P1

Suppose we have M simulated values x_i with different uncertainties σ_i. Assume also that the x_i are Gaussian, that is $x_i = N(\mu, \sigma_i^2)$, a normal random variable w/ mean μ + variance σ_i^2. Let the x_i have a common mean $E(x_i) = \mu$

What is the "best" way to estimate μ, given x_i, σ_i

The arithmetic mean $\bar{x} = \frac{1}{M} \sum_i x_i$ is unbiased

i.e. $E(\bar{x}) = \mu$

But by rules of addition for normal random variables

$$\bar{x} = \frac{1}{M} \sum_i x_i = \frac{1}{M} \sum_{i=1}^{M} \mu + N(0, \sigma_i^2)$$

$$= \mu + \frac{1}{M} \sum_{i=1}^{M} N(0, \sigma_i^2)$$

$$= \mu + N\left(0, \sum_{i=1}^{M} \frac{\sigma_i^2}{M^2}\right)$$

Thus, the "error" or uncertainty associated with this estimate is

$$\varepsilon = \frac{1}{M} \left\{ \sum_i \sigma_i^2 \right\}^{1/2}$$

P2

If instead, we construct the weighted average of the form

$$\tilde{X} = \frac{1}{\sum_i \sigma_i^{-2}} \sum_i \frac{x_i}{\sigma_i^2} \Rightarrow E(\tilde{X}) = \mu \text{ as before, so } \tilde{X} \text{ is unbiased}$$

But $\tilde{X} = \mu + \frac{1}{\sum_i \sigma_i^{-2}} \sum_i \frac{N(0, \sigma_i^2)}{\sigma_i^2}$

$$= \mu + \frac{1}{\sum_i \sigma_i^{-2}} \sum_i N(0, \sigma_i^{-2})$$

$$= \mu + \frac{1}{\sum_i \sigma_i^{-2}} N\left(0, \sum_i \sigma_i^{-2}\right)$$

$$= \mu + N\left(0, \left(\sum_i \sigma_i^{-2}\right)^{-1}\right)$$

Now, the question becomes, which is smaller?

$$\frac{1}{M^2} \sum_i \sigma_i^2 \quad \text{or} \quad \left(\sum_i \sigma_i^{-2}\right)^{-1} = \frac{1}{\sum_i \frac{1}{\sigma_i^2}}$$

(P3)

To resolve this, note the following

$$\frac{1}{M^2}\sum_i \sigma_i^2 = \frac{1}{M}\sum_i \frac{\sigma_i^2}{M}$$, which is $\frac{1}{M}$ times the arithmetic mean of variances

Also $$\frac{1}{\sum_i \frac{1}{\sigma_i^2}} = \frac{1}{M}\frac{M}{\sum_i \frac{1}{\sigma_i^2}}$$, which is $\frac{1}{M}$ times the harmonic mean of variances.

But the well-known inequality (assuming at least one σ_i^2 differs from the others)

$$\frac{M}{\sum_i \frac{1}{\sigma_i^2}} < \sum_i \frac{\sigma_i^2}{M} \Rightarrow \left(\sum_i \sigma_i^{-2}\right)^{-1} < \sum_i \frac{\sigma_i^2}{M^2}$$

Thus, the weighted average has a correspondingly smaller "error".

In this case, a reasonable estimate of uncertainty in the true mean is thus

$$\varepsilon = \left\{\sum_i \sigma_i^{-2}\right\}^{-1/2}$$