Sporadic points on modular curves

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Slides available at `http://www.math.emory.edu/~dzb/slides/`

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Mazur's Theorem

Theorem (Mazur, 1978)

Let $E/\mathbb{Q}$ be an elliptic curve. Then $E(\mathbb{Q})_{tors}$ is isomorphic to one of the following groups.

$$\begin{aligned} &\mathbb{Z}/N\mathbb{Z}, && \text{for } 1 \leq N \leq 10 \text{ or } N = 12, \\ &\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2N\mathbb{Z}, && \text{for } 1 \leq N \leq 4. \end{aligned}$$

Via geometry, let

- $Y_1(N)$ be the curve paramaterizing (E, P), where P is a point of exact order N on E, and let
- $Y_1(M, N)$ (with $M \mid N$) be the curve paramaterizing E/K such that $E(K)_{\text{tors}}$ contains $\mathbb{Z}/M\mathbb{Z} \oplus \mathbb{Z}/N\mathbb{Z}$.

Then $Y_1(N)(\mathbb{Q}) \neq \varnothing$ and $Y_1(2, 2N)(\mathbb{Q}) \neq \varnothing$ iff N are as above.

Modular curves via Tate normal form

Example ($N = 9$)

$E(K) \cong \mathbb{Z}/9\mathbb{Z}$ if and only if there exists $t \in K$ such that E is isomorphic to

$$y^2 + (t - rt + 1)xy + (rt - r^2 t)y = x^3 + (rt - r^2 t)x^2$$

where r is $t^2 - t + 1$. The torsion point is $(0, 0)$.

Example ($N = 11$)

$E(K) \cong \mathbb{Z}/11\mathbb{Z}$ correspond to $a, b \in K$ such that

$$a^2 + (b^2 + 1)a + b;$$

in which case E is isomorphic to

$$y^2 + (s - rs + 1)xy + (rs - r^2 s)y = x^3 + (rs - r^2 s)x^2$$

where r is $ba + 1$ and s is $-b + 1$.

Rational Points on $X_1(N)$ and $X_1(2,2N)$

Let $X_1(N)$ and $X_1(M,N)$ be the smooth compactifications of $Y_1(N)$ and $Y_1(M,N)$. We can restate the results of Mazur's Theorem as follows.

- $X_1(N)$ and $X_1(2,2N)$ have genus 0 for **exactly** the N appearing in Mazur's Theorem. (So in particular, there are **infinitely many** $E/\mathbb{Q}$ with such torsion structure.)
- If $g(X_1(N))$ (resp. $g(X_1(2,2N))$) is greater than 0, then $X_1(N)(\mathbb{Q})$ (resp. $X_1(2,2N)(\mathbb{Q})$) consists only of cusps.

So, in a sense, the simplest thing that could happen does happen for these modular curves.

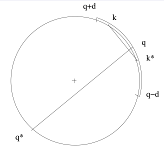
Higher Degree Torsion Points

Theorem (Merel, 1996)

For every integer $d \geq 1$, there is a constant $N(d)$ such that for all $K/\mathbb{Q}$ of degree at most d and all E/K,

$$\#E(K)_{tors} \leq N(d).$$

Expository reference: Darmon, Rebellodo (Clay summer school, 2006)



Problem

Fix $d \geq 1$. Classify all groups which can occur as $E(K)_{tors}$ for $K/\mathbb{Q}$ of degree d. Which of these occur infinitely often?

Quadratic Torsion

Theorem (Kamienny–Kenku–Momose, 1980's)

Let E be an elliptic curve over a quadratic number field K. Then $E(K)_{tors}$ is one of the following groups.

$$\begin{array}{ll} \mathbb{Z}/N\mathbb{Z}, & \text{for } 1 \leq N \leq 16 \text{ or } N = 18, \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2N\mathbb{Z}, & \text{for } 1 \leq N \leq 6, \\ \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3N\mathbb{Z}, & \text{for } 1 \leq N \leq 2, \text{ or} \\ \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}. & \end{array}$$

In particular, the corresponding curves $X_1(M, N)$ all have $g \leq 2$, which guarantees that they have infinitely many quadratic points.

Sporadic Points

Let $X/\mathbb{Q}$ be a curve and let $P \in X(\overline{\mathbb{Q}})$. The **degree** of P is $[\mathbb{Q}(P) : \mathbb{Q}]$.

The set of degree d points of X is infinite if

- X admits a degree d map $X \to \mathbb{P}^1$;
- X admits a degree d map $X \to E$, where rank $E(\mathbb{Q}) > 0$; or
- Jac_X contains a positive rank abelian subvariety A such that $A + D \subset W^d(X)$ for some D.

- Most $\overline{\mathbb{Q}}$ points arise in the fashion. We call outliers **isolated**.
- When X is a modular curve, cusps and CM points give rise to many isolated points; we call an isolated point **sporadic** if it is not cuspidal or CM.

See Bianca Viray's CNTA talk, linked here.

Cubic Torsion

Theorem (Jeon–Kim–Schweizer, 2004)

Let E be an elliptic curve over a cubic number field K. Then the subgroups which arise as $E(K)_{tors}$ infinitely often are exactly the following.

$$\begin{array}{ll} \mathbb{Z}/N\mathbb{Z}, & \text{for } 1 \leq N \leq 20,\ N \neq 17, 19, \text{ or} \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2N\mathbb{Z}, & \text{for } 1 \leq N \leq 7. \end{array}$$

Minimalist conjecture

Conjecture

A modular curve X admits a non cuspidal, non CM point of degree d if and only if

- X admits a degree d map $X \to \mathbb{P}^1$; or
- X admits a degree d map $X \to E$, where rank $E(\mathbb{Q}) > 0$; or
- Jac_X contains a positive rank abelian subvariety such that...

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$$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2N\mathbb{Z}, \quad \text{for } 1 \leq N \leq 7.$$

Theorem (Najman, 2014)

The elliptic curve 162b1 has a 21-torsion point over $\mathbb{Q}(\zeta_9)^+$.

Remark

Parent showed that the largest prime that can divide $E(K)_{\text{tors}}$ in the cubic case is $p = 13$.

Classification of Cubic Torsion

Theorem (Etropolski–Morrow–ZB–Derickx–van Hoeij)

The only torsion subgroups which appear for an elliptic curve over a cubic field are

$$\begin{aligned} &\mathbb{Z}/N\mathbb{Z}, && \text{for } 1 \leq N \leq 21,\ N \neq 17, 19, \text{ and} \\ &\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2N\mathbb{Z}, && \text{for } 1 \leq N \leq 7. \end{aligned}$$

The only sporadic point is the elliptic curve `162b1` *over* $\mathbb{Q}(\zeta_9)^+$.

Modular curves

Definition

- $X(N)(K) := \{(E/K, P, Q) : E[N] = \langle P, Q\rangle\} \cup \{\text{cusps}\}$
- $X(N)(K) \ \ni \ (E/K, P, Q) \Leftrightarrow \rho_{E,N}(G_K) = \{I\}$

Definition

$\Gamma(N) \subset H \subset \mathrm{GL}_2(\widehat{\mathbb{Z}})$ (finite index)

- $X_H := X(N)/H$
- $X_H(K) \ni (E/K, \iota) \Leftrightarrow H(N) \subset H \mod N$

Stacky disclaimer

This is only true up to twist; there are some subtleties if

1. $j(E) \in \{0, 12^3\}$ (plus some minor group theoretic conditions), or
2. if $-I \in H$.

Example - torsion on an elliptic curve

If E has a K-rational **torsion point** $P \in E(K)[n]$ (of exact order n) then:

$$H(n) \subset \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$$

since for $\sigma \in G_K$ and $Q \in E(\overline{K})[n]$ such that $E(\overline{K})[n] \cong \langle P, Q \rangle$,

$$\begin{aligned} \sigma(P) &= P \\ \sigma(Q) &= a_\sigma P + b_\sigma Q \end{aligned}$$

Example - Isogenies

If E has a K-rational, **cyclic isogeny** $\phi\colon E \to E'$ with $\ker\phi = \langle P \rangle$ then:

$$H(n) \subset \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

since for $\sigma \in G_K$ and $Q \in E(\overline{K})[n]$ such that $E(\overline{K})[n] \cong \langle P, Q \rangle$,

$$\begin{aligned} \sigma(P) &= a_\sigma P \\ \sigma(Q) &= b_\sigma P + c_\sigma Q \end{aligned}$$

Example - other maximal subgroups

Normalizer of a split Cartan:

$$N_{\mathrm{sp}} = \left\langle \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle$$

$H(n) \subset N_{\mathrm{sp}}$ and $H(n) \not\subset C_{\mathrm{sp}}$ iff

- there exists an unordered pair $\{\phi_1, \phi_2\}$ of cyclic isogenies,
- whose kernels intersect trivially,
- neither of which is defined over K,
- but which are both defined over some quadratic extension of K,
- and which are Galois conjugate.

Example - other maximal subgroups

Normalizer of a non-split Cartan:

$$C_{\text{ns}} = \text{im}\left(\mathbb{F}_{p^2}^* \to \text{GL}_2(\mathbb{F}_p)\right) \subset N_{\text{ns}}$$

$H(n) \subset N_{\text{ns}}$ and $H(n) \not\subset C_{\text{ns}}$ iff

E admits a "necklace" (Rebolledo, Wuthrich)

A typical subgroup (from Rouse–ZB)

"Jenga"

$$\begin{array}{ccccccc}
\ker\phi_4 & \subset & H(32) & \subset & \mathrm{GL}_2(\mathbb{Z}/32\mathbb{Z}) & & \dim_{\mathbb{F}_2}\ker\phi_4 = 4 \\
 & & \downarrow{\scriptstyle\phi_4} & & \downarrow & & \\
\ker\phi_3 & \subset & H(16) & \subset & \mathrm{GL}_2(\mathbb{Z}/16\mathbb{Z}) & & \dim_{\mathbb{F}_2}\ker\phi_3 = 3 \\
 & & \downarrow{\scriptstyle\phi_3} & & \downarrow & & \\
\ker\phi_2 & \subset & H(8) & \subset & \mathrm{GL}_2(\mathbb{Z}/8\mathbb{Z}) & & \dim_{\mathbb{F}_2}\ker\phi_2 = 2 \\
 & & \downarrow{\scriptstyle\phi_2} & & \downarrow & & \\
\ker\phi_1 & \subset & H(4) & \subset & \mathrm{GL}_2(\mathbb{Z}/4\mathbb{Z}) & & \dim_{\mathbb{F}_2}\ker\phi_1 = 3 \\
 & & \downarrow{\scriptstyle\phi_1} & & \downarrow & & \\
 & & H(2) & = & \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z}) & &
\end{array}$$

$$\ker\phi_i \subset I + \ell^i M_2(\mathbb{F}_\ell) \cong \mathbb{F}_\ell^4$$

Non-abelian entanglements

(from Brau–Jones)

There exists a surjection $\theta\colon \mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$.

$$\begin{tikzcd}
 & H(6) := \Gamma_\theta \arrow[ddl] \arrow[ddr] & \\
 & \mathrm{GL}_2(\mathbb{Z}/6\mathbb{Z}) \arrow[u, phantom, "\cap"] \arrow[dl] \arrow[dr] & \\
\mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z}) & & \mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z})
\end{tikzcd}$$

$$\operatorname{im}\rho_{E,6} \subset H(6) \Leftrightarrow j(E) = 2^{10}3^3t^3(1-4t^3) \Rightarrow K(E[2]) \subset K(E[3])$$
$$X_H \cong \mathbb{P}^1 \xrightarrow{j} X(1)$$

Rational Points on modular curves

Mazur's program B

Compute $X_H^{(d)}(\mathbb{Q})$ for all H.

Remark

- Sometimes $X_H \cong \mathbb{P}^1$ or elliptic with rank $X_H(\mathbb{Q}) > 0$.
- Some X_H have *sporadic* points.
- Can compute $g(X_H)$ group theoretically (via Riemann–Hurwitz).
- Can compute $\#X_H(\mathbb{F}_q)$ via moduli and enumeration [Sutherland].

Fact

$$g(X_H), \gamma(X_H) \to \infty \text{ as } \left[\mathrm{SL}_2(\widehat{\mathbb{Z}}) : H \cap \mathrm{SL}_2(\widehat{\mathbb{Z}})\right] \to \infty.$$

Najman's example

Theorem (Najman, 2014)

The elliptic curve 162b1 has a 21-torsion point over $\mathbb{Q}(\zeta_9)^+$.

- Let $H := \rho_{E,21}(G_{\mathbb{Q}})$.
- Then H contains an index 3 subgroup H' such that $H' \subset \langle \left(\begin{smallmatrix} 1 & * \\ 0 & * \end{smallmatrix}\right) \rangle$
- Thus we have a degree 3 map

$$X_{H'} \to X_H$$

and an induced map

$$X_H \to \operatorname{Sym}^3 X_{H'} \to \operatorname{Sym}^3 X_1(21)$$

Mazur - Rational Isogenies of Prime Degree (1978)

Let N be a positive integer. Examples of elliptic curves over $\mathbf{Q}$ possessing rational cyclic N-isogenies are known for the following values of N:

N	g	ν	N	g	ν	N	g	ν
$\leqq 10$	0	∞	11	1	3	27	1	1
12	0	∞	14	1	2	37	2	2
13	0	∞	15	1	4	43	3	1
16	0	∞	17	1	2	67	5	1
18	0	∞	19	1	1	163	13	1
25	0	∞	21	1	4			

Sporadic points on $X_H(\ell)$, $H \subset \mathrm{GL}_2(\mathbb{F}_\ell)$

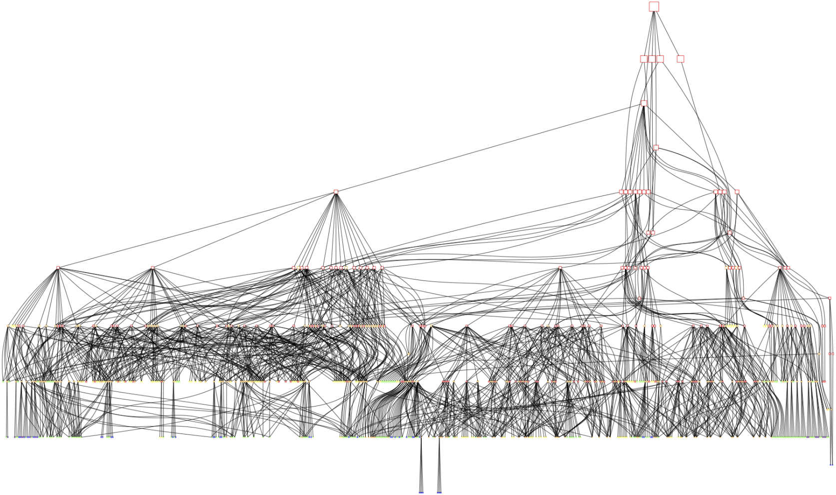
Zywina, "On the Possible Images of the Mod ℓ Representations Associated to..."

7	$3^3 \cdot 5 \cdot 7^5/2^7$	$H \subsetneq N_{ns}(7)$	Sutherland 2012
11	-11^2 $-11 \cdot 131^3$	$\left\langle \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right), \begin{pmatrix}2^2&0\\0&2^9\end{pmatrix} \right\rangle$ $\left\langle \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right), \begin{pmatrix}2^4&0\\0&2^7\end{pmatrix} \right\rangle$	$g(X_0(11)) = 1$
13	$\frac{2^4 \cdot 5 \cdot 13^4 \cdot 17^3}{3^{13}}$ $-\frac{2^{12} \cdot 5^3 \cdot 11 \cdot 13^4}{3^{13}}$ $\frac{2^{18} \cdot 3^3 \cdot 13^4 \cdot 127^3 \cdot 139^3 \cdot 157^3 \cdot 283^3 \cdot 929}{5^{13} \cdot 61^{13}}$	$\widetilde{H} \subset S_4 \subset \mathrm{PGL}_2(\mathbb{F}_{13})$	BDMTV Annals 2019 $g = r = 3$
17	$-17 \cdot 373^3/2^{17}, \; -17^2 \cdot 101^3/2$	$H_i \subsetneq B(17)$	$g(X_0(17)) = 1$
37	$-7 \cdot 11^3, \; -7 \cdot 137^3 \cdot 2083^3$	$H_i \subsetneq B(37)$	$\exists\, \iota \neq w_{37}$

2-adic sporadic points; $H \subset \mathrm{GL}_2(\mathbb{Z}/32\mathbb{Z})$, index 96 or 64

j-invariant	level of H	Generators of image	
2^{11}	16	$\begin{bmatrix}7 & 14\\ 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 5\\ 6 & 11\end{bmatrix}, \begin{bmatrix}3 & 0\\ 0 & 7\end{bmatrix}$	hyperelliptic
$2^4 \cdot 17^3$	16	$\begin{bmatrix}7 & 0\\ 0 & 3\end{bmatrix}, \begin{bmatrix}3 & 5\\ 14 & 7\end{bmatrix}, \begin{bmatrix}7 & 7\\ 2 & 1\end{bmatrix}$	genus 3
$\frac{4097^3}{2^4}$	16	$\begin{bmatrix}3 & 5\\ 6 & 3\end{bmatrix}, \begin{bmatrix}3 & 5\\ 14 & 7\end{bmatrix}, \begin{bmatrix}7 & 7\\ 2 & 1\end{bmatrix}$	rank 1
$\frac{257^3}{2^8}$	16	$\begin{bmatrix}7 & 14\\ 0 & 1\end{bmatrix}, \begin{bmatrix}5 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 5\\ 6 & 3\end{bmatrix}$	
$-\frac{857985^3}{62^8}$	32	$\begin{bmatrix}25 & 18\\ 2 & 7\end{bmatrix}, \begin{bmatrix}25 & 25\\ 2 & 7\end{bmatrix}, \begin{bmatrix}1 & 0\\ 8 & 1\end{bmatrix}, \begin{bmatrix}25 & 11\\ 2 & 7\end{bmatrix}$	not hyperelliptic
$\frac{919425^3}{496^4}$	32	$\begin{bmatrix}29 & 0\\ 4 & 1\end{bmatrix}, \begin{bmatrix}31 & 27\\ 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 4\\ 0 & 1\end{bmatrix}, \begin{bmatrix}31 & 31\\ 2 & 1\end{bmatrix}$	genus 3, rank 3
$-\frac{3\cdot 18249920^3}{17^{16}}$	16	$\begin{bmatrix}4 & 7\\ 15 & 12\end{bmatrix}, \begin{bmatrix}7 & 14\\ 7 & 9\end{bmatrix}, \begin{bmatrix}2 & 1\\ 11 & 9\end{bmatrix}$	$g(X_{ns}(16)) = 2$
$-\frac{7\cdot 1723187806080^3}{79^{16}}$			rank 2

Subgroups of $\mathrm{GL}_2(\mathbb{Z}_2)$



Sporadic points on X_H, $H \subset \mathrm{GL}_2(\mathbb{Z}_\ell), \ell > 2$

Rouse–Sutherland–Zureick-Brown, in progress

Label = level.index.genus.tiebreaker

Theorem (Balakrishnan–Dogra–Müller–Tuitman–Vonk)

There are sporadic points if H has label `25.50.2.1` *and* `25.75.2.1`

See their recent (2021) paper "**Quadratic Chabauty For Modular Curves: Algorithms And Examples**"

Theorem (Rouse–Sutherland–Zureick-Brown)

- *No other sporadic rational points for* $\ell = 3, 5, 7, 11$, *unless*
- $H = N_{ns}(3^3), N_{ns}(5^2), N_{ns}(7^2)$, *or* $N_{ns}(11^5)$ *or*
- *H has label* `49.147.9.1` *or* `49.196.9.1`.

See Jeremy Rouse's CNTA talk, linked here.

Application: isolated points with rational j-invariant

Bourdon–Gill–Rouse–Watson, 2020

(Application) Classification of all odd degree isolated points on $X_1(N)$ with rational j-invariant:

$$j = -3^3 \cdot 5^6/2^3, \text{ or } 3^3 \cdot 13/2^2.$$

The first is the Najman cubic example, and the second corresponds to a degree 8 point on $X_1(28)$, found by Najman and González-Jiménez.

(Morrow) $H_1 \times H_2 \subset \mathrm{GL}_2(\mathbb{Z}/2^m\mathbb{Z}) \times \mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$

$$\emptyset$$

if Δ_E is a square

Camacho-Navarro–Li–Morrow–Petok–Zureick-Brown

$H_1 \times H_2 \subset \mathrm{GL}_2(\mathbb{Z}/p^m\mathbb{Z}) \times \mathrm{GL}_2(\mathbb{Z}/q^n\mathbb{Z})$ (Genus 1)

$3B^0 - 3a$	$4A^0 - 4a$	109503/64, -35937/4
$3B^0 - 3a$	$4D^0 - 4a$	-35937/4, 109503/64
$3B^0 - 3a$	$5A^0 - 5a$	-316368, 432
$3B^0 - 3a$	$5B^0 - 5a$	-25/2, -349938025/8, -121945/32, 46969655/32768
$3B^0 - 3a$	$7B^0 - 7a$	3375/2, -189613868625/128 -140625/8, -1159088625/2097152
$3C^0 - 3a$	$4A^0 - 4a$	3375/64
$3C^0 - 3a$	$5B^0 - 5a$	1331/8, -1680914269/32768
$4A^0 - 4a$	$5B^0 - 5a$	-1723025/4, 1026895/1024

Camacho-Navarro–Li–Morrow–Petok–Zureick-Brown

$H_1 \times H_2 \subset \mathrm{GL}_2(\mathbb{Z}/p^m\mathbb{Z}) \times \mathrm{GL}_2(\mathbb{Z}/q^n\mathbb{Z})$ (Genus ≥ 2)

label 1	label 2	sporadic j-invariants
4A0-4a	7B0-7a	$-38575685889/16384,\ 351/4$
4D0-4a	5A0-5a	$-36,\ -64278657/1024$
5B0-5a	9A0-9a	$-23788477376,\ 64.$
5E0-5a	2A0-8a	-5000
4A0-4a	5E0-5a	(genus 3)

More Sporadic Points on $X_1(N)$, via Derickx–van Hoeij

d \ N	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
20	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
19	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
18	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
17	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	12	∞	∞	∞
16	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	11	∞	∞	∞
15	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	3	∞	10	∞
14	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	1	∞	∞	∞
13	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	4	∞	1	∞	3	26
12	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	3	∞	3	∞
11	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	3	∞	9	∞	1	∞				4
10	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	2	∞	1	∞	∞	∞	2	∞	1	3	1	2
9	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	1	∞	1	8	1	4		∞			1	1
8	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞		∞		∞	1	3	1	∞			1	1
7	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	1	∞	∞	∞		∞			1			2				
6	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞		∞	1	∞	∞	∞		∞							1			
5	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞		∞	∞			1		2										
4	∞	∞	∞	∞	∞	∞	∞	∞		∞	∞	∞		∞																
3	∞	∞	∞	∞	∞	∞		∞		∞	1																			
2	∞	∞	∞	∞	∞	∞		∞																						
1		∞																												

Classification of Cubic Torsion

Theorem (Etropolski–Morrow–ZB–Derickx–van Hoeij)

The only torsion subgroups which appear for an elliptic curve over a cubic field are

$$\mathbb{Z}/N\mathbb{Z}, \qquad \text{for } 1 \leq N \leq 21,\ N \neq 17, 19, \text{ and}$$

$$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2N\mathbb{Z}, \qquad \text{for } 1 \leq N \leq 7.$$

The only sporadic point is the elliptic curve `162b1` *over* $\mathbb{Q}(\zeta_9)^+$.

Formal immersions

Previous work

- (Parent) handles $p > 13$.
- (Momose) $N = 27, 64$.
- (Wang) $N = 77, 91, 143, 169$
- (Bruin–Najman) $N = 40, 49, 55$

Main technique

- If N is large, then there are no elliptic curves mod small $\ell \nmid 2N$ with an N torsion point (e.g., by the Hasse bound).
- Thus a non cuspidal point of $X_1(N)$ reduces mod ℓ to a cusp.
- Fiddle with conditions on ℓ, N so that the formal immersion criterion works. (E.g., need to worry about cusps splitting.)

Sporadic cubic torsion: summary of arguments

Level	Genus	Method of proof	Genus of quotient
32	17	Maps to another curve on this table	$g(X_1(2,16)) = 5$
36	17	Maps to another curve on this table	$g(X_1(2,18)) = 7$
22	6	Local methods at $p = 3$ (§6.1)	N/A
25	12	Local methods at $p = 3$	N/A
21	5	Direct analysis over $\mathbb{Q}$ (§6.2)	N/A
26	10	Direct analysis over $\mathbb{F}_3$	N/A
30	9	Direct analysis over $\mathbb{Q}$ on $X_0(30)$ (§6.4)	$g(X_0(30)) = 3$
33	21	Direct analysis over $\mathbb{Q}$ on $X_0(33)$	$g(X_0(33)) = 3$
35	25	Direct analysis over $\mathbb{Q}$ on $X_0(35)$	$g(X_0(35)) = 3$
39	33	Direct analysis over $\mathbb{Q}$ on $X_0(39)$	$g(X_0(39)) = 3$
(2,16)	5	Hecke bound + direct analysis over $\mathbb{F}_3$ (§6.5)	N/A
(2,18)	7	Hecke bound + direct analysis over $\mathbb{F}_5$	N/A
28	10	Hecke bound + direct analysis over $\mathbb{F}_3$ (§6.6)	N/A
24	5	Hecke bound + additional argument (§4.15) + direct analysis over $\mathbb{F}_5$	N/A
45	41	Hecke bound + direct analysis over $\mathbb{Q}$ on $X_H(45)$ (§6.7)	$g(X_H(45)) = 5$
65	121	Formal immersion criteria (§7.3)	$g(X_0(65)) = 5$
121	526	Formal immersion criteria (§7.1)	$g(X_0(121)) = 6$

Good fortune – many small level ranks are zero

Let

$$S_0 = \{1, \dots, 36, 38, \dots, 42, 44, \dots, 52, 54, 55, 56, 59, 60, 62, 63, 64, 66, 68, 69, 70, 71, 72, 75, 76, 78, 80, 81, 84, 87, 90, 94, 95, 96, 98, 100, 104, 105, 108, 110, 119, 120, 126, 132, 140, 144, 150, 168, 180\},$$

$$S_1 = \{1, \dots, 21, 24, 25, 26, 27, 30, 33, 35, 36, 42, 45\}.$$

Theorem (Etropolski–Morrow–ZB–Derickx–van Hoeij)

1. rank $J_0(N)(\mathbb{Q}) = 0$ *if and only if* $N \in S_0$.
2. rank $J_1(N)(\mathbb{Q}) = 0$ *if and only if* $N \in S_0 - \{63, 80, 95, 104, 105, 126, 144\}$.
3. rank $J_1(2, 2N)(\mathbb{Q})$ *if and only if* $N \in S_1$.

Computing cubic points when $\#J_1(N)(\mathbf{Q}) < \infty$

Consider $X_1(21)$ which has genus 5 and gonality 4.

Known points: 6 rational cusps, 2 quadratic cusps, 2 cubic points D_0 and D_0' (these are the sporadic torsion points)

Compute that $J_1(21)(\mathbb{Q}) = \langle D \rangle \cong \mathbb{Z}/364\mathbb{Z}$ with $D = [D_0 - 3 \cdot \infty]$.

Define an Abel–Jacobi map

$$\iota \colon X_1(21)^{(3)}(\mathbb{Q}) \to J_1(21)(\mathbb{Q}), E \mapsto E - 3 \cdot \infty.$$

For each $nD \in J_1(21)(\mathbb{Q})$, $nD \in \operatorname{im} \iota \to |nD + 3 \cdot \infty| \neq \varnothing$.

- `Magma`'s intrinsic `RiemannRochSpace` can check this and determine the effective divisor E s.t $|nD + 3 \cdot \infty| = \{E\}$.

Compute the list of such n and check that they correspond to the above known points.

The Mordell–Weil Sieve

For a finite set S of primes of good reduction, we have the following commutative diagram.

$$\begin{tikzcd}
X^{(d)}(\mathbb{Q}) \arrow[r, "\iota"] \arrow[d] & J(\mathbb{Q}) \arrow[d, "\alpha"] \\
\prod_{p \in S} X^{(d)}(\mathbb{F}_q) \arrow[r, "\beta"] & \prod_{p \in S} J(\mathbb{F}_p)
\end{tikzcd}$$

Compare the images of α and β.

Computing torsion on modular Jacobians

The reduction map $A(\mathbb{Q})_{\text{tors}} \to A(\mathbb{F}_p)$ is injective for $p > 2$ (Katz).

The GCD of $\#A(\mathbb{F}_p)$ gives a naive upper bound on $A(\mathbb{Q})_{\text{tors}}$.

Better: compute the "GCD" of the groups $A(\mathbb{F}_p)$

Example ($A = J_1(21)$)

- $A(\mathbb{F}_5) \cong \mathbb{Z}/2184\mathbb{Z}$
- $A(\mathbb{F}_{11}) \cong \mathbb{Z}/14\mathbb{Z} \times \mathbb{Z}/6916\mathbb{Z}$

- $2184 = 728 \cdot 3$
- $6916 = 364 \cdot 19$

- $\text{GCD}(2184, 6916) = 728$
- $\text{GCD}(A(\mathbb{F}_5), A(\mathbb{F}_{11})) = \mathbb{Z}/364\mathbb{Z}$.

Computing torsion on modular Jacobians

For $J_1(N)$, let $q \nmid 2N$ be prime, let T_q be the qth Hecke operator.

By **Eichler–Shimura**

$$\ker(T_q - q\langle q\rangle - 1\colon J_1(N)(\overline{\mathbb{Q}})_{\mathrm{tors}} \to J_1(N)(\overline{\mathbb{Q}})_{\mathrm{tors}})$$

contains prime-to-q torsion on $J_1(N)(\mathbb{Q})$.

Also, $\tau - 1$ vanishes on $J_1(N)(\mathbb{Q})_{\mathrm{tors}}$, where τ is complex conjugation.

"Hecke Bound"

For a finite set of primes $q_1, \ldots, q_n$, define $M_N :=$

$$J_1(N)(\overline{\mathbb{Q}})_{\mathrm{tors}}[T_{q_1} - q_1\langle q_1\rangle - 1, \ldots, T_{q_n} - q_n\langle q_n\rangle - 1, \tau - 1].$$

Then $J_1(N)(\mathbb{Q})_{\mathrm{tors}} \subset M_N$, which we call the **Hecke bound**.

This M_N is easy to compute via **modular symbols** in Sage.

Computing torsion - modular symbols and cusps

Modular symbols

Under the uniformization

$$J_H(N)(\mathbb{C}) \cong H_1(X_H(N)(\mathbb{C}), \mathbb{C})/H_1(X_H(N)(\mathbb{C}), \mathbb{Z})$$

we can identify the geometric torsion as

$$J_H(N)(\overline{\mathbb{Q}})_{\text{tors}} \cong H_1(X_H(N)(\mathbb{C}), \mathbb{Q})/H_1(X_H(N)(\mathbb{C}), \mathbb{Z}).$$

Conjecture (Conrad–Edixhoven–Stein; DEvHMZB)

$$\mathrm{Cl}^{\text{cusp},0}\, X_1(N) = J_1(N)(\mathbb{Q})_{\text{tors}}.$$

Theorem (DEvHMZB)

This is true for $N \leq 55$, $N \neq 54$.

Thanks!

Thank you!