

Diophantine and tropical geometry

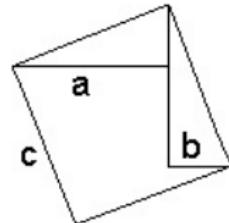
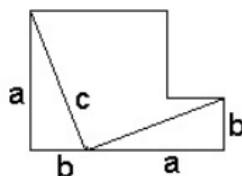
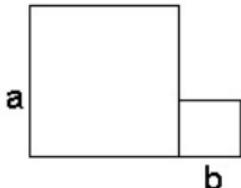
David Zureick-Brown

joint with Eric Katz (Waterloo) and Joe Rabinoff (Georgia Tech)

Slides available at <http://www.mathcs.emory.edu/~dzb/slides/>

University of Colorado-Boulder Colloquium
March 10, 2015

$$a^2 + b^2 = c^2$$



Basic Problem (Solving Diophantine Equations)

Analysis

Let $f_1, \dots, f_m \in \mathbb{Z}[x_1, \dots, x_n]$ be polynomials.

Let R be a ring (e.g., $R = \mathbb{Z}, \mathbb{Q}$).

Problem

Describe the set

$$\{(a_1, \dots, a_n) \in R^n : \forall i, f_i(a_1, \dots, a_n) = 0\}.$$

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Fact

Solving diophantine equations is hard.

Hilbert's Tenth Problem

The ring $R = \mathbb{Z}$ is especially hard.

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Theorem (Davis-Putnam-Robinson 1961, Matijasevič 1970)

There does not exist an algorithm solving the following problem:

input: $f_1, \dots, f_m \in \mathbb{Z}[x_1, \dots, x_n]$;

output: YES / NO according to whether the set

$$\{(a_1, \dots, a_n) \in \mathbb{Z}^n : \forall i, f_i(a_1, \dots, a_n) = 0\}$$

is non-empty.

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This is *still open* for many other rings (e.g., $R = \mathbb{Q}$).

Fermat's Last Theorem

Theorem (Wiles et. al)

The only solutions to the equation

$$x^n + y^n = z^n, n \geq 3$$

are multiples of the triples

$$(0, 0, 0), \quad (\pm 1, \mp 1, 0), \quad \pm(1, 0, 1), \quad (0, \pm 1, \pm 1).$$



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Qualitative:

- Does there **exist** a solution?
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- How can we explicitly **find** all solutions? (With proof?)

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Implicit question

- Why do equations **have** (or fail to have) solutions?
- Why do some have **many** and some have **none**?
- What **underlying mathematical structures** control this?

Example: Pythagorean triples

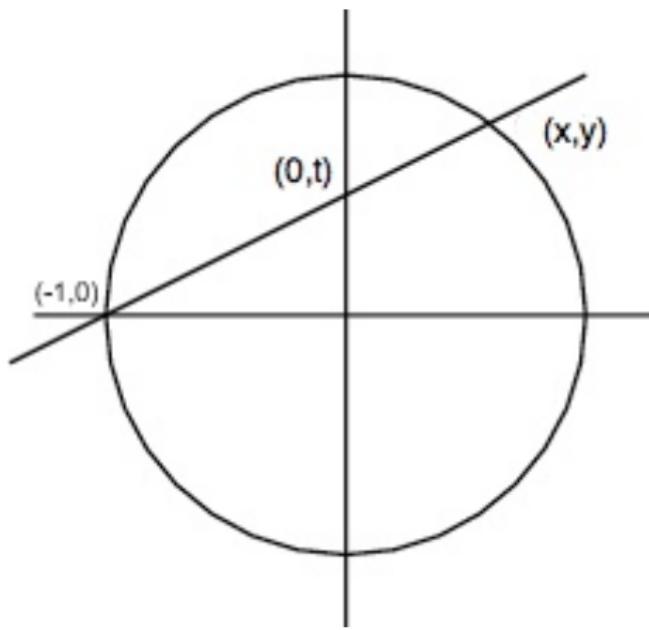
Lemma

The equation

$$x^2 + y^2 = z^2$$

has infinitely many non-zero coprime solutions.

Pythagorean triples



$$\text{Slope} = t = \frac{y}{x+1}$$

$$x = \frac{1-t^2}{1+t^2}$$

$$y = \frac{2t}{1+t^2}$$

Pythagorean triples

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The solutions to

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The Mordell Conjecture

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For $n \geq 5$, the equation

$$y^2 = f(x)$$

has only finitely many solutions if $f(x)$ is squarefree, with degree > 4 .

Fermat Curves

Question

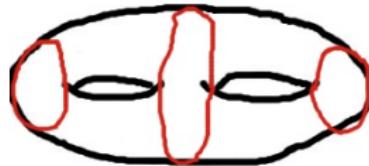
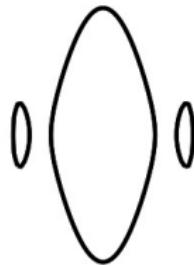
Why is Fermat's last theorem believable?

- ① $x^n + y^n - z^n = 0$ looks like a surface (3 variables)
- ② $x^n + y^n - 1 = 0$ looks like a curve (2 variables)

Mordell Conjecture

Example

$$y^2 = (x^2 - 1)(x^2 - 2)(x^2 - 3)$$



This is a cross section of a two holed torus. The **genus** is the number of holes.

Conjecture (Mordell)

A curve of genus $g \geq 2$ has only finitely many rational solutions.

Fermat Curves

Question

Why is Fermat's last theorem believable?

- ① $x^n + y^n - 1 = 0$ is a curve of genus $(n - 1)(n - 2)/2$.
- ② Mordell implies that for **fixed** $n > 3$, the n th Fermat equation has only finitely many solutions.

Fermat Curves

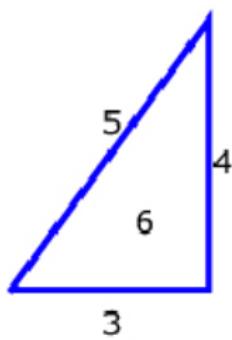
Question

What if $n = 3$?

- ① $x^3 + y^3 - 1 = 0$ is a curve of genus $(3 - 1)(3 - 2)/2 = 1$.
- ② We were lucky; $Ax^3 + By^3 = Cz^3$ can have infinitely many solutions.

Congruent number problem

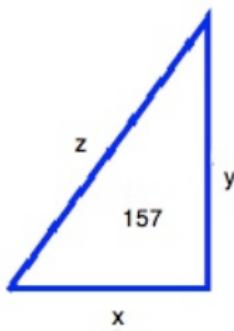
$$x^2 + y^2 = z^2, xy = 2 \cdot 6$$



$$3^2 + 4^2 = 5^2, \quad 3 \cdot 4 = 2 \cdot 6$$

Congruent number problem

$$x^2 + y^2 = z^2, xy = 2 \cdot 157$$



Congruent number problem

The pair of equations

$$x^2 + y^2 = z^2, \quad xy = 2 \cdot 157$$

has **infinitely many** solutions. **How large** is the smallest solution? How many **digits** does the smallest solution have?

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$$x = \frac{157841 \cdot 4947203 \cdot 52677109576}{2 \cdot 3^2 \cdot 5 \cdot 13 \cdot 17 \cdot 37 \cdot 101 \cdot 17401 \cdot 46997 \cdot 356441}$$

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“Next” solution has **176 digits**!

Back of the envelope calculation

$$x^2 + y^2 = z^2, xy = 2 \cdot 157$$

- Num, den(x, y, z) $\leq 10 \sim 10^6$ many, **1 min** on Emory's computers.

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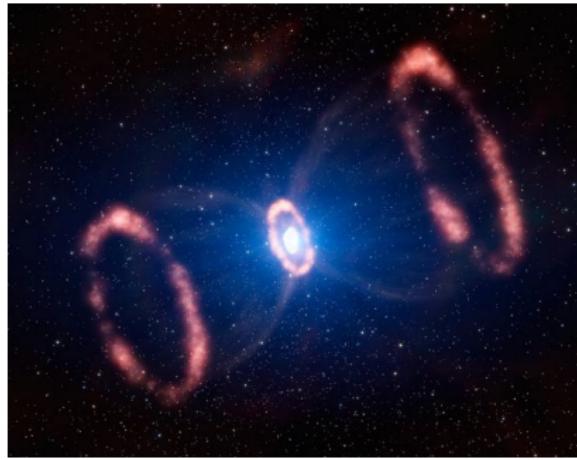
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- 10^9 many computers in the world – so **10^{243} years**
- Expected time of 'heat death' of universe – **10^{100} years**.



Fermat Surfaces

Conjecture

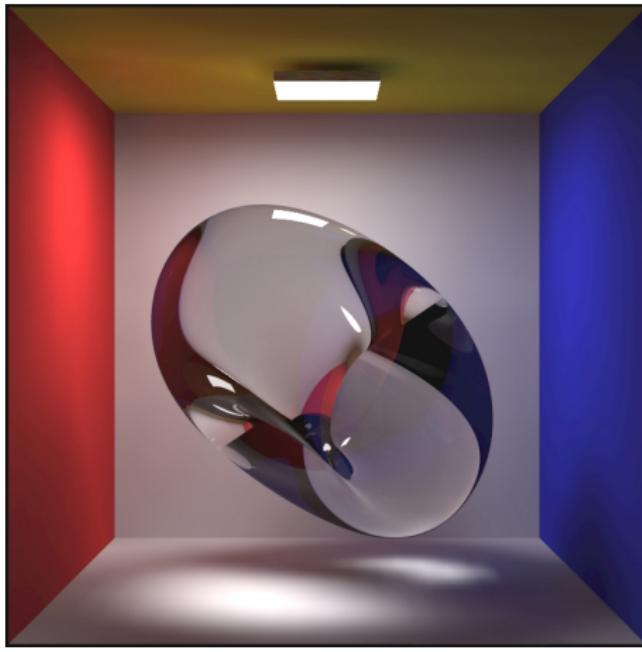
The only solutions to the equation

$$x^n + y^n = z^n + w^n, n \geq 5$$

satisfy $xyzw = 0$ or lie on the lines ‘lines’ $x = \pm y$, $z = \pm w$ (and permutations).

The Swinnerton-Dyer K3 surface

$$x^4 + 2y^4 = 1 + 4z^4$$



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- Two ‘obvious’ solutions – $(\pm 1 : 0 : 0)$.
- The next smallest solutions are $(\pm \frac{1484801}{1169407}, \pm \frac{1203120}{1169407}, \pm \frac{1157520}{1169407})$.

Problem

Find another solution.

Remark

- ➊ **10^{16} years** to find via brute force.
- ➋ Age of the universe – **$13.75 \pm .11$ billion years** (roughly 10^{10}).

Fermat-like equations

Theorem (Poonen, Schaefer, Stoll)

The coprime integer solutions to $x^2 + y^3 = z^7$ are the 16 triples

$$(\pm 1, -1, 0), \quad (\pm 1, 0, 1), \quad \pm(0, 1, 1),$$

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Generalized Fermat Equations

Problem

What are the solutions to the equation $x^a + y^b = z^c$?

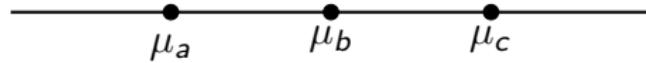
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Theorem (Darmon and Granville)

Fix $a, b, c \geq 2$. Then the equation $x^a + y^b = z^c$ has only finitely many coprime integer solutions iff $\chi = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1 \leq 0$.



Known Solutions to $x^a + y^b = z^c$

The ‘known’ solutions with

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$$

are the following:

$$1^p + 2^3 = 3^2$$

$$2^5 + 7^2 = 3^4, 7^3 + 13^2 = 2^9, 2^7 + 17^3 = 71^2, 3^5 + 11^4 = 122^2$$

$$17^7 + 76271^3 = 21063928^2, 1414^3 + 2213459^2 = 65^7$$

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Problem (Beal’s conjecture)

These are all solutions with $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1 < 0$.

Generalized Fermat Equations – Known Solutions

Conjecture (Beal, Granville, Tijdeman-Zagier)

This is a complete list of coprime non-zero solutions such that

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...or even for a counterexample.

Examples of Generalized Fermat Equations

Theorem (Poonen, Schaefer, Stoll)

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$$\frac{1}{2} + \frac{1}{3} + \frac{1}{7} - 1 = -\frac{1}{42} < 0$$

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$$\frac{1}{2} + \frac{1}{3} + \frac{1}{6} - 1 = 0$$

Examples of Generalized Fermat Equations

Theorem (Darmon, Merel)

Any pairwise coprime solution to the equation

$$x^n + y^n = z^2, n > 4$$

satisfies $xyz = 0$.

$$\frac{1}{n} + \frac{1}{n} + \frac{1}{2} - 1 = \frac{2}{n} - \frac{1}{2} < 0$$

Other applications of the modular method

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Theorem (Bugeaud, Mignotte, Siksek 2006)

The only Fibonacci numbers that are perfect powers are

$$F_0 = 0, F_1 = F_2 = 1, F_6 = 8, F_{12} = 144.$$

Examples of Generalized Fermat Equations

Theorem (Klein, Zagier, Beukers, Edwards, others)

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$$(T/2)^2 + H^3 + (f/12^3)^5$$

- ① $f = st(t^{10} - 11t^5s^5 - s^{10})$,
- ② $H = \text{Hessian of } f$,
- ③ $T = \text{a degree 3 covariant of the dodecahedron}$.

(p, q, r) such that $\chi < 0$ and the solutions to $x^p + y^q = z^r$ have been determined.

$\{n, n, n\}$	Wiles, Taylor-Wiles, building on work of many others
$\{2, n, n\}$	Darmon-Merel, others for small n
$\{3, n, n\}$	Darmon-Merel, others for small n
$\{5, 2n, 2n\}$	Bennett
$(2, 4, n)$	Ellenberg, Bruin, Ghioca $n \geq 4$
$(2, n, 4)$	Bennett-Skinner; $n \geq 4$
$\{2, 3, n\}$	Poonen-Shaefer-Stoll, Bruin. $6 \leq n \leq 9$
$\{2, 2\ell, 3\}$	Chen, Dahmen, Siksek; primes $7 < \ell < 1000$ with $\ell \neq 31$
$\{3, 3, n\}$	Bruin; $n = 4, 5$
$\{3, 3, \ell\}$	Kraus; primes $17 \leq \ell \leq 10000$
$(2, 2n, 5)$	Chen $n \geq 3^*$
$(4, 2n, 3)$	Bennett-Chen $n \geq 3$
$(6, 2n, 2)$	Bennett-Chen $n \geq 3$
$(2, 6, n)$	Bennett-Chen $n \geq 3$

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$(2, 4, n)$	Ellenberg, Bruin, Ghioca $n \geq 4$
$(2, n, 4)$	Bennett-Skinner; $n \geq 4$
$\{2, 3, n\}$	Poonen-Shaefer-Stoll, Bruin. $6 \leq n \leq 9$
$\{2, 2\ell, 3\}$	Chen, Dahmen, Siksek; primes $7 < \ell < 1000$ with $\ell \neq 31$
$\{3, 3, n\}$	Bruin; $n = 4, 5$
$\{3, 3, \ell\}$	Kraus; primes $17 \leq \ell \leq 10000$
$(2, 2n, 5)$	Chen $n \geq 3^*$
$(4, 2n, 3)$	Bennett-Chen $n \geq 3$
$(6, 2n, 2)$	Bennett-Chen $n \geq 3$
$(2, 6, n)$	Bennett-Chen $n \geq 3$
$(2, 3, 10)$	ZB

Faltings' theorem / Mordell's conjecture

Theorem (Faltings, Vojta, Bombieri)

Let X be a smooth curve over \mathbb{Q} with genus at least 2. Then $X(\mathbb{Q})$ is finite.

Example

For $g \geq 2$, $y^2 = x^{2g+1} + 1$ has only finitely many solutions with $x, y \in \mathbb{Q}$.

Uniformity

Problem

- ① Given X , compute $X(\mathbb{Q})$ exactly.
- ② Compute bounds on $\#X(\mathbb{Q})$.

Conjecture (Uniformity)

There exists a constant $N(g)$ such that every smooth curve of genus g over \mathbb{Q} has at most $N(g)$ rational points.

Theorem (Caporaso, Harris, Mazur)

Lang's conjecture \Rightarrow uniformity.

Uniformity numerics

g	2	3	4	5	10	45	g
$B_g(\mathbb{Q})$	642	112	126	132	192	781	$16(g + 1)$

Remark

Elkies studied K3 surfaces of the form

$$y^2 = S(t, u, v)$$

with lots of rational lines, such that S restricted to such a line is a perfect square.

Coleman's bound

Theorem (Coleman)

Let X be a curve of genus g and let $r = \text{rank}_{\mathbb{Z}} \text{Jac}_X(\mathbb{Q})$. Suppose $p > 2g$ is a prime of **good reduction**. Suppose $r < g$. Then

$$\#X(\mathbb{Q}) \leq \#X(\mathbb{F}_p) + 2g - 2.$$

Remark

- ① A modified statement holds for $p \leq 2g$ or for $K \neq \mathbb{Q}$.
- ② Note: **this does not prove uniformity** (since the first good p might be large).

Tools

p -adic integration and Riemann–Roch

Chabauty's method

(**p -adic integration**) There exists $V \subset H^0(X_{\mathbb{Q}_p}, \Omega_X^1)$ with $\dim_{\mathbb{Q}_p} V \geq g - r$ such that,

$$\int_P^Q \omega = 0 \quad \forall P, Q \in X(\mathbb{Q}), \omega \in V$$

(**Coleman, via Newton Polygons**) Number of zeroes in a residue disc D_P is $\leq 1 + n_P$, where $n_P = \#(\text{div } \omega \cap D_P)$

(**Riemann-Roch**) $\sum n_P = 2g - 2$.

(**Coleman's bound**) $\sum_{P \in X(\mathbb{F}_p)} (1 + n_P) = \#X(\mathbb{F}_p) + 2g - 2$.

Example (from McCallum-Poonen's survey paper)

Example

$$X: y^2 = x^6 + 8x^5 + 22x^4 + 22x^3 + 5x^2 + 6x + 1$$

- ① Points reducing to $\tilde{Q} = (0, 1)$ are given by

$$x = p \cdot t, \text{ where } t \in \mathbb{Z}_p$$

$$y = \sqrt{x^6 + 8x^5 + 22x^4 + 22x^3 + 5x^2 + 6x + 1} = 1 + x^2 + \dots$$

- ② $\int_{(0,1)}^{P_t} \frac{xdx}{y} = \int_0^t (x - x^3 + \dots) dx$

Chabauty's method

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(**Coleman's bound**) $\sum_{P \in X(\mathbb{F}_p)} (1 + n_P) = \#X(\mathbb{F}_p) + 2g - 2$.

Stoll's hyperelliptic uniformity theorem

Theorem (Stoll)

Let X be a *hyperelliptic* curve of genus g and let $r = \text{rank}_{\mathbb{Z}} \text{Jac}_X(\mathbb{Q})$. Suppose $r < g - 2$.

Then

$$\#X(\mathbb{Q}) \leq 8(r + 4)(g - 1) + \max\{1, 4r\} \cdot g$$

Tools

p -adic integration on *annuli*

comparison of different analytic continuations of p -adic integration

Main Theorem (partial uniformity for curves)

Theorem (Katz, Rabinoff, ZB)

Let X be *any* curve of genus g and let $r = \text{rank}_{\mathbb{Z}} \text{Jac}_X(\mathbb{Q})$. Suppose $r \leq g - 2$. Then

$$\#X(\mathbb{Q}) \leq C(g, r)$$

Tools

p-adic integration on annuli

comparison of different analytic continuations of *p*-adic integration

Non-Archimedean (Berkovich) structure of a curve [BPR]

Combinatorial restraints coming from the Tropical canonical bundle

Main Theorem (partial uniformity for curves)

Theorem (Katz, Rabinoff, ZB)

Let X be *any* curve of genus g and let $r = \text{rank}_{\mathbb{Z}} \text{Jac}_X(\mathbb{Q})$. Suppose $r \leq g - 2$. Then

$$\#X(\mathbb{Q}) \leq (3g)2^{(2g-2)!}(3g-3)!7^{2g^2+g+1} \exp(2g-2)$$

Tools

p-adic integration on annuli

comparison of different analytic continuations of *p*-adic integration

Non-Archimedean (Berkovich) structure of a curve [BPR]

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Comments

Corollary ((Partially) effective Manin-Mumford)

There is an effective constant $N(g)$ such that if $g(X) = g$, then

$$\#(X \cap \text{Jac}_{X,\text{tors}})(\mathbb{Q}) \leq N(g)$$

Corollary

*There is an effective constant $N'(g)$ such that if $g(X) = g > 3$ and X/\mathbb{Q} has **totally degenerate, trivalent** reduction mod 2, then*

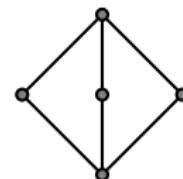
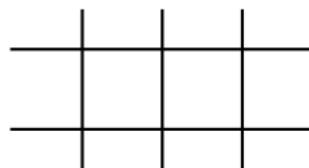
$$\#(X \cap \text{Jac}_{X,\text{tors}})(\mathbb{C}) \leq N'(g)$$

The second corollary is a big improvement

- ① It requires working over a **non-discretely valued** field.
- ② The bound **only depends on the reduction type**.
- ③ Integration over **wide opens** (c.f. Coleman) instead of discs and annuli.

Baker-Payne-Rabinoff and the slope formula

(Dual graph Γ of $X_{\mathbb{F}_p}$)



(Contraction Theorem) $\tau: X^{\text{an}} \rightarrow \Gamma$.

(Combinatorial harmonic analysis/potential theory)

f a meromorphic function on X^{an}

$F := (-\log |f|) \big|_{\Gamma}$ associated tropical, piecewise linear function

$\text{div } F$ combinatorial record of the slopes of F

(Slope formula) $\tau_* \text{div } f = \text{div } F$