Math 280: Graph Theory

Instructor: David Zureick-Brown ("DZB")

All assignments

Last updated: October 30, 2024

Gradescope code: VD5BZK

Show all work for full credit!
Proofs should be written in full sentences whenever possible.

Gradescope Instructions for submitting work in Math 280

You will be using the online Gradescope progam to submit your homework and exams. These instructions tell you how to sign up initially, and how to submit your written work.

Signing up for Gradescope the first time.

If you haven't used Gradescope for an **Amherst College** course before:

- Go to `http://www.gradescope.com`, click on "Sign up for free" (which may auto-scroll you to the bottom of the page), and select Sign up as [a] "Student".
- In the signup box:
  - Use the course entry code **VD5BZK**
  - Use your full name
  - Use your **Amherst College email** address. Or, if you are a Five-College student, use your email address from your own school.
  - Leave the "Student ID" entry blank.
- You will probably get an email asking to set a password for your account, so check your `amherst.edu` email inbox. (Or your email inbox through your own school, for Five-College students.)

Adding Math 280 to Gradescope.

If you **have** used Gradescope for an Amherst course before, and so you already have an account through your `amherst.edu` email, you still need to add Math 280, so:

- Go to `http://www.gradescope.com` and log in.
- Go to your Account Dashboard (click the Gradescope logo at upper left), and click "Add Course" at bottom right.
- Use the course code **VD5BZK**

(submission instructions on next page)

Submitting written work

First write it out on paper as you would normally. Then **scan it** to create a PDF. One method for scanning is the smartphone app **DropBox**. It makes nice clear scans, and it saves them directly into a folder so that you can have all your assignments in one place. **CamScanner** is another free scanning App, and there are others, too. **Gradescope** now has its own scanning app. You can also use a printer/scanner if you prefer.

Please be kind to our dear graders and make sure your submission is legible !

In particular, please leave some spacing between separate problems.

If you have a tablet computer, you may write your work there (instead of on paper) and save it as a PDF.

Some of you may know the math formatting package LaTeX and may want to use it in Math 280. That's fine, too; if so, you may write up your work in LaTeX and save the resulting PDF.

In short, any method is fine as long as it creates a legible **PDF** file and **NOT a photo**.

For example, if you use the DropBox app, then in your created *Math 280 Homework* Dropbox folder, you can select create (+) at the bottom of the screen and click the *Scan Document* option. Snap a shot of the first page of your homework, and then click [+] to snap shots of any subsequent pages. Do **not** use the *Take Photo* option.

After you have scanned/saved your work as a PDF, submit it on Gradescope as follows:

- Go to `http://www.gradescope.com` and log in.
- Select the course "Math 280, Fall 2024" and the appropriate assignment.
- Select "submit pdf" to submit your work in PDF format. Browse to find your PDF and upload.
- Now it is time to **tag** your problems. This is an **important step** , where you are telling Gradescope which problems are on which page(s).

  For each problem, select the pages of your submission where your written solution appears.

  I think the easiest thing to do is to click on the page of **your** homework upload where you wrote the given problem, and then click on the assigned problem listed. Repeat for each problem.

  You must tag the problems or else you will not get credit for your work.

  Gradescope will give you a warning when you go to submit your assignment if you have not selected the pages correctly. If you tag a problem incorrectly, you can fix it by clicking "More" and "Reselect Pages".
- Click Save or Submit.

After your assignment is graded, you will be able to see your score on the written problems, along with comments, on Gradescope. You should receive an email notifying you when each homework set is graded.

Assignment 1: Introduction to the course

Due by 9:55am (section 02) / 12:55pm (section 01), eastern, on Thursday, Sept 12

Suggested readings for this problem set:

- Syllabus: https://dmzb.github.io/teaching/2024Fall280/syllabus-math-280-spring-2024.pdf
- Gradescope instructions (previous page)
- Sections 1.1.1 and 1.1.2 and start 1.1.3. Here is a link to a pdf of the first few subsections of the book.

All readings are from Harris, Hirst, and Mossinghoff, *Combinatorics and Graph Theory*.

Assignment: due Thursday, Sept 12, 9:55am (section 02) / 12:55pm (section 01), via Gradescope (VD5BZK):

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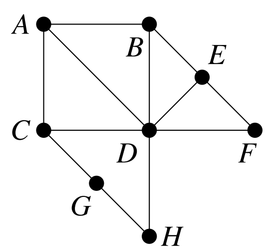
1. Let G be the following graph:



   (a) Write out (as sets) the vertex set and edge set of G.

   (b) Find the degree sequence of G.

   (c) Let $\overline{G}$ be the complement of G. (See the textbook, Section 1.1.3, item 3.) Draw a picture of $\overline{G}$.

2. Let G be a graph of order n and size t.

   (a) What is the maximum possible size of G? (That is, what's the maximum possible number of edges G could have?)

   (b) Let $\overline{G}$ be the complement of G. (See the textbook, Section 1.1.3, item 3.) Find the order and size of $\overline{G}$.

   Don't forget to justify your answer! You don't need to give a formal proof on this more computational problem, but you need to explain why you know your answers are correct.

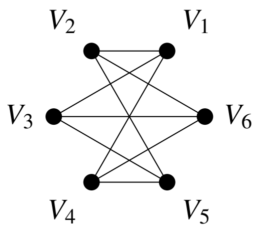
3. For which of the sequences below does there exist a **simple** graph on 4 vertices with that degree sequence? If there is such a graph, draw it, and if there is not such a graph, justify why there is no such graph.

   (a) $(0, 0, 0, 0)$ (c) $(2, 1, 1, 1)$ (e) $(3, 3, 1, 1)$

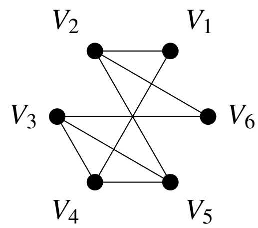
   (b) $(3, 2, 1, 0)$ (d) $(4, 3, 2, 1)$ (f) $(3, 3, 2, 2)$

4. (a) Below are 5 graphs. Which graphs are isomorphic, and which are not? For the ones that are isomorphic, say what the isomorphism is (but you do not need to prove that it is an isomorphism). For the ones that are not isomorphic, give a short explanation of why they are not. (Note: since there are 5 graphs, there are 10 pairs of graphs; make sure that you give explanations for each pair.)

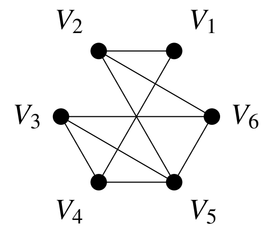
(A)



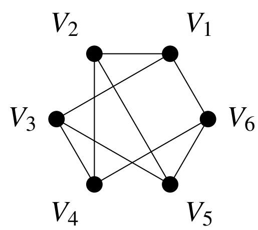
(B)



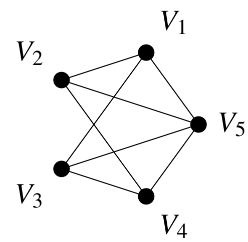
(C)



(D)



(E)



(b) Give an example of a graph with the same degree sequence as the first graph from part (a), but which is not isomorphic to that graph. (Don't prove that they are not isomorphic, just give the graph.)

5. Let G be a graph of order $n \geq 2$. Prove that the degree sequence of G has at least one pair of repeated entries.

(Click here for a hint.)

6. Let G be a graph of odd order. Suppose that all the vertices of G have the same degree r. Prove that r is an even number.

(Click here for a hint.)

7. There are n Amherst students participating in a meeting. Among any group of 4 participants, there is one who knows the other three members of the group. Prove that there is one participant who knows all other participants.

(Click here for a hint.)

8. Let G be a graph and let S, T be subsets of $V(G)$.

   (a) Prove that if $S \subseteq T$ then $N(S) \subseteq N(T)$.

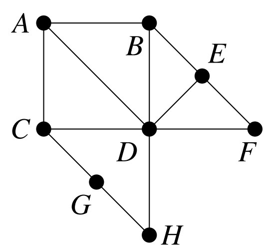
   (b) Is the converse true? If so, prove it. If not, give a counterexample (i.e., an example of a graph G and subseteqs $S, T \subseteq V(G)$ such that $N(S) \subseteq N(T)$ but $S \not\subseteq T$.

Assignment 2: Basic properties, e.g., connectedness

Suggested readings for this problem set: Section 1.1.1-1.1.3
All readings are from Harris, Hirst, and Mossinghoff, *Combinatorics and Graph Theory*.

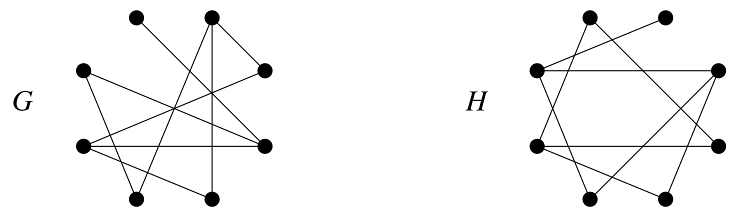
Assignment: due Thursday, Sept 19, 9:55am (section 02) / 12:55pm (section 01), via Gradescope (VD5BZK):

1. Let G be the following graph:



   (a) Find the longest path in X. Justify why it is the longest. (You don't need to give a formal proof, but say in one sentence how you know it is the longest.)
   (b) Find the longest trail in X. Justify why it is the longest. (You don't need to give a formal proof, but say in one sentence how you know it is the longest.)
   (c) What is $k(G)$? (Recall that $k(G)$ is the connectivity of G.)

2. Consider the following two graphs:



   Verify (but do not include in your solution) that G and H have the same order, same size, and same degree sequence.

   Then prove that in spite of that, G and H are *not* isomorphic.

3. Let G and H be isomorphic graphs. Prove that their complements $\overline{G}$ and $\overline{H}$ are also isomorphic.

4. Prove that the complement of a disconnected graph is connected.

5. Let n be a positive integer and let $i \in \{0, \ldots, n-1\}$. Prove that there exists a graph G with order n and $k(G) = i$.

   (Click here for a hint.)

6. Prove that if u is a vertex of odd degree in a graph, then there exists a vertex $v \neq u$ of the graph such that v also has odd degree and such that there is a path from u to v.

7. Prove that every closed walk of odd length in a graph contains a cycle of odd length.

8. Let G be a 2-connected graph. Prove that G contains at least one cycle.

   (2-connected means that if you delete any one vertex v, the subgraph $G - v$ is still connected; i.e., for any two of the remaining vertices, there's a path between them that avoids v.)

Assignment 3: More connectedness; bipartite and regular graphs; distance

Suggested readings for this problem set: Section 1.1.3–1.2.2

All readings are from Harris, Hirst, and Mossinghoff, *Combinatorics and Graph Theory*.

Assignment: due Thursday, Sept 26, 9:55am (section 02) / 12:55pm (section 01), via Gradescope (VD5BZK):

1. (a) Give an example of a graph G with 8 vertices such that G is isomorphic to its complement. (No need to prove that it is isomorphic, just give the example.)

   (b) Let G be a graph with 7 vertices. Prove that G cannot be isomorphic to its complement.

2. Let n and m be positive integers. Prove that $K_{n,m}$ is regular if and only if $n = m$.

3. Let $n, m \geq 3$. Prove that K_n is not a subgraph of $K_{m,m}$.

   (Click here for a hint.)

4. Suppose G is a simple graph that has ten edges, and six vertices $v_1, v_2, \ldots v_6$ with degrees $2, 2, 3, 4, 4, n$, respectively, for some integer n.

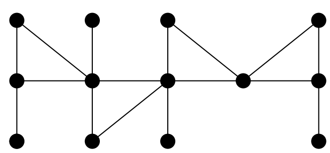
   (a) What is the integer n, i.e., $\deg(v_6)$?

   (b) Is G connected? (Yes, no, or maybe?) If yes or no, explain why. If "maybe," give an example of such a graph G that is connected, and another that is not connected.

5. Prove that a connected regular bipartite graph with at least 3 vertices is 2-connected, i.e., it does not contain any vertex whose deletion results a disconnected graph.

   (Click here for a hint.)

6. Find the radius, diameter, and center of the following graph:



7. Let G be a graph, and let $u, v \in V(G)$ be adjacent vertices. Prove that their eccentricities $\mathrm{ecc}(u)$ and $\mathrm{ecc}(v)$ differ by at most 1.

8. Prove that if $\mathrm{diam}(G) \geq 3$, then $\mathrm{diam}(\overline{G}) \leq 3$.

Assignment 4: More distance, Trees, Prüfer sequences

Due by 9:55am (section 02) / 12:55pm (section 01), eastern, on Thursday, Oct 3

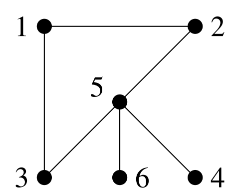
Suggested readings for this problem set:

- Sections 1.3.1–1.3.4

All readings are from Harris, Hirst, and Mossinghoff, *Combinatorics and Graph Theory*.

Assignment: due Thursday, Oct 3, 9:55am (section 02) / 12:55pm (section 01), via Gradescope (VD5BZK):

1. Let G be the following graph:



(a) Find the adjacency matrix A of G.

(b) List all of the walks of length three from vertex 1 to vertex 4. Compute A^3, and use this to verify that you have the correct number of walks.

2. Let G be a graph with $V(G) = \{v_1, \ldots, v_n\}$ and with adjacency matrix A. For each $j = 1, \ldots, n$, prove that the (j, j) entry of A^2 is $\deg(v_j)$.

(Click here for a hint.)

3. Prove that if x belongs to the periphery of a graph G and $d(x, y) = ecc(x)$, then y belongs to the periphery of G.

4. Draw all unlabeled trees of order 7.

(More precisely: Find a set of trees of order 7 so that *every* tree of order 7 is isomorphic to one in your set, and so that no two in your set are isomorphic to each other.)

(Click here for a hint.)

5. Let T be a tree, and let $u, v \in V(T)$. Prove that there is *exactly one* path from u to v.

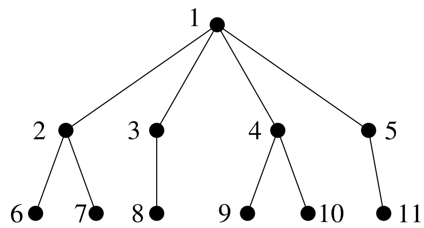
6. (a) Let T be a tree of order $n \geq 2$. Prove that T is bipartite.

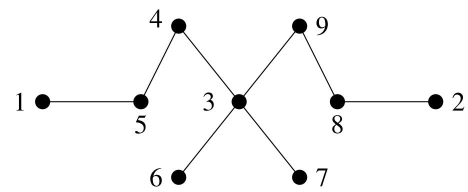
   (Click here for a hint.)

   (b) Which trees are complete bipartite graphs? Prove that your answer is correct.

7. Let T be a tree that has an even number of edges. Prove that at least one vertex of T has even degree.

8. (a) Use Prüfer's method to find the Prüfer sequences of the following two trees.





(b) Use Prüfer's method (see 1.3.4 of the book) to draw and label a tree with Prüfer sequence 5,4,3,5,4,3,5,4,3.

Assignment 5: Prüfer; Spanning trees; Matrix Tree THM; Eulerian/Hamiltonian

Due by 9:55am (section 02) / 12:55pm (section 01), eastern, on Thursday, Oct 10

Suggested readings for this problem set:

- Sections 1.3.4 and 1.4.1-1.4.4

All readings are from Harris, Hirst, and Mossinghoff, *Combinatorics and Graph Theory*.

Assignment: due Thursday, Oct 10, 9:55am (section 02) / 12:55pm (section 01), via Gradescope (VD5BZK):

1. How many forests are there on 4 vertices? Justify your answer.

2. (a) Use Prüfer's method to draw and label the trees with Prüfer sequences 1,1,1,1,1 and 3,3,3,3.
   (b) Inspired by your answers in part (a), make a conjecture about which trees have constant Prüfer sequences.
   (c) Prove your conjecture from part (b).

3. Let T be a tree of order $n \geq 2$, and suppose that none of the vertices of T have degree 2. Prove that T has more than $n/2$ leaves.

4. Let T be a labeled tree, and let σ be its Prüfer sequence. For each vertex $v \in V(T)$, prove that v appears in σ exactly $\deg(v) - 1$ times.

   (*Suggestion*: Do an induction on $n \geq 2$, where n is the order of the tree.)

   (*Note*: As a special case, this means that none of the leaves of T appear in the sequence σ at all. The textbook states that as a separate fact, but since it's just a special case of the above statement, you only need to prove the above statement.)

5. For both of the following two graphs, write down its Laplacian matrix, and then use the Matrix Tree Theorem to find its number of spanning trees.

   P_4 $\qquad\qquad$ $K_{2,3}$

6. For each of the following, draw an Eulerian graph that satisfies the conditions, or prove that no such graph exists.

   (a) An even number of vertices, and an even number of edges.
   (b) An even number of vertices, and an odd number of edges.
   (c) An odd number of vertices, and an even number of edges.
   (d) An odd number of vertices, and an odd number of edges.

7. Determine the values of $m, n \geq 1$ such that the complete bipartite graph $K_{m,n}$ is Eulerian. Prove your answer.

8. Determine the precise set of values of $m, n \geq 1$ such that the complete bipartite graph $K_{m,n}$ has an Eulerian trail. Prove your answer.

Midterm 1 study guide

Take home exam, Thursday, Oct 17. Submit your exam via Gradescope.

The exam will be released at 2:15pm on Thursday, and will be due at 9:55am on Tuesday, October 22. The exam will be available on gradescope, and should be submitted via Gradescope.

The *types* of problems will include a subset of

1. Computations
2. Proofs
3. Algorithms
4. True False
5. Bonus problem

Problems with extremely long proofs or that involved some unusual trick will not be on the exam.

Since this is a take home exam, none of the problems will be identical to homework problems, but many problems will be minor variations of homework or of problems we worked in class.

A good way to prepare is to:

1. Know all of the definitions and terminology;
2. Know all of the statements of theorems, and examples of how we use the theorems;
3. Make a list of all of the different *proof techniques* from class and from the homework and review how those techniques are used in proofs and problems;
4. Practice doing problems "from scratch" and use your solutions as "hints" when you get stuck.

Additionally:

1. You are allowed to use the textbook, lecture notes and any materials from the course website.
2. Using Google or any other online resources is not always a reliable source. Please do not use Chat GPT or any other AI assistants.
3. You are allowed to use Theorems, lemmas, etc from the book or from class as part of your solutions, and you are not required to reprove these during the exam. Please do cite them (e.g., "Proposition 1.3.4 from our book") or refer to them by name, if they have a special name (e.g., "by the Matrix Tree Theorem").
4. Do not discuss the problems or their solutions with your classmates.
5. You can always ask me (the instructor) if you have clarifying questions, but asking for hints or asking if a proof is correct is not allowed.

A typical exam will have a few questions from each week of the course and will cover **assignments 1-4**. The problems will be similar to the homework problems, and the proofs will use the same techniques.

Assignment 6: More Eulerian and Hamiltonian graphs; independence

Due by 9:55am (section 02) / 12:55pm (section 01), eastern, on Thursday, Oct 24

Suggested readings for this problem set:

- Sections 1.3.4 and 1.4.2, and start 1.4.3

All readings are from Harris, Hirst, and Mossinghoff, *Combinatorics and Graph Theory*.

Assignment: due Thursday, Oct 24, 9:55am (section 02) / 12:55pm (section 01), via Gradescope (VD5BZK):

1. When does $K_{a,b,c}$ have an Eulerian Circuit? What about $K_{n_1,\dots,n_r}$? (These are complete *multipartite* graphs; see the index of our book to find a definition of multipartite.)

2. Prove that the independence number is a graph isomorphism invariant. (I.e, prove that if G and H are isomorphic, then $\alpha(G) = \alpha(H)$.

3. Let G be a graph with n vertices and at least $(n-1)(n-2)/2 + 2$ edges. Prove that G has a Hamiltonian cycle.

   (Click here for a hint.)

4. Let G be a graph with at least one cycle. Suppose that for every cycle C, and for every vertex v not in the cycle, v has at least three neighbors on the cycle. Prove that G contains a Hamiltonian cycle.

   (Click here for a hint.)

Assignment 7: Planar graphs

Due by 9:55am (section 02) / 12:55pm (section 01), eastern, on Thursday, Oct 31

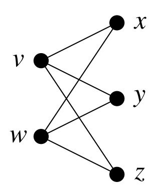
Suggested readings for this problem set:

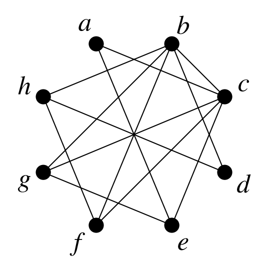
- Sections 1.5.1-1.5.4 ,

All readings are from Harris, Hirst, and Mossinghoff, *Combinatorics and Graph Theory*.

Assignment: due Thursday, Oct 31, 9:55am (section 02) / 12:55pm (section 01), via Gradescope (VD5BZK):

1. Find planar representations of each of the following graphs:

(a): 

(b): 

2. Draw a planar graph in which every vertex has degree exactly 5

3. Let G be a connected planar graph of order 24, and suppose that G is regular of degree 3. How many regions are there in a planar representation of G?

4. Give an example of planar graphs G_1 and G_2 that have the same number n of vertices, the same number q of edges, and the same number r of regions, **but** which are not isomorphic.

5. Prove (without using Kuratowski's theorem or the graph minor theorem) that every tree is planar.

6. Let G be a connected planar graph of order $n \geq 3$, and suppose that G is K_3-free. (That is, G has no cycles of length 3.) Prove that the number q of edges of G satisfies $q \leq 2n - 4$.

7. Let G be a bipartite planar graph. Prove that $\delta(G) \leq 3$.

   (Click here for a hint.)

8. Let G be of order $n \geq 11$. Prove that at least one of G or $\overline{G}$ is nonplanar.

Assignment 8: Graph coloring; chromatic number

Due by 9:55am (section 02) / 12:55pm (section 01), eastern, on Thursday, Nov 7

Suggested readings for this problem set:

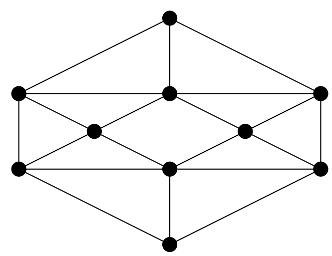
- Sections 1.6.1-1.6.4,

All readings are from Harris, Hirst, and Mossinghoff, *Combinatorics and Graph Theory*.

Assignment: due Thursday, Nov 7, 9:55am (section 02) / 12:55pm (section 01), via Gradescope (VD5BZK):

1. Determine the chromatic number of the Petersen graph. As always, don't forget to justify your answer.

2. Determine the chromatic number of the Birkhoff Diamond, shown below. As always, don't forget to justify your answer.



3. Let G be a graph, let $e \in E(G)$, and let $G' = G - e$. Prove that $\chi(G') \leq \chi(G) \leq \chi(G') + 1$.

4. Prove that for any $n \geq 2$ and any graph G of order n, we have $\chi(G) = n$ if and only if G is complete.

5. Prove that a graph G has at least $\chi(G)(\chi(G) - 1)/2$ edges.

   (Click here for a hint.)

6. Let G be a graph with n vertices and $\chi(G) = n - 1$.

   (a) Use the theorem that "$\chi(G) \leq r$ iff G is r-partite" to prove that $\omega(G) \geq n - 2$.

   (b) Using part (a), prove that G is isomorphic to $K_n - e$ for some edge e.

   (Click here for a hint.)

7. Recall that $\alpha(G)$ and $\chi(G)$ are the independence number and chromatic number of G, respectively. Prove that for any $n \geq 1$ and any graph G of order n, we have

$$\frac{n}{\alpha(G)} \leq \chi(G) \leq n + 1 - \alpha(G).$$

   (Click here for a hint.)

8. Show that $\chi(G) + \chi(\overline{G}) \leq n + 1$ for any graph G of n vertices.

   (Click here for a hint.)

IN PROGRESS! Check back later for the final assignment.

Midterm 2 study guide

Take home exam, Tuesday, Nov 12. Submit your exam via Gradescope.

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2. Using Google or any other online resources is not always a reliable source.
3. You are allowed to use Theorems, lemmas, etc from the book or from class as part of your solutions, and you are not required to reprove these during the exam.
4. Do not discuss the problems or their solutions with your classmates.
5. You can always ask me (the instructor) if you have clarifying questions, but asking for hints or asking if a proof is correct is not allowed.

A typical exam will have a few questions from each relevant week of the course and will cover **assignments 5-8**. You can expect problems about following:

Assignment 9: Chromatic polynomial

Due by 9:55am (section 02) / 12:55pm (section 01), eastern, on Thursday, Nov 21

Suggested readings for this problem set:

- Sections 1.7.1-1.7.4 ,

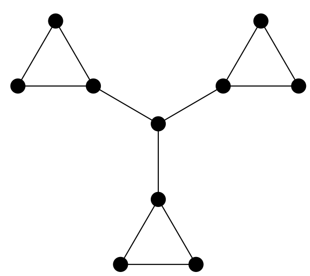
All readings are from Harris, Hirst, and Mossinghoff, *Combinatorics and Graph Theory*.

Assignment: due Thursday, Nov 21, 9:55am (section 02) / 12:55pm (section 01), via Gradescope (VD5BZK):

1. For any $n \geq 2$, let $H = K_n - e$ be the graph K_n with one edge removed. Prove that the chromatic polynomial of H is is $c_H(k) = k(k-1)\cdots(k-n+3)(k-n+2)^2$.

   (*Suggestion*: Don't use induction. Instead, use the result of Problem 5 above, Theorem 1.48, and the known formula for the chromatic polynomial of K_n.)

2. Prove that the following graph G has no perfect matching.



3. Prove that for any tree T of order n, the chromatic polynomial of T is $c_T(k) = k(k-1)^{n-1}$.

   (*Suggestion*: Use Theorem 1.48 and induction on n.)

4. Prove that for any $n \geq 3$, the chromatic polynomial of the cycle graph C_n is $c_{C_n}(k) = (k-1)((k-1)^{n-1} + (-1)^n)$.

   (*Suggestion*: Use the result of Problem 3 above, Theorem 1.48, and induction on n.)

5. Let $n \geq 2$, and let e be any edge of the complete graph K_n. Prove that K_n/e is isomorphic to K_{n-1}.

6. (a) Find a perfect matching of C_{12}.

   (b) Find the minimum size of a maximal matching of C_{12}. That is, find a maximal matching M of C_{12} that has some number m of edges, and then prove that any *other* matching M' with $m-1$ or fewer edges cannot be maximal.

IN PROGRESS! Check back later for the final assignment.

IN PROGRESS! Check back later for the final assignment.

IN PROGRESS! Check back later for the final assignment.

Assignment 10: Matchings; Hall's theorem

Due by 9:55am (section 02) / 12:55pm (section 01), eastern, on Thursday, Dec 5

Suggested readings for this problem set:

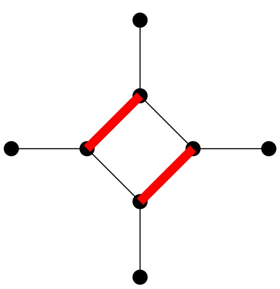
- Textbook Sections 1.7.1-1.7.4 ,

All readings are from Harris, Hirst, and Mossinghoff, *Combinatorics and Graph Theory*.

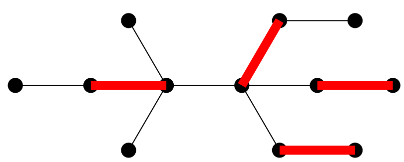
Assignment: due Thursday, Dec 5, 9:55am (section 02) / 12:55pm (section 01), via Gradescope (VD5BZK):

1. For each of the following graphs, with matchings M as shaded, find an M-augmenting path, and use it to obtain a bigger matching.

(a)



(b)



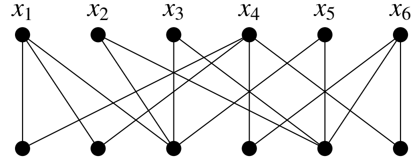
2. For each of the following families of sets, explicitly and carefully check whether the conditions of Theorem 1.52 are met. If so, then find an SDR, saying exactly which element is chosen from each set. If not, then show how the hypotheses are violated.

   (a) $\{1,2,3\}$, $\{2,3,4\}$, $\{3,4,5\}$, $\{4,5\}$, $\{1,2,5\}$

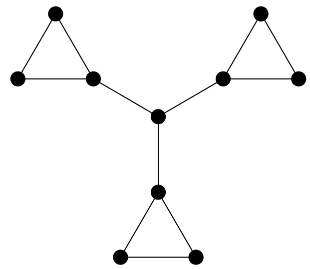
   (b) $\{1,2,4\}$, $\{2,4\}$, $\{2,3\}$, $\{1,2,3\}$

   (d) $\{1,2,5\}$, $\{1,5\}$, $\{1,2\}$, $\{2,5\}$

3. Use Hall's Theorem to prove that the following bipartite graph does not have a perfect matching.



4. Find a maximum matching of the following graph G, and prove that it is indeed a *maximum* matching.



5. Find and draw a connected, 3-regular graph that has both a cut vertex and a perfect matching.

   Don't forget to (briefly) verify that your graph has all these properties. (3-regular, has a cut vertex, and has a perfect matching.)

6. Let G be a graph with connected components $H_1, \ldots, H_k$. Prove that G has a perfect matching if and only if every component H_i has a perfect matching.

7. Let T be a tree. Prove that T has at most one perfect matching.

   (*Suggestion*: Use strong induction on the number of vertices.)

IN PROGRESS! Check back later for the final assignment.

IN PROGRESS! Check back later for the final assignment.

Assignment 11: Ramsey Theory

Due by 9:55am (section 02) / 12:55pm (section 01), eastern, on Thursday[, Dec 11

Suggested readings for this problem set:

- Sections 1.8.1–1.8.3

All readings are from Harris, Hirst, and Mossinghoff, *Combinatorics and Graph Theory*.

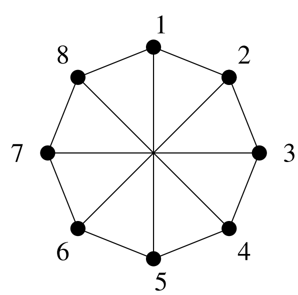
Assignment: due Thursday, Dec 11, 9:55am (section 02) / 12:55pm (section 01), via Gradescope (VD5BZK):

1. Find and prove a formula for the number of 2-colorings of the edges of K_n, for each $n \geq 1$.

2. Give a full proof that for any integer $k \geq 2$, we have $R(2, k) = k$.

3. Give a full proof that for any integers $p, q \geq 2$, we have $R(p, q) = R(q, p)$.

4. Prove that the following graph G of order 8 satisfies $\omega(G) \leq 2$ and $\omega(\overline{G}) \leq 3$. (That is, prove there are no K_3's in G, and no K_4's in $\overline{G}$.)



[**Note**: This is a variant of Figure 1.126, which the book points to but skips the analysis of in proving Theorem 1.62. It may help to note that each vertex i has edges to $i-1$, $i+1$, and $i+4$, if we consider these integers modulo 8.]

5. Consider the graph G on 13 vertices $\{1, 2, \ldots, 13\}$ where each vertex i has four edges, connecting it to the vertices $i-1$, $i+1$, $i-5$, and $i+5$, where we consider these integers modulo 13. (See Figure 1.131 in the textbook.)

   Prove that $\omega(G) \leq 2$.

6. Let G be the graph on 13 vertices from Problem 5 above.

   (a) Prove that for any vertex j, there is no subgraph of $\overline{G}$ that contains vertices j and $j+3$ and is a copy of K_5. (As before, consider $j+3$ modulo 13.)

(b) Prove that for any vertex j, there is no subgraph of $\overline{G}$ that contains vertices j and $j+6$ and is a copy of K_5. (As before, consider $j+6$ modulo 13. You may use the result of part (a).)

(c) Use the results of parts (a) and (b) to prove that $\omega(\overline{G}) \leq 4$.

7. Use Theorem 1.64 and the results of Problems 5 and 6 to prove that $R(3,5) = 14$.

Final exam study guide

Final exam is a take home exam, released on **May 11, tentative** and due May 16 (tentative). Submit your exam via Gradescope.

The **last day of class** is Tuesday, December 10.

More info to come

The *types* of problems will include a subset of

1. Computations
2. Proofs
3. Algorithms
4. True False
5. Bonus problem

Problems with extremely long proofs or that involved some unusual trick will not be on the exam.
Since this is a take home exam, none of the problems will be identical to homework problems, but many problems will be minor variations of homework or of problems we worked in class.
A good way to prepare is to:

1. Know all of the definitions and terminology;
2. Know all of the statements of theorems, and examples of how we use the theorems;
3. Make a list of all of the different *proof techniques* from class and from the homework and review how those techniques are used in proofs and problems;
4. Practice doing problems "from scratch" and use your solutions as "hints" when you get stuck.

n Additionally:

1. You are allowed to use the textbook, lecture notes and any materials from the course website.
2. Using Google or any other online resources is not always a reliable source.
3. You are allowed to use Theorems, lemmas, etc from the book or from class as part of your solutions, and you are not required to reprove these during the exam.
4. Do not discuss the problems or their solutions with your classmates.
5. You can always ask me (the instructor) if you have clarifying questions, but asking for hints or asking if a proof is correct is not allowed.

A typical exam will have a few questions from each week of the course and will cover **assignments 1-4**. You can expect problems about following:

- **TBA**

Hints

1.5. What degrees are possible in such a graph G? Feel free to use the pigeonhole principle in your justification, which is Theorem 2.1 from our book.

1.6. Remember that you are allowed to used theorems we proved in class and theorems from the book to help you with homework problems.

1.7. Consider the vertex of largest degree, and argue that it has degree $n-1$.

2.2. For each such n and i, give an example of a graph with order n and connectivity i. First try $i = n-2$, then $i = n-3$, and so on. Final hint: start with a disconnected graph, then add as many vertices and edges as possible.

3.3. First prove that K_3 is not a subgraph of $K_{3,3}$. Then see if you can generalize your proof.

3.5. Proceed by contradiction. Suppose that v is a cut vertex, and then count the number of edges in each connected component and in each part of the bipartition, to get a contradiction. (Do some examples to help figure out the argument.)

4.2. You can use the theorem that the ij entry of A^k is the number of walks of length k from v_i to v_i.

4.4. There are 11 of them. Careful not to draw the same one twice in a different way! You don't need to give a formal proof that your set is complete; just draw 11 truly different trees of order 7. Make sure to draw **clearly**; unclear graphs will be marked wrong.

Also, it is helpful to think about the possible degree sequences. (There might be multiple trees with the same degree sequence, but not every degree sequence is possible for a tree.)

4.5. What theorems do we know about when a graph is bipartite?

6.3. Show that the graph satisfies the hypothesis of Ore's theorem (Theorem 1.23 in our book)

6.4. This one is weird: if the graph isn't hamiltonian, consider the **smallest** cycle; use the hypothesis of the problem to produce a smaller cycle unless the smallest cycle is a triangle; then, if the smallest cycle is a triangle, argue that the graph is a complete graph.

7.7. Use the result of the previous problem.

8.5. Use the theorem that "$\chi(G) \leq r$ iff G is r-partite"

8.6. Use the theorem that "$\chi(G) \leq r$ iff G is r-partite"

8.7. Hint will appear later.

8.8. Use the theorem that "$\chi(G) \leq r$ iff G is r-partite". For a second hint, review the proof that $\chi(G) \cdot \chi(\overline{G}) \geq n$ from class. Third hint: prove first that if $V(G)$ has a partition $\{V_1, \ldots, V_k\}$ such that, for each $1 \leq i < j \leq k$, there exists an $x \in V_i$ and a $y \in V_j$ which are non-adjacent, then $\chi(G) \leq n - k + 1$.