Introduction to Mathematical Thinking

Test-Flight Peer-Assessment Solutions

by

Dileep Nackathaya

Q1. Say whether the following is true or false and support your answer by a proof. $(\exists m \in \mathcal{N})(\exists n \in \mathcal{N})(3m + 5n = 12)$

PROOF The statement is false. The smallest $m, n \in \mathcal{N}$ that we can try are 1 and 1.

- Substituting these in the above equation: $3 * 1 + 5 * 1 = 8 < 12$.
- Let us try the next possible value for m, which is 2: $3 * 2 + 5 * 1 = 11 < 12$.
- For $m = 3, n = 1$, and $m = 1, n = 2$, and all other combinations of m and n, $3m + 5n > 12$.
- Therefore, $\neg[(\exists\ m \in \mathcal{N})(\exists\ n \in \mathcal{N})(3m + 5n = 12)]$.
- Or, $(\forall \text{ m} \in \mathcal{N})(\forall \text{ n} \in \mathcal{N})(3m + 5n \neq 12)$.

This completes the proof. ■

Q2. Say whether the following is true or false and support your answer by a proof: The sum of any five consecutive integers is divisible by 5 (without remainder).

PROOF Assume the above statement is true.

- If the sum of any five consecutive integers - call it s - is divisible by 5, then, there is an integer p such that $s = 5p$.
- Consider an integer n. The next four consecutive integers are: $n + 1$, $n + 2$, $n + 3$, and $n + 4$.
- The sum of these five consecutive integers, $s = n + (n + 1) + (n + 2) + (n + 3) + (n + 4)$, which is equal to $5n + 10$.
- Simplifying, $s = 5(n + 2)$. $(n + 2)$ is an integer - call it p. So, $s = 5p$. Therefore, s is divisible by 5.

Hence, the statement is true. This completes the proof. ■

Q3. Say whether the following is true or false and support your answer by a proof: For any integer n, the number $n^2 + n + 1$ is odd.

PROOF Assume the statement is true. We know that any odd number can be expressed in the form 2q + 1, and any even number can be expressed in the form 2q, where q $\in \mathcal{Z}$. Consider three cases: 1. n is odd, 2. n is even, and 3. n is zero.

- Case 1: If n is odd, n^2+n+1= $(2q+1)^2+(2q+1)+1$. Simplifying the RHS: $(2q+1)((2q+1)+1)+1$ (factoring out $(2q+1)$ from the first two terms). Further simplifying, RHS $= 2(2q+1)(q+1)+1$ (factoring out 2 from the second term). $(2q+1)(q+1)$ is an integer - let it be s. So, $n^2+n+1 = 2s+1$. Therefore, n^2+n+1 is odd when n is odd.
- Case 2: If n is even, n^2+n+1= $(2q)^2+2q+1$. Simplifying the RHS: $2q(2q+1)+1$. $q(2q+1)$ is an integer - let it be s. Therefore, $n^2+n+1 = 2s+1$. This shows that n^2+n+1 is odd when n is even.
- Case 3: If n is zero, n^2+n+1 reduces to 1, which is odd.

Therefore, for any integer n, n^2+n+1 is odd. This completes the proof. ■

Q4. Prove that every odd natural number is of one of the forms $4n+1$ or $4n+3$, where n is an integer.

PROOF We know from the Division Theorem that any natural number, m, can be expressed in the form $4n+r$, where $n, r \in \mathcal{N}$and $0 \leq r \leq 3$. So, any natural number can be expressed in one of the following forms: $4n, 4n+1, 4n+2$, or, $4n+3$. $4n$ and $4n+2$ are clearly even.

Hence, any odd natural number can be expressed either in the form $4n+1$ or in the form $4n+3$. This completes the proof. ■

Q5. Prove that for any integer n, at least one of the integers $n, n+2, n+4$ is divisible by 3.

PROOF If a number is divisble by 3, it can be expressed in the form $3p$, where $p \in \mathcal{Z}$. Consider an arbitrary integer n.

- If n is divisible by 3, the statement is true and the proof is complete. If n is not divisible by 3, it can be expressed in one of two forms (from the Division Theorem): $3p+1$ or $3p+2$, where $p \in \mathcal{Z}$.
- If n is of the form $3p+1$, $n+2$ simplifies to $(3p+1)+2 = 3(p+1)$. $(p+1)$ is an integer, therefore, $n+2$ is divisible by 3 if n is of the form $3p+1$.
- If n is of the form $3p+2$, $n+4$ simplifies to $(3p+2)+4 = 3(p+2)$. $(p+2)$ is an integer, therefore, $n+4$ is divisible by 3 if n is of the form $3p+2$.

Hence, for any integer n, at least one of $n, n+2$, or $n+4$ is divisible by 3. This completes the proof. ■

Q6. A classic unsolved problem in number theory asks if there are infinitely many pairs of 'twin primes', pairs of primes separated by 2, such as 3 and 5, 11 and 13, or 71 and 73. Prove that the only prime triple (i.e. three primes, each 2 from the next) is 3, 5, 7.

PROOF A number, n, is said to be prime if it is greater than 1 and divisible only by itself and 1. Or, it can not be expressed in the form $n = pq$, where $p, q \in \mathcal{Z}$. Let n be a prime number. Then, a prime triplet would be $n, n+2, n+4$.

1. If $n = 3$, then the prime triplet is $3, 5, 7$.
2. If $n > 3$, there can not be a prime triplet because at least one of $n, n+2, n+4$, where $n \in \mathcal{Z}$ is divisible by 3. The proof for this is as follows:

- If a number is divisble by 3, it can be expressed in the form $3p$, where $p \in \mathcal{Z}$. Consider an arbitrary integer s.
- If s is divisible by 3, the statement is true and the proof is complete. If s is not divisible by 3, it can be expressed in one of two forms: $3p + 1$ or $3p + 2$, where $p \in \mathcal{Z}$.
- If s is of the form $3p + 1$, $s + 2$ simplifies to $(3p + 1) + 2 = 3(p + 1)$. $(p + 1)$ is an integer, therefore, $s + 2$ is divisible by 3 if s is of the form $3p + 1$.
- If s is of the form $3p + 2$, $s + 4$ simplifies to $(3p + 2) + 4 = 3(p + 2)$. $(p + 2)$ is an integer, therefore, $s + 4$ is divisible by 3 if s is of the form $3p + 2$.
- Hence, for any integer s, at least one of $s, s + 2$, or $s + 4$ is divisible by 3.

Therefore, there is no prime triplet other than 3, 5, and 7. This completes the proof. ■

Q7. Prove that for any natural number n, $2 + 2^2 + 2^3 + ... + 2^n = 2^{n+1} - 2$.

PROOF By mathematical induction:

1. For $n = 1$, the identity reduces to $2^1 = 2^{1+1} - 2$, which simplifies to $2 = 2$. The identity is true for $n = 1$.
2. Assume the identity is true for n. Therefore, $2 + 2^2 + 2^3 + ... + 2^n = 2^{n+1} - 2$.
3. Add 2^{n+1} to the LHS: $(2 + 2^2 + 2^3 + ... + 2^n) + 2^{n+1}$. But, we know $2 + 2^2 + 2^3 + ... + 2^n = 2^{n+1} - 2$ from (2).
4. Simplifying the LHS in (3): $(2^{n+1} - 2) + 2^{n+1} = 2 * 2^{n+1} - 2 = 2^{n+2} - 2$, which is the result for $n + 1$.

Hence, by the principle of mathematical induction, the identity is true. ■

Q8. Prove (from the definition of a limit of a sequence) that if the sequence $\{a_n\}_{n=1}^{\infty}$ tends to limit L as $n \to \infty$, then for any fixed number $M > 0$, the sequence $\{Ma_n\}_{n=1}^{\infty}$ tends to the limit ML.

PROOF Let $\epsilon > 0$ be given. The sequence $\{a_n\}_{n=1}^{\infty}$ tends to the limit L, so, we can find a p such that for $m \geq p$: $|a_m - L| < \epsilon/M$, where $m, p \in \mathcal{N}$.

1. Simplifying the limit definition mentioned above: $M|a_m - L| < \epsilon$.
2. Simplifying the expression in (1) further: $|Ma_m - ML| < \epsilon$.
3. The expression in (2) is the definition of limit for the sequence $\{Ma_n\}_{n=1}^{\infty}$, and it is evident that it tends to the limit ML.

This completes the proof. ■

Q9. Given an infinite collection $A_n, n = 1, 2, ...$ of intervals of the real line, their *intersection* is defined to be $\bigcap\limits_{n=1}^{\infty} A_n = \{x|(\forall n)(x \in A_n)\}$. Give an example of a family of intervals $A_n, n = 1, 2, ...$, such that $A_{n+1} \subset A_n$ for all n and $\cap_{n=1}^{\infty} A_n = \emptyset$. Prove that your example has the stated property.

PROOF Consider the family of open intervals: $A_n = (0, 1/n)$ on the real line, where $n \in \mathcal{N}$. Then, A_1 is the interval $(0, 1)$, A_2 is the interval $(0, 1/2)$, and so on.

- Therefore, A_{n+1} will be the interval $(0, 1/n + 1)$.
- Clearly, $A_2 \subset A_1, A_3 \subset A_2, ..., A_{n+1} \subset A_n$ since all the elements in each of these intervals also belong to the preceding interval.
- As $n \to \infty$, it is evident that the limit of $1/n \to 0$. Therefore, as $n \to \infty$, the interval A_n approaches $(0, 0)$.
- We know that an open interval, $(a, b) = \{x|a < x < b\}$. So, $(0, 0) = \{x|0 < x < 0\}$, which is the empty set, $\emptyset$.
- Therefore, it is clear that $\bigcap\limits_{n=1}^{\infty} A_n = \emptyset$ (since there is no element that appears in all of these sets).

Hence, the family of intervals $A_n = (0, 1/n), n = 1, 2, ...,$ satisfies the above property. This completes the proof. ■

Q10. Give an example of a family of intervals $A_n, n = 1, 2, ...$, such that $A_{n+1} \subset A_n$ for all n and $\cap_{n=1}^{\infty} A_n$ consists of a single real number. Prove that your example has the stated property.

PROOF Consider the family of closed intervals: $A_n = [0, 1/n]$ on the real line, where $n \in \mathcal{N}$. Then, A_1 is the interval $[0, 1]$, A_2 is the interval $[0, 1/2]$, and so on.

- Therefore, A_{n+1} will be the interval $[0, 1/n + 1]$.
- Clearly, $A_2 \subset A_1, A_3 \subset A_2, ..., A_{n+1} \subset A_n$ since all the elements in each of these intervals also belong to the preceding interval.
- As $n \to \infty$, it is evident that the limit of $1/n \to 0$. Therefore, as $n \to \infty$, the interval A_n approaches $[0, 0]$.
- We know that a closed interval, $[a, b] = \{x | a \leq x \leq b\}$. So, $[0, 0] = \{x | 0 \leq x \leq 0\}$, which is a set with a single real number: $\{0\}$.
- Therefore, it is clear that $\bigcap\limits_{n=1}^{\infty} A_n = \{0\}$ (since 0 is the only element that belongs to all the sets).

Hence, the family of intervals $A_n = [0, 1/n], n = 1, 2, ...$, satisfies the above property. This completes the proof. ■