Math 375: Representation Theory
Instructor: David Zureick-Brown ("DZB")

# All assignments

Last updated: April 23, 2025

Gradescope code: J7PV4B

**Show all work for full credit!**
*Proofs should be written in full sentences whenever possible.*

## Gradescope Instructions for submitting work in Math 375

You will be using the online Gradescope progam to submit your homework and exams. These instructions tell you how to sign up initially, and how to submit your written work.

### Signing up for Gradescope the first time.

If you haven't used Gradescope for an **Amherst College** course before:

- Go to `http://www.gradescope.com`, click on "Sign up for free" (which may auto-scroll you to the bottom of the page), and select Sign up as [a] "Student".
- In the signup box:
  - Use the course entry code **J7PV4B**
  - Use your full name
  - Use your **Amherst College email** address. Or, if you are a Five-College student, use your email address from your own school.
  - Leave the "Student ID" entry blank.
- You will probably get an email asking to set a password for your account, so check your `amherst.edu` email inbox. (Or your email inbox through your own school, for Five-College students.)

### Adding Math 375 to Gradescope.

If you **have** used Gradescope for an Amherst course before, and so you already have an account through your `amherst.edu` email, you still need to add Math 375, so:

- Go to `http://www.gradescope.com` and log in.
- Go to your Account Dashboard (click the Gradescope logo at upper left), and click "Add Course" at bottom right.
- Use the course code **J7PV4B**

(submission instructions on next page)

## Submitting written work

First write it out on paper as you would normally. Then **scan it** to create a PDF. One method for scanning is the smartphone app **DropBox**. It makes nice clear scans, and it saves them directly into a folder so that you can have all your assignments in one place. **CamScanner** is another free scanning App, and there are others, too. **Gradescope** now has its own scanning app. You can also use a printer/scanner if you prefer.

**Please be kind to our dear graders and make sure your submission is legible !**

*In particular, please leave some spacing between separate problems.*

If you have a tablet computer, you may write your work there (instead of on paper) and save it as a PDF.

Some of you may know the math formatting package LaTeX and may want to use it in Math 375. That's fine, too; if so, you may write up your work in LaTeX and save the resulting PDF.

In short, any method is fine as long as it creates a legible **PDF** file and **NOT a photo**.

For example, if you use the DropBox app, then in your created *Math 375 Homework* Dropbox folder, you can select create (+) at the bottom of the screen and click the *Scan Document* option. Snap a shot of the first page of your homework, and then click [+] to snap shots of any subsequent pages. Do **not** use the *Take Photo* option.

After you have scanned/saved your work as a PDF, submit it on Gradescope as follows:

- Go to `http://www.gradescope.com` and log in.
- Select the course "Math 375, Spring 2025" and the appropriate assignment.
- Select "submit pdf" to submit your work in PDF format. Browse to find your PDF and upload.
- Now it is time to **tag** your problems. This is an **important step** , where you are telling Gradescope which problems are on which page(s).

  For each problem, select the pages of your submission where your written solution appears.

  I think the easiest thing to do is to click on the page of **your** homework upload where you wrote the given problem, and then click on the assigned problem listed. Repeat for each problem.

  **You must tag the problems or else you will not get credit for your work.**

  Gradescope will give you a warning when you go to submit your assignment if you have not selected the pages correctly. If you tag a problem incorrectly, you can fix it by clicking "More" and "Reselect Pages".
- Click Save or Submit.

After your assignment is graded, you will be able to see your score on the written problems, along with comments, on Gradescope. You should receive an email notifying you when each homework set is graded.

Assignment 1: Introduction to the course; review of groups and linear algebra

*Due by 9:55am, eastern, on Tuesday, February 4*

- ***Suggested readings for this problem set:*** **Chapter 1**
- Syllabus: https://dmzb.github.io/teaching/2025Spring375/syllabus-math-375-S25.pdf
- Gradescope instructions (previous page)

All readings are from Robinson, *A Course in the Theory of Groups*.

***Assignment:*** due Tuesday, February 4, 9:55am, via Gradescope (J7PV4B):

1. (a) If $N \trianglelefteq G$, $H \leq G$, then $NH \leq G$. (I.e., if $N$ is a normal subgroup of $G$ and $H$ is any subgroup of $G$, then $NH$ is also a subgroup of $G$.)
   (b) Is $NH$ normal? Prove or disprove.[1]

2. Let $G$ be a group. A bijective function $\psi\colon G \to G$ is called an *automorphism* if

$$(\forall a\,,\, b \in G) \quad \psi(ab) = \psi(a)\psi(b)$$

   holds. (I.e., an automorphism is a bijective homomorphism.)

   The set of all automorphisms is a group with respect to composition; it is called the *automorphism group* of $G$ and is denoted by $\mathrm{Aut}(G)$.

   For any $g \in G$ let $c_g\colon G \to G$ be defined by $c_g(x) := gxg^{-1}$, and let $\mathrm{Inn}(G)= \{c_g \mid g \in G\}$.[2]

   Let $\phi_G\colon G \to \mathrm{Aut}\,G$ be the map given by $g \mapsto c_g$.[3].

   (a) Prove that $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.
   (b) Give an example of a group $G$ such that $\mathrm{Inn}(G) \neq \mathrm{Aut}(G)$. (No proof necessary.)
   (c) Give an example of a nontrivial group $G$ such that $\mathrm{Inn}(G) = \mathrm{Aut}(G)$. (No proof necessary.)
   (d) Describe the kernel and image of $\phi_G$. What can you deduce about this from the first isomorphism theorem?

3. (a) What is the center $Z(S_n)$ of the symmetric group? Prove that your answer is correct.
   (b) Let $c = (1, 2)(3, 4) \in S_n$, $n > 4$. Determine $|\mathrm{C}_G(c)|$.
   (c) Let $d = (1, 2, \ldots, n) \in S_n$. Determine $|\mathrm{C}_G(d)|$.

[1] For a disproof, give a counterexample
[2] Verify on your own that $c_g$ is an automorphism (i.e., don't submit a proof that $c_g$ is an automorphism).
[3] Verify on your own that $\phi_G$ is a homomorphism

Assignment 2: Simple groups

***Suggested readings for this problem set:*** **Chapter 1; Chapter 3**
All readings are from Robinson, *A Course in the Theory of Groups*.

***Assignment:*** due Tuesday, February 11, 9:55am, via Gradescope (J7PV4B):

1. Let $G$ be a group.

   (a) Show that $C_G(Z(G)) = G$ and $N_G(Z(G)) = G$.

   (b) Let $H \subset G$ be a subgroup. Suppose that $H$ has cardinality 2. Prove that $N_G(H) = C_G(H)$. Deduce that if $H$ is normal, then it is also central. (Central means that $H$ is a subgroup of $Z(G)$.)

   (c) Let $N$ be a proper normal subgroup of $G$. Show that $G/N$ is simple if and only if $N$ is a maximal normal subgroup of $G$.

   (d) Assume that $G/\mathrm{Z}(G)$ is cyclic. Prove that $G$ is abelian.

   (Click here for a hint.)

2. Let $G$ be the set of upper triangular $3 \times 3$ matrices over the field $\mathbb{F}_3$, whose diagonal elements are 1.

$$G := \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \text{ s.t. } a, b, c \in \mathbb{F}_3 \right\}$$

   Verify on your own that $G$ is a group with respect to matrix multiplication.

   (a) Verify that every nonidentity element of $G$ is of order 3.

   (b) Calculate the center of the group $G$.

   (c) What is the Jordan–Holder series of $G$?

3. Let $G$ be a group. The subgroup $D = \{(g, g) \mid g \in G\}$ of $G \times G$ is called the *diagonal subgroup* of $G$.

   (a) Prove that $D$ is a normal subgroup in $G \times G$ if and only if $G$ is abelian

   (b) A subgroup $M$ of a group $H$ is called a *maximal subgroup* if $M \subsetneqq G$ and there is no subgroup $K$ satisfying $M \subsetneqq K \subsetneqq H$. Prove that $D$ is a maximal subgroup of $G \times G$ if and only if $G$ is a simple group, i.e. it has no proper normal subgroup.

Assignment 3: Centrality; Group actions

***Suggested readings for this problem set:*** **Chapter 3; Chapter 1, Section 1.6**
All readings are from Robinson, *A Course in the Theory of Groups*.

***Assignment:*** due Tuesday, February 18, 9:55am, via Gradescope (J7PV4B):

**Disclaimer:** we'll talk a bit more about stabilizers etc. on Thursday. If there is another snow day, then I'll extend the due date of this assignment. (But in the meantime, feel free to look up the definitions of the terms in the second problem.)

1. Let $F$ be a field. Compute (with proof) the center of $\mathrm{GL}_3(F)$.

2. Consider the following group actions.[4] [5] [6]

   (a) $\mathrm{GL}_n(F)$ acts on $F^n$ by multiplication (i.e., for $M \in \mathrm{GL}_n(F)$ and $\overline{v} \in F^n$, $M$ acts on $\overline{v}$ by $M\overline{v}$).

   What is the kernel of this action? For $\overline{v} \in F^n$, what is the stabilizer of $\overline{v}$? What is the orbit of $\overline{v}$?

   (b) $\mathrm{GL}_n(F)$ acting on $M_n(F)$ by conjugation (i.e., for $A \in \mathrm{GL}_n(F)$ and $M \in M_n(F)$, $A$ acts on $M$ by $AMA^{-1}$).

   What is the kernel of this action? For $M \in M_n(F)$, what is the stabilizer of $M$? (I.e., what other concept from this course is the stabilizer equal to?) If $M$ is diagonalizable, what is the orbit of $M$?[7]

   (c) A group $G$ acting on itself by left multiplication.

   What is the kernel of this action? For $g \in G$, what is the stabilizer of $g$?

   (d) The symmetric group $S_n$ acting on a polynomial ring $F[x_1, \dots, x_n]$ by permuting the variables. (I.e., for $\sigma \in S_n$, $\sigma(x_i) = x_{\sigma(i)}$.).

   What is the kernel of this action? Is this action transitive? What is the orbit of $x_1$ under this action? What is the orbit of $x_i x_j$ under this action?

[4] Verify for yourself that these are group actions and make sure you know which ones are left vs right actions (no need to prove this).

[5] It is also ok in these problems to assume that $F$ is $\mathbb{R}$ if you are more comfortable with that; it won't really change the answers to this problem, but in the future it is helpful to become more comfortable with $F = \mathbb{C}$ or a finite field $F = \mathbb{F}_p$.

[6] Several of these subquestions might require cases, i.e., might have different answers depending on what e.g. $\overline{v}$ is.

[7] If you know "advanced" linear algebra, also consider what the orbit of a non-diagonalizable matrix is; but don't write that part up.

Assignment 4: More group actions; Orbit–Stabilizer and applications; Class equation

*Due by 9:55am, eastern, on Tuesday, February 25*

***Suggested readings for this problem set:*** **Chapter 1, Section 1.6**
All readings are from Robinson, *A Course in the Theory of Groups*.

***Assignment:*** due Tuesday, February 25, 9:55am, via Gradescope (J7PV4B):

1. Suppose that $G$ acts on a set $S$. We say that this action is **$k$-transitive** if for every distinct $x_1, \ldots, x_k \in X$ and distinct $y_1, \ldots, y_k \in X$, there exists $g \in G$ such that $gx_i = y_i$. (Note that while $x_i \neq x_j$ and $y_i \neq y_j$ for $i \neq j$, the $x$'s and $y$'s are not necessarily distinct from each other; we could have $x_i = y_i$.)

   (a) Show that the action of $S_n$ on $X = \{1, \ldots, n\}$ is $n$-transitive. Is there a proper subgroup $H$ of $S_n$ which acts $n$-transitively on $X$?

   (b) ~~Show that the action of $\mathrm{GL}_n(F)$ on $X = F^n - \{\bar{0}\}$ is $n$-transitive. Is it $n+1$-transitive? Is there a proper subgroup $H$ of $\mathrm{GL}_n(F)$ which acts $n$-transitively on $X$?~~

   The statement of this problem was incorrect. Giving the timing, I am just removing this problem from the problem set.

   (c) Suppose that $G$ acts on a finite set $X$ of size $n$. Prove that if this action is $k$ transitive, then

   $$n(n-1)\ldots(n-k+1) \mid |G|.$$

2. The following problems all have short solutions.

   (a) Find (with proof) all finite groups which have exactly two conjugacy classes.

   (b) Let $p$ and $q$ be distinct primes. Assume $G$ is a non-abelian group of order $pq$. Prove that $|Z(G)| = 1$

   (c) Let $G$ be a non-abelian group of order $p^3$. Show that $|Z(G)| = p$.

   (d) Let $G$ be a group of order $p^a$. Prove that for every $0 \leq b \leq a$, there exists a subgroup $H$ of $G$ such that $|H| = p^b$.

   (e) If the center of $G$ is of index $n$, prove that every conjugacy class has at most $n$ elements.

   (Click here for a hint)

3. Let $G$ be a finite group and let $H$ be a subgroup. Suppose that $H$ intersects every conjugacy class of $G$. (I.e., suppose that for every $g_1 \in G$, there exists $g_2 \in G$ such that $g_2 g_1 g_2^{-1} \in H$.)

   Prove that $H = G$.

Assignment 5: Sylow Theorems

*Due by 9:55am, eastern, on Tuesday, March 4*

***Suggested readings for this problem set:*** **Chapter 1, Section 1.6**
All readings are from Robinson, *A Course in the Theory of Groups*.

***Assignment:*** due Tuesday, March 4, 9:55am, via Gradescope (J7PV4B):

(Problems that are not about the Sylow Theorems.)

1. Let $G$ be a finite group and $H$ and $K$ are subgroups of $G$. Assume $|G : H|$ and $|G : K|$ are coprime. Prove that $HK = G$

   (Click here for a hint)

2. (a) Let $p$ be the smallest prime divisor of $|G|$. Suppose that $H$ is a subgroup of $G$ of index $p$. Prove that $H$ is normal.
   (b) Let $G$ be a group of order $4k + 2$, $k \geq 1$. Prove that $G$ is not simple.

   (Click here for a hint)

3. (Sylow Theorem problems).
   (a) Let $G$ be a group of order $3^2 \cdot 5$. Prove that $G$ is not simple.
   (b) Let $G$ be a group of order $3 \cdot 5 \cdot 7$. Prove that $G$ is not simple.
   (c) Let $G$ be a group of order $2 \cdot 3 \cdot 5^2$. Prove that $G$ is not simple.
   (d) Let $p < q$ be primes. Let $G$ be a group of order $pq$.
      i. Prove that $G$ is not simple using the Sylow Theorems.
      ii. Prove that $G$ is not simple without using the Sylow Theorems, and instead using Cauchy's theorem (1.6.17) and problem 1(a) from this assignment. (Note that this second proof also works for groups of order $pq^i$.)

      (Click here for a hint)

   Bonus (0 points, do not turn in):
   (a) If $G = pq$, are all subgroups of $G$ normal?
   (b) Think of some examples of $|G|$ where your arguments for parts (abcd) do not work.

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Midterm study guide

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*In person oral exam, Tuesday March 4 and Thursday, March 6.*

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**Please go to this link to sign up for a 15 minute slot for your exam.**

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- This will be a 10-15 minute oral exam with no notes.
- There will be no class during exam week.
- The exam will be in our usual classroom
- The intent is that everyone will get an A.
- I will ask you to do one problem at the board from a list of problems that I give you ahead of time.
- The list of problems will mostly be problems from homework, class, or some theorems and propositions.
- As you explain your solution, I'll ask some questions (e.g., "what is the definition of a stabilizer?", or "what is an example of a group action?")
- The only thing you need to do to prepare is to keep up with the course (i.e., do the homework every week, and make sure that you understand the content being presented in class).
- The exam will cover all of the material leading up to the exam date, with the exception of the most recent lecture.
- Here is a list of questions for the exam. Pick one of these to prepare ahead of time I'll ask you to present a solution at the board:
  - The Orbit–Stabilizer,
  - Cauchy's theorem that if $p \mid |G|$, then $G$ has an element of order $p$,
  - Groups of order $p^n$ are not simple unless $n \leq 1$
  - The Sylow Theorems.
  - Groups of order $pq$ are not simple.
- Your solution does not need to match my solution from class.
- If we run out of time and don't finish the proof(e.g., if you want to prove the Sylow theorems) that's fine!

~~The week before the exam, I will post a sign up sheet for 15 minute timeslots.~~ It is posted now (see above).

Assignment 6: Semidirect products

*Due by 9:55am, eastern, on Tuesday, March 25*

***Suggested readings for this problem set:*** **Chapter 8**
All readings are from Robinson, *A Course in the Theory of Groups*.

***Assignment:*** due Tuesday, March 25, 9:55am, via Gradescope (J7PV4B):

1. (Examples of Semidirect prodcts)
   (a) Prove that $S_n$ is isomorphic to $A_n$ semidirect $\mathbb{Z}/2\mathbb{Z}$.
   (b) Let $F$ be a field and prove that $\mathrm{GL}_n(F)$ is isomorphic to $\mathrm{SL}_n(F)$ semidirect $F^*$.
   (c) Give an example of a group that is not a semidirect product.
   (d) Give a second example that is fundamentally different than your first example.
2. (Recognition of semidirect products)
   (a) Prove that every group of order $pq$ is a semidirect product of two cyclic groups.
   (b) Prove that every group of order $pq$ is abelian if and only if $p$ does not divide $q - 1$.
   (c) Classify all groups of order 28. (There are two abelian, and two that are semidirect products. Find them all, and prove that any group of order 28 is isomorphic to one of these 4 groups.)
3. (Bonus problem, do not submit)[8] A bracelet is made by sliding coloured beads to a string and tying its ends. How many different bracelets can you make with 6 red and 6 blue beads? Note that you can put the bracelet into your wrist in two possible ways and you can rotate it.

   (Click here for a hint)

[8] I cut this problem from an earlier assignment. I decided not to assign it later, but it is a nice application of group actions.

Assignment 7: Linear representations

*Due by 9:55am, eastern, on Tuesday, April 1*

***Suggested readings for this problem set:*** **Chapter 8**
All readings are from Robinson, *A Course in the Theory of Groups*.

***Assignment:*** due Tuesday, April 1, 9:55am, via Gradescope (J7PV4B):

1. Let $n \geq 3$[9] and let $D_{2n}$ be the dihedral group. This has the presentation $\langle r, s \mid r^n = s^2 = e, srs = r^{-1} \rangle$. Give an example of an irreducible and faithful 2 dimensional real representation, and prove that it is irreducible and faithful. Give 2 examples of 1 dimensional representations of $D_{2n}$ which are not isomorphic to each other.
2. Let $V$ be an irreducible representation of a group $G$. Explain why $V \times V$ is reducible.
3. Let $G$ be a finite group and let $V$ be a representation of $G$ over a field $F$. Prove that if $\dim_F V > |G|$, then $V$ is reducible.

[9] The original version of this problem was is incorrect for $n = 2$.

Assignment 8: Maschke's Theorem; Schur's Lemma;

*Due by 9:55am, eastern, on Tuesday, April 8*

***Suggested readings for this problem set:*** **Chapter 8**
All readings are from Robinson, *A Course in the Theory of Groups*.

***Assignment:*** due Tuesday, April 8, 9:55am, via Gradescope (J7PV4B):

1. Let $n \geq 2$. Describe $n$ distinct, one-dimensional complex representations of the cyclic group with $n$ elements and explain why they are pairwise non-isomorphic. Give an example of a finite cyclic group $G$ and a two dimensional irreducible real representation.

2. Let $G$ be a finite simple group. Prove that $G$ has an irreducible faithful representation.

3. Let $V$ and $W$ be a vector spaces over $F = \mathbb{R}$ (or $\mathbb{C}$) and let $(\rho, V)$ and $(\rho', W)$ be representations of $G$.

   Define the set of invariants in $V$ under $G$, denoted $V^G$, to be the set of all $v \in V$ such that $\rho(g)v = v$ for all $g \in G$.

   Recall that $\mathrm{Hom}_F(V, W)$ is the set of $F$-linear maps from $V$ to $W$. This is an $F$-vector space.

   It is also a representation of $G$ via the action $g \cdot T = g^{-1}Tg$, i.e., $g \cdot T$ is the function $\overline{v} \mapsto g^{-1}T(g\overline{v})$.

   Denote by $\mathrm{Hom}_G(V, W) \subset \mathrm{Hom}_F(V, W)$ the subspace of $G$-equivariant linear transformations (i.e., the subspace of $T$ such that for all $g \in G$ and $\overline{v} \in V$, $T(g\overline{v}) = gT(\overline{v})$). Verify for yourself that this is a subspace.

   (a) Prove that $(\rho, V^G)$ is a subrepresentation of $V$.

   (b) Let $V$ be the standard 3-dimensional representation of $S_3$. Compute $\mathrm{Hom}_{S_3}(V, V)$.

   (c) Prove that $(\mathrm{Hom}_F(V, W))^G = \mathrm{Hom}_G(V, W)$, the subspace of intertwining operators from $V$ to $W$.

   (d) If $(\rho, V)$ is an irreducible complex representation, prove that $\dim \mathrm{Hom}_G(V^{\oplus m}, V^{\oplus n}) = mn$. Note that $V^{\oplus m} = V \oplus \cdots \oplus V$ ($m$ times).

Assignment 9: Characters, Orthogonality relations

*Due by 9:55am, eastern, on Tuesday,*

***Suggested readings for this problem set:*** **Chapter 8**
All readings are from Robinson, *A Course in the Theory of Groups*.

***Assignment:*** due Tuesday, April 15, 9:55am, via Gradescope (J7PV4B):

After we prove the orthogonality relations, I recommend reading

https://mathoverflow.net/questions/2795/why-are-characters-so-well-behaved/2808#2808

1. Let $V$ be the (complex) regular representation of $S_3$. Explicitly decompose $V$ into its irreducible subrepresentations.

2. Compute the character tables of the groups $C_2 \times C_2$, $C_n$, and $A_4$.

3. (a) Suppose that a finite group $G$ has only two isomorphism classes of irreducible representations. Show that $G$ is isomorphic to $\mathbb{Z}/2$.

   (b) Prove that the character table of a finite group is an invertible matrix.

Assignment 10: Proof of orthogonality relations; new representations from old

*Due by 9:55am, eastern, on Tuesday, May 6*

***Suggested readings for this problem set:*** **Chapter 8**
All readings are from Robinson, *A Course in the Theory of Groups*.

***Assignment:*** due Tuesday, May 6, 9:55am, via Gradescope (J7PV4B):

1. Let $G$ be a finite group. Let $V = \mathbb{C}^n$ and let $\rho\colon G \to \mathrm{GL}(V)$ be a representation. Prove the following.
   (a) Prove that $\langle \chi_\rho, \chi_\rho \rangle$ is a nonzero integer.
   (b) Prove that $\rho$ is irreducible if and only if $\langle \chi_\rho, \chi_\rho \rangle = 1$.
   (c) Prove that distinct columns of the character table are orthogonal.
   (d) Let $\chi$ be a character of $G$. Show that $\overline{\chi}$ is also a character of $G$.
   (e) Let $V$ be the standard 3 dimensional irreducible representation of $S_4$. What are the irreducible factors and multiplicities of the representation $\mathrm{Hom}(V, V)$? (You do not need to give me explicit subspaces!)
2. Compute (with proof) the character table $A_5$. (For fun, also do $S_5$, and find explicitly all of the irreducible representations. But don't turn this part in.)
3. Assuming the Feit–Thompson and Burnside theorems and that $A_5$ and $\mathrm{PSL}_2(\mathbb{F}_7)$ are simple, find all non-abelian simple groups of order less than or equal to 200.

## Final exam (oral) study guide

- The **last day of class** is Tuesday, May 6.
- The exam will take place during finals week (May 9-15).
- This will be a 30 minute oral exam.
- I plan to ask a few very simple questions (e.g., "what is the definition of a representation?", or "what is an example of a character?") and will ask you to do one problem at the board from a list of problems that I give you ahead of time.
- The list of problems will mostly be problems from homework, class, or some theorems and propositions.
- The only thing you need to do to prepare is to keep up with the course (i.e., do the homework every week, and make sure that you understand the content being presented in class).
- The exam will be comprehensive. Once we are closer to the date, I will post more specific details.
- Here is a list of questions for the exam. Pick two of these to prepare ahead of time, and I'll ask you to present a solution at the board:
  - Maschke's theorem
  - Schur's lemma
  - Orthogonality of characters
  - Characters form a basis for the space of class functions.
  - Compute the character table of $A_5$.

Please go here and leave your availability for the exam.

Hints

2.1. There is no "trick" to this problem; you can do it by "writing out what everything means".

4.2. All of these problems can either be solved using

   1. just the definitions,
   2. the orbit–stabilizer theorem,
   3. the class equation,
   4. the theorem that a group of order $p^a$ has a nontrivial center, or
   5. a previous homework problem involving the center.

5.1. Try to calculate the order of $HK$. Also, you might need to use the second isomorphism theorem.

5.2. For (a), let $G$ act on $G/H$ (the cosets of $H$) and think about the map $G \to S_n$ (where $n = |G/H|$) and its kernel.
For (b), let $G$ act on $G$ by left translation and again think about the map $G \to S_n$ (where $n = |G|$). What is the intersection of $A_n$ with $G$? Can you find an element of $g \in G$ that is odd?

5.3. For (c), you really have to use every piece of information you get out of the Sylow theorems and counting elements. For example, for $P \neq P' \in \mathrm{Syl}_5$, the intersection $P \cap P'$ could have order 1 or 5. If all such intersections have order 5, and have the same five elements, what goes wrong?

6.1. Burnside's Lemma.