

Random Dieudonné Modules and the Cohen-Lenstra Heuristics

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Slides available at <http://www.mathcs.emory.edu/~dzb/slides/>

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How often does p **divide** $h(-D)$?

What is

$$P(p \mid h(-D)) = \lim_{X \rightarrow \infty} \frac{\#\{0 \leq D \leq X \text{ s.t. } p \mid h(-D)\}}{\#\{0 \leq D \leq X\}}?$$

Guess: Random Integer?

$$P(p \mid h(-D)) = P(p \mid D) = \frac{1}{p}$$

$$\begin{aligned}P(p \mid h(-D)) &\approx \frac{1}{p} + \frac{1}{p^2} - \frac{1}{p^5} - \frac{1}{p^7} + \cdots && (p \text{ odd}) \\&= 1 - \prod_{i \geq 1} \left(1 - \frac{1}{p^i}\right) \\&= \mathbf{0.43 \dots \neq 1/3} && (p = 3) \\&= \mathbf{0.23 \dots \neq 1/5} && (p = 5)\end{aligned}$$

$$P(\text{Cl}(-D)_3 \cong \mathbb{Z}/9\mathbb{Z}) \approx \mathbf{0.070}$$

$$P(\text{Cl}(-D)_3 \cong (\mathbb{Z}/3\mathbb{Z})^2) \approx \mathbf{0.0097}$$

Random finite abelian groups

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$$(i) \quad \sum_{G \in \underline{G}_p} \frac{1}{\# \text{Aut } G} = \prod_i \left(1 - \frac{1}{p^i}\right)^{-1} = C_p^{-1}$$

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- (iii) $\text{Avg}(\#G[p]) = \text{Avg}(p^{r_p(G)}) = 2$

Cohen and Lenstra's conjecture

Let $f: \underline{G}_p \rightarrow \mathbb{Z}$ be a function.

Definition

$$\text{Avg } f = \sum_{G \in \underline{G}_p} \frac{C_p}{\# \text{Aut } G} \cdot f(G)$$

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- (iii) $P(\text{Cl}(-D)_p \cong G) = \frac{C_p}{\# \text{Aut } G}.$

- Kohnen-Ono – $N_{p \nmid h}(X) \gg \frac{x^{\frac{1}{2}}}{\log x}$
- Heath-Brown – $N_{p|h}(X) \gg \frac{x^{\frac{9}{10}}}{\log x}$
- Byeon – $N_{\text{Cl}_p \cong (\mathbb{Z}/p\mathbb{Z})^2}(X) \gg \frac{x^{\frac{1}{p}}}{\log x}$
- Davenport-Heilbronn – $\text{Avg Cl}(-D)[3] = 2$

$$\mathrm{Cl}(-D) = \mathrm{Pic}(\mathrm{Spec} \mathcal{O}_K)$$

VS

$$0 \rightarrow \mathrm{Pic}^0(C) \rightarrow \mathrm{Pic}(C) \xrightarrow{\deg} \mathbb{Z} \rightarrow 0$$

Basic Question over $\mathbb{F}_q(t)$, $\ell \neq p$

Fix $G \in \underline{G}_\ell$.

What is

$$P(\mathrm{Pic}^0(C)_\ell \cong G)?$$

(Limit is taken as $\deg f \rightarrow \infty$, where $C: y^2 = f(x)$.)

Main Tool over $\mathbb{F}_q(t)$ – Tate Module

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- $\text{Frob} \in \text{Gal}_{\mathbb{F}_q} \rightarrow \text{Aut } T_\ell(\text{Jac}_C) \cong \mathbb{Z}_\ell^{2g}$
- $\text{coker}(\text{Frob} - \text{Id}) \cong \text{Jac}_C(\mathbb{F}_q)_p = \text{Pic}^0(C)$

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Basic question – what is

$$P(p \mid \# \text{Jac}_C(\mathbb{F}_p))?$$

$$T_\ell(\text{Jac}_C) \cong \mathbb{Z}_\ell^r, 0 \leq r \leq g$$

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Complication

As C varies, r varies. Need to know the distribution of p -ranks, or find a better algebraic gadget than $T_\ell(\text{Jac}_C)$.

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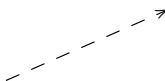
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 \end{array}$$

$V^{-1}: df \mapsto \frac{d(f^p)}{p}$

Invariants

- (i) $p\text{-rank}(\text{Jac}_C) = \dim F^\infty(M \otimes \mathbb{F}_p).$
- (ii) $a(\text{Jac}_C) = \dim \text{Hom}(\alpha_p, \text{Jac}_C[p]) = \dim (\ker V \cap \ker F).$
- (iii) $\text{Jac}_C(\mathbb{F}_p)_p = \text{coker}(F - \text{Id})|_{F^\infty(M \otimes \mathbb{F}_p)}.$

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Theorem (Cais, Ellenberg, ZB)

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- C plane curve, $p = 2$ – **NO!?!**

C plane curve, $p = 2$

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Proof – theta characteristics.

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$$- P(a(\text{Jac}_C(\mathbb{F}_p)) = 0) = \lim_{g \rightarrow \infty} \frac{\#\mathcal{H}_g^{\text{ord}}(\mathbb{F}_p)}{\#\mathcal{H}_g(\mathbb{F}_p)}.$$

Rational points on Moduli Spaces

- $P(a(\text{Jac}_C(\mathbb{F}_p)) = 0) = \lim_{g \rightarrow \infty} \frac{\#\mathcal{H}_g^{\text{ord}}(\mathbb{F}_p)}{\#\mathcal{H}_g(\mathbb{F}_p)}.$
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- One can access this through cohomology and the Weil conjectures.
- Our data suggests that $\mathcal{H}_g^{\text{ord}}$ has cohomology that **does not arise by pulling back** from \mathcal{H}_g .

Thank you

Thank You!