
MATH 220, Mathematical Reasoning and Proof
MWF 1 - 1:50

All assignments
Last updated: August 31, 2023
Gradescope code: 7DVWGG

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Assignment 1

Topics: Introduction to the course. Mathematical reasoning. Logic.

Reading: 1.1, 1.2, 1.5, 2.1, 2.2

TODO Suggested problems (do not hand in)

- With answers:

- 1.1:
- 1.2:
- 1.5:
- 2.1:
- 2.2:

- Without answers: [Handout 1](#)

Assignment 1, due Friday, Sep 15, via Gradescope TODO:

1. Add boolean algebra problem, truth table problem (for contrapositive); “which make mathematical sense”, see V for more
2. Suppose that n is an even integer, and let m be any integer. Prove that nm is even.
3. Suppose that n is an odd integer. Prove that n^2 is an odd integer. (Hint: an integer n is odd if and only if there exists an integer k such that $n = 2k + 1$.)
4. Prove that if n^2 is even, then n is even. (Hint: page 67.)
5. Write the negation of each of the following statements.
 - (a) All triangles are isosceles.
 - (b) Every door in the building was locked.
 - (c) Some even numbers are multiples of three.
 - (d) Every real number is less than 100.
 - (e) Every integer is positive or negative.
 - (f) If f is a polynomial function, then f is continuous at 0.
 - (g) If $x^2 > 0$, then $x > 0$.
 - (h) There exists a $y \in \mathbf{R}$ such that $xy = 1$.
 - (i) $(2 > 1)$ and $(\forall x, x^2 > 0)$
 - (j) $\forall \epsilon > 0, \exists \delta > 0$ such that if $|x| < \delta$, then $|f(x)| < \epsilon$.

Assignment 2

Topics: “Direct” proofs, proof by cases, and divisibility problems.

Reading TODO:

- Finish reading chapter 1.
- Section 5.3

TODO Suggested problems (do not hand in)

1. With answers: Section 5.3, #1(a), 4(a), 6(ac)
2. Without answers: Section 5.3, #2, 4 (without induction), 5 (without induction)
3. [Handout 2](#)

Assignment, due Friday, Sep 22, via Gradescope TODO:

1. Prove that if x is an integer, then $x^2 + 2$ is not divisible by 4. (Hint: there are two cases: x is even, x is odd. Also, feel free to use basic facts about even or odd, e.g., “odd + odd = even”, without additional proof.)
2. Prove that the product of three consecutive integers is divisible by 6. (It suffices to prove that it is divisible by 2 and 3 separately.)
3. Show that for all integers a and b ,
$$a^2b^2(a^2 - b^2)$$
is divisible by 12. (It suffices to prove that it is divisible by 4 and 3 separately.)
4. Find all positive integers n such that $n^2 - 1$ is prime. Prove that your answer is correct.

Assignment 3

Topics: Proof by contradiction. Unsolvability of equations. Irrationality.

Reading TODO:

- Section 1.4, p. 41-42 (stop at Historical Comments)
- Section 5.4

TODO Suggested problems (do not hand in)

1. Without answers: Section 1.4 #21
2. Without answers: Section 5.4 #6, 7, 10(a), 15, 18,
3. [Handout 3](#)

Assignment, due Friday, Sep 29, via Gradescope TODO:

1. Prove that $2^{1/3}$ is irrational.
2. Prove that there are no positive integer solutions to the equation $x^2 - y^2 = 10$.
3. Let a, b, c be integers satisfying $a^2 + b^2 = c^2$. Show that abc must be even. (Harder problem, just for fun: show that a or b must be even.)
4. Suppose that a and n are integers that are both at least 2. Prove that if $a^n - 1$ is prime, then $a = 2$ and n is a prime. (Primes of the form $2^n - 1$ are called Mersenne primes.)

Assignment 4

Topics: Induction.

Reading TODO: Section 5.2, p. 159-163

Fun Video: Vi Hart; “Doodling in Math: Spirals, Fibonacci, and Being a Plant”

<https://www.youtube.com/watch?v=ahXIMUkSXX0>

TODO Suggested problems (do not hand in)

1. With answers: Section 5.2 #1(a), 4(a), 8(ad), 9(a), 29
2. Without answers: Section 5.2 #2-9, 13
3. [Handout 4](#)
4. [Handout 5](#)

Assignment, due Friday, Oct 06, via Gradescope TODO:

1. Prove that for every positive integer n ,

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

2. Let a_n be defined recursively by $a_1 = 1$ and $a_n = \sqrt{1 + a_{n-1}}$. Prove that for all positive integers n , $a_n < 2$.
3. Prove by induction that if b_1, b_2, \dots, b_n are even integers, then $b_1 + b_2 + \cdots + b_n$ is even.
4. Let $F_1, F_2, F_3, \dots = 1, 1, 2, 3, 5, 8, \dots$ be the Fibonacci sequence. Prove that $F_1^2 + \cdots + F_n^2 = F_n F_{n+1}$.

Assignment 5

Topics: Basics of set theory. Basic operations. Proofs with sets.

Reading TODO:

1. Section 2.1, p. 49-57;
2. Section 2.2, p. 61-65 (stop at DeMorgan's laws)

TODO Suggested problems (do not hand in)

1. With answers (many of these are calculations; do as many as you need to do to understand the definitions):
 - (a) Section 2.1, #1(adg), 2(adg), 4(adg), 5(a), 7(a), 8(ae), 9(adf), 10(a), 18(acf), 19(ad), 20(ae), 21
 - (b) Section 2.2, #1(adgj), 2(ad), 4(ad), 5(ad), 7(a), 9(ad), 14(a),
2. Without answers:
 - (a) Section 2.1, 13, 14, 15, 16,
 - (b) Section 2.2, #1-12
3. [Handout 6](#)

Assignment 6, due Friday, Oct 20, via Gradescope TODO:

1. Let $A = \{n \in \mathbb{Z} | n \text{ is a multiple of } 4\}$ and $B = \{n \in \mathbb{Z} | n^2 \text{ is a multiple of } 4\}$
 - (a) Prove or disprove: $A \subseteq B$.
 - (b) Prove or disprove: $B \subseteq A$.
2. Prove that $A \cup (A \cap B) = A$.
3. Let A, B and C be sets.
 - (a) Prove that $(A \subseteq C) \wedge (B \subseteq C) \Rightarrow A \cup B \subseteq C$.
 - (b) State the contrapositive of part (a).
 - (c) State the converse of part (a). Prove or disprove it.
4. Let n and m be integers. Prove that if $n\mathbb{Z} \subseteq m\mathbb{Z}$ then m divides n .

Midterm

Topics: Wednesday, October 11 will be an in class Exam review. We will not cover any new material; in class, I will answer whatever questions you have. **Please show up with questions.** There will be the usual office hours on Thursday, October 12; instead there will be office hours Wednesday, October 18, 2:30-3:30.

The exam is on Friday, October 13.

Content: The questions will all be either

1. homework problems,
2. suggested problems,
3. problems we worked in class, or
4. minor variations of one of these.

A typical exam will have one or two questions from each week of the course. You can expect problems like the following:

- Negations
- Give definitions
- Contrapositive
- Contradiction
- Induction

Assignment 6

Topics: More proofs with sets. DeMorgan's laws. Cartesian Products. Power sets

Reading TODO:

1. Section 2.2, p. 65-66;
2. Section 2.3, p. 72, just the part about power sets.

TODO Suggested problems (do not hand in)

1. With answers:
 - (a) Section 2.2, 13(a), 16(a)
 - (b) Section 2.3, #1(a), 3, 5(adg),
2. Without answers:
 - (a) Section 2.2, 14, 16-19, 21, 23-27
 - (b) Section 2.3, #1(b), 2,4
3. [Handout 7](#)

Assignment 7, due Thursday, Oct 27, via Gradescope TODO:

1. Let A and B be sets. Prove that $(A \cup B) \cap \overline{A} = B - A$.
2. Let A and B be sets. Prove that $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$.
3. Let $A = \{0, 1, 2\}$. Which of the following statements are true? (No justification is needed.)
 - (a) $\{0\} \subseteq P(A)$;
 - (b) $\{1, 2\} \in P(A)$;
 - (c) $\{1, \{1\}\} \subseteq P(A)$.
 - (d) $\{\{0, 1\}, \{1\}\} \subseteq P(A)$;
 - (e) $\emptyset \in P(A)$;
 - (f) $\emptyset \subseteq P(A)$;
 - (g) $\{\emptyset\} \in P(A)$.
 - (h) $\{\emptyset\} \subseteq P(A)$;
4. Let A and B be sets. Prove that if $A \subseteq B$, then $P(A) \subseteq P(B)$. State the converse of this and prove or disprove it.

Assignment 7

Topics: Introduction to functions; images and surjectivity

Reading TODO:

1. Section 3.1, p. 81-90 (stop at “Inverse Image”);
2. Section 3.2, p. 97-100 (stop at Injective Functions).

TODO Suggested problems (do not hand in)

1. With answers:
 - (a) Section 3.1, #1(adg), 4(ace), 5(a), 8(a), 10(a), 12(1d)
 - (b) Section 3.2, #1(adgj), 2(ad)
2. Without answers:
 - (a) Section 3.1, 1-4,6-13
 - (b) Section 3.2, 1-6
 - (c) [Handout 9](#)

Assignment 8, due Friday, Nov 03, via Gradescope TODO:

1. Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be the function defined by $f(x) = 6x + 5$.
 - (a) Prove that $f(\mathbf{R}) = \mathbf{R}$.
 - (b) Compute $f([1, 4])$. Prove your answer.
2. Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be the function defined by $x^4 + x^2$.
 - (a) Compute the image of f . Prove that your answer is correct.
 - (b) Compute $f([-1, 2])$. Prove that your answer is correct.
3. Let A and B be sets and let X and Y be subsets of A . Let $f: A \rightarrow B$ be a function. Prove or disprove each of the following. When giving a disproof, please give a counterexample.
 - (a) $f(X \cap Y) \subseteq f(X) \cap f(Y)$.
 - (b) $f(X \cap Y) \supset f(X) \cap f(Y)$.
 - (c) $f(X) - f(Y) \subseteq f(X - Y)$.
 - (d) $f(X) - f(Y) \supset f(X - Y)$.

Assignment 8

Topics: Inverse Image (or “Preimage”).

Reading TODO: Section 3.1, p. 90-92 (stop at the Historical Comments. Or don’t.)

TODO Suggested problems (do not hand in)

1. With answers: Section 3.1, #17(ad), #18(adg), #19(a), #21(a)
2. Without answers: 17-21
3. [Handout 10](#)

Due to a conflict, **office hours** will be 4:00-5pm on Friday, November 8 (over Zoom). This is after the homework assignment is due; due to this inconvenience, you are welcome to turn in the assignment late (anytime before Thursday, November 10). (Gradescope will mark this as late, but I will not deduct any points.)

Assignment 9, due Friday, Nov 10, via Gradescope TODO:

1. Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be the function defined by $f(x) = 3x + 1$.
 - (a) Compute $f^{-1}(\{1, 5, 8\})$ (do not give a proof).
 - (b) Compute $f^{-1}(W)$, where $W = (4, \infty)$, and give a proof that your answer is correct.
 - (c) Compute $f^{-1}(\mathbf{E})$, where \mathbf{E} is the set of even integers, and give a proof that your answer is correct.
2. Let $f: \mathbf{Z} \rightarrow \mathbf{Z}$ be the function defined by $f(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ 2n + 4, & \text{if } n \text{ is odd.} \end{cases}$

Compute $f^{-1}(\mathbf{E})$. Prove that your answer is correct. (Reminder: \mathbf{E} is the set of even integers.)
3. Let A and B be sets and let X be a subset of B . Let $f: A \rightarrow B$ be a function. Prove or disprove the following. (For a disproof, please give an explicit counterexample.)
 - (a) $X \subseteq f(f^{-1}(X))$.
 - (b) $X \supset f(f^{-1}(X))$.
4. Let A and B be sets. Let $S \subseteq A$ and let $T \subseteq B$. Let $f: A \rightarrow B$ be a function. Prove or disprove the following. (For a disproof, please give an explicit counterexample.)
 - (a) $f(S) \subseteq T \Rightarrow S \subseteq f^{-1}(T)$.
 - (b) $S \subseteq f^{-1}(T) \Rightarrow f(S) \subseteq T$.

Assignment 9

Topics: Injectivity.

Reading TODO: Section 3.2, p. 100-105

TODO Suggested problems (do not hand in)

1. With answers: 3.2, #12(adg), #13(bd)
2. Without answers: 3.2 #9-14, 19(abc)
3. [Handout 11](#)

Assignment 10, due Friday, Nov 17, via Gradescope TODO:

1. Which of the following functions $f: \mathbf{R} \rightarrow \mathbf{R}$ are injective? If the function is injective, give a proof. If it is not injective, give a counterexample.
 - (a) $f(x) = x^4 + x^2$;
 - (b) $f(x) = x^3 + x^2$;
 - (c) $f(x) = \begin{cases} -x - 1, & \text{if } x > 0 \\ x^2, & \text{if } x \leq 0. \end{cases}$
2. Let A and B be sets and let X and Y be subsets of A . Let $f: A \rightarrow B$ be an injective function. Prove that $f(X \cap Y) = f(X) \cap f(Y)$.
3. Let $f: A \rightarrow B$ be a function. Which of the followings statements are equivalent to the statement ‘ f is injective’? (No proof necessary.)
 - (a) $f(a) = f(b)$ if $a = b$;
 - (b) $f(a) = f(b)$ and $a = b$ for all $a, b \in A$;
 - (c) If a and b are in A and $f(a) = f(b)$, then $a = b$;
 - (d) If a and b are in A and $a = b$, then $f(a) = f(b)$;
 - (e) If a and b are in A and $f(a) \neq f(b)$, then $a \neq b$;
 - (f) If a and b are in A and $a \neq b$, then $f(a) \neq f(b)$.
4. We define a function $f: [a, b] \rightarrow \mathbf{R}$ to be **decreasing** if for all $x_1, x_2 \in [a, b]$, if $x_1 < x_2$, then $f(x_1) > f(x_2)$.
 - (a) Negate the definition of decreasing.
 - (b) Prove that a decreasing function is injective.

Assignment 10

Topics: Composition of functions.

Reading TODO: Section 3.3, p. 110-113

TODO Suggested problems (do not hand in)

1. With answers: 3.3, #1(a), 2(a), 3(ad), 7(a)
2. Without answers: 3.3 #1-7, 9
3. [Handout 12](#)

Assignment 11, due Friday, Dec 01, via Gradescope TODO:

1. Let A, B and C be sets and let $f: A \rightarrow B$ and $g: B \rightarrow C$ be functions. Prove or disprove each of the following.
 - (a) If $g \circ f$ is an injection, then g is an injection.
 - (b) If $g \circ f$ is a surjection, then f is a surjection.
 - (c) If $g \circ f$ is a surjection, then g is a surjection.
2. Let A and B be sets and let $f: A \rightarrow B$ and $g: B \rightarrow A$ be functions. Prove that if $g \circ f$ and $f \circ g$ are bijective, then so are f and g .
3. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ be functions. Suppose that f and g are both decreasing. Prove that $g \circ f$ is increasing.

Midterm

Topics: Friday, Nove will be an in class Exam review. We will not cover any new material; in class, I will answer whatever questions you have. **Please show up with questions.**

There will be the usual office hours on Thursday, October 12; instead there will be office hours Wednesday, October 18, 2:30-3:30.

The exam is on Friday, October 13.

Content: The questions will all be either

1. homework problems,
2. suggested problems,
3. problems we worked in class, or
4. minor variations of one of these.

A typical exam will have one or two questions from each week of the course. You can expect problems like the following:

- Negations
- Give definitions
- Contrapositive
- Contradiction
- Induction

Assignment 11

Thanksgiving break is Thursday, November 24 and Friday, November 25;
There will no class these days.

Office hours are Wednesday, November 30, and the assignment will be due Thursday, December 1.

Topics: Inverse functions.

Reading TODO: Section 3.3, p. 114-116

TODO Suggested problems (do not hand in)

1. With answers: 3.3 #10(adgj), 11(a)
2. Without answers: 3.3 #10, 12, 14, 15, 17, 18, 19, 22
3. [Handout 13](#)

Assignment 12, due Thursday, Dec 08, via Gradescope TODO:

1. Define $f: \mathbf{R} - \{1\} \rightarrow \mathbf{R} - \{1\}$ by $f(x) = \frac{x+1}{x-1}$. Prove that f is a bijection. Find a formula for the inverse $f^{-1}(x)$, and prove that it is correct.
2. Let A, B and C be sets and let $f: A \rightarrow B$ and $g: B \rightarrow C$ be functions. Prove that if f and g are invertible, then so is $g \circ f$, and prove that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.
3. Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be the function $f(x) = x^3 + x$. Prove that f is invertible without finding a formula for f^{-1} .
4. Let A and B be sets and let $f: A \rightarrow B$ be a function. Suppose that f has a *left inverse* g ; that is, suppose that there exists a function $g: B \rightarrow A$ such that $g \circ f = id_A$. Prove that f is injective.

Assignment 12

Topics: Relations.

Reading TODO: Section 4.2, p. 139-144 (stop at the proof of Theorem 4.2.6)

TODO Suggested problems (do not hand in)

1. With answers: Section 4.2 #1(a), 3(ad), 4(a), 5(a), 12(a)
2. Without answers: Section 4.2 #1, 3, 4
3. [Handout 14](#)

Assignment 13, due Friday, Dec 13, via canvas:

1. Let $A = \{1, 2, 3\}$ and define a relation on A by $a \sim b$ if $a + b \neq 3$. Determine whether this relation is reflexive, symmetric, transitive.
2. Define a relation on \mathbf{Z} given by $a \sim b$ if $a - b$ is divisible by 4.
 - (a) Prove that this is an equivalence relation.
 - (b) What integers are in the equivalence class of 18? (No proof necessary.)
 - (c) What integers are in the equivalence class of 31? (No proof necessary.)
 - (d) How many distinct equivalence classes are there? What are they? (No proof necessary.)
3. Define a relation on \mathbf{Z} given by $a \sim b$ if $a^2 - b^2$ is divisible by 4.
 - (a) Prove that this is an equivalence relation.
 - (b) How many distinct equivalence classes are there? What are they? (No proof necessary.)
4. Let A be a set, and let $P(A)$ be the power set of A . Assume that A is not the empty set. Define a relation on $P(A)$ by $X \sim Y$ if $X \subseteq Y$. Is this relation reflexive, symmetric, and/or transitive? In each case give a proof, or disprove with a counterexample. (For a counterexample, give an example of A , X , and Y that disproves the statement.)

Assignment 13

Topics: Relations.

Reading TODO: Section 4.2, p. 139-144 (stop at the proof of Theorem 4.2.6)

TODO Suggested problems (do not hand in)

1. With answers: Section 4.2 #1(a), 3(ad), 4(a), 5(a), 12(a)
2. Without answers: Section 4.2 #1, 3, 4
3. [Handout 14](#)

Assignment 13, due Friday, Dec 13, via canvas:

1. Let $A = \{1, 2, 3\}$ and define a relation on A by $a \sim b$ if $a + b \neq 3$. Determine whether this relation is reflexive, symmetric, transitive.
2. Define a relation on \mathbf{Z} given by $a \sim b$ if $a - b$ is divisible by 4.
 - (a) Prove that this is an equivalence relation.
 - (b) What integers are in the equivalence class of 18? (No proof necessary.)
 - (c) What integers are in the equivalence class of 31? (No proof necessary.)
 - (d) How many distinct equivalence classes are there? What are they? (No proof necessary.)
3. Define a relation on \mathbf{Z} given by $a \sim b$ if $a^2 - b^2$ is divisible by 4.
 - (a) Prove that this is an equivalence relation.
 - (b) How many distinct equivalence classes are there? What are they? (No proof necessary.)
4. Let A be a set, and let $P(A)$ be the power set of A . Assume that A is not the empty set. Define a relation on $P(A)$ by $X \sim Y$ if $X \subseteq Y$. Is this relation reflexive, symmetric, and/or transitive? In each case give a proof, or disprove with a counterexample. (For a counterexample, give an example of A , X , and Y that disproves the statement.)

Final Exam

Final exam is TBA

The **last day of class** is Wednesday, December 13.

There will be **office hours** on TBA. I will send out a survey to find a time that works for everyone who is planning to attend.

The final exam will not be comprehensive, and will only cover content introduced after the midterm. Still, while I won't give you problems that are "just" about induction, contradiction, negations, etc. (so for example, I will not ask any irrationality questions) you will still need to use those techniques in some of your proofs.

The exam will be, roughly 8-10 questions, with multiple parts. Some questions will be "prove or disprove". For disproofs, please write out a counterexample as your disproof.

A typical exam will have one or two questions from each week of the course. You can expect a subset of the following:

- Images
- preimages
- Injectivity
- Surjectivity
- Compositions
- Invertibility
- Relations
- Problems from handouts 9-14