Greatest hits in Ramsey theory

Alan Li[1]
advised by Prof. David Zureick-Brown[2]

[1]Amherst College
[2]Amherst College

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Introduction

What is Ramsey theory?

- "Finding ordered substructures in large structures"

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What is Ramsey theory?

- "Finding ordered substructures in large structures"
- Given substructure Y, how large must structure X be until it is forced to contain Y?

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Primer on Graph Theory

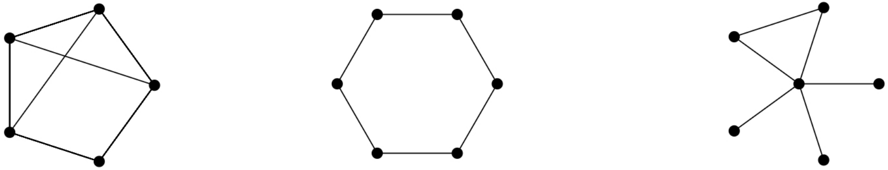

Figure: Examples of graphs.

Complete graph K_n: a graph with n vertices and an edge between every pair of vertices. As a subgraph of a graph, also known as a *clique*.

Students at a Party

There are n students at a party, where any two students are either friends or strangers. Must there be three students who are all either mutually friends or mutually strangers?

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Equivalently, given a 2-colored K_n, must there be a monochromatic K_3?

Students at a Party

$n = 5$:

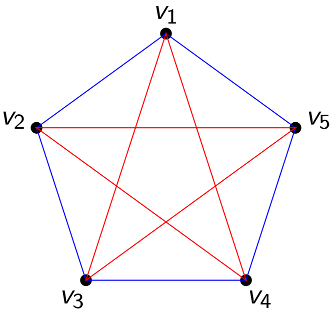

Figure: Example of 5 people, with no 3 people friends or strangers

Students at a Party

$n = 6$:

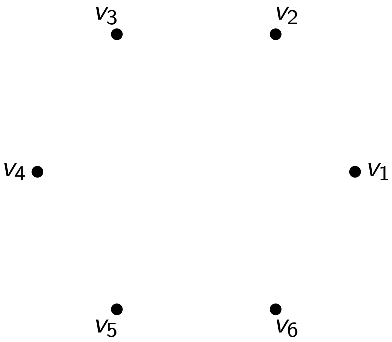

Figure: Proof that with 6 people, there exists 3 people either friends or strangers

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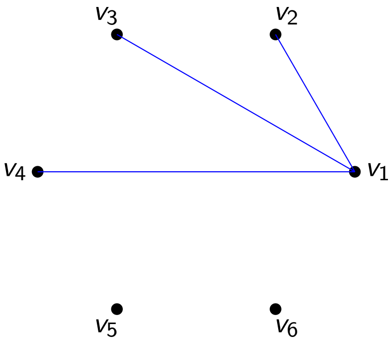

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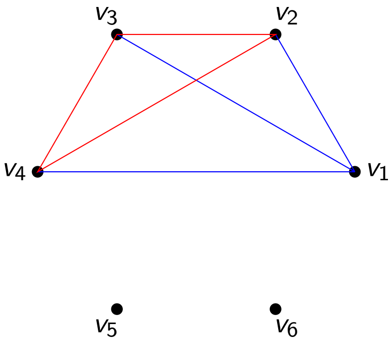

Figure: Proof that with 6 people, there exists 3 people either friends or strangers

Students at a Party

Generalization: What if we want to find ℓ mutual friends or strangers, or equivalently a monochromatic K_ℓ? What if we have r categories (instead of friends/strangers), or equivalently an r-colored K_n?

Ramsey's Theorem

Ramsey's Theorem

Let r be any positive integer. For any $\ell_1, ..., \ell_r \in \mathbb{Z}^+$, there exists an $n \in \mathbb{Z}^+$ such that any r-coloring (with colors $c_1, ..., c_r$) of K_n contains an ℓ_i-clique of color c_i for some $1 \leq i \leq n$. Denote the smallest such n to be $R(\ell_1, ..., \ell_r)$, known as *Ramsey numbers*.

Ramsey's Theorem

Proof of Ramsey's Theorem (when $r = 2$):

- Base case: $R(1, \ell)$ and $R(\ell, 1)$.

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- Consider 2-coloring on complete graph with $R(\ell_1 - 1, \ell_2) + R(\ell_1, \ell_2 - 1)$ vertices.

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- Consider 2-coloring on complete graph with $R(\ell_1 - 1, \ell_2) + R(\ell_1, \ell_2 - 1)$ vertices.
- Choose $x \in G$; either x has $\geq R(\ell_1 - 1, \ell_2)$ red neighbours or $\geq R(\ell_1, \ell_2 - 1)$ blue neighbours.

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- Choose $x \in G$; either x has $\geq R(\ell_1 - 1, \ell_2)$ red neighbours or $\geq R(\ell_1, \ell_2 - 1)$ blue neighbours.

Note that this also shows that $R(\ell_1, \ell_2) \leq R(\ell_1 - 1, \ell_2) + R(\ell_1, \ell_2 - 1)$.

Bounds on Ramsey numbers

Corollary

For all $\ell_1, \ell_2 \in \mathbb{Z}^+$, we have:

$$R(\ell_1, \ell_2) \leq \binom{\ell_1 + \ell_2}{\ell_1}.$$

Bounds on Ramsey numbers

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Proof:

- Base case: $R(1, \ell)$ and $R(\ell, 1)$.
- Inductive step:

$$\begin{aligned} R(\ell_1, \ell_2) &\leq R(\ell_1 - 1, \ell_2) + R(\ell_1, \ell_2 - 1) \\ &\leq \binom{\ell_1 + \ell_2 - 1}{\ell_1 - 1} + \binom{\ell_1 + \ell_2 - 1}{\ell_2 - 1} \\ &= \binom{\ell_1 + \ell_2}{\ell_1}. \end{aligned} \tag{1}$$

Bounds on Ramsey numbers

There is more interest in the case where $\ell_1 = \ell_2$, known as the *diagonal Ramsey numbers*.
In this case, we have $R(\ell, \ell) \leq \binom{2\ell}{\ell} \leq 4^\ell$.
Can we do better?

Bounds on Ramsey numbers

Marcelo Campos, Simon Griffiths, Robert Morris, and Julian Sahasrabudhe [2023]

For all sufficiently large $\ell \in \mathbb{Z}^+$, we have:

$$R(\ell, \ell) \leq (4 - \epsilon)^\ell$$

for some $\epsilon > 2^{-7}$.

In the following slides we will give a high level overview of the proof.

Erdos–Szekeres Algorithm

Given $G = (V, E)$ and $\ell_1, \ell_2 \in \mathbb{Z}^+$, the algorithm is as follows:

- Initiate $a = \ell_1$, $b = \ell_2$, $X = V$, and $A, B = \emptyset$.
- IF $a = 0$: RETURN A.
- IF $b = 0$: RETURN B.
- ELSE: Pick any $v \in X$, and let $N_R(v)$ be its red neighbours.
  - IF $|N_R(v) \cap X| \geq \frac{a}{a+b}|X|$: add v to A, replace X with $N_R(x) \cap X$, and decrement ℓ_1.
  - ELSE IF $|N_R(v) \cap X| < \frac{a}{a+b}|X|$: add v to B, replace X with $N_B(x) \cap X$, and decrement ℓ_2.
- Repeat.

Bounds on Ramsey numbers

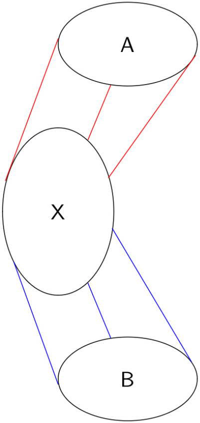

Figure: Schematic diagram for the Erdos–Szekeres Algorithm: at any point the set A forms a red clique, the set B forms a blue clique, and the set X is only connected to A with red edges and B with blue edges.

Bounds on Ramsey numbers

This algorithm is inefficient: this paper lowers the upper bound by improving the algorithm.

Definition

Given a 2-coloring of a graph $G = (V, E)$, a **book** is a disjoint pair (S, T) where $S, T \in V$, and where S forms a monochromatic clique (WLOG of the color red), and all edges between S and T are red.

The key observation is that if T contains a red clique of size $\ell - |S|$, then we have a red clique of size ℓ by "adding" it to S.
So we want a modified algorithm (the **Book Algorithm**) that keeps track of a red book.

Bounds on Ramsey numbers

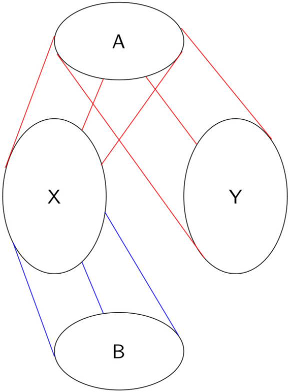

Figure: Schematic diagram for our modified algorithm: at any point the set A forms a red clique, the set B forms a blue clique, and the set X is only connected to A with red edges and B with blue edges, and the set Y is only connected to A with red edges.

Bounds on Ramsey numbers

Book algorithm

Given a graph $G = (V, E)$, a 2-coloring on G, $\ell_1, \ell_2 \in \mathbb{Z}^+$, and $\mu \in (\frac{1}{2}, 1)$, the **book algorithm** is as follows:

0) Initialize sets X, Y as equipartition of V, and initialize $A, B = \emptyset$.

1) **Degree Regularization**
2) **Big blue step**
3) **Red step**
4) **Density boost**

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1) **Degree Regularization**
While X is nonempty, replace X with
$\{x \in X : |N_R(x) \cap Y| \geq (p - \epsilon^{-\frac{1}{2}}\alpha_h)|Y|\}$, where ϵ and α_h are carefully chosen values, and p is the current red density between X and Y.
2) **Big blue step**
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Bounds on Ramsey numbers

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1) **Degree Regularization**

2) **Big blue step**

If "many" vertices of X have "high" blue degree in X, then there exists a "large" blue book (S, T) in X. Replace B with $B \cup S$, and replace X with T. Then, go back to 1). Otherwise, skip to 3).

3) **Red step**

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Given a graph $G = (V, E)$, a 2-coloring on G, $\ell_1, \ell_2 \in \mathbb{Z}^+$, and $\mu \in (\frac{1}{2}, 1)$, the **book algorithm** is as follows:

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1) **Degree Regularization**
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3) **Red step**
Choose a "specific" $x \in X$ such that $|N_B(x) \cap X| \leq \mu|X|$. If the red density between $X \cap N_R(x)$, $Y \cap N_R(x)$ is "high enough", then put x into A, and replace X, Y with $X \cap N_R(x)$, $Y \cap N_R(x)$ respectively. Go back to 1). Else: skip to 5).
4) **Density boost**

Bounds on Ramsey numbers

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put x into B, and replace X, Y with $X \cap N_B(x), Y \cap N_R(x)$ respectively. Go back to 1).

What can go wrong with the Book Algorithm?

- $N_R(x) \cap X$ is too small, so that X shrinks too fast.

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- the red density between $N_R(x) \cap X$ and $N_R(x) \cap Y$ is "too low".

Bounds on Ramsey numbers

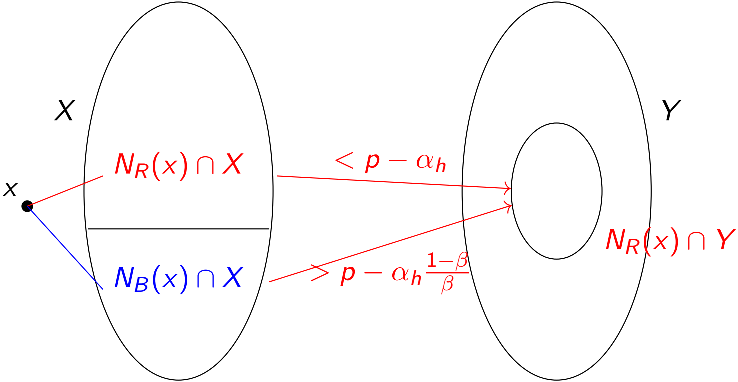

Figure: Schematic diagram for the "Density boost" step

Bounds on Ramsey numbers

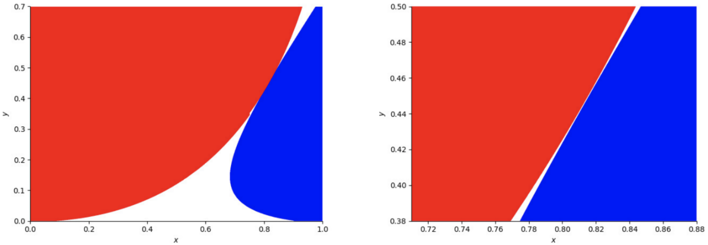

Figure: The blue region represents when $n \geq (4 + o(1))^{\ell}$, and the red region represents when outside of which there is either a red ℓ-clique or a blue ℓ-clique. Note that they do not overlap: so we have an exponential improvement.

Bounds on Ramsey numbers

Lower bounds:

Erdős [1947]

For all $k \geq 3$, we have
$$R(\ell, \ell) > \lfloor 2^{\frac{\ell}{2}} \rfloor.$$

Proof was instrumental to the development of the Probabilistic method!

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Arithmetic Progressions

Ramsey theory is not just about graphs!
Can also look at arithmetic progressions in subsets of $\mathbb{Z}^+$ and $\mathbb{F}_p^n$.

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Van der Waerden's Theorem

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Ellenberg and Gijswijt [2017]

Let $r_3(\mathbb{F}_3^n)$ denote the largest 3-AP free subset of $\mathbb{F}_3^n$. Then $r_3(\mathbb{F}_3^n) = O(2.76^n)$.

Van der Waerden's Theorem

For all $k, r \in \mathbb{Z}^+$, there exists $W(k, r) \in \mathbb{Z}^+$ such that if $[W(k, r)]$ is r-colored, then there exists a monochromatic k-AP.

Proof.
$W(2, r)$ and $W(k, 1)$ is trivial, so we first try to prove the existence of $W(3, 2)$.

Outline of Proof of $W(3, 2)$

- $A = \{1, 2, 3, 4, 5\} \subset \mathbb{Z}^+$

Van der Waerden's Theorem

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- $A = \{1,2,3,4,5\} \subset \mathbb{Z}^+$
- There exists $a, a+d, a+2d \in A$ such that $a, a+d$ is red, and $a+2d$ is blue; this true for all translates $A+n$ where $n \in \mathbb{Z}^+$.

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- Define an induced 32-coloring of $\mathbb{Z}^+$ where n is colored based on the coloring of $A+n$.

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- Define an induced 32-coloring of $\mathbb{Z}^+$ where n is colored based on the coloring of $A+n$.
- $W(32,2)$ exists $\Rightarrow$ exists $n, d_1 \in \mathbb{Z}^+$ where $A+n$ and $A+n+d_1$ have the same coloring.

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Outline of Proof of $W(3,2)$

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- Define an induced 32-coloring of $\mathbb{Z}^+$ where n is colored based on the coloring of $A+n$.
- $W(32,2)$ exists $\Rightarrow$ exists $n, d_1 \in \mathbb{Z}^+$ where $A+n$ and $A+n+d_1$ have the same coloring.
- Then either $\{a+n, (a+d)+n+d_1, (a+2d)+n+2d_1\}$ or $\{(a+2d)+n, (a+2d)+n+d_1, (a+2d)+n+2d_1\}$ form a monochromatic 3-AP!

Van der Waerden's Theorem

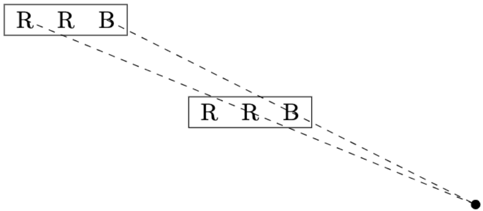

Figure: This is the schematic proof for $W(3,2)$. Each rectangular block denotes (a translate of) A, with a 3-AP consisting of red, red, and blue. Given that we can find two such identical rectangular blocks, we can find a blue 2-AP and a red 2-AP that "converges" to the same point.

Van der Waerden's Theorem

This method generalizes to proving the existence of $W(k, r)$:

- Induct from $W(k-1, r^*) \forall r^* \in \mathbb{Z}^+$ to $W(k, r)$

Van der Waerden's Theorem

This method generalizes to proving the existence of $W(k, r)$:

- Induct from $W(k-1, r^*) \forall r^* \in \mathbb{Z}^+$ to $W(k, r)$
- Use "induced coloring" idea to iteratively "stack" monochromatic $(k-1)$-APs until we a force a monochromatic k-AP.

Van der Waerden's Theorem

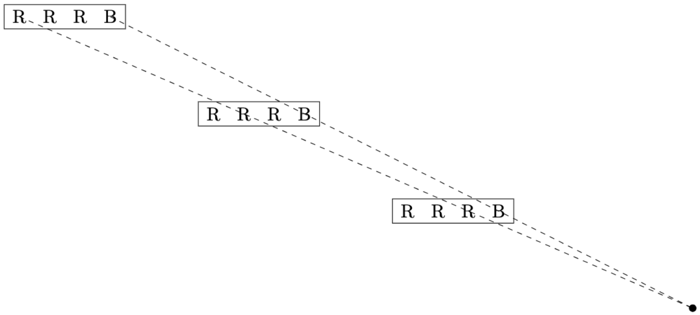

Figure: This is the schematic proof for $W(4, 2)$...

The End

Thank you!