

# Progress on Mazur's Program B

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Slides available at <http://www.mathcs.emory.edu/~dzb/slides/>

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$$G_{\mathbb{Q}} := \text{Aut}(\overline{\mathbb{Q}}/\mathbb{Q})$$
$$E[n](\overline{\mathbb{Q}}) \cong (\mathbb{Z}/n\mathbb{Z})^2$$

$$\rho_{E,n}: G_{\mathbb{Q}} \rightarrow \text{Aut } E[n] \cong \text{GL}_2(\mathbb{Z}/n\mathbb{Z})$$

$$\rho_{E,\ell^\infty}: G_{\mathbb{Q}} \rightarrow \text{GL}_2(\mathbb{Z}_\ell) = \varprojlim_n \text{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z})$$

$$\rho_E: G_{\mathbb{Q}} \rightarrow \text{GL}_2(\widehat{\mathbb{Z}}) = \varprojlim_n \text{GL}_2(\mathbb{Z}/n\mathbb{Z})$$

# Background - Galois Representations

$$\rho_{E,n}: G_{\mathbb{Q}} \twoheadrightarrow H(n) \hookrightarrow \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$$

$$\left\{ \begin{array}{c} \overline{\mathbb{Q}} \\ \downarrow \\ \overline{\mathbb{Q}}^{\ker \rho_{E,n}} = \mathbb{Q}(E[n]) \\ \downarrow \\ \mathbb{Q} \end{array} \right\} H(n)$$

Problem (Mazur's "program B")

*Classify all possibilities for  $H(n)$ .*

## Example - torsion on an elliptic curve

If  $E$  has a  $K$ -rational **torsion point**  $P \in E(K)[n]$  (of exact order  $n$ ) then:

$$H(n) \subset \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$$

since for  $\sigma \in G_K$  and  $Q \in E(\overline{K})[n]$  such that  $E(\overline{K})[n] \cong \langle P, Q \rangle$ ,

$$\sigma(P) = P$$

$$\sigma(Q) = a_\sigma P + b_\sigma Q$$

## Example - Isogenies

If  $E$  has a  $K$ -rational, **cyclic isogeny**  $\phi: E \rightarrow E'$  with  $\ker \phi = \langle P \rangle$  then:

$$H(n) \subset \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

since for  $\sigma \in G_K$  and  $Q \in E(\overline{K})[n]$  such that  $E(\overline{K})[n] \cong \langle P, Q \rangle$ ,

$$\sigma(P) = a_\sigma P$$

$$\sigma(Q) = b_\sigma P + c_\sigma Q$$

## Example - other maximal subgroups

### Normalizer of a split Cartan:

$$N_{\text{sp}} = \left\langle \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle$$

$H(n) \subset N_{\text{sp}}$  and  $H(n) \not\subset C_{\text{sp}}$  iff

- there exists an unordered pair  $\{\phi_1, \phi_2\}$  of cyclic isogenies,
- neither of which is defined over  $K$
- but which are both defined over some quadratic extension of  $K$
- and which are Galois conjugate.

# Background - Galois Representations

$$\rho_{E,n}: G_{\mathbb{Q}} \twoheadrightarrow H(n) \hookrightarrow \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$$

$$\left\{ \begin{array}{c} \overline{\mathbb{Q}} \\ \downarrow \\ \overline{\mathbb{Q}}^{\ker \rho_{E,n}} = \mathbb{Q}(E[n]) \\ \downarrow \\ \mathbb{Q} \end{array} \right\} H(n)$$

Problem (Mazur's "program B")

*Classify all possibilities for  $H(n)$ .*

# Modular curves

## Definition

- $X(N)(K) := \{(E/K, P, Q) : E[N] = \langle P, Q \rangle\} \cup \{\text{cusps}\}$
- $X(N)(K) \ni (E/K, P, Q) \Leftrightarrow \rho_{E,N}(G_K) = \{I\}$

## Definition

$\Gamma(N) \subset H \subset \text{GL}_2(\widehat{\mathbb{Z}})$  (finite index)

- $X_H := X(N)/H$
- $X_H(K) \ni (E/K, \iota) \Leftrightarrow H(N) \subset H \pmod{N}$

## Stacky disclaimer

This is only true up to twist; there are some subtleties if

- 1  $j(E) \in \{0, 12^3\}$  (plus some minor group theoretic conditions), or
- 2 if  $-I \in H$ .



# Rational Points on modular curves

## Mazur's program B

Compute  $X_H(\mathbb{Q})$  for all  $H$ .

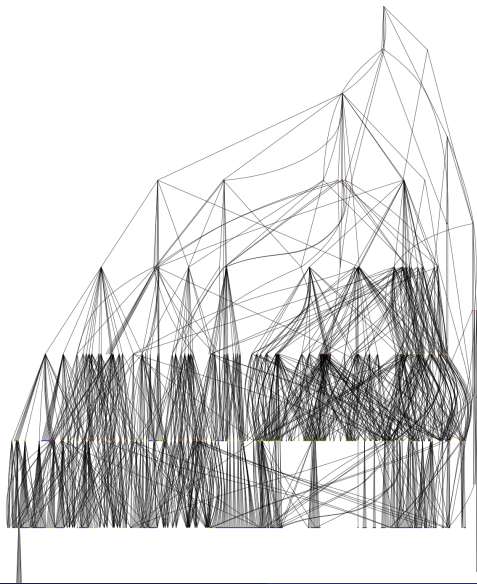
## Remark

- Sometimes  $X_H \cong \mathbb{P}^1$  or elliptic with rank  $X_H(\mathbb{Q}) > 0$ .
- Some  $X_H$  have *sporadic* points.
- Can compute  $g(X_H)$  group theoretically (via Riemann–Hurwitz).

## Fact

$g(X_H), \gamma(X_H) \rightarrow \infty$  as  $[\mathrm{GL}_2(\widehat{\mathbb{Z}}) : H] \rightarrow \infty$ .

# Gratuitous picture – subgroups of $GL_2(\mathbb{Z}_2)$



# Sample subgroup (Serre)

$$\begin{array}{ccccc}
 \ker \phi_2 & \subset & H(8) & \subset & \mathrm{GL}_2(\mathbb{Z}/8\mathbb{Z}) & \dim_{\mathbb{F}_2} \ker \phi_2 = 3 \\
 & & \downarrow \phi_2 & & \downarrow & \\
 I + 2M_2(\mathbb{Z}/2\mathbb{Z}) & \subset & H(4) & = & \mathrm{GL}_2(\mathbb{Z}/4\mathbb{Z}) & \dim_{\mathbb{F}_2} \ker \phi_1 = 4 \\
 & & \downarrow \phi_1 & & \downarrow & \\
 & & H(2) & = & \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z}) & 
 \end{array}$$

$$\chi: \mathrm{GL}_2(\mathbb{Z}/8\mathbb{Z}) \rightarrow \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/8\mathbb{Z})^* \rightarrow \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/8\mathbb{Z})^* \cong \mathbb{F}_2^3.$$

$$\chi = \mathrm{sgn} \times \det$$

$$H(8) := \chi^{-1}(G), \quad G \subset \mathbb{F}_2^3.$$

# Sample subgroup (Dokchitser<sup>2</sup>)

$$\begin{array}{ccccc} \langle I + 2E_{1,1}, I + 2E_{2,2} \rangle & \subset & H(4) & \subset & \mathrm{GL}_2(\mathbb{Z}/4\mathbb{Z}) & \dim_{\mathbb{F}_2} \ker \phi_1 = 2 \\ & & \downarrow & & \downarrow & \\ & & H(2) & = & \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z}) & \end{array}$$

$$H(2) = \left\langle \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\rangle \cong \mathbb{F}_3 \rtimes D_8.$$

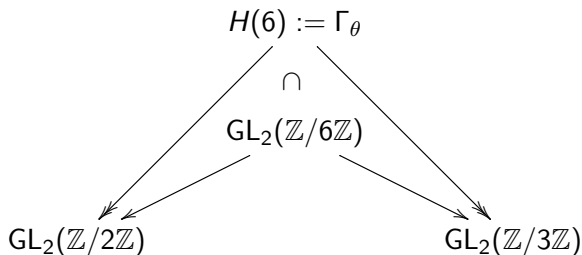
$$\begin{aligned} \mathrm{im} \rho_{E,4} \subset H(4) &\Leftrightarrow j(E) = -4t^3(t+8). \\ X_H &\cong \mathbb{P}^1 \xrightarrow{j} X(1). \end{aligned}$$

# A typical subgroup

|               |           |                     |           |                                 |                                       |
|---------------|-----------|---------------------|-----------|---------------------------------|---------------------------------------|
| $\ker \phi_4$ | $\subset$ | $H(32)$             | $\subset$ | $GL_2(\mathbb{Z}/32\mathbb{Z})$ | $\dim_{\mathbb{F}_2} \ker \phi_4 = 4$ |
|               |           | $\downarrow \phi_4$ |           | $\downarrow$                    |                                       |
| $\ker \phi_3$ | $\subset$ | $H(16)$             | $\subset$ | $GL_2(\mathbb{Z}/16\mathbb{Z})$ | $\dim_{\mathbb{F}_2} \ker \phi_3 = 3$ |
|               |           | $\downarrow \phi_3$ |           | $\downarrow$                    |                                       |
| $\ker \phi_2$ | $\subset$ | $H(8)$              | $\subset$ | $GL_2(\mathbb{Z}/8\mathbb{Z})$  | $\dim_{\mathbb{F}_2} \ker \phi_2 = 2$ |
|               |           | $\downarrow \phi_2$ |           | $\downarrow$                    |                                       |
| $\ker \phi_1$ | $\subset$ | $H(4)$              | $\subset$ | $GL_2(\mathbb{Z}/4\mathbb{Z})$  | $\dim_{\mathbb{F}_2} \ker \phi_1 = 3$ |
|               |           | $\downarrow \phi_1$ |           | $\downarrow$                    |                                       |
|               |           | $H(2)$              | $=$       | $GL_2(\mathbb{Z}/2\mathbb{Z})$  |                                       |

# Non-abelian entanglements

There exists a surjection  $\theta: \mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z}) \rightarrow \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$ .



$$\mathrm{im} \rho_{E,6} \subset H(6) \Leftrightarrow K(E[2]) \subset K(E[3])$$

# Classification of Images - Mazur's Theorem

## Theorem

*Let  $E$  be an elliptic curve over  $\mathbb{Q}$ . Then for  $\ell > 11$ ,  $E(\mathbb{Q})[\ell] = \{0\}$ .*

In other words, for  $\ell > 11$  the mod  $\ell$  image is not contained in a subgroup conjugate to

$$\begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}.$$

# Classification of Images - Mazur; Bilu, Parent

## Theorem (Mazur)

*Let  $E$  be an elliptic curve over  $\mathbb{Q}$  without CM. Then for  $\ell > 37$  the mod  $\ell$  image is not contained in a subgroup conjugate to*

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$$

## Theorem (Bilu, Parent)

*Let  $E$  be an elliptic curve over  $\mathbb{Q}$  without CM. Then for  $\ell > 13$  the mod  $\ell$  image is not contained in a subgroup conjugate to*

$$\left\langle \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle.$$



# Main conjecture

## Conjecture

Let  $E$  be an elliptic curve over  $\mathbb{Q}$  without CM. Then for  $\ell > 37$ ,  $\rho_{E,\ell}$  is surjective.

# Serre's Open Image Theorem

## Theorem (Serre, 1972)

*Let  $E$  be an elliptic curve over  $K$  without CM. The image of  $\rho_E$*

$$\rho_E(G_K) \subset \mathrm{GL}_2(\hat{\mathbb{Z}})$$

*is open.*

Note:

$$\mathrm{GL}_2(\hat{\mathbb{Z}}) \cong \prod_p \mathrm{GL}_2(\mathbb{Z}_p)$$

# “Vertical” image conjecture

## Conjecture

There exists a constant  $N$  such that for every  $E/\mathbb{Q}$  without CM

$$\left[ \rho_E(G_{\mathbb{Q}}) : \mathrm{GL}_2(\hat{\mathbb{Z}}) \right] \leq N.$$

## Remark

This follows from the “ $\ell > 37$ ” conjecture.

## Problem

*Assume the “ $\ell > 37$ ” conjecture and compute  $N$ .*

# Main Theorems

## Rouse, ZB (2-adic)

The index of  $\rho_{E,2^\infty}(G_{\mathbb{Q}})$  divides 64 or 96; all such indices occur.

## Zywina (mod $\ell$ )

Classifies  $\rho_{E,\ell}(G_{\mathbb{Q}})$  (modulo some conjectures).

## Zywina (all possible indices)

The **index** of  $\rho_{E,N}(G_{\mathbb{Q}})$  divides 220, 336, 360, 504, 864, 1152, 1200, 1296 or 1536.

## Morrow (composite level)

Classifies  $\rho_{E,2 \cdot \ell}(G_{\mathbb{Q}})$ .

## Camacho–Li–Morrow–Petok–ZB (composite level)

Classifies  $\rho_{E,\ell_1^n \cdot \ell_2^m}(G_{\mathbb{Q}})$  (partially).

# Main Theorems continued

## Zywina–Sutherland

Parametrizations in all **prime power** level,  $g = 0$  and  $g = 1, r > 0$  cases.

## Gonzalez–Jimenez, Lozano–Robledo

Classify  $E/\mathbb{Q}$  with  $\rho_{E,n}(G_{\mathbb{Q}})$  abelian.

## Brau–Jones, Jones–McMurdy (in progress)

Equations for  $X_H$  for entanglement groups  $H$ .

## Rouse–ZB for other primes (in progress)

Partial progress; e.g. for  $N = 3^n$ .

## Derickx–Etropolski–Morrow–van Hoejk–ZB (in progress)

Classify possibilities for cubic torsion.

# Some applications and complements

## Theorem (R. Jones, Rouse, ZB)

- 1 **Arithmetic dynamics:** let  $P \in E(\mathbb{Q})$ .
- 2 How often is the order of  $\tilde{P} \in E(\mathbb{F}_p)$  odd?
- 3 Answer depends on  $\rho_{E,2^\infty}(G_{\mathbb{Q}})$ .
- 4 Examples: 11/21 (generic), 121/168 (maximal), 1/28 (minimal)

## Theorem (Various authors)

Computation of  $S_{\mathbb{Q}}(d)$  and  $S(d)$  for particular  $d$ .

## Theorem (Daniels, Lozano-Robledo, Najman, Sutherland)

Classification of  $E(\mathbb{Q}(3^\infty))_{tors}$

# More applications

## Theorem (Sporadic points)

*Najman's example  $X_1(21)^{(3)}(\mathbb{Q})$ ; "easy production" of other examples.*

## Theorem (Jack Thorne)

*Elliptic curves over  $\mathbb{Q}_\infty$  are modular.*

*(One step is to show  $X_0(15)(\mathbb{Q}_\infty) = X_0(15)(\mathbb{Q}) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ .)*

## Theorem (Zywina)

*Constants in the Lang–Trotter conjecture.*

## Index, # of isogeny classes

1 , 727995

2 , 7281

3 , 175042

4 , 1769

6 , 57500

8 , 577

12 , 29900

16 , 235

24 , 5482

32 , 20

48 , 1544

64 , 0 (two examples)

96 , 241 (first example -  $X_0(15)$ )

CM , 1613



**Index, # of isogeny classes**

64 , 0

$$j = -3 \cdot 2^{18} \cdot 5 \cdot 13^3 \cdot 41^3 \cdot 107^3 \cdot 17^{-16}$$

$$j = -2^{21} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13^3 \cdot 23^3 \cdot 41^3 \cdot 179^3 \cdot 409^3 \cdot 79^{-16}$$

Rational points on  $X_{ns}^+(16)$  (Heegner, Baran)

# Fun 2-adic facts

- ① All indices dividing 96 occur infinitely often; 64 occurs only twice.
- ② The 2-adic image is determined by the mod 32 image
- ③ 1208 different images can occur for non-CM elliptic curves
- ④ There are 8 “sporadic” subgroups.

# More fun 2-adic facts

If  $E/\mathbb{Q}$  is a non-CM elliptic curve whose mod 2 image has index

- 1, the 2-adic image can have index as large as 64.
- 2, the 2-adic image has index 2 or 4.
- 3, the 2-adic image can have index as large as 96.
- 6, the 2-adic image can have index as large as 96;
- (although some quadratic twist of  $E$  must have 2-adic image with index less than 96).

## Definition

- $H \subset H' \Leftrightarrow X_H \rightarrow X_{H'}$
- Say that  $H$  is **minimal** if
  - ①  $g(X_H) > 1$  and
  - ②  $H \subset H' \Leftrightarrow g(X_{H'}) \leq 1$
- Every modular curve maps to a minimal or genus  $\leq 1$  curve.

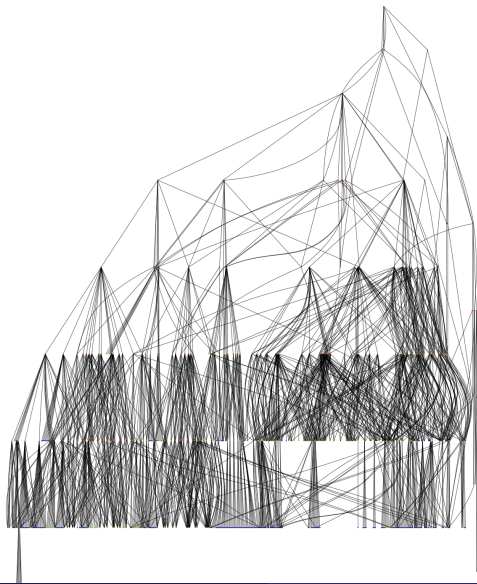
## Definition

We say that  $H$  is **arithmetically minimal** if

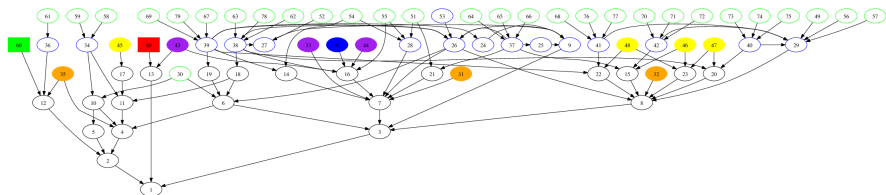
- ①  $\det(H) = \widehat{\mathbb{Z}}^*$ , and
- ② a few other conditions.

- 1 Compute all arithmetically minimal  $H \subset \mathrm{GL}_2(\mathbb{Z}_2)$
- 2 Compute equations for each  $X_H$
- 3 Find (with proof) all rational points on each  $X_H$ .

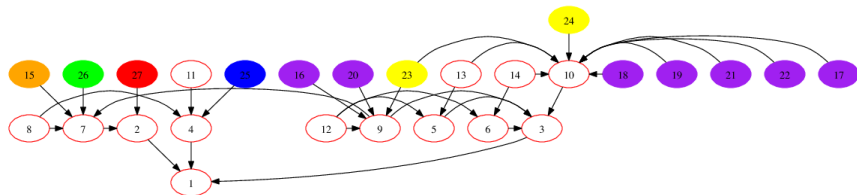
# Gratuitous picture – subgroups of $GL_2(\mathbb{Z}_2)$



# Gratuitous picture – subgroups of $GL_2(\mathbb{Z}_3)$

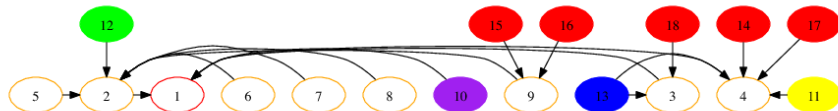


## Gratuitous picture – subgroups of $GL_2(\mathbb{Z}_5)$





# Gratuitous picture – subgroups of $GL_2(\mathbb{Z}_{11})$



318 curves  $X_H$  with  $-I \in H$  (excluding pointless conics)

|        |     |    |    |    |    |   |
|--------|-----|----|----|----|----|---|
| Genus  | 0   | 1  | 2  | 3  | 5  | 7 |
| Number | 175 | 52 | 57 | 18 | 20 | 4 |

# Finding Equations – Basic idea

- 1 The canonical map  $C \hookrightarrow \mathbb{P}^{g-1}$  is given by  $P \mapsto [\omega_1(P) : \cdots : \omega_g(P)]$ .
- 2 For a general curve, this is an embedding, and the relations are quadratic.
- 3 For a modular curve,

$$M_k(H) \cong H^0(X_H, \Omega^1(\Delta)^{\otimes k/2})$$

given by

$$f(z) \mapsto f(z) dz^{\otimes k/2}.$$

## Equations – Example: $X_1(17) \subset \mathbb{P}^4$

$$q - 11q^5 + 10q^7 + O(q^8)$$

$$q^2 - 7q^5 + 6q^7 + O(q^8)$$

$$q^3 - 4q^5 + 2q^7 + O(q^8)$$

$$q^4 - 2q^5 + O(q^8)$$

$$q^6 - 3q^7 + O(q^8)$$

$$xu + 2xv - yz + yu - 3yv + z^2 - 4zu + 2u^2 + v^2 = 0$$

$$xu + xv - yz + yu - 2yv + z^2 - 3zu + 2uv = 0$$

$$2xz - 3xu + xv - 2y^2 + 3yz + 7yu - 4yv - 5z^2 - 3zu + 4zv = 0$$

# Equations – general

- ①  $H' \subset H$  of index 2,  $X_{H'} \rightarrow X_H$  degree 2.
- ② Given equations for  $X_H$ , compute equations for  $X_{H'}$ .
- ③ Compute a new modular form on  $H'$ , compute (quadratic) relations between this and modular forms on  $H$ .
- ④ **Main technique** – if  $X_{H'}$  has “new cusps”, then write down Eisenstein series which vanish at “one new cusp, not others”.

# Rational points rundown, $\ell = 2$

318 curves (excluding pointless conics)

| Genus            | 0   | 1  | 2  | 3  | 5  | 7  |
|------------------|-----|----|----|----|----|----|
| Number           | 175 | 52 | 56 | 18 | 20 | 4  |
| Rank of Jacobian |     |    |    |    |    |    |
| 0                |     | 25 | 46 | –  | –  | ?? |
| 1                |     | 27 | 3  | 9  | 10 | ?? |
| 2                |     |    | 7  | –  | –  | ?? |
| 3                |     |    |    | 9  | –  | ?? |
| 4                |     |    |    |    | –  | ?? |
| 5                |     |    |    |    | 10 | ?? |

# More 2-adic facts

- ① There are 8 “sporadic” subgroups
  - ① Only one genus 2 curve has a sporadic point
  - ② Six genus 3 curves each have a single sporadic point
  - ③ The genus 1, 5, and 7 curves have no sporadic points
- ② Many accidental isomorphisms of  $X_H \cong X_{H'}$ .
- ③ There is one  $H$  such that  $g(X_H) = 1$  and  $X_H \in X_H(\mathbb{Q})$ .

# Rational Points rundown: $\ell = 3$

3  $g = 0$     Handled by Sutherland-Zywina

$g = 1$     all rank zero

$g = 4$     map to  $g = 1$

$g = 2$     Chabauty works

$g = 4$     no 3-adic points

---

$g = 3$     Picard curves; map to rank 0 AV

$g = 4$     Admits étale triple cover

$g = 6$     Admits étale triple cover

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$g = 12$     gonality  $\leq 9$ , plane model, degree 121

$g = 43$     New ideas needed



$$X_H: -x^3y + x^2y^2 - xy^3 + 3xz^3 + 3yz^3 = 0$$

# Rational Points rundown: $\ell = 5$

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|   |                                  |                                |
|---|----------------------------------|--------------------------------|
| 5 | $g = 0$ (10 level 5, 3 level 25) | All level 5 curves are genus 0 |
|   | $g = 4$ (4 level 25)             | No 5-adic points               |

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|  |   |  |
|--|---|--|
|  | $g = 2$ (2 level 25)                    | Rank 2, $A_5$ mod 2 image  |
|  | $g = 4$ (3 level 25)                    | All isomorphic.<br>Each has 5 rational points<br>Each admits an order 5 aut<br>Simple Jacobian |
|  | $g = 8, 14, 22, 36$ (levels 25 and 125) | No models (or ideas, yet)  |

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## Rational Points rundown: $\ell = 7$

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|   |                                 |  |
|---|---------------------------------|--|
| 7 | $g = 1, 3$                      | $[Z, 4.4]$ handles these, $X_H(\mathbb{Q})$ is finite. |
|   | $g = 19, 26$ , level 49         | Maps to one of the 6 above                             |
|   | $g = 1$ , level 49              | $[SZ]$ handles this one (rank 0)                       |
|   | $g = 3, 19, 26$ , level 49, 343 | Map to curve on previous line                          |
|   | $g = 12$ , level 49             | Handled by   |
|   |                                 | Greenberg–Rubin–Silverberg–Stoll                       |

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|  |                     |                           |
|--|---------------------|---------------------------|
|  | $g = 9, 12, 69, 94$ | No models (or ideas, yet) |
|--|---------------------|---------------------------|

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# Rational Points rundown: $\ell = 11$

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11 all maximal are genus one

only positive rank is  $X_{ns}(11)$

All but one are ruled out by Zywina    some have sporadic points;  
[Z, Theorem 1.6]

$g = 5$ , level 11    [Z, Lemma 4.5]

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$g = 5776$ , level 121    “Challenge. . .”

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# Rational Points rundown: $\ell = 13$

Zywina handles all level 13 except for the cursed curve

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13  $g = 2, 3$ , level 13 (8 total)

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$g = 8$ , level 169

$X_0(13^2)$ , handled by Kenku

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$X_{ns}(13)$

Cursed. Genus 3, rank 3.

No torsion. Some points

Probably has maximal mod 2 image

Solved by Balakrishnan, Müller

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# Explicit methods: highlight reel

- Local methods
- Chabauty
- Elliptic Chabauty
- Mordell–Weil sieve
- étale descent
- Pryms
- **Equationless descent via group theory.**
- **New techniques for computing** Aut  $C$ .

# Pryms

$$\begin{array}{ccc}
 D & \xrightarrow{\iota - \text{id} - (\iota(P) - P)} & \ker_0(J_D \rightarrow J_C) =: \text{Prym}(D \rightarrow C) \\
 \text{et} \downarrow \circlearrowleft \iota & & \\
 C & & 
 \end{array}$$

Example (Genus  $C = 3 \Rightarrow$  Genus  $D = 5$ )

- $C: Q(x, y, z) = 0$
- $Q = Q_1 Q_3 - Q_2^2$ .

$$D_\delta: Q_1(x, y, z) = \delta u^2$$

$$Q_2(x, y, z) = \delta uv$$

$$Q_3(x, y, z) = \delta v^2$$

- $\text{Prym}(D_\delta \rightarrow C) \cong \text{Jac}_{H_\delta}$ ,
- $H_\delta: y^2 = -\delta \det(M_1 + 2xM_2 + x^2M_3)$ .

Thank you!