6x + 1$$

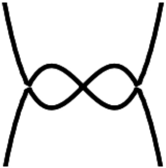
1. Points reducing to $\widetilde{Q} = (0,1)$ are given by

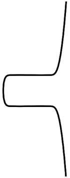
$$\begin{aligned} x &= \quad p \cdot t, \text{ where } t \in \mathbb{Z}_p \\ y &= \quad \sqrt{x^6 + 8x^5 + 22x^4 + 22x^3 + 5x^2 + 6x + 1} = 1 + x^2 + \cdots \end{aligned}$$

2. $\displaystyle\int_{(0,1)}^{P_t} \frac{xdx}{y} = \int_0^t (x - x^3 + \cdots)dx$

Models – semistable example

$$\begin{aligned} y^2 &= (x(x-1)(x-2))^3 - 5 \\ &= (x(x-1)(x-2))^3 \mod 5. \end{aligned}$$

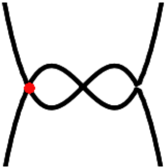


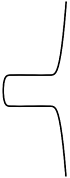


Note: no point can reduce to $(0,0)$. Local equation looks like $xy = 5$

Models – semistable example (not regular)

$$
\begin{aligned}
y^2 &= (x(x-1)(x-2))^3 - 5^{4} \\
&= (x(x-1)(x-2))^3 \mod 5
\end{aligned}
$$

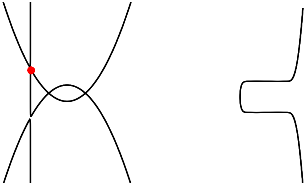




Now: $(0, 5^2)$ reduces to $(0, 0)$. Local equation looks like $xy = 5^4$

Models – semistable example

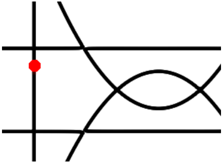
$$y^2 = (x(x-1)(x-2))^3 - 5^{4}$$
$$= (x(x-1)(x-2))^3 \mod 5$$

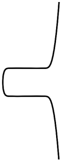


Blow up. Local equation looks like $xy = 5^3$

Models – semistable example (regular at (0,0))

$$
\begin{aligned}
y^2 &= (x(x-1)(x-2))^3 - 5^{\color{red}4} \\
&= (x(x-1)(x-2))^3 \mod 5
\end{aligned}
$$





Blow up. Local equation looks like $xy = 5$

Analytic continuation of integrals

(**Residue Discs.**)

$$P \in \mathcal{X}^{\mathrm{sm}}(\mathbb{F}_p),\ t\colon D_P \cong p\mathbb{Z}_p,\ \omega|_{D_P} = f(t)dt$$

(**Integrals on a disc.**)

$$Q, R \in D_P,\ \int_Q^R \omega := \int_{t(Q)}^{t(R)} f(t)dt.$$

(**Integrals between discs.**)

$$Q \in D_{P_1},\ R \in D_{P_2},\ \int_Q^R \omega := ?$$

Analytic continuation of integrals via Abelian varieties

(**Integrals between discs.**)

$$Q \in D_{P_1},\ R \in D_{P_2},\ \int_Q^R \omega := ?$$

(**Albanese map.**)

$$\iota \colon X \hookrightarrow \mathrm{Jac}_X,\ Q \mapsto [Q - \infty]$$

(**Abelian integrals via functorality and additivity.**)

$$\int_Q^R \iota^*\omega = \int_{\iota(Q)}^{\iota(R)} \omega = \int_{[Q-\infty]}^{[R-\infty]} \omega = \int_0^{[R-Q]} \omega = \frac{1}{n}\int_0^{n[R-Q]} \omega$$

Analytic continuation of integrals via Frobenius

(**Integrals between discs.**)

$$Q \in D_{P_1},\ R \in D_{P_2},\ \int_Q^R \omega := ?$$

(**Abelian integrals via functorality and Frobenius.**)

$$\int_Q^R \omega = \int_Q^{\phi(Q)} \omega + \int_{\phi(Q)}^{\phi(R)} \omega + \int_{\phi(R)}^R \omega$$

(**Very clever trick (Coleman)**)

$$\int_{\phi(Q)}^{\phi(R)} \omega_i = \int_Q^R \phi^* \omega_i = df_i + \sum_j \int_Q^R a_{ij} \omega_j$$

Comparison of integrals

Facts

1. For X with good reduction, the **Abelian** and **Coleman** integrals agree.
2. ~~A mystery~~. The associated Berkovich curve is contractable.
3. For X with bad reduction they differ.

Theorem (Stoll; Katz-Rabinoff-Zureick–Brown)

There exist linear functions $a(\omega), c(\omega)$ such that

$$\oint_Q^R \omega - \int_Q^R \omega = a(\omega)\left(\log(t(R)) - \log(t(Q))\right) + c(\omega)\left(t(Q) - t(R)\right)$$

Why bother? Integration on Annuli (a trade off)

Assumption

Assume $\mathcal{X}/\mathbb{Z}_p$ is **stable**, but not regular.

(**Residue Discs.**)

$$P \in \mathcal{X}^{\mathrm{sm}}(\mathbb{F}_p),\ t\colon D_P \cong p\mathbb{Z}_p,\ \omega|_{D_P} = f(t)dt$$

(**Residue Annuli.**)

$$P \in \mathcal{X}^{\mathrm{sing}}(\mathbb{F}_p),\ t\colon D_P \cong p\mathbb{Z}_p - p^r\mathbb{Z}_p, \omega|_{D_P} = f(t, t^{-1})dt$$

(**Integrals on an annulus are multivalued.**)

$$\int_Q^R \omega := \int_{t(Q)}^{t(R)} f(t, t^{-1})dt = \cdots + a(\omega)\log t + \cdots$$

(**Cover the annulus with discs**)

Each analytic continuation implicitly chooses a branch of log.

Why bother? Integration on Annuli (a trade off)

(**Abelian integrals.**) Analytically continue via Albanese.

$$\oint_Q^R \omega = 0 \text{ if } R, Q \in X(\mathbb{Q}), \omega \in V$$

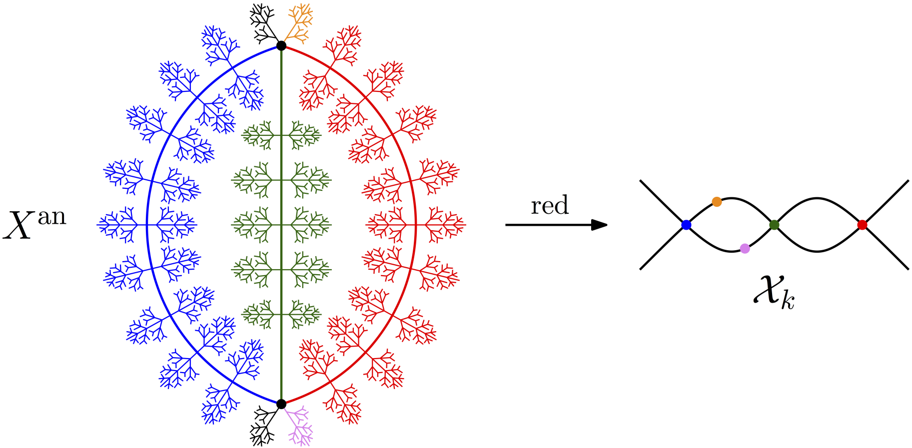
(**Berkovich-Coleman integrals.**) Analytically continue via Frobenius.

$$\int_Q^R \omega := \int_{t(Q)}^{t(R)} f(t, t^{-1})dt = \cdots + a(\omega) \log_{\mathrm{Col}} t + \cdots$$

(**Stoll's theorem.**)

$$\oint_Q^R \omega - \int_Q^R \omega = a(\omega) \left(\log_{\mathrm{ab}}(r(R)) - \log_{\mathrm{ab}}(t(Q))\right) + c(\omega) \left(t(Q) - t(R)\right)$$

Berkovich picture



Stoll's comparison theorem, tropical geometry edition

Theorem (Katz, Rabinoff, ZB)

The difference $\log_{Col} - \log_{ab}$ is the unique homomorphism that takes the value

$$\int_\gamma \omega$$

on $Trop(\gamma)$, where $Trop\colon G(\mathbb{K}) \to T(\mathbb{K})/T(\mathcal{O})$.

$$\begin{tikzcd} & T \arrow[d] & \\ \Lambda \arrow[r] & G \arrow[r] \arrow[d] & (\mathrm{Jac}_X)^{\mathrm{an}} \\ & B & \end{tikzcd}$$

$T =$ torus, $\Lambda =$ discrete, and $B =$ Abelian w/ good reduction.