9. Given an infinite collection $A_n, n = 1, 2, ...$ of intervals of the real line, their *intersection* is defined to be

$$\bigcap_{n=1}^{\infty} A_n = \{x | (\forall n)(x \in A_n)\}$$

Give an example of a family of intervals $A_n, n = 1, 2, ...,$ such that $A_{n+1} \subset A_n$ for all n and $\cap_{n=1}^{\infty} A_n = \emptyset$. Prove that your example has the stated property.

PROOF Consider the family of open intervals: $A_n = (0, 1/n)$ on the real line, where $n \in \mathcal{N}$. Then, A_1 is the interval $(0, 1)$, A_2 is the interval $(0, 1/2)$, and so on.

- Therefore, A_{n+1} will be the interval $(0, 1/n + 1)$.
- Clearly, $A_2 \subset A_1, A_3 \subset A_2, ..., A_{n+1} \subset A_n$ since all the elements in each of these intervals also belong to the preceding interval.
- As $n \to \infty$, it is evident that the limit of $1/n \to 0$. Therefore, as $n \to \infty$, the interval A_n approaches $(0, 0)$.
- We know that an open interval, $(a, b) = \{x | a < x < b\}$. So, $(0, 0) = \{x | 0 < x < 0\}$, which is the empty set, $\emptyset$.
- Therefore, it is clear that $\bigcap_{n=1}^{\infty} A_n = \emptyset$ (since there is no element that appears in all of these sets).

Hence, the family of intervals $A_n = (0, 1/n), n = 1, 2, ...,$ satisfies the above property. This completes the proof. ■