Some results about group

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I found some results really interesting about group which listed at the book *Basic Algebra by NATHAN JACOBSON*

1 Invertiblity in a monoid

Let M donates a monoid,any of the following conditions

1. $ab = ca = 1$
2. $aba = a, ab^2a = 1$

can conclude that a is invertible with b as inverse.
prove:

1. if $ab = ca = 1$ then,
$$b = 1 \cdot b = cab = c \cdot 1 = c$$
thus
$$ab = ba = 1$$
end the prove.
2. if $aba = a, ab^2a = 1$ then,
$$1 = ab^2a = abab^2a = ab$$
$$1 = ab^2a = abba = ba$$
end the prove.

2 Semigroup with some properities

Let G be a semigroup having following properites

(a) G contains right unit 1_r,thus $\forall a \in G, a1_r = a$

(b) $\forall a \in G$ have a right inverse to 1_r $(ab = 1_r)$.

then G is a group.
prove: $\forall a \in G$ we have $b \in G$ such thar

$$ab = 1_r$$

then there exists $c \in G$ that

$$bc = 1_r$$

hence we have

$$a = a1_r = abc = 1_r c = 1_r 1_r c = 1_r a$$

so we have $\forall a \in G$

$$a1_r = 1_r a = a$$

thus 1_r is the unit in G and G is a monoid.Using conclusion in previous chapter,we know that a is invertible for all $a \in G$.So G is a group.

3 Semigroup with solvable system

G is a monoid,if the equation $ax = b$ and $ya = b$ is solvable for any $a, b \in G$ then G is a group.
prove:
we random choose $b \in G$ and we have x_0 such that

$$bx_0 = b$$

$\forall a \in G$ we have

$$a = yb = ybx_0 = ax_0$$

$$a = x_0 b = x_0 x_0 b = x_0 a$$

hence we have G have a unit $1 = x_0$.
$\forall a \in G$ there exists x, y such that

$$ax = ya = 1$$

we know a is invertible from first chapter.End our prove.

4 Finite semigroup with cancellation law

Let G donate a finite semigroup, is G satisfies

$$ax = ay \Rightarrow x = y$$

and

$$xa = ya \Rightarrow x = y$$

G is a group.
prove: $\forall a \in G$ we consider a map $L_a \colon G \to G$

$$L_a(g) = ag \text{ for all } g \in G$$

since $ax = ay \Rightarrow x = y$ we know that L_a is injective, $|G|$ is finite,we have L_a is bijective.
Similiarly,we can define a bijective map R_a.
Thus G is a Semigroup with solvable system which refer to be a group by previos chapter.