9. Given an infinite collection $A_n, n = 1, 2, ...$ of intervals of the real line, their *intersection* is defined to be

$$\bigcap_{n=1}^{\infty} A_n = \{x | (\forall n)(x \in A_n)\}$$

Give an example of a family of intervals $A_n, n = 1, 2, ...$, such that $A_{n+1} \subset A_n$ for all n and $\cap_{n=1}^{\infty} A_n = \emptyset$. Prove that your example has the stated property.

PROOF Let A_1be the interval $[0, 1]$of the real line. A_2be the interval $[1, 2]$, A_3be the interval $[2, 3]$ and so on until we have an infinite collection of intervals.

- Therefore, A_n will be the interval $[\max(A_{n-1}), n]$ and A_{n+1}will be the interval $[\max(A_n), n + 1]$.
- Clearly, $A_2 \subset A_1, A_3 \subset A_2, ..., A_{n+1} \subset A_n$ since each of these intervals have one element from the preceding interval.
- *Therefore, the family of intervals* $A_n = [\max(A_{n-1}), n], n = 1, 2, ...$ *satisfies the above property.*
- But, there is no element that appears in all the intervals. For example, $\cap_{n=1}^{3} A_n = A_1 \cap A_2 \cap A_3 = \emptyset$.
- Extending the same logic, it is clear that $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

This completes the proof. ■