MATH 220, Mathematical Reasoning and Proof
MWF 1 - 1:50

All assignments
Last updated: October 13, 2023
Gradescope code: 7DVWGG

Contents

Assignment 1

Topics: Introduction to the course. Mathematical reasoning. Logic.

Reading: 1.1, 1.2, 1.5, 2.1, 2.2

Suggested problems (do not hand in; these are just for extra practice): Handout 1

Assignment 1, due Friday, Sep 15, 12:55pm via Gradescope:

1. Write the negation of each of the following statements.

   (a) All triangles are isosceles.

   (b) Every door in the building was locked.

   (c) Some even numbers are multiples of three.

   (d) Every real number is less than 100.

   (e) Every integer is positive or negative.

   (f) If f is a polynomial function, then f is continuous at 0.

   (g) If $x^2 > 0$, then $x > 0$.

   (h) There exists a $y \in \mathbf{R}$ such that $xy = 1$.

   (i) $(2 > 1)$ and $(\forall x, x^2 > 0)$

   (j) $\forall \epsilon > 0$, $\exists \delta > 0$ such that if $|x| < \delta$, then $|f(x)| < \epsilon$.

2. Write the converse, contrapositive, and negation of each of the following implications.

   (a) If a quadrilateral is a rectangle, then it has two pairs of parallel sides.

   (b) $(P \wedge \neg Q) \Rightarrow R$

   (c) $P \Rightarrow (R \Rightarrow \forall x, Q(x))$

3. Let P and Q be statements. Write the truth table for

   (a) $(\neg P) \vee Q$

   (b) $(P \wedge (\neg Q)) \Rightarrow Q$

4. Are the statements $(P \vee Q) \wedge R$ and $P \vee (Q \wedge R)$ equivalent? If so, give a proof. If not, explain why by giving a counterexample.

   (Two statement forms are equivalent if they have the same truth tables, and here, a counterexample simply means some choice of truth values for P, Q, and R such that the two statement forms give different outputs.)

5. Let P and Q be statements.

   (a) Prove that $\neg(P \Rightarrow Q)$ is equivalent to $P \wedge \neg Q$.

   (b) Prove that $\neg(P \Rightarrow Q)$ is *not* equivalent to $\neg P \wedge Q$.

(c) Give an example of statements P and Q such that $\neg P \Rightarrow \neg Q$ is true and $\neg(P \Rightarrow Q)$ is false.

6. Suppose that n is an even integer, and let m be any integer. Prove that nm is even.

7. Suppose that n is an odd integer. Prove that n^2 is an odd integer. (Hint: an integer n is odd if and only if there exists an integer k such that $n = 2k + 1$.)

8. Prove that if n^2 is even, then n is even. (Hint: ~~page 67~~ contrapositive.)

Assignment 2

Topics: "Direct" proofs, proof by cases, and divisibility problems.

Reading:

- 3.3, from Definition 3.3.6
- 6.4, just Definition 6.4.1
- see Index to find definitions like prime, etc

Suggested problems (do not hand in; these are just for extra practice)

1. Handout 2

Assignment, due Friday, Sep 22, 12:55pm via Gradescope:

1. Suppose that $a \mid b$. Prove that for all $n \in \mathbb{Z}_{>0}$, $a^n \mid b^n$.
2. Suppose that there exists an integer $n \in \mathbb{Z}_{>0}$ such that $a \mid b^n$. Is it true that $a \mid b$? Prove or disprove your answer. (For a disproof, please give a counterexample that demonstrates that the statement is false.)
3. Prove that for all $a \in \mathbb{Z}$ and for $n \in \mathbb{Z}_{\geq 0}$, $a - 1$ divides $a^n - 1$.
4. Prove that for all integers n, n and $n + 1$ have no common divisors other than ± 1.
5. Prove that if x is an integer, then $x^2 + 2$ is not divisible by 4. (Hint: there are two cases: x is even, x is odd. Also, feel free to use basic facts about even or odd, e.g., "odd + odd = even", without additional proof.)
6. Prove that the product of three consecutive integers is divisible by 6. (It suffices to prove that it is divisible by 2 and 3 separately.)
7. Show that for all integers a and b,
$$a^2 b^2 (a^2 - b^2)$$
is divisible by 12. (It suffices to prove that it is divisible by 4 and 3 separately.)
8. Find all positive integers n such that $n^2 - 1$ is prime. Prove that your answer is correct.

Assignment 3

Topics: Proof by contradiction. Unsolvability of equations. Irrationality.

Reading: 3.2
Suggested problems (do not hand in; these are just for extra practice)

1. Handout 3

Assignment, due Friday, Sep 29, 12:55pm via Gradescope:

1. Prove that there do not exist integers a, and b such that $21a + 30b = 1$.
2. Prove that $2^{1/3}$ is irrational.
3. Suppose that x is a real number such that $0 \leq x \leq \pi/2$. Prove that $\sin x + \cos x \geq 1$. (Hint: at some point in your proof, use that $(\sin x)^2 + (\cos x)^2 = 1$.)
4. Prove that there are no positive integer solutions to the equation $x^2 - y^2 = 10$.
5. Let a, b, c be integers satisfying $a^2 + b^2 = c^2$. Show that abc must be even.
6. Suppose that a and n are integers that are both at least 2. Prove that if $a^n - 1$ is prime, then $a = 2$ and n is a prime. (Primes of the form $2^n - 1$ are called Mersenne primes.)
7. Suppose that $a, b \in \mathbb{Z}$. Prove that $a^2 - 4b \neq 2$.
8. Prove that $\log_{10} 7$ is irrational

Assignment 4

Topics: Induction.

Reading: Chapter 6

Fun Video (optional): Vi Hart; "Doodling in Math: Spirals, Fibonacci, and Being a Plant"
https://www.youtube.com/watch?v=ahXIMUkSXX0

Suggested problems (do not hand in; these are just for extra practice)

1. Handout 4
2. Handout 5

Assignment, due Friday, Oct 06, 12:55pm via Gradescope:

1. Prove that for every positive integer n,

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

2. Let a_n be defined recursively by $a_1 = 1$ and $a_n = \sqrt{1 + a_{n-1}}$. Prove that for all positive integers n, $a_n < 2$.

3. Prove by induction that if $b_1, b_2, \ldots, b_n$ are even integers, then $b_1 + b_2 + \cdots + b_n$ is even.

4. Let $F_1, F_2, F_3, \ldots = 1, 1, 2, 3, 5, 8, \ldots$ be the Fibonacci sequence. Prove that $F_1^2 + \cdots + F_n^2 = F_n F_{n+1}$.

5. Prove that $n! > 2^n$ for all $n \geq 4$.

6. *Bernoulli's inequality*: let $\beta \in \mathbb{R}$ be a real number such that $\beta > -1$ and $\beta \neq 0$. Prove that for all integers $n \geq 2$, $(1 + \beta)^n > 1 + n\beta$.

7. Prove that for all integers $n \geq 1$,

$$1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{n}.$$

8. Prove (using induction) that for all integers $n \geq 1$, $2^{2n} - 1$ is divisible by 3.

Assignment 5

Topics: Basics of set theory. Basic operations. Proofs with sets.

Reading: 1.3, 1.4, 2.3
Suggested problems (do not hand in; these are just for extra practice): Handout 6

Assignment 5, due Friday, Oct 13, 12:55pm via Gradescope:

1. Let $A = \{n \in \mathbb{Z} | n \text{ is a multiple of } 4\}$ and $B = \{n \in \mathbb{Z} | n^2 \text{ is a multiple of } 4\}$
   (a) Prove or disprove: $A \subseteq B$.
   (b) Prove or disprove: $B \subseteq A$.
2. Prove that $A \cup (A \cap B) = A$.
3. Let A, B and C be sets.
   (a) Prove that $(A \subseteq C) \wedge (B \subseteq C) \Rightarrow A \cup B \subseteq C$.
   (b) State the contrapositive of part (a).
   (c) State the converse of part (a). Prove or disprove it.
4. Let n and m be integers. Prove that if $n\mathbb{Z} \subseteq m\mathbb{Z}$ then m divides n.

Midterm study guide

Topics: Friday, October 13 will be an in class Exam.

Content: The questions will all be either

1. homework problems,
2. suggested problems,
3. problems we worked in class, or
4. minor variations of one of these.

Problems with very long proofs or that involved some unusual trick will not be on the exam.

You are allowed to use any previous problem from class or from the homework (e.g., "additivity of divisibility" or "the 2 out of 3 rule") on the exam without reproving it, unless otherwise noted on the exam. (E.g., if I ask you to prove "additivity of divisibility" on the exam, you will need to prove this using only the definition of divisibility, and I will remind you of this in the statement of the problem.)

A typical exam will have one or two questions from each week of the course. You can expect problems about following:

- Negations
- Give definitions (e.g., divides, rational, subset)
- Direct proofs
- Proof by contrapositive
- Divisibility problems
- Contradiction
- Induction
- Proofs with sets.

There will be a negation problem, at least one definition, and around 4-5 problems involving proofs (possibly including "prove or disprove" problems).

For sets: I will ask one problem, verbatim, from the homework or from class.

For definitions, I want a definition, in prose (complete sentences), and I want "just" the definition, and not any additional facts about the definition. (E.g., if you give the definition of rational, do not include that a rational number can be written in reduced form; that is a fact about rational numbers not part of the definition of rational.)

Assignment 6

Topics: More proofs with sets. DeMorgan's laws. Cartesian Products. Power sets

Suggested problems (do not hand in; these are just for extra practice)

1. Handout 7

Assignment 6, due Thursday, Oct 20, 12:55pm via Gradescope:

1. Recall that $(a, b) = \{x : x \in \mathbb{R} \mid a < x < b\}$. Prove or disprove each of the following:
   (a) $(-1, 1) \subseteq (-2, 2)$.
   (b) $(-1, 2) \subseteq (-2, 1)$.
2. Let A, B, and C be sets. Prove or disprove the following.
   (a) If $A \not\subseteq B$ and $B \not\subseteq C$, then $A \not\subseteq C$.
   (b) If $A \subseteq B$ and $B \not\subseteq C$, then $A \not\subseteq C$.
3. Let A, B be sets. Prove each of the following:
   (a) Suppose that $B \subseteq C$. Prove that $A - C \subseteq A - B$.
   (b) $A \subseteq B$ if and only if $A \cap B = A$.
4. Let A, B and C be sets. Prove or disprove each of the following. (For a disproof, please give an explicit counterexample; i.e., give an example of sets A, B and C demonstrating that the statement is false.)
   (a) $(A \cap B) \cup C = A \cap (B \cup C)$.
   (b) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.
5. Let A and B be sets. Prove that $(A \cup B) \cap \overline{A} = B - A$.
6. Let A and B be sets. Prove that $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$.
7. Let $A = \{0, 1, 2\}$. Which of the following statements are true? (No justification is needed.)
   (a) $\{0\} \subseteq P(A)$;
   (b) $\{1, 2\} \in P(A)$;
   (c) $\{1, \{1\}\} \subseteq P(A)$.
   (d) $\{\{0, 1\}, \{1\}\} \subseteq P(A)$;
   (e) $\emptyset \in P(A)$;
   (f) $\emptyset \subseteq P(A)$;

(g) $\{\emptyset\} \in P(A)$.

(h) $\{\emptyset\} \subseteq P(A)$;

8. Let A and B be sets. Prove that if $A \subseteq B$, then $P(A) \subseteq P(B)$. State the converse of this and prove or disprove it.