18.305 Lecture Notes: Stability of Mathieu's Equation via Classical and RG Methods

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November 4, 2015

Abstract

Instability and chaos are familiar phenomena in studies of nonlinear dynamical systems. What is perhaps surprising is that similar phenomena can appear in certain *linear* systems. An example is *Mathieu's equation*, a deceptively simple linear ODE that arises in various places in science and engineering. Analyzing the stability of Mathieu's equation was a successful application of the method of multiple scales in the late 20th century; as is true for so many analyses of this era, it was subsequently shown that the same conclusions may be derived—with arguably greater ease—using the 21st-century method of the renormalization group. In these notes we present both approaches, leaving it to readers to decide which is mechanically, philosophically and/or aesthetically superior.

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1 Mathieu's equation

Mathieu's equation is the following[1] seemingly innocuous linear ODE:

$$\frac{d^2u}{dt^2} + \Big[\omega_0^2 + \epsilon\omega_1^2\cos\omega_1 t\Big]u = 0. \tag{1}$$

Naïvely, this equation describes a simple harmonic oscillator whose frequency is itself a periodic function of time. There are multiple physical situations in which such an equation might arise:

- The one-dimensional motion of a mass on a spring in which the spring constant oscillates in time with frequency ω_1 around an average value which corresponds to a bare oscillator frequency of ω_0.
- The one-dimensional propagation of a time-harmonic electromagnetic field through a material whose dielectric permittivity varies sinusoidally in space.
- The quantum mechanics of a particle in a periodically varying potential, such as might be found in a one-dimensional crystal lattice.

We will be interested in the asymptotic (large-t) behavior of solutions to (1) in the small-ϵ limit, and specifically in this question: *For which sets of parameter values $\{\omega_0, \omega_1, \epsilon\}$ is Mathieu's equation **stable**—that is, characterized by the property that its solutions remain bounded[2] for all times?*

Our full, rigorous analysis of this question using the method of multiple scales and the renormalization group starts in Section 2, but before we get there let's pause to think about equation (1) from a couple of elementary perspectives.

[1] In class I wrote this equation with the prefactor of the $\cos\omega_1 t$ set equal simply to ϵ, in which case ϵ is a parameter with units of [frequency]^2. Equation (1) is just factoring out a quantity with units of [frequency]^2 to make ϵ dimensionless. Eventually we will work in units such that $\omega_1 = 1$.

[2] More technically, we will say that Mathieu's equation is stable for a set of parameter values if, given any real-valued pair of initial conditions $u(0), \dot{u}(0)$, there exists some constant U such that the solution to (1) satisfies $|u(t)| < U$ for all $t \in [-\infty, \infty]$. Note that this excludes solutions whose amplitude is exponentially decaying as $t \to \infty$, since those solutions will blow up as $t \to -\infty$.

1.1 Mathieu's Equation in Floquet Theory

Mathieu's equation is an instance of the branch of classical analysis known as *Floquet theory.* As we will see, Floquet theory actually doesn't help us much in the department of quantitative stability analysis, but we will present it here to put our work in context and obtain a useful language in which to characterize our results.

Mathieu's equation may be written in the form

$$\mathcal{L}(t)u(t) = 0 \tag{2}$$

where $\mathcal{L}(t)$ is the time-dependent differential operator

$$\mathcal{L}(t) = \frac{d^2}{dt^2} + \omega_0^2 + \epsilon\omega_1^2 \cos\omega_1 t.$$

Like any linear second-order differential equation, equation (2) has two linearly independent solutions; call them $u_1(t), u_2(t)$. Then any solution of (2) may be written uniquely as a linear combination of u_1, u_2.

Floquet theory begins by noticing that $\mathcal{L}$ is invariant under a certain discrete time translation:

$$\mathcal{L}\left(t + \frac{2\pi}{\omega_1}\right) = \mathcal{L}(t) \tag{3}$$

We can use this observation as follows: Consider the function v_1 obtained by time-translating $u_1(t)$ through a time offset Δt:

$$v_1(t) \equiv u_1\left(t - \frac{2\pi}{\omega_1}\right).$$

Operating on $v_1(t)$ with $\mathcal{L}$, we find

$$\mathcal{L}(t)v_1(t) = \mathcal{L}(t)u_1\left(t - \frac{2\pi}{\omega_1}\right)$$

Change time variables to $t' = t - \frac{2\pi}{\omega_1}$:

$$= \mathcal{L}\left(t' + \frac{2\pi}{\omega_1}\right)u_1(t')$$

Use (3):

$$\begin{aligned} &= \mathcal{L}(t')u_1(t') \\ &= 0 \end{aligned}$$

by the definition of u_1.

So this says that $v_1(t)$ is a solution of Mathieu's equation. But this means that it is a linear combination of $u_1(t), u_2(t)$, since *any* solution of Mathieu's equation is a linear combination of $u_1(t)$ and $u_2(2)$. Similarly, the function obtained by time-translating v_2 is also a linear combination of u_1 and u_2. In other words,

$$v_1(t) \equiv u_1\left(t + \frac{2\pi}{\omega_1}\right) = Au_1(t) + Bu_2(t)$$
$$v_2(t) \equiv u_2\left(t + \frac{2\pi}{\omega_1}\right) = Cu_1(t) + Du_2(t)$$

or, putting $\mathbf{u} = \binom{u_1}{u_2}$,

$$\mathbf{u}\left(t + \frac{2\pi}{\omega_1}\right) = \mathbf{\Lambda u}(t)$$

where $\mathbf{\Lambda}$ is some 2×2 matrix. If the eigenpairs of this matrix are $\{\lambda_p, \mathbf{w}_p\}$ for $p = 1, 2$ [where $\mathbf{w}_p = \binom{w_{p1}}{w_{p2}}$ is a 2-dimensional vector] we can construct linear combinations[3] of the u_n functions that are eigenfunctions of the time-translation operator, i.e.

$$f_1(t) = w_{11}^* u_1(t) + w_{12}^* u_2(t) \quad \Longrightarrow \quad f_1\left(t + \frac{2\pi}{\omega_1}\right) = \lambda_1 f_1(t).$$
$$f_2(t) = w_{21}^* u_1(t) + w_{22}^* u_2(t) \quad \Longrightarrow \quad f_2\left(t + \frac{2\pi}{\omega_1}\right) = \lambda_2 f_2(t).$$

It is conventional to write the eigenvalues of $\mathbf{\Lambda}$ in the form

$$\lambda_p = e^{\mu_p + i\beta_p} \quad (p = 1, 2)$$

where μ_p, β_p are real-valued; the real part of the exponent μ_p is called the *Lyapunov* exponent. The functions we have constructed above then satisfy, for any integer m, the translation properties

$$f_1\left(t + \frac{2m\pi}{\omega_1}\right) = e^{m\mu_1 + im\beta_1} f_1(t),$$
$$f_2\left(t + \frac{2m\pi}{\omega_1}\right) = e^{m\mu_2 + im\beta_2} f_2(t).$$

and we can phrase the question of stability in terms of the Lyapunov exponents: The solutions to Mathieu's equation will be stable (no explosive growth of amplitude as $t \to \pm\infty$) if both Lyapunov exponents are zero. (Alternatively, if we were to redefine our notion to stability to require only that solutions be exponentially decaying as $t \to +\infty$, then we would also allow Lyapunov exponents that are negative.)

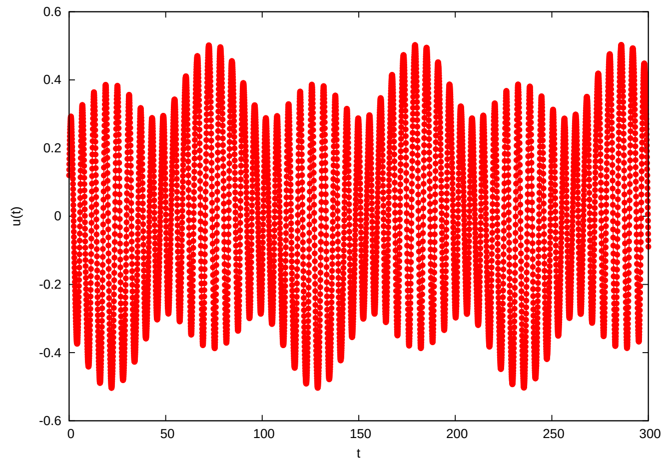
[3]Proof: In terms of the unitary matrix $\mathbf{W} \equiv \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}$ we have $\mathbf{\Lambda} = \mathbf{WLW}^\dagger$ where $\mathbf{L} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. Then the definition $\mathbf{f} = \binom{f_1}{f_2} \equiv \mathbf{W}^\dagger \mathbf{u}$ ensures ($\Delta \equiv 2\pi/\omega_1$)

$$\mathbf{f}(t + \Delta) = \mathbf{W}^\dagger \mathbf{u}(t + \Delta) = \mathbf{W}^\dagger \mathbf{WLW}^\dagger \mathbf{u}(t) = \mathbf{Lf}(t).$$

Although this is a nice language in which to discuss questions of stability, it doesn't say anything about how to determine the matrix $\mathbf{\Lambda}$ or its eigenvalues, and thus it doesn't actually help us much in determining the parameter values for which Mathieu's equation is stable; to accomplish that we must use more sophisticated analytical methods or numerical methods.

1.2 Numerical studies of Mathieu's equation

If you came across equation (1) in numerical work, you might think of using a numerical ODE integrator to investigate the solutions $u(t)$ for various different values of the parameters $\omega_0, \omega_1, \epsilon$. What you would find is that for some combinations of parameter values the solutions are perfectly well-behaved oscillations with bounded amplitudes, while for other combinations of parameter values the amplitudes of the solutions grow exponentially in time. For example, here are plots of two solutions with $\omega_1 = 1$, $\epsilon = 0.5$; in the upper plot we have $\omega_0 = 1.08$, and in the lower plot we have $\omega_0 = 1.03$.



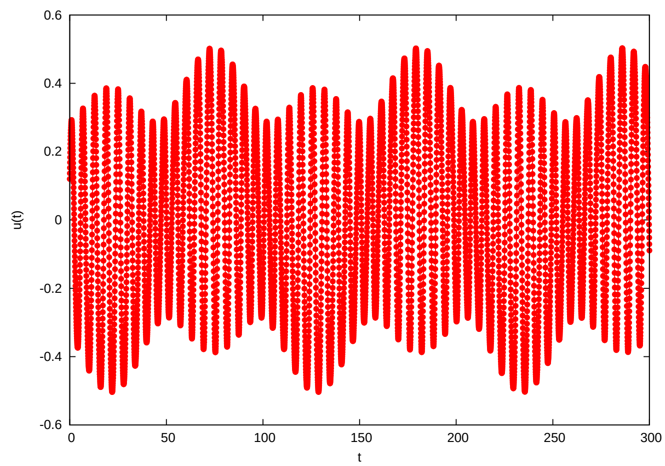


Figure 1: Top: Solution of Mathieu's equation with parameter values $\{\omega_0 = 1.08, \omega_1 = 1, \epsilon = 0.5.\}$ Bottom: Same, except now $\omega_0 = 1.03$

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In the upper case of Figure 1, the system oscillates forever with bounded amplitude; the amplitude itself varies sinusoidally at a frequency on the order of $20\times$ smaller than the oscillation frequency. (Although the figure shows the solution for a particular choice of initial conditions, with these parameter values we get bounded amplitudes for all choices of initial conditions.) In the language of the previous section, this is a case in which the Lyapunov exponents are zero.

On the other hand, in the lower case, the amplitude of the oscillation grows without bound, blowing up exponentially with a characteristic time scale on the order of 50× the oscillation period. This corresponds to a positive Lyapunov exponent.

How could we get such qualitatively different behavior from such a small change in parameter values? For what parameter values will the oscillations be of bounded amplitude, and for what values will they blow up? The following plot (generated by the JULIA code `PlotMathieuStability` included at the end of these lecture notes) shows the result of a numerical attempt to answer these questions. To generate this plot, for various choices of the parameter values $\{a, \epsilon\}$ (where $a \equiv \omega_0^2/\omega_1^2$) I numerically evolve Mathieu's equation subject to random initial conditions. If the amplitude of the solution blows up within a finite time, I place a blue dot at the point (a, ϵ) in the plot; otherwise I place a red dot there.

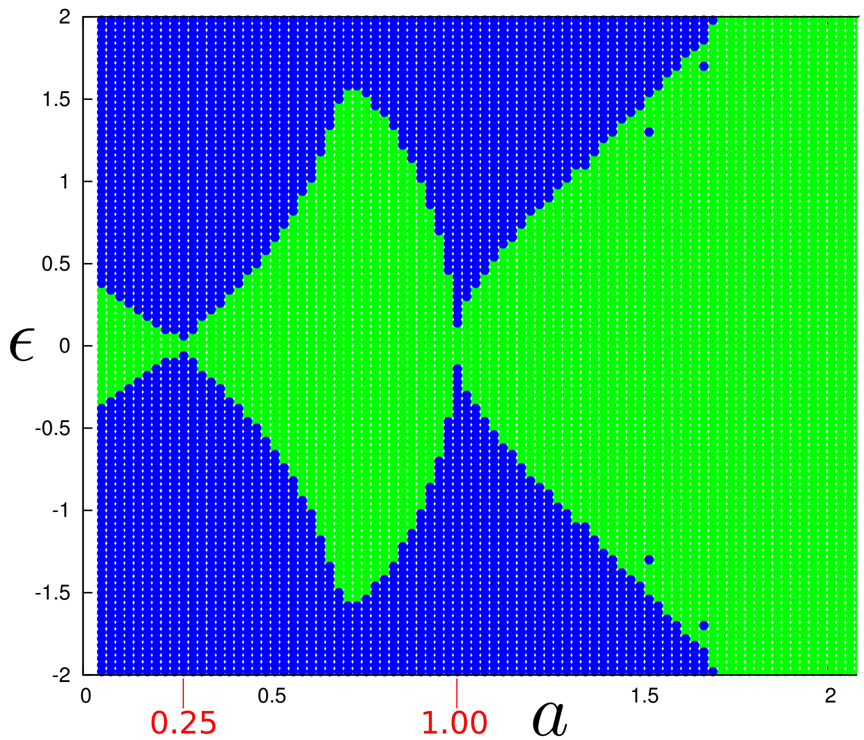

Figure 2: Stable and unstable regions of the (a, ϵ) plane for Mathieu's equation (1). Red (blue) regions correspond to stable (unstable) parameter values. Generated by the JULIA code listed at the end of these notes.

The key things to take away from this plot are the following:

- For $\epsilon = 0$ (the horizontal a axis) the points are green for all $a > 0$. This makes sense: With $\epsilon = 0$ equation (1) is just the simple harmonic oscillator equation, whose solutions (as long as $\omega_0^2 > 0$, i.e. $a > 0$) are sinusoids with amplitudes that remain bounded for all time.

- For most values of the a parameter (that is, for most combinations of the frequencies ω_0, ω_1) there is a nonzero range of small ϵ values around $\epsilon = 0$ for which Mathieu's equation is stable. For example, near the point $a = 0.75$ I can choose ϵ to be as large as ± 1.5 and still have stability. (Of course, for any value of a the equation will eventually be unstable if I take ϵ sufficiently large.)

- However, for certain special values of a, the stability window seems to close; in particular, for $a = \frac{1}{4}$ there appears to be *no* nonzero value of ϵ for which the equation is stable. Something similar appears to be happening at $a = 1$. (In fact, although the resolution of this plot is not adequate to show it, the stability window shrinks to zero for $a = 1$ just as it does for $a = \frac{1}{4}$.)

- Near the special values of a, the boundary between the stable and unstable regions is defined by certain *curves* in the $\{a, \epsilon\}$ plane, which we might think of as single-variable functions $a(\epsilon)$ [or $\epsilon(a)$], or, equivalently as the zero locus of a two-variable function $F(a, \epsilon)$. Can we write down analytical expressions for these functions?

Item (1) here is elementary, and item (2) makes intuitive sense. The origins of item (3) may not be immediately obvious, but in the following subsection we will see how to understand this phenomenon in fairly simple terms.

On the other hand, obtaining an answer to the question posed by item (4) will require the full apparatus of singular perturbation theory. We will discuss this in Section 2.

Note: For a higher-resolution version of the stability diagram of Figure 2, see Figure 11.11 of Bender & Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (available online from a link on the course website).

1.3 Mathieu's equation as an attempt to avoid resonant feedback

One way to think about the $\cos \omega_1 t$ term in Mathieu's equation is as an attempt to avoid resonant terms when we feed back a portion of the oscillator's output to its input. Thus, we have studied a number of ODEs of the form

$$\ddot{u} + \omega_0^2 u = \epsilon F[u, \dot{u}] \tag{4}$$

which we view as a simple harmonic oscillator driven by a forcing function that itself depends on the oscillator solution; we can think of this as a feedback mechanism, in which a small portion (proportional to ϵ) of the output of an oscillator system is fed back into the input after being modified by some operation F. Some of the possible operations we have considered include

$$\begin{aligned} F &= -u^3 && \text{(Duffing's equation)} \\ F &= -\dot{u}\left[1 - \tfrac{1}{3}\dot{u}^2\right] && \text{(Rayleigh's oscillator)} \\ F &= -\left[\omega_1^2 \cos \omega_1 t\right] u && \text{(Mathieu's equation)} \end{aligned}$$

Our approach to solving equations like (4) is to look for the solution in the form of a perturbation series,

$$u(t) = u_0(t) + \epsilon u_1(t) + \epsilon^2 u_2(t) + \cdots \tag{5}$$

and organize the resulting equation by powers of ϵ. This yields a hierarchy of equations that begins

$$\frac{d^2 u_0}{dt^2} + \omega_0^2 u_0 = 0 \tag{6a}$$

$$\frac{d^2 u_1}{dt^2} + \omega_0^2 u_1 = F[u_0, \dot{u}_0] \tag{6b}$$

$$\vdots \quad = \quad \vdots \tag{6c}$$

The solution to (6a) is a sinusoid at frequency ω_0. This typically means that the forcing function in (6b) is a sum of sinusoids at various frequencies including ω_0. Such resonant forcing terms result in terms in $u_1(t)$ whose amplitude grows linearly with t, eventually invalidating the perturbative expansion (5) because terms like ϵt are not small at times $t \gtrsim \frac{1}{\epsilon}$.

Mathieu's equation can be viewed as a new twist on this idea in which the effect of the modification $F[u]$ is to effect a *frequency shift* on the solution u, sending it off-resonance. Indeed, suppose we take the solution of (6a) to be

$$u_0(t) = A_0 e^{i\omega_0 t}.$$

Then, for the particular case of Mathieu's equation, (6b) reads

$$\frac{d^2 u_1}{dt^2} + \omega_0^2 u_1 = -A_0 \omega_1^2 \left[\cos \omega_1 t\right] e^{i\omega_0 t} \tag{7}$$

$$= -\frac{A_0 \omega_1^2}{2}\left[e^{i(\omega_0+\omega_1)t} + e^{i(\omega_0-\omega_1)t}\right] \tag{8}$$

The effect of the $\cos \omega_1 t$ term has been to shift the frequencies of the forcing function *off resonance*, thus avoiding solutions with unboundedly growing amplitudes and restoring the validity of a valid perturbation series (5).

Or, at least, that is the *hope*. A closer look at (8) reveals one way in which this naïve hope may break down: If $\omega_1 = 2\omega_0$, then the frequency of the second term on the RHS in (8) is $(\omega_0 - 2\omega_1) = -\omega_0$, which is now back *on* resonance! This suggests that curious things will happen when $\frac{\omega_0}{\omega_1} = \frac{1}{2}$, or when the parameter $a \equiv \frac{\omega_0}{\omega_1}$ takes the value $a = \frac{1}{4}$. This explains the special behavior visible at the point $a = \frac{1}{4}$ in Figure 2.

To understand the special behavior at $a = 1$, and to work out the functional forms of the curves bounding the stable and unstable regions near $a = 1/4$ and $a = 1$, requires the machinery of singular perturbation theory, to which we now turn.

2 Stability of Mathieu's equation

To investigate the stability of Mathieu's equation we begin by agreeing to measure time in units of $1/\omega_1$, in terms of which (1) reads

$$\frac{d^2u}{dt^2} + \Big[a + \epsilon \cos t\Big]u = 0 \tag{9}$$

where $a = \left(\frac{\omega_0}{\omega_1}\right)^2$. Our task is to determine the pairs of parameter values (a, ϵ) for which (9) is unstable.

Unstable values of a

The first thing to notice is that there are some values of a for which (9) is unstable for *any* nonzero ϵ. This is exemplified by the analysis of Section 1.3, in which we found the particular case $\omega_1 = 2\omega_0$ (or $a = \frac{1}{4}$) to be problematic. As it turns out, this is only one of an infinite sequence of a values for which the equation is unstable for any ϵ.

To understand this, recall that the instability at $a = \frac{1}{4}$ arose because—for that particular value of a—the frequency shift effected by the cos term had the effect of shifting $\pm\omega_0 \to \mp\omega_0$, thus yielding resonant terms on the RHS of the equation for u_1 in the perturbation series (5). The value $a = \frac{1}{4}$ is the unique value for which this phenomenon can occur at first order in perturbation theory.

However, other values of a yield instability at higher orders in perturbation theory. Indeed, consider pth order in perturbation theory: the RHS of the equation defining u_p in (5) will contain terms of the form $[\cos^p(\omega_1 t)]e^{i\omega_0 t}$, corresponding to sinusoids with frequencies ranging from $\omega_0 + p\omega_1$ down to $\omega_0 - p\omega_1$. We will have instability if this p-fold frequency shift has the effect of shifting $\pm\omega_0 \to \mp\omega_0$, i.e. if

$$\omega_0 - p\omega_1 = -\omega_0 \qquad \Longrightarrow \qquad a = \left(\frac{\omega_0}{\omega_1}\right)^2 = \frac{p^2}{4}, \qquad p = 1, 2, \cdots$$

This explains the closing of the stability window near $a = \frac{1}{4}$ and $a = 1$ in Figure (2), with the next such phenomenon taking place at $a = \frac{9}{4}$.

Behavior near unstable values of a

Thus, precisely at the values $a = \left(\frac{p^2}{4}\right)$ for integer p, Mathieu's equation is unstable for any finite ϵ, as we see[4] in Figure 2. However, if we move slightly to the right or left of one of these points on the a axis, then Figure 2 seems to be saying that there is a finite stability window—that is, for each $a \neq \frac{p^2}{4}$ there is a nonzero value $\epsilon^{\max}(a) > 0$ with the property that Mathieu's equation with

[4]If the low resolution of my numerical plot in Figure 2 does not convince you, I encourage you to look at higher-resolution versions of this figure available in Bender and Orszag (reference cited above) or elsewhere

parameters $\{a, \epsilon\}$ is stable for $|\epsilon| < \epsilon^{\max}(a)$. We would like to determine the function[5] $\epsilon^{\max}(a)$. [In what follows I will drop the superscript "max" and just denote this function by $\epsilon(a)$.]

Since we know $\epsilon(a)$ has the value 0 at $a = \frac{p^2}{4}$, we might look for a Taylor series around $a = \frac{p^2}{4}$ of the form

$$\epsilon(a) = C_1\left(a - \frac{p^2}{4}\right) + C_2\left(a - \frac{p^2}{4}\right)^2 + \cdots \tag{10}$$

with the quantity $\left(a - \frac{p^2}{4}\right)$ considered small. We could then try to figure out the values of the constants $C_1, C_2, \cdots$ that seem to lie at the threshold between stable and unstable behavior.

This turns out to be a perfectly valid way to analyze stability in the vicinity of $a = \frac{1}{4}$, as I encourage you to work out on your own. However, in the vicinity of other points—such as $a = 1$—this procedure breaks down. (For example, in the vicinity of $a = 1$ the function $\epsilon(a)$ behaves like $\sim \frac{1}{\sqrt{1-a}}$, which has no Taylor expansion around $a = 1$.)

A procedure that turns out to work more generally is to invert the sense of (10) and solve instead for the function $a(\epsilon)$. After all, a curve in the $\{a, \epsilon\}$ plane may be described—at least locally—equally well by a rule of the form $\epsilon = \epsilon(a)$ or by a rule of the form $a = a(\epsilon)$; the latter just corresponds to rotating the coordinate axes by 90 degrees so that we think of ϵ as the independent variable.)

Thus we think of ϵ as the independent variable and suppose a is given by a series of the form

$$a = a_0 + \epsilon a_1 + \epsilon^2 a_2 + \cdots \tag{11}$$

where a_0 is a number of the form $a_0 = \frac{p^2}{4}$; the details of the analysis will depend on which of these numbers we consider.

For a given value of a_0, the coefficients a_1, a_2, will now be determined by playing a game of perturbative analysis analogous to several similar games we have played previously. As we will see, once $a_0 = \frac{p^2}{4}$ is fixed, the coefficient $a_1, a_2, \cdots, a_{p-1}$ will all vanish; in other words, the perturbation series for a in the vicinity of $\frac{1}{4}$ and 1 take the form

$$\text{Near } a = \frac{1}{4}: \qquad a(\epsilon) = \frac{1}{4} + \epsilon a_1 + \epsilon^2 a_2 + \cdots$$

$$\text{Near } a = 1: \qquad a(\epsilon) = 1 + \epsilon^2 a_2 + \epsilon^3 a_3 + \cdots$$

The value of the nonvanishing a_p coefficients can be determined using the machinery of singular perturbation theory, including the techniques of the method of multiple scales and the renormalization group.

[5] This function will be precisely the function defining the *curve* in the $\{a, \epsilon\}$ plane separating the stable and unstable regions, as discussed previously. Of course, there are 4 different branches of this curve depending on whether we venture from $a = \frac{p^2}{4}$ in the direction of positive or negative a and positive or negative ϵ.

Substituting (11) and (5) into (1) and separately demanding the vanishing of the coefficient of each power of ϵ yields the following hierarchy of equations:

$$\ddot{u}_0 + a_0 u_0 = 0 \tag{12a}$$

$$\ddot{u}_1 + a_0 u_1 = -\Big[a_1 + \cos t\Big] u_0 \tag{12b}$$

$$\ddot{u}_2 + a_0 u_2 = -\Big[a_1 + \cos t\Big] u_1 - a_2 u_0 \tag{12c}$$

$$\ddot{u}_3 + a_0 u_3 = -\Big[a_1 + \cos t\Big] u_2 - a_3 u_0 - a_2 u_1 \tag{12d}$$

$$\vdots \;=\; \vdots \tag{12e}$$

The solution[6] to (12a) is

$$u_0 = A_0 e^{i\alpha_0 t} + \text{CC}, \qquad \alpha_0 \equiv \sqrt{a_0} \tag{13}$$

whereupon equation (12b) reads

$$\ddot{u}_1 + a_0 u_1 = -a_1 A_0 e^{i\alpha_0 t} - \frac{A_0}{2}\Big[e^{i\alpha_1} + e^{i\alpha_{-1}}\Big] + \text{CC} \tag{14}$$

where I defined

$$\alpha_p \equiv \alpha_0 + p.$$

Our analysis now bifurcates depending on the value of a_0, i.e. near which of the points $\{\frac{1}{4}, 1, \frac{9}{4}, \cdots\}$ we are working:

- If $a_0 = \frac{1}{4}$, then the RHS of equation (14) contains resonant terms coming from the $\omega_{\pm 1}$ sector, and we must adjust the value of a_1 to eliminate them.
- If $a_0 \neq \frac{1}{4}$, then the only resonant term in (14) is the one proportional to a_1; thus the simple choice $a_1 = 0$ suffices to avoid resonance at this order in perturbation theory.

The first case is treated in Section 2.1; here we press on with the second case.

Analysis for $a_0 \neq \frac{1}{4}$

If $a_0 \neq \frac{1}{4}$, then the choice $a_1 = 0$ eliminates resonant terms from the RHS of (14). With this choice, (14) reads

$$\ddot{u}_1 + a_0 u_1 = -\frac{A_0}{2}\Big[e^{i\alpha_1} + e^{i\alpha_{-1}}\Big] + \text{CC}$$

[6] In what follows I will introduce a proliferation of shorthand notation to keep the equations as concise as possible. The starting point is $\alpha_0 = \sqrt{a_0}$, not to be confused with the symbol $\omega_0 = \sqrt{a}$. The symbol ω_0 is actually retired in this section in favor of $a = a_0 + \epsilon a_1 + \epsilon^2 a_2 + \cdots$. The quantity $\alpha_0 = \sqrt{a_0}$ is not a new quantity, but is introduced here because writing $\sqrt{a_0}$ all the time gets cumbersome.

with solution[7]

$$u_1 = -\frac{A_0}{2}\left[\frac{1}{L_1}e^{i\alpha_1 t} + \frac{1}{L_{-1}}e^{i\alpha_{-1}t}\right] + \gamma A_0 e^{i\alpha_0 t} + \text{CC} \tag{15}$$

where I defined[8]

$$\begin{aligned} L_p &\equiv a_0 - \alpha_p^2 \\ &= \alpha_0^2 - (\alpha_0 + p)^2 \\ &= -2p\alpha_0 - p^2. \end{aligned}$$

The RHS of equation (12c) then reads (remembering $a_1 = 0$)

$$\begin{aligned} \ddot{u}_2 + a_0 u_2 = \frac{A_0}{4}&\left[\frac{1}{L_1}\left(e^{i\alpha_2 t} + e^{i\alpha_0 t}\right) + \frac{1}{L_{-1}}\left(e^{i\alpha_0 t} + e^{i\alpha_{-2}t}\right)\right] \\ &- \frac{\gamma A_0}{2}\left[e^{i\alpha_1 t} + 1\right] - a_2 A_0 e^{i\alpha_0 t} + \text{CC} \end{aligned} \tag{16}$$

[7]In (19), the term with amplitude γA_0 is the homogeneous solution, added to ensure the satisfaction of boundary conditions. I don't want to bother to impose boundary conditions and determine the value of the coefficient here right now; however, whatever it is I know it's going to be proportional to A_0, so I have written in that dependence explicitly with γ a number to be determined. My goal is to avoid having to determine γ as long as possible and possibly forever; by giving it a name here I can achieve the dual goals of **(a)** not spending any time right now figuring out its value, and **(b)** reminding myself that it's there for future reference in case I run into situations where having forgotten to include this term will yield expressions that don't make sense.

[8]Note that $L_p = a_0 - \alpha_p^2$ is just the usual denominator I get when solving the inhomogeneous linear ODE $\left(\frac{d^2}{dt^2} + a_0\right)u = e^{i\alpha_p t}$ for an off-resonance forcing function.

2.1 Analysis for $a_0 = \frac{1}{4}$

For $a_0 = \frac{1}{4}$ we have $\alpha_0 = \frac{1}{2}$ and $\alpha_{-1} = -\frac{1}{2} = -\alpha_0$. Then equation (14) reads[9]

$$\ddot{u}_1 + a_0 u_1 = -\left[a_1 A_0 + \frac{1}{2}A_0^*\right] e^{i\frac{t}{2}} - \frac{A_0}{2} e^{i\frac{3t}{2}} + \text{CC} \tag{17}$$

I can now proceed in one of two ways: using the method of multiple scales or using the method of the renormalization group.

Determination of a_1 by method of multiple scales

In the method of multiple scales, we think of the various terms in (5) as functions of both t and the new time variable $\tau = \epsilon t$; this has the effect of imparting τ dependence to the quantity A_0 in (13) and (15). The RHS of equation (15) is augmented by a new term $-2\frac{\partial^2 u_0}{\partial\tau\partial t}$ and reads

$$\ddot{u}_1 + a_0 u_1 = -\left[a_1 A_0 + \frac{1}{2}A_0^*\right] e^{i\frac{t}{2}} - \frac{A_0}{2} e^{i\frac{3t}{2}} - \underbrace{2\frac{\partial^2 u_0}{\partial\tau\partial t}}_{i\frac{\partial A_0}{\partial\tau}e^{i\frac{t}{2}}} + \text{CC}$$

or

$$\ddot{u}_1 + a_0 u_1 = -\left[a_1 A_0 + \frac{1}{2}A_0^* + i\frac{\partial A_0}{\partial\tau}\right] e^{i\frac{t}{2}} - \frac{A_0}{2} e^{i\frac{3t}{2}} + \text{CC}.$$

Following the spirit of the method of multiple scales, I now adjust the τ dependence of A_0 to make the coefficient of the resonant terms vanish. This yields the equation

$$\frac{\partial A_0}{\partial\tau} = i a_1 A_0 + \frac{i}{2}A_0^* \tag{18}$$

or, writing $A_0 = X_0 + iY_0$ and separating real and imaginary parts,

$$\frac{\partial X}{\partial\tau} = \left(a_1 - \frac{1}{2}\right) Y, \qquad \frac{\partial Y}{\partial\tau} = \left(a_1 + \frac{1}{2}\right) X. \tag{19}$$

Combining these into a single second-order differential equation for X yields

$$\frac{\partial^2 X}{\partial\tau^2} = \left(a_1^2 - \frac{1}{4}\right) X$$

with solutions

$$X(\tau) = e^{\pm\left[a_1^2 - \frac{1}{4}\right]^{1/2}\tau}.$$

[9] Caution: You have to look inside the CC to find one of the contributions to the $e^{it/2}$ sinusoid here. This arises because $\alpha_{-1} = -\alpha_0$ so the complex conjugate of the $e^{i\alpha_{-1}t}$ term is a term involving $e^{+i\alpha_0 t} = e^{it/2}$.

Restoring $\tau = \epsilon t$, this means that the envelope of the oscillatory solution to the ODE will be given by

$$X(t) = e^{\pm\left[a_1^2 - \frac{1}{4}\right]^{1/2}\epsilon t}. \tag{20}$$

I now argue thusly:

- If $a_1 > \frac{1}{2}$, equation (18) describes an envelope that exponentially grows or decays in time.
- If $a_1 < \frac{1}{2}$, equation (18) describes an envelope that oscillates sinusoidally in time.

Evidently the value $a_1 = \frac{1}{2}$ is the separatrix that delineates stable from unstable behavior, and thus I have picked off the first nonvanishing coefficient in the function defining the stability boundary curve:[10]

$$a(\epsilon) = \frac{1}{4} + \frac{1}{2}\epsilon + O(\epsilon^2)$$

Determination of a_1 by RG techniques

Alternatively, I could use the method of the renormalization group to derive the same conclusions. In this approach, I solve (15)—with the resonant terms in the RHS forcing function present—to obtain[11]

$$u_1(t) = i\left[a_1 A_0 + \frac{1}{2}A_0^*\right](t - t_0)e^{i\frac{t}{2}} + \frac{A_0}{6}e^{i\frac{3}{2}t} + \gamma A_0 e^{i\frac{t}{2}} \tag{21}$$

where t_0 is the usual time in the distant past at which I chose to impose the boundary conditions. The full solution (5) to first order in ϵ then reads

$$u(t) = \left\{A_0 + i\epsilon\left[a_1 A_0 + \frac{1}{2}A_0^*\right](t - t_0)\right\}e^{i\frac{t}{2}} + \epsilon\frac{A_0}{6}e^{i\frac{3}{2}t} + \epsilon\gamma A_0 e^{i\frac{t}{2}} + O(\epsilon^2) + \text{CC} \tag{22}$$

Now invoke the usual RG philosophy:

- The term in (20) that grows like $\epsilon(t - t_0)$ will invalidate the perturbative nature of the solution for large times on the order of $(t - t_0) \sim \frac{1}{\epsilon}$.
- But it seems somehow wrong that anything in my solution should be depending so sensitively on the point at which I imposed the initial conditions; I think of that point as being in the distant past, a murky era enshrouded in mystery that is difficult to probe experimentally. Instead, what I have before me is the oscillator solution at times in the present

[10] The values of numerical coefficients such as $a_1 = \frac{1}{2}$ are dependent on the conventions I used when writing down (1); some authors, including Bender and Orszag, write the equation with my ϵ replaced by 2ϵ, in which case the coefficient a_1 would come out equal to 1 instead of my $\frac{1}{2}$.

[11] The term with amplitude γA_0 is the homogeneous solution; see footnote regarding this in the discussion of the $a \neq \frac{1}{4}$ case above.

(not the distant past), and it should be possible to understand the growth or decay of the oscillator's envelope based solely on measurements of this temporally local behavior.

- Thus I imagine replacing the distant-past time offset t_0 with a time τ that I expect to be closer to the present time t. Of course, when I replace t_0 with τ I have to adjust A_0 appropriately; this promotes $A_0 \to A(\tau)$ into a function of τ.

Making the replacements $\{t_0, A_0\} \to \{\tau, A(\tau)\}$ in (20), I find

$$u(t) = \left\{ A + i\epsilon \left[a_1 A + \frac{1}{2} A^* \right] (t - \tau) \right\} e^{i\frac{t}{2}} + \epsilon \frac{A}{6} e^{i\frac{3}{2}t} + \epsilon\gamma A e^{i\frac{t}{2}} + O(\epsilon^2) + \text{CC} \quad (23)$$

But nothing can depend on my arbitrary time offset τ, whereupon I impose the good ol' RG equation:

$$\boxed{\frac{\partial u}{\partial \tau} = 0}$$

Differentiating (21) and recalling that the time dependence of A_0 starts at order ϵ yields

$$\frac{\partial A}{\partial \tau} = i\epsilon \left[a_1 A + \frac{1}{2} A^* \right]$$

Putting $A = X + iY$, solving for $X(\tau)$ as we did in the MOMS case, and putting $\tau = t$ then recovers the result of equation (18), i.e.

$$X(t) = e^{\pm\left[a_1^2 - \frac{1}{4}\right]^{1/2} \epsilon t}. \quad (24)$$

Thus the RG analysis recovers the result: to first order in ϵ the stability boundary curve in the vicinity of $\frac{1}{4}$ is given by

$$a(\epsilon) = \frac{1}{4} + \frac{1}{2}\epsilon + a_2\epsilon^2 + O(\epsilon^3)$$

Determination of a_2 by RG techniques

A JULIA code for numerical studies of Mathieu stability

```
##################################################
# julia code for investigating stability of solutions
# of Mathieu's equation
# homer reid 11/2015
##################################################
include("IntegrateODE.jl");

#(u,tV,uV)=IntegrateODE( (t,u) -> fMathieu(t,u,w0,w1,Eps),
#                        u0, t0, tMax, dt);

##################################################
# RHS function for Mathieu's equation written as
# a first-order system d \vec{u}/dt = f[\vec{u}]
##################################################
function fMathieu(t, u, w0, w1, Eps)
  [ u[2]; -(w0*w0 + Eps*cos(w1*t))*u[1] ];
end

##################################################
# test the stability of the Mathieu equation
#   d^2 u/dt^2 + [w0^2 + Eps*cos(w1 t)]*u = 0
# by returning false if unstable, true if stable
##################################################
##################################################
function TestMathieuStability(w0, w1, Eps)

  TLong  = 2.0*pi / min(w0,w1,abs(w0-w1));
  TShort = 2.0*pi / max(w0,w1,abs(w0+w1));

  t0=0.0;
  tMax=10.0*TLong;
  dt = TShort/100.0;   # time step
  u0=rand(2);          # random initial conditions
  u0[2]*=w0;

  # integrate the ODE over 20 periods of the slower
  # oscillation using 4th-order Runge-Kutta; call it
  # unstable if the amplitude ever exceeds 10x its
  # initial value
  t=t0;
```

```
  u=u0;
  Threshold=10.0 * abs(u0[1]);
  while t < tMax
    du=RK4Step( (t,u)->fMathieu(t,u,w0,w1,Eps),t,u,dt);
    t+=dt;
    u+=du;
    if abs(u[1]) > Threshold
      return false;
    end
  end

  return true;

end

##################################################
##################################################
##################################################
function PlotMathieuStability()

  w1=1.0;

  aMin=0.05;
  aMax=2.5;
  aStep=(aMax-aMin)/100.0;
  aVector=collect(aMin:aStep:aMax);

  EpsMin=0.0;
  EpsMax=+2.0;
  EpsStep=(EpsMax-EpsMin)/100.0;
  EpsVector=collect(EpsMin+EpsStep:EpsStep:EpsMax);

  OutFile=open("MathieuStability.dat","w");
  for a in aVector, Eps in EpsVector

    Stable=TestMathieuStability(sqrt(a)*w1, w1, Eps);
    if (!Stable) # second opinion to avoid false positives
     Count=-1;
     for n=1:6
       nStable=TestMathieuStability(sqrt(a)*w1, w1, Eps);
       Count += (nStable ? 1 : -1);
     end
     Stable=(Count>0);
    end
```

```
    @printf(OutFile,"%e %e %i\n",a,Eps,Stable ? 1 : 0);
    @printf("%e %e %i\n",a,Eps,Stable ? 1 : 0);
  end
  close(OutFile);

end
```