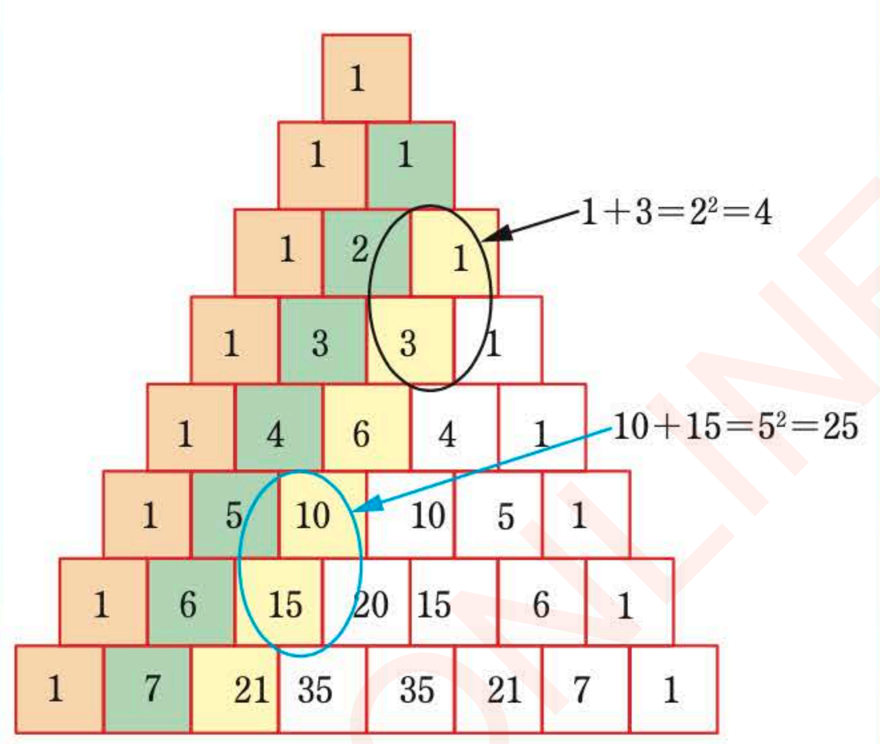
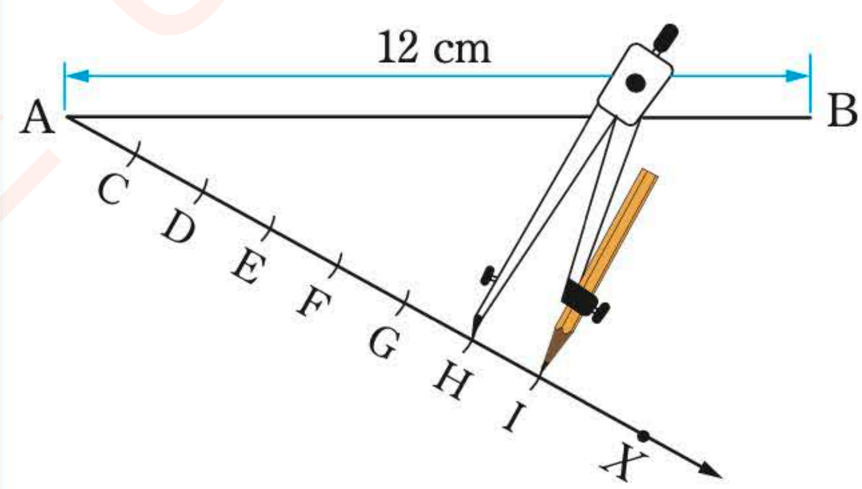
Additional Mathematics

for Secondary Schools

Student's Book

Form One





$$\theta_i = \frac{(n-2)\times 180^\circ}{n}$$

Tanzania Institute of Education

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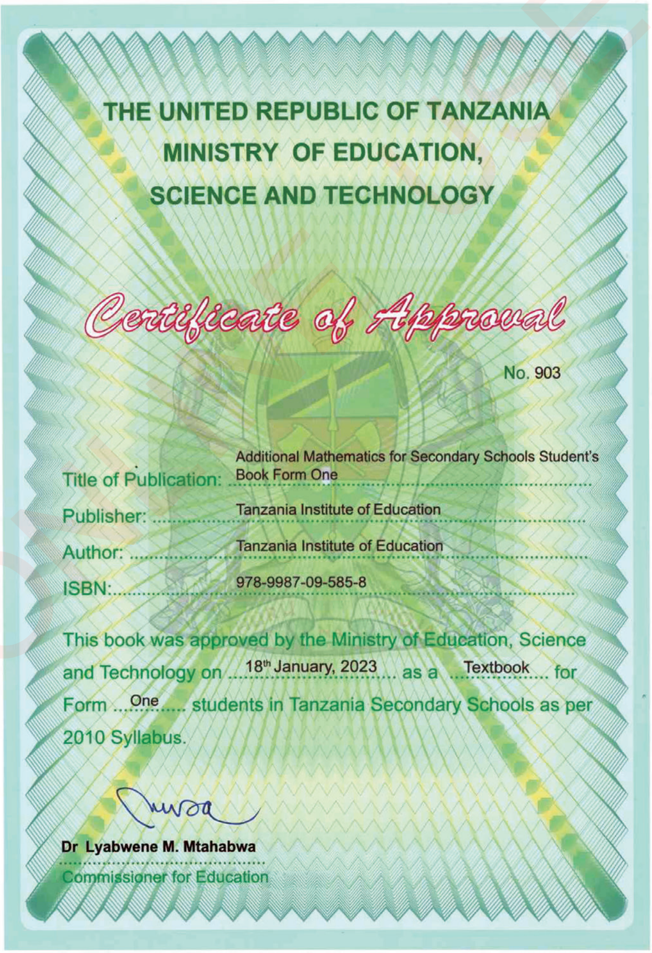
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Preface

This book, *Additional Mathematics for Secondary Schools Student's Book Form One* is written specifically for Form One students in the United Republic of Tanzania. It is written in accordance with the 2010 *Additional Mathematics Syllabus for Ordinary Level Secondary Schools Education Form I – IV,* issued by the then Ministry of Education and Vocational Training. The book consists of five chapters, namely; Numbers, Symmetry, Algebra, Geometrical constructions, and Coordinate geometry. Each chapter contains illustrations, activities, and exercises. Answers to exercises are provided. However, answers to questions involving proofs, verifications, and some illustrations are not provided. You are encouraged to do all activities and exercises together with other assignments provided. Doing so, will enable you to develop the intended competencies.

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Chapter One

Numbers

Introduction

In daily life, people try to count quantities such as goods, ornaments, jewels, animals, trees, and many others. The way of having a count of quantities led to the development of a common way of describing these quantities. The value given to these descriptions is known as a number. Numbers can be manipulated to make patterns. In this chapter, you will learn number patterns and rules of divisibility. The competencies developed will help you in classifying quantities of various things using appropriate number patterns and verifying divisibility of various quantities among individuals.

Number patterns

Number patterns describe relationships between numbers in a particular order of arrangement. The following activity demonstrates how number patterns can be recognised.

Activity: Recognising number patterns

Individually or in a group, perform the following tasks:

1. Choose any whole number from 1 to 9, record its value, and call it the first number.
2. Compute the second number by multiplying the first number obtained in task 1 by 3.
3. Compute the third number by multiplying the second number obtained in task 2 by 3.
4. Repeat the procedures in tasks 2 and 3 to get the values of the fourth, fifth, and sixth numbers.
5. List down the numbers obtained in tasks 1 to 4 in order of their arrangement separated by commas.
6. Share the list of numbers obtained in task 5 with others through discussion for more inputs.

From the Activity, it can be observed that the listed numbers make a pattern in which the next number is obtained by multiplying the preceding number by 3. A list of numbers following a certain rule to form a pattern is called a number pattern. Each number in a pattern is called a term of that pattern. A term number is a number that tells its position in a pattern. For instance, the pattern 2, 3, 4, 6, 8, 10, ... can be presented in tabular form as shown in Table 1.1.

Table 1.1: *Number pattern*

Term number	Terms of the pattern
1^{st} term	2
2^{nd} term	4
3^{rd} term	6
4^{th} term	8
5^{th} term	10

The number pattern in Table 1.1 shows that, each term is obtained by multiplying the term number by 2, that is, $1\times2=2$, $2\times2=4$, $3\times2=6$, and so on. Similarly, $-1, 2, -4, 8, -16, \ldots$ is a pattern of numbers in which the second number is obtained by multiplying the first number by -2, the third number is obtained by multiplying the second number by -2, and so on.

Generation of number patterns

There are various rules for generating different number patterns. Every pattern has a specific rule, and hence there are different types of number patterns such as ascending or descending number patterns, multiples of a certain number, powers of a certain number, or a specific formula describing the pattern.

Number patterns by rules

Solving a problem of number patterns means finding a rule which established the pattern. In order to identify the rule, it is necessary to have at least three consecutive terms of the pattern. Observing the behaviour of consecutive terms helps to understand the relationship between the terms in a pattern.

Example 1.1

Find the 10^{th} term in the pattern $1, 3, 5, 7, 9, 11, \ldots$

Solution

Given $1, 3, 5, 7, 9, 11, \ldots$

The first term is 1.

The second term is 3, which is obtained by adding 2 to the first term.

The third term is 5, which is obtained by adding 2 to the second term.

Thus, 2 is a constant or fixed number that is added to a preceding term to get the next term.

$$\begin{aligned}
\text{Hence, } 7^{th}\text{ term} &= 6^{th}\text{ term} + 2\\
&= 11 + 2 = 13,\\
8^{th}\text{ term} &= 7^{th}\text{ term} + 2\\
&= 13 + 2 = 15,\\
9^{th}\text{ term} &= 8^{th}\text{ term} + 2\\
&= 15 + 2 = 17,\\
10^{th}\text{ term} &= 9^{th}\text{ term} + 2\\
&= 17 + 2 = 19.
\end{aligned}$$

Therefore, the 10^{th} term is 19.

Example 1.2

Determine the next term in each of the following number patterns:

(a) 3, 6, 12, ____

(b) 24, 12, 6, ____

(c) 25, 23, 21, ____

(d) 1, 4, 27, 256, ____

Solution

(a) Given the pattern 3, 6, 12, ____.

In this pattern, 2 is a fixed (constant) number multiplied by the preceding term to get the next term.

Thus, the next term is $12 \times 2 = 24$.

Therefore, the next term is 24.

(b) Given the pattern 24, 12, 6, _____.

In this pattern, $\frac{1}{2}$ is a fixed number multiplied by the preceding term to get the next term.

Thus, the next term is $6 \times \frac{1}{2} = 3$.

Therefore, the next term is 3.

(c) Given 25, 23, 21, ______.

In this pattern, -2 is a fixed number added to the preceding term to get the next term.

Thus, the next term is $21 + (-2) = 19$.

Therefore, the next term is 19.

(d) Given the pattern 1, 4, 27, 256, ____.

The first term is $1 = 1^1$.

The second term is $4 = 2^2$.

The third term is $27 = 3^3$.

The fourth term is $256 = 4^4$.

Thus, this pattern is formed by powers of natural numbers.

Hence, the fifth tem is $3\,125 = 5^5$.

Therefore, the next tem is 3 125.

Note that; In some cases, patterns do not exhibit a fixed number either to be added or multiplied. In this regard, two or more steps are required to determine the specific rule.

Example 1.3

Determine the next term in each of the following number patterns by using step rules:

(a) 3, 6, 13, 24, 39, ______ (b) 1, 5, 14, 30, 55, 91, _____ (c) 6, 11, 13, 12, 8, 1, _____

Solution

(a) Pattern 3 6 13 24 39 $\boxed{58}$

Difference 3 7 11 15 $\boxed{19}$

Difference 4 4 4 $\boxed{4}$

Therefore, the next term of the pattern is 58.

(b) Pattern $\quad 1 \quad 5 \quad 14 \quad 30 \quad 55 \quad 91 \quad \boxed{140}$

Difference $\quad 4 \quad 9 \quad 16 \quad 25 \quad 36 \quad \boxed{49}$

Difference $\quad 5 \quad 7 \quad 9 \quad 11 \quad \boxed{13}$

Difference $\quad 2 \quad 2 \quad 2 \quad \boxed{2}$

Therefore, the next term of the pattern is 140.

(c) Pattern $\quad 6 \quad 11 \quad 13 \quad 12 \quad 8 \quad 1 \quad \boxed{-9}$

Difference $\quad 5 \quad 2 \quad -1 \quad -4 \quad -7 \quad \boxed{-10}$

Difference $\quad -3 \quad -3 \quad -3 \quad -3 \quad \boxed{-3}$

Therefore, the next term of the pattern is -9.

Example 1.4

David formulated a number pattern using a two-step rule. He started with 5 to develop the following number pattern; 5, 10, 8, 16, 14, 28, 26, ...

(a) Describe the two-step rule.

(b) Find the next three terms of the pattern using the two-step rule developed in (a).

Solution

(a) The two-step rule is described as follows:

Multiplication and then subtraction. That is, the second number is obtained by multiplying the first number by 2 and then 2 is subtracted from the second number to get the third number, and so on.

(b) The next three terms are:

$26\times2=52$, $52-2=50$,

$50\times2=100$.

Therefore, the next three terms of the number pattern are 52, 50, and 100.

Example 1.5

Khadija formulated a number pattern using a two-step rule. She started with 1 as her first number and ended up forming a number pattern 1, 4, 8, 32, 36, ...

(a) Describe the two-step rule.

(b) Find the next four terms of the pattern using the two-step rule described in (a).

Solution

Given 1, 4, 8, 32, 36, ...

(a) The two-step rule of the given number pattern involves multiplication and then addition. That is, the second number is obtained by multiplying the first term by 4 and the third number is obtained by adding 4 to the second term. The fourth term is obtained by multiplying the third term by 4, and so on.

(b) The next four terms of the pattern are obtained as follows:

$36 \times 4 = 144, \quad 144 + 4 = 148,$

$148 \times 4 = 592, \quad 592 + 4 = 596$

Therefore, the next four terms of the pattern are 144, 148, 592, and 596.

Exercise 1.1

1. Find the missing terms in each of the following number patterns:

(a) $-2,\ 2,\ 6,\ 10,$ ___, ___, ___, 26

(b) $-3,\ -4,\ -4,\ -3,\ -1,\ 2,\ 6,$ ___, ___, ___, 32

(c) $-27, -8, -1, 0, 1,$ ___, ___, ___, 125

(d) $27, -9, 3, -1,$ ___, ___, ___, $-\frac{1}{81}$

(e) $10, 5, 7, 2, 4, -1,$ ___, ___, ___, -7

(f) 16, 9, 13, 6, 10, 3, 7, ___, ___, ___, 1

2. Find the next four terms in each of the following number patterns:

(a) $\frac{2}{4}, \frac{4}{9}, \frac{7}{15}, \frac{11}{22},$ ___, ___, ___, ___

(b) $6, 11, 13, 12, 8, 1, -9,$ ___, ___, ___, ___

(c) $-13, -2, 6, 11, 13,$ ___, ___, ___, ___

(d) $-14, -13, -9, 0, 16,$ ___, ___, ___, ___

3. Find the first three terms in each of the following number patterns:

(a) ___, ___, ___, 27, 64, 125, 216

(b) ___, ___, ___, 20, 30, 42, 56, 72

(c) ___, ___, ___, $0, \frac{16}{31}, \frac{41}{35}, \frac{77}{38}, \frac{126}{40}$

(d) ___, ___, ___, $\frac{12}{13}, \frac{8}{6}, \frac{1}{10}, -\frac{9}{3}, -\frac{22}{7}$

4. Describe each of the following number patterns:
   (a) 3, 10, 17, 24, 31, ...
   (b) 3, 6, 11, 18, 27, ...

5. A student recorded the temperature variations in one of the cold rooms of the Chemistry laboratory. Initially, the temperature was $8\,°\mathrm{C}$ before it started decreasing by $2\,°\mathrm{C}$ in every 30 seconds.
   (a) What was the temperature after:
      (i) 180 seconds?
      (ii) 8 minutes?
      (iii) 5 minutes and 30 seconds?
   (b) How long did it take for the temperature to be:
      (i) $0\,°\mathrm{C}$?
      (ii) $-6\,°\mathrm{C}$?
      (iii) $-14\,°\mathrm{C}$?

Number patterns by formulas

There are occasions whereby a number pattern is given and it is required to find a formula which can be used to generate the number pattern. This formula is called the n^{th} term. For instance, $\frac{1}{3}, \frac{3}{4}, \frac{5}{5}, \frac{7}{6}, \ldots$ is a number pattern generated by the formula $T_n = \frac{2n-1}{n+2}$, where n is a counting number. On the other hand, a formula can be given in order to generate the number pattern.

Example 1.6

Find the n^{th} term for each of the following number patterns:

(a) 1, 3, 5, 7, 9, 11, ...

(b) $-1, 1, -1, 1, -1, 1, \ldots$

(c) $6, 2, -2, -6, -10, -14, \ldots$

Solution

(a) Given 1, 3, 5, 7, 9, 11, ...

Thus, 1^{st} term: $n = 1;\; 1 = 1\times 2 - 1,$

2^{nd} term: $n = 2;\; 3 = 2\times 2 - 1,$

3^{rd} term: $n = 3;\; 5 = 3\times 2 - 1,$

4^{th} term: $n = 4;\; 7 = 4\times 2 - 1,$

$\vdots \qquad\qquad \vdots$

n^{th} term: $a_n = n\times 2 - 1.$

Thus, $a_n = 2n - 1.$

Therefore, the n^{th} term formula is $a_n = 2n - 1$, where n is a natural number.

(b) Given $-1, 1, -1, 1, -1, 1, \ldots$

Since the terms -1 and 1 are alternating (interchanging), then the n^{th} term is $b_n = (-1)^n$.

Therefore, the n^{th} term is $b_n = (-1)^n$. where n is a natural number.

(c) Given $6, 2, -2, -6, -10, -14, \ldots$

Pattern $\quad 6 \quad 2 \quad -2 \quad -6 \quad -10 \quad -14 \ldots$

Difference $\quad -4 \quad -4 \quad -4 \quad -4 \quad -4$

That is, 1^{st} term: $n=1;\ 6=-4\times1+10,$

2^{nd} term: $n=2;\ 2=-4\times2+10,$

3^{rd} term: $n=3;\ -2=-4\times3+10,$

4^{th} term: $n=4,\ -6=-4\times4+10,$

$\vdots \qquad \vdots$

n^{th} term: $a_n=-4\times n+10$

Thus, $a_n=-4n+10.$

Therefore, the n^{th} term is $a_n=-4n+10,$ where n is a natural number.

Example 1.7

Find the n^{th} term for each of the following number patterns:

(a) 5, 9, 13, 17, ...

(b) $-7, -4, -1, 2, ...$

(c) $6, 4, 2, 0, -2, ...$

Solution

(a) Given 5, 9, 13, 17, ...

Pattern 5 9 13 17...

Difference 4 4 4

That is, 1^{st} term: $n=1;\ 5=4\times1+1,$

2^{nd} term: $n=2;\ 9=4\times2+1,$

3^{rd} term: $n=3;\ 13=4\times3+1,$

4^{th} term: $n=4,\ 17=4\times4+1,$

$\vdots \qquad \vdots$

n^{th} term: $a_n=4\times n+1$

Thus, $a_n=4n+1.$

Therefore, the n^{th} term is $a_n=4n+1,$ where n is a natural number.

(b) Given $-7, -4, -1, 2, ...$

Pattern $-7 \quad -4 \quad -1 \quad 2 ...$

Difference 3 3 3

That is,

1^{st} term: $n=1;\ -7=3\times1-10,$

2^{nd} term: $n=2;\ -4=3\times2-10,$

3^{rd} term: $n=3;\ -1=3\times3-10,$

4^{th} term: $n=4,\ 2=3\times4-10,$

$\vdots \qquad \vdots$

n^{th} term: $a_n=3\times n-10$

Thus, $a_n=3n-10.$

Therefore, the n^{th} term is $a_n=3n-10,$ where n is a natural number.

(c) Given $6, 4, 2, 0, -2, ...$

Pattern $6 \quad 4 \quad 2 \quad 0 \quad -2 ...$

Difference $-2 \quad -2 \quad -2 \quad -2$

That is, 1^{st} term: $n=1;\ 6=-2\times1+8,$

2^{nd} term: $n=2;\ 4=-2\times2+8,$

3^{rd} term: $n=3;\ 2=-2\times3+8,$

4^{th} term: $n=4,\ 0=-2\times4+8,$

5^{th} term: $n=5,\ -2=-2\times5+8,$

$\vdots \qquad \vdots$

n^{th} term: $a_n=-2\times n+8$

Thus, $a_n=-2n+8.$

Therefore, the n^{th} term is $a_n = -2n + 8$, where n is a natural number.

Example 1.8

Generate the number pattern by using the formula $I_n = \frac{n^2+1}{2n+1}$, where n is a natural number.

Solution

Given the formula $I_n = \frac{n^2+1}{2n+1}$.

The natural numbers are 1, 2, 3, 4, 5, ...

The first term is obtained when $n = 1$.

Thus, $I_1 = \frac{1^2+1}{(2\times1)+1} = \frac{2}{3}$.

The second term is obtained when $n = 2$.

Thus, $I_2 = \frac{2^2+1}{(2\times2)+1} = \frac{5}{5}$.

The third term is obtained when $n = 3$.

Thus, $I_3 = \frac{3^2+1}{(2\times3)+1} = \frac{10}{7}$.

The fourth term is obtained when $n = 4$.

Thus, $I_4 = \frac{4^2+1}{(2\times4)+1} = \frac{17}{9}$.

The next terms can be obtained by substituting the values of n in the given formula in a similar way.

Therefore, the number pattern is

$$\frac{2}{3}, \frac{5}{5}, \frac{10}{7}, \frac{17}{9}, \ldots$$

Example 1.9

Determine the number pattern generated by using the formula $s_n = \frac{3n^2-4}{2n-2}$, where n is a positive even number.

Solution

Given the formula $s_n = \frac{3n^2-4}{2n-2}$.

The positive even numbers are 2, 4, 6, 8, ...

The first term is obtained when $n = 2$.

Thus, the first term, $s_2 = \frac{3(2)^2-4}{2(2)-2}$

$$= 4.$$

The second term is obtained when $n = 4$.

Thus, the second term, $s_4 = \frac{3(4)^2-4}{2(4)-2}$

$$= \frac{22}{3}.$$

The third term is obtained when $n = 6$.

Thus, the third term, $s_6 = \frac{3(6)^2-4}{2(6)-2}$

$$= \frac{52}{5}.$$

Other terms can be obtained in a similar way.

Therefore, the number pattern is

$$4, \frac{22}{3}, \frac{52}{5}, \frac{94}{7}, \frac{148}{9}, \ldots$$

Exercise 1.2

1. Find the n^{th} term for each of the following number patterns:
   (a) 1, 2, 4, 8, ...
   (b) 2, 4, 6, 8, ...
   (c) 3, 9, 19, 33, 51, ...
   (d) 12, 15, 18, 21, ...
   (e) 2, 5, 10, 17, ...
2. Find the first three terms of the pattern generated by the formula $T_n = \frac{3n-1}{4n+2}$, where n is an odd number.
3. Find the first four terms of the pattern defined by the formula $S_n = \frac{n^3+1}{5-2n}$, where n is a whole number.
4. Generate the last four terms of the number pattern defined by the formula $I_n = \frac{3n^2+n}{1-3n}$, where n is a whole number less than ten.
5. Determine the first five terms of the number pattern generated by the formula $a_n = \frac{n^3+n}{2n+1}$, where n is a negative integer greater than -10.
6. Given the pattern 1, 4, 7, 10, 13. Find the sum of the last two terms generated by the formula $b_n = 4n^2 - n$, using the given pattern as the values of n.

Fibonacci sequence

The Fibonacci sequence is a number pattern in which each term is the sum of the two preceding terms. The Fibonacci sequence is commonly denoted by F_n such that $F_n = F_{n-1} + F_{n-2}$, where n is a natural number greater than or equal to 2. To generate the Fibonacci sequence, it is necessary to state two initial numbers and then the formula can be applied to generate the next terms. For instance, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, … is the Fibonacci sequence with initial numbers, $F_0 = 1$ and $F_1 = 1$. Numbers generated in the Fibonacci sequence are called Fibonacci numbers.

Example 1.10

Find the next four terms in each of the following Fibonacci sequences:

(a) -1, 1, 0, ____, ____, ____, ____.

(b) 2, 5, 7, 12,____, ____, ____, ____.

Solution

(a) Given the sequence -1, 1, 0, ____, ____, ____, ____.

From the Fibonacci sequence, the terms are obtained as;

$-1+1=0,\ 1+0=1,\ 0+1=1,$

$1+1=2,\ 1+2=3.$

Therefore, the next four terms are; 1, 1, 2, and 3.

(b) Given the sequence 2, 5, 7, 12, ____, ____, ____, ____.

From the Fibonacci sequence, the next four terms are;

$7+12=19,\ 12+19=31,\ 19+31=50,$
$31+50=81.$

Therefore, the next four terms are; 19, 31, 50, and 81.

Example 1.11

Using Fibonacci numbers, verify that $F_0+F_1+F_2+F_3+F_4+F_5+F_6=F_8-1.$

Solution

Let $F_0=1$ and $F_1=1$ be the two initial numbers of the Fibonacci sequence. It follows that, the next Fibonacci numbers are:

$F_2=1+1=2,$

$F_3=F_2+F_1$
$=2+1=3,$

$F_4=F_3+F_2$
$=3+2=5,$

$F_5=F_4+F_3$
$=5+3=8,$

$F_6=F_5+F_4$
$=8+5=13,$

$F_7=F_6+F_5$
$=13+8=21,$

$F_8=F_7+F_6$
$=21+13=34.$

Thus, $F_0+F_1+F_2+F_3+F_4+F_5+F_6=$
$1+1+2+3+5+8+13=33.$

On the other hand,
$F_8-1=34-1=33.$

Therefore,
$F_0+F_1+F_2+F_3+F_4+F_5+F_6=F_8-1.$

Example 1.12

Show that for any Fibonacci sequence whose first two terms are $F_0=F_1=1$, F_3 divides F_{11}.

Solution

Given $F_0=1$ and $F_1=1$. It implies that, the Fibonacci sequence is 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, …
From the Fibonacci sequence,
$F_3=3$ and $F_{11}=144,$ such that

$$\frac{F_{11}}{F_3}=\frac{144}{3}=48,$$ which is an integer.

Therefore, F_3 divides F_{11}.

Exercise 1.3

1. In each of the following Fibonacci sequences, fill in the missing numbers to complete the sequence:
   (a) $-2,\ 2,\ 0,\ 2,$ ___, ___, ___, ___, 16
   (b) $-6,\ 2,\ -4,$ ___, ___, ___, ___, -22
   (c) 2, 4, 6, 10, 16, ___, ___, ___, ___, 178

(d) $-\frac{3}{4}, -\frac{5}{4}, -2, -\frac{13}{4}$, ___, ___, ___, ___, -36

(e) $-0.5, -0.7, -1.2, -1.9$, ___, ___, ___, ___, -21.2

2. Find the next four terms in each of the following patterns:

(a) $3x, 3x+2y, 6x+2y$, ___, ___, ___, ___

(b) $-x,\ x+y,\ y$, ___, ___, ___, ___

(c) $-2x-7y, 3x-y, x-8y$, ___, ___, ___, ___

(d) $\frac{1}{7}a+b, -\frac{2}{3}a-\frac{4}{7}b, -\frac{11}{21}a+\frac{3}{7}b$, ___, ___, ___, ___

3. An Ankole cow matures at the age of 2 years and reproduces every year after the first calf. A school dairy project started with one matured Ankole cow. Suppose the cow reproduces female calves only.

(a) How many cows (including calves) will there be after five years?

(b) How long will it take to have 610 cows (including calves)?

Pascal's triangle

The Pascal's triangle is an infinite triangular array of numbers in which the first number at the top is 1 and each term below the first two rows is the sum of the two terms above it.

The following are the steps for constructing the Pascal's triangle:

1. Write 1 at the top and with 1s running down the two sides of the triangle as shown in the following pattern.

$$\begin{array}{ccc} & 1 & \\ 1 & & 1 \end{array}$$

2. Generate new rows to build up a triangle of numbers. Each new row must begin and end with 1 as shown in the following pattern.

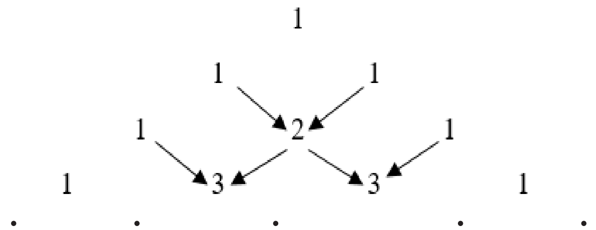
$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & \square & & 1 & \\ 1 & & \square & & \square & & 1 \\ \cdot & \cdot & & \cdot & & \cdot & \cdot \end{array}$$

3. The remaining numbers in each row are obtained by adding the two numbers in the row above which lie above left and above right of the intended number. Thus, adding the two 1s in the second row gives 2, and this resulting number goes in the vacant space in the third row as shown in the following pattern.

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & \square & & \square & & 1 \\ \cdot & \cdot & & \cdot & & \cdot & \cdot \end{array}$$

4. The two vacant spaces in the fourth row are each found by adding together the two numbers in the

third row which lie above left and above right of the intended number, that is, $1 + 2 = 3$ and $2 + 1 = 3$ as shown in the following pattern.



5. Continue building up the triangle in this way to obtain many rows as possible. The following illustration shows the first eight rows of the Pascal's triangle.

First row:								1							
Second row:							1		1						
Third row:						1		2		1					
Fourth row:					1		3		3		1				
Fifth row:				1		4		6		4		1			
Sixth row:			1		5		10		10		5		1		
Seventh row:		1		6		15		20		15		6		1	
Eighth row:	1		7		21		35		35		21		7		1

Properties of the Pascal's triangle

1. All the numbers in the Pascal's triangle are positive.
2. The initial and final numbers in each row are 1s.
3. The numbers are symmetrical about a vertical line through the apex of the triangle. The initial row with a single 1 on it is symmetric. This means that, for a number generated on the left, there is a corresponding number on the right.
4. The sum of numbers in the r^{th} row of the Pascal's triangle is given by 2^{r-1}, where $r = 1$ for the 1st row, $r = 2$ for the 2nd row, $r = 3$ for the 3rd row, and so on.
5. The numbers in the third diagonal when read from top right or left to bottom left or right, respectively, are called triangular numbers. For instance, 1, 3, 6, 10, 15, … are triangular numbers.
6. The numbers in the fourth diagonal when read from top right or top left to bottom left or bottom right, respectively, are called tetrahedral numbers. For instance, 1, 4, 10, 20, 35, … are tetrahedral numbers.

Example 1.13

Find the sum of numbers in the fifth row of the Pascal's triangle.

Solution

In the 5^{th} row, $r = 5$.

Thus, sum of entries $= 2^{r-1}$

$= 2^{5-1} = 2^4$

$= 16.$

Therefore, the sum of numbers in the fifth row of the Pascal's triangle is 16.

Example 1.14

List down the numbers in the seventh row of the Pascal's triangle and hence, find their sum.

Solution

The Pascal's triangle with seven rows is given as follows:

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

Numbers in the seventh row are 1, 6, 15, 20, 15, 6, and 1.

Sum of entries $=$

$1 + 6 + 15 + 20 + 15 + 6 + 1 = 64.$

Alternatively,

Sum of entries $= 2^{r-1}$, where $r = 7$

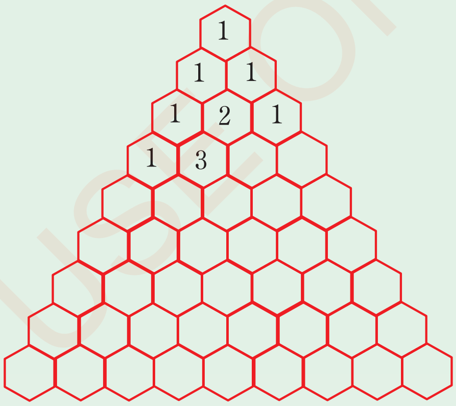
$= 2^{7-1}$

$= 2^6 = 64.$

Therefore, the numbers in the seventh row are 1, 6, 15, 20, 15, 6, 1 and their sum is 64.

Exercise 1.4

1. Construct the Pascal's triangle with six rows and use it to:
   (a) List down the numbers in the 4^{th} row.
   (b) Find the sum of the numbers in the 6^{th} row.
2. (a) Complete the following Pascal's triangle:



   (b) From the Pascal's triangle in (a), what do you observe in the first two diagonals when numbers are read from the top right to the bottom left?
   (c) List down all triangular numbers from the Pascal's triangle in (a).
   (d) List down all tetrahedral numbers from the Pascal's triangle in (a).
3. Study carefully the following number pattern and then answer the questions that follow:

Row	Number pattern							
1	1							
2	1	1						
3	1	2	1					
4	1	3	3	1				
5	1	4	6	4	1			
6	1	5	10	10	5	1		
7	1	6	15	20	15	6	1	
8	1	7	21	35	35	21	7	1

(a) Name the pattern.

(b) How is the entry 35 in the 8th row obtained?

(c) Determine the triangular numbers.

(d) List the numbers in the 9th and 10th rows.

(e) What is the name of the figure formed by the pattern?

Number charts

A number chart is a table containing numbers in a numerical order. The general outlook of the number chart depicts a pattern. There are different types of number charts such as; the number chart of hundreds, odd numbers chart, even numbers chart, prime numbers chart, blank number chart, addition number chart, and multiplication number chart, among many others. For instance, a multiplication chart of numbers from 1 to 10 as given in Table 1.2 is a table showing the products of two numbers. Often, one list of numbers is written to the left column and another list to the top row, and a list of products is presented in a rectangular array. From the multiplication number chart in Table 1.2, some patterns may be directly observed. For instance, a diagonal of perfect square numbers; 1, 4, 9, 16, 25, 36, 49, 64, 81, and 100.

Table 1.2: *Multiplication number chart*

×	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Example 1.15

Study carefully the following number chart and then answer the questions that follow.

2	3	4	5	6	7	8	9
3	4	5	6	7	8	9	10
4	5	6	7	8	9	10	11
5	6	7	8	9	10	11	12
6	7	8	9	10	11	12	13
7	8	9	10	11	12	13	14
8	9	10	11	12	13	14	19
9	10	11	12	13	14	15	16

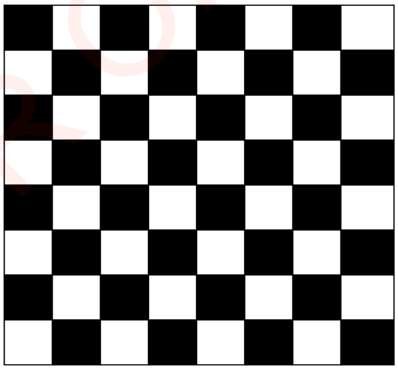
(a) Shade all squares containing even numbers and delete all odd numbers.

(b) Which traditional game is depicted by the result in (a)?

(c) What is that name of the instrument used to play the game in (b)?

Solution

(a) The resulting table after shading all squares containing even numbers and deleting all odd numbers is as follows.



(b) A traditional game depicted is a checkers game or draughts.

(c) The instrument used to play the game in (a) is a checkerboard or draughtboard.

Example 1.16

Copy the following number chart and then fill the missing entries such that the sum of numbers in each column or row is forty five and no number occurs more than once.

	19	
		13
16	11	

Solution

From the given chart, starting with the bottom row which misses only one number gives, $45-(16+11)=18$. Hence, the bottom row is now complete with the numbers 16, 11, and 18.

Similarly, the missing number in the middle column is $45-(19+11)=15$. All other missing numbers can now be obtained in a similar way.

Therefore, the complete number chart is given as follows.

sum	45	45	45	sum
45	12	19	14	45
45	17	15	13	45
45	16	11	18	45
sum	45	45	45	sum

Exercise 1.5

1. Prepare a multiplication number chart with 256 as the largest number in the chart and then answer the following questions:

(a) Write down all perfect squares from the chart.

(b) Write down all perfect cubes from the chart.

(c) Write down the number of diagonals containing even numbers only.

(d) Write down the number of diagonals containing even and odd numbers.

(e) How many diagonals are in the chart?

2. Fill in the missing numbers in the following chart such that the sum in each row and column is eighty-seven and each number occurs only once.

	24	28	
27			25
	30		
23			

3. Fill in the missing numbers in the following chart such that the sum in each row and column is thirty-four and each number occurs only once.

	11	14	
13			12
	16		
			15

4. Construct a number chart of 3 rows and 3 columns with entries $-5, -4, -3, -2, -1, 0, 3, 5,$ and 7 such that the sum in each row and each column is zero.

5. The following is a number chart consisting of another number chart. Fill in the numbers between 9 and 17 inclusive such that the sum in each row and each column in the inner chart is thirty-nine and the sum in each row and each column in the outer number chart is sixty-five.

23	1	2	20	19
22				4
5				21
8				18
7	25	24	6	3

6. Use counting numbers less than 10 to fill in a number chart with 3 rows and 3 columns such that 5 appears at the centre of the diagonals and the sum in each row, each column, and in the main diagonals is fifteen.

Applications of number patterns

A number pattern is more than a beautiful design. It follows a predictable rule which allows to predict what will come next. Number patterns are helpful in developing geometrical patterns, predicting sales in business, and arranging similar objects in a good order among many other applications.

Example 1.17

Lutome runs a samosa stall at Kingalu market. He sells each samosa at Tsh 300. Instead of calculating how much each order costs, he decides to make the following table.

Number of samosa	1	2	3	4	5	6	7	8	9
Cost (in Tsh)	300	600	900	1 200					

(a) Complete the table made by Lutome.

(b) Asha wants to buy 10 samosa. Describe two different methods which can help Lutome to find out the cost.

(c) Which of the two methods described in (b) will help Lutome most if Asha wants to buy 55 samosa?

Solution

(a) The complete table made by Lutome is as follows.

Number of samosa	1	2	3	4	5	6	7	8	9
Cost (in Tsh)	300	600	900	1 200	1 500	1 800	2 100	2 400	2 700

(b) (i) He can extend the patterns by adding 1 to the last number in the first row and 300 to the last number in the second row. That is,

Number of samosa	1	2	3	4	5	6	7	8	9	10
Cost (in Tsh)	300	600	900	1 200	1 500	1 800	2 100	2 400	2 700	3 000

Thus, the cost is Tsh $(2\,700+300)$ $=$ Tsh $3\ 000$.

(ii) He can find the required number in the second row by multiplying the number in the first row by 300, that is, $10\times300=3\ 000$.

Therefore, the cost for 10 samosa is Tsh 3 000.

(c) The second method will help Lutome most.

Multiplying the number of samosa by 300 gives $55\times300=16\ 500$.

Therefore, the total cost for 55 samosa is Tsh 16 500.

Exercise 1.6

1. Regina, Habiba, and Rebecca attended a clinic at Buziku health centre for family planning advise. They were advised to give birth after every three years and should stop at 54 years of age. Suppose that all of them gave birth in 2021 at the age of 21, 27, and 30 years, respectively.
   (a) If no death and all other factors are held constant, what is the number of children for all mothers at the end of 2045?
   (b) How many children will Habiba have after the stop age?
   (c) If Regina's first child was a girl who also gave birth at her 21 years and followed the same plan, how many grandchildren will Regina have at 54?
2. Khalifa and Joan were playing a game on numbers. Khalifa posed the following table and asked Joan to suggest any possible pair of row titles of the table. Suggest Joan's responses and fill the responses in the spaces provided.

	1	2	3	4	5	6	7	8	9	10
	7	14	21	28	35	42	49	56	63	70

3. Record the following numerals in your exercise book, write the numbers in ascending order, and then answer the questions that follow.

22 99 33 55 66 88 11 44 77

   (a) What is the common property of all the numbers?
   (b) What is the rule for this number pattern?
   (c) What would be the twentieth number in this number pattern? How did you obtain it?
4. Using a graph paper, draw rectangles with dimensions of 1 cm × 3.5 cm, 2 cm × 3.5 cm, 3 cm × 3.5 cm, and 4 cm × 3.5 cm.
   (a) Determine and record in ascending order the areas of the rectangles.
   (b) What do you notice about the areas of these rectangles in relation to the length of their sides?

Rules for divisibility

The quality of a number being divided by another number, without a remainder is referred to as divisibility. The rules for divisibility suggest the approach on how different numbers can be operated by dividing and multiplying. A divisibility rule is a kind of shortcut that helps in identifying if a given integer is divisible by another integer by examining its digits, without performing the whole division process. A divisor of a number is an integer that completely divides the given number. Multiple divisibility rules can be applied to the same number which can quickly determine its prime factorization. Divisibility rules help to determine if a given number is prime or non-prime or composite number.

A prime number is a number which is divisible by 1 and itself. In other words, it has only two factors, namely; 1 and itself. For instance, 11 has only two factors which are 1 and 11. Thus, 11 is a prime number.

A composite number is a number which has more than two factors. For instance, 24 has 1, 2, 3, 4, 6, 8, 12, and 24 as its factors. Thus, 24 is a composite number. Divisibility rules for natural numbers are very useful because they help us to quickly determine if a number can be divided without performing a long division. This is useful especially when the numbers are large. In general, a natural number x divides another natural number y if and only if there exists a whole number n such that $n \times x = y$. There are some well-known standard rules for divisibility of natural numbers. Table 1.3 shows some divisibility rules and corresponding examples.

Table 1.3: *Divisibility rules*

No.	Divisibility rules	Examples
1.	Divisibility of numbers by 2: A number is divisible by 2 if the last digit (a digit in its ones position) is an even number or 0.	124 is divisible by 2 because the last digit (4) is even. 110 is divisible by 2 because its digit in ones position is 0.
2.	Divisibility of numbers by 3: A number is divisible by 3 if the sum of its digits is divisible by 3.	1 290 is divisible by 3 because $1+2+9+0=12$, and 12 is divisible by 3, that is, $12 \div 3 = 4$.
3.	Divisibility of numbers by 4: A number is divisible by 4 if the number formed by its last two digits (ones and tens) is divisible by 4 or the last two digits are zeros.	1 084 is divisible by 4 because 84 is divisible by 4, that is, $84 \div 4 = 21$. 1 500 is divisible by 4 because its last two digits are zeros.
4.	Divisibility of numbers by 5: A number is divisible by 5 if it has either 0 or 5 in its ones position.	1 267 350 is divisible by 5 because the digit in its ones position is 0. 175 is divisible by 5 because the digit in its ones position is 5.

No.	Divisibility rules	Examples
5.	Divisibility of numbers by 6: A number is divisible by 6 if it is divisible by both 2 and 3, and it is an even number.	1 290 is divisible by 6 because: (i) It is even, thus, it is divisible by 2. (ii) It is divisible by 3 since the sum of the digits is divisible by 3.
6.	Divisibility of numbers by 7: A number is divisible by 7 if the difference between twice the last digit and the remaining part of the given number is either 0 or a multiple of 7. **Note:** If a number is very large, repeat the process until a two-digit number is obtained.	2 975 is divisible by 7 because the difference between twice the last digit and the number formed by other digits is a multiple of 7, that is, $297-(5\times 2) = 297 - 10 = 287$. $28-2(7)=14$, which is divisible by 7. Hence, 2 975 is divisible by 7.
7.	Divisibility of numbers by 8: A number is divisible by 8 if the number formed by its last three digits is divisible by 8.	1 240 is divisible by 8 because 240 is divisible by 8, that is, $240 \div 8 = 30$. 737 056 is divisible by 8 because the last three digits are 056, which gives 56; a number divisible by 8.
8.	Divisibility of numbers by 9: A number is divisible by 9 if the sum of its digits is divisible by 9.	504 is divisible by 9 because $5+0+4=9$, and 9 is divisible by 9, that is, $9 \div 9 = 1$.
9.	Divisibility of numbers by 10: A number is divisible by 10 if it has 0 in its ones position.	1 230 is divisible by 10 because its digit in ones position is 0.
10.	Divisibility of numbers by 11: A number is divisible by 11 if the difference between the sum of the digits at odd and even places in the given number is either 0 or a multiple of 11.	3 784 is divisible by 11 because: The digits at even place are 7 and 4, and their sum is $7+4=11$. The digits at odd place are 3 and 8, and their sum is $3+8=11$. The difference of the sums is $11-11=0$. Hence, 3 784 is divisible by 11.

No.	Divisibility rules	Examples
11.	Divisibility of numbers by 12: A number is divisible by 12 if it is divisible by both 3 and 4, that is, the last two digits are divisible by 4 and the sum of its digits is divisible by 3.	462 157 692 is divisible by 12 because: The last 2 digits are 92, thus, divisible by 4. The sum is $4+6+2+1+5+7+6+9+2=42$, which is divisible by 3. Therefore, 462 157 692is divisible by 12.
12.	Divisibility of numbers by 13: Multiply the last digit by 4 and add the result to the remaining number. The result must be divisible by 13. **Note that;** If a given number is very large, repeat the process until a two-digit number is obtained.	9 009 is divisible by 13 because the last digit is 9 and the remaining number is 900. Thus, $900+(9\times4)=936$. The last digit is 6. Thus, $93+(6\times4)=117$. The last digit is 7. Thus, $11+(7\times4)=39$. Since 39 is divisible by 13, then 9 009 is divisible by 13.
13.	Divisibility of numbers by 14: A number is divisible by 14 if the number is divisible by 2 and 7 because 2 and 7 are prime factors of 14.	1 064 is divisible by 14 becuse: 1 064 is even, hence it is divisible by 2. $106-(4\times2)=98$, which is a multiple of 7, hence 1 064 is divisible by 14.
14.	Divisibility of numbers by 15: A number is divisible by 15 if it is divisible by both 3 and 5 because 3 and 5 are prime factors of 15.	118 395 is divisible by 15 becuse: Sum of its digits is $1+1+8+3+9+5=27$, which is divisible by 3. 118 395 ends with 5. Hence, it is divisible by 5.
15.	Divisibility of numbers by 16: A number is divisible by 16 if the number formed by the last three digits is divisible by 16. **Note that;** If a digit in hundreds is zero, then a number formed by the last four digits in the given number must be divisible by 16.	7 852 176 is divisible by 16 because 176 is divisible by 16. 735 737 056 is divisible by 16 because 7 056 is divisible 16.

No.	Divisibility rules	Examples
16.	Divisibility of numbers by 17: If the product of the last digit of the given number and 5 is subtracted from the number formed by the remaining digits and the results is 0 or a multiple of 17, then it is divisible by 17. If the number is still large, repeat the process.	867 is divisible by 17. $86-7\times5=51$ Since 51 is a multiple of 17, then 867 is divisible by 17. 1 615 is divisible by 17 since $161-5\times5=136$. Repeating the process, $13-30=-17$, which is divisible by 17.
17.	Divisibility of numbers by 18: A number is divisible by 18 if it is divisible by both 2 and 9, that is if the given number is even and the sum of its digits is divisible by 9.	1 728 is divisible by 18. That is, 1 728 is even, hence it is divisible by 2. Sum of its digits is 1 + 7 + 2 + 8 = 18, which is divisible by 9.
18.	Divisibility of numbers by 19: If the last digit in the given number is multiplied by 2 and the result is added to the remaining number such the result is divisible by 19, then the given number is divisible by 19. If the number is still large, repeat the process.	741 is divisible by 19. The last digit is 1. Thus, $74+(1\times2)=76$. Since 76 is a multiple of 19, then 741 is divisible by 19.
19.	Divisibility of numbers by 20: A number is divisible by 20 if the number ends with either 00, 20, 40, 60, or 80. Also, a number is divisible by 20 if the number formed by the last two digit in the given number is divisible by 20.	19 700 is divisible by 20 because the last two digits are 00. 179 240 is divisible by 20 because the number formed by the last two digits, that is, 40 is divisible by 20.

Example 1.18

Without actual division, check if the following numbers are divisible by 2, 3, 4, 5, and 7.

(a) 3 501 804 (b) 3 185 (c) 26 110

Solution

(a) Given 3 501 804.

Divisibility by 2:

The number is an even number. Hence, it is divisible by 2.

Divisibility by 3:

Sum of its digits is

$3+5+0+1+8+0+4=21$, which is divisible by 3.

Hence, 3 501 804 is divisible by 3.

Divisibility by 4:

The number formed by the last two digits is 04 or 4, which is divisible by 4.

Hence, 3 501 804 is divisible by 4.

Divisibility by 5:

3 501 804 does not end with 0 or 5. Hence, it is not divisible by 5.

Divisibility by 7:

The last digit is 4. Thus,

$350\ 180-(4\times2)=350\ 172$.

The last digit is 2. Thus,

$35\ 017-(2\times2)=35\ 013$.

The last digit is 3. Thus,

$3\ 501-(3\times2)=3\ 495$.

The last digit is 5. Thus,

$349-(5\times2)=339$.

The last digit is 9. Thus,

$33-(9\times2)=15$.

Since 15 is not divisible by 7, then 350 180 is not divisible by 7.

Therefore, 3 501 804 is divisible by 2, 3, and 4, but not 5 and 7.

(b) Given 3 185.

Divisibility by 2:

3 185 is not divisible by 2 because it not an even number.

Divisibility by 3:

Sum of the digits is $3+1+8+5=17$, which is not divisible by 3.

Hence, 3 185 is not divisible by 3.

Divisibility by 4:

3 185 is not divisible by 4 because the number formed by the last two digits is 85, which is not divisible by 4.

Divisibility by 5:

The number is divisible by 5 because the last digit is 5.

Divisibility by 7:

The last digit is 5. Thus,

$318-(5\times2)=308$.

The last digit is 8. Thus,
$30-(8\times 2)=14$.

Since 14 is divisible by 7, then 3 185 is divisible by 7.

Therefore, 3 185 is divisible by 5 and 7 but not 2, 3, and 4.

(c) Given 26 110.

Divisibility by 2:

26 110 is divisible by 2 because it is an even number.

Divisibility by 3:

Sum of the digits is $2+6+1+1+0=10$ which is not divisible by 3.

Hence, 26 110 is not divisible by 3.

Divisibility by 4:

The number formed by the last two digits is 10, which is not divisible by 4.

Hence, 26 110 is not divisible by 4.

Divisibility by 5:

26 110 is divisible by 5 because the last digit is 0.

Divisibility by 7:

The last digit is 0. Thus,
$2\ 611-(0\times 2)=2\ 611$.

The last digit of 2 611 is 1. Thus,
$261-(1\times 2)=259$.

The last digit of 259 is 9. Thus,
$25-(9\times 2)=7$.

Since 7 is divisible by 7, then 26 110 is divisible by 7.

Therefore, 26 110 is divisible by 2, 5, and 7, but not 3 and 4.

Example 1.19

If 6 is a factor of both 12 066 and 49 320, check whether it is also a factor of $49\ 320+12\ 066$ and $49\ 320-12\ 066$.

Solution

Given $49\ 320 + 12\ 066 = 61\ 386$.

For 6 to be a factor of 61 386, it follows that 2 and 3 must also be the factors. 61 386 is divisible by 2 because it is an even number.

61 386 is divisible by 3 because the sum of its digits is $6+1+3+8+6=24$, which is divisible by 3. Hence, 6 is a factor of $49\ 320 + 12\ 066$.

Given $49\ 320 - 12\ 066 = 37\ 254$.

37 254 is divisible by 2 because it is an even number.

37 254 is divisible by 3 because the sum of its digits is $3+7+2+5+4=21$, which is divisible by 3.

Therefore, 6 is a factor of
$49\ 320 - 12\ 066$.

Exercise 1.7

1. List down the possible four-digit numbers with 5 and 3 in hundreds and tens positions, respectively, such that the numbers formed are divisible by both 4 and 9.
2. Use divisibility rules to determine which among the following list of numbers are divisible by 3, 7, and 9.

   9 876, 98 768, 1 234 869, 123 456 789, 21 756.
3. Masanja wanted to buy m oranges with an intention of giving them to either his 3 nephews or 6 siblings or 8 uncles. If m is divisible by all categories, find the lowest possible number of oranges he could buy.
4. Using the divisibility rules, determine which of the numbers, 204, 147 497, 1 134, 448 100, and 4 913 are divisible by:

   (a) 3, 5, 6, 12, 17, and 19.

   (b) more than one factor/number.

Chapter summary

1. A number pattern is a list of numbers following a certain rule.
2. A pattern rule establishes the numbers or objects in a pattern by identifying initials and how the pattern continues.
3. The Fibonacci sequence is a number pattern such that each term in the pattern after the second, is the sum of the two previous terms.
4. The Pascal's triangle is an infinite triangular array of numbers beginning with a 1 at the top.
5. The numbers in the third diagonals of the Pascal's triangle are called triangular numbers.
6. The numbers in the fourth diagonals of the Pascal's triangle are called tetrahedral numbers.
7. A number chart is a table containing numbers in a numerical order.
8. A divisibility rule is a shortcut approach that helps in identifying if a given integer is divisible by a divisor by examining its digits, without performing the whole division process.

Revision exercise 1

1. Temu and Hassan had an argument on the following number pattern that was given as a homework problem: 5, 3, 8, 6, 11. Temu suggested that the pattern rule is to add 5 to the first number and then subtract 2 to get the next number. Was Temu correct? If not, what is the correct response?
2. Neema has Tsh 49 000. If she spends Tsh 8 000 everyday, show the pattern that represents the amount of money left over in her spending, and hence find the amount of money that she will remain with after 5 days.
3. Find the values of the 12^{th} and 13^{th} Fibonacci numbers if the 9^{th} and 10^{th} numbers in the pattern are 21 and 34, respectively.
4. Ashura wants to generate a Fibonacci sequence. The first and second terms of the sequence are 3 and 4, respectively.
   - **(a)** Find the next three terms of the sequence.
   - **(b)** Ashura thinks that the sum of the first ten terms is equal to 11 times the 7^{th} term of the sequence. Show whether she is correct or not.
5. Find the n^{th} term for each of the following number patterns:
   - **(a)** 0, 3, 8, 15, 24, …
   - **(b)** $-3, 0, 5, 12, 21, ...$
6. Find the first four terms of the pattern defined by the formula $S_n = \frac{2n^2 - 3}{1 - 2n^2}$, where n is a natural number.
7. Find the rule that establishes the number pattern 0, 2, 6, 12, 20, 30, 42, …
8. Find the first four terms of the pattern generated by the formula $T_n = \frac{4n^2 + 1}{2n - 6}$, where n is a perfect square.
9. Find the last three terms in ascending order of the pattern generated by the rule $n^3 - 2n$, where n is a whole number from 0 to 9.
10. A fixed number added to the preceding number to get the next number in a pattern is 3 and the 20^{th} term (T_{20}) is 64.
    - **(a)** Determine the 22^{nd} term (T_{22}).
    - **(b)** Which term in the pattern will be equal to $3T_5 - 2$?
11. Shadrack has Tsh 3 000 and he earns Tsh 800 for an hour, for drawing pictures. If he wants to buy a watch that costs Tsh 8 600, how many hours does he need to work?
12. **(a)** Find the sum of numbers in the first few rows of Pascal's triangle to complete the following table.

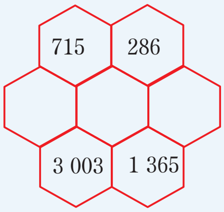
Row	1	2	3	4	5	6
Row sum	1	2	4			

   (b) What is the pattern of the row sum in 12 (a)?

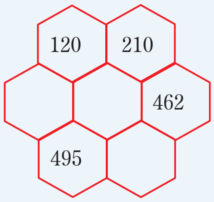
(c) How could you relate the row number to the row sum?

13. The following are the portions of the Pascal's triangle. Fill in the missing numbers in each portion:

(a)



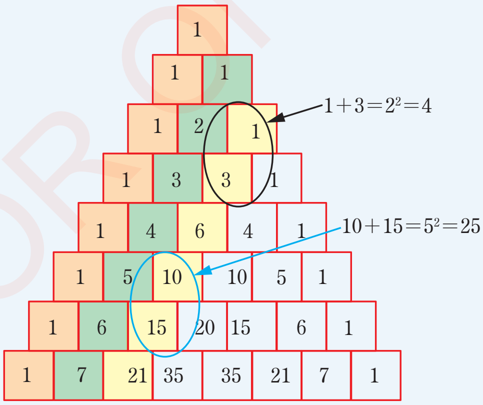
(b)



14. How would you express the sum of numbers in each of the following rows of the Pascal's triangle?

(a) 13th row **(b)** 20th row

15. Study the patterns in the following Pascal's triangle and then determine the values of a and b in the equation $a+b=81$, where a and b are triangular numbers. The circled numbers are given as examples.



16. Complete the following chart by filling in the missing numbers with integers from −12 to 12 without repeating any integer such that the sum in each row and each column is zero.

3	−6			1
			7	
		0		
	−7			11
−1		4	−12	

17. Construct a number chart of 4 rows and 4 columns with numbers starting with 20 and ending with 35, and then verify that the sum in the main diagonals is 110.

18. By using counting numbers up to sixteen, construct a number chart of 4 rows and 4 columns such that the sum of numbers in the main diagonals, each row, and each column is thirty-four without repeating any number.

19. Using the divisibility rules:

(a) Verify that 41 295 is divisible by 15.

(b) Determine whether or not 2 464 is divisible by 18.

(c) Verify that 1 452 is divisible by 11.

(d) Determine whether or not 945 724 is divisible by 4 and 8.

20. List the numbers that are divisible by 3 from the following list.

4 257, 2 798, 2 512, 3 492, 659

Chapter Two
Symmetry

Introduction

Symmetry is a fundamental part of nature that can be visible in many aspects of life including art, architecture, and design. Artists use symmetry to make beautiful and decorative materials. Architects use symmetry to produce a sense of balance in buildings, bridges, and other structures. Symmetry can also be observed in some animals, plants, and mechanical objects. In this chapter, you will learn about symmetrical figures and shapes, lines of symmetry, and patterns with shapes. The competencies developed will help you in designing and constructing different symmetrical figures and shapes.

Symmetrical figures and shapes

In geometry, a shape is considered to be an outer boundary or outer surface of an object. A figure is an object represented on a plane or in space. Symmetry is the property of a figure or shape being made up of exactly similar parts facing each other around an axis. A figure or shape is said to be symmetrical if it can be folded or divided into two halves along a line such that the two halves match exactly.

Figure 2.1 shows some examples of symmetrical figures while Figure 2.2 shows some examples of figures that are not symmetrical.

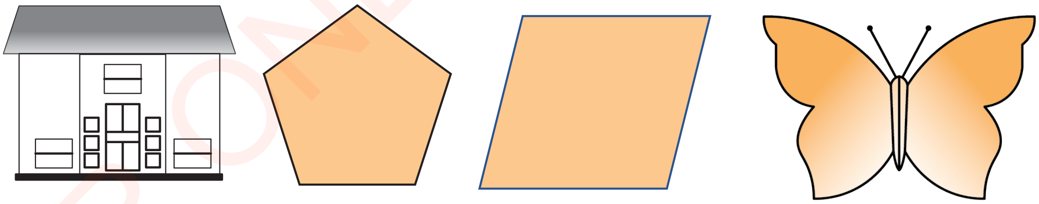


Figure 2.1: *Symmetrical figures*

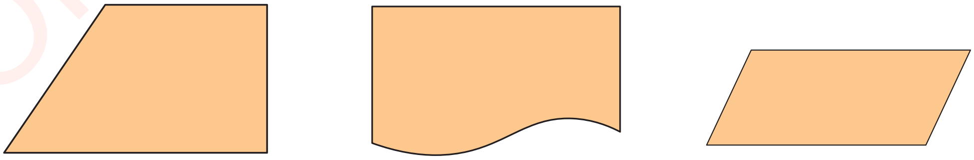


Figure 2.2: *Non-symmetrical figures*

Identification of symmetrical figures and shapes

A symmetrical figure or shape can be identified by observing whether it can be folded or divided into two identical parts or shapes. Activity 2.1 enriches the ability to identify symmetrical objects.

Activity 2.1: Identifying symmetrical figures and shapes

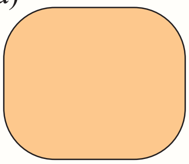
Individually or in a group, perform the following tasks:

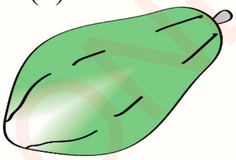
1. Write down all English alphabets in capitals letters.
2. Identify all alphabets in task 1 that can be folded or divided into two halves that match exactly and those that cannot.
3. Identify other objects around your school or home that can be divided into two identical halves.
4. Share your results with other students through discussion for more inputs.

In Activity 2.1, it can be observed that some of the English alphabets, for instance, A, H, M, O, and T can be folded or divided into halves which are exactly the same. All alphabets which can be folded or divided such that their halves are exactly the same are called symmetrical alphabets. Otherwise, they are called non-symmetrical alphabets. Symmetrical objects which can be found at school or home include; windows, doors, chalkboards, tables, white boards, plane papers, pens, pairs of spectacles, classrooms, cups, plates, jugs, and many others.

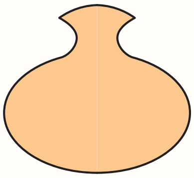
Example 2.1

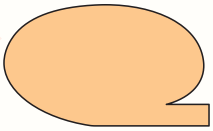
State whether each of the following shapes is symmetrical or not.

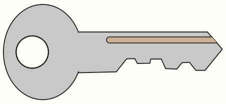
(a)


(b)


(c)


(d)


(e)


(f)


(g)


Solution

(a) The shape is symmetrical.
(b) The shape is symmetrical.
(c) The shape is symmetrical.
(d) The shape is symmetrical.
(e) The shape is not symmetrical.
(f) The shape is not symmetrical.
(g) The shape is not symmetrical.

Example 2.2

State whether or not each of the following letters is symmetrical:

(a) D (b) B (c) R

(d) X (e) N (f) M

Solution

(a) D is symmetrical

(b) B is symmetrical

(c) R is not symmetrical

(d) X is symmetrical

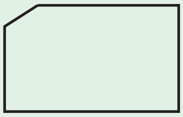
(e) N is not symmetrical

(f) M is symmetrical

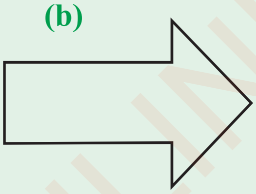
Exercise 2.1

1. In each of the following shapes, state whether or not the shape is symmetrical:

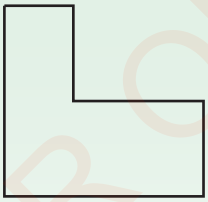
(a)



(b)



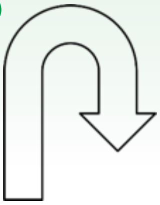
(c)



(d)

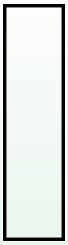
(e)

(f)



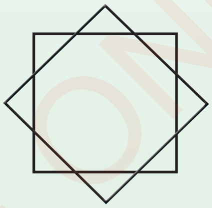
(g)

(h)

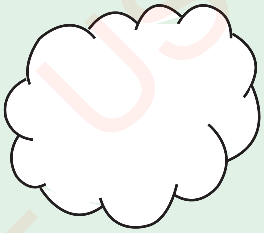


(i)

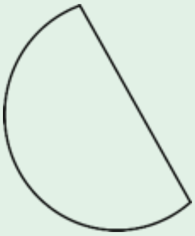
(j)



(k)



(l)



2. Which odd numbers between 2 and 10 are not symmetrical?
3. How many even numbers between 1 and 10 are symmetrical?
4. Which of the letters in the word ZEBRA are symmetrical?
5. Which of the letters in the word SCHEME are not symmetrical?

Lines of symmetry

A line of symmetry of a shape or figure is an imaginary line that divides the shape or figure into two equal and symmetrical parts. A line of symmetry is also called the axis of symmetry or mirror line because it presents two reflections of an image that coincide. The dotted lines in Figure 2.3 represent lines of symmetry.

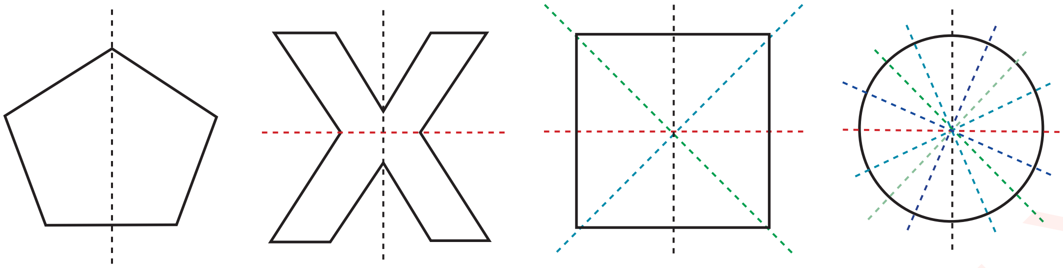


Figure 2.3: *Lines of symmetry*

Drawing lines of symmetry of symmetrical shapes

A line of symmetry is usually drawn using a dotted line. A symmetrical figure or shape can have more than one line of symmetry. For instance; a square, rhombus, circle, and pentagon have more than one line of symmetry. Figures or shapes that are not symmetrical do not have lines of symmetry. Activity 2.2 demonstrates the construction of lines of symmetry.

Activity 2.2: Drawing lines of symmetry

Individually or in a group, perform the following tasks:

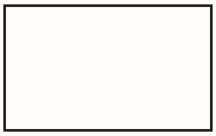
1. Cut a square piece of paper of a reasonable length by using a ruler and a pair of scissors or razor blade.
2. Fold the paper in task 1 vertically such that its one half lies exactly over the other half.
3. Unfold the paper and then use a ruler and pen or sharp pencil to draw a dotted line along a folding line in task 2.
4. Repeat tasks 2 and 3 to exhaust all the possibilities of making symmetrical parts and mark their corresponding lines of symmetry.
5. Count the number of lines of symmetry drawn in tasks 3 and 4.
6. Share your results with other students through discussion for more inputs.

From Activity 2.2, it can be observed that, the dotted lines form the lines of symmetry.

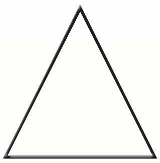
Example 2.3

Draw the lines of symmetry for each of the following figures, if any:

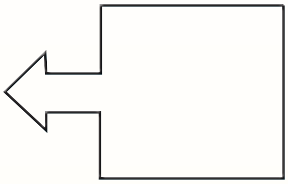
(a)



(b)

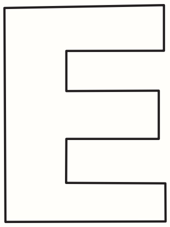


(c)

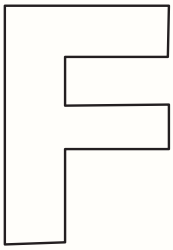


(d)

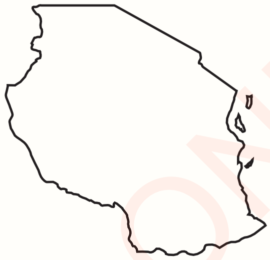
(e)



(f)

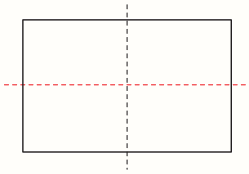


(g)

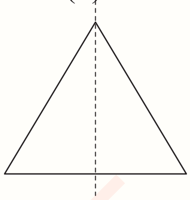


Solution

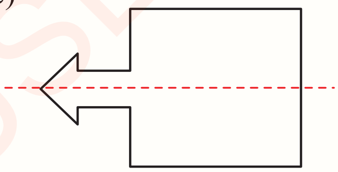
(a)



(b)

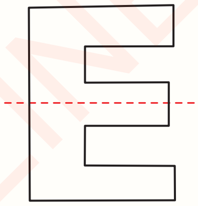


(c)



(d) The figure has no line of symmetry.

(e)



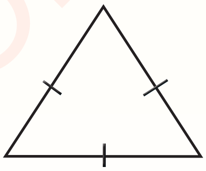
(f) The figure has no line of symmetry.

(g) The figure has no line of symmetry.

Example 2.4

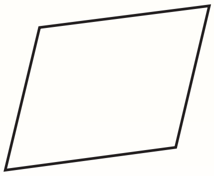
Draw the lines of symmetry for each of the following shapes, if any:

(a) Equilateral triangle



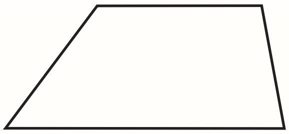
(b) Cocoyam

(c) Rhombus

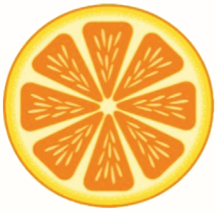


(d) Square

(e) Trapezium

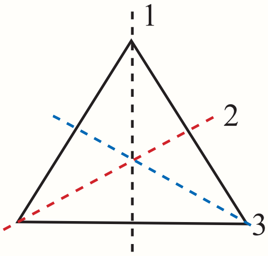


(f) Orange



Solution

(a) Equilateral triangle

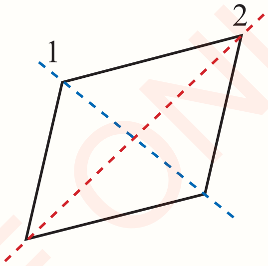


An equilateral triangle has three lines of symmetry.

(b) Cocoyam

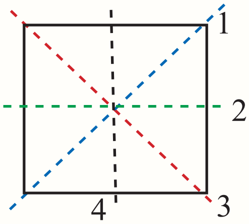
A cocoyam has no line of symmetry.

(c) A rhombus



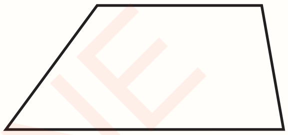
A rhombus has two lines of symmetry.

(d) Square



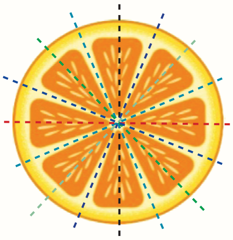
A square has four lines of symmetry.

(e) Trapezium



A trapezium has no line of symmetry.

(f) Orange

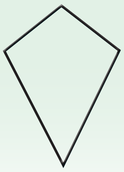


The orange has an infinite number of lines of symmetry. Some lines of symmetry are indicated.

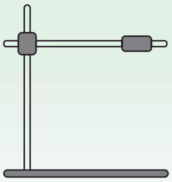
Exercise 2.2

1. State the number of lines of symmetry for each the following shapes:

(a) Kite



(b) Retort stand



(c) Five thousand Tanzania banknote

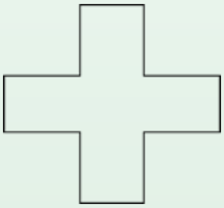
(d) Parallelogram

(e) 50 shillings Tanzanian coin

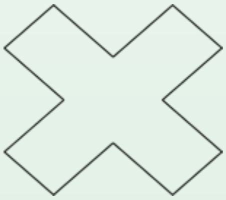
(f) Jack fruit

2. Draw the lines of symmetry for each of the following figures, if any.

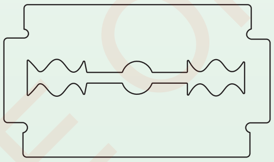
(a)



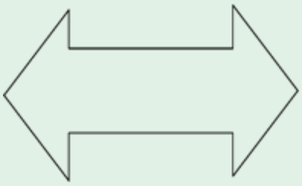
(b)



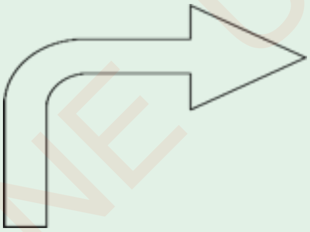
(c)



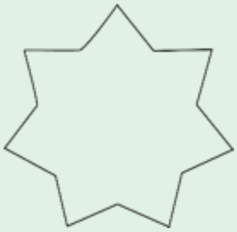
(d)



(e)



(f)



(g)

(h)

3. Which English alphabets in capital letters have:

(a) One line of symmetry?

(b) Two or more lines of symmetry?

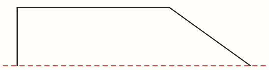
(c) No line of symmetry?

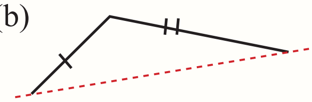
Construction of symmetrical shapes or figures

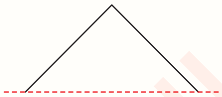
A symmetrical shape or figure can be drawn by observing its lines of symmetry.

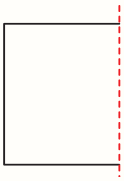
Example 2.5

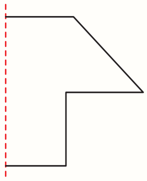
Complete each of the following shapes so that the dotted line becomes a line of symmetry.

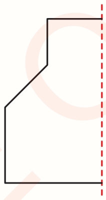
(a) 

(b) 

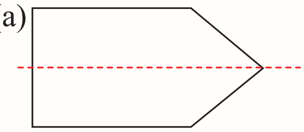
(c) 

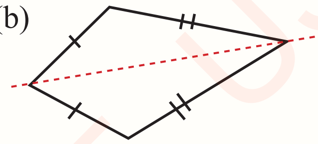
(d) 

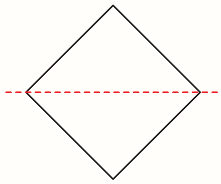
(e) 

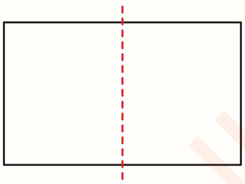
(f) 

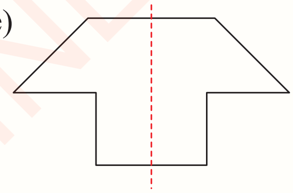
Solution

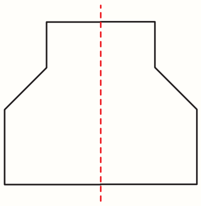
(a) 

(b) 

(c) 

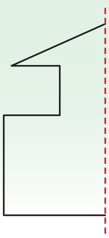
(d) 

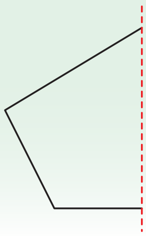
(e) 

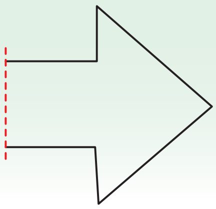
(f) 

Exercise 2.3

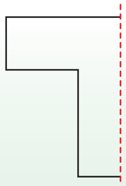
1. Complete each of the following figures so that the dotted line becomes the line of symmetry:

(a) 

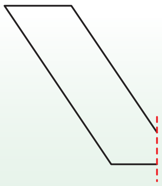
(b) 

(c) 

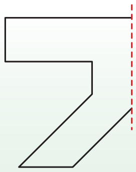
(d)



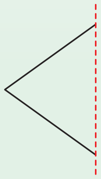
(e)



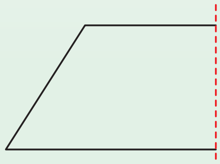
(f)



(g)



(h)

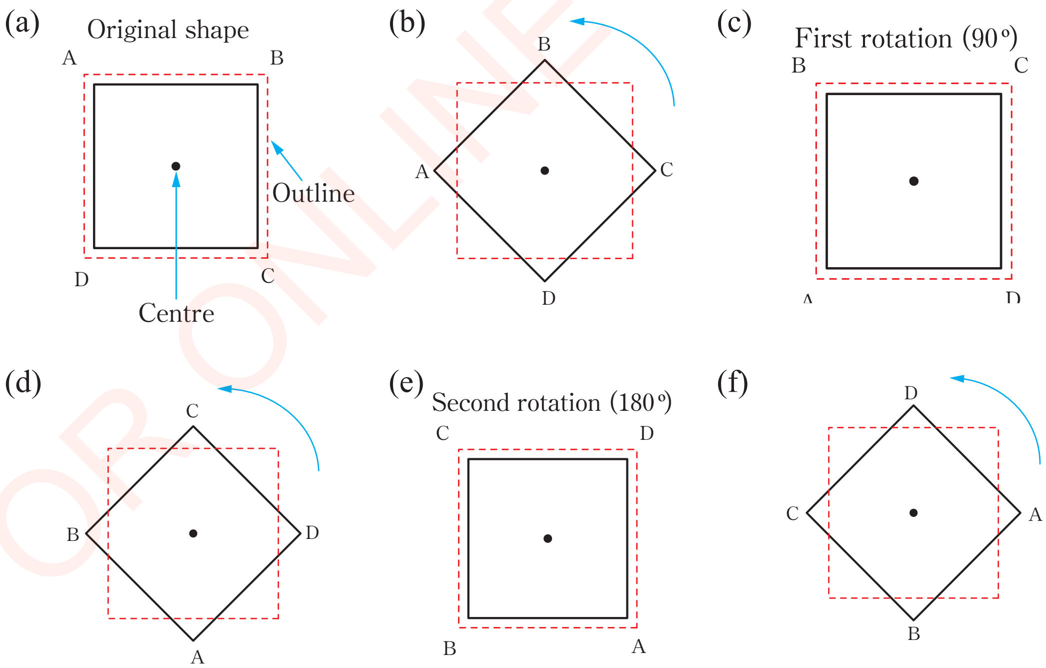


2. Draw and indicate the lines of symmetry of each of the following shapes:

(a) An oval (b) A rectangular tile (c) A diamond
(d) A regular heptagon (e) A four-point star (f) Isosceles trapezium
(g) Heart

Rotational symmetry

Rotational symmetry of a shape occurs when an object is rotated about a fixed point called point of rotation and the shape of the object looks the same. The rotation is done at an angle more than $0°$ and less than or equal to $360°$ in either clockwise or anticlockwise direction. In figure 2.4, the square ABCD is rotated through $90°$, $180°$, $270°$, and $360°$.



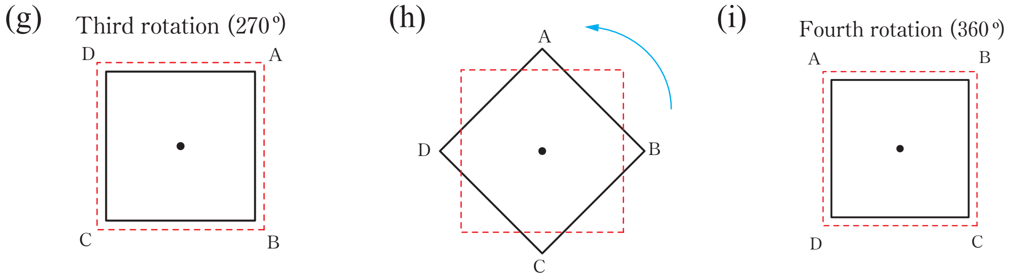


Figure 2.4: *Rotations of a square* ABCD

From Figure 2.4, it can be observed that, after rotations by 90°, 180°, 270°, and 360°, the square ABCD fitted exactly in its outline and the rotated shape is exactly the same as the original square. Thus, the square has four rotational symmetry.

Order of rotational symmetry

The number of positions in which a shape can be rotated and still appears exactly the same as it was before the rotation, is called the order of rotational symmetry or simply, order of rotation. For instance, a rectangle can be rotated two times about the centre in which its shape is invariant. Hence, its order of rotation is 2. Activity 2.3 highlights how to determine order of rotational symmetry of geometrical figures.

Activity 2.3: Determining the order of rotational symmetry

Individually or in a group, perform the following tasks:

1. Construct a square of a reasonable size using hard materials available in your environment such as a box or a card board.
2. Mark one corner of the square in task 1 with ink.
3. Rotate the square on its outline until the marked corner returns to its original position while counting the number of rotations the square fits exactly in its outline and looks the same as the original shape.
4. Write down the number of rotations observed in task 3.
5. Share your results with other students through discussion for more inputs.

From Activity 2.3, it can be observed that, the number of rotations in which the square will look exactly the same is 4.

To determine the order of rotational symmetry of a geometrical shape, rotate the shape while determining how many times the shape looks exactly the same as the original shape after a complete turn of 360°. Thus, the order of rotational symmetry is the number of times a shape looks exactly the same when rotated through 360°. Table 2.1 shows the number of rotations of some plane figures.

Table 2.1: *Rotations of some plane figures*

Plane figure	Rotations
Rectangle	2 rotations ($180°$ and $360°$)
Square	4 rotations ($90°$, $180°$, $270°$, and $360°$)
Rhombus	2 rotations ($180°$ and $360°$)
Parallelogram	2 rotations ($180°$ and $360°$)
Trapezium	1 rotation ($360°$)
Equilateral triangle	3 rotations ($120°$, $240°$, and $360°$)
Circle	Infinite number of rotations

From Table 2.1, any figure or shape can only be rotated up to a maximum of $360°$ with the minimum order of rotational symmetry equal to one. However, a circle will always fit in its outline at every angle it rotates. Hence, the order of rotational symmetry of a circle is infinity.

Angle of rotational symmetry

The angle turned during rotation of a shape that has a rotational symmetry is called the angle of rotational symmetry, or angle of rotation. Consider an equilateral triangle ABC in Figure 2.5.

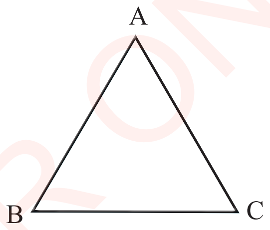


Figure 2.5: *Equilateral triangle* ABC

The equilateral triangle ABC in Figure 2.5 can be rotated in a clockwise direction as illustrated in Figure 2.6.

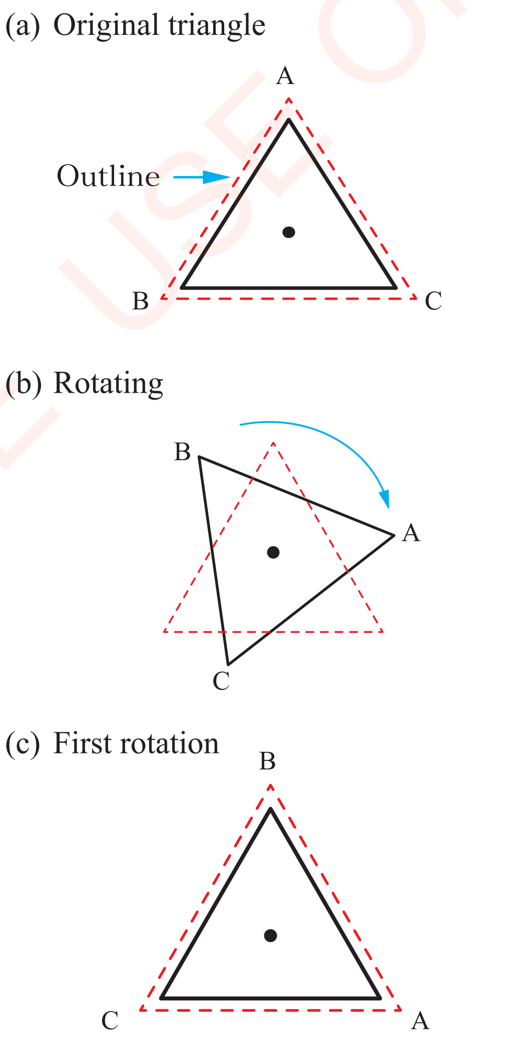


Figure 2.6: *Rotation of equilateral triangle* ABC

From Figure 2.5, the vertex A was initially at 0°. After the first rotation, the vertex A of the equilateral triangle ABC is at 120° as shown in Figure 2.7.

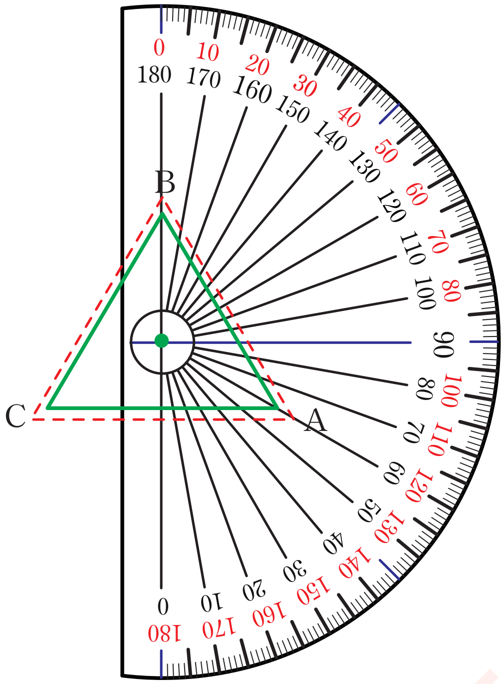


Figure 2.7: *Angle of rotational symmetry of an equilateral triangle* ABC

Thus, an equilateral triangle fits in its outline for the first time when it is rotated through 120°. This implies that, the angle of rotational symmetry of an equilateral triangle is 120°. A square has four order of rotational symmetry and its angle of rotational symmetry is 90°.

Generally, the angle of rotational symmetry of a shape is determined by the formula;

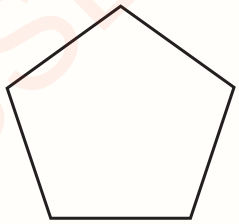
$$\text{Angle of rotational symmetry} = \frac{360°}{n},$$

where n is the order of rotational symmetry.

Note that; Some shapes such as trapezium, parallelogram, scalene triangle, micrometer screw gauge, vernier caliper, spatula, fork spoon, and many others do not have lines of symmetry but they have a rotational symmetry.

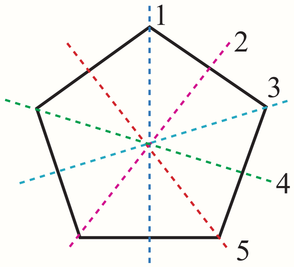
Example 2.6

Draw and state the number of lines of symmetry of the following shape. Hence, find the order and angle of rotational symmetry.



Solution

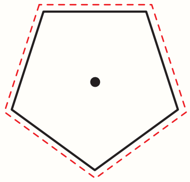
The lines of symmetry can be drawn as follows:



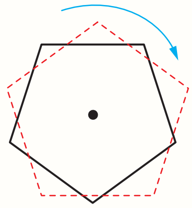
Thus, the shape has five lines of symmetry.

The shape can be rotated in a clockwise direction at any angle as follows.

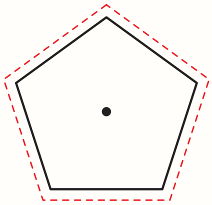
(a) Original shape



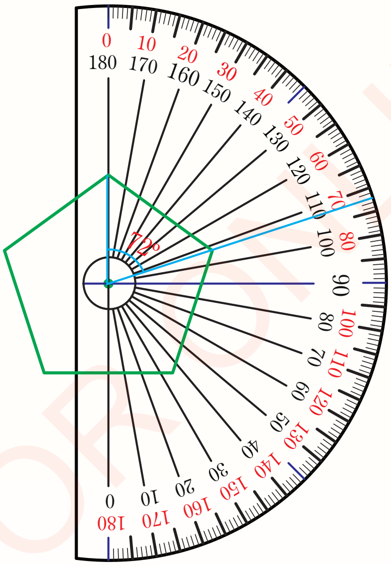
(b) Rotating



(c) First rotation



After rotation, the shape will fit in its outline for the first time as shown in the following figure:



Thus, the order of rotational symmetry is 5.

Let n be the order of rotational symmetry. It implies that, $n = 5$.

$$\text{Angle of rotational symmetry} = \frac{360°}{n}$$

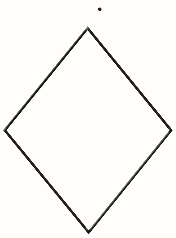
$$= \frac{360°}{5} = 72°.$$

Thus, the angle of rotational symmetry is $72°$.

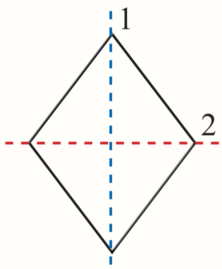
Therefore, the given figure has 5 lines of symmetry, 5 order of rotational symmetry, and the angle of rotational symmetry is $72°$.

Example 2.7

Determine the number of axis of symmetry and hence, find the order and angle of rotational symmetry of the following rhombus.



Solution



There are two axis of symmetry in the rhombus.

If the rhombus is rotated $360°$, it fits in its outline two times.

Thus, its order of rotational symmetry is 2 $(n = 2)$. From the angle of rotation formula,

$$\text{Angle of rotational symmetry} = \frac{360°}{n} = \frac{360°}{2} = 180°.$$

Therefore, the order of rotational symmetry is 2 and angle of rotational symmetry is $180°$.

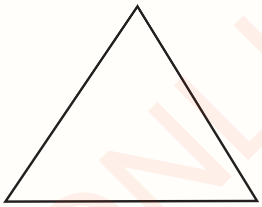
If the equilateral triangle is rotated $360°$, it will be exactly the same as the original triangle three times. Thus, its order of rotational symmetry is 3, that is, $n = 3$, so that its angle of rotation is given by,

$$\text{Angle of rotational symmetry} = \frac{360°}{n} = \frac{360°}{3} = 120°.$$

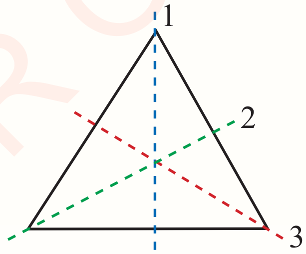
Therefore, an equilateral triangle has 3 order of rotational symmetry and the angle of rotation is $120°$.

Example 2.8

Draw the lines of symmetry, hence determine the order and angle of rotational symmetry of the following equilateral triangle.



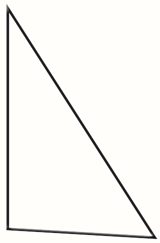
Solution



Thus, the given equilateral triangle has three lines of symmetry.

Example 2.9

Find the order and angle of rotational symmetry of a scalene triangle. Hence, state the number of lines of symmetry.



Solution

If a scalene triangle is rotated $360°$, it will be exactly the same as the original triangle only once.

Thus, it has 1 order of rotational symmetry. That is, $n = 1$. Hence,

$$\text{Angle of rotational symmetry} = \frac{360°}{n}$$

$$= \frac{360°}{1} = 360°.$$

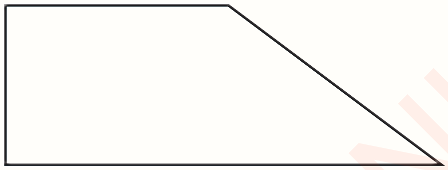
Therefore, the order of rotational symmetry is 1 and the angle of rotational symmetry is 360°.

A scalene triangle has no line of symmetry.

Example 2.10

Complete the following diagram so that the resulting figure has a rotational symmetry but no line of symmetry.

Solution

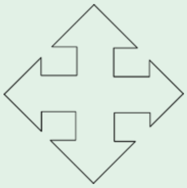


The resulting figure has no line of symmetry, but its order of rotational symmetry is one.

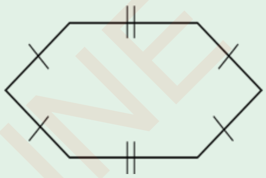
Exercise 2.4

Find the order and angle of rotational symmetry for each of the following shapes:

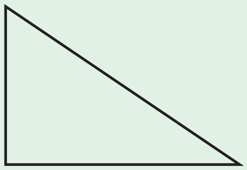
1.



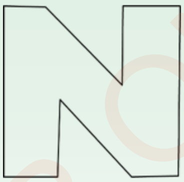
2.



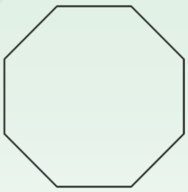
3.



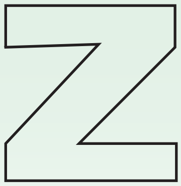
4.



5.



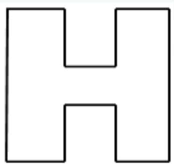
6.



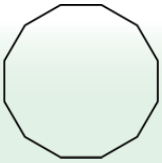
7.

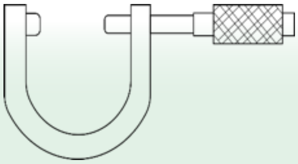
8.

9.



10. 

11. 

12. 

Patterns with shapes

When symmetrical shapes are arranged in a repeatedly order, they make a pattern. For instance; building walls, roofs, floors, and carpets may be designed using patterns. Patterns are made beautiful by using materials with symmetrical shapes. In industries, patterns in many products such as clothes, tyre threads, and tiles are made using symmetrical shapes. Figure 2.8 illustrates some examples of patterns made from symmetrical shapes.

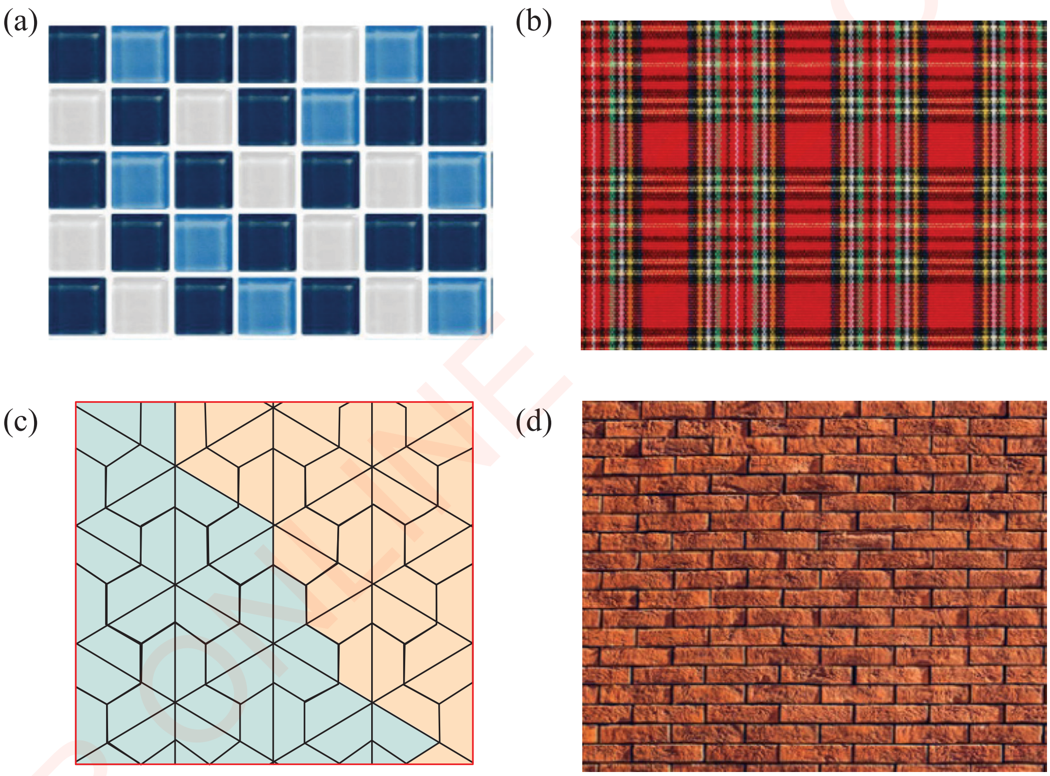


Figure 2.8: *Patterns in differents objects*

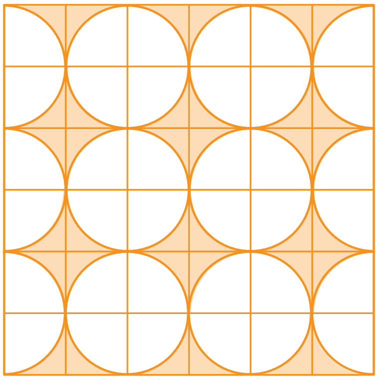
Formation of patterns with shapes

Patterns can be formed from symmetrical shapes such as triangles, quadrilaterals (rectangles, squares, trapeziums, parallelograms, rhombuses, kites), and circles.

In Figure 2.8, patterns in (a) are made up of squares with different colours, (b) are combination of squares and rectangles, (c) are combination of trapeziums and triangles, and (d) are combination of coloured rectangles and squares.

In Figure 2.9, patterns in (a) are made up of squares and quarter circles, and in (b) are made up of circles, semicircles, and quarter circles. Complex patterns are created by extending simple patterns.

(a)



(b)

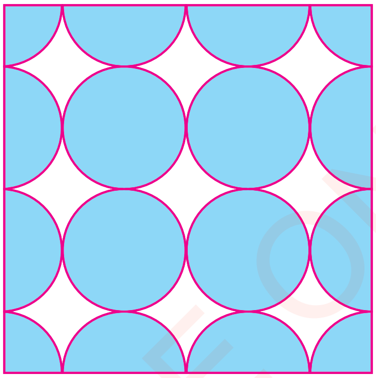
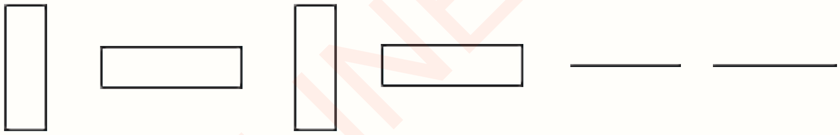


Figure 2.9: *Examples of patterns*

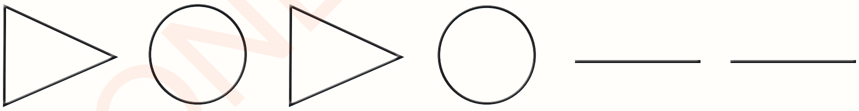
Example 2.11

Draw the next two shapes in each of the following patterns:

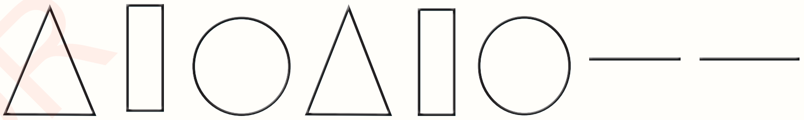
(a)



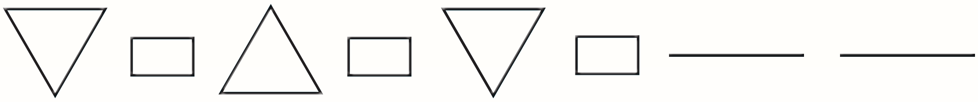
(b)



(c)



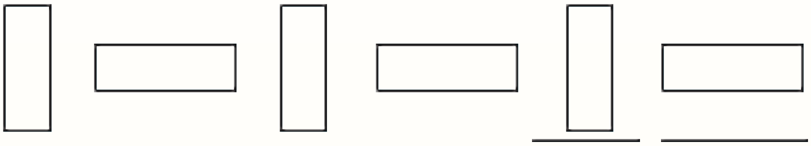
(d)



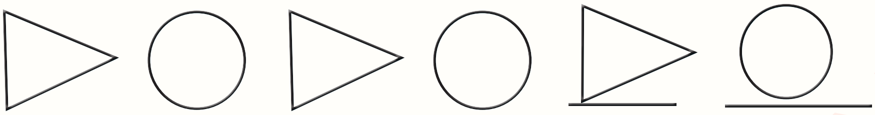
Solution

By observing the order of arrangement of the shapes, the next two shapes in each pattern are as follows:

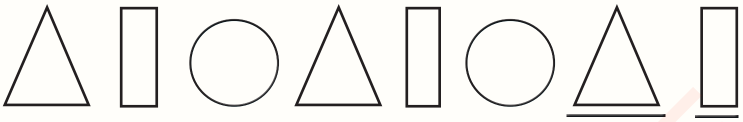
(a)



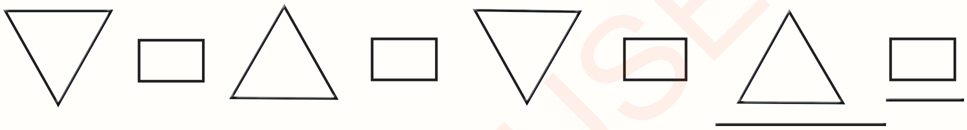
(b)



(c)

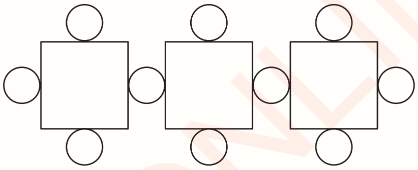


(d)



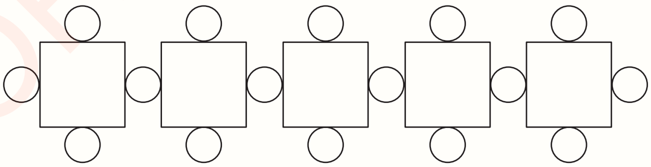
Example 2.12

Complete the following pattern so that it has five squares and sixteen circles.



Solution

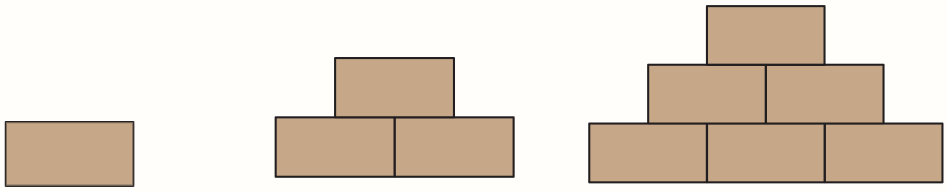
The pattern is completed as follows:



Example 2.13

Study the following patterns carefully and then draw patterns 4 and 5.

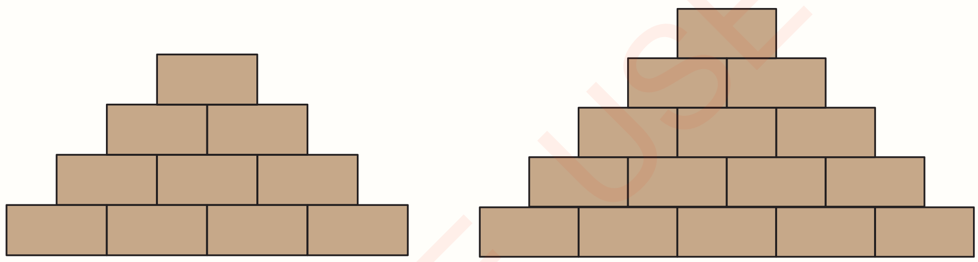
Pattern 1 Pattern 2 Pattern 3



Solution

By observing the order of arrangements of the rectangles, patterns 4 and 5 are drawn as follows:

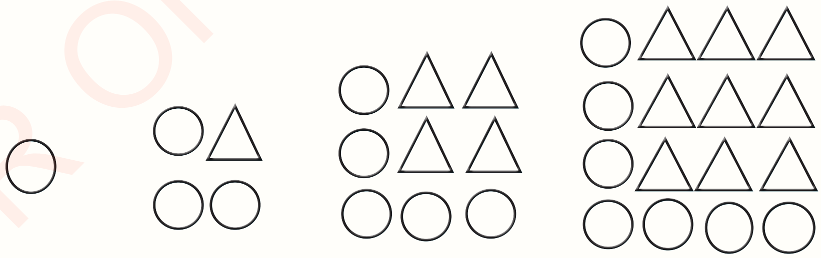
Pattern 4 Pattern 5



Example 2.14

Study the following patterns carefully and then answer the questions that follow.

Pattern 1 Pattern 2 Pattern 3 Pattern 4



(a) How many circles and triangles are there in pattern 5?

(b) Draw pattern 5.

Solution

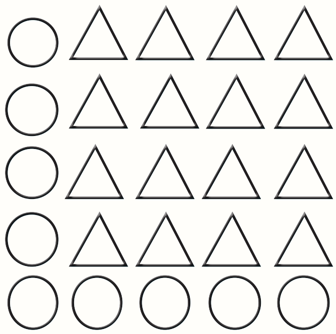
By using the concept of number patterns, the following table can be used to determine the number of circles and triangles in the patterns:

Pattern	Number of circles	Number of triangles
1	1	0
2	3	1
3	5	4
4	7	9
5	9	16

Therefore,

(a) There are 9 circles and 16 triangles in pattern 5.

(b)



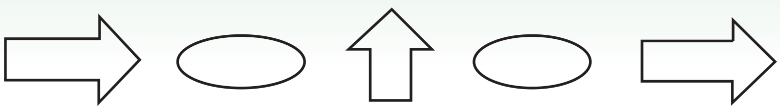
Exercise 2.5

1. Draw the next three symmetrical shapes in each of the following patterns:

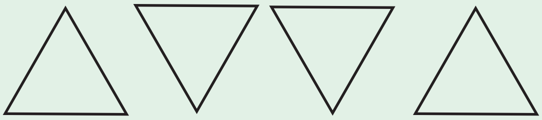
(a)

(b)

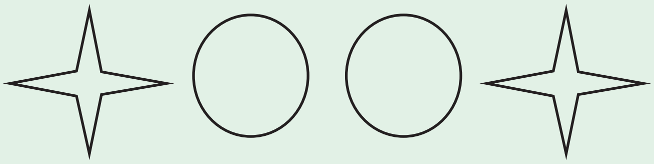
(c)



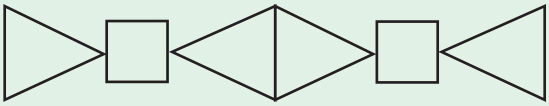
(d)



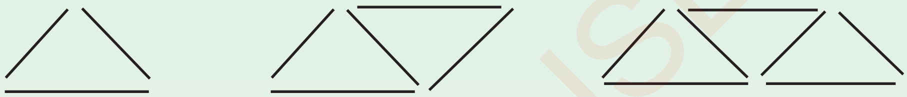
(e)



2. Complete the following pattern so that it has four squares and eight triangles.



3. The following patterns are made from sticks. Use the patterns to answer the questions that follow.



Pattern 1 Pattern 2 Pattern 3

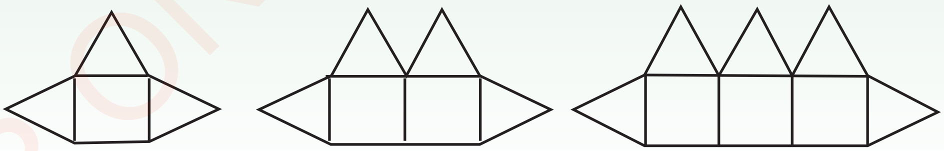
(a) Draw pattern 4.

(b) Draw pattern 5.

(c) How many sticks are there in pattern 6?

4. The following patterns are formed by squares and triangles. Use the patterns to answer the questions that follow:

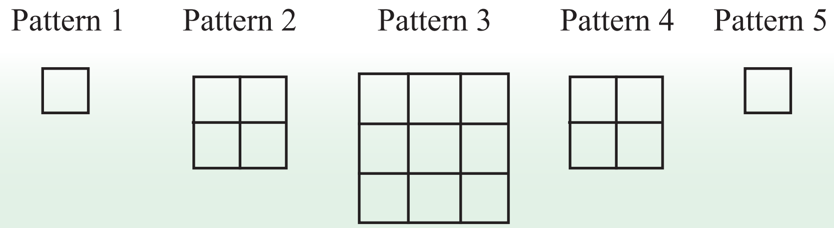
Pattern 1 Pattern 2 Pattern 3



(a) How many triangles are there in pattern 5?

(b) How many squares are there in pattern 6?

5. The following patterns are made of squares. Use the patterns to answer the questions that follow:



(a) Draw pattern 7.

(b) Draw pattern 8.

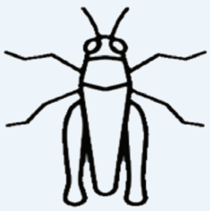
Chapter summary

1. An object or shape is said to be symmetrical if it can be folded or divided into two halves along a line such that the two halves match exactly.
2. A line of symmetry is the axis or imaginary line that divides a shape or figure into two equal and symmetrical parts.
3. A plane shape has a rotational symmetry if and only if it can be rotated more than $0°$ and less than or equal to $360°$ about a fixed point (centre of rotation) so that it remains invariant.
4. The number of positions in which a figure can be rotated and remain invariant is called the order of rotational symmetry.
5. The angle turned by a figure or shape with rotational symmetry in such a way that the figure or shape is invariant is called the angle of rotational symmetry.
6. The angle of rotational symmetry is given by $\frac{360°}{n}$, where n is the order of rotational symmetry.
7. When symmetrical shapes are designed in a repeatedly order, they form a pattern.

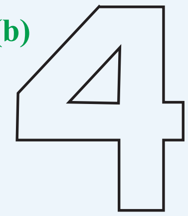
Revision exercise 2

1. State whether each of the following shapes is symmetrical or not.

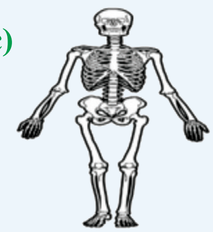
(a)

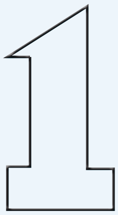
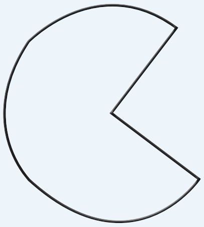


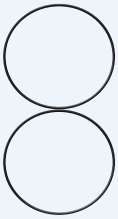
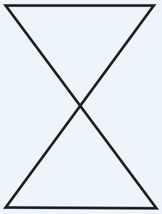
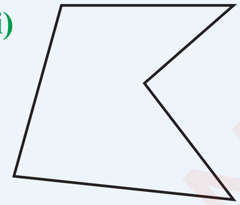
(b)



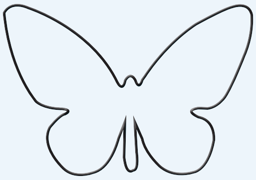
(c)

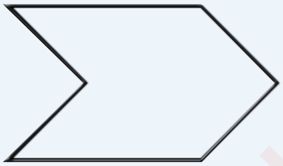
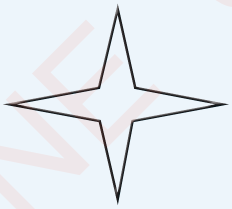
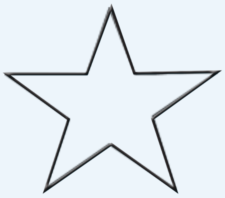


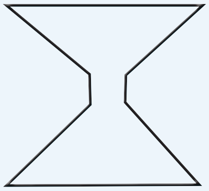
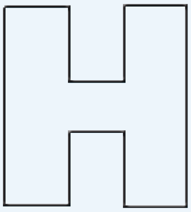
(d)  **(e)**  **(f)** 

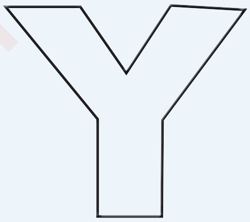
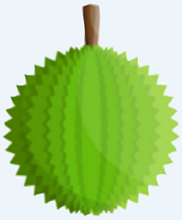
(g)  **(h)**  **(i)** 

2. How many lines of symmetry does each of the following shapes have?

(a)  **(b)**  **(c)** 

(d)  **(e)**  **(f)** 

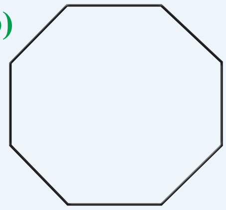
(g)  **(h)**  **(i)** 

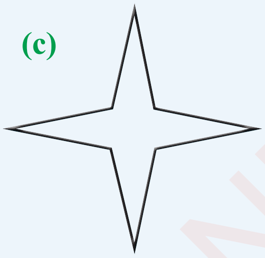
(j)  **(k)**  **(l)** 

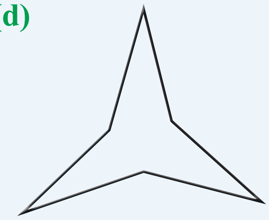
3. Two different quadrilaterals A and B have only one line of symmetry each. In quadrilateral A, the line of symmetry is a diagonal. In quadrilateral B, the line of symmetry is not a diagonal. Draw each of the quadrilaterals showing the line of symmetry, and write their specific names.

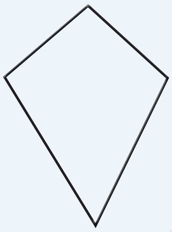
4. Determine the order and angle of rotational symmetry of each of the following figures:

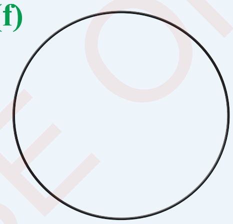
(a)


(b)


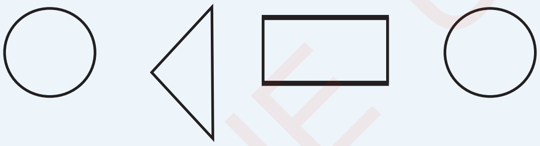
(c)


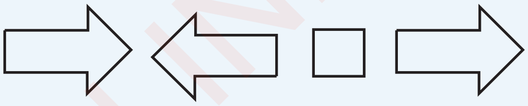
(d)


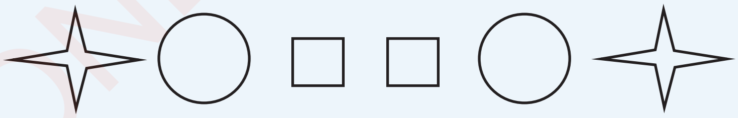
(e)


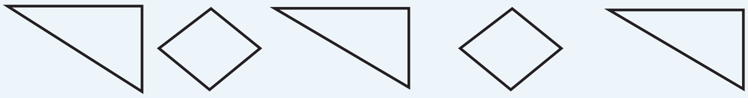
(f)


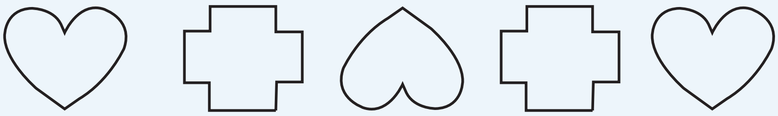
5. Draw the next four symmetrical shapes in each of the following patterns:

(a)


(b)


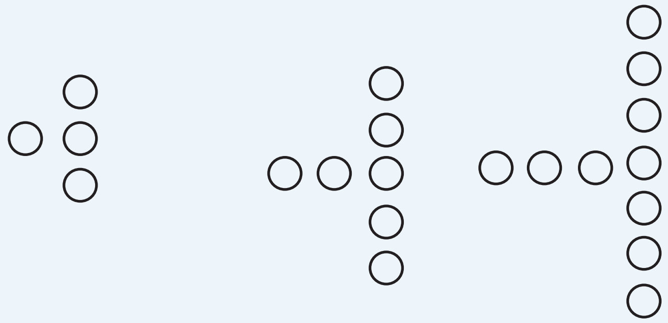
(c)


(d)


(e)


6. Use the following patterns to answer the questions that follow:

Pattern 1 Pattern 2 Pattern 3

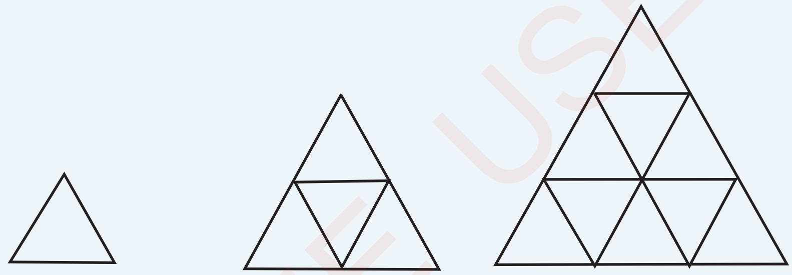


(a) Draw pattern 4.

(b) How many small circles are there in pattern 6?

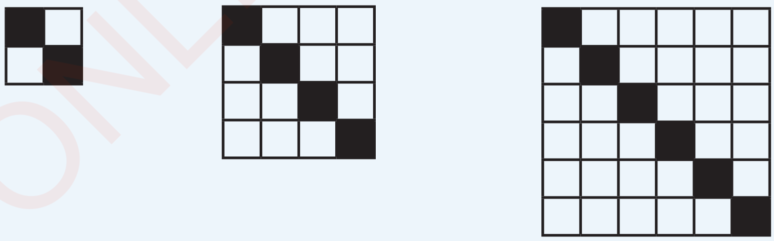
7. Study the following patterns carefully and then draw patterns 4 and 5:

Pattern 1 Pattern 2 Pattern 3



8. Study the following patterns carefully and then answer the questions that follow:

Pattern 1 Pattern 2 Pattern 3



(a) Draw pattern 4.

(b) How many shaded squares are there in pattern 5?

Chapter Three
Algebra

Introduction

Algebra deals with mathematical statements that describe the relationship between variables in our daily life situations or problems. In algebra, letters and symbols are usually used to represent quantities. The letters and symbols used in algebra are called variables. In this chapter, you will learn about algebraic expressions, algebraic equations, transposition of formulae, and inequalities. The study of algebra helps in developing logical thinking and enables a person to break down a real life problem first and then find its solution easily. The competencies developed will help you to easily solve daily life problems in business and financial management, cooking, technology, sports, early life child growth, health and fitness, outdoor landscaping, and home improvement, among many other applications.

Algebraic expressions

An algebraic expression is a mathematical phrase that contains variables, coefficients, and mathematical operations (addition, subtraction, multiplication, and division) without an equal sign. Expressions are made up of terms. Simplification of an algebraic expression is a handy mathematical skill that allows to transform expressions into simpler and compact forms.

The basic unit of an algebraic expression is called a term. In general, a term is either a number, a product of a number and one or more variables, or a product of variables only. For instance, terms in the expression $4x-2yz+xy$ are $4x$, $-2yz$, and xy. Terms containing the same variable or a variable which is raised to the same exponent in an expression are like terms, otherwise they are unlike terms. For instance, the expression $3a+7b+2a-b$ has four terms which are; $3a$, $7b$, $2a$, and $-b$, with coefficients 3, 7, 2, and -1, respectively. In this expression, $3a$ and $2a$ are like terms. Similarly, $7b$ and $-b$ are like terms. On the other hand, in the expression $3x-xy+2x^2$, all terms involved are unlike. An algebraic expression can be written in its simplest form by collecting like terms. In order to simplify an algebraic expression by collecting like terms, the following steps are used:

1. Remove any grouping symbols such as brackets by multiplying factors.
2. Identify the like terms in the expression.
3. Combine the like terms by adding or subtracting the coefficients.

Note that; In simplifying an expression containing different operations, the order of performing the operations follows BODMAS rule [Brackets (B), Orders of powers or roots (O), Division (D), Multiplication (M), Addition (A), and Subtraction (S)].

Example 3.1

Simplify $7ac-3ab-7ac+9ab$.

Solution

Given $7ac-3ab-7ac+9ab$.

Collecting like terms gives,

$$\begin{aligned}7ac-3ab-7ac+9ab &= 7ac-7ac\\ &\quad -3ab+9ab\\ &= -3ab+9ab\\ &= 6ab.\end{aligned}$$

Therefore, $7ac-3ab-7ac+9ab=6ab$.

Example 3.2

Simplify $5+(4b-2a)+3b$.

Solution

Given $5+(4b-2a)+3b$.

Opening brackets and collecting like terms give,

$$\begin{aligned}5+(4b-2a)+3b &= 5+4b-2a+3b\\ &= 4b+3b-2a+5\\ &= 7b-2a+5.\end{aligned}$$

Therefore,

$5+(4b-2a)+3b=7b-2a+5$.

Example 3.3

Simplify $2m+2n-3(m+n)$.

Solution

Given $2m+2n-3(m+n)$.

Opening brackets and collecting like terms give,

$$\begin{aligned}2m+2n-3(m+n) &= 2m+2n-3m-3n\\ &= 2m-3m+2n-3n\\ &= -m-n.\end{aligned}$$

Therefore, $2m+2n-3(m+n)=-m-n$.

Example 3.4

Simplify $p(-2q+5r)-3(2pq-4pr)$.

Solution

Given $p(-2q+5r)-3(2pq-4pr)$.

Opening brackets and collecting like terms give,

$$\begin{aligned}&p(-2q+5r)-3(2pq-4pr)\\ &= -2pq+5pr-6pq+12pr\\ &= -2pq-6pq+5pr+12pr\\ &= -8pq+17pr\\ &= 17pr-8pq.\end{aligned}$$

Therefore,

$p(-2q+5r)-3(2pq-4pr)=17pr-8pq$.

Exercise 3.1

Simplify each of the algebraic expressions in questions 1 to 14:

1. $10a-4(2a+3b)$
2. $4-3n+4n+3$
3. $m(2p-3q)-m(2p-q)-5(mp-mq)$
4. $(8x+3y+5z)-(2z+y)$
5. $2(a-b)+3(2a+4)$
6. $6x-2x-x$
7. $5k+3+2(k+9)$
8. $11c+7d-5c-10d$
9. $7p-2q+3r-q+4r$
10. $2(4w-5)-2(w-7)$
11. $10a-4(2a+3b)$
12. $2(8x+y-3z)-3(2x+z-y)$
13. $6mn-5(pq-2mn)+16pq$
14. $4xyz-2pqr+6(xyz-5pqr)$
15. Simplify the expression $4(2ad-6ef+gh)-3(2ef-3gh+5ad)$, hence, state:
    (a) The number of terms.
    (b) The coefficient of gh.

Algebraic equations

An equation is a mathematical statement which shows that two mathematical expressions are equal. Thus, an equation is obtained by connecting two expressions by an equal sign. A solution to the given equation is an assignment of values to the unknown variables that makes the equality in the equation true. Solving equation is simply finding the solution that satisfies the conditions stated in the given equation.

Solving equations involving absolute values

In some cases, the magnitude of a number is desired without considering its sign. In this case, the value representing the magnitude of a number is called an absolute value of the given number. The absolute value of a number x is denoted as $|x|$ and defined as; $|x|=\begin{cases} x \text{ if } x\geq 0, \\ -x \text{ if } x<0. \end{cases}$

This means, if x is non-negative, it remains unchanged. But if x is negative, then it is multiplied by -1 to get an absolute value. For instance, $|3|=3$, since $3\geq 0$ and $|-3|=-1\times(-3)=3$ because $-3<0$.

Example 3.5

Solve the equation $|x|+1=5$.

Solution

Given that $|x|+1=5$.

This implies, $x+1=5$ or $-x+1=5$.

Thus, $x=5-1$ or $-x=5-1$, which gives $x=4$ or $x=-4$.

Therefore, $x=4$ or $x=-4$.

Example 3.6

Solve the equation $2|x-1|-1=5$.

Solution

Given $2|x-1|-1=5$, which implies that, $2|x-1|=6$.

Dividing both sides by 2 gives,

$|x-1|=3$.

The value of $|x-1|$ can be written as

$$|x-1|=\begin{cases} x-1 & \text{if } x-1\geq 0 \\ -(x-1) & \text{if } x-1<0 \end{cases}$$

Thus, if $x-1\geq 0$, then $|x-1|=x-1$.

Since $|x-1|=3$, then $x-1=3$.

Hence, $x=3+1=4$.

Similarly, if $x-1<0$, then

$|x-1|=-(x-1)$.

Thus, $-(x-1)=3$, implying that

$-x+1=3$.

Solving for x gives,

$x=1-3=-2$.

Therefore, $x=-2$ or $x=4$.

Example 3.7

Given that $2t-|3t+1|=3t+1$, find the values of t.

Solution

Given $2t-|3t+1|=3t+1$.

It implies that, $-|3t+1|=3t+1-2t$.

Thus, $|3t+1|=-t-1$.

From the definition of an absolute value,

$$|3t+1|=\begin{cases} 3t+1 & \text{if } 3t+1\geq 0 \\ -(3t+1) & \text{if } 3t+1<0 \end{cases}$$

If $3t+1\geq 0$, then $3t+1=-t-1$.

Collecting like terms gives, $4t=-2$.

Hence, $t=-\frac{1}{2}$.

Similarly, if $3t+1<0$, then

$-(3t+1)=-t-1$, which implies that,

$3t+1=t+1$.

Thus, $2t=0$, implying that, $t=0$.

Therefore, $t=-\frac{1}{2}$ or $t=0$.

Example 3.8

Find the values of x given that

$|2x-1|=|4x+9|$.

Solution

Given $|2x-1|=|4x+9|$.

Both sides of the equation involve absolute values. In this case, the two sides are equal if and only if the quantities inside the absolute value symbol are equal or have opposite signs. Thus,

$2x-1=4x+9$ or $2x-1=-(4x+9)$.

Collecting like terms and simplifying give, $-2x=10$ or $6x=-8$.

Solving for values of x gives,

$$x=-5 \text{ or } x=-\frac{4}{3}.$$

Therefore, $x=-5$ or $x=-\frac{4}{3}$.

Example 3.9

Solve the equation $\left|\frac{1}{2}y+4\right|=|4y-6|$.

Solution

Given $\left|\frac{1}{2}y+4\right|=|4y-6|$.

Both sides of the equation involve absolute values. In this case, the two sides are equal if and only if the quantities inside the absolute bars are equal or have opposite signs. Thus,

$$\frac{1}{2}y+4=4y-6 \text{ or } \frac{1}{2}y+4=-(4y-6).$$

Collecting like terms and simplifying give,

$-7y=-20$ or $9y=4$.

Solving for values of x gives,

$$y=\frac{20}{7} \text{ or } y=\frac{4}{9}.$$

Therefore, $y=\frac{20}{7}$ or $y=\frac{4}{9}$.

Exercise 3.2

Solve each of the following equations:

1. $2+|2x-5|=7$
2. $|x+1|+9=18$
3. $2|5-3x|=12$
4. $4x-3=|x+6|$
5. $\left|\frac{x-1}{x+1}\right|=3$
6. $\left|\frac{3x-7}{2x+3}\right|=1$
7. $|5-3x|=\frac{1}{2}|2x+3|$
8. $|5x+3|=|2x-1|$
9. $|3x-4|=|2x+3|$
10. $\left|\frac{4x-2}{5}\right|=\left|\frac{6x+3}{2}\right|$
11. $2x-|4x-6|=x-7$
12. $|8x-5|=2|3x-20|$

Word problems on linear simultaneous equations

Linear simultaneous equations involve two or more linear equations with two or more variables that have to be solved together in order to find the values of the unknown quantities which are true for each of the equations. In this subsection, simultaneous equations involving two linear equations will be discussed. Linear simultaneous equations can be formed from word problems. The following are steps used when solving word problems involving simultaneous equations:

1. Read the problem and identify the unknown quantities.
2. Assign variables to the uknown quantities.
3. Form equations by expressing the conditions in the given problem using the variables assigned in step 2.
4. Solve the resulting equations simultaneously.
5. State the answer using a clear statement.

Example 3.10

The difference between two numbers is 2 and their sum is 14. Find the two numbers.

Solution

Let x and y be the two numbers in the given problem such that x is greater than y.

Thus, the difference of the two numbers is expressed as $x-y=2$.

The sum of the two numbers is expressed as $x+y=14$.

Thus, the pair of linear simultaneous equations is given as,

$$\begin{cases} x-y=2 \text{.................................(i)} \\ x+y=14 \text{...............................(ii)} \end{cases}$$

Adding equatons (i) and (ii) gives,

$$+\begin{cases} x-y=2 \\ x+y=14 \end{cases}$$
$$2x=16$$

Solving for the value of x gives, $x=8$.

Using equation (ii), it implies that, $8+y=14$.

Solving for the value of y gives, $y=14-8=6$.

Therefore, the two numbers are 6 and 8.

Example 3.11

If twice the age of the son is added to the age of his father, the sum is 56. But, if twice the age of the father is added to the age of his son, the sum is 82. Find the ages of the father and his son.

Solution

Let x be the age of the father and y be the age of his son.

Thus, the equation representing twice the age of the son plus the age of the father is $2y+x=56$. Also, the equation representing twice the age of the father plus the age of the son is $2x+y=82$.

Thus, the pair of linear simultaneous equation is given as,

$$\begin{cases} 2y+x=56 \text{.................................(i)} \\ y+2x=82 \text{.................................(ii)} \end{cases}$$

Multiplying equation (i) by 2 gives,

$4y+2x=112$.............................(iii)

Subtracting equation (ii) from equation

(iii) gives,

$$-\begin{cases}4y+2x=112\\ y+2x=82\end{cases}$$
$$3y=30$$

Solving for the value of y gives, $y=10$.

Substituting $y=10$ into equation (i) gives,
$2(10)+x=56.$

This implies that,
$20+x=56$.

Solving for x gives,
$$x=56-20$$
$$=36.$$

Therefore, the age of the father is 36 years and the age of his son is 10 years.

Example 3.12

Three girls and four boys can do a piece of work in fourteen days, while four girls and six boys with similar abilities can do the same piece of work in ten days. How long would it take for a boy to finish the same work?

Solution

Let x be the number of days in which a girl finishes the piece of work and y be the number of days in which a boy finishes the piece of work.

It follows that, a girl's one day work is $\frac{1}{x}$ and a boy's one day work is $\frac{1}{y}$.

Given that 3 girls and 4 boys can do the work for 14 days.

Thus, 3 × (girl's one day work) + 4 × (boys's one day work) $=\frac{1}{14}$.

It implies that,

$$3\left(\frac{1}{x}\right)+4\left(\frac{1}{y}\right)=\frac{1}{14} \quad \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

Also, 4 girls and 6 boys can do the piece of work in 10 days.

Thus, 4 × (girl's one day work) + 6 × (boys' one day work) $=\frac{1}{10}$.

It implies that,

$$4\left(\frac{1}{x}\right)+6\left(\frac{1}{y}\right)=\frac{1}{10} \quad \ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

Multiplying equation (i) by 4 and equation (ii) by 3 gives the following system of linear equations.

$$\begin{cases}\frac{12}{x}+\frac{16}{y}=\frac{2}{7} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)}\\ \frac{12}{x}+\frac{18}{y}=\frac{3}{10} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iv)}\end{cases}$$

Substracting equation (iii) from equation (iv) gives,

$$-\begin{cases}\frac{12}{x}+\frac{18}{y}=\frac{3}{10}\\ \frac{12}{x}+\frac{16}{y}=\frac{2}{7}\end{cases}$$
$$\frac{2}{y}=\frac{1}{70}$$

Solving for y gives, $y=140$.

Therefore, it would take one hundred and forty days for a boy to finish the same work.

Exercise 3.3

1. The sum of two numbers is 50 and their difference is 16. Find the numbers.
2. The sum of two numbers is 43. If the larger number is doubled and the smaller is tripled, their difference is 36. Find the numbers.
3. Ten percent of the blue balls were added to twenty percent of the red balls to obtain 24 balls. Yet, three times the number of the blue balls exceeded the number of red balls by 20 balls. How many blue balls and red balls were there?
4. If the numerator and denominator of a fraction are increased by two and one, respectively, it becomes three quarters. If its numerator and denominator are decreased by two and one, respectively, it becomes a half. Find the fraction.
5. Twice the first number minus three times the second number is equal to 2. If the sum of the two numbers is 11, find the numbers.
6. The sum of ages of a mother and her daughter is 52 years. Eight years ago, the age of the mother was five times that of her daughter. Find the present ages of the mother and her daughter.
7. The sum of the ages of a father and his son is 110 years. The difference in their ages is 30 years. Determine their ages.
8. The difference between the length and width of the rectangle is 12 centimetres. The perimeter of the rectangle is 56 centimetres. Determine the dimensions of the rectangle.
9. In a game competition, 2 000 tickets were sold. An adult's ticket costs 5 000 Tanzanian shillings while a child's ticket costs 1 000 Tanzanian shillings. If a total of 5 200 000 Tanzanian shillings was collected, how many tickets of each kind were sold?
10. The cost of 4 pens and 5 exercise books together is 4 800 Tanzanian shillings, while the cost of 10 pens and 12 exercise books together is 11 600 Tanzanian shillings. Find the cost of a pen and that of an exercise book.
11. Two girls and seven boys can do a piece of work in four days. The same piece of work can be done in three days by four girls and four boys. How long would it take one girl and one boy to do the same piece of work?
12. If 6 women and 8 men can complete a certain task in 10 days, while 26 women and 48 men can complete the same task in 2 days, how long would it take 15 women and 20 men to complete the same type of task?

Transposition of formulae

Formulae are used to relate two or more quantities. Rearranging one quantity in terms of the other quantities is essential for easy computation. This rearrangement process is called transposition of the formula. Thus, transposition of formula is the process of expressing one variable in terms of other variables. Transposition of the formula is also known as making one variable the subject of the formula. In making one variable the subject of the formula, the equality in the given formula must be maintained in the sense that, any operation performed on one side must also be performed on the other side of the formula.

Example 3.13

Transpose s in the formula

$v^2 = u^2 + 2as$.

Solution

Given that $v^2 = u^2 + 2as$.

Subtracting u^2 from both sides of the equation gives,

$v^2 - u^2 = u^2 + 2as - u^2$.

Simplifying the equation gives,

$v^2 - u^2 = 2as$.

Dividing both sides of the equation by $2a$ gives, $\frac{v^2 - u^2}{2a} = s$.

Therefore, $s = \frac{v^2 - u^2}{2a}$.

Example 3.14

Make a the subject of the formula

$R = \frac{\rho l}{a+b}$.

Solution

Given $R = \frac{\rho l}{a+b}$.

Multiplying both sides by $a+b$ gives, $(a+b)R = \rho l$.

Dividing both sides of the equation by R gives, $a+b = \frac{\rho l}{R}$.

Subtracting b from both sides gives,

$a = \frac{\rho l}{R} - b$.

Therefore, $a = \frac{\rho l}{R} - b$.

Example 3.15

The volume of a sphere is given by the formula $V = \frac{4}{3}\pi r^3$. Transpose r in terms of other variables and then find the value of r if $V = \frac{2048\pi}{3}$ cm^3.

Solution

Given that $V = \frac{4}{3}\pi r^3$.

Multiplying both sides of the equation by 3 gives,

$3V = 4\pi r^3$

Dividing both sides of the equation by 4π gives,

$$\frac{3V}{4\pi} = r^3$$

Taking the cube root both sides of the equation gives,

$$\sqrt[3]{\frac{3V}{4\pi}} = r$$

Thus, $r = \sqrt[3]{\frac{3V}{4\pi}}$.

Given $V = \frac{2048\pi}{3}\ \text{cm}^3$.

Then, $r = \sqrt[3]{\frac{3 \times 2048\pi\ \text{cm}^3}{3 \times 4\pi}}$

$$= \sqrt[3]{512}\ \text{cm}$$

$$= 8\ \text{cm}$$

Therefore, $r = \sqrt[3]{\frac{3V}{4\pi}}$ and

when $V = \frac{2048\pi}{3}\ \text{cm}^3$, $r = 8$ cm.

Example 3.16

Express g as the subject of the formula

$$T = 2\pi\sqrt{\frac{l}{g}}.$$

Solution

Given $T = 2\pi\sqrt{\frac{l}{g}}$.

Dividing both sides of the equation by 2π gives,

$$\frac{T}{2\pi} = \sqrt{\frac{l}{g}}.$$

Squaring both sides of the equation gives,

$\left(\frac{T}{2\pi}\right)^2 = \left(\sqrt{\frac{l}{g}}\right)^2$, implying that,

$$\left(\frac{T}{2\pi}\right)^2 = \frac{l}{g}.$$

Multiplying both sides of the equation by g gives,

$\left(\frac{T}{2\pi}\right)^2 \times g = \frac{l}{g} \times g$. Thus, $\frac{T^2 g}{4\pi^2} = l$.

Multiplying both sides of the equation by $4\pi^2$ gives,

$$T^2 g = 4\pi^2 l.$$

Dividing both sides of the equation by T^2 gives,

$$g = \frac{4\pi^2 l}{T^2}.$$

Therefore, $g = \frac{4\pi^2 l}{T^2}$.

Example 3.17

The focal length, f of a convex lens is given by the formula $\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$, where u and v are the distances of the object and image from the lens, respectively. Make v the subject of the formula and evaluate v, given that $f = 8$ cm and $u = 40$ cm.

Solution

Given that, $\frac{1}{f}=\frac{1}{u}+\frac{1}{v}$.

Subtracting $\frac{1}{u}$ from both sides of the equation gives,

$$\frac{1}{f}-\frac{1}{u}=\frac{1}{u}+\frac{1}{v}-\frac{1}{u}$$

Simplification of the equation gives,

$$\frac{1}{f}-\frac{1}{u}=\frac{1}{v}$$

Using the common factor of f and u on the left-hand side of the equation gives,

$$\frac{u-f}{fu}=\frac{1}{v}$$

Multiplying by fuv and dividing by $u-f$ both sides of the equation gives,

$$v=\frac{fu}{u-f}.$$

Given $f=8$ cm and $u=40$ cm, then

$$v=\frac{8\text{ cm}\times 40\text{ cm}}{40\text{ cm}-8\text{ cm}}$$

$$=\frac{320\text{ cm}^2}{32\text{ cm}}$$

$$=10\text{ cm}.$$

Therefore, $v=\frac{fu}{u-f}$ and when $f=8$ cm and $u=40$ cm, v=10 cm.

Exercise 3.4

1. Express the given letter as the subject of the formula in each of the following:

   (a) $V=\frac{Er}{R+r}$, r

   (b) $s=ut+\frac{1}{2}at^2$, a

   (c) $\frac{1}{f}=\frac{1}{u}+\frac{1}{v}$, f

   (d) $A=2\pi r(r+h)$, h

   (e) $\frac{t}{2t-s}=3s$, t

   (f) $A=\left(\frac{t+1}{t-1}\right)^2$, t

   (g) $r\sqrt{\frac{y^2-t}{k}}=\frac{r^2}{a}$, y

   (h) $\frac{D}{d}=\sqrt[3]{\frac{3x^2-y^2}{r}}$, y

2. The final length, l_2 of a piece of wire heated through θ °C is given by the formula $l_2=l_1(1+\alpha\theta)$, where l_1 is the initial length and α is the coefficient of expansion. Make α the subject of the formula.

3. Express the given letter as the subject of the formula and hence evaluate its value by using the values provided:

   (a) h in $A=2\pi r^2+2\pi rh$, given $A=90\pi$, $r=3$

(b) x in $t-\frac{x}{5}=11$, given $t=6$

(c) F in $C=(F-32°)\times\frac{4}{9}$, given $C=26°$

(d) x in $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, given $y=3,\ a=1,\ b=5$

(e) a in $ay^2=x^3$, given $x=3,\ y=1$

(f) b in $\frac{p}{q}=\sqrt{\frac{a+2b}{a-2b}}$, given $a=4,\ p=3,\ q=1$

4. The height of water flowing in a pipe is given by the formula $h=\frac{0.03Lv^2}{2dg}$, where L, v, d, and g are the length of the pipe, velocity of water, diameter of the pipe, and acceleration due to gravity, respectively.

(a) Express v as the subject of the formula.

(b) Evaluate v given that $h=10,\ L=19.62,\ d=0.3$, and $g=9.81$ (units).

5. The quantity of heat, Q of a copper wire is determined by the formula $Q=mc(t_2-t_1)$, where m is the mass of the wire, c is the specific heat capacity, t_1 and t_2 are initial and final temperatures of the wire, respectively. Express t_2 as the subject of the formula, hence evaluate t_2 when $m=10,\ c=389,\ Q=1\ 600$, and $t_1=15$.

6. If $T=2a\sqrt{\frac{r-np}{np+r}}$, verify that $r=\frac{nP\left(T^2+4a^2\right)}{4a^2-T^2}$. Hence, find the value of r when $a=2,\ n=4,\ T=2$, and $P=3$.

7. The surface area of a sphere is given by the formula $A=4\pi r^2$, where r is the radius of the sphere. Make r the subject of the formula and then evaluate r when $A=256\pi$.

8. Given the formula $s=uv+\frac{1}{2}gt^2$, where t is time.

(a) Show that $t=\sqrt{\frac{2(s-uv)}{g}}$.

(b) Find the value of t when $g=10,\ u=60,\ v=2$, and $s=840$.

Inequalities

An inequality is a mathematical statement which compares two quantitites. The comparison of quantities is expressed by a mathematical statement of an order relationship which involves one of the signs; greater than, greater than or equal to, less than, and less than or equal to, between the quantities. The difference between equations and inequalities is that, in equations, the solutions are given as exact values while in inequalities, the solutions are given as range or set of values. Table 3.1 gives comparison of numbers a and b using inequality signs and their meaning.

Table 3.1: *Inequality signs and their meaning*

S/N	Inequality sign	Mathematical statement	Meaning
1.	$>$	$a > b$	a is greater than b
2.	$\geq$	$a \geq b$	a is greater than or equal to b
3.	$<$	$a < b$	a is less than b
4.	$\leq$	$a \leq b$	a is less than or equal to b

For instance:

(a) $x < 3$ is read as "x is less than 3". This means that, the variable x can be any number smaller than 3 such as ..., $-2, -1, 0, 2, 2.5, 2.9, 2.999$.

(b) $x > 5$ is read as "x is greater than 5". This means that the variable x can be any number greater than 5 such as 5.001, 5.2, 6, …

(c) $x \leq 4$ is read as "x is less than or equal to 4". This means that, the variable x can be any number smaller than or equal to 4 such as ..., $-6, -2, 0, 1, 2, 3.5, 3.99, 4$.

(d) $x \geq -2$ is read as "x is greater than or equal to -2". This means that, the variable x can be any number greater than or equal to -2 such as $-2, -1.999, -1.5, 0, 2, 5, \ldots$

(e) $5 > 2$ is read as "5 is greater than 2". In other words, 2 is less than 5. So, $5 > 2$ is the same as $2 < 5$.

Note that; If $x \leq a$ and $x \geq a$, then $x = a$.

Solution of linear inequalities

Linear inequalities can be manipulated using similar rules used in equations with the following important exceptions:

(a) The inequality sign is unchanged if the same number is added on both sides of the inequality.

(b) The inequality sign is unchanged if the same number is subtracted from both sides of the inequality.

(c) The inequality sign is unchanged if the same positive integer is multiplied or divided on both sides of the inequality.

(d) The inequality sign is reversed if both sides of the inequality are multiplied or divided by a negative number.

Example 3.18

Find the solution of the inequality $x+3>2$.

Solution

Given that $x+3>2$.

Subtracting 3 from both sides gives,

$x+3-3>2-3$

Thus, $x>-1$.

Therefore, $x>-1$.

Example 3.19

Solve for a in the inequality $3-5a \leq 2(a+5)$.

Solution

Given that $3-5a \leq 2(a+5)$.

Opening brackets gives,

$3-5a \leq 2a+10$.

Collecting like terms gives,

$-5a-2a \leq 10-3$

Simplifying gives, $-7a \leq 7$

Dividing both sides by -7 gives,

$a \geq -1$.

Therefore, $a \geq -1$.

Example 3.20

Solve for k in the inequality $\frac{k}{10} \geq 6+\frac{2k}{5}$.

Solution

Given that $\frac{k}{10} \geq 6+\frac{2k}{5}$.

Multiplying each term by 10 gives,

$$\frac{k}{10} \times 10 \geq 6 \times 10+\frac{2k}{5} \times 10$$

Thus, $k \geq 60+4k$.

Collecting like terms gives,

$k-4k \geq 60$, which implies that,

$-3k \geq 60$.

Dividing both sides by -3 gives,

$k \leq -20$.

Therefore, $k \leq -20$.

Exercise 3.5

Determine the solution of each of the following inequalities:

1. $\frac{3b-9}{6}>1$
2. $2-x \leq -\frac{x}{3}$
3. $11 \leq \frac{2m+3}{9}$
4. $6x \geq 4+2x$
5. $\frac{x}{4}+3 \leq 8$
6. $5-2x \geq 11$
7. $3(2x-4)+4x<4(3x-7)+8$
8. $24 \geq -6(m-6)$
9. $-r-5(r-6)<-18$
10. $\frac{6+x}{12} \leq -1$

Representation of linear inequalities on a number line

Since an inequality is satisfied by a range of values (solution set), it is often useful to draw a picture which represents the solutions of the inequality on a number line. When locating the solution set to an inequality on a number line, the following steps can be used:

1. Draw a number line with a reasonable positive and negative integers.
2. Draw a small circle on the number obtained as a starting point of the solution to the inequality on a number line. A small circle can also be drawn below or above the number.
3. The small circle should be unshaded when > or < sign is used and shaded when ≥ or ≤ sign is used. Unshaded circle means that the number is not included where as a shaded circle means that the number is included in the solution set.
4. Draw an arrow in the positive direction (to the right) if the variable is greater than or in the negative direction (to the left) if the variable is less than.

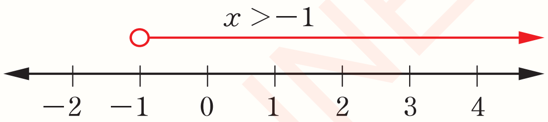
Example 3.21

Locate each of the following inequalities on a number line:

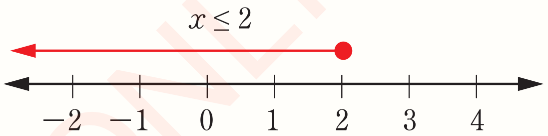
(a) $x > -1$ (b) $x \leq 2$ (c) $m \geq -5$ (d) $-5 \leq r < 2$

Solution

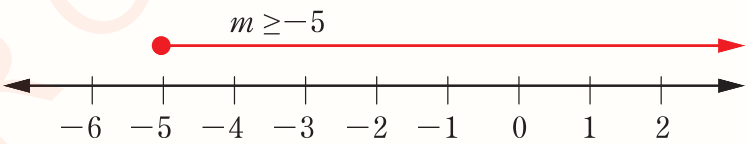
(a)



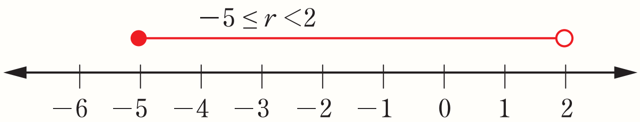
(b)



(c)



(d)



Example 3.22

Solve the inequality $\frac{2x-6}{4}<1$ and represent its solution on a number line.

Solution

Given $\frac{2x-6}{4}<1$.

Multiplying both sides by 4 gives,

$2x-6<4$.

Adding 6 to both sides gives,

$2x<4+6$.

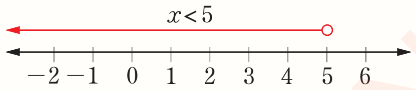
Thus, $2x<10$.

Dividing both sides by 2 gives,

$x<5$.

Therefore, the solution is $x<5$.

On a number line, $x<5$ is located as follows.



Example 3.23

Solve the inequality $5-6x\geq -19$ and represent its solution on a number line.

Solution

Given that $5-6x\geq -19$.

Collecting like terms gives,

$-6x\geq -19-5$

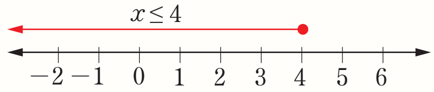
Thus, $-6x\geq -24$.

Dividing both sides by -6 gives,

$x\leq 4$.

Therefore, the solution is $x\leq 4$.

On a number line, $x\leq 4$ is represented as follows.



Example 3.24

Find the solution to $-1\leq 3x+2<17$ and represent it on a number line.

Solution

Given $-1\leq 3x+2<17$.

This inequality can be written as,

$-1\leq 3x+2$ and $3x+2<17$.

Thus, $-1-2\leq 3x$ and $3x<17-2$.

Simplify to obtain,

$-3\leq 3x$ and $3x<15$.

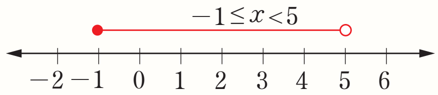
Dividing both sides by 3 gives,

$-1\leq x$ and $x<5$.

Thus, the solution is $-1\leq x$ and $x<5$.

Therefore, the combined solution is written as $-1\leq x<5$.

This solution is represented on a number line as follows.



Example 3.25

Solve the inequality

$-4x-3\leq -x+6<10-2x$ and

represent its solution on a number line.

Solution

Given $-4x-3\leq -x+6<10-2x$.

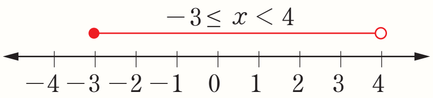
The inequality can be written as $-4x-3\leq -x+6$ and $-x+6<10-2x$.

Thus, $-4x+x\leq 6+3$ and $-x+2x<10-6$, implying that, $-3x\leq 9$ and $x<4$.

Hence, the solution set is $x\geq -3$ and $x<4$.

Therefore, the combined solution is written as $-3\leq x<4$.

The solution is represented on a number line as follows.



Example 3.26

Solve the inequality $|-2-4y|>14$ and represent its solution on a number line.

Solution

Given $|-2-4y|>14$.

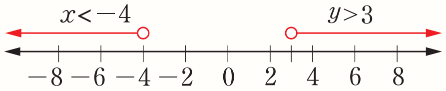
The inequality can be written as $-2-4y>14$ or $-(-2-4y)>14$.

This implies that, $-4y>14+2$ or $4y>14-2$, which gives, $-4y>16$ or $4y>12$.

Thus, $y<-4$ or $y>3$.

Therefore, $y<-4$ or $y>3$.

This solution is represented on a number line as follows:



Example 3.27

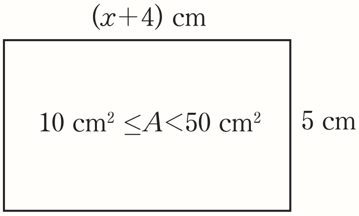
The surface area of a rectangle with length $(x+4)$ cm and width 5 cm is less than 50 cm^2 but greater than or equal to 10 cm^2. Form an inequality representing this information and hence represent its solution on a number line.

Solution

Let l be the length of the rectangle, w be the width, and A be the area.

Given $l=(x+4)$ cm, $w=5$ cm, and $10\text{ cm}^2\leq A<50\text{ cm}^2$.

Thus, the rectangle is sketched as follows.



The area of a rectangle is the product of length and width.

Thus, $A=l\times w$.

This implies that, $A=5(x+4)\text{ cm}^2$.

Thus, $10\text{ cm}^2\leq 5(x+4)\text{cm}^2<50\text{ cm}^2$.

Dividing by $5\,\text{cm}^2$ throughout the inequality gives,

$$\frac{10\text{ cm}^2}{5\,\text{cm}^2} \le (x+4)\frac{5\text{ cm}^2}{5\,\text{cm}^2} < \frac{50\text{ cm}^2}{5\,\text{cm}^2},$$

which simplifies to, $2 \le x+4 < 10$.

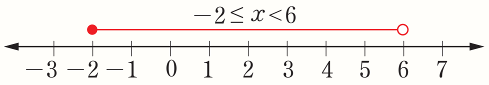
This inequality can be written as, $2 \le x+4$ and $x+4 < 10$.

Thus, $2-4 \le x$ and $x < 10-4$.

Hence, $-2 \le x$ and $x < 6$, which implies that, $-2 \le x < 6$.

Therefore, the inequality is $-2 \le x < 6$.

The solution is represented on a number line as follows.



Exercise 3.6

Locate the solution to each of the following linear inequalities on a number line:

1. $10 \le 2x-4 < 12$

2. $20-8x < 4$

3. $-\frac{3}{4}x \ge -\frac{5}{12}$

4. $13p < 7p-42$

5. $8x+3(x-12) > 7x-28$

6. $\frac{v-9}{-4} \le 2$

7. $-60 \ge -4(-6x-3)$

8. $-4 < 8-3m \le 11$

9. $-3 \le x-1 < 1$

10. $-11 \le n-9 \le -5$

11. $\left|\frac{2}{3}x-\frac{5}{6}\right| \le \frac{4}{3}$

12. The perimeter of a triangle with dimensions $(x+1)$ units, $(x+2)$ units, and $(x+3)$ units, is greater than 21 units but less than or equal to 30 units. Form the inequality representing this problem and hence represent its solution on a number line.

Chapter summary

1. An algebraic expression is a mathematical phrase that contains variables, coefficients, and mathematical operations without an equal sign.
2. An equation is a mathematical statement which shows that two mathematical expressions are equal.
3. The absolute value of a number x is defined as; $|x| = \begin{cases} x \text{ if } x \ge 0, \\ -x \text{ if } x < 0. \end{cases}$
4. Transposition of formula is the process of expressing one variable in terms of other variables in the given formula.

5. An inequality is a mathematical statement of order relationship between two algebraic expressions connected by the signs; greater than $(>)$, greater than or equal to $(\geq)$, less than $(<)$, less than or equal to $(\leq)$.
6. The inequality sign is reversed if both sides of the inequality are multiplied or divided by a negative number.

Revision exercise 3

1. Simplify each of the following expressions:

(a) $3(a+2b)+a-b$

(b) $6t+(5a-2t)$

(c) $p(8n-3t)+6(4pn-5pt)$

(d) $2(p+3)+5$

(e) $3(x+2)+(x-2)$

(f) $\dfrac{3y+4p}{4}-\dfrac{3p+2y}{6}-\dfrac{y-p}{12}$

2. Determine the solution of each of the following equations:

(a) $6-|2x-1|=3$

(b) $|2x-1|=2$

(c) $|x|-2=6$

(d) $\dfrac{|5x+15|}{6}=5$

(e) $\left|\dfrac{2}{3}x-\dfrac{5}{6}\right|=|x|$

(f) $\dfrac{|3x+7|}{|2x+3|}=1$

(g) $\dfrac{|2m+5|}{|3m-1|}=|-2|$

(h) $|4-3y|=\dfrac{2}{3}|4y-3|$

3. In a certain school, there are 880 students. If there are 80 more boys than girls, find the number of boys and girls in that school.

4. After six years, the age of a man will be three times the age of his son. Three years ago, the man's age was nine times as old as his son's age. Find their present ages.

5. The sum of two numbers is 209. If the difference between one number and twice the other number is 7, find the numbers.

6. Express the given letter as the subject of the formula in each of the following:

a) $t=2\pi\sqrt{\dfrac{h+k}{g}},\quad h$

(b) $F=\dfrac{9}{5}C+32,\quad C$

(c) $y=\dfrac{\lambda(x-d)}{d},\quad x$

(d) $A=\dfrac{3(F-f)}{L},\quad f$

(e) $I=\dfrac{E-e}{R+r},\quad R$

7. Given $L=\frac{1}{2}\rho v^2 ac$ as a lift force equation on an aircraft, where ρ is the density, v is the velocity, a is the area, and c is the lift coefficient. Make v the subject of the formula.

8. Express the given letter as the subject of the formula in each of the following formulas. Hence, evaluate its value for the given values of other letters:

   (a) d in $I=A+(n-1)d$ given $I=10,\ A=2,\ n=5$

   (b) b in $p=2(l+b)$ given $p=36,\ l=10$

   (c) h in $V=\frac{1}{3}\pi r^2 h$ given $V=56.4,\ r=3,\ \pi=\frac{22}{7}$

   (d) b in $A=\frac{1}{2}bh$ given $A=75,\ h=10$

   (e) d in $s=\frac{n}{2}(2a+(n-1)d)$ given $s=128,\ n=4,\ a=2$

9. Find the solution to each of the following inequalities:

   (a) $10\le 3x-(2x-4)<15$

   (b) $2\ge 2x+3\ge -10$

   (c) $-3<5-2x$

   (d) $\frac{3}{4}x-1\le 5$

   (e) $3<\frac{3}{5}x-1\le 5$

10. Solve and locate the solution to each of the following inequalities on a number line:

    (a) $4>-\frac{1}{2}(x-7)$

    (b) $5x-4<16$

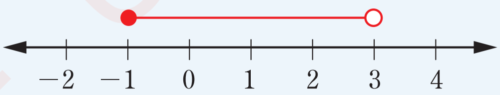
    (c) $2(x-3)\ge 3(x+1)$

    (d) $3\le -2x-7$

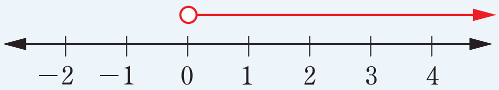
    (e) $\frac{2}{5}x<-1$

11. Determine in terms of x the inequality represented by each of the following number lines:

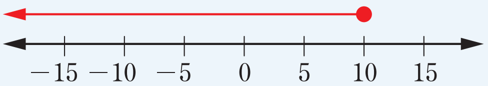
    (a)



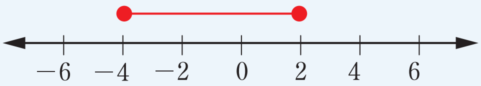
    (b)



    (c)



    (d)



12. The area of a right-angled triangle with height 6 cm and base $(2y+1)$ cm is greater or equal to 33 cm^2

and less than 42 cm^2. Formulate the inequality and then represent its solution on a number line.

13. Kitoronyi has a shop nearby a school. He sells soda and pieces of cake. Each piece of cake costs Tsh 1 500 and each bottle of soda costs Tsh 500. On a certain day, he made a total of Tsh 78 500 by selling a total of 87 pieces of cake and bottles of soda altogether. How many pieces of cake and bottles of soda were sold?

14. Zubeda and John went to the market to buy tomatoes and onions. Zubeda had Tsh 5 200 and bought 12 tomatoes and 20 onions. John had Tsh 4 000 and bought 10 tomatoes and 15 onions. What was the price of each item if they spent all the total amount of money?

15. The cost of 10 calculators and 8 textbooks together is 370 000 Tanzanian shillings, while the cost of 7 calculators and 4 textbooks together is 235 000 Tanzanian shillings. Find the cost of one calculator and one textbook.

Chapter Four: Geometrical constructions

Introduction

In most places such as homes, schools, industries, and construction sites, figures with lines and angles are used. Presentation of lines and angles in different figures on a piece of paper using mathematical instruments can be easily achieved through the use of knowledge of geometrical constructions. The study of geometrical constructions deals with the use of geometrical instruments to construct geometrical figures. In this chapter, you will learn about division of a line segment, angles of a polygon, and construction of polygons. The competencies developed will help you to appreciate the construction of buildings, gardens, dams, roads, and many other structures.

Division of a line segment

A line is a collection of points extending in both directions without an end. It can be considered as a straight path which can be extended indefinitely in both directions. It is represented by two arrow-heads in opposite directions as shown in Figure 4.1. Since it has no end points, a line does not have any fixed length. A line passing through two points A and B is denoted by $\overleftrightarrow{AB}$ or $\overleftrightarrow{BA}$.

Figure 4.1: *A line* AB

In Figure 4.1, if the points A and B on a line are considered as the end points of the length between points A and B, they form a line segment. Thus, a line segment is a piece or part of a line having two end points whose length is fixed. The length of a line segment is the distance between its end points.

A line segment AB is denoted as $\overline{AB}$ or $\overline{BA}$. An example of a line segment is shown in Figure 4.2.

Figure 4.2: *A line segment* AB

The length of a line segment can be measured in metric units of length such as millimetres, centimetres, metres, or kilometres, among other metric units.

A ray is a part of a line that has one fixed point and the other point does not have an end. If A and B are two points, then a ray which starts from point A through point B is denoted as $\overrightarrow{AB}$. An example of a ray is shown in Figure 4.3.

Figure 4.3: *A ray* AB

Proportion division of a line segment using geometrical instruments

A line segment can be divided into n equal parts by using a set square, ruler, and pair of compasses, where n is any natural number. Activity 4.1 demonstrates the proportion division of a line segment.

Activity 4.1: Dividing a line segment into equal parts

Individually or in a group, perform the following tasks:

1. Draw a line segment AB of convenient length on a piece of paper using a ruler and pencil.
2. Draw a ray AX which makes an acute angle with $\overline{AB}$, make it a little longer than $\overline{AB}$.
3. Put a tip of pair of compasses fixed with a sharp pencil on point A.
4. Along $\overrightarrow{AX}$, extend the compasses a little bit and draw a small arc which crosses $\overrightarrow{AX}$ and mark this point as P.
5. Without altering this distance, shift the tip of the pair of compasses to point P and draw another arc along $\overrightarrow{AX}$ and mark this point as Q.
6. Draw a line segment to join points B and Q using a ruler.
7. Draw a parallel line to BQ starting from point P to $\overline{AB}$ using a set square. Mark the point where the line from point P meets $\overline{AB}$ as M.
8. Measure the lengths of $\overline{AM}$ and $\overline{MB}$.
9. Repeat tasks 7 and 8 with up to 6 equal divisions along $\overrightarrow{AX}$.
10. Share the results with other students through discussion for more inputs.

The method of dividing a line segment into proportions using a ruler, a set square, and a pair of compasses as performed in Activity 4.1 is more accurate than simply measuring and marking points. This is because, sometimes the lengths cannot be measured precisely and therefore, the points cannot be marked correctly.

Example 4.1

Divide a line segment of length 12 cm into seven equal parts.

Solution

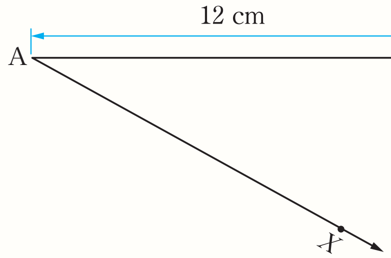
Let the line segment of 12 cm be $\overline{AB}$.

Thus, $\overline{AB}$ can be divided into seven equal parts using the following steps:

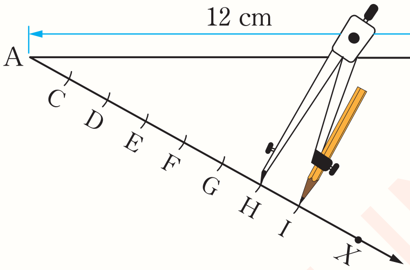
1. Draw $\overline{AB}$ as shown in the following figure.

2. Draw a ray AX which makes an acute angle with $\overline{AB}$ (make it a

little bit longer than $\overline{AB}$) as shown in the following figure.

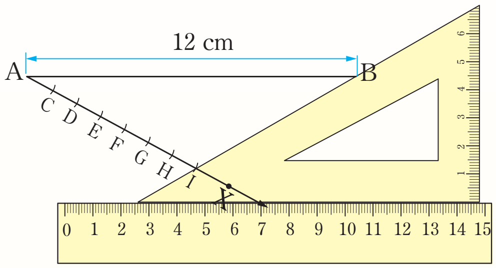


3. Since seven parts are needed, then the number of points to be located on $\overrightarrow{AX}$ should be 7. Now, mark the points C, D, E, F, G, H, and I, such that $\overline{AC} = \overline{CD} = \overline{DE} = \overline{EF} = \overline{FG} = \overline{GH} = \overline{HI}$ by using a pair of compasses fixed with a sharp pencil as shown in the following figure.



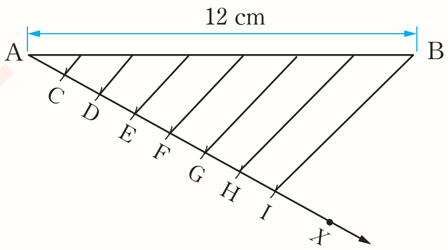
4. Join the points B and I by using a ruler. Using a set square, a ruler, and a pencil, draw parallel line segments to $\overline{BI}$ from point H back to point C, all line segments ending at $\overline{AB}$. In order to ensure that the line segments are parallel, the following procedures are used:

   (a) Place a set square on $\overline{BI}$ with its base on a ruler as shown in the following figure.



Make sure that the ruler is fixed in such a way that the base of the set square is not changing.

   (b) Slide the set square with its base on the ruler to points H, G, F, E, D, and C on $\overline{AI}$ while drawing the line segments joining the points on $\overline{AI}$ and $\overline{AB}$. The line segments drawn will indeed be parallel to $\overline{BI}$ if the base of the set square moves on the fixed ruler exactly as it is placed as shown in the following figure.



Therefore, the line segment AB has been divided into seven equal parts.

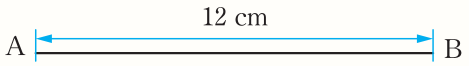
Example 4.2

Divide a line segment of length 12 cm into five equal parts.

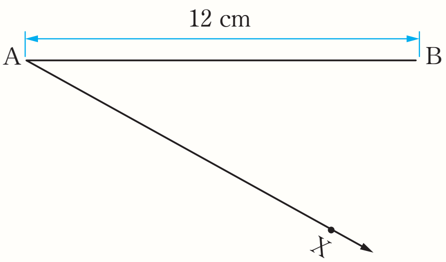
Solution

Divide the line segment into five parts using the following steps:

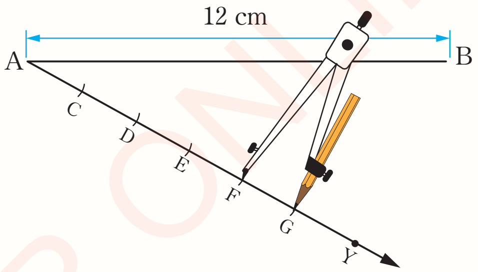
1. Draw a line segment, say AB as shown in the following figure.



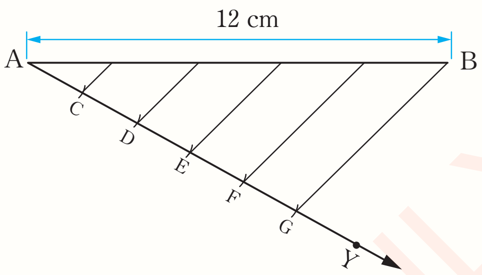
2. Draw a ray AY which makes an acute angle with $\overline{AB}$ (make it a little bit longer than $\overline{AB}$) as shown in the following figure.



3. Since five equal parts are needed, then locate five points on $\overrightarrow{AY}$. Now, mark the points C, D, E, F, and G, such that $\overline{AC} = \overline{CD} = \overline{DE} = \overline{EF} = \overline{FG}$ by using a pair of compasses fixed with a sharp pencil.



4. Join the points B and G by using a ruler. Using a set square, a ruler, and a pencil, draw line segments parallel to $\overline{BG}$ each from the points F, E, D, and C to the line segment AB by sliding the set square with its lower edge on the edge of the ruler as shown in the following figure.



Therefore, the line segment AB has been divided into five equal parts.

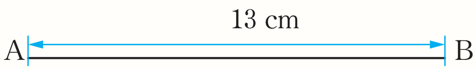
Example 4.3

Divide a line segment of length 13 cm into 8 equal parts.

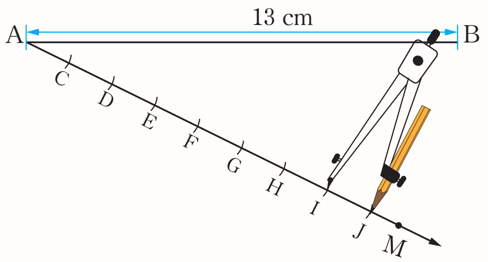
Solution

Divide the line segment into 8 equal parts as follows.

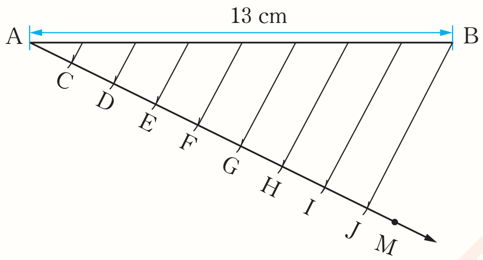
1. Draw the given line segment, say AB as shown in the following figure.



2. Draw a ray AM which makes an acute angle with $\overline{AB}$, make it a little bit longer than $\overline{AB}$. Mark 8 points C, D, E, F, G, H, I, and J on $\overrightarrow{AM}$ by using a pair of compasses fixed with a sharp pencil such that $\overline{AC} = \overline{CD} = \overline{DE} = \overline{EF} = \overline{FG} = \overline{GH} = \overline{HI} = \overline{IJ}$ as shown in the following figure.



3. Join the points B and J by using a ruler. Using a set square, a ruler, and a pencil, draw line segments parallel to $\overline{BJ}$ each from the points I, H, G, F, E, D, and C to the line segment AB by sliding the set square with its lower edge on the edge of the ruler as shown in the following figure.



Therefore, the line segment AB has been divided into 8 equal parts.

Exercise 4.1

1. By using a set square, a ruler, and a pair of compasses, draw a line segment, 8 cm long and divide it into the following parts:
   (a) Six equal parts
   (b) Ten equal parts
2. Draw a line segment of length 11 cm and divide it into the given number of parts:
   (a) 6 equal parts
   (b) 4 equal parts

Angles of polygons

A polygon is a closed two-dimensional or plane figure that has a finite number of sides and angles. The sides of a polygon are made up of line segments connected to each other end to end without intersecting. Figure 4.4 represents some of the polygons.

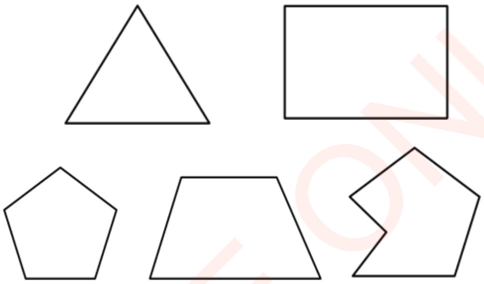


Figure 4.4: *Polygons*

The line segments of a polygon are called sides or edges.

A circle is also a plane figure but it is not a polygon because it has a curved shape and does not have sides and angles. Thus, it can be concluded that all polygons are closed two-dimensional figures but not all two-dimensional figures are polygons.

A point where two line segments meet is called a vertex or a corner and it forms an interior angle as shown in Figure 4.5.

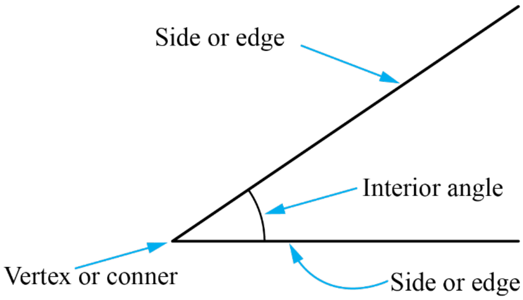


Figure 4.5: *Interior angle*

If the length of a side of a polygon is extended outwards, then an exterior angle is formed as shown in Figure 4.6.

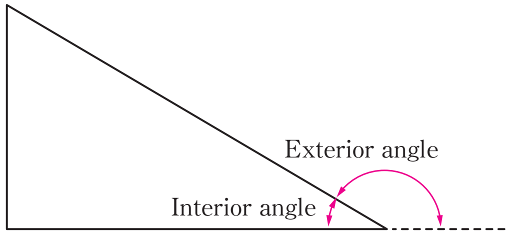


Figure 4.6: *Interior and exterior angles*

Regular polygons

If all sides of a polygon are equal (equilateral) and the measure of interior angles are equal (equiangular), then the polygon is known as a regular polygon. Equivalently, a regular polygon is an equilateral and equiangular polygon. Examples of regular polygons include squares, regular pentagons, regular hexagons, and equilateral triangles as shown in Figure 4.7.

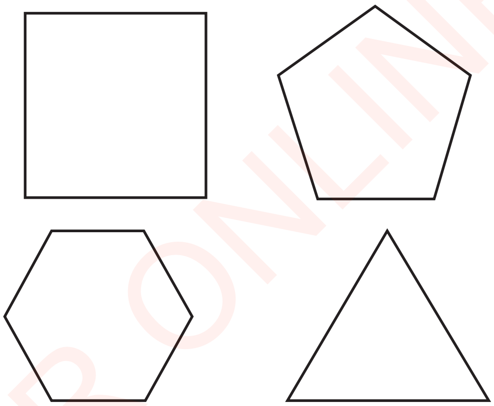


Figure 4.7: *Regular polygons*

Irregular polygons

A polygon whose sides or measure of interior angles are not equal is called an irregular polygon. Examples of irregular polygons include, scalene triangles, rectangles, kites, and trapeziums as shown in Figure 4.8.

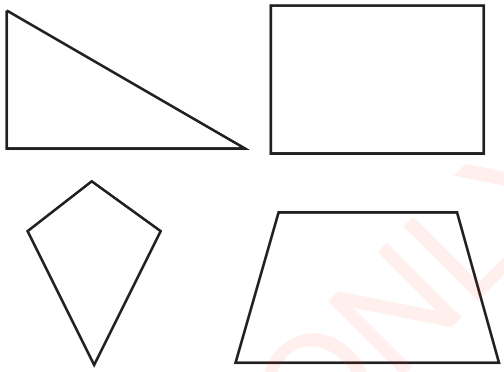
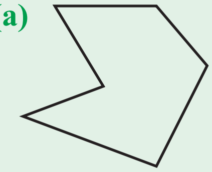


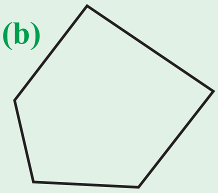
Figure 4.8: *Irregular polygons*

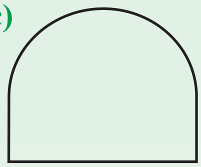
Exercise 4.2

From the following figures, identify:

1. Non-polygons
2. Regular polygons
3. Irregular polygons

(a)


(b)


(c)


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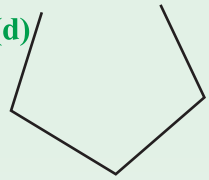
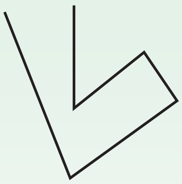
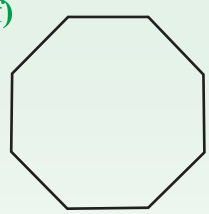
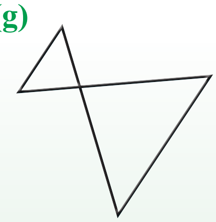


(e)


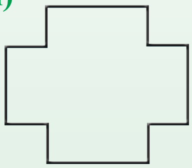
(f)


(g)

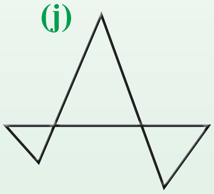


(h)

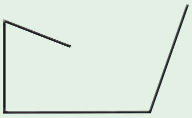
(i)



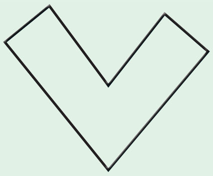
(j)



(k)

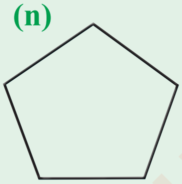


(l)

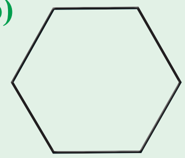


(m)

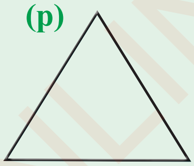
(n)



(o)



(p)



Sum of interior angles of polygons

A triangle is a simple polygon made up of three line segments. There are three interior angles in a triangle. The sum of all three interior angles in the triangle is equal to 180°. Activity 4.2 guides on how to find the sum of interior angles of a polygon.

Activity 4.2: Finding the sum of interior angles of a polygon

Individually or in a group, perform the following tasks:

1. Locate any three points on a piece of paper such that the points are not aligned on a straight line.
2. Using a ruler, draw the line segments connecting pairs of points in task 1 such that the line segments meet at a point but do not cross each other.
3. Complete task 2 by working on all the points in task 1 such that the resulting figure is a polygon.
4. Using a protractor, measure all the interior angles made in task 2.
5. Compute the sum of all interior angles in task 4 and record its measure.
6. Repeat tasks 1 to 5 by changing the number of points from three to four.
7. What conclusion can you make from the performed tasks?
8. Share your results with other students through discussion for more inputs.

From Activity 4.2, it can be concluded that the sum of interior angles of any three-sided polygon is 180° and that of a four-sided polygon is 360°. Thus, the sum of interior angles of other polygons can be obtained by dividing the given polygon into a number of distinct triangles without crossing the lines.

Therefore, the sum of interior angles of a polygon is obtained by multiplying by 180° the number of triangles formed as demonstrated in Activity 4.3.

Activity 4.3: Finding the sum of interior angles of a polygon by number of triangles

Individually or in a group, perform the following tasks:

1. Locate any four points on a piece of paper such that the points are not aligned on a straight line.
2. Using a ruler, draw the line segments connecting pairs of points in task 1 such that the line segments meet at a point but do not cross each other.
3. Complete task 2 by working on all the points in task 1 such that the resulting figure is a polygon.
4. Choose any vertex and draw a line segment connecting another vertex such that a triangle is formed.
5. Repeat task 4 by drawing line segments from the vertex of your choice to other vertices making different triangles where line segments made are not crossing each other.
6. Count and record the number of distinct triangles formed in task 5.
7. Compute the product of number of triangles recorded in task 6 and 180°, where 180° is the sum of interior angles of a triangle.
8. Record the value obtained in task 7.
9. Share your results with other students through discussion for more inputs.

Note that:

(a) The lines inside a given polygon may be drawn in different ways but the total number of triangles will always be the same.

(b) Lines should not cross each other.

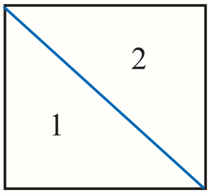
(c) Many lines can meet at one vertex.

Example 4.4

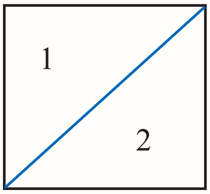
Find the sum of interior angles of a square.

Solution

A square can be divided into two triangles as follows:



or



Hence, the sum of interior angles in a square is the product of number of triangles and 180°.

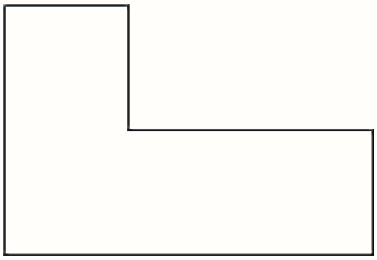
Thus,

$$\text{the sum of interior angle} = 2\times180° = 360°.$$

Therefore, the sum of interior angles of a square is 360°.

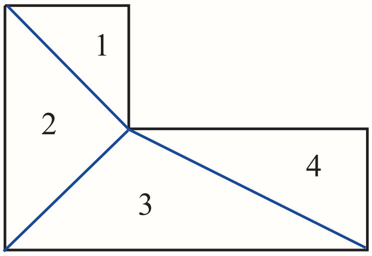
Example 4.5

Find the sum of interior angles of the following figure.

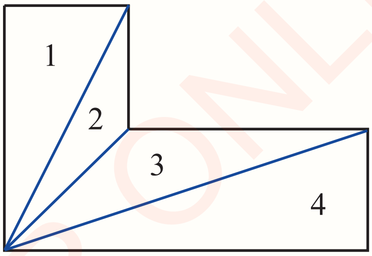


Solution

The lines inside the given figure can be drawn in two alternative ways as follows.



or



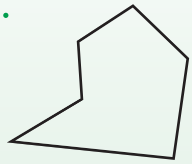
From the two figures, four triangles are formed in each case.

Therefore, the sum of interior angles of a given polygon is $4 \times 180° = 720°$.

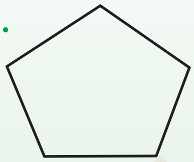
Exercise 4.3

Find the sum of interior angles in the following polygons by dividing each of the polygons into distinct triangles.

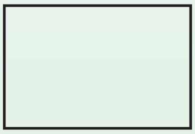
1.



2.

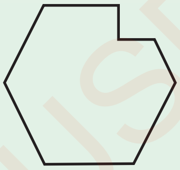


3.

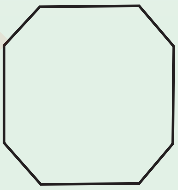


4.

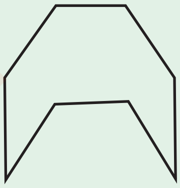
5.



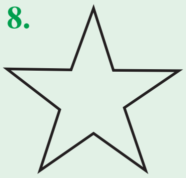
6.



7.



8.



Formula for the sum of interior angles of a polygon

The method of finding the sum of interior angles of a polygon by dividing a given polygon into a number of distinct triangles is suitable for polygons with few number of sides. This method can be tedious when the given polygon has a large number of sides. From the triangle method approach, it is noted that the number of sides of a polygon exceeds the number of distinct triangles formed by 2.

Table 4.1 represents different types of polygons, number of sides, corresponding number of triangles formed by drawing lines inside the polygon without crossing each other, and the sum of interior angles obtained by multiplying the number of triangles by 180°.

Table 4.1: *Sum of interior angles*

Name of a polygon	Number of sides (n)	Number of triangles	Sum of interior angles
Triangle	3	1	$1\times180°=180°$
Quadrilateral	4	2	$2\times180°=360°$
Pentagon	5	3	$3\times180°=540°$
Hexagon	6	4	$4\times180°=720°$
Heptagon	7	5	$5\times180°=900°$
Octagon	8	6	$6\times180°=1080°$
Nonagon	9	7	$7\times180°=1260°$
Decagon	10	8	$8\times180°=1440°$
Hendecagon	11	9	$9\times180°=1620°$
Dodecagon	12	10	$10\times180°=1800°$
For an n-gon	n	$n-2$	$(n-2)\times180°$

Therefore, for an n-gon, there are $n-2$ distinct triangles and the sum of its interior angles is given by $180°(n-2)$.

Example 4.6

Find the sum of interior angles of a thirteen-sided polygon.

Solution

Given that the polygon has 13 sides, so $n=13$.

The sum of interior angles $=180°(n-2)$.

This implies that,

Sum of interior angles $=180°(13-2)$

$=180°\times11$

$=1980°$.

Therefore, the sum of interior angles of a thirteen-sided polygon is 1980°.

Example 4.7

Find the number of sides of a polygon whose sum of interior angles is 1260°.

Solution

Given that the sum of interior angles $=1260°$.

The sum of interior angles $=180°(n-2)$.

This implies that,

$$1260°=180°(n-2)$$

Thus, $\frac{1260°}{180°} = n-2$. It implies that,

$$n = 7+2$$
$$= 9.$$

Therefore, the number of sides of the polygon is 9.

Measure of interior angles of regular polygons

The measure of an interior angle of a regular polygon is obtained by dividing the sum of interior angles by the number of angles. But the number of angles in a regular polygon is equal to the number of sides.

Therefore, the measure of interior angle,

$$\theta_i = \frac{\text{Sum of interior angles}}{\text{Number of sides}}$$
$$= \frac{(n-2)\times 180°}{n}.$$

Example 4.8

Find the measure of an interior angle of a regular hexagon.

Solution

A regular hexagon has six sides.

Thus, $n = 6$.

From the measure of interior angle,

$$\theta_i = \frac{\text{Sum of interior angles}}{\text{Number of sides}}.$$
$$= \frac{180°(n-2)}{n}$$
$$= \frac{180°(6-2)}{6}$$
$$= \frac{180°\times 4}{6}$$

Thus, $\theta_i = 120°$.

Therefore, the measure of an interior angle of a regular hexagon is $120°$.

Example 4.9

A regular polygon has 40 sides.

Determine:

(a) The total number of interior angles.

(b) The number of triangles that can be formed in the polygon.

(c) The measure of its interior angle.

Solution

(a) Since the given polygon has 40 sides, it has 40 corners (vertices). Therefore, the total number of interior angles is 40.

(b) Given $n = 40$.

Number of triangles $= n-2$

$$= 40-2 = 38.$$

Therefore, the number of triangles in the given polygon is 38.

(c) The measure of interior angle,

$$\theta_i = \frac{180°(n-2)}{n}$$
$$= \frac{180°\times 38}{40}$$
$$= 171°.$$

Therefore, the measure of interior angle of the given polygon is $171°$.

Sum of exterior angles of regular polygons

Consider a regular hexagon in Figure 4.9.

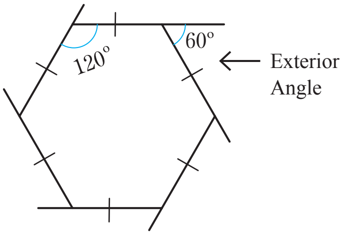
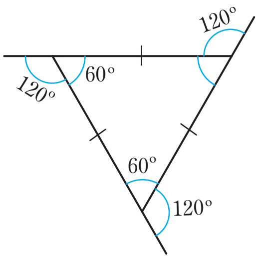
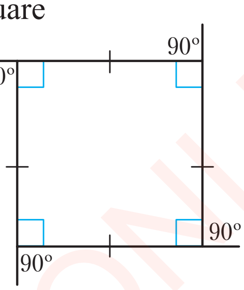
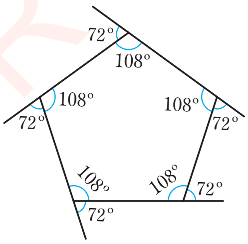


Figure 4.9: *Regular hexagon*

There are six exterior angles in a regular hexagon, each measuring $60°$ as shown in Figure 4.9. The measure of an exterior angle of any polygon is obtained by subtracting the size of interior angle from $180°$. Thus, the sum of exterior angles of a regular hexagon is $6\times60° = 360°$. Table 4.2 shows the sum of exterior angles for some regular polygons.

Table 4.2: *Sum of exterior angles for some regular polygons*

Type of a regular polygon	Measure of exterior angle (θ_e)	Sum of exterior angles $(n\theta_e)$
(a) Equilateral triangle 	$\theta_e = 180° - \theta_i$ $= 180° - 60°$ $= 120°$	Sum of exterior angles $= 120°\times3$ $= 360°$
(b) Square 	$\theta_e = 180° - \theta_i$ $= 180° - 90°$ $= 90°$	Sum of exterior angles $= 90°\times4$ $= 360°$
(c) Regular pentagon 	$\theta_e = 180° - \theta_i$ $= 180° - 108°$ $= 72°$	Sum of exterior angles $= 72°\times5$ $= 360°$

From Table 4.2, it can be observed that the sum of exterior angles of regular polygons is the same. Thus, it can be concluded that, the sum of exterior angles of any regular polygon is $360°$. The measure of exterior angle of a regular polygon is obtained by dividing the sum of exterior angles by number of sides.

That is, the measure of exterior angle,

$$\theta_e = \frac{\text{Sum of exterior angles}}{\text{Number of sides}}.$$

But the sum of exterior angles for any regular polygon is $360°$.

Therefore, $\theta_e = \dfrac{360°}{n}$, where n is the number of sides of the given regular polygon.

Example 4.10

If a regular polygon has an exterior angle of $18°$, find the number of sides of the polygon.

Solution

Given $\theta_e = 18°$.

From $\theta_e = \dfrac{360°}{n}$, it follows that,

$$n\theta_e = 360°.$$

Thus, $n = \dfrac{360°}{\theta_e}$.

Hence, $n = \dfrac{360°}{18°} = 20.$

Therefore, the number of sides of the regular polygon is 20.

Example 4.11

The interior angle of a regular polygon is four times the exterior angle. Find:

(a) The measure of the exterior angle.

(b) The measure of the interior angle.

(c) The number of sides of the polygon.

Solution

(a) Given that $\theta_i = 4\theta_e$, where θ_i is an interior angle and is an exterior angle.

But $\theta_i + \theta_e = 180°$. Thus,

$$4\theta_e + \theta_e = 180°. \text{ It implies that,}$$

$$5\theta_e = 180°.$$

Hence, $\theta_e = \dfrac{180°}{5} = 36°$.

Therefore, the measure of exterior angle is $36°$.

(b) The interior angle is $\theta_i = 4\theta_e$.

Thus, $\theta_i = 4 \times 36° = 144°$.

$$= 144°.$$

Therefore, the measure of interior angle is $144°$.

(c) From $\theta_e = \dfrac{360°}{n}$, it implies that,

$$n = \frac{360°}{\theta_e}$$

$$= \frac{360°}{36°}$$

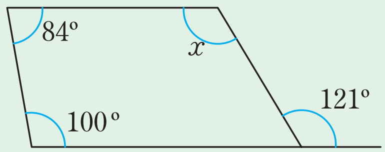
$$= 10.$$

Therefore, the number of sides of the regular polygon is 10.

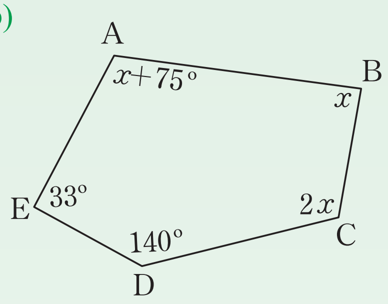
Exercise 4.4

1. Find the value of x in each of the following figures:

(a)



(b)



2. Find the measure of interior and exterior angles of a regular polygon with the following number of sides.

(a) 10 **(b)** 9

3. If a regular polygon has an exterior angle of $40°$, find:

(a) The number of sides.

(b) Size of interior angle.

(c) The sum of interior angles.

4. What are the measures of interior and exterior angles of a regular heptagon?

5. How many sides does a regular polygon with interior angle measure of $168°$ have?

6. How many sides does a regular polygon with exterior angle measure of $20°$ have?

Constructions of regular polygons

Regular polygons can be constructed by using geometrical instruments such as a ruler, a pair of compasses fixed with a sharp pencil, and a protractor.

Activity 4.4 provides a guide on constructions of regular polygons.

Activity 4.4: Constructing regular polygons

Individually or in a group, perform the following tasks:

1. Locate any two points on a piece of paper such that there is a convenient distance between them and label them as A and B.
2. Use a ruler to draw a line segment AB connecting the two points in task 1.
3. Use a protractor to measure angle $\theta = 60°$ to the line AB at point A.
4. Adjust a pair of compasses such that the tip rests on point A and a tip of a sharp pencil rests on point B.
5. Without altering the adjustments made on the pair of compasses in task 4, locate point C such that the line segment AC makes an angle $\theta = 60°$ to the line segment AB.
6. Repeat tasks 3, 4, and 5 until a regular polygon is made.
7. Repeat tasks 1 to 6 by changing the size of angle from $\theta = 60°$ to $\theta = 120°$ and make a polygon.
8. Share your results with other students through discussion for more inputs.

Example 4.12

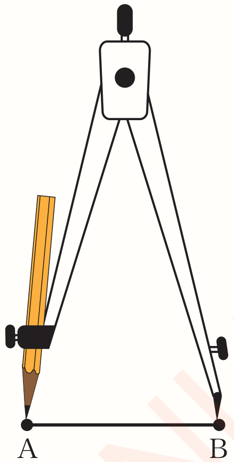
Construct an equilateral triangle.

Solution

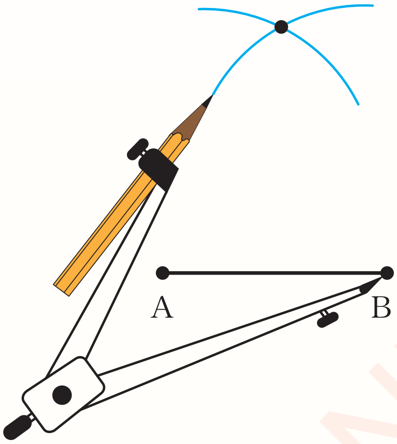
The following steps are used to construct an equilateral triangle:

1. Draw a line segment AB using a ruler and a sharp pencil as shown in the following figure.

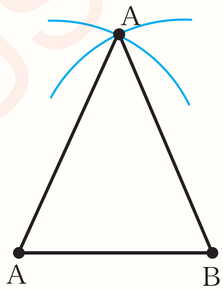
2. Using a pair of compasses fixed with a sharp pencil, copy the length of a line segment AB as shown in the following figure.



3. Put a tip of the pair of compasses on point B and draw an arc either above or below the line segment AB.
4. Without altering the width of the compass, shift the tip to point A and draw another arc intersecting the former arc. Mark this point as C as shown in the following figure.



5. Use a ruler and a sharp pencil to draw line segments AC and BC to form an equilateral triangle as shown in the following figure.



Verification: Use a ruler to verify that the lengths of the lines are equal and a protractor to verify that the interior angles are equal.

Example 4.13

Construct a regular pentagon with side length 4 cm.

Solution

By using geometrical instruments, a regular pentagon of side length

4 cm can be constructed through the following steps:

1. Find the interior angle of the given regular polygon. For the case of a pentagon, $n = 5$. Thus, interior angle

$$\theta_i = \frac{(n-2)\times 180°}{n}$$

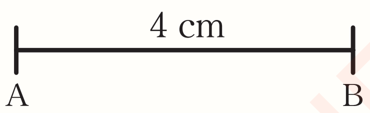
$$= \frac{(5-2)\times 180°}{5}$$

$$= 108°.$$

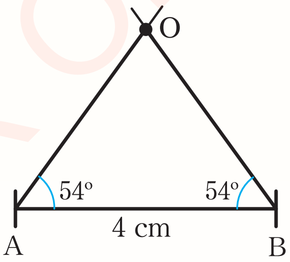
2. Divide the angle obtained by 2 to get the base angles of an isosceles triangle. That is,

$$\frac{108°}{2} = 54°.$$

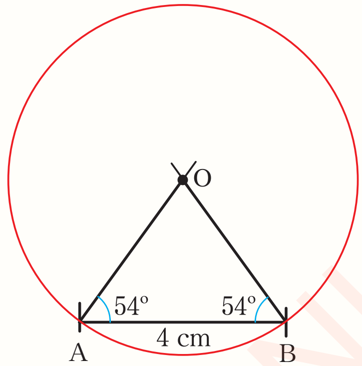
3. Draw a line segment AB with length 4 cm as shown in the following figure.



4. By using a protractor, measure the angle obtained in step 2 from both ends of the line segment, and then connect the lines so that they intersect at O as shown in the following figure.

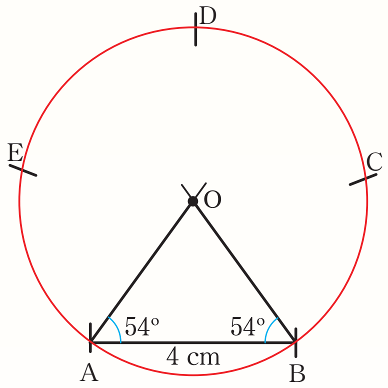


5. Put a tip of a pair of compasses at point O and a pencil at point A or point B and then draw a circle as shown in the following figure.

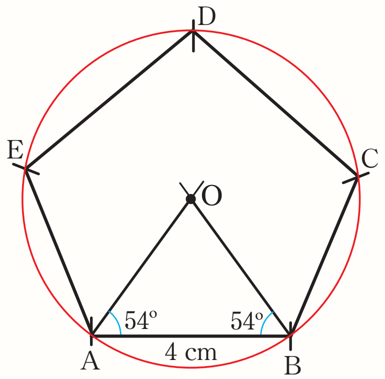


6. Copy the distance of the line segment AB by putting a tip of a pair of compasses at point B and a pencil at point A.

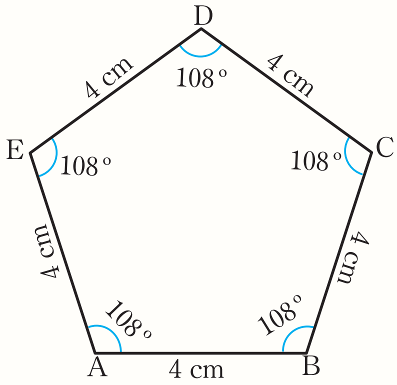
7. Without altering the adjustment made, while the tip is at B, mark point C on the circumference of the circle, and then shift the tip to C and put another mark D on the circumference. Proceed marking on the circumference until you get back to point A as shown in the following figure.



8. Draw the line segments BC, CD, DE, and EA as shown in the following figure.



9. Erase all unwanted lines, curves, and centre to remain with a clear polygon as in the following figure.



Note that; This method can be used to construct any regular polygon.

Alternatively; the given polygon can be constructed as follows:

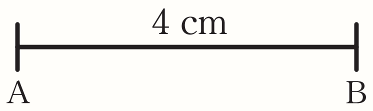
1. Find the interior angle of a given regular pentagon:

$$\theta_i = \frac{(n-2)\times 180^\circ}{n}$$

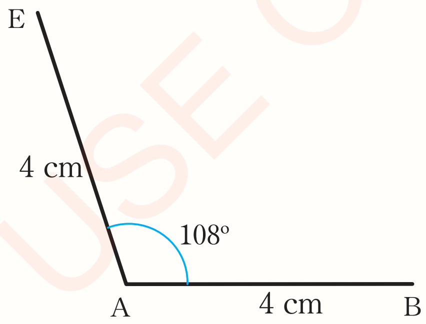
$$= \frac{(5-2)\times 180^\circ}{5}$$

$$= 108^\circ$$

2. Draw a line segment AB of length 4 cm as shown in the following figure.

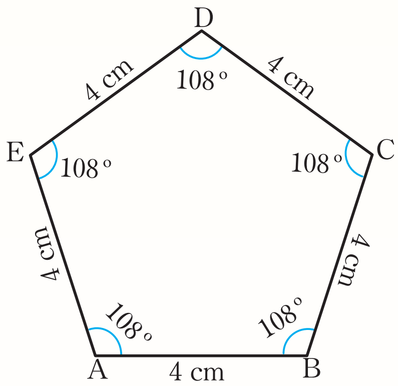


3. Using a protractor, measure the angle obtained in step 1 and then draw a line segment of length 4 cm connecting A as shown in the following figure.



4. Repeat step 2 until the fifth line segment connects to point B.

   Therefore, the required regular polygon is constructed as it is shown in the following figure.



Exercise 4.5

1. Construct each of the following polygons by using a pair of compasses, ruler, and pencil:
   (a) A square with side length 5 cm.
   (b) A regular hexagon with side length 4 cm.
   (c) A regular nonagon with side length 3 cm.
   (d) A regular pentadecagon (15-sided figure) with side length 3 cm.
2. Construct each of the following polygons by using a protractor, ruler, and pencil:
   (a) A regular heptagon with side length 5 cm.
   (b) A regular decagon with side length 4 cm.
   (c) A regular hendecagon (11-sided figure) with side length 3 cm.

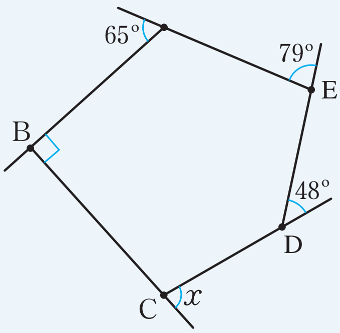
Chapter summary

1. The sum of interior angles of a polygon is given by $180°(n-2)$, where n is the number of sides.
2. The measure of interior angle of a regular polygon is given by $\theta_i = \dfrac{180°(n-2)}{n}$.
3. The measure of exterior angle of a regular polygon is given by $\theta_e = \dfrac{360°}{n}$.
4. The interior and exterior angles of a regular polygon are related by $\theta_i + \theta_e = 180°$.
5. The sum of interior angles of a regular polygon is given by $n\theta_i$.
6. The sum of exterior angles of a regular polygon is $360°$. That is, $n\theta_e = 360°$.

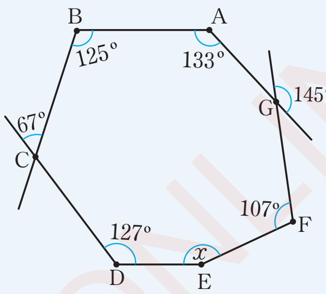
Revision exercise 4

1. Draw a line segment of 9.9 cm and divide it into 7 equal parts.
2. Draw a line segment of 12 cm and divide it into the following number of parts:
   (a) 5 equal parts.
   (b) 14 equal parts.
   (c) 10 equal parts.
3. Find the value of x in each of the following irregular polygons:

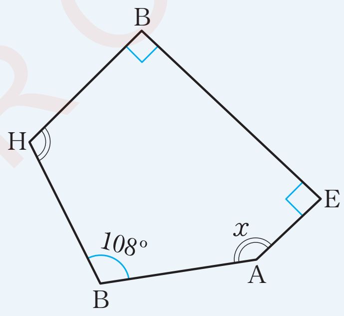
(a)



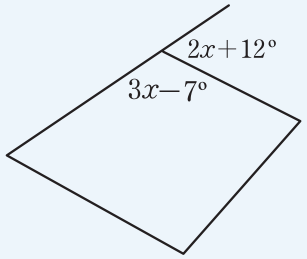
(b)



(c)



4. Determine the number of sides of regular polygons with the following measure of interior angles:
   (a) 140° (b) 156°
5. Find the number of sides of regular polygons with the following exterior angles:
   (a) 30° (b) 12°
6. Determine the number of sides and measure of interior angles of regular polygons with the following sum of interior angles.
   (a) 2160° (b) 6120° (c) 4140°
7. Find the number of sides of a regular polygon whose interior angle is 157.5°.
8. Determine the value of x and measure of the exterior angle in the following figure.



9. Construct a 12-sided regular polygon with a side length of 2.5 cm.
10. Construct a 15-sided regular polygon of a side length 3 cm, hence use a protractor to find:
    (a) The measure of its interior angle.

(b) The measure of its exterior angle.

(c) The sum of its interior angles.

11. Find the sum of interior angles of a regular polygon given that each interior angle is y and each exterior angle is given by $\frac{(y-36°)}{3}$. Hence, construct triangles in the polygon.

12. The interior angle of a regular polygon exceeds the exterior angle by 100°. Find the sum of exterior angles and number of triangles that can be constructed in the polygon.

13. If the interior angle of a regular polygon is five times its exterior angle, find the sum of its interior angles.

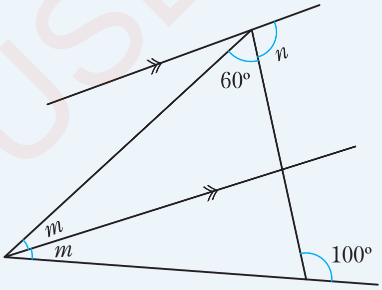
14. A regular polygon has an interior angle which is $6\frac{1}{2}$ times the exterior angle. How many triangles can be obtained from the polygon?

15. A regular polygon has an exterior angle 22.5°. Find the sum of interior angles of the polygon.

16. The interior angle of a regular polygon is 120° more than the exterior angle. Find the number of triangles that can be constructed from the polygon, hence determine the sum of interior angles.

17. Find the measure of interior angle of a polygon if the sum of its interior angles is 2160°.

18. Find the values of m and n in the following figure.



Chapter Five: Coordinate geometry

Introduction

The relationship of quantities or items can be expressed in terms of equations or graphs. Graphs are used to illustrate how quantities are related and behave. A branch of geometry which describes the positions of points on the coordinate plane using ordered pairs of numbers is called coordinate geometry. In this chapter, you will learn about graphs of linear equations, collinear points, parallel lines, and perpendicular lines. The competencies developed will help you to solve real life problems in various aspects such as linear models used in economics and business, prediction, engineering, geographical locations, descriptions of physical features, and many other applications.

Graphs of linear equations

An equation is a mathematical statement with an equal sign (=) connecting two algebraic expressions. Examples of equations are $2x+4y=2$ and $5x=8$.

A linear equation in two variables is the equation with a general form $Ax+By+C=0$, where x and y are variables, A and B are coefficients of x and y, respectively such that they are not both zero, and C is a constant. For instance, $7x+9y+4=0$ is a linear equation in two variables, x and y, where 7 is the coefficient of x, 9 is the coefficient of y, and 4 is a constant. If A or B is zero, then the linear equation remains with one variable. For instance, $2x+9=0$ is a linear equation in one variable x and $y-4=0$ is a linear equation in one variable y. A linear equation can be represented graphically by connecting points which satisfy the equation by a straight line. A graph of a linear equation can be drawn by using either a table of values or intercepts.

Sketching graphs of linear equations by using table of values

When drawing a graph of a linear equation using the table of values, use the following steps:

1. Make y the subject of the given equation.
2. Choose a set of x-values (at least two points are required).

3. Find the corresponding values of y by substituting the values of x in the equation formed in step 1.
4. Plot the points obtained in step 3 on the xy-plane.
5. Connect the points to obtain a straight line.
6. Label the line drawn for the given equation.

Activity 5.1 helps to determine the nature of the graph of a linear equation.

Activity 5.1: Recognising the graph of a linear equation

Individually or in a group, perform the following tasks:

1. Write down any equation of the form $ax+by+c=0$, where a, b, and c are non-zero real numbers.
2. Make y the subject of the written equation in task 1.
3. Choose the set of x-values (at least 9 points).
4. Prepare a table of values by finding the corresponding values of y in the equation formulated in task 2.
5. Use a graph paper with an appropriate scale to draw the graph of the equation on the xy-plane using the table of values in task 4.
6. Label the graph in task 5 by the linear equation written in task 1.
7. Share your results with other students through discussion for more inputs.

From Activity 5.1, it can be observed that when a pair of coordinate points obtained in tasks 1 to 4 are plotted on a graph paper, they all lie on the same straight line. Thus, the graph of a linear equation is always a straight line.

Example 5.1

Use a table of values to sketch the graph of $2x+y=5$.

Solution

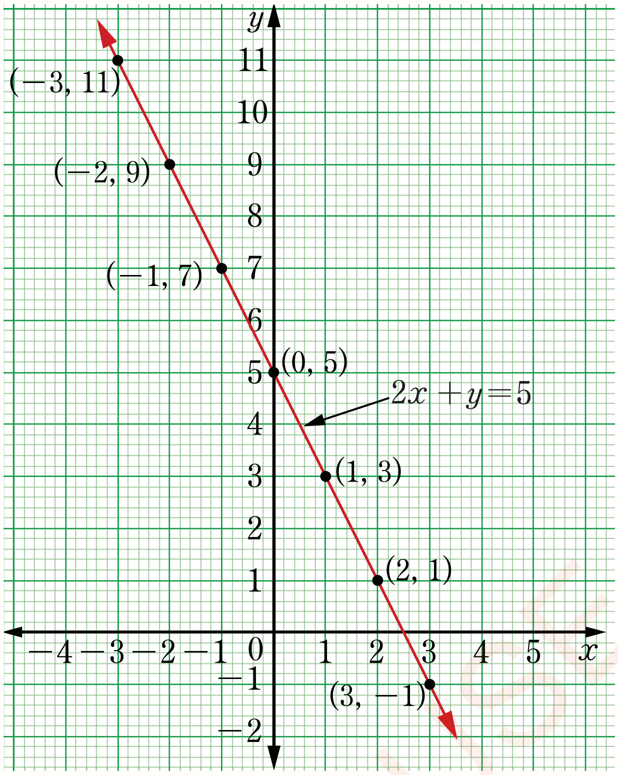
Given $2x+y=5$.

Making y the subject of the formula gives, $y=5-2x$.

The table of values of $y=5-2x$ is as follows.

x	-3	-2	-1	0	1	2	3
$y=5-2x$	11	9	7	5	3	1	-1

The graph of $2x+y=5$ is presented as follows.



Example 5.2

Use a table of values to sketch the graph of $y = x - 5$.

Solution

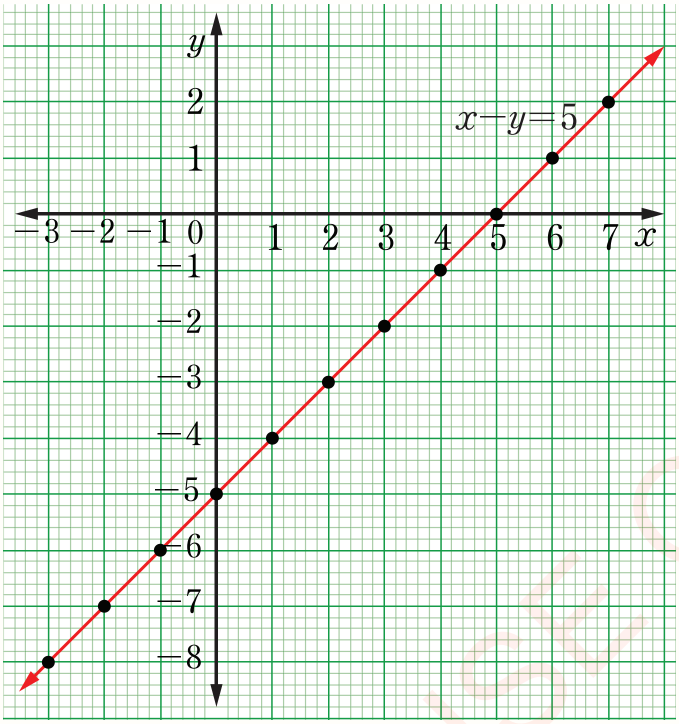
Given $x - y = 5$.

Rewriting y in terms of x gives, $y = x - 5$.

The table of values for $y = x - 5$ is as follows.

x	-3	-2	-1	0	1	2	3	4	5	6	7
$y = x - 5$	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2

The graph of $y = x - 5$ is presented in the following figure.



Example 5.3

Use a table of values to sketch the graph of $x + y - 4 = 0$.

Solution

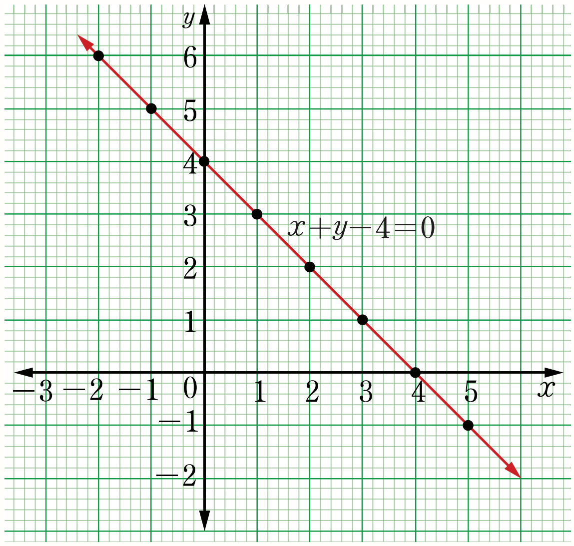
Given $x + y - 4 = 0$.

Making y the subject of the formula gives, $y = 4 - x$.

The table of values for $y = 4 - x$ is as follows.

x	-2	-1	0	1	2	3	4	5
$y = 4 - x$	6	5	4	3	2	1	0	-1

The graph of $x + y - 4 = 0$ is drawn as follows.



Exercise 5.1

Use a table of values to sketch the graph of each of the equations in questions 1 to 6:

1. $2x+3y+6=0$
2. $y=\frac{x-6}{3}$
3. $y=x$
4. $x=0$
5. $y=-10x$
6. $\frac{1}{2}y=\frac{1}{3}x$
7. Complete the following table of values and hence sketch the corresponding graph given that $2x-5=2y$.

x	-2		0			2	
y		-3.5		-1.5	2		0.5

8. If $10x-2y=4$, then:
   (a) Make y the subject of the equation.
   (b) Construct a table of values in the interval $-2\le x\le 2$.
   (c) Sketch the graph using the table of values constructed in part (b).
9. Use the following table of values to sketch the corresponding graph.

x	-3	-2	-1	0	1	2	3
y	-5	-4	-3	-2	-1	0	1

10. Given $2x+y-4=0$.
    (a) Complete the following table of values.

x		-1			2	
y	8		4			-2

    (b) Use the table in (a) to sketch the graph of the equation.

Sketching graphs of linear equations by using intercepts

The graph of a linear equation can be sketched using at least two points. The simplest method of finding the required points is by using the intercepts of the axes. The points of intercepts can be used to sketch the required graph. Activity 5.2 provides a guide in determining intercepts.

Activity 5.2: Recognising intercepts of linear equations

A student developed a computer program which returns an output value to any input value. Output values corresponding to some selected input values were recorded in the following table.

Input	3	4	5	6	7
Output	-1	-3	-5	-7	-9

Individually or in a group, perform the following tasks:

1. Sketch the graph to represent the values from the given table such that the x and y- axes represent the inputs and outputs, respectively.
2. Extend the straight line drawn in task 1 such that it crosses both x and y- axes.
3. Identify the point where the extended line in task 2 crosses the x- axis and label it as point A.
4. Identify the point where the extended line in task 2 crosses the y- axis and label it as point B.
5. State the names of points A and B in tasks 3 and 4.
6. Share your results with other students through discussion for more inputs.

From Activity 5.2, it can be observed that the points A and B where the line crosses the x and y- axes are called x- intercept and y- intercept, respectively.

All linear equations in the form $Ax + By + C = 0$, where $A \neq 0$ and $B \neq 0$ have y- intercept and x- intercept. The y- intercept is the value of y when $x = 0$, whereas the x- intercept is the value of x when $y = 0$.

Example 5.4

Sketch the graph of $2x + 5y = 10$.

Solution

Given the line $2x + 5y = 10$.

The graph of the equation can be sketched by using intercepts.

First, find the y- intercept by substituting $x = 0$ into the equation $2x + 5y = 10$ to obtain $2(0) + 5y = 10$.

This implies that, $5y = 10$.

Solving for y gives, $y = \frac{10}{5} = 2$, implying that, the y- intercept is 2.

Thus, the point where the line crosses the y- axis is (0, 2).

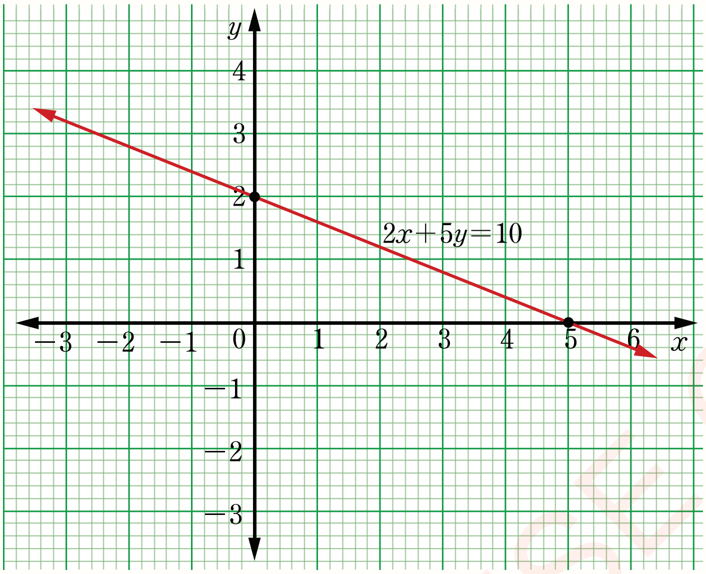
Next, find the x- intercept by substituting $y = 0$ into the equation $2x + 5y = 10$ to obtain $2x + 5(0) = 10$.

This implies that, $2x = 10$.

Thus, $x = \frac{10}{2} = 5$, implying that the x- intercept is 5.

Thus, the point where the line crosses the x- axis is (5, 0).

Therefore, the graph of $2x+5y=10$ is sketched by connecting the points (0, 2) and (5, 0) using a straight line as follows.



Example 5.5

Sketch the graph of the line
$-\frac{2}{3}x+\frac{1}{2}y-2=0.$

Solution

Given $-\frac{2}{3}x+\frac{1}{2}y-2=0.$

The graph of the given equation can be sketched using the intercepts as follows.

Find the y- intercept by substituting $x=0$ into the given equation to obtain

$$-\frac{2}{3}(0)+\frac{1}{2}y-2=0.$$

Thus, $\frac{y}{2}-2=0$. It implies that,

$$y-4=0.$$

Hence, $y=4$.

It follows that, the y-intercept is 4 and the point where the line crosses the y-axis is (0, 4).

Substitute $y=0$ into the given equation to obtain the x- intercept.

That is, $-\frac{2}{3}x+\frac{1}{2}(0)-2=0.$

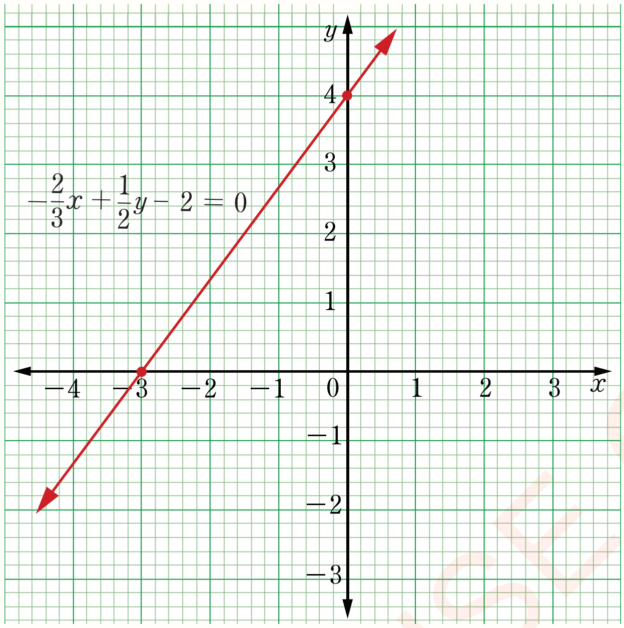
Simplification gives,

$$-2x=6.$$

Thus, $x=-3$.

Hence, the x- intercept is -3 and the point where the line crosses the x- axis is $(-3,0)$.

Therefore, the graph of the line $-\frac{2}{3}x+\frac{1}{2}y-2=0$ is presented as follows.



Note that; A linear equation in one variable is either a vertical line or a horizontal line.

For instance, $x=4$ is a vertical line which crosses the x-axis at 4, $x=0$ is a vertical line through the y-axis, $y=-3$ is a horizontal line which crosses the y-axis at -3, and $y=0$ is a horizontal line through the x-axis.

Example 5.6

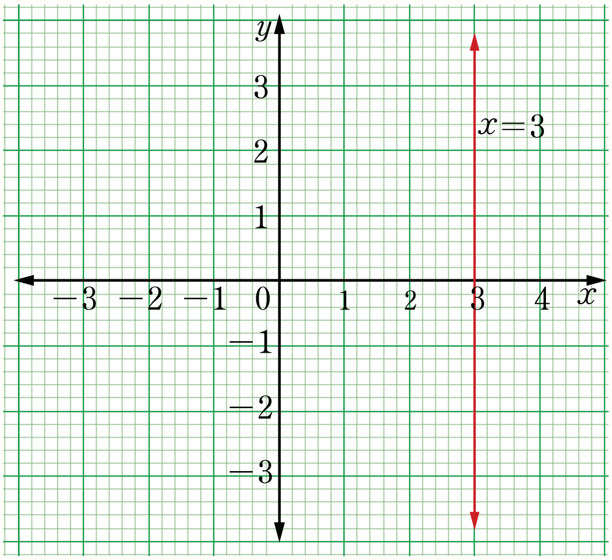
Sketch the graph of the line $2x-6=0$.

Solution

Given $2x-6=0$.

Solving for x gives $x=3$. This is a vertical line which crosses the x-axis at $x=3$.

The graph of $2x-6=0$ is presented as follows.

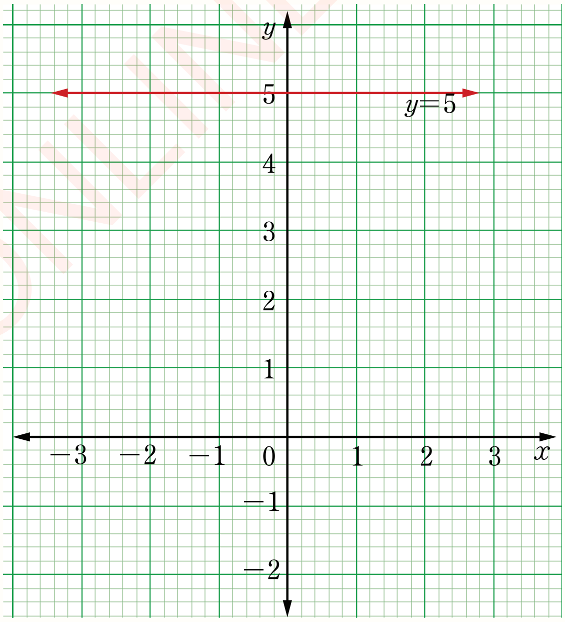


Example 5.7

Sketch the graph of the line $y = 5$.

Solution

Given the line $y = 5$. Its graph is a horizontal line passing through the point $(0,5)$ as shown in the following figure



Exercise 5.2

In questions 1 to 8, sketch the graphs of the equations by using intercepts.

1. $4x-5y=20$
2. $7y-4x=28$
3. $\frac{x}{7}+\frac{y}{5}=1$
4. $5x-3y=15$
5. $y=-\frac{4}{5}x+2$
6. $y+7x+3=0$
7. $4x-y=2$
8. $x-6y=-2$
9. Given the line $3y+6=12x$.
   (a) Determine its intercepts.
   (b) Use the intercepts in (a) to sketch its graph.
10. Given the line $x-\frac{y}{2}=3$.
   (a) Determine its intercepts.
   (b) Use the intercepts in (a) to sketch the graph of the line.

Gradient of a line

A gradient or slope of a line is the measure of the steepness of a straight line. It is usually denoted by a letter m and is given as,

$$\text{Gradient, } m=\frac{\text{vertical change}}{\text{horizontal change}}.$$

In the xy-plane, the vertical change is the change in y values and it is denoted by Δy. The horizontal change is the change in x values and it is denoted by Δx.

Thus, the gradient of a straight line can also be expressed as,

$$m=\frac{\text{change in } y}{\text{change in } x}=\frac{\Delta y}{\Delta x}.$$

Consider a straight line passing through the points $A(x_1, y_1)$ and $B(x_2, y_2)$ as shown in Figure 5.1.

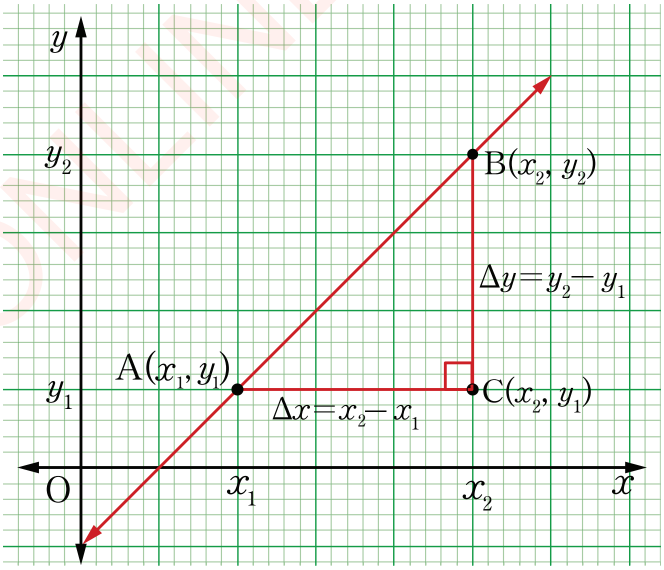


Figure 5.1: *Gradient of a straight line* AB

From Figure 5.1, $m=\frac{\Delta y}{\Delta x}$, where $\Delta y = y_2 - y_1$ and $\Delta x = x_2 - x_1$.

Thus, $m=\frac{y_2-y_1}{x_2-x_1}$ or $m=\frac{y_1-y_2}{x_1-x_2}$.

Therefore, the gradient of a straight line can be determined by $m=\frac{y_2-y_1}{x_2-x_1}$ or $m=\frac{y_1-y_2}{x_1-x_2}$.

The following steps can be used to find the gradient of a straight line joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$:

1. Write the formula for the gradient, $m=\frac{y_2-y_1}{x_2-x_1}$.
2. Assign the values to (x_1, y_1) and (x_2, y_2).
3. Substitute the values in the formula to obtain the gradient m.

Example 5.8

Find the gradient of a straight line passing through the points $(2,2)$ and $(3,6)$.

Solution

Let $(x_1, y_1)=(2,2)$ and

$(x_2, y_2)=(3,6)$.

From $m=\frac{y_2-y_1}{x_2-x_1}$, substituting the values of x_1, x_2, y_1, and y_2 results to

$$m=\frac{6-2}{3-2}$$
$$=4.$$

Therefore, the gradient of the line passing through the points $(2,2)$ and $(3,6)$ is 4.

The geometrical implication of this gradient is that, for every one unit of length in the horizontal movement, there are 4 units of length in the vertical movement.

Note that:

(a) When the gradient of a line is positive, it implies that the points move upward from the left to the right or downward from the right to the left as shown in Figure 5.2.

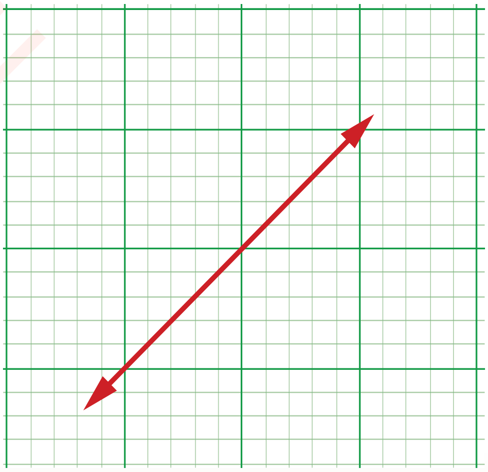


Figure 5.2: *Positive gradient of a line*

(b) When the gradient of a line is negative, it implies that the points move upward from the right to the left or downward from the left to the right as shown in Figure 5.3.

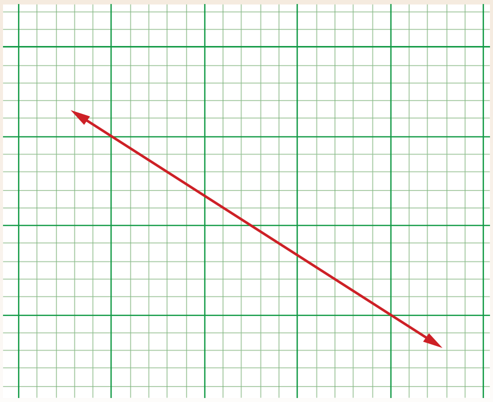


Figure 5.3: *Negative gradient of a line*

(c) When the gradient of a line is zero, it implies that the line is horizontal whose vertical values (y-values) are constant. That is, change in y-values is zero as shown in Figure 5.4.

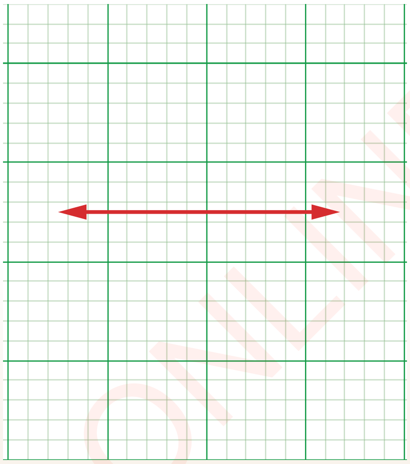


Figure 5.4: *Gradient of a line is zero*

(d) When the gradient of a line is infinity, it implies that, the line is vertical whose horizontal value (x-value) is constant. That is, change in x-values is zero as shown in Figure 5.5.

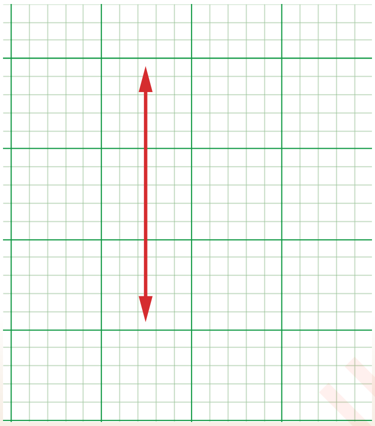


Figure 5.5: *Gradient of a line is infinity*

Forming an equation from a given linear graph

Given a linear graph, an equation of a straight line can be formed. The following steps are useful:

1. Identify at least two points on the line.
2. Use the identified points to find the gradient of the line.
3. Use a variable point (x, y) and one of the identified points to find the gradient.
4. Equate the obtained gradients in steps 2 and 3 to get the required linear equation.
5. Rewrite the linear equation in any of the following forms:
   (a) General form; $ax+by+c=0$, where a and b are coefficients of x and y, respectively, such that a and b are not both zero, and c is a constant.
   (b) Standard form; $y=mx+c$, where m is the gradient and c is the y-intercept.

(c) Intercepts form; $\frac{x}{p}+\frac{y}{q}=1$,

where p and q are the x and y-intercepts, respectively.

Activity 5.3 demonstrates how to find the equation of a line from a given graph.

Activity 5.3: Determining the equation of a line from a linear graph

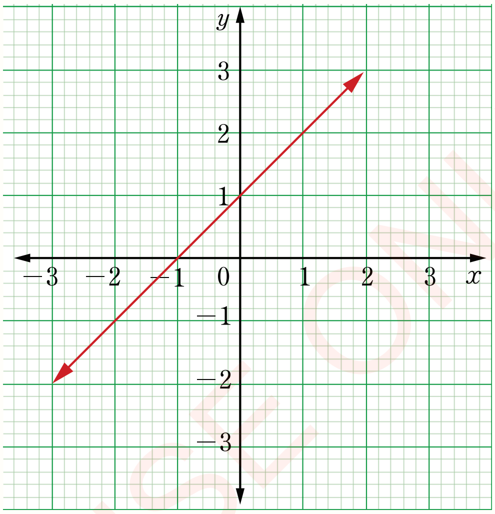
Individually or in a group, perform the following tasks:

1. Using a graph paper, sketch the graph of a line whose gradient is either positive or negative and intersecting the axes.
2. From task 1, identify any two points which lie on the line and label them as A and B.
3. From task 2, use the identified points to find the gradient of the line AB and name the gradient as m_1.
4. From task 1, identify another point which lies on the line AB and label it as point $P(x, y)$.
5. Use the formula of a gradient of a line, one point from task 2, and the gradient obtained in task 4 to get the required equation of the line.
6. Share your results with other students through discussion for more inputs.

From Activity 5.3, it can be observed that, any pair of points chosen from the linear graph gives the same gradient and eventually gives the same equation of the line.

Example 5.9

Find the equation of a line represented by the following graph in the form $ax+by+c=0$.



Solution

From the graph, it can be observed that, the line passes through the points $(-1, 0)$ and $(1, 2)$.

Let $(x_1, y_1)=(-1, 0)$ and $(x_2, y_2)=(1, 2)$. It follows that, the gradient of the graph is given by,

$$m=\frac{y_2-y_1}{x_2-x_1}$$

$$=\frac{2-0}{1-(-1)}=1.$$

Thus, $m=1$.

Let (x, y) be any other point on the straight line.

Using (x, y) and (x_1, y_1), the gradient is given by

$m=\frac{y-y_1}{x-x_1}$, implying that $m=\frac{y-0}{x-(-1)}$.

Equating the two gradients gives,

$1=\frac{y}{x+1}$.

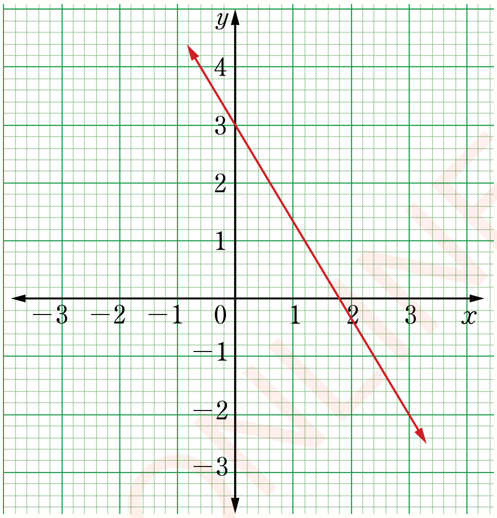
Thus, $y=x+1$.

Rewriting this equation in the form $ax+by+c=0$ gives $x-y+1=0$.

Therefore, the equation of the line is $x-y+1=0$.

Example 5.10

Determine the equation of the line represented by the following graph in the form $y=mx+c$.



Solution

From the graph, it can be observed that, the line passes through the points $(3,-2)$ and $(0,3)$.

Let $(x_1,y_1)=(3,-2)$ and

$(x_2,y_2)=(0,3)$ such that the gradient,

$$m=\frac{y_2-y_1}{x_2-x_1}$$

$$=\frac{3-(-2)}{0-3}$$

$$=-\frac{5}{3}.$$

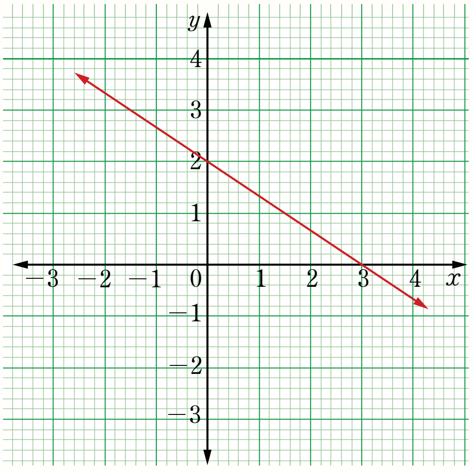
Thus, the gradient, $m=-\frac{5}{3}$.

The given line crosses the y- axis at 3, which means that the y- intercept is 3.

Therefore, the equation of the line is $y=-\frac{5}{3}x+3$.

Example 5.11

Find the equation of a line represented by the following graph in the form $\frac{x}{p}+\frac{y}{q}=1$.



Solution

Let p and q be the x and y -intercepts, respectively.

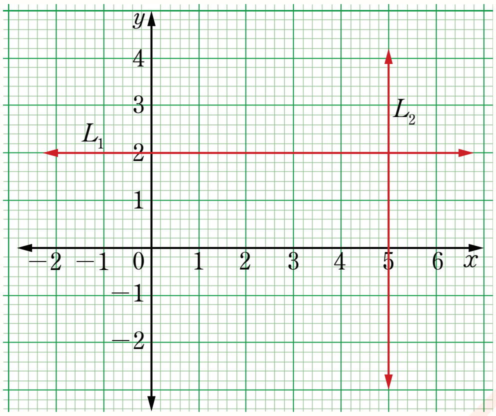
From the graph, the line passes through the points $(3,0)$ and $(0,2)$.

Thus, $p = 3$ and $q = 2$.

Therefore, the required equation is $\frac{x}{3}+\frac{y}{2}=1$.

Example 5.12

Determine the equations of the lines L_1 and L_2 in the following graph.



Solution

Consider the line L_1. It implies that, L_1 passes through the points $(-1,2)$ and $(2,2)$. Let $(x_1, y_1) = (-1,2)$ and $(x_2, y_2) = (2,2)$.

The gradient of L_1 is given by,

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{2-2}{2-(-1)}$$

$$= \frac{0}{3} = 0.$$

Thus, the gradient of L_1 is 0.

Let (x, y) be any other point on L_1.

Using (x, y) and $(x_1, y_1) = (-1,2)$ gives,

$m_1 = \frac{y - y_1}{x - x_1}$, where $m_1 = 0$.

Thus, $0 = \frac{y-2}{x-(-1)}$, implying that,

$$0 = \frac{y-2}{x+1}$$

Cross multiplication gives,

$$0 = y - 2.$$

Hence, $y = 2$.

Therefore, the equation of the line L_1 is $y = 2$.

Consider the line L_2. It follows that, L_2 passes through the points $(5,1)$ and $(5,3)$. Let $(x_1, y_1) = (5,1)$ and $(x_2, y_2) = (5,3)$.

The gradient of the line L_2 is given by

$$m_2 = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{3-1}{5-5}$$

$= \frac{2}{0}$, which is infinity (∞).

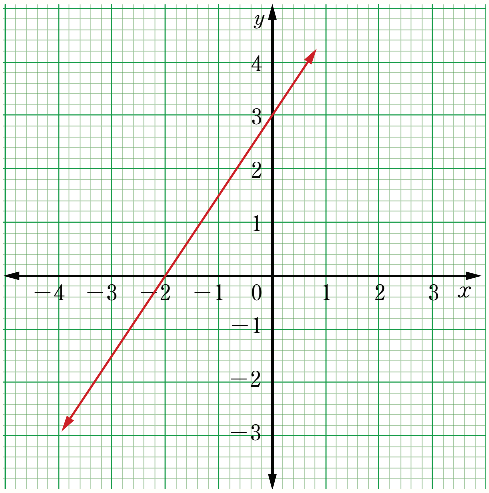
Thus, the gradient of the line L_2 is infinity, implying that L_2 is a vertical line.

Generally, the equation of a vertical line takes the form $x = a$, where a is the x-intercept.

Therefore, the equation of the line L_2 is $x = 5$.

Example 5.13

Find the equation of the a line represented by the following graph in general form.



Solution

The line passes through the points $(-2,0)$ and $(0,3)$. Let $(x_1, y_1)=(-2,0)$ and $(x_2, y_2)=(0,3)$. It implies that,

$$m=\frac{y_2-y_1}{x_2-x_1}$$

$$=\frac{3-0}{0-(-2)}$$

$$=\frac{3}{2}.$$

Thus, the gradient of the line is $\frac{3}{2}$.

Let (x, y) be any other point on the straight line.

Using (x, y) and (x_2, y_2), the gradient of the line is given by $m=\frac{y-y_2}{x-x_2}$,

where $m=\frac{3}{2}$, which implies that,

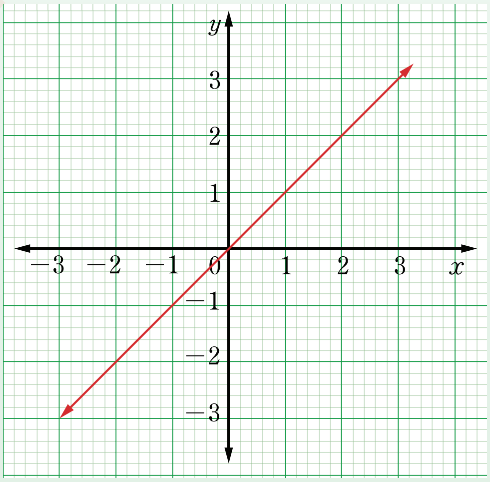
$$\frac{3}{2}=\frac{y-3}{x-0}.$$

Thus, $3x-2y+6=0$.

Therefore, the general equation of the line is $3x-2y+6=0$.

Exercise 5.3

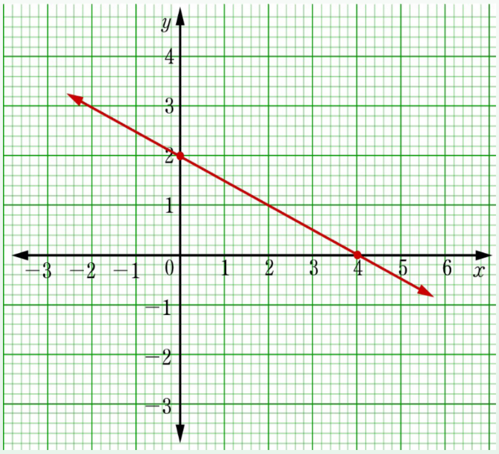
1. A straight line passes through the points $(-4,1)$ and $(3,-1)$.
   (a) Find its gradient.
   (b) Find its equation in the form $ax+by+c=0$.
   (c) Determine the intercepts of the line.
2. Find the equation of the line represented by the following graph.



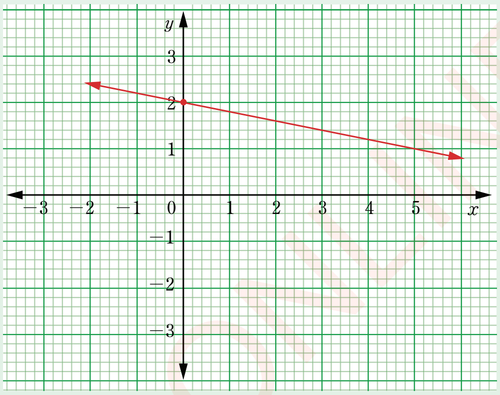
3. Sketch the graph of a line which passes through the points $(2,4)$ and $(-2,-2)$. Hence, find its equation.

4. Find the equation of a line whose y-intercept is -6 and x-intercept is -2 and then sketch its graph.

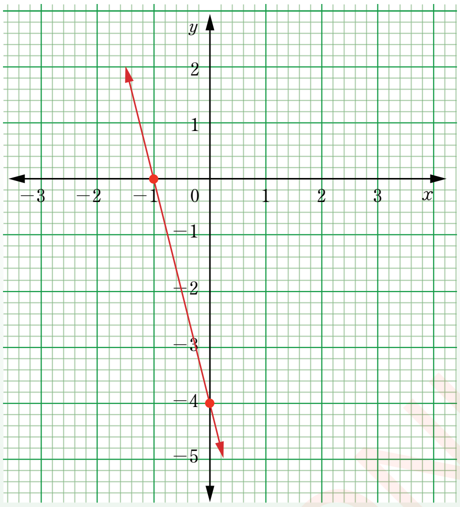
5. Find the equation of a line represented by the following graph in the form $ax+by+c=0$.



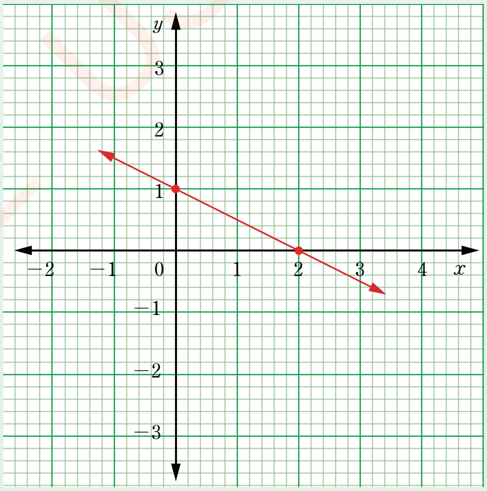
6. At what point will the line in the following graph cross the x-axis?



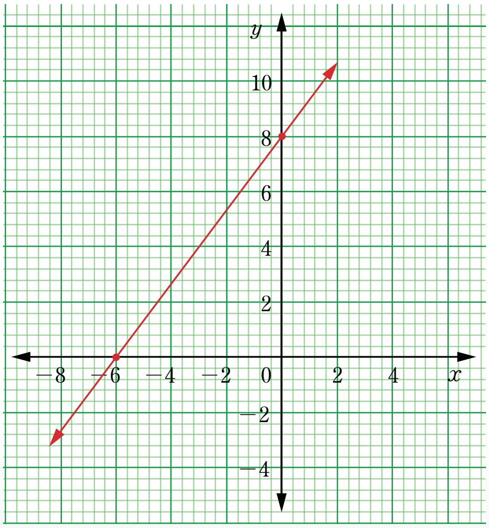
7. Find the equation of a line represented by the following graph in general form.



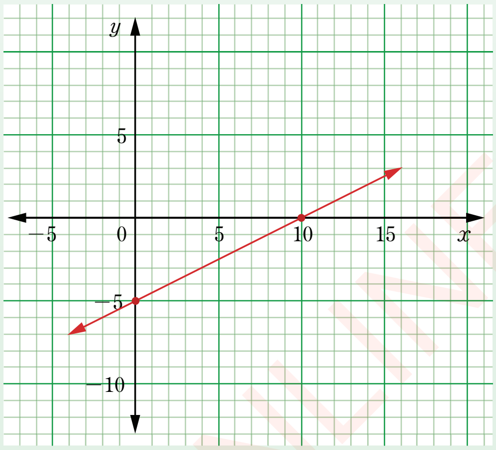
8. Find the equation of a line represented by the following graph in the form $\frac{x}{p}+\frac{y}{q}=1$.



9. Determine the equation of a line represented by the following graph in the form $y=mx+c$.



10. Find the equation of a line represented by the following graph in the form $ax + by + c = 0$.



Solution of linear simultaneous equations by graphical method

When solving a system of linear simultaneous equations graphically, the graphs of the equations are drawn on the same xy-plane and the coordinates of the point of intersection gives the solution of the system.

Simultaneous equations are solved graphically through the following steps:

1. Using graph papers with appropriate scales, draw the lines of the given linear simultaneous equations on the same xy-plane.
2. Extend the lines until they intersect each other.
3. Determine the point of intersection.

The point of intersection of the lines is a solution to the system of linear simultaneous equations.

Example 5.14

Solve the linear simultaneous equations $x + y = 6$ and $2x + y = 8$ graphically.

Solution

Given $\begin{cases} x + y = 6 \\ 2x + y = 8. \end{cases}$

The intercepts of each equation can be tabulated as follows.

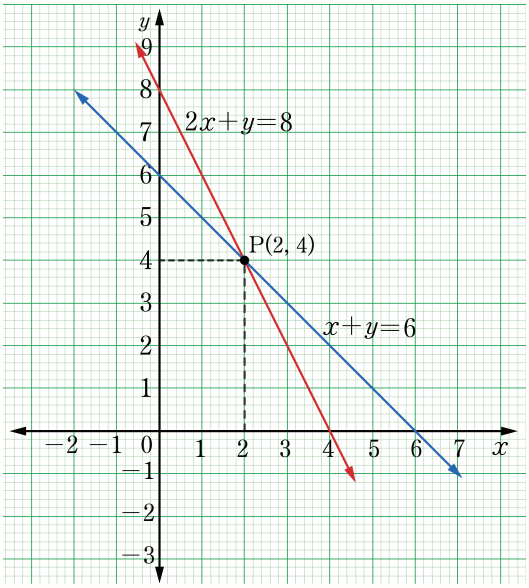
$x + y = 6$

x	0	6
y	6	0

$2x + y = 8$

x	0	4
y	8	0

The graphs of $x + y = 6$ and $2x + y = 8$ are sketched on the same xy-plane as follows.



From the graph, the point of intersection is (2,4).

Therefore, the solution to the simultaneous equations is $x=2$ and $y=4$, which can also be written as $(x, y)=(2,4)$.

Example 5.15

Solve the following system of simultaneous equations graphically.

$$\begin{cases} 3x-y=4 \\ x+4y=-3 \end{cases}$$

Solution

Given $\begin{cases} 3x-y=4 \\ x+4y=-3. \end{cases}$

From the equation $3x-y=4$, making y the subject of the equation gives, $y=3x-4$.

The table of values for $y=3x-4$ is given as follows.

x	-2	-1	0	1	2	3
$y=3x-4$	-10	-7	-4	-1	2	5

From the equation $x+4y=-3$, making y the subject of the equation gives,

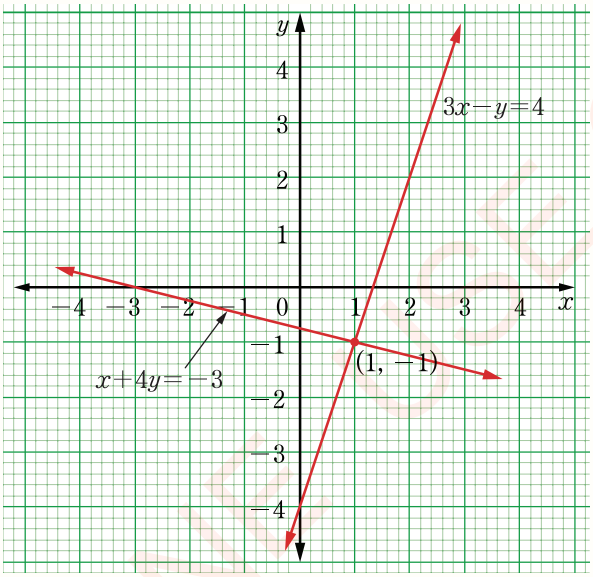
$y=-\left(\frac{x+3}{4}\right)$.

The table of values for $y=-\left(\frac{x+3}{4}\right)$ is given as follows.

x	-4	-3	-2	-1	0	1	2	3
$y=-\left(\frac{x+3}{4}\right)$	0.25	0	-0.25	-0.5	-0.75	-1	-1.25	-1.5

In order to avoid wrong approximation of decimal numbers, $(-3,0)$ and $(1,-1)$ can be used to sketch the line.

The two equations are drawn on the same *xy*-plane as follows.



From the graph, the point of intersection is $(1,-1)$.

Therefore, the solution of the system is $x=1$ and $y=-1$.

Example 5.16

Use the graphical method to determine the solution of the following simultaneous equations.

$$\begin{cases} x+2y=-4 \\ 2x+y=1 \end{cases}$$

Solution

Given $\begin{cases} x+2y=-4 \\ 2x+y=1 \end{cases}$

Consider the first equation $x+2y=-4$.

Making y the subject of the equation gives, $y=-\left(\frac{x+4}{2}\right)$.

The table of values for $y=-\left(\frac{x+4}{2}\right)$ is as follows.

x	-3	-2	-1	0	1	2	3
$y=-\left(\frac{x+4}{2}\right)$	-0.5	-1	-1.5	-2	-2.5	-3	-3.5

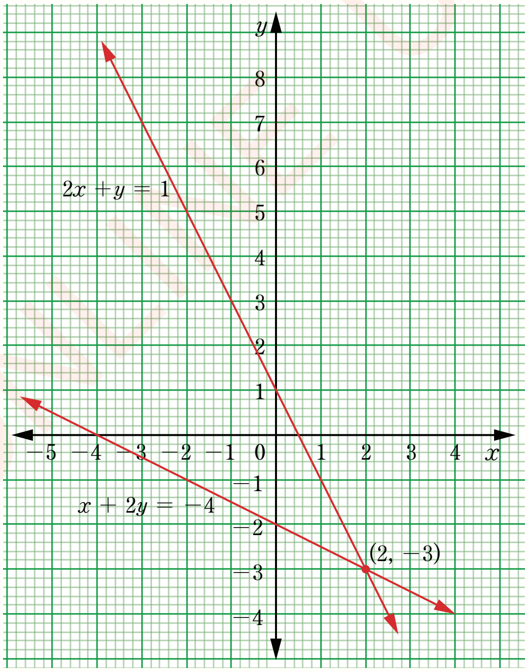
Now, consider the second equation $2x+y=1$.

Making y the subject of the equation gives, $y=1-2x$.

The table of values for $y=1-2x$ is as follows.

x	-3	-2	-1	0	1	2	3
$y=1-2x$	7	5	3	1	-1	-3	-5

The graphs of $x+2y=-4$ and $2x+y=1$ are presented on the same xy-plane as follows.



From the graphs, the point of intersection is $(2,-3)$.

Therefore, the solution to the system is $x=2$ and $y=-3$.

Exercise 5.4

In questions 1 to 6, use the graphical method to find the solution to the linear simultaneous equations.

1. $\begin{cases} x+y=3 \\ x-y=5 \end{cases}$

2. $\begin{cases} x-y=2 \\ 2x+3y=9 \end{cases}$

3. $\begin{cases} 4x+3y-14=0 \\ 3x+4y-14=0 \end{cases}$

4. $\begin{cases} y=\dfrac{1}{2}x+1 \\ y=-x+1 \end{cases}$

5. $\begin{cases} x-2y=2 \\ x+y=5 \end{cases}$

6. $\begin{cases} 3x+2y=6 \\ -x+2y=2 \end{cases}$

7. Draw the line passing through the point $(1,0)$ whose gradient is -2 and the line passing through the point $(-2,-2)$ with gradient $\dfrac{1}{2}$ on the same pair of axes. Hence, determine the point of intersection.

8. Determine graphically the point of intersection of the line passing through the points $(0,-3)$ and $(-2,0)$ and another line passing through the points $(-2,-2)$ and $(1,1)$.

9. Draw the lines $2x-5y=10$ and $4x+5y=20$ on the same *xy*-plane. Hence, locate the point of intersection.

10. Complete the following table of values and use it to sketch the graphs of the given equations on the same *xy*-plane. Hence, use the graphs to determine the solution to the system of linear equations.

x	-2	-1	0	1	2	3	4	5
$y=\dfrac{3}{2}x-6$	-9				-3			
$y=\dfrac{1}{2}x-2$			-2				0	

Collinear points

Three or more points are said to be collinear if they lie on the same straight line, that is, the straight lines through any two points among the three points have the same gradient. Otherwise, the points are non-collinear. Consider the three points A, B, and C located on the line in Figure 5.6.

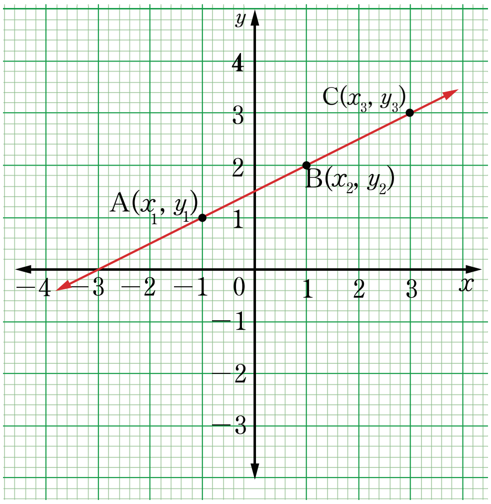


Figure 5.6: *Collinear points*

Let m_1 be the gradient of the line passing through the points $A(x_1, y_1)$ and $B(x_2, y_2)$ such that,

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-1}{1-(-1)} = \frac{1}{2}.$$

Thus, the gradient of the line between the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $\frac{1}{2}$.

Let m_2 be the gradient of the line passing through the points $B(x_2, y_2)$ and $C(x_3, y_3)$ such that,

$$m_2 = \frac{y_3 - y_2}{x_3 - x_2} = \frac{3-2}{3-1} = \frac{1}{2}.$$

Thus, the gradient of the line through the points $B(x_2, y_2)$ and $C(x_3, y_3)$ is $\frac{1}{2}$.

Let the gradient of the line through the points $A(x_1, y_1)$ and $C(x_3, y_3)$ be m_3.

This implies that, $m_3 = \frac{y_3 - y_1}{x_3 - x_1}$

$$= \frac{3-1}{3-(-1)} = \frac{1}{2}.$$

Thus, the gradient of the line through the points $A(x_1, y_1)$ and $C(x_3, y_3)$ is $\frac{1}{2}$.

Since $m_1 = m_2 = m_3$, then the points A, B, and C in Figure 5.6 are collinear.

Example 5.17

Show that the points $E(-5,18)$, $F(-2,12)$, and $G(7,-6)$ are collinear.

Solution

Given the points $E(-5,18)$, $F(-2,12)$, and $G(7,-6)$

Let $(x_1, y_1) = (-5,18)$,
$(x_2, y_2) = (-2,12)$, and
$(x_3, y_3) = (7,-6)$.

For the points to be collinear, the gradients of $\overleftrightarrow{EF}$, $\overleftrightarrow{FG}$, and $\overleftrightarrow{EG}$ must be equal.

Let m_1, m_2, and m_3 be the gradients of $\overleftrightarrow{EF}$, $\overleftrightarrow{FG}$, and $\overleftrightarrow{EG}$, respectively.

It implies that,

$$m_1=\frac{y_2-y_1}{x_2-x_1}=\frac{12-18}{-2-(-5)}=-2$$

Thus, $m_1=-2$.

$$m_2=\frac{y_3-y_2}{x_3-x_2}=\frac{-6-12}{7-(-2)}=-2$$

Thus, $m_2=-2$.

$$m_3=\frac{y_3-y_1}{x_3-x_1}=\frac{-6-18}{7-(-5)}=-2$$

Thus, $m_3=-2$.

Hence, $m_1=m_2=m_3=-2$.

Therefore, the points $E(-5,18)$, $F(-2,12)$, and $G(7,-6)$ are collinear.

Example 5.18

Determine the value of t for the points $(-2, t)$, $(3, 5)$, and $(6, 6)$ to be collinear.

Solution

For collinear points, the gradient determined by using any two points should be equal.

Let

$(x_1, y_1)=(-2, t)$, $(x_2, y_2)=(3, 5)$, and $(x_3, y_3)=(6,6)$. The gradients m_1 and m_2 are given by,

$$m_1=\frac{y_2-y_1}{x_2-x_1}=\frac{5-t}{3-(-2)}=\frac{5-t}{5}.$$

Thus, $m_1=\dfrac{5-t}{5}$.

$$m_2=\frac{y_3-y_2}{x_3-x_2}=\frac{6-5}{6-3}=\frac{1}{3}$$

Thus, $m_2=\dfrac{1}{3}$.

Since the points are collinear, then $m_1=m_2$, implying that

$$\frac{5-t}{5}=\frac{1}{3}.$$

Cross multiplication gives,

$15-3t=5$.

Thus, $t=\dfrac{10}{3}$.

Therefore, the value of t is $\dfrac{10}{3}$.

Exercise 5.5

1. Determine whether or not each of the following points are collinear:
   - (a) $(3,2)$, $(-1,5)$ and $(7,3)$
   - (b) $(-10,-12)$, $(4,2)$, and $(-3,-5)$
   - (c) $(-4,-1)$, $(6,4)$, and $(10,6)$
   - (d) $(-2,-4)$, $(0,0)$, and $(-5,10)$
   - (e) $(-6,8)$, $(5,-6)$, and $(0,5)$

2. Find the value of t in each of the following collinear points:
   - (a) $(-2,-2)$, $(3, 4)$, and $(5, t)$
   - (b) $(-7, 1)$, $(4, 3)$, and $(t, 7)$
   - (c) $(4, 2)$, $(4, 3)$, and $(t, -10)$
   - (d) $(-2, 3)$, $(-7, 4)$, and $(3, t)$

Parallel and perpendicular lines

Two or more lines in the xy-plane may relate to each other in different ways. There are lines which intersect and form different angles and others do not intersect at all.

Parallel lines

Two or more lines are said to be parallel if they do not intersect even when extended indefinitely in either directions. In Figure 5.7, the lines L_1 and L_2 are parallel.

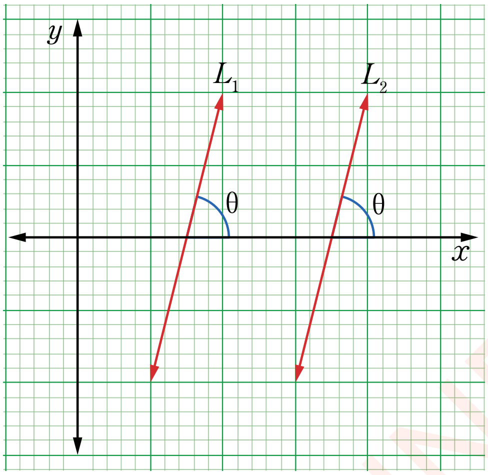


Figure 5.7: *Parallel lines*

Activity 5.4 provides a guide in identifying the conditions for parallel lines.

Activity 5.4: Determining the condition for two lines to be parallel

Individually or in a group, perform the following tasks:

1. Draw a straight line on the xy-plane and name it as L_1.
2. Identify any two points lying on the line L_1.
3. Determine the gradient of the line L_1 using the points identified in task 2 and name it as m_1.
4. Draw another line on the same xy-plane which is parallel to the L_1 and name it as L_2.
5. Identify any two points lying on the line L_2.
6. Determine the gradient of the line L_2 using the points identified in task 5 and name it as m_2.
7. Compare the gradients of the lines L_1 and L_2, then give comments.
8. Share your results with other students through discussion for more inputs.

From Activity 5.4, if the parallel lines are well-drawn, then the following hold:

(a) The lines have a constant distance apart.

(b) The lines never intersect.

(c) The lines have the same angle of inclination to the horizontal (x-axis) or vertical (y-axis).

(d) The lines have the same gradient (slope), that is $m_1 = m_2$.

(e) The angle between the lines is zero.

Example 5.19

Show that the line passing through the points $(0, 2)$ and $(-3, 4)$ is parallel to the line passing through $(6, 0)$ and $(3, 2)$.

Solution

Let $(x_1, y_1) = (0, 2)$, $(x_2, y_2) = (-3, 4)$, and m_1 be the gradient of the line passing through the points. That is,

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{4-2}{-3-0}$$
$$= -\frac{2}{3}.$$

Thus, the gradient of the line passing through the points $(0, 2)$ and $(-3, 4)$ is $-\frac{2}{3}$.

Similarly, let $(x_3, y_3) = (6, 0)$, $(x_4, y_4) = (3, 2)$, and m_2 be the gradient of the line through the points.

That is,

$$m_2 = \frac{y_4 - y_3}{x_4 - x_3}$$
$$= \frac{2-0}{3-6}$$
$$= -\frac{2}{3}.$$

Thus, the gradient of the line passing through the points $(6, 0)$ and $(3, 2)$ is $-\frac{2}{3}$.

Hence, the lines have the same gradient, that is, $m_1 = m_2 = -\frac{2}{3}$.

Therefore, the two lines are parallel.

Example 5.20

Find the equation of a line passing through the points $(-3, 1)$ which is parallel to $5x - 3y + 10 = 0$.

Solution

Let m_1 be the gradient of the line passing through the point $(-3, 1)$ and m_2 be the gradient of the line $5x - 3y + 10 = 0$.

Making y the subject of the equation $5x - 3y + 10 = 0$ gives, $y = \frac{5}{3}x + \frac{10}{3}$.

Thus, $m_2 = \frac{5}{3}$.

Since the two lines are parallel, then $m_1 = m_2$.

Thus, $m_1 = \frac{5}{3}$.

Using $m_1 = \frac{5}{3}$ and the point $(x_0, y_0) = (-3, 1)$, the equation of the line is given by, $m_1 = \frac{y - y_0}{x - x_0}$.

This implies that, $\frac{5}{3} = \frac{y-1}{x-(-3)}$.

Cross multiplication gives,

$5(x+3) = 3(y-1)$.

Thus, $5x - 3y + 18 = 0$.

Therefore, the equation of the line passing through the point $(-3, 1)$ is $5x - 3y + 18 = 0$.

Example 5.21

Find the value of k, if the line passing through the points $(1,k)$ and $(2,1)$ is parallel to the line passing through $(-1,-2)$ and $(3,4)$.

Solution

Let $(x_0,y_0)=(1,k)$, $(x_1,y_1)=(2,1)$ be the points on the first line and $(x_2,y_2)=(-1,-2)$, $(x_3,y_3)=(3,4)$ be the points on the second line.

Since the two lines are parallel, then their gradients are equal. That is,

$$\frac{y_1-y_0}{x_1-x_0}=\frac{y_3-y_2}{x_3-x_2}.$$

Thus, $\dfrac{1-k}{2-1}=\dfrac{4-(-2)}{3-(-1)}$.

Simplification gives,

$$1-k=\frac{6}{4}.$$

Hence,

$$k=1-\frac{6}{4}$$

$$=-\frac{1}{2}.$$

Therefore, the value of k is $-\frac{1}{2}$.

Exercise 5.6

1. Verify whether or not the line defined by each of the following pairs of equations are parallel:
   (a) $3x-4y+1=0$ and $6x-8y=-6$
   (b) $\frac{1}{3}x+\frac{4}{7}y=8$ and $7x+12y=6$
   (c) $3x+4y-3=0$ and $3x-y+4=0$
   (d) $9x=7y+0.5$ and $y=\frac{9}{7}x-8$
2. Find the equation of a line passing through the point $(-3,4)$ and parallel to $\frac{y}{2}-\frac{x}{7}=1$.
3. Find the equation of the line that passes through the point $(2,4)$ and parallel to $8x+7y-4=0$.
4. Find the equation of the line passing through the point $(0,0)$ and parallel to the line passing through $(-3,-3)$ and $(-2,5)$.
5. Find the value of k for which the line $2x-ky+2=0$ will be parallel to $4x+3y=10$.
6. Find the value of r so that the line joining the points (2, 2) and (4, 6) is parallel to the line passing through the points (4, 4) and $(-2, r)$.
7. Show that the points $A(-2,-3)$, $B(4, 6)$, and $C(-8,-12)$ are collinear.
8. State the conditions at which the line represented by the equation $ax+by+c=0$ is:

(a) parallel to the x-axis,

(b) parallel to the y-axis.

9. Find the value of λ so that the points $P(-8,-12)$, $Q(-2,-3)$, and $R\left(-\lambda,-\frac{3}{4}\right)$ are collinear.

10. Determine the equation of the line which passes through the point $N(-2,-4)$ and parallel to the line $\frac{2}{3}x+\frac{3}{5}y=1$.

Perpendicular lines

Two lines are said to be perpendicular if they intersect at a right angle $(90°)$.

In Figure 5.8, the lines L_1 and L_2 are perpendicular as they intersect at a right angle.

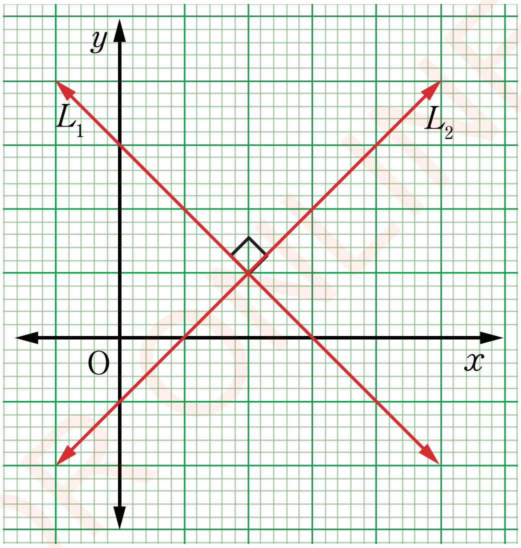


Figure 5.8: *Perpendicular lines*

Activity 5.5 provides a guide in identifying the condition for perpendicular lines.

Activity 5.5: Determining the condition for two lines to be perpendicular

Individually or in a group, perform the following tasks:

1. Draw a straight line in the xy-plane which is neither a vertical nor a horizontal line and name it as L_1.
2. Identify any two points which lie on the line L_1 in task 1, and name the points as A and B.
3. Determine the gradient using the points A and B identified in task 2 and name it as m_1.
4. Draw another straight line L_2 such that it makes $90°$ with the line L_1 at point A.
5. Identify any other point lying on the line L_2 apart from A and name it as C.
6. Determine the gradient of the line L_2 by using the point identified in task 5 and point A and name it as m_2.
7. Find the product of the gradients obtained in tasks 3 and 6, and record the result.
8. What have you observed in task 7?
9. Share your results with other students through discussion for more inputs.

In Activity 5.5, if the lines drawn are perpendicular and the gradients are well calculated, then it can be observed that the product of the gradients is -1, that is, $m_1m_2=-1$.

Generally, if two lines are perpendicular, then the product of their gradients is -1 or the gradients are negative reciprocal to each other, that is, $m_1 m_2 = -1$ or $m_2 = -\frac{1}{m_1}$ or $m_1 = -\frac{1}{m_2}$.

Example 5.22

Show that the line passing through the points $(0,5)$ and $(-2,7)$ and the line passing through $(4,1)$ and $(-2,-5)$ are perpendicular.

Solution

Let m_1 be the gradient of the line passing through the points $(x_1, y_1) = (0,5)$ and $(x_2, y_2) = (-2,7)$ and m_2 be the gradient of the line passing through the points $(x_3, y_3) = (4,1)$ and $(x_4, y_4) = (-2,-5)$.

Thus, $m_1 = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{7-5}{-2-0}$

$= -1.$

Similarly,

$m_2 = \frac{y_4 - y_3}{x_4 - x_3},$

$= \frac{-5-1}{-2-4}$

$= 1.$

Hence, the product of the gradients of the two lines is -1. That is,

$m_1 m_2 = (-1) \times 1 = -1.$

Therefore, the given lines are perpendicular.

Example 5.23

Find the equation of the line which passes through the point $(-3,1)$ and is perpendicular to the line $3x + 4y - 1 = 0$.

Solution

Let m_1 be the gradient of the line passing through the point $(x_1, y_1) = (-3,1)$ and m_2 be the gradient of the line $3x + 4y - 1 = 0$.

Writing the given equation in the form of $y = m_2 x + c$ gives,

$$y = -\frac{3}{4}x + \frac{1}{4}.$$

Hence, $m_2 = -\frac{3}{4}$.

For perpendicular lines, $m_1 = -\frac{1}{m_2}$.

Thus, $m_1 = -\frac{1}{-\frac{3}{4}}$

$= \frac{4}{3}.$

Using the gradient and the given point, the equation of the line is obtained as follows.

$m_1 = \frac{y - y_1}{x - x_1}$, where $m_1 = \frac{4}{3}$.

Thus, $\frac{4}{3}=\frac{y-1}{x-(-3)}$.

Cross multiplication gives,
$4x+12=3y-3$.

Hence, $4x-3y+15=0$.

Therefore, the equation of the line is $4x-3y+15=0$.

Example 5.24

Find the value of v, if the line passing through the points $(-1, 2v)$ and $(2,3)$ is perpendicular to the line passing through $(5,3)$ and $(6,7)$.

Solution

Let m_1 be the gradient of line passing through the points $(-1,2v)$ and $(2,3)$ and m_2 be the gradient of the line passing through $(5,3)$ and $(6,7)$.

Since the two lines are perpendicular, then $m_1m_2=-1$. It follows that,

$$\left(\frac{2v-3}{-1-2}\right)\left(\frac{7-3}{6-5}\right)=-1$$

Simplification gives,

$$\frac{8v-12}{-3}=-1.$$

Thus, $8v-12=3$.

Hence, $v=\frac{15}{8}$.

Therefore, the value of v is $\frac{15}{8}$.

Example 5.25

A triangle has vertices $P(5,7)$, $Q(2,2)$, and $R(15,1)$.

(a) Use the concept of gradient to show that it is a right-angled triangle.

(b) Sketch the triangle PQR.

(c) Determine the equations of the line segments QR and PQ.

Solution

(a) Given the vertices $P(5,7)$, $Q(2,2)$, and $R(15,1)$. For a right-angled triangle, the line segments of two sides meet at a right angle. That is, two line segments are perpendicular. Thus, two line segments among $\overline{PQ}$, $\overline{PR}$, and $\overline{QR}$ are perpendicular.

Let $(x_1, y_1)=(5,7)$,

$(x_2, y_2)=(2,2)$ and

$(x_3, y_3)=(15,1)$.

The gradient of the line segment PQ is given by:

$$m_{\overline{PQ}}=\frac{y_2-y_1}{x_2-x_1}.$$

Thus, $m_{\overline{PQ}}=\frac{2-7}{2-5}$

$$=\frac{5}{3}.$$

The gradient of the line segment PR is given by:

$$m_{\overline{PR}} = \frac{y_3 - y_1}{x_3 - x_1}.$$

Thus, $m_{\overline{PR}} = \dfrac{1-7}{15-5}$

$$= -\frac{3}{5}.$$

The gradient of the line segment QR is given by:

$$m_{\overline{QR}} = \frac{y_3 - y_2}{x_3 - x_2}.$$

Thus, $m_{\overline{QR}} = \dfrac{1-2}{15-2}$

$$= -\frac{1}{13}.$$

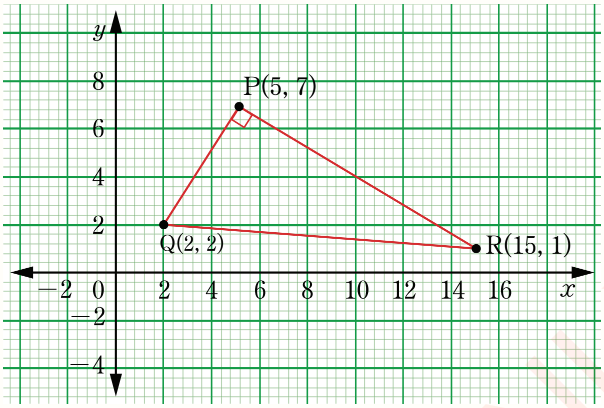
It follows that,

$$m_{\overline{PQ}} m_{\overline{PR}} = \frac{5}{3} \times \left(-\frac{3}{5}\right) = -1.$$

Since the condition for perpendicular lines is satisfied, then $\overline{PQ}$ and $\overline{PR}$ make a right angle.

Therefore, the points P, Q, and R from a right-angled triangle with the right angle at vertex P.

(b) The triangle PQR is sketched as follows.



(c) The gradient of the line segment QR is $-\dfrac{1}{13}$.

The equation of $\overline{QR}$ is given by,

$$m = \frac{y - y_1}{x - x_1}.$$

Thus, $-\dfrac{1}{13} = \dfrac{y-2}{x-2}$.

Cross multiplication gives,

$$-x + 2 = 13y - 26.$$

Hence, the equation of $\overline{QR}$ is

$$x + 13y - 28 = 0.$$

The gradient of $\overline{PQ}$ is $\dfrac{5}{3}$.

From the equation of the line,

$m = \dfrac{y - y_1}{x - x_1}$, it follows that,

$$\frac{5}{3} = \frac{y-2}{x-2}.$$

Cross multiplication gives,

$5x-10=3y-6.$

Thus, $5x-3y-4=0.$

Hence, the equation of $\overline{PQ}$ is $5x-3y-4=0.$

Therefore, the equations of the line segments QR and PQ are $x+13y-28=0$ and $5x-3y-4=0,$ respectively.

Exercise 5.7

1. Determine the gradient of the line which is perpendicular to the given line:

(a) $4x+7y-9=0$

(b) $8y-5x=9$

(c) $\frac{1}{6}x+\frac{5}{7}y=0$

(d) $0.5x+3.2y=-8$

2. The line which passes through the point $(5,1)$ is perpendicular to the line $3x+12y-1=0$. Find their point of intersection.

3. Find the equation of a line whose y-intercept is -8 and it is perpendicular to the line $y=\frac{5}{7}x+2.$

4. Determine the equation of the line passing through the point $(-2,7)$ and it is perpendicular to the line $y=4x+6$.

5. Find the equation of the line with y- intercept 3 and perpendicular to the line $x=4$.

6. Show that the points $A(-3,2)$, $M(5,6)$, and $K(2,-8)$ are the vertices of a right-angled triangle.

7. The points Q, R, S, and T are $(5,1)$, $(-5,6)$, $(8,3)$, and $(p,-13)$, respectively, where p is a real number.

(a) Find the equation of the line passing through Q and R.

(b) Given that $\overline{ST}$ is perpendicular to $\overline{QR}$, find the value of p.

8. Find the equation of a line passing through the point $(-4,-6)$ and it is perpendicular to the line $2x-6y+15=0.$

9. Determine the equation of the line passing through the point $(4,-2)$ and it is perpendicular to the line joining $(1,2)$ and $(6,-4).$

10. Show that the line passing through the points $P(-1,2)$ and $Q(1,-2)$ is perpendicular to the line passing through $R(2,1)$ and $S(-2,-1).$

Chapter summary

1. Linear equations can be represented graphically by connecting the set of points which satisfy the equations of lines.
2. Gradient of a line is the measure of its steepness.
3. The general form of the linear equation is $ax+by+c=0$, where a, b, and c are any real number such that a and b are not both zero.
4. The standard form of a linear equation is $y=mx+c$, where m is the gradient and c is the y-intercept.
5. The intercepts form of a linear equation is $\frac{x}{p}+\frac{y}{q}=1$, where p and q are the x and y - intercepts, respectively.
6. Three or more points are collinear if they lie on the same line.
7. Parallel lines never intersect to each other even when extended indefinitely in either directions.
8. Two lines are said to be perpendicular if they intersect at the right angle.

Revision exercise 5

1. Find the equation of a line with the gradient $-\frac{1}{2}$ and y-intercept 4.
2. Rewrite each of the following equations in the form $\frac{x}{p}+\frac{y}{q}=1$:
   (a) $4x-7y-28=0$
   (b) $x+\frac{3}{11}y=3$
   (c) $\frac{7}{2}x-\frac{5}{3}y=8$
3. If a line passing through the points $(-3k, 6)$ and $(9, 5-k)$ has the gradient of $-\frac{2}{9}$:
   (a) Find the value of k.
   (b) Find its equation.
   (c) State the intercepts of the equation obtained in (b).
4. Sketch the graph of a line that passes through the point $(1,2)$ and whose gradient is $\frac{7}{3}$.
5. Find the value of x, if the points $(x+1, 3)$, $(5,4)$, and $(9,1)$ are collinear.
6. Verify that the points $A(4,2)$, $B(2,4)$, $C(3,6)$, and $D(5,4)$ form a parallelogram. Hence, find the equations of the lines of the sides of the parallelogram formed.

7. If the gradient of the line which passes through the points $\left(\frac{t}{9}, 5\right)$ and $(-7, 0)$ is -5, find the value of t.

8. Find the equation of a line that passes through the point $(2, 4)$ and it is perpendicular to $8x + 7y - 4 = 0$.

9. Find the value of k for which the line $2kx + 6y = 3$ is perpendicular to $3x - 7y + 7 = 0$.

10. Show that the points $A(0, 5)$, $B(9, 12)$, and $C(3, 14)$ are vertices of a right-angled triangle. Hence, find the equation of the line segment AB.

11. A line segment AB is contained in a line $2x - 11y + 2 = 0$. Find the values of k and q given that the points of the line segment are $A(k, 2)$ and $B(-1, q)$.

12. Show that the condition for a line $Ax + By + C = 0$ to be perpendicular to the line $ax + by + c = 0$ is $Aa + Bb = 0$.

13. Given the lines $(2 + n)x + 4y = 10$ and $5x - 2y = 12$. Find the value of n if they are:
    (a) parallel
    (b) perpendicular

14. Show that the condition for a line $Ax + By + C = 0$ to be parallel to the line $ax + by + c = 0$ is $Ab - Ba = 0$.

15. Solve each of the following systems of linear simultaneous equations graphically:

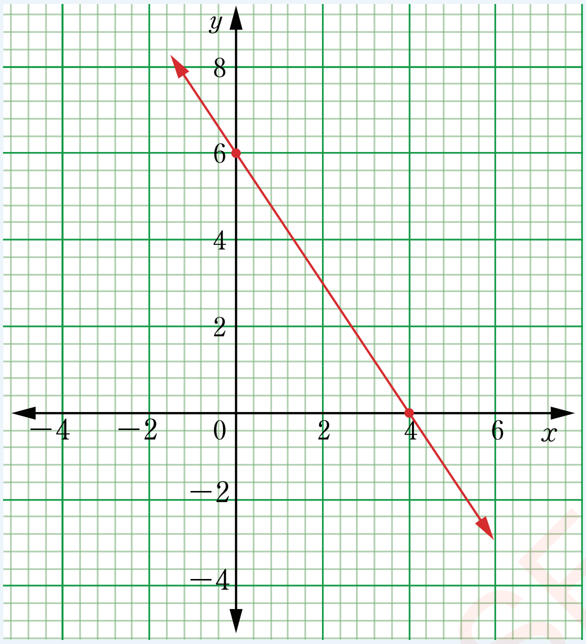
    (a) $\begin{cases} \frac{1}{7}x + \frac{1}{3}y = 3 \\ 4x - y = 22 \end{cases}$ (b) $\begin{cases} x - 2y - 4 = 0 \\ 2x + y - 3 = 0 \end{cases}$ (c) $\begin{cases} x = 4 - \frac{3}{2}y \\ -3x + \frac{y}{2} = 8 \end{cases}$

16. Two lines intersect at $(3, 3)$. The first line has a gradient of 3 and the second line has y-intercept -9. Find the equations of the lines.

17. Two parallel lines with gradient -2 cross the y-axis at $(0, 7)$ and $(0, -2)$. At what points will the lines cross the x-axis?

18. Find the general equation of a line which passes through the point (a, b) and it is perpendicular to $y = \frac{4}{3}x - 7$.

19. Determine the equation of a line represented by the following graph.



20. A triangle has vertices D(4, 6), E(4, 1), and F(12, 1).

(a) Use the concept of gradient to verify that the vertices form a right-angled triangle.

(b) Determine the equations of the line segments DE and EF.

Answers

Chapter One

Exercise 1.1

1. (a) 14, 18, 22
(b) 11, 17, 24
(c) 8, 27, 64
(d) $\frac{1}{3}, -\frac{1}{9}, \frac{1}{27}$
(e) 1, -4, -2
(f) 0, 4, -3

2. (a) $\frac{16}{30}, \frac{22}{39}, \frac{29}{49}, \frac{37}{60}$
(b) $-22, -38, -57, -79$
(c) 12, 8, 1, -9
(b) 41, 77, 126, 190

3. (a) 0, 1, 8
(b) 2, 6, 12
(c) $-\frac{14}{5}, -\frac{13}{13}, -\frac{9}{20}$
(d) $\frac{6}{12}, \frac{11}{16}, \frac{13}{9}$

4. (a) Add 7 to the preceding term
(b) Add 2 more than the preceding difference

5. (a) (i) $-4\,°C$
(ii) $-24\,°C$
(iii) $-14\,°C$
(b) (i) 2 minutes
(ii) 3 minutes and 30 seconds
(iii) 5 minutes and 30 seconds

Exercise 1.2

1. (a) 2^{n-1}
(b) $2n$
(c) $2n^2+1$
(d) $3n+9$
(e) n^2+1

2. $\frac{1}{3}, \frac{4}{7}, \frac{7}{11}$

3. $\frac{1}{5}, \frac{2}{3}, 9, -28$

4. $-\frac{114}{17}, -\frac{77}{10}, -\frac{200}{23}, -\frac{126}{13}$

5. $\frac{738}{17}, \frac{520}{15}, \frac{350}{13}, \frac{222}{11}, \frac{130}{9}$

6. 1 053

Exercise 1.3

1. (a) 2, 4, 6, 10
(b) $-2, -6, -8, -14$
(c) 26, 42, 68, 110
(d) $-\frac{21}{4}, -\frac{17}{2}, -\frac{55}{4}, -\frac{89}{4}$
(e) $-3.1, -5, -8.1, -13.1$

2. (a) $9x+4y, 15x+6y, 24x+10y, 39x+16y$
(b) $x+2y, x+3y, 2x+5y, 3x+8y$
(c) $4x-9y, 5x-17y, 9x-26y, 14x-43y$
(d) $-\frac{25}{21}a-\frac{1}{7}b, -\frac{12}{7}a+\frac{2}{7}b, -\frac{61}{21}a+\frac{1}{7}b, -\frac{97}{21}a+\frac{3}{7}b$

3. (a) 13 cows (b) 13 years

Exercise 1.4

1.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

(a) 1, 3, 3, 1 (b) 32

2. (a)

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

(b) (i) The first diagonal consists of only 1s.

(ii) The second diagonal consists of counting numbers ranging from 1 to 8.

(c) 1, 3, 6, 10, 15, 21, 28

(d) 1, 4, 10, 20, 35, 56

3. (a) Pascal's triangle

(b) 35 is obtained by adding 15 and 20

(c) 1, 3, 6, 10, 15, 21

(d) 9^{th} row: 1, 8, 28, 56, 70, 56, 28, 8, 1.

10^{th} row: 1, 9, 36, 84, 126, 126, 84, 36, 9, 1

(e) Right-angled triangle

Exercise 1.5

1. (a) 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256

(b) 1, 8, 27, 64

(c) 30

(d) 28

(e) 60

2.

21	24	28	14
27	15	20	25
16	30	22	19
23	18	17	29

3.

8	11	14	1
13	2	7	12
3	16	9	6
10	5	4	15

4.

5	−1	−4
−5	−2	7
0	3	−3

Note: Different charts can be constructed.

5.

23	1	2	20	19
22	16	9	14	4
5	11	13	15	21
8	12	17	10	18
7	25	24	6	3

6.

8	1	6
3	5	7
4	9	2

Exercise 1.6

1. (a) 27 children
   (b) 10 children
   (c) 5 grand children
2. Possible pair:

n
$7n$

3. (a) The numbers are divisible by 11
   (b) Item number times 11
   (c) 220 (obtained after multiplying 20 by 11)
4. (a) $3.5\,\text{cm}^2$, $7\,\text{cm}^2$, $10.5\,\text{cm}^2$, and $14\,\text{cm}^2$
   (b) The areas are divisible by the length of one side, that is, divisible by 3.5 cm

Exercise 1.7

1. Possible numbers are 4 536 and 8 532
2. 9 876, 1 234 869, 123 456 789, and 21 756 are divisible by 3;
   21756 is divisible by 7;
   123 456 789 is divisible by 9.
3. 24
4. (a) 204 and 1 134 are divisible by 3;
   448 100 is divisible by 5;
   204 and 1 134 are divisible by 6;
   204 is divisible by 12;
   4 913 and 204 are divisible by 17;
   147 497 is divisible by 19.
   (b) 204, 1 134, 4 913

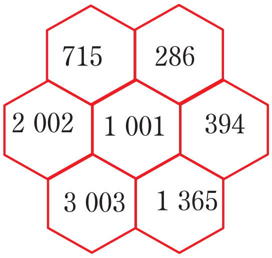
Revision exercise 1

1. Temu's response was wrong. The correct response is subtract 2 from the first number then add 5.
2. The pattern is 49 000, 41 000, 33 000, 25 000, 17 000, 9 000. After five days, she will be left with Tsh 9 000.
3. 89, 144
4. (a) 7, 11, 18
   (b) The number pattern is 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, …
   Sum of these first ten terms is 517.
   The seventh term is 47, then $47\times11=517$, showing that Ashura is correct.
5. (a) n^2-1 (b) n^2-4
6. $1, -\frac{5}{7}, -\frac{15}{17}, -\frac{29}{31}$
7. n^2-n, where n a natural number
8. $-\frac{5}{4}, \frac{65}{2}, \frac{325}{12}, \frac{2501}{44}$
9. 329, 496, 711
10. (a) $T_{22}=70$ (b) $T_{17}=55$
11. 7 hours
12. (a)

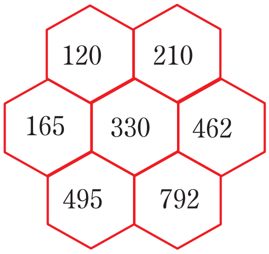
Row	1	2	3	4	5	6
Row sum	1	2	4	8	16	32

   (b) Preceding term times 2
   (c) Row sum $=2^n$, where n is the n^{th} row and $n\geq0$

13. (a)



(b)



14. (a) Sum of 13th row $= 2^{12}$

(b) Sum of 20th row $= 2^{19}$

15. $a = 36, \quad b = 45$

16.

3	−6	−4	6	1
5	9	−10	7	−11
−2	−8	0	8	2
−5	−7	10	−9	11
−1	12	4	−12	−3

17.

20	21	22	23
24	25	26	27
28	29	30	31
32	33	34	35

18.

16	2	3	13
5	6	7	8
9	10	11	12
4	14	15	1

19. (a) Divisible

(b) Not divisible

(c) Divisible

(d) Divisible by 4 but not by 8

20. 3 492, 4 257

Chapter Two

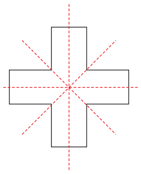
Exercise 2.1

1. (a) Not symmetrical
   (b) Symmetrical
   (c) Not symmetrical
   (d) Symmetrical
   (e) Symmetrical
   (f) Not symmetrical
   (g) Symmetrical
   (h) Symmetrical
   (i) Symmetrical
   (j) Symmetrical
   (k) Not symmetrical
   (l) Symmetrical
2. 3
3. 1
4. E, B, A
5. S

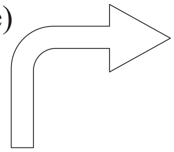
Exercise 2.2

1. (a) 1 (d) 0
(b) 0 (e) 7
(c) 2 (f) 0

2. (a)

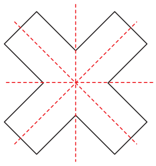


(e)

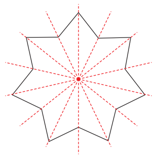


No line of symmetry

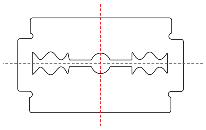
(b)



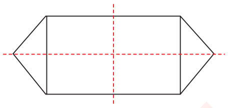
(f)



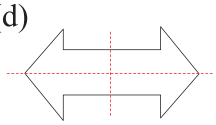
(c)



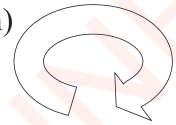
(g)



(d)



(h)

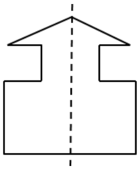


No line of symmetry

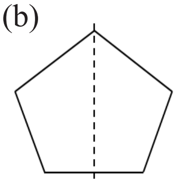
3. (a) A, B, C, D, E, K, M, T, U, V, W, Y
(b) H, I, O, X
(c) F, G, J, L, N, P, Q, R, S, Z

Exercise 2.3

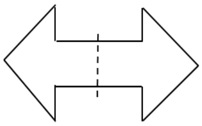
1. (a)



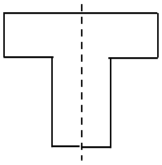
(b)



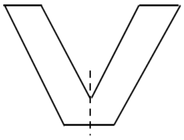
(c)



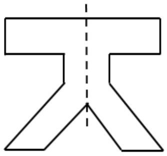
(d)



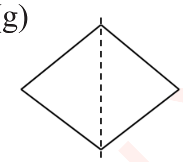
(e)



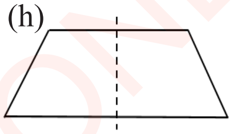
(f)



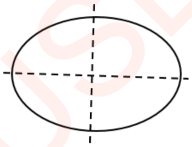
(g)



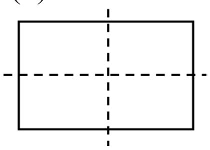
(h)



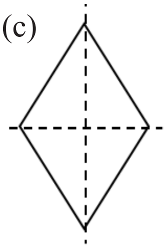
2. (a)



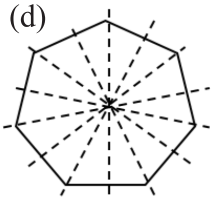
(b)



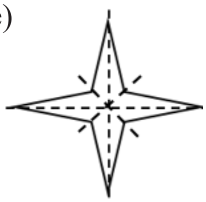
(c)



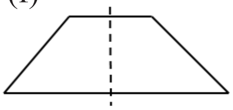
(d)



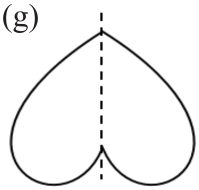
(e)



(f)



(g)



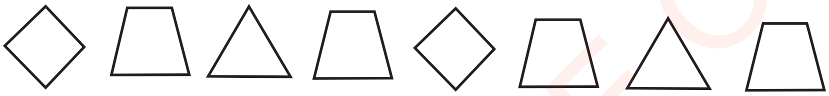
Exercise 2.4

1. 4, 90°	**2.** 2, 180°	**3.** 1, 360°
4. 2, 180°	**5.** 8, 45°	**6.** 2, 180°
7. 5, 72°	**8.** 1, 360°	**9.** 2, 180°
10. 2, 180°	**11.** 12, 30°	**12.** 1, 360°

Exercise 2.5

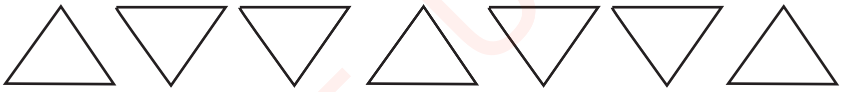
1. (a)

(b)

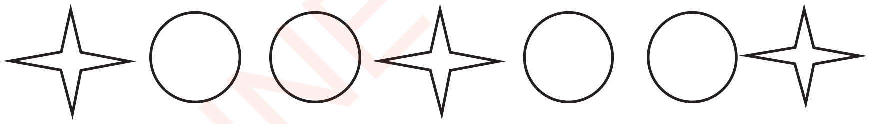


(c)

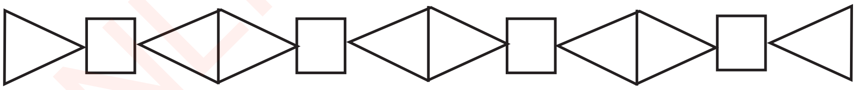
(d)



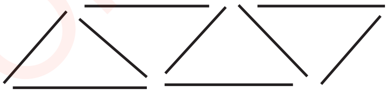
(e)



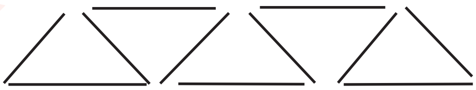
2.



3. (a)



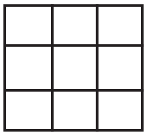
(b)



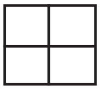
(c) 13

4. (a) 7 (b) 6

5. (a)



(b)

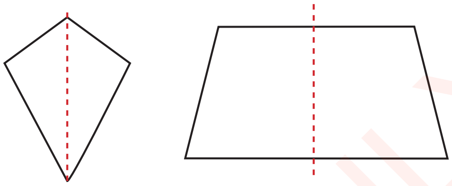


Revision exercise 2

1. (a) Symmetrical
(b) Not symmetrical
(c) Symmetrical
(d) Symmetrical
(e) Not symmetrical
(f) Symmetrical
(g) Symmetrical
(h) Symmetrical
(i) Not symmetrical

2. (a) 1 (e) 4 (i) 2
(b) 2 (f) 5 (j) 1
(c) 1 (g) 1 (k) 1
(d) 1 (h) 2 (l) 1

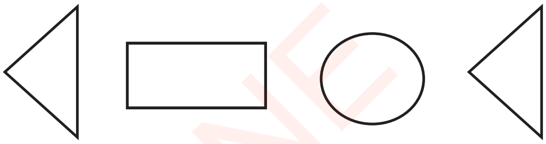
3.



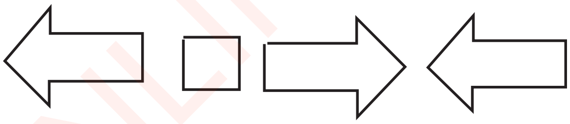
A Kite
B Isosceles trapezium

4. (a) 3, 120°
(b) 8, 45°
(c) 4, 90°
(d) 3, 120°
(e) 1, 360°
(f) Infinity, $0° < \theta \leq 360°$

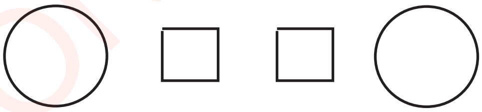
5. (a)



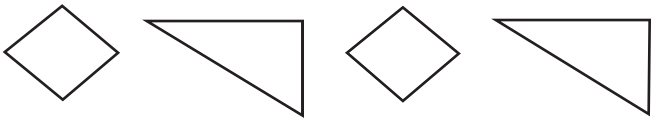
(b)



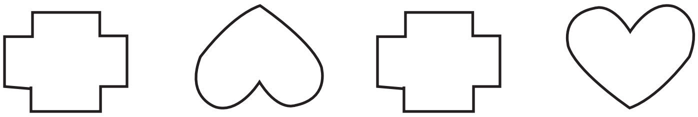
(c)



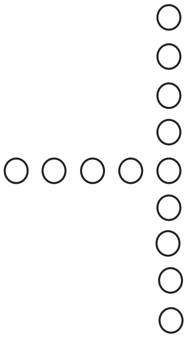
(d)



(e)

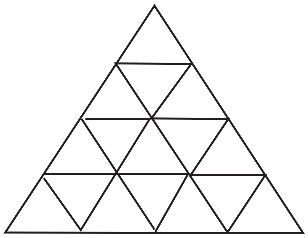


6. (a)

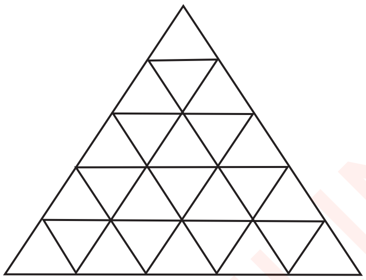


(b) 19 cirles

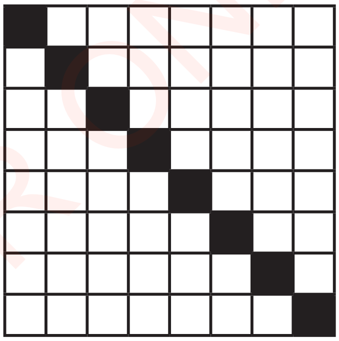
7. Pattern 4



Pattern 5



8. (a)



(b) 10 squares

Chapter Three

Exercise 3.1

1. $2a-12b$

2. $7+n$

3. $-5pm+3mq$

4. $8x+2y+3z$

5. $8a-2b+12$

6. $3x$

7. $7k+21$

8. $6c-3d$

9. $7p-3q+7r$

10. $6w+4$

11. $2a-12b$

12. $10x+5y-9z$

13. $16mn+11pq$

14. $10xyz-32pqr$

15. $-7ad-30ef+13gh$

(a) 3 (b) 13

Exercise 3.2

1. 0, 5

2. -10, 8

3. $-\frac{1}{3}, \frac{11}{3}$

4. $-\frac{3}{5}, 3$

5. $-2, -\frac{1}{2}$

6. $\frac{4}{5}$, 10

7. $\frac{7}{8}, \frac{13}{4}$

8. $-\frac{4}{3}, -\frac{2}{7}$

9. $7, \frac{1}{5}$

10. $-\frac{19}{22}, -\frac{11}{38}$

11. $-\frac{1}{5}, \frac{13}{3}$

12. $-\frac{35}{2}, \frac{45}{14}$

Exercise 3.3

1. 33, 17

2. 33, 10

3. 40 blue, 100 red

4. $\frac{7}{11}$

5. 7, 4

6. 38 years, 14 years,

7. 70 years, 40 years

8. 20 cm, 8 cm

9. 800, 1 200

10. 200 Tanzanian shillings, 800 Tanzanian shillings

11. 12 days

12. 4 days

Exercise 3.4

1. (a) $r=\frac{VR}{E-V}$ (b) $a=\frac{2(s-ut)}{t^2}$

(c) $f=\frac{uv}{u+v}$ (d) $h=\frac{A}{2\pi r}-r$

(e) $t=\frac{3s^2}{6s-1}$ (f) $t=\frac{\sqrt{A}+1}{\sqrt{A}-1}$

(g) $y=\pm\sqrt{\frac{kr^2}{a^2}+t}$

(h) $y=\pm\sqrt{3x^2-\frac{rD^3}{d^3}}$

2. $\alpha=\frac{l_2-l_1}{l_1\theta}$

3. (a) $h=\frac{A-2\pi r^2}{2\pi r}$, 12

(b) $x=5t-55$, -25

(c) $F=\frac{9}{4}C+32°$, $90.5°$

(d) $x=\pm\frac{a\sqrt{b^2-y^2}}{b}$, $\pm\frac{4}{5}$

(e) $a=\frac{x^3}{y^2}$, 27

(f) $b=\frac{a\left(p^2-q^2\right)}{2\left(p^2+q^2\right)}$, $\frac{8}{5}$

4. $v=\pm\sqrt{\frac{2dgh}{0.03L}}$, ±10

5. $t_2 = t_1 + \dfrac{Q}{mc}$, 15.41

6. $r = 20$

7. $r = \sqrt{\dfrac{A}{4\pi}}$, $r = 8$

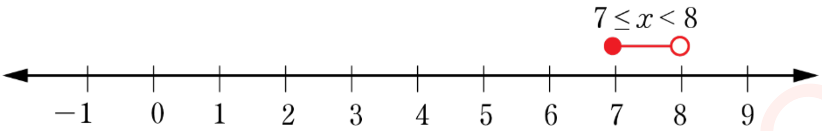
8. $t = 12$

Exercise 3.5

1. $b > 5$
2. $x \geq 3$
3. $m \geq 48$
4. $x \geq 1$
5. $x \leq 20$
6. $x \leq -3$
7. $x > 4$
8. $m \geq 2$
9. $r > 8$
10. $x \leq -18$

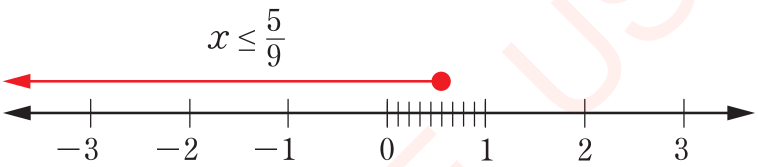
Exercise 3.6

1.

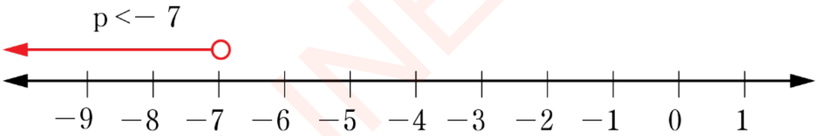


2.

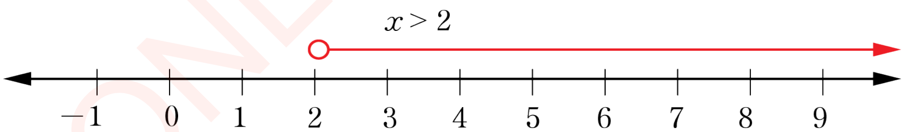
3.



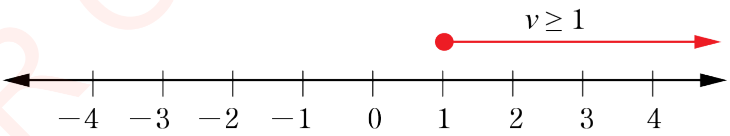
4.



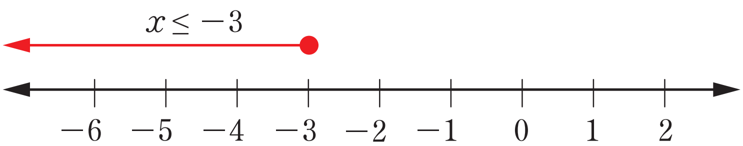
5.



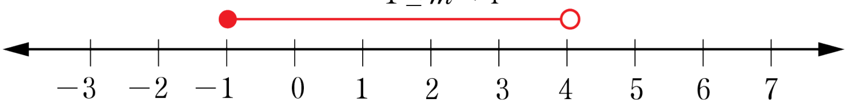
6.



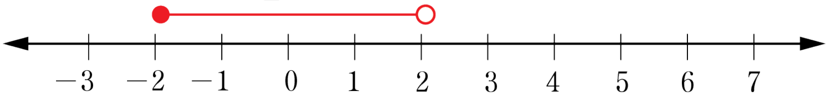
7.



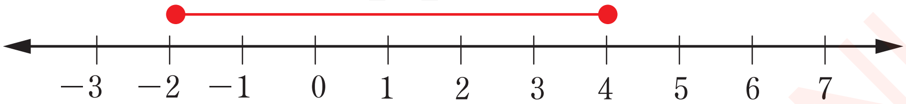
8. $-1 \leq m < 4$



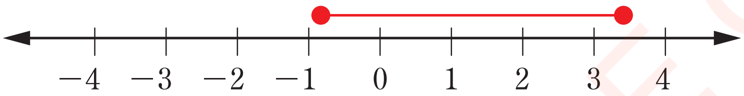
9. $-2 \leq x < 2$



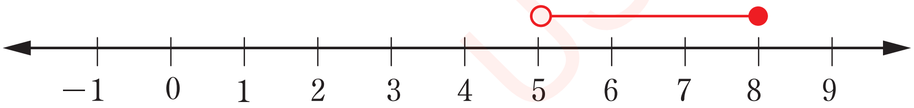
10. $-2 \leq n \leq 4$



11. $-\frac{3}{4} \leq x \leq \frac{13}{4}$



12. $5 < x \leq 8$



Revision exercise 3

1. (a) $4a+5b$

(b) $4t+5a$

(c) $32pn-33pt$

(d) $2p+11$

(e) $4x+4$

(f) $\frac{1}{3}y+\frac{7}{12}p$

2. (a) $-1,\ 2$

(b) $-\frac{1}{2},\ \frac{3}{2}$

(c) $-8,\ 8$

(d) $-9,\ 3$

(e) $-\frac{5}{2},\ \frac{1}{2}$

(f) $-4,\ -2$

(g) $-\frac{3}{8},\ \frac{7}{4}$

(h) $6,\ \frac{18}{17}$

3. 480 boys, 400 girls

4. 30 years, 6 years

5. 137, 72

6. (a) $h=\frac{t^2g}{4\pi^2}-k$

(b) $C=\frac{5}{9}(F-32)$

(c) $x=\frac{d}{\lambda}(y+\lambda)$

(d) $f=\frac{3F-AL}{3}$

(e) $R=\frac{E-e}{I}-r$

7. $v=\pm\sqrt{\frac{2L}{\rho ac}}$

8. (a) $d=\frac{I-A}{n-1}$, 2

(b) $b=\frac{p-2l}{2}$, 8

(c) $h=\frac{3V}{\pi r^2}$, 5.98

(d) $b=\frac{2A}{h}$, 15

(e) $d=\frac{2(s-na)}{n(n-1)}$, 20

9. (a) $6\le x<11$

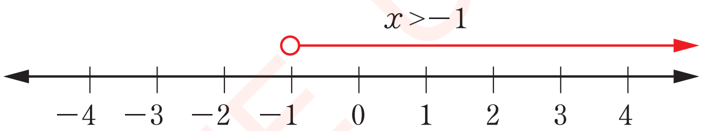
(b) $-\frac{13}{2}\le x\le-\frac{1}{2}$

(c) $x<4$

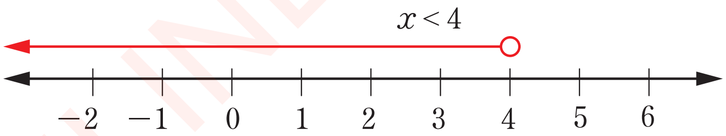
(d) $x\le 8$

(e) $\frac{20}{3}<x\le 10$

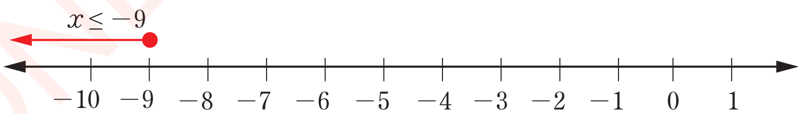
10. (a) $x>-1$



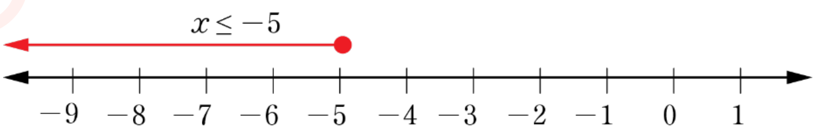
(b) $x<4$



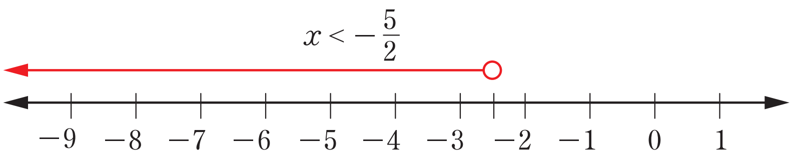
(c) $x\le-9$



(d) $x\le-5$

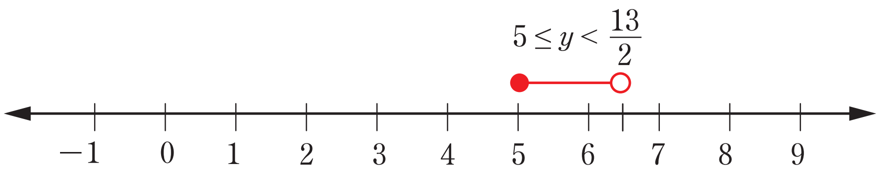


(e) $x<-\frac{5}{2}$



11. (a) $-1 \leq x < 3$ (b) $x > 0$ (c) $x \leq 10$ (d) $-4 \leq x \leq 2$

12. $5 \leq y < \frac{13}{2}$



13. 35 pieces of cake, 52 bottles of soda

14. 100 Tanzanian shillings, 200 Tanzanian shillings.

15. 25 000 Tanzanian shillings, 15 000 Tanzanian shillings

Chapter Four

Exercise 4.2

1. (c), (d), (e), (g), (h), (j), (k)

2. (f), (m), (n), (o), (p)

3. (a), (b), (i), (l)

Exercise 4.3

1. 720°

2. 540°

3. 360°

4. 360°

5. 1080°

6. 1080°

7. 1080°

8. 1440°

Exercise 4.4

1. (a) 117° (b) 73°

2. (a) 144°, 36° (b) 140°, 40°

3. (a) 9 (b) 140°

(c) 1260°

4. 128.57°, 51.43°

5. 30

6. 18

Revision exercise 4

3. (a) $x = 78°$ (b) $x = 150°$

(c) $x = 126°$

4. (a) 9 (b) 15

5. (a) 12 (b) 30

6. (a) 14, 154.28° (b) 36, 170°

(c) 25, 165.6°

7. 16

8. $x = 35°$, $\theta_e = 82°$

10. (a) 156° (b) 24°

(c) 2340°

11. 1440°

12. 360°, 7

13. 1800°

14. 13

15. 2520°

16. 10, 1800°

17. 154.3°

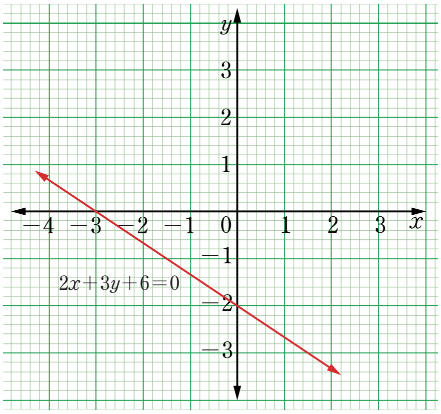
18. $m = 20°$, $n = 100°$

Additional Maths Form I.indd 141 24/01/2023 16:30

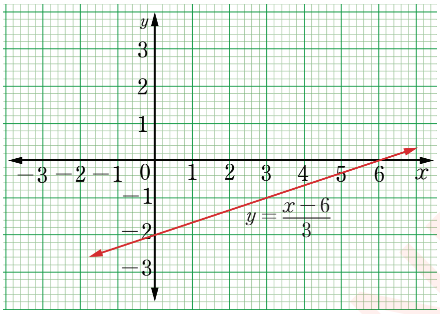
Chapter Five

Exercise 5.1

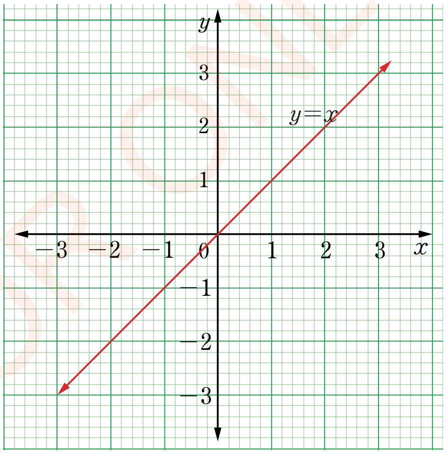
1.



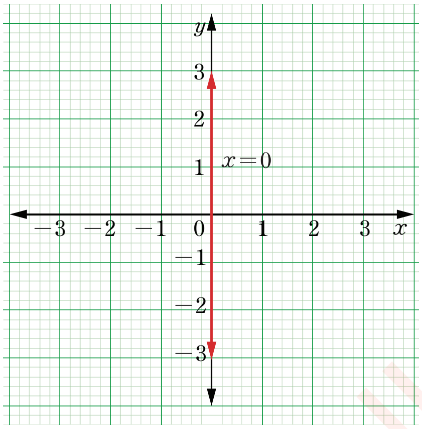
2.



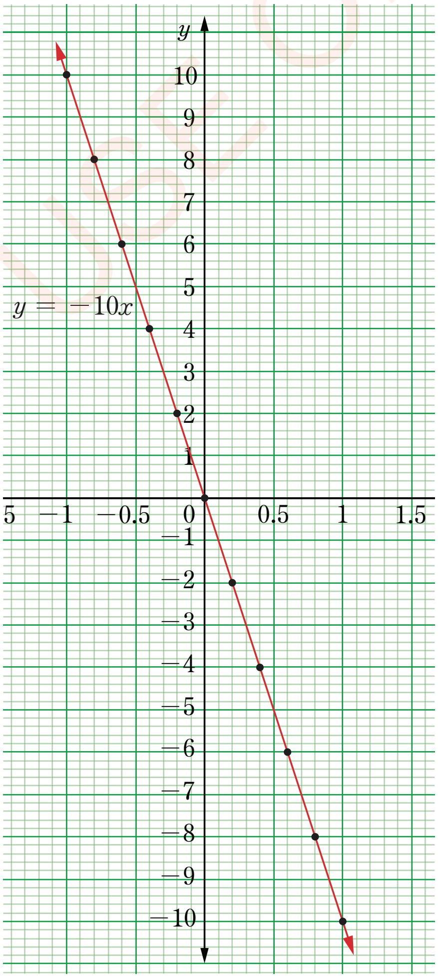
3.



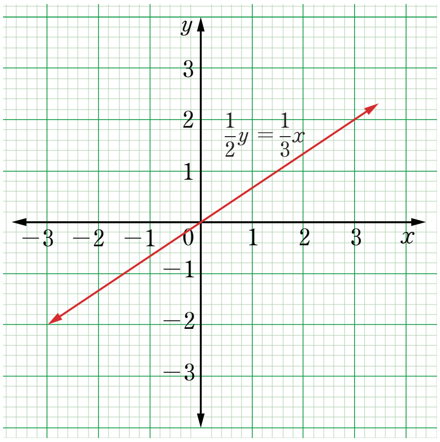
4.



5.

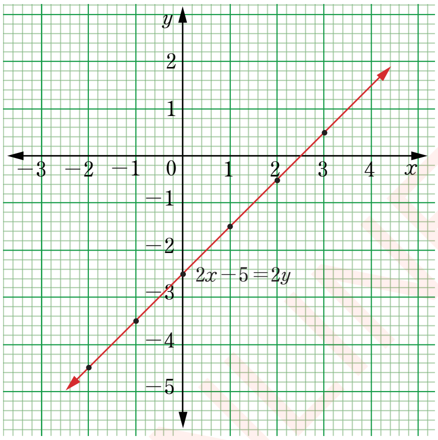


6.



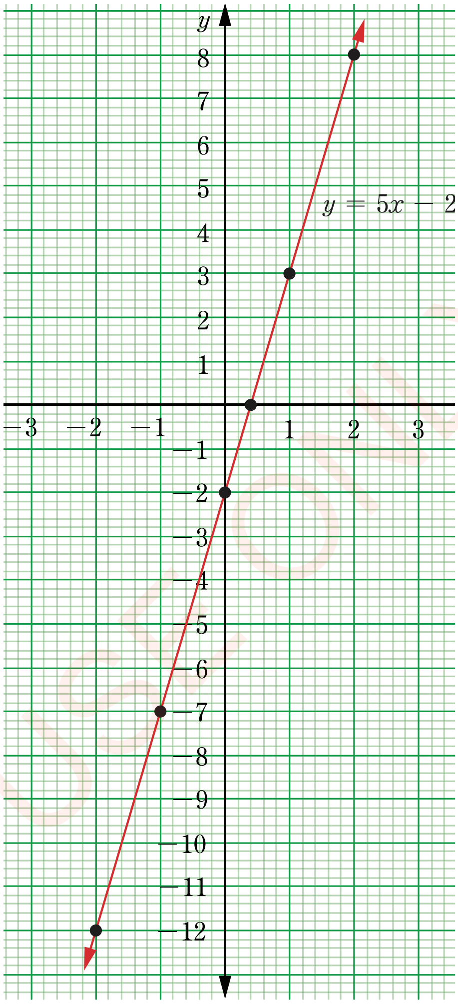
7.

x	-2	-1	0	1	2	3
y	-4.5	-3.5	-2.5	-1.5	-0.5	0.5

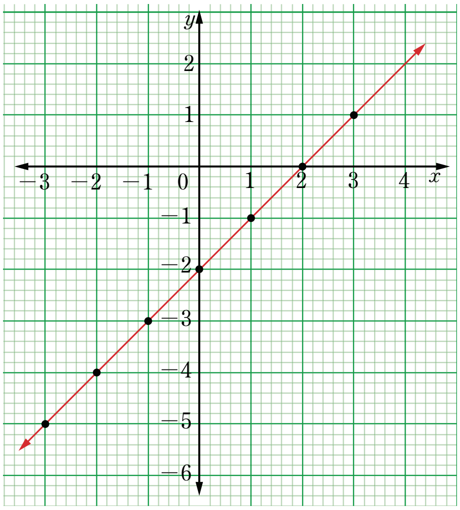


8. $y=5x-2$

x	-2	-1	0	1	2
y	-12	-7	-2	3	8



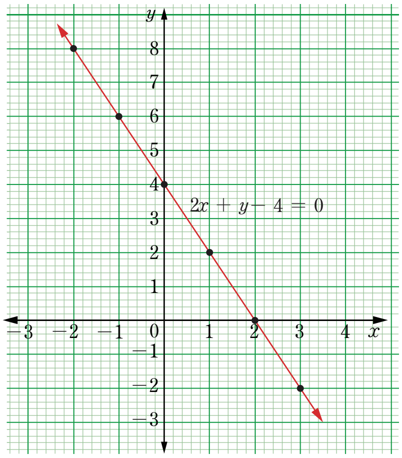
9.



10. (a)

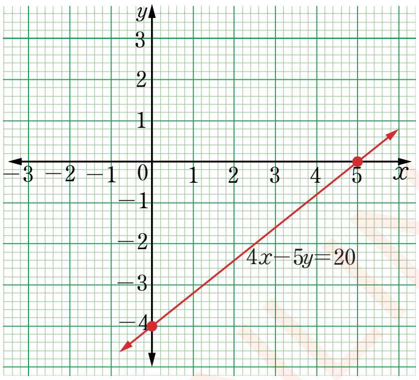
x	–2	–1	0	1	2	3
y	8	6	4	2	0	–2

(b)

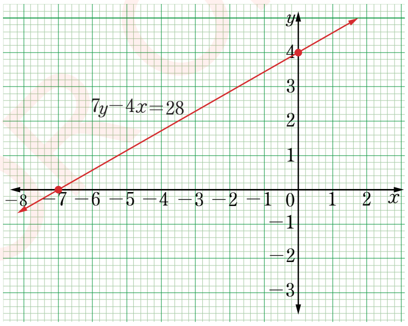


Exercise 5.2

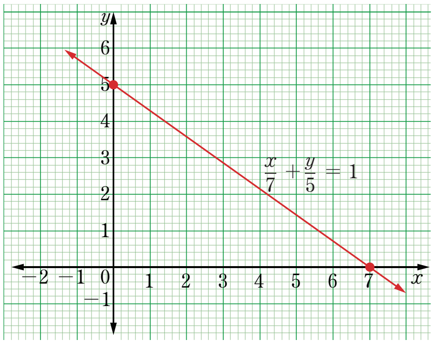
1.



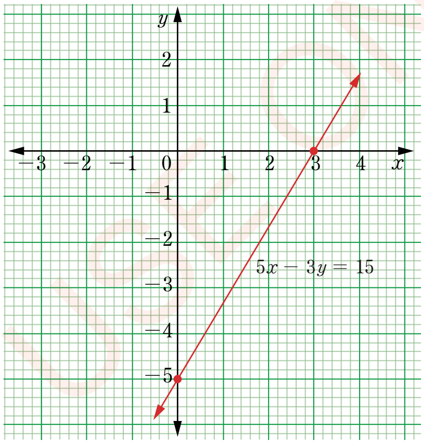
2.



3.



4.



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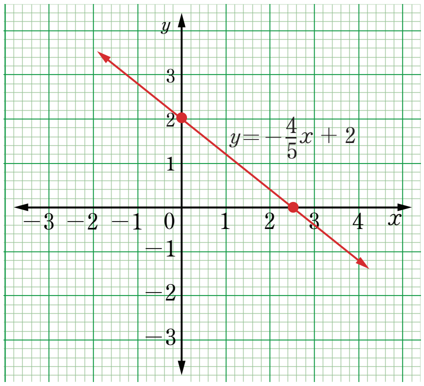

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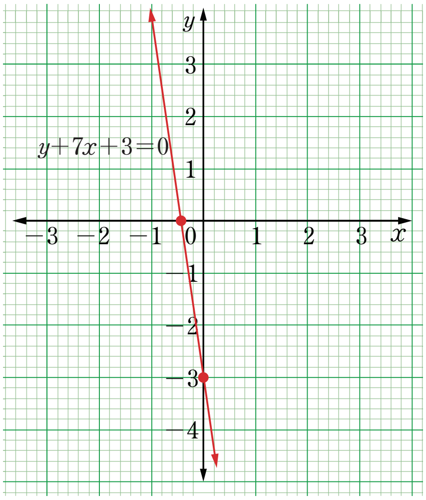

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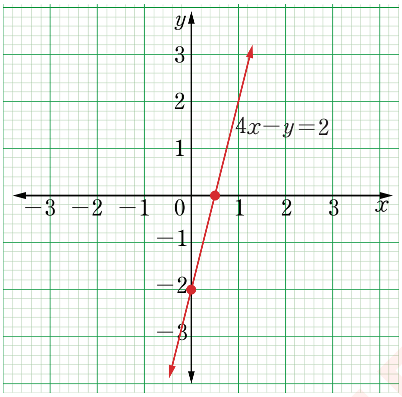
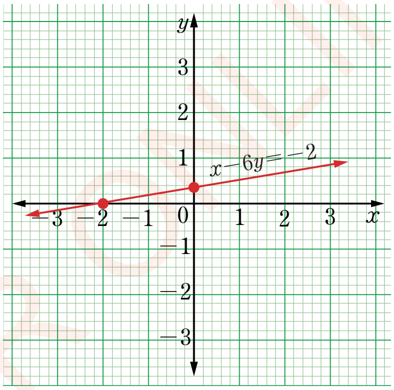


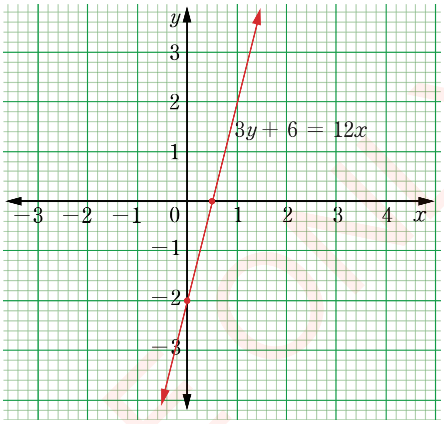
8.



9. (a) y-intercept is -2,

x-intercept is $\frac{1}{2}$

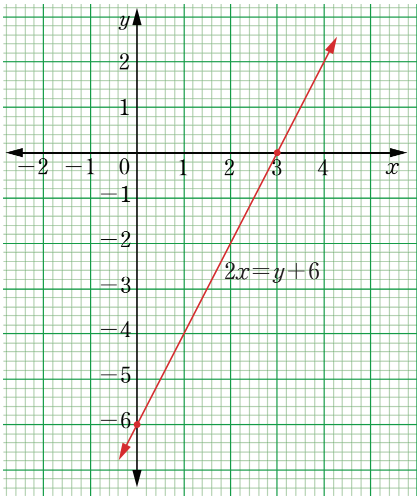
(b)



10. (a) y-intercept is -6,

x-intercept is 3

(b)



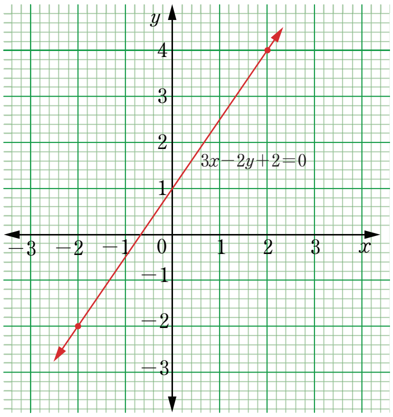
Exercise 5.3

1. (a) $-\frac{2}{7}$ (b) $2x+7y+1=0$

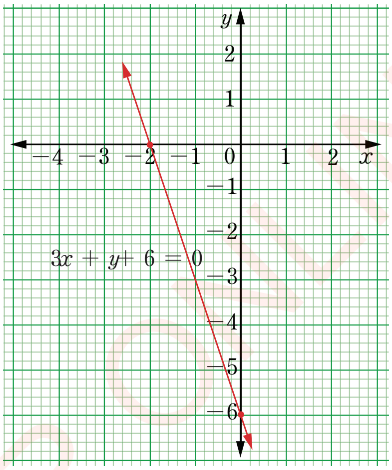
(c) y-intercept is $-\frac{1}{7}$, x-intercept is $-\frac{1}{2}$

2. $y=x$

3.



4. $y+3x+6=0$



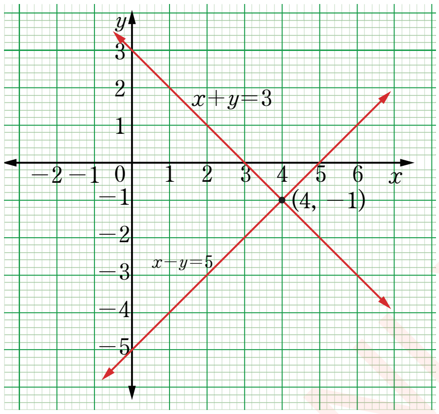
5. $x+2y-4=0$ **6.** $(10,0)$

7. $4x+y+4=0$ **8.** $\frac{y}{1}+\frac{x}{2}=1$

9. $y=\frac{4}{3}x+8$ **10.** $x-2y-10=0$

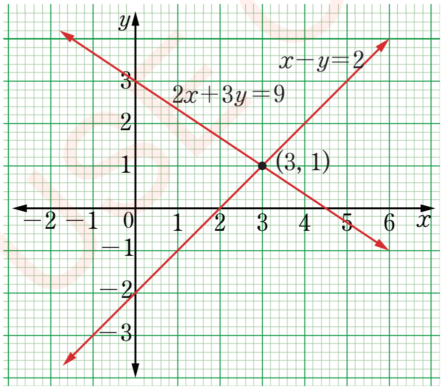
Exercise 5.4

1.



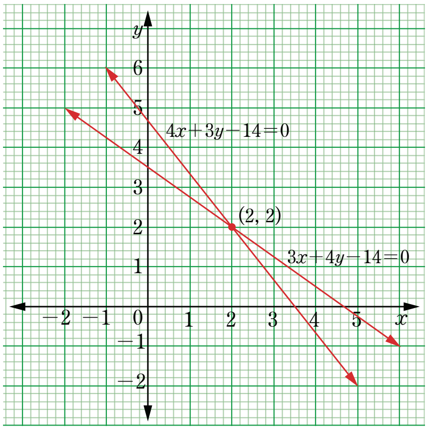
$(x,y)=(4,-1)$

2.



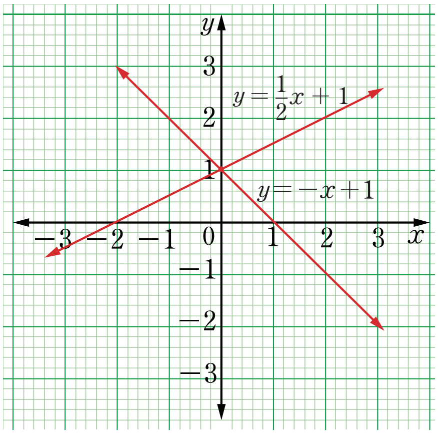
$(x,y)=(3,1)$

3.



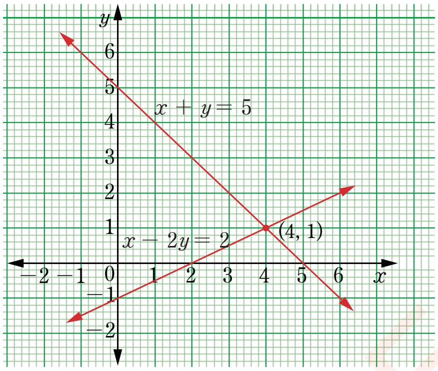
$(x,y)=(2,2)$

4.



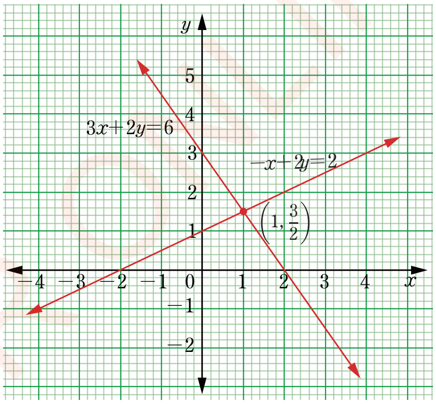
$(x, y) = (0, 1)$

5.



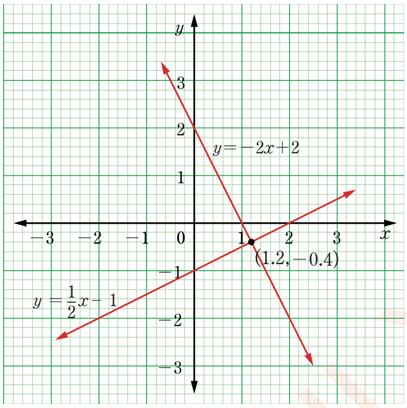
$(x, y) = (4, 1)$

6.



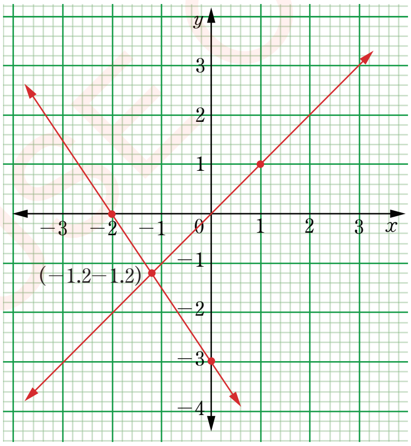
$(x, y) = \left(1, \frac{3}{2}\right)$

7.



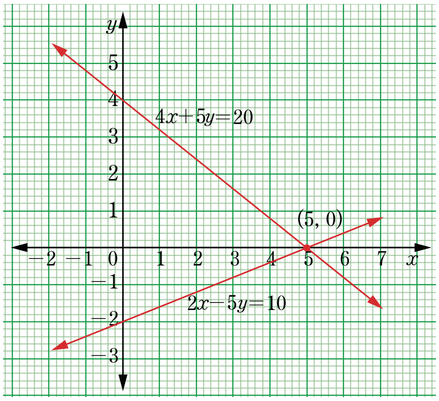
$(x, y) = (1.2, -0.4)$

8.



$(x, y) = (-1.2, -1.2)$

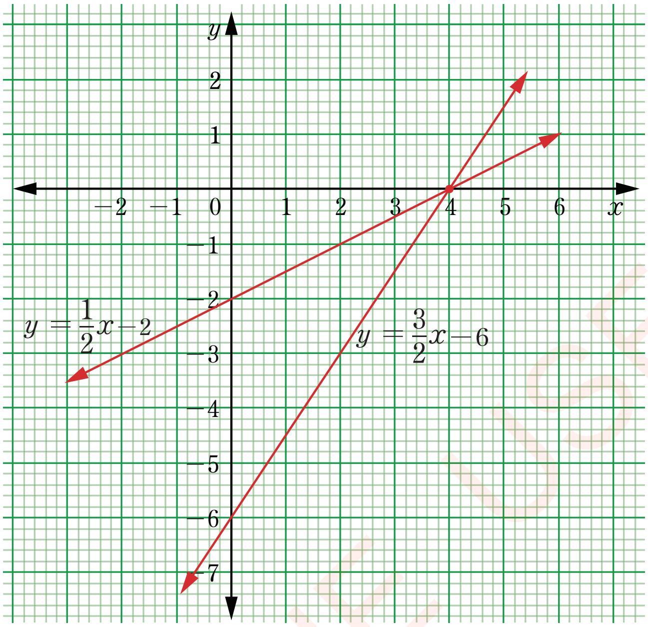
9.



$(x, y) = (5, 0)$

10.

x	-2	-1	0	1	2	3	4	5
$y=\frac{3}{2}x-6$	-9	-7.5	-6	-4.5	-3	-1.5	0	1.5
$y=\frac{1}{2}x-2$	-3	-2.5	-2	-1.5	-1	-0.5	0	0.5



$(x, y) = (4, 0)$

Exercise 5.5

1. (a) Non-collinear
   (b) Collinear
   (c) Collinear
   (d) Non-collinear
   (e) Non-collinear

2. (a) $t=\frac{32}{5}$ (c) $t=4$
   (b) $t=26$ (d) $t=2$

Exercise 5.6

1. (a) Parallel lines
   (b) Parallel lines
   (c) Not parallel
   (d) Parallel lines
2. $2x-7y+34=0$
3. $8x+7y-44=0$
4. $y=8x$
5. $k=-\frac{3}{2}$
6. $r=-8$

8. (a) $c \neq 0,\ a = 0,\ b \neq 0$

(b) $c \neq 0,\ a \neq 0,\ b = 0$

9. $\lambda = \frac{1}{2}$

10. $9y + 10x + 56 = 0$

Exercise 5.7

1. (a) $\frac{7}{4}$ (c) $\frac{30}{7}$

(b) $-\frac{8}{5}$ (d) $\frac{32}{5}$

2. $\left(\frac{229}{51}, -\frac{53}{51}\right)$

3. $y = -\frac{7}{5}x - 8$

4. $y = -\frac{1}{4}x + \frac{13}{2}$

5. $y = 3$

7. (a) $x + 2y = 7$ (b) $p = 0$

8. $y + 3x + 18 = 0$

9. $6y - 5x + 32 = 0$

Revision exercise 5

1. $y = -\frac{1}{2}x + 4$

2. (a) $\frac{x}{7} - \frac{y}{4} = 1$ (b) $\frac{x}{3} + \frac{y}{11} = 1$

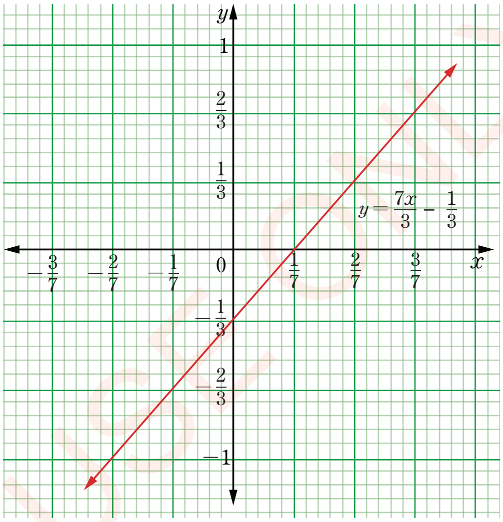
(c) $\frac{7x}{16} - \frac{5y}{24} = 1$

3. (a) $k = 3$ (b) $\frac{x}{18} + \frac{y}{4} = 1$

(c) y-intercept = 4,

x-intercept = 18

4. $y = \frac{7}{3}x - \frac{1}{3}$



5. $x = \frac{16}{3}$

6. $x + y = 6,\ x + y = 9,\ 2x - y = 0,$
$2x - y = 6$

7. $t = -72$

8. $7x - 8y + 18 = 0$

9. $k = 7$

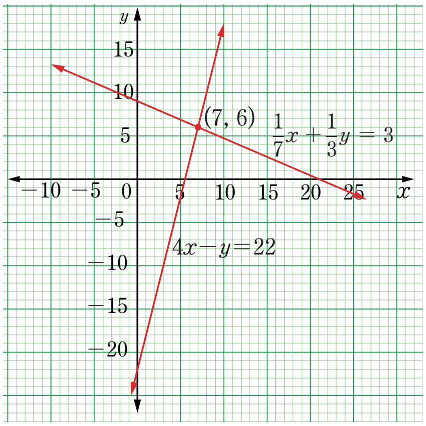
10. $\overline{AB}$: $7x - 9y + 45 = 0$

11. $k = 10,\ q = 0$

13. (a) $n = -12$

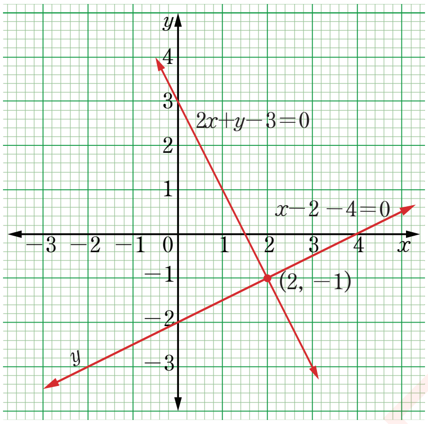
(b) $n = -\frac{2}{5}$

15. (a)



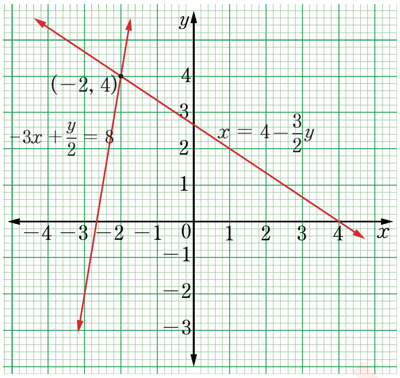
$(x, y) = (7, 6)$

(b)



$(x, y) = (2, -1)$

(c)



$(x, y) = (-2, 4)$

16. $y = 4x - 9,\ y = 3x - 6$

17. $(-1, 0),\ (3.5, 0)$

18. $4y + 3x - (3a + 4b) = 0$

19. $3x + 2y - 12 = 0$

20. $\overline{DE}:\ x = 4,\quad \overline{EF}:\ y = 1$

Glossary

Absolute value	an absolute value of a real number x, denoted by $\lvert x \rvert$ is a non-negative value of x. In other words, the absolute value of a number describes the distance from zero to that number on a number line without considering direction
Coordinates	a set of values, two or more numbers or sometimes a letter and a number which helps to show or locate the exact position of a point on a grid known as coordinate plane
Draughts	a checkerboard game for two players who each have 12 pieces; the object is to jump over and so capture the opponent's pieces
Formula	a group of symbols that make a mathematical statement
Geometry	a study concerned with properties of space such as distance, shape, size, and relative position of figures
Invariant	a feature or property that remains unchanged when a particular transformation is applied on it
Outline	a line that appears to bound an object
Pair of compass	a geometrical tool used to draw geometrical figures or shapes
Protractor	a geometrical instrument used to measure angles

Rotation	a motion of a certain space that preserves at least one point
Sequence	an arrangement of numbers in a particular order
Set squares	geometrical instruments usually triangular in shape, which are used to draw parallel lines, perpendicular lines, and make some standard angles
Solution set	a set of values that give a true statement when substituted into an equation

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