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Combinatorics and Graph Theory

Second Edition

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Combinatorics and Graph Theory

Second Edition

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To

Priscilla, Sophie, and Will,

Holly,

Kristine, Amanda, and Alexandra

Preface to the Second Edition

> *There are certain rules that one must abide by in order to create a successful sequel.*
>
> — Randy Meeks, from the trailer to *Scream 2*

While we may not follow the precise rules that Mr. Meeks had in mind for successful sequels, we have made a number of changes to the text in this second edition. In the new edition, we continue to introduce new topics with concrete examples, we provide complete proofs of almost every result, and we preserve the book's friendly style and lively presentation, interspersing the text with occasional jokes and quotations. The first two chapters, on graph theory and combinatorics, remain largely independent, and may be covered in either order. Chapter 3, on infinite combinatorics and graphs, may also be studied independently, although many readers will want to investigate trees, matchings, and Ramsey theory for finite sets before exploring these topics for infinite sets in the third chapter. Like the first edition, this text is aimed at upper-division undergraduate students in mathematics, though others will find much of interest as well. It assumes only familiarity with basic proof techniques, and some experience with matrices and infinite series.

The second edition offers many additional topics for use in the classroom or for independent study. Chapter 1 includes a new section covering distance and related notions in graphs, following an expanded introductory section. This new section also introduces the adjacency matrix of a graph, and describes its connection to important features of the graph. Another new section on trails, circuits, paths, and cycles treats several problems regarding Hamiltonian and Eulerian paths in graphs, and describes some elementary open problems regarding paths in graphs, and graphs with forbidden subgraphs.

Several topics were added to Chapter 2. The introductory section on basic counting principles has been expanded. Early in the chapter, a new section covers multinomial coefficients and their properties, following the development of the binomial coefficients. Another new section treats the pigeonhole principle, with applications to some problems in number theory. The material on Pólya's theory of counting has now been expanded to cover de Bruijn's more general method of counting arrangements in the presence of one symmetry group acting on the objects, and another acting on the set of allowed colors. A new section has also been added on partitions, and the treatment of Eulerian numbers has been significantly expanded. The topic of stable marriage is developed further as well, with three interesting variations on the basic problem now covered here. Finally, the end of the chapter features a new section on combinatorial geometry. Two principal problems serve to introduce this rich area: a nice problem of Sylvester's regarding lines produced by a set of points in the plane, and the beautiful geometric approach to Ramsey theory pioneered by Erdős and Szekeres in a problem about the existence of convex polygons among finite sets of points in the plane.

In Chapter 3, a new section develops the theory of matchings further by investigating marriage problems on infinite sets, both countable and uncountable. Another new section toward the end of this chapter describes a characterization of certain large infinite cardinals by using linear orderings. Many new exercises have also been added in each chapter, and the list of references has been completely updated.

The second edition grew out of our experiences teaching courses in graph theory, combinatorics, and set theory at Appalachian State University, Davidson College, and Furman University, and we thank these institutions for their support, and our students for their comments. We also thank Mark Spencer at Springer-Verlag. Finally, we thank our families for their patience and constant good humor throughout this process. The first and third authors would also like to add that, since the original publication of this book, their families have both gained their own second additions!

May 2008

John M. Harris
Jeffry L. Hirst
Michael J. Mossinghoff

Preface to the First Edition

Three things should be considered: problems, theorems, and applications.

— Gottfried Wilhelm Leibniz,
Dissertatio de Arte Combinatoria, 1666

This book grew out of several courses in combinatorics and graph theory given at Appalachian State University and UCLA in recent years. A one-semester course for juniors at Appalachian State University focusing on graph theory covered most of Chapter 1 and the first part of Chapter 2. A one-quarter course at UCLA on combinatorics for undergraduates concentrated on the topics in Chapter 2 and included some parts of Chapter 1. Another semester course at Appalachian State for advanced undergraduates and beginning graduate students covered most of the topics from all three chapters.

There are rather few prerequisites for this text. We assume some familiarity with basic proof techniques, like induction. A few topics in Chapter 1 assume some prior exposure to elementary linear algebra. Chapter 2 assumes some familiarity with sequences and series, especially Maclaurin series, at the level typically covered in a first-year calculus course. The text requires no prior experience with more advanced subjects, such as group theory.

While this book is primarily intended for upper-division undergraduate students, we believe that others will find it useful as well. Lower-division undergraduates with a penchant for proofs, and even talented high school students, will be able to follow much of the material, and graduate students looking for an introduction to topics in graph theory, combinatorics, and set theory may find several topics of interest.

Chapter 1 focuses on the theory of finite graphs. The first section serves as an introduction to basic terminology and concepts. Each of the following sections presents a specific branch of graph theory: trees, planarity, coloring, matchings, and Ramsey theory. These five topics were chosen for two reasons. First, they represent a broad range of the subfields of graph theory, and in turn they provide the reader with a sound introduction to the subject. Second, and just as important, these topics relate particularly well to topics in Chapters 2 and 3.

Chapter 2 develops the central techniques of enumerative combinatorics: the principle of inclusion and exclusion, the theory and application of generating functions, the solution of recurrence relations, Pólya's theory of counting arrangements in the presence of symmetry, and important classes of numbers, including the Fibonacci, Catalan, Stirling, Bell, and Eulerian numbers. The final section in the chapter continues the theme of matchings begun in Chapter 1 with a consideration of the stable marriage problem and the Gale–Shapley algorithm for solving it.

Chapter 3 presents infinite pigeonhole principles, König's Lemma, Ramsey's Theorem, and their connections to set theory. The systems of distinct representatives of Chapter 1 reappear in infinite form, linked to the axiom of choice. Counting is recast as cardinal arithmetic, and a pigeonhole property for cardinals leads to discussions of incompleteness and large cardinals. The last sections connect large cardinals to finite combinatorics and describe supplementary material on computability.

Following Leibniz's advice, we focus on problems, theorems, and applications throughout the text. We supply proofs of almost every theorem presented. We try to introduce each topic with an application or a concrete interpretation, and we often introduce more applications in the exercises at the end of each section. In addition, we believe that mathematics is a fun and lively subject, so we have tried to enliven our presentation with an occasional joke or (we hope) interesting quotation.

We would like to thank the Department of Mathematical Sciences at Appalachian State University and the Department of Mathematics at UCLA. We would especially like to thank our students (in particular, Jae-Il Shin at UCLA), whose questions and comments on preliminary versions of this text helped us to improve it. We would also like to thank the three anonymous reviewers, whose suggestions helped to shape this book into its present form. We also thank Sharon McPeake, a student at ASU, for her rendering of the Königsberg bridges.

In addition, the first author would like to thank Ron Gould, his graduate advisor at Emory University, for teaching him the methods and the joys of studying graphs, and for continuing to be his advisor even after graduation. He especially wants to thank his wife, Priscilla, for being his perfect match, and his daughter Sophie for adding color and brightness to each and every day. Their patience and support throughout this process have been immeasurable.

The second author would like to thank Judith Roitman, who introduced him to set theory and Ramsey's Theorem at the University of Kansas, using an early draft

of her fine text. Also, he would like to thank his wife, Holly (the other Professor Hirst), for having the infinite tolerance that sets her apart from the norm.

The third author would like to thank Bob Blakley, from whom he first learned about combinatorics as an undergraduate at Texas A & M University, and Donald Knuth, whose class *Concrete Mathematics* at Stanford University taught him much more about the subject. Most of all, he would like to thank his wife, Kristine, for her constant support and infinite patience throughout the gestation of this project, and for being someone he can always, well, count on.

September 1999

John M. Harris
Jeffry L. Hirst
Michael J. Mossinghoff

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1
Graph Theory

"Begin at the beginning," the King said, gravely, "and go on till you come to the end; then stop."

— Lewis Carroll, *Alice in Wonderland*

The Pregolya River passes through a city once known as Königsberg. In the 1700s seven bridges were situated across this river in a manner similar to what you see in Figure 1.1. The city's residents enjoyed strolling on these bridges, but, as hard as they tried, no resident of the city was ever able to walk a route that crossed each of these bridges exactly once. The Swiss mathematician Leonhard Euler learned of this frustrating phenomenon, and in 1736 he wrote an article [98] about it. His work on the "Königsberg Bridge Problem" is considered by many to be the beginning of the field of graph theory.

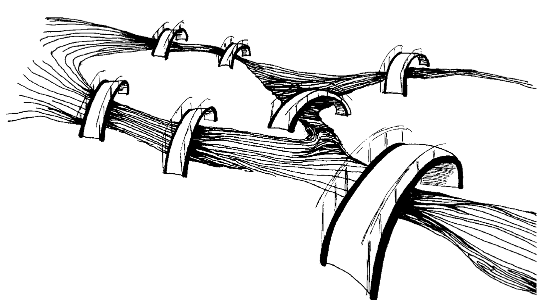


FIGURE 1.1. The bridges in Königsberg.

J.M. Harris et al., *Combinatorics and Graph Theory*, DOI: 10.1007/978-0-387-79711-3_1,

At first, the usefulness of Euler's ideas and of "graph theory" itself was found only in solving puzzles and in analyzing games and other recreations. In the mid 1800s, however, people began to realize that graphs could be used to model many things that were of interest in society. For instance, the "Four Color Map Conjecture," introduced by DeMorgan in 1852, was a famous problem that was seemingly unrelated to graph theory. The conjecture stated that four is the maximum number of colors required to color any map where bordering regions are colored differently. This conjecture can easily be phrased in terms of graph theory, and many researchers used this approach during the dozen decades that the problem remained unsolved.

The field of graph theory began to blossom in the twentieth century as more and more modeling possibilities were recognized — and the growth continues. It is interesting to note that as specific applications have increased in number and in scope, the theory itself has developed beautifully as well.

In Chapter 1 we investigate some of the major concepts and applications of graph theory. Keep your eyes open for the Königsberg Bridge Problem and the Four Color Problem, for we will encounter them along the way.

1.1 Introductory Concepts

A definition is the enclosing a wilderness of idea within a wall of words.

— Samuel Butler, *Higgledy-Piggledy*

1.1.1 Graphs and Their Relatives

A *graph* consists of two finite sets, V and E. Each element of V is called a *vertex* (plural *vertices*). The elements of E, called *edges*, are unordered pairs of vertices. For instance, the set V might be $\{a, b, c, d, e, f, g, h\}$, and E might be $\{\{a, d\}$, $\{a, e\}$, $\{b, c\}$, $\{b, e\}$, $\{b, g\}$, $\{c, f\}$, $\{d, f\}$, $\{d, g\}$, $\{g, h\}\}$. Together, V and E are a graph G.

Graphs have natural visual representations. Look at the diagram in Figure 1.2. Notice that each element of V is represented by a small circle and that each element of E is represented by a line drawn between the corresponding two elements of V.

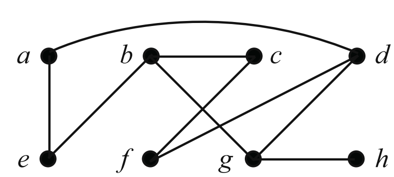


FIGURE 1.2. A visual representation of the graph G.

As a matter of fact, we can just as easily define a graph to be a diagram consisting of small circles, called vertices, and curves, called edges, where each curve connects two of the circles together. When we speak of a graph in this chapter, we will almost always refer to such a diagram.

We can obtain similar structures by altering our definition in various ways. Here are some examples.

1. By replacing our set E with a set of *ordered* pairs of vertices, we obtain a *directed graph*, or *digraph* (Figure 1.3). Each edge of a digraph has a specific orientation.

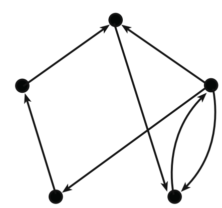


FIGURE 1.3. A digraph.

2. If we allow repeated elements in our set of edges, technically replacing our set E with a multiset, we obtain a *multigraph* (Figure 1.4).

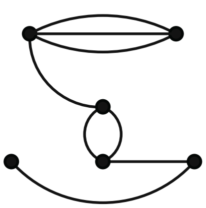


FIGURE 1.4. A multigraph.

3. By allowing edges to connect a vertex to itself ("loops"), we obtain a *pseudograph* (Figure 1.5).

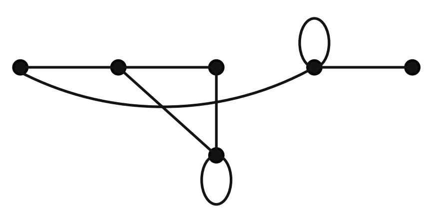


FIGURE 1.5. A pseudograph.

4. Allowing our edges to be arbitrary subsets of vertices (rather than just pairs) gives us *hypergraphs* (Figure 1.6).

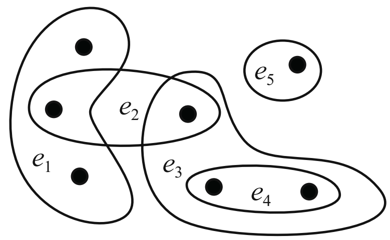


FIGURE 1.6. A hypergraph with 7 vertices and 5 edges.

5. By allowing V or E to be an infinite set, we obtain *infinite graphs*. Infinite graphs are studied in Chapter 3.

In this chapter we will focus on finite, simple graphs: those without loops or multiple edges.

Exercises

1. Ten people are seated around a circular table. Each person shakes hands with everyone at the table except the person sitting directly across the table. Draw a graph that models this situation.

2. Six fraternity brothers (Adam, Bert, Chuck, Doug, Ernie, and Filthy Frank) need to pair off as roommates for the upcoming school year. Each person has compiled a list of the people with whom he would be willing to share a room.

   Adam's list: Doug
   Bert's list: Adam, Ernie
   Chuck's list: Doug, Ernie
   Doug's list: Chuck
   Ernie's list: Ernie
   Frank's list: Adam, Bert

   Draw a digraph that models this situation.

3. There are twelve women's basketball teams in the Atlantic Coast Conference: Boston College (B), Clemson (C), Duke (D), Florida State (F), Georgia Tech (G), Miami (I), NC State (S), Univ. of Maryland (M), Univ. of North Carolina (N), Univ. of Virginia (V), Virginia Tech (T), and Wake Forest Univ. (W). At a certain point in midseason,

   B has played I, T*, W

   C has played D*, G

D has played C*, S, W

F has played N*, V

G has played C, M

I has played B, M, T

S has played D, V*

M has played G, I, N

N has played F*, M, W

V has played F, S*

T has played B*, I

W has played B, D, N

The asterisk(*) indicates that these teams have played each other twice. Draw a multigraph that models this situation.

4. Can you explain why no resident of Königsberg was ever able to walk a route that crossed each bridge exactly once? (We will encounter this question again in Section 1.4.1.)

1.1.2 The Basics

Your first discipline is your vocabulary;

— Robert Frost

In this section we will introduce a number of basic graph theory terms and concepts. Study them carefully and pay special attention to the examples that are provided. Our work together in the sections that follow will be enriched by a solid understanding of these ideas.

The Very Basics

The vertex set of a graph G is denoted by $V(G)$, and the edge set is denoted by $E(G)$. We may refer to these sets simply as V and E if the context makes the particular graph clear. For notational convenience, instead of representing an edge as $\{u, v\}$, we denote this simply by uv. The *order* of a graph G is the cardinality of its vertex set, and the *size* of a graph is the cardinality of its edge set.

Given two vertices u and v, if $uv \in E$, then u and v are said to be *adjacent*. In this case, u and v are said to be the *end vertices* of the edge uv. If $uv \notin E$, then u and v are *nonadjacent*. Furthermore, if an edge e has a vertex v as an end vertex, we say that v is *incident* with e.

The *neighborhood* (or *open neighborhood*) of a vertex v, denoted by $N(v)$, is the set of vertices adjacent to v:

$$N(v) = \{x \in V \mid vx \in E\}.$$

The *closed neighborhood* of a vertex v, denoted by $N[v]$, is simply the set $\{v\} \cup N(v)$. Given a set S of vertices, we define the neighborhood of S, denoted by $N(S)$, to be the union of the neighborhoods of the vertices in S. Similarly, the closed neighborhood of S, denoted $N[S]$, is defined to be $S \cup N(S)$.

The *degree* of v, denoted by $\deg(v)$, is the number of edges incident with v. In simple graphs, this is the same as the cardinality of the (open) neighborhood of v. The *maximum degree* of a graph G, denoted by $\Delta(G)$, is defined to be

$$\Delta(G) = \max\{\deg(v) \mid v \in V(G)\}.$$

Similarly, the *minimum degree* of a graph G, denoted by $\delta(G)$, is defined to be

$$\delta(G) = \min\{\deg(v) \mid v \in V(G)\}.$$

The *degree sequence* of a graph of order n is the n-term sequence (usually written in descending order) of the vertex degrees.

Let's use the graph G in Figure 1.2 to illustrate some of these concepts: G has order 8 and size 9; vertices a and e are adjacent while vertices a and b are nonadjacent; $N(d) = \{a, f, g\}$, $N[d] = \{a, d, f, g\}$; $\Delta(G) = 3$, $\delta(G) = 1$; and the degree sequence is 3, 3, 3, 2, 2, 2, 2, 1.

The following theorem is often referred to as the First Theorem of Graph Theory.

Theorem 1.1. *In a graph G, the sum of the degrees of the vertices is equal to twice the number of edges. Consequently, the number of vertices with odd degree is even.*

Proof. Let $S = \sum_{v \in V} \deg(v)$. Notice that in counting S, we count each edge exactly twice. Thus, $S = 2|E|$ (the sum of the degrees is twice the number of edges). Since S is even, it must be that the number of vertices with odd degree is even. $\square$

Perambulation and Connectivity

A *walk* in a graph is a sequence of (not necessarily distinct) vertices $v_1, v_2, \ldots, v_k$ such that $v_i v_{i+1} \in E$ for $i = 1, 2, \ldots, k-1$. Such a walk is sometimes called a v_1–v_k *walk*, and v_1 and v_k are the *end vertices* of the walk. If the vertices in a walk are distinct, then the walk is called a *path*. If the edges in a walk are distinct, then the walk is called a *trail*. In this way, every path is a trail, but not every trail is a path. Got it?

A *closed path*, or *cycle*, is a path $v_1, \ldots, v_k$ (where $k \geq 3$) together with the edge $v_k v_1$. Similarly, a trail that begins and ends at the same vertex is called a *closed trail*, or *circuit*. The *length* of a walk (or path, or trail, or cycle, or circuit) is its number of edges, counting repetitions.

Once again, let's illustrate these definitions with an example. In the graph of Figure 1.7, a, c, f, c, b, d is a walk of length 5. The sequence b, a, c, b, d represents a trail of length 4, and the sequence d, g, b, a, c, f, e represents a path of length 6.

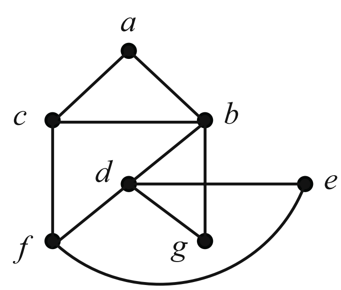


FIGURE 1.7.

Also, g, d, b, c, a, b, g is a circuit, while e, d, b, a, c, f, e is a cycle. In general, it is possible for a walk, trail, or path to have length 0, but the least possible length of a circuit or cycle is 3.

The following theorem is often referred to as the Second Theorem in this book.

Theorem 1.2. *In a graph G with vertices u and v, every u–v walk contains a u–v path.*

Proof. Let W be a u–v walk in G. We prove this theorem by induction on the length of W. If W is of length 1 or 2, then it is easy to see that W must be a path. For the induction hypothesis, suppose the result is true for all walks of length less than k, and suppose W has length k. Say that W is

$$u = w_0, w_1, w_2, \ldots, w_{k-1}, w_k = v$$

where the vertices are not necessarily distinct. If the vertices are in fact distinct, then W itself is the desired u–v path. If not, then let j be the smallest integer such that $w_j = w_r$ for some $r > j$. Let W_1 be the walk

$$u = w_0, \ldots, w_j, w_{r+1}, \ldots, w_k = v.$$

This walk has length strictly less than k, and therefore the induction hypothesis implies that W_1 contains a u–v path. This means that W contains a u–v path, and the proof is complete. □

We now introduce two different operations on graphs: *vertex deletion* and *edge deletion*. Given a graph G and a vertex $v \in V(G)$, we let $G - v$ denote the graph obtained by removing v and all edges incident with v from G. If S is a set of vertices, we let $G - S$ denote the graph obtained by removing each vertex of S and all associated incident edges. If e is an edge of G, then $G - e$ is the graph obtained by removing only the edge e (its end vertices stay). If T is a set of edges, then $G - T$ is the graph obtained by deleting each edge of T from G. Figure 1.8 gives examples of these operations.

A graph is *connected* if every pair of vertices can be joined by a path. Informally, if one can pick up an entire graph by grabbing just one vertex, then the

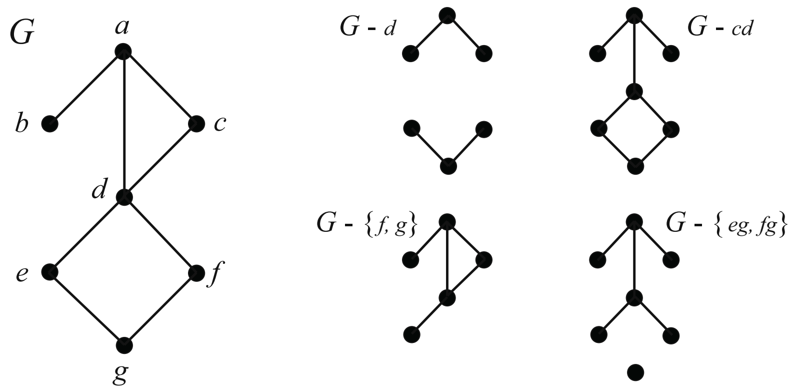


FIGURE 1.8. Deletion operations.

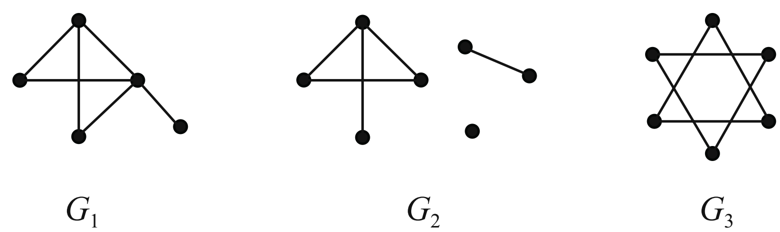


FIGURE 1.9. Connected and disconnected graphs.

graph is connected. In Figure 1.9, G_1 is connected, and both G_2 and G_3 are not connected (or *disconnected*). Each maximal connected piece of a graph is called a *connected component*. In Figure 1.9, G_1 has one component, G_2 has three components, and G_3 has two components.

If the deletion of a vertex v from G causes the number of components to increase, then v is called a *cut vertex*. In the graph G of Figure 1.8, vertex d is a cut vertex and vertex c is not. Similarly, an edge e in G is said to be a *bridge* if the graph $G - e$ has more components than G. In Figure 1.8, the edge ab is the only bridge.

A proper subset S of vertices of a graph G is called a *vertex cut set* (or simply, a *cut set*) if the graph $G - S$ is disconnected. A graph is said to be *complete* if every vertex is adjacent to every other vertex. Consequently, if a graph contains at least one nonadjacent pair of vertices, then that graph is not complete. Complete graphs do not have any cut sets, since $G - S$ is connected for all proper subsets S of the vertex set. Every non-complete graph has a cut set, though, and this leads us to another definition. For a graph G which is not complete, the *connectivity* of G, denoted $\kappa(G)$, is the minimum size of a cut set of G. If G is a connected, non-complete graph of order n, then $1 \leq \kappa(G) \leq n - 2$. If G is disconnected, then $\kappa(G) = 0$. If G is complete of order n, then we say that $\kappa(G) = n - 1$.

Further, for a positive integer k, we say that a graph is k-*connected* if $k \leq \kappa(G)$. You will note here that "1-connected" simply means "connected."

Here are several facts that follow from these definitions. You will get to prove a couple of them in the exercises.

i. A graph is connected if and only if $\kappa(G) \geq 1$.

ii. $\kappa(G) \geq 2$ if and only if G is connected and has no cut vertices.

iii. Every 2-connected graph contains at least one cycle.

iv. For every graph G, $\kappa(G) \leq \delta(G)$.

Exercises

1. If G is a graph of order n, what is the maximum number of edges in G?

2. Prove that for any graph G of order at least 2, the degree sequence has at least one pair of repeated entries.

3. Consider the graph shown in Figure 1.10.

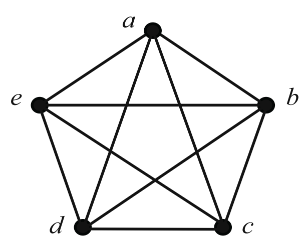


FIGURE 1.10.

   (a) How many different paths have c as an end vertex?

   (b) How many different paths avoid vertex c altogether?

   (c) What is the maximum length of a circuit in this graph? Give an example of such a circuit.

   (d) What is the maximum length of a circuit that does not include vertex c? Give an example of such a circuit.

4. Is it true that a finite graph having exactly two vertices of odd degree must contain a path from one to the other? Give a proof or a counterexample.

5. Let G be a graph where $\delta(G) \geq k$.

   (a) Prove that G has a path of length at least k.

   (b) If $k \geq 2$, prove that G has a cycle of length at least $k + 1$.

6. Prove that every closed odd walk in a graph contains an odd cycle.

7. Draw a connected graph having at most 10 vertices that has at least one cycle of each length from 5 through 9, but has no cycles of any other length.

8. Let P_1 and P_2 be two paths of maximum length in a connected graph G. Prove that P_1 and P_2 have a common vertex.

9. Let G be a graph of order n that is not connected. What is the maximum size of G?

10. Let G be a graph of order n and size strictly less than $n - 1$. Prove that G is not connected.

11. Prove that an edge e is a bridge of G if and only if e lies on no cycle of G.

12. Prove or disprove each of the following statements.

    (a) If G has no bridges, then G has exactly one cycle.

    (b) If G has no cut vertices, then G has no bridges.

    (c) If G has no bridges, then G has no cut vertices.

13. Prove or disprove: If every vertex of a connected graph G lies on at least one cycle, then G is 2-connected.

14. Prove that every 2-connected graph contains at least one cycle.

15. Prove that for every graph G,

    (a) $\kappa(G) \leq \delta(G)$;

    (b) if $\delta(G) \geq n - 2$, then $\kappa(G) = \delta(G)$.

16. Let G be a graph of order n.

    (a) If $\delta(G) \geq \frac{n-1}{2}$, then prove that G is connected.

    (b) If $\delta(G) \geq \frac{n-2}{2}$, then show that G need not be connected.

1.1.3 Special Types of Graphs

until we meet again . . .

— from *An Irish Blessing*

In this section we describe several types of graphs. We will run into many of them later in the chapter.

1. Complete Graphs

   We introduced complete graphs in the previous section. A complete graph of order n is denoted by K_n, and there are several examples in Figure 1.11.

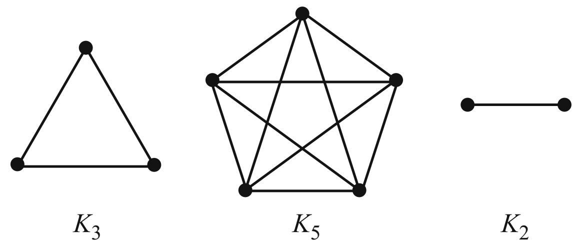


FIGURE 1.11. Examples of complete graphs.

2. Empty Graphs

   The *empty graph* on n vertices, denoted by E_n, is the graph of order n where E is the empty set (Figure 1.12).

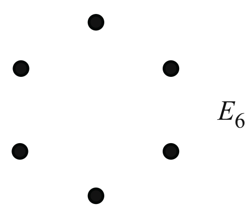


FIGURE 1.12. An empty graph.

3. Complements

   Given a graph G, the *complement* of G, denoted by $\overline{G}$, is the graph whose vertex set is the same as that of G, and whose edge set consists of all the edges that are *not* present in G (Figure 1.13).

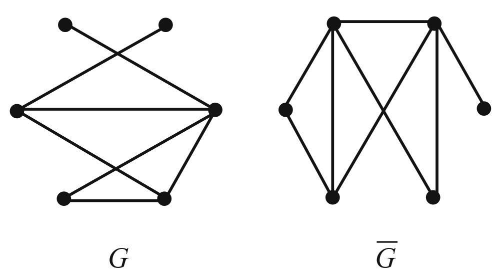


FIGURE 1.13. A graph and its complement.

4. Regular Graphs

   A graph G is *regular* if every vertex has the same degree. G is said to be *regular of degree* r (or r-*regular*) if $\deg(v) = r$ for all vertices v in G. Complete graphs of order n are regular of degree $n - 1$, and empty graphs are regular of degree 0. Two further examples are shown in Figure 1.14.

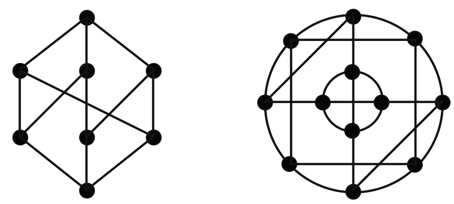


FIGURE 1.14. Examples of regular graphs.

5. Cycles

   The graph C_n is simply a cycle on n vertices (Figure 1.15).

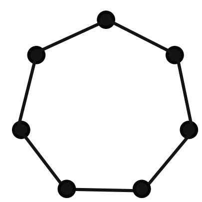


FIGURE 1.15. The graph C_7.

6. Paths

   The graph P_n is simply a path on n vertices (Figure 1.16).

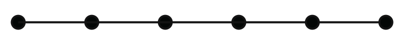


FIGURE 1.16. The graph P_6.

7. Subgraphs

   A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. In this case we write $H \subseteq G$, and we say that G contains H. In a graph where the vertices and edges are unlabeled, we say that $H \subseteq G$ if the vertices *could* be labeled in such a way that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. In Figure 1.17, H_1 and H_2 are both subgraphs of G, but H_3 is not.

8. Induced Subgraphs

   Given a graph G and a subset S of the vertex set, the *subgraph of G induced by S*, denoted $\langle S\rangle$, is the subgraph with vertex set S and with edge set $\{uv \mid u, v \in S \text{ and } uv \in E(G)\}$. So, $\langle S\rangle$ contains all vertices of S and all edges of G whose end vertices are *both* in S. A graph and two of its induced subgraphs are shown in Figure 1.18.

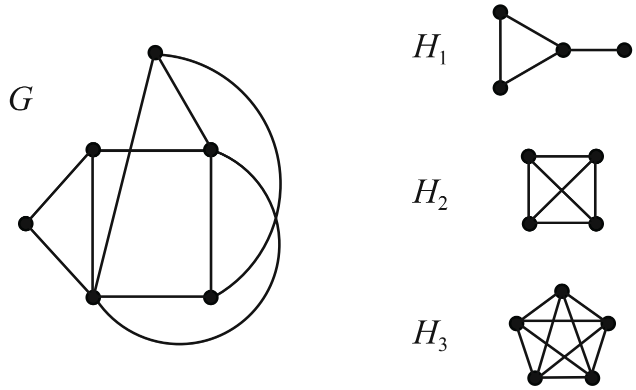


FIGURE 1.17. H_1 and H_2 are subgraphs of G.

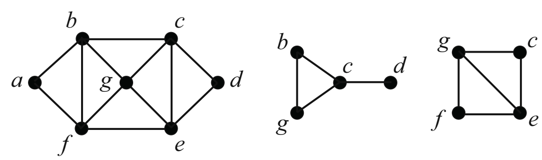


FIGURE 1.18. A graph and two of its induced subgraphs.

9. Bipartite Graphs

   A graph G is *bipartite* if its vertex set can be partitioned into two sets X and Y in such a way that every edge of G has one end vertex in X and the other in Y. In this case, X and Y are called the *partite sets*. The first two graphs in Figure 1.19 are bipartite. Since it is not possible to partition the vertices of the third graph into two such sets, the third graph is not bipartite.

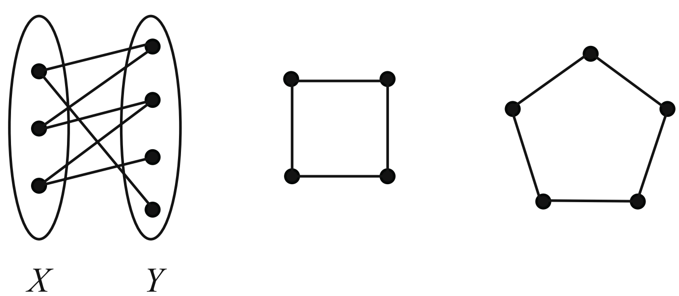
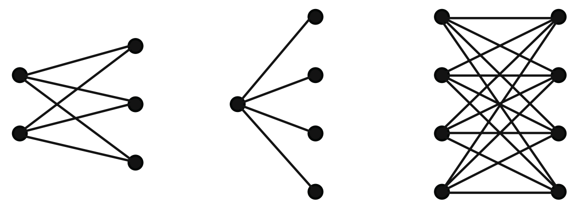


FIGURE 1.19. Two bipartite graphs and one non-bipartite graph.

A bipartite graph with partite sets X and Y is called a *complete bipartite graph* if its edge set is of the form $E = \{xy \mid x \in X, y \in Y\}$ (that is, if every possible connection of a vertex of X with a vertex of Y is present in the graph). Such a graph is denoted by $K_{|X|,|Y|}$. See Figure 1.20.



$K_{2,3}$ $K_{1,4}$ $K_{4,4}$

FIGURE 1.20. A few complete bipartite graphs.

The next theorem gives an interesting characterization of bipartite graphs.

Theorem 1.3. *A graph with at least two vertices is bipartite if and only if it contains no odd cycles.*

Proof. Let G be a bipartite graph with partite sets X and Y. Let C be a cycle of G and say that C is $v_1, v_2, \ldots, v_k, v_1$. Assume without loss of generality that $v_1 \in X$. The nature of bipartite graphs implies then that $v_i \in X$ for all odd i, and $v_i \in Y$ for all even i. Since v_k is adjacent to v_1, it must be that k is even; and hence C is an even cycle.

For the reverse direction of the theorem, let G be a graph of order at least two such that G contains no odd cycles. Without loss of generality, we can assume that G is connected, for if not, we could treat each of its connected components separately. Let v be a vertex of G, and define the set X to be

$$X = \{x \in V(G) \mid \text{the shortest path from } x \text{ to } v \text{ has even length}\},$$

and let $Y = V(G) \setminus X$.

Now let x and x' be vertices of X, and suppose that x and x' are adjacent. If $x = v$, then the shortest path from v to x' has length one. But this implies that $x' \in Y$, a contradiction. So, it must be that $x \neq v$, and by a similar argument, $x' \neq v$. Let P_1 be a path from v to x of shortest length (a shortest v–x path) and let P_2 be a shortest v–x' path. Say that P_1 is $v = v_0, v_1, \ldots, v_{2k} = x$ and that P_2 is $v = w_0, w_1, \ldots, w_{2t} = x'$. The paths P_1 and P_2 certainly have v in common. Let v' be a vertex on both paths such that the v'–x path, call it P_1', and the v'–x' path, call it P_2', have only the vertex v' in common. Essentially, v' is the "last" vertex common to P_1 and P_2. It must be that P_1' and P_2' are shortest v'–x and v'–x' paths, respectively, and it must be that $v' = v_i = w_i$ for some i. But since x and x' are adjacent, $v_i, v_{i+1}, \ldots, v_{2k}, w_{2t}, w_{2t-1}, \ldots, w_i$ is a cycle of length $(2k - i) + (2t - i) + 1$, which is odd, and that is a contradiction.

Thus, no two vertices in X are adjacent to each other, and a similar argument shows that no two vertices in Y are adjacent to each other. Therefore, G is bipartite with partite sets X and Y. □

We conclude this section with a discussion of what it means for two graphs to be the same. Look closely at the graphs in Figure 1.21 and convince yourself that one could be re-drawn to look just like the other. Even though these graphs

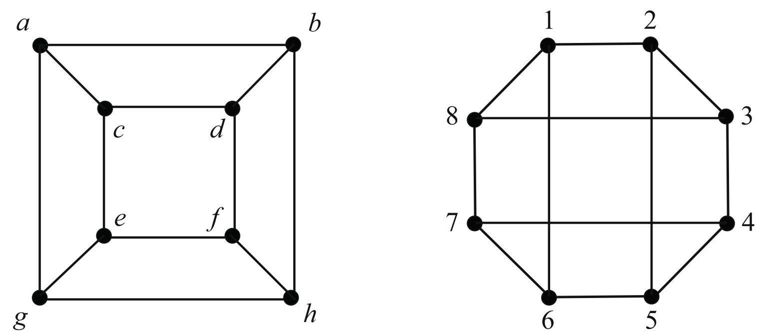


FIGURE 1.21. Are these graphs the same?

have different vertex sets and are drawn differently, it is still quite natural to think of these graphs as being the same. The idea of isomorphism formalizes this phenomenon.

Graphs G and H are said to be *isomorphic* to one another (or simply, isomorphic) if there exists a one-to-one correspondence $f : V(G) \to V(H)$ such that for each pair x,y of vertices of G, $xy \in E(G)$ if and only if $f(x)f(y) \in E(H)$. In other words, G and H are isomorphic if there exists a mapping from one vertex set to another that preserves adjacencies. The mapping itself is called an *isomorphism*. In our example, such an isomorphism could be described as follows:

$$\{(a,1),(b,2),(c,8),(d,3),(e,7),(f,4),(g,6),(h,5)\}.$$

When two graphs G and H are isomorphic, it is not uncommon to simply say that $G = H$ or that "G is H." As you will see, we will make use of this convention quite often in the sections that follow.

Several facts about isomorphic graphs are immediate. First, if G and H are isomorphic, then $|V(G)| = |V(H)|$ and $|E(G)| = |E(H)|$. The converse of this statement is not true, though, and you can see that in the graphs of Figure 1.22. The vertex and edge counts are the same, but the two graphs are clearly not iso-

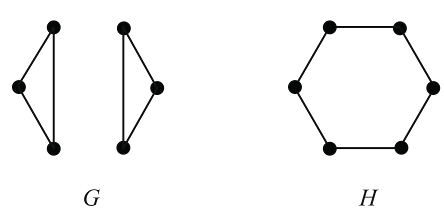


FIGURE 1.22.

morphic.

A second necessary fact is that if G and H are isomorphic then the degree sequences must be identical. Again, the graphs in Figure 1.22 show that the converse of this statement is not true. A third fact, and one that you will prove in Exercise 8, is that if graphs G and H are isomorphic, then their complements $\overline{G}$ and $\overline{H}$ must also be isomorphic.

In general, determining whether two graphs are isomorphic is a difficult problem. While the question is simple for small graphs and for pairs where the vertex counts, edge counts, or degree sequences differ, the general problem is often tricky to solve. A common strategy, and one you might find helpful in Exercises 9 and 10, is to compare subgraphs, complements, or the degrees of adjacent pairs of vertices.

Exercises

1. For $n \geq 1$, prove that K_n has $n(n-1)/2$ edges.

2. If K_{r_1,r_2} is regular, prove that $r_1 = r_2$.

3. Determine whether K_4 is a subgraph of $K_{4,4}$. If yes, then exhibit it. If no, then explain why not.

4. Determine whether P_4 is an induced subgraph of $K_{4,4}$. If yes, then exhibit it. If no, then explain why not.

5. List all of the unlabeled connected subgraphs of C_{34}.

6. The concept of complete bipartite graphs can be generalized to define the *complete multipartite graph* $K_{r_1,r_2,\ldots,r_k}$. This graph consists of k sets of vertices $A_1, A_2, \ldots, A_k$, with $|A_i| = r_i$ for each i, where all possible "interset edges" are present and no "intraset edges" are present. Find expressions for the order and size of $K_{r_1,r_2,\ldots,r_k}$.

7. The *line graph* $L(G)$ of a graph G is defined in the following way: the vertices of $L(G)$ are the edges of G, $V(L(G)) = E(G)$, and two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G share a vertex.

   (a) Let G be the graph shown in Figure 1.23. Find $L(G)$.

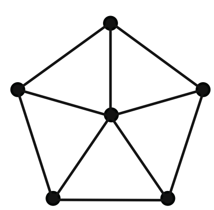


FIGURE 1.23.