

ℓ -adic images of Galois for elliptic curves over \mathbb{Q}

David Zureick-Brown (Amherst College)
Jeremy Rouse (Wake Forest University)
Drew Sutherland (MIT)

Harvard Number Theory Seminar

February 18, 2026

Slides available at <https://dmzb.github.io/>

Galois Representations

$$\begin{aligned}\mathbb{Q} &\subset K \subset \overline{\mathbb{Q}} \\ G_K &:= \text{Aut}(\overline{K}/K) \\ E[n](\overline{K}) &\cong (\mathbb{Z}/n\mathbb{Z})^2\end{aligned}$$

$$\begin{aligned}\rho_{E,n}: G_K &\rightarrow \text{Aut } E[n] \cong \text{GL}_2(\mathbb{Z}/n\mathbb{Z}) \\ \rho_{E,\ell^\infty}: G_K &\rightarrow \text{GL}_2(\mathbb{Z}_\ell) = \varprojlim_n \text{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z}) \\ \rho_E: G_K &\rightarrow \text{GL}_2(\widehat{\mathbb{Z}}) = \varprojlim_n \text{GL}_2(\mathbb{Z}/n\mathbb{Z})\end{aligned}$$

Serre's Open Image Theorem

Theorem (Serre, 1972)

Let E be an elliptic curve over K without CM. The image

$$\rho_E(G_K) \subset \mathrm{GL}_2(\widehat{\mathbb{Z}})$$

of ρ_E is open.

Note:

$$\mathrm{GL}_2(\widehat{\mathbb{Z}}) \cong \prod_{\ell} \mathrm{GL}_2(\mathbb{Z}_{\ell})$$

Thus ρ_{E,ℓ^∞} is surjective for all but finitely many ℓ .

For CM curves, see Lozano-Robledo's [paper](#) and work by Bourdon, Clark, and Pollack.

Image of Galois

$$\rho_{E,n}: G_{\mathbb{Q}} \twoheadrightarrow H(n) \hookrightarrow \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$$

$$\left\{ \begin{array}{c} \overline{\mathbb{Q}} \\ | \\ \overline{\mathbb{Q}}^{\ker \rho_{E,n}} = \mathbb{Q}(E[n]) \\ | \\ \mathbb{Q} \end{array} \right\} H(n)$$

Problem (Mazur's "program B")

Classify all possibilities for $H(n)$.

Mazur's Program B

As presented at Modular functions in one variable V in Bonn

Theorem 1 also fits into a general program:

B. Given a number field K and a subgroup H of $GL_2(\hat{\mathbb{Z}}) = \prod_p GL_2(\mathbb{Z}_p)$ classify
all elliptic curves E/K whose associated Galois representation on torsion points
maps $Gal(\bar{K}/K)$ into $H \subset GL_2(\hat{\mathbb{Z}})$.

Mazur - Rational points on modular curves (1977)

Example - torsion on an elliptic curve

If E has a K -rational **torsion point** $P \in E(K)[n]$ (of exact order n) then:

$$H(n) \subset \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$$

since for $\sigma \in G_K$ and $Q \in E(\overline{K})[n]$ such that $E(\overline{K})[n] \cong \langle P, Q \rangle$,

$$\begin{aligned}\sigma(P) &= P \\ \sigma(Q) &= a_\sigma P + b_\sigma Q\end{aligned}$$

Example - Isogenies

If E has a K -rational, **cyclic isogeny** $\phi: E \rightarrow E'$ with $\ker \phi = \langle P \rangle$ then:

$$H(n) \subset \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

since for $\sigma \in G_K$ and $Q \in E(\overline{K})[n]$ such that $E(\overline{K})[n] \cong \langle P, Q \rangle$,

$$\begin{aligned} \sigma(P) &= a_\sigma P \\ \sigma(Q) &= b_\sigma P + c_\sigma Q \end{aligned}$$

Example - other maximal subgroups

Normalizer of a split Cartan:

$$N_{\text{sp}} = \left\langle \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle$$

$H(n) \subset N_{\text{sp}}$ and $H(n) \not\subset C_{\text{sp}}$ iff

- there exists an unordered pair $\{\phi_1, \phi_2\}$ of cyclic isogenies,
- whose kernels intersect trivially,
- neither of which is defined over K ,
- but which are both defined over some quadratic extension of K ,
- and which are Galois conjugate.

Example - other maximal subgroups

$\mathbb{F}_{p^2}^*$ acts on $\mathbb{F}_{p^2} \cong \mathbb{F}_p \times \mathbb{F}_p$

Normalizer of a non-split Cartan:

$$C_{\text{ns}} = \text{im} \left(\mathbb{F}_{p^2}^* \rightarrow \text{GL}_2(\mathbb{F}_p) \right) \subset N_{\text{ns}}$$

$H(n) \subset N_{\text{ns}}$ and $H(n) \not\subset C_{\text{ns}}$ iff

E admits a “necklace” (Rebolledo, Wuthrich)

Image of Galois

$$\rho_{E,n}: G_{\mathbb{Q}} \twoheadrightarrow H(n) \hookrightarrow \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$$

$$\left\{ \begin{array}{c} \overline{\mathbb{Q}} \\ | \\ \overline{\mathbb{Q}}^{\ker \rho_{E,n}} = \mathbb{Q}(E[n]) \\ | \\ \mathbb{Q} \end{array} \right\} H(n)$$

Problem (Mazur's "program B")

Classify all possibilities for $H(n)$.

Modular curves

Definition

- $X(N)(K) := \{(E/K, P, Q) : E[N] = \langle P, Q \rangle\} \cup \{\text{cusps}\}$
- $X(N)(K) \ni (E/K, P, Q) \Leftrightarrow \rho_{E,N}(G_K) = \{I\}$

Let $\Gamma(N) \subset H \subset \text{GL}_2(\widehat{\mathbb{Z}})$. The minimal such N is the **level** of H .

Definition

$X_H := X(N)/H(N)$ (where $H(N)$ is the image of H in $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})$)

$$X_H(K) \ni (E/K, \iota) \Leftrightarrow \rho_{E,N}(G_K) \subset H(N)$$

Stacky disclaimer

This is only true up to twist; there are some subtleties if

- 1 $j(E) \in \{0, 12^3\}$ (plus some minor group theoretic conditions), or
- 2 if $-I \in H$.

Rational Points on modular curves

Mazur's program B

Compute $X_H(\mathbb{Q})$ for all H .

Remark

- Sometimes $X_H \cong \mathbb{P}^1$ or elliptic with rank $X_H(\mathbb{Q}) > 0$.
- Some X_H have **exceptional** points (i.e, non-cusp non-CM points).
- Can compute $g(X_H)$ group theoretically (via Riemann–Hurwitz).

Fact

$$g(X_H), \gamma(X_H) \rightarrow \infty \text{ as } [\mathrm{GL}_2(\widehat{\mathbb{Z}}) : H] \rightarrow \infty.$$

(Serre) Sample subgroup $H \subset \mathrm{GL}_2(\widehat{\mathbb{Z}})$

$$\ker \phi_2 \subset H(8) \subset \mathrm{GL}_2(\mathbb{Z}/8\mathbb{Z}) \quad \dim_{\mathbb{F}_2} \ker \phi_2 = 3$$

$$\begin{array}{ccccc} & & \downarrow \phi_2 & & \downarrow \\ I + 2M_2(\mathbb{Z}/2\mathbb{Z}) & \subset & H(4) & = & \mathrm{GL}_2(\mathbb{Z}/4\mathbb{Z}) \end{array} \quad \dim_{\mathbb{F}_2} \ker \phi_1 = 4$$

$$\begin{array}{ccc} & \downarrow \phi_1 & \downarrow \\ & H(2) & = \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z}) \end{array}$$

$$\chi: \mathrm{GL}_2(\mathbb{Z}/8\mathbb{Z}) \rightarrow \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/8\mathbb{Z})^* \rightarrow \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/8\mathbb{Z})^* \cong \mathbb{F}_2^3.$$

$$\chi = \mathbf{sgn} \times \det$$

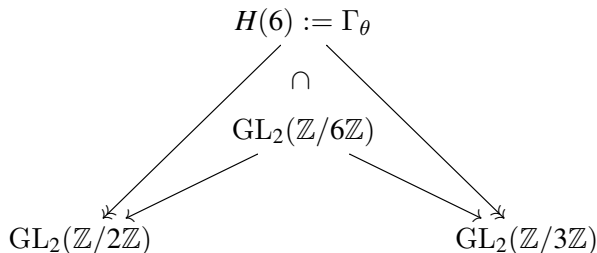
$$H(8) := \chi^{-1}(G), \quad G \subset \mathbb{F}_2^3.$$

A typical subgroup $H \subset \mathrm{GL}_2(\widehat{\mathbb{Z}})$

$\ker \phi_4$	\subset	$H(32)$	\subset	$\mathrm{GL}_2(\mathbb{Z}/32\mathbb{Z})$	$\dim_{\mathbb{F}_2} \ker \phi_4 = 4$
		$\downarrow \phi_4$		\downarrow	
$\ker \phi_3$	\subset	$H(16)$	\subset	$\mathrm{GL}_2(\mathbb{Z}/16\mathbb{Z})$	$\dim_{\mathbb{F}_2} \ker \phi_3 = 3$
		$\downarrow \phi_3$		\downarrow	
$\ker \phi_2$	\subset	$H(8)$	\subset	$\mathrm{GL}_2(\mathbb{Z}/8\mathbb{Z})$	$\dim_{\mathbb{F}_2} \ker \phi_2 = 2$
		$\downarrow \phi_2$		\downarrow	
$\ker \phi_1$	\subset	$H(4)$	\subset	$\mathrm{GL}_2(\mathbb{Z}/4\mathbb{Z})$	$\dim_{\mathbb{F}_2} \ker \phi_1 = 3$
		$\downarrow \phi_1$		\downarrow	
		$H(2)$	$=$	$\mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$	

Non-abelian entanglements

There exists a surjection $\theta: \mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z}) \rightarrow \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$.



Brau–Jones

$$\mathrm{im} \rho_{E,6} \subset H(6) \Leftrightarrow j(E) = 2^{10} 3^3 t^3 (1 - 4t^3) \Rightarrow K(E[2]) \subset K(E[3])$$
$$X_H \cong \mathbb{P}^1 \xrightarrow{j} X(1)$$

Main conjecture

Conjecture (Serre)

Let E be an elliptic curve over \mathbb{Q} without CM. Then for $\ell > 37$, $\rho_{E,\ell}$ is surjective.

In other words, conjecturally, $\rho_{E,\ell^\infty} = \mathrm{GL}_2(\mathbb{Z}_\ell)$ for $\ell > 37$.

“Vertical” image conjecture

Conjecture

There exists a constant N such that for every E/\mathbb{Q} without CM

$$\left[\mathrm{GL}_2(\widehat{\mathbb{Z}}) : \rho_E(G_{\mathbb{Q}}) \right] \leq N.$$

Remark

This follows from the “ $\ell > 37$ ” conjecture.

Problem

Assume the “ $\ell > 37$ ” conjecture and compute N .

Labeling subgroups of $\mathrm{GL}_2(\widehat{\mathbb{Z}})$ up to conjugacy

Definition

When $\det(H) = \widehat{\mathbb{Z}}^\times$ these labels have the form $N.i.g.n$, where N is the level, i is the index, g is the genus, and n is a tiebreaker given by ordering the subgroups of $\mathrm{GL}_2(N)$.

Example

- The Borel subgroup $B(13)$ has label $13.14.0.1$.
- The normalizer of the split Cartan $N_{\mathrm{sp}}(13)$ has label $13.91.3.1$.
- The normalizer of the nonsplit Cartan $N_{\mathrm{ns}}(13)$ has label $13.78.3.1$.
- The maximal S_4 exceptional group $S_4(13)$ has label $13.91.3.2$.

Obligatory XKCD cartoon

HOW STANDARDS PROLIFERATE:
(SEE: A/C CHARGERS, CHARACTER ENCODINGS, INSTANT MESSAGING, ETC.)

SITUATION:
THERE ARE
14 COMPETING
STANDARDS.

14?! RIDICULOUS!
WE NEED TO DEVELOP
ONE UNIVERSAL STANDARD
THAT COVERS EVERYONE'S
USE CASES.



SOON:

SITUATION:
THERE ARE
15 COMPETING
STANDARDS.

Obligatory XKCD cartoon

HOW STANDARDS PROLIFERATE:
(SEE: A/C CHARGERS, CHARACTER ENCODINGS, INSTANT MESSAGING, ETC.)

SITUATION:

~~14~~ THERE ARE
COMPETING
STANDARDS.

15

14?! RIDICULOUS!
WE NEED TO DEVELOP
ONE UNIVERSAL STANDARD
THAT COVERS EVERYONE'S
USE CASES.



~~GOON:~~

yesterday

SITUATION:

~~15~~ THERE ARE
COMPETING
STANDARDS.

16

Main Theorem

Definition

A point $(E, \iota) \in X_H(K)$ is **exceptional** if $X_H(K)$ is finite and $\text{End } E = \mathbb{Z}$.

Theorem (Rouse–Sutherland–ZB 2021)

Let ℓ prime, E/\mathbb{Q} be a non-CM elliptic curve, and $H = \rho_{E, \ell^\infty}(G_{\mathbb{Q}})$.

Then exactly one of the following is true:

- ① $X_H(\mathbb{Q})$ is infinite and H is listed in (Sutherland–Zywina 2017);
- ② X_H has a rational exceptional point listed in Table 1;
- ③ $H \leq N_{\text{ns}}(3^3), N_{\text{ns}}(5^2), N_{\text{ns}}(7^2), N_{\text{ns}}(11^2)$, or $N_{\text{ns}}(\ell)$ for some $\ell > 13$;
- ④ H is a subgroup of $49.179.9.1$ or $49.196.9.1$.

We conjecture that cases (3) and (4) never occur.

If they do, the exceptional points have **extraordinarily** large heights (e.g. $10^{10^{200}}$ for $X_{\text{ns}}^+(11^2)(\mathbb{Q})$).

label	level	notes	j -invariants/models of exceptional points
16.64.2.1	2 ⁴	$N_{\text{ns}}(16)$	$-2^{18} \cdot 3 \cdot 5^3 \cdot 13^3 \cdot 41^3 \cdot 107^3 / 17^{16}$ $-2^{21} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13^3 \cdot 23^3 \cdot 41^3 \cdot 179^3 \cdot 409^3 / 79^{16}$
16.96.3.335	2 ⁴	$H(4) \subsetneq N_{\text{sp}}(4)$	$257^3 / 2^8$
16.96.3.343	2 ⁴	$H(4) \subsetneq N_{\text{sp}}(4)$	$17^3 \cdot 241^3 / 2^4$
16.96.3.346	2 ⁴	$H(4) \subsetneq N_{\text{sp}}(4)$	$2^4 \cdot 17^3$
16.96.3.338	2 ⁴	$H(4) \subsetneq N_{\text{sp}}(4)$	2^{11}
32.96.3.230	2 ⁵	$H(4) \subsetneq N_{\text{sp}}(4)$	$-3^3 \cdot 5^3 \cdot 47^3 \cdot 1217^3 / (2^8 \cdot 31^8)$
32.96.3.82	2 ⁵	$H(8) \subsetneq N_{\text{sp}}(8)$	$3^3 \cdot 5^6 \cdot 13^3 \cdot 23^3 \cdot 41^3 / (2^{16} \cdot 31^4)$
25.50.2.1	5 ²	$H(5) = N_{\text{ns}}(5)$	$2^4 \cdot 3^2 \cdot 5^7 \cdot 23^3$
25.75.2.1	5 ²	$H(5) = N_{\text{sp}}(5)$	$2^{12} \cdot 3^3 \cdot 5^7 \cdot 29^3 / 7^5$
7.56.1.2	7	$\subsetneq N_{\text{ns}}(7)$	$3^3 \cdot 5 \cdot 7^5 / 2^7$
7.112.1.2	7	$-I \notin H$	$y^2 + xy + y = x^3 - x^2 - 2680x - 50053$ $y^2 + xy + y = x^3 - x^2 - 131305x + 17430697$
11.60.1.3	11	$\subsetneq B(11)$	$-11 \cdot 131^3$
11.120.1.8	11	$-I \notin H$	$y^2 + xy + y = x^3 + x^2 - 30x - 76$
11.120.1.9	11	$-I \notin H$	$y^2 + xy = x^3 + x^2 - 2x - 7$
11.60.1.4	11	$\subsetneq B(11)$	-11^2
11.120.1.3	11	$-I \notin H$	$y^2 + xy = x^3 + x^2 - 3632x + 82757$
11.120.1.4	11	$-I \notin H$	$y^2 + xy + y = x^3 + x^2 - 305x + 7888$
13.91.3.2	13	$S_4(13)$	$2^4 \cdot 5 \cdot 13^4 \cdot 17^3 / 3^{13}, -2^{12} \cdot 5^3 \cdot 11 \cdot 13^4 / 3^{13}$ $2^{18} \cdot 3^3 \cdot 13^4 \cdot 127^3 \cdot 139^3 \cdot 157^3 \cdot 283^3 \cdot 929 / (5^{13} \cdot 61^{13})$
17.72.1.2	17	$\subsetneq B(17)$	$-17 \cdot 373^3 / 2^{17}$
17.72.1.4	17	$\subsetneq B(17)$	$-17^2 \cdot 101^3 / 2$
37.114.4.1	37	$\subsetneq B(37)$	$-7 \cdot 11^3$
37.114.4.2	37	$\subsetneq B(37)$	$-7 \cdot 137^3 \cdot 2083^3$

Table 1. All known exceptional groups, j -invariants, and points of prime power level.

UNSOLVED *mysteries*

Arithmetically maximal level ℓ^n groups with $\ell \leq 13$ with $X_H(\mathbb{Q})$ **unknown**.

label	level	group	genus
27.243.12.1	3^3	$N_{\text{ns}}(3^3)$	12
25.250.14.1	5^2	$N_{\text{ns}}(5^2)$	14
49.1029.69.1	7^2	$N_{\text{ns}}(7^2)$	69
49.147.9.1	7^2	$\langle \left(\begin{smallmatrix} 16 & 6 \\ 20 & 45 \end{smallmatrix} \right), \left(\begin{smallmatrix} 20 & 17 \\ 40 & 36 \end{smallmatrix} \right) \rangle$	9
49.196.9.1	7^2	$\langle \left(\begin{smallmatrix} 42 & 3 \\ 16 & 31 \end{smallmatrix} \right), \left(\begin{smallmatrix} 16 & 23 \\ 8 & 47 \end{smallmatrix} \right) \rangle$	9
121.6655.511.1	11^2	$N_{\text{ns}}(11^2)$	511

Each has **rank = genus**, **rational CM points**, **no rational cusps**, and **no known exceptional points**.

UNSOLVED *mysteries*

Arithmetically maximal level ℓ^n groups with $\ell \leq 13$ with $X_H(\mathbb{Q})$ **unknown**.

label	level	group	genus
27.243.12.1	3^3	$N_{\text{ns}}(3^3)$	12
25.250.14.1	5^2	$N_{\text{ns}}(5^2)$	14
49.1029.69.1	7^2	$N_{\text{ns}}(7^2)$	69
49.147.9.1	7^2	$\langle \left(\begin{smallmatrix} 16 & 6 \\ 20 & 45 \end{smallmatrix} \right), \left(\begin{smallmatrix} 20 & 17 \\ 40 & 36 \end{smallmatrix} \right) \rangle$	9
49.196.9.1	7^2	$\langle \left(\begin{smallmatrix} 42 & 3 \\ 16 & 31 \end{smallmatrix} \right), \left(\begin{smallmatrix} 16 & 23 \\ 8 & 47 \end{smallmatrix} \right) \rangle$	9
121.6655.511.1	11^2	$N_{\text{ns}}(11^2)$	511

Each has **rank = genus**, **rational CM points**, **no rational cusps**, and **no known exceptional points**.

Summary of ℓ -adic images of Galois for non-CM E/\mathbb{Q} .

ℓ	2	3	5*	7*	11*	13	17	37*	other*
subgroups	1208	47	25	17	8	12	3	3	1
exceptional	7	0	2	2	6	1	2	2	0
unexceptional	1201	47	23	15	2	11	1	1	1
max level	32	27	25	7	11	13	17	37	1
max index	96	72	120	112	120	91	72	114	1
max genus	3	0	2	1	1	3	1	4	0

Summary of $H \leq \mathrm{GL}_2(\mathbb{Z}_\ell)$ which occur as $\rho_{E,\ell^\infty}(G_\mathbb{Q})$ for some non-CM E/\mathbb{Q} .

Starred primes are conjectural.

In particular, we conjecture that there are 1207, 46, 24, 16, 7, 11, 2, 2 proper subgroups of $\mathrm{GL}_2(\mathbb{Z}_\ell)$ that arise as $\rho_{E,\ell^\infty}(G_\mathbb{Q})$ for non-CM E/\mathbb{Q} for $\ell = 2, 3, 5, 7, 11, 13, 17, 37$ and none for any other ℓ .

Applications

Theorem (R. Jones, Rouse, ZB)

- 1 **Arithmetic dynamics:** let $P \in E(\mathbb{Q})$.
- 2 How often is the order of $\tilde{P} \in E(\mathbb{F}_p)$ odd?
- 3 Answer depends on $\rho_{E,2^\infty}(G_{\mathbb{Q}})$.
- 4 Examples: 11/21 (generic), 121/168 (maximal), 1/28 (minimal)

Theorem (Daniels, Lozano-Robledo, Najman, Sutherland)

Classification of $E(\mathbb{Q}(3^\infty))_{\text{tors}}$

Theorem (Gonzalez-Jimenez, Lozano-Robledo)

Classify E/\mathbb{Q} with $\rho_{E,N}(G_{\mathbb{Q}})$ abelian.

Theorem (Rouse–Sutherland–ZB)

Improved algorithms for computing $\rho_{E,n}(G_{\mathbb{Q}})$.

Arithmetically maximal groups

Definition

We say that an open subgroup $H \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}})$ is **arithmetically maximal** if

- 1 $\det(H) = \widehat{\mathbb{Z}}^\times$ (necessary for \mathbb{Q} -points),
- 2 a conjugate of $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ lies in H (necessary for \mathbb{R} -points),
- 3 $j(X_H(\mathbb{Q}))$ is finite but $j(X_{H'}(\mathbb{Q}))$ is infinite for $H \subsetneq H' \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}})$.

Arithmetically maximal groups H arise as maximal subgroups of an H' with $X_{H'}(\mathbb{Q})$ infinite.

Arithmetically maximal groups

Definition

We say that an open subgroup $H \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}})$ is **arithmetically maximal** if

- 1 $\det(H) = \widehat{\mathbb{Z}}^\times$ (necessary for \mathbb{Q} -points),
- 2 a conjugate of $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ lies in H (necessary for \mathbb{R} -points),
- 3 $j(X_H(\mathbb{Q}))$ is finite but $j(X_{H'}(\mathbb{Q}))$ is infinite for $H \subsetneq H' \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}})$.

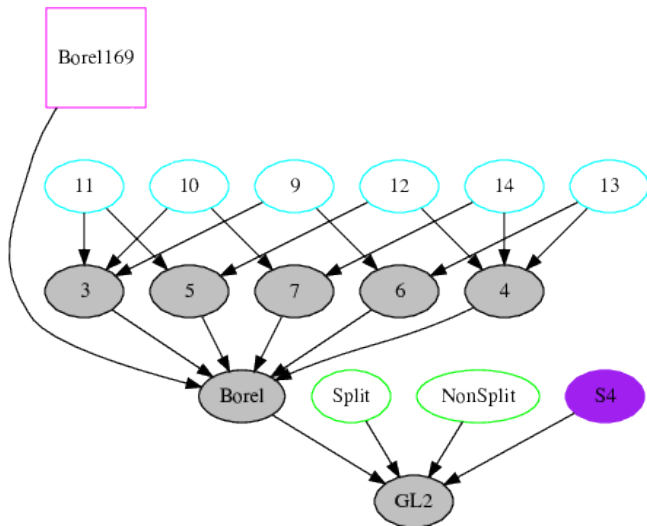
Arithmetically maximal groups H arise as maximal subgroups of an H' with $X_{H'}(\mathbb{Q})$ infinite.

Theorem (Sutherland–Zywina 2017)

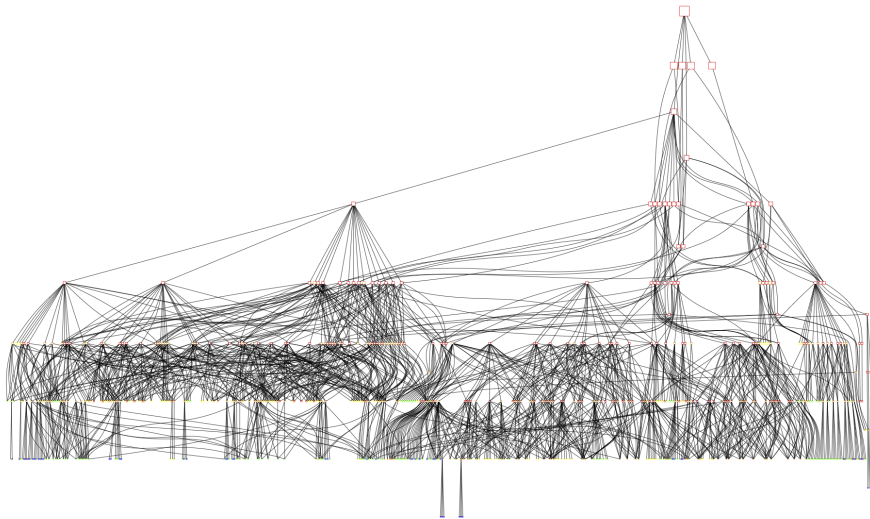
For $\ell = 2, 3, 5, 7, 11, 13$ there are 1208, 47, 23, 15, 2, 11 subgroups $H \leq \mathrm{GL}_2(\widehat{\mathbb{Z}})$ of ℓ -power level with $X_H(\mathbb{Q})$ infinite, and only $H = \mathrm{GL}_2(\widehat{\mathbb{Z}})$ for $\ell > 13$.

This allows us to compute explicit upper bounds on the level and index of arithmetically maximal subgroup of prime power level ℓ and we can then exhaustively enumerate them.

Subgroups of $GL_2(\mathbb{Z}_{13})$



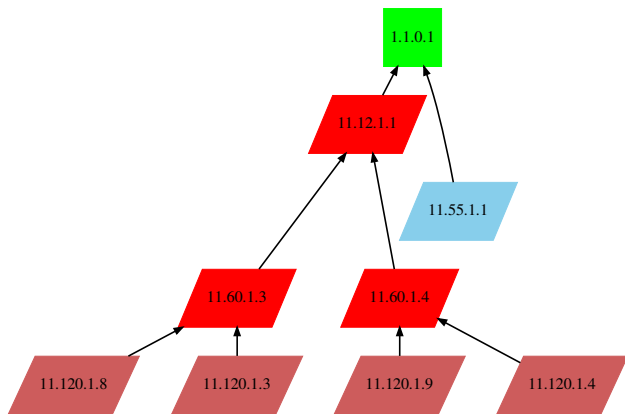
Subgroups of $GL_2(\mathbb{Z}_2)$



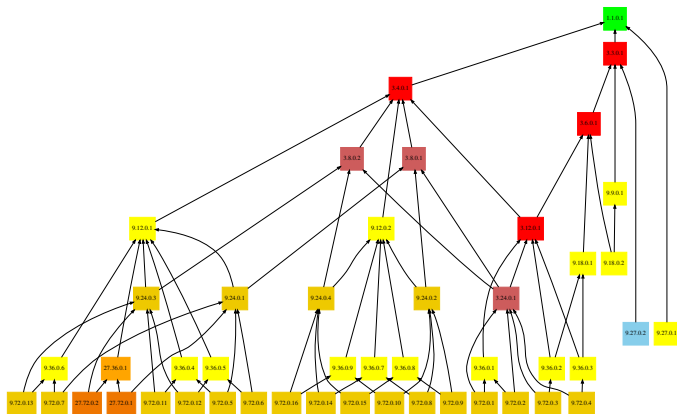
Steps of the proof

- 1 Compute the set \mathcal{S} of **arithmetically maximal** subgroups of ℓ -power level for $\ell \leq 37$ (for all $\ell > 37$ we already know $N_{\text{ns}}(\ell)$ is the only possible exceptional group).
- 2 For $H \in \mathcal{S}$ check for **local obstructions** and compute the **isogeny decomposition** of the Jacobian of X_H and the analytic ranks of all its simple factors.
- 3 For $H \in \mathcal{S}$ **compute equations** for X_H and $j_H: X_H \rightarrow X(1)$ (if needed). In several cases we can prove $X_H(\mathbb{Q})$ is empty without a model for X_H .
- 4 For $H \in \mathcal{S}$ with $-I \in H$ **determine the rational points** in $X_H(\mathbb{Q})$ (if possible). In several cases we are able to exploit recent progress by others ($\ell = 13$ for example).
- 5 For $H \in \mathcal{S}$ with $-I \notin H$ **compute equations** for the universal curve $\mathcal{E} \rightarrow U$, where $U \subseteq X_H$ is the locus with $j(P) \neq 0, 1728, \infty$.

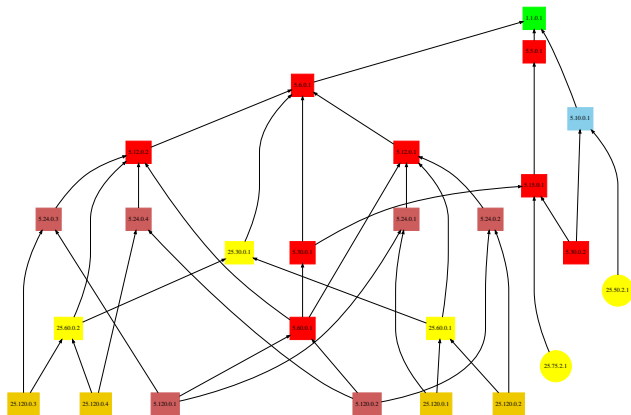
Subgroups of $GL_2(\mathbb{Z}_{11})$



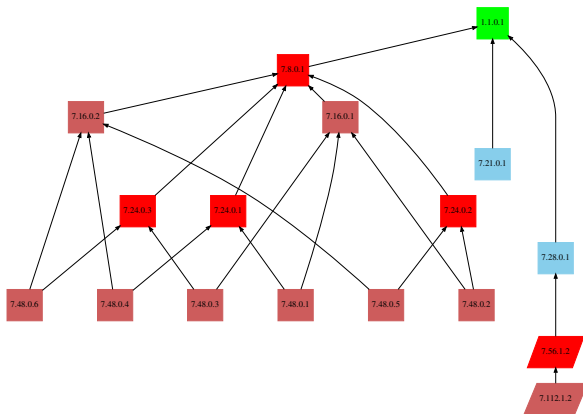
Subgroups of $GL_2(\mathbb{Z}_3)$



Subgroups of $GL_2(\mathbb{Z}_5)$



Subgroups of $GL_2(\mathbb{Z}_7)$



Finding Equations for X_H – Basic idea

- 1 The canonical map $C \hookrightarrow \mathbb{P}^{g-1}$ is given by $P \mapsto [\omega_1(P) : \cdots : \omega_g(P)]$.
- 2 For a general curve, this is an embedding, and the relations are quadratic.
- 3 For a modular curve,

$$M_k(H) \cong H^0(X_H, \Omega^1(\Delta)^{\otimes k/2})$$

given by

$$f(z) \mapsto f(z) dz^{\otimes k/2}.$$

Equations – Example: $X_1(17) \subset \mathbb{P}^4$

Cusp forms

$$q - 11q^5 + 10q^7 + O(q^8)$$

$$q^2 - 7q^5 + 6q^7 + O(q^8)$$

$$q^3 - 4q^5 + 2q^7 + O(q^8)$$

$$q^4 - 2q^5 + O(q^8)$$

$$q^6 - 3q^7 + O(q^8)$$

$$xu + 2xv - yz + yu - 3yv + z^2 - 4zu + 2u^2 + v^2 = 0$$

$$xu + xv - yz + yu - 2yv + z^2 - 3zu + 2uv = 0$$

$$2xz - 3xu + xv - 2y^2 + 3yz + 7yu - 4yv - 5z^2 - 3zu + 4zv = 0$$

Computing models of modular curves

- We introduce a variety of improvements and tricks to compute models of various X_H .
- See Rouse's [VaNTAGe talk](#) for more details and interesting examples.
- To compute $j_H: X_H \rightarrow X(1)$ we represent E_4 and E_6 as ratios of elements of the canonical ring.
- We show that E_4 is a rational function of an element of weight k and weight $k - 4$ if

$$k \geq \frac{2e_\infty + e_2 + e_3 + 5g - 4}{2(g - 1)}$$

- We used this method to compute canonical models for many curves of large genus.
- See Assaf's [recent paper](#) and Zywinina's [BIRS talk](#) for other efficient approaches.

Explicit methods: highlight reel

- Local methods
 - Chabauty and Elliptic Chabauty
 - Mordell–Weil sieve
 - étale descent
 - Pryms
 - *Equationless étale descent via group theory*
 - *New techniques for computing $\text{Aut } C$*
-
- *Nonabelian Chabauty*
 - **“Equationless” local methods** and **Mordell–Weil sieve**
 - **Greenberg Transforms** (and big computations)
 - **Novel variants of existing techniques**
 - **Modularity of isogeny factors of J_H** (w/ Voight)

Computing $X_H(\mathbb{F}_p)$ “via moduli”

Idea: one can compute $\#X_1(N)(\mathbb{F}_p)$ by enumerating elliptic curves over \mathbb{F}_p , then computing their N torsion subgroups.

Deligne–Rapoport 1973

The **modular curves** X_H and Y_H are coarse spaces for the stacks \mathcal{M}_H and \mathcal{M}_H^0 that parameterize elliptic curves E with **H -level structure**, by which we mean an equivalence class $[\iota]_H$ of isomorphisms $\iota: E[N] \rightarrow \mathbb{Z}(N)^2$, where $\iota \sim \iota'$ if $\iota = h \circ \iota'$ for some $h \in H$.

- $Y_H(\bar{k}) = \{(j(E), \alpha) : \alpha = Hg\mathcal{A}_E\}$ with $\mathcal{A}_E := \{\varphi_N : \varphi \in \text{Aut}(E_{\bar{k}})\}$, and $Y_H(k) = Y_H(\bar{k})^{G_k}$.
- $X_H^\infty(k) = \{\alpha \in H \backslash \text{GL}_2(N)/U(N) : \alpha^{\chi_N(G_K)} = \alpha\}$ where $U(N) := \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, -1 \rangle$.
- For $k = \mathbb{F}_q$, to compute $\#X_H(k) = \#Y_H(k) + \#X_H^\infty(k)$ count double cosets fixed by G_k .
- See Drew's **Slides** for a nice summary of the implementation.

Arithmetically maximal H of ℓ -power level for which
 $X_H(\mathbb{F}_p) = \emptyset$ for some $p \neq \ell \leq 37$

label	level	generators	p	rank	genus
16.48.2.17	2^4	$\begin{pmatrix} 11 & 9 \\ 4 & 13 \end{pmatrix}, \begin{pmatrix} 13 & 5 \\ 4 & 11 \end{pmatrix}, \begin{pmatrix} 1 & 9 \\ 12 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 9 \\ 0 & 5 \end{pmatrix}$	3, 11	0	2
27.108.4.5	3^3	$\begin{pmatrix} 4 & 25 \\ 6 & 14 \end{pmatrix}, \begin{pmatrix} 8 & 0 \\ 3 & 1 \end{pmatrix}$	7, 31	0	4
25.150.4.2	5^2	$\begin{pmatrix} 7 & 20 \\ 20 & 7 \end{pmatrix}, \begin{pmatrix} 22 & 2 \\ 13 & 22 \end{pmatrix}$	2	0	4
25.150.4.7	5^2	$\begin{pmatrix} 24 & 24 \\ 0 & 18 \end{pmatrix}, \begin{pmatrix} 2 & 5 \\ 0 & 23 \end{pmatrix}$	3, 23	4	4
25.150.4.8	5^2	$\begin{pmatrix} 8 & 4 \\ 0 & 23 \end{pmatrix}, \begin{pmatrix} 16 & 7 \\ 0 & 8 \end{pmatrix}$	2	0	4
25.150.4.9	5^2	$\begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix}, \begin{pmatrix} 3 & 18 \\ 0 & 14 \end{pmatrix}$	2	0	4
49.168.12.1	7^2	$\begin{pmatrix} 39 & 6 \\ 36 & 24 \end{pmatrix}, \begin{pmatrix} 11 & 9 \\ 24 & 2 \end{pmatrix}$	2	3	12
13.84.2.2	13	$\begin{pmatrix} 3 & 7 \\ 0 & 8 \end{pmatrix}, \begin{pmatrix} 12 & 4 \\ 0 & 12 \end{pmatrix}$	2	0	2
13.84.2.3	13	$\begin{pmatrix} 9 & 2 \\ 0 & 7 \end{pmatrix}, \begin{pmatrix} 4 & 4 \\ 0 & 7 \end{pmatrix}$	3	0	2
13.84.2.4	13	$\begin{pmatrix} 8 & 12 \\ 0 & 10 \end{pmatrix}, \begin{pmatrix} 8 & 3 \\ 0 & 9 \end{pmatrix}$	2	0	2
13.84.2.6	13	$\begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 11 & 3 \\ 0 & 10 \end{pmatrix}$	3	0	2

Decomposing the Jacobian of X_H

Let H be an open subgroup of $\mathrm{GL}_2(\widehat{\mathbb{Z}})$ of level N .

Let J_H denote the Jacobian of X_H .

Theorem (Rouse–Sutherland–Voight–ZB 2021)

Each **simple factor** A of J_H is **isogenous** to A_f for a weight-2 eigenform f on $\Gamma_0(N^2) \cap \Gamma_1(N)$.

Corollary (Kolyvagin's theorem)

If A is an **isogeny factor** of X_H , and if the **analytic rank** of A is **zero**, then $A(\mathbb{Q})$ is finite.

Corollary (Decomposition)

We can **decompose** J_H up to isogeny using linear algebra and point-counting.

Mordell–Weil sieve

- Let X be a **curve** and A be an **abelian variety**.

$$\begin{array}{c} X(\mathbb{Q}) \\ \downarrow \\ X(\mathbb{F}_p) \end{array}$$

- If $X(\mathbb{F}_p)$ is **empty** for some p then $X(\mathbb{Q})$ is **empty**.

Mordell–Weil sieve

- Let X be a **curve** and A be an **abelian variety**.

$$\begin{array}{ccc} X(\mathbb{Q}) & \longrightarrow & A(\mathbb{Q}) \\ \downarrow & & \downarrow \beta \\ X(\mathbb{F}_p) & \xrightarrow{\pi} & A(\mathbb{F}_p). \end{array}$$

- If $X(\mathbb{F}_p)$ is **empty** for some p then $X(\mathbb{Q})$ is **empty**.
- If $\text{im } \pi \cap \text{im } \beta$ is **empty** then $X(\mathbb{Q})$ is **empty**.

Mordell–Weil sieve

- Let X be a **curve** and A be an **abelian variety**.

$$\begin{array}{ccc} X(\mathbb{Q}) & \longrightarrow & A(\mathbb{Q}) \\ \downarrow & & \downarrow \beta \\ \prod_{p \in S} X(\mathbb{F}_p) & \xrightarrow{\pi_S} & \prod_{p \in S} A(\mathbb{F}_p). \end{array}$$

- If $X(\mathbb{F}_p)$ is **empty** for some p then $X(\mathbb{Q})$ is **empty**.
- If $\text{im } \pi \cap \text{im } \beta$ is **empty** then $X(\mathbb{Q})$ is **empty**.
- This is explicit and is implemented in Magma.

An equationless sieve for the group $121 \cdot 605 \cdot 41 \cdot 1$

The curve X_H has **local points everywhere**, and analytic **rank = genus = 41**.

$H(11) \subset N_{\text{ns}}(11)$, so X_H maps to $X_{\text{ns}}^+(11)$, which is an elliptic curve of rank 1.

$$\begin{array}{ccc} X_H(\mathbb{Q}) & \longrightarrow & X_{\text{ns}}^+(11)(\mathbb{Q}) \stackrel{=}{=} \langle R \rangle \cong \mathbb{Z} \\ \downarrow & & \downarrow \beta \\ \prod_{p \in S} X_H(\mathbb{F}_p) & \xrightarrow{\pi_S} & \prod_{p \in S} X_{\text{ns}}^+(11)(\mathbb{F}_p) \end{array}$$

We can compute $\text{im } \pi_S$ without *equations* for X_H or π_S

- A point of $X_{\text{ns}}^+(11)(\mathbb{F}_p)$ corresponds to E with $\rho_{E,11}(G_{\mathbb{F}_p}) \subset N_{\text{ns}}(11)$ and lifts to a point of $X_H(\mathbb{F}_p)$ if and only if $\rho_{E,121}(G_{\mathbb{F}_p}) \subset H(121)$.
- For $p = 13$ the image of any point in $Y_H(\mathbb{Q})$ maps to nR with $n \equiv 1, 5 \pmod{7}$.
- For $p = 307$ any point in $Y_H(\mathbb{Q})$ maps to nR with $n \equiv 2, 3, 4, 7, 10, 13 \pmod{14}$.
- Therefore $Y_H(\mathbb{Q}) = \emptyset$ (and in fact $X_H(\mathbb{Q}) = \emptyset$; there are no rational cusps).

Gargantuan models of modular curves¹

- We computed canonical models (over \mathbb{Q}) for $27.729.43.1$ (resp. $25.625.36.1$).
- We use these models to prove that X_H has no \mathbb{Q}_3 (resp. \mathbb{Q}_5) as follows.
- These models have very bad reduction at $p = 3$ (resp. 5). (They're not even flat.)
- $X_H(\mathbb{F}_p) \neq \emptyset$ for all p , but $X_H(\mathbb{Z}/p^2\mathbb{Z}) = \emptyset$ for $p = 3$ (resp. 5).
- The “Greenberg transform” (i.e., the “Wittferential tangent space” of Buium) is adjoint to Witt vectors: $X_H^{(1)}(\mathbb{F}_p) = X_H(\mathbb{Z}/p^2\mathbb{Z})$.
- The fibers of the map $X_H^{(1)} \rightarrow X_H$ have no \mathbb{F}_p points.

¹We give thanks to Poonen and Zywinia

Subgroups of $GL_2(\mathbb{Z}_2)$

