

# Counting points, counting fields, and heights on stacks.

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Slides available at <http://www.mathcs.emory.edu/~dzb/slides/>

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Let  $K$  be a number field or function field of a curve.

Let  $X \hookrightarrow \mathbb{P}_K^N$  be a projective variety.

## Conjecture (Batyrev–Manin)

There exists a nonempty open subscheme  $U \subset X$  and constants  $a, b, c$  such that

$$N_U(B) \sim cB^a (\log B)^b.$$

Let  $G \subset S_n$  be a transitive subgroup.

## Conjecture (Malle)

There exists constants  $a, b, c$  such that

$$N_{G,K}(B) \sim cB^a (\log B)^b.$$

- ① Let  $\mathcal{X}$  be a proper Artin stack with finite diagonal.
- ② Let  $V \in \text{Vect } \mathcal{X}$  be a (Northcott) vector bundle.

## Conjecture (Ellenberg–Satriano–ZB)

There exists a non-empty open substack  $\mathcal{U} \subset \mathcal{X}$  and constants  $a, b, c$  such that

$$N_{\mathcal{U}, V}(B) \sim cB^a (\log B)^b.$$

# Why bother?

$$BG = [\mathrm{Spec} \mathbb{Z}/G]$$

$$BG(K) \leftrightarrow L \supset K \text{ with } \mathrm{Gal}(L/K) \cong G$$

## Question

Is there an intrinsic notion of height on  $BG$ ?

# 99 problems

There does not exist an embedding  $\mathcal{X} \hookrightarrow \mathbb{P}^N$ .

The coarse space of  $BG$  is a point.

- 1  $\text{Vect } BG \cong \text{Rep } G \Rightarrow$
- 2  $\text{Pic } BG$  is torsion  $\Rightarrow$
- 3  $\text{ht}_V$  cannot be additive

# 99 problems

Let  $R$  be a DVR with fraction field  $K$ .

## Problem

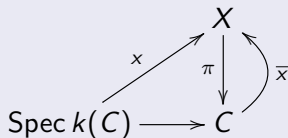
$\mathcal{X}(R) \rightarrow \mathcal{X}(K)$  is not surjective

$$\begin{array}{ccccc} \mathrm{Spec} L & \longrightarrow & P & \longrightarrow & \star \\ \downarrow & & \downarrow \scriptstyle \text{ét} & & \downarrow \\ \mathrm{Spec} K & \longrightarrow & T & \longrightarrow & BG \end{array}$$

One must deal with non-tame Artin stacks.

# Geometric heights

- Let  $K = k(C)$ , where  $C$  is a smooth proper curve over  $k$ .
- Let  $X$  be a proper variety over  $C$ .
- Let  $L \in \text{Pic } X$ .
- Let  $x \in X(K)$ ,
- with extension  $\bar{x}: C \rightarrow X$



$$\text{ht}_L(x) := \deg \bar{x}^* L$$

(Also true for varieties over number fields if  $L$  is metrized and  $\deg$  is the Arakelov degree.)

# Tuning stacks

Let  $\mathcal{X}$  be a proper Artin stack with finite diagonal over  $C$  (either a smooth proper curve over a field or  $\mathrm{Spec} \mathcal{O}_K$ , with function field  $K$ ).

Let  $x \in \mathcal{X}(K)$ .

## Theorem

*There exists a stack  $\mathcal{C}$  and a commutative diagram*

$$\begin{array}{ccccc} & & x & & \\ & \text{Spec } K & \xrightarrow{\quad} & \mathcal{C} & \xrightarrow{\quad \bar{x} \quad} & \mathcal{X} \\ & \searrow & & \downarrow \pi & \swarrow p & \\ & & & \mathcal{C} & & \end{array}$$

*such that  $\pi$  is a birational moduli space morphism.*

We call such a  $\mathcal{C}$  a **tuning stack** for  $x$ , and we call a terminal such  $\mathcal{C}$  a “universal” tuning stack.



$$\begin{array}{ccccc} \mathrm{Spec} k(H) & \longrightarrow & H & \longrightarrow & \star \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec} k(t) & \longrightarrow & \mathcal{C} & \xrightarrow{\bar{x}} & B_{\mu_2} \\ & \searrow & \downarrow \pi & & \\ & & \mathbb{P}^1 & & \end{array}$$

$$\mathrm{ht}_L(x) = -(g+1)$$

- $\text{Vect } BG \cong \text{Rep } G$
- $p: \star \rightarrow BG$
- Let  $V = p_* \mathcal{O}_\star$
- (corresponds to the regular representation of  $G$ )

Let  $x \in BG(\mathbb{Q})$  be a rational point, corresponding to a  $G$ -extension  $L \supset \mathbb{Q}$ .

## Proposition

$$\text{ht}_V(x) = \frac{\Delta_L}{2}$$

## $BG$ redux: $H \subset G$

- $\text{Vect } BG \cong \text{Rep } G$
- $p: BH \rightarrow BG$
- Let  $V = p_* \mathcal{O}_{BH}$
- (corresponds to the permutation representation of  $G$  on  $G/H$ )

Let  $x \in BG(\mathbb{Q})$  be a rational point, corresponding to a  $G$ -extension  $L \supset \mathbb{Q}$ .

### Proposition

$$\text{ht}_V(x) = \frac{\Delta_{L^H}}{2}$$

- ① Let  $\mathcal{X}$  be a proper Artin stack with finite diagonal.
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## Conjecture (Ellenberg–Satriano–ZB)

There exists a non-empty open substack  $\mathcal{U} \subset \mathcal{X}$  and constants  $a, b, c$  such that

$$N_{\mathcal{U}, V}(B) \sim cB^a (\log B)^b.$$

Thank you!