The following notebook demonstrates the use of recurrence plots in determining if and when multimode forcing characteristics are damped out in a non-linear damped oscillator of the form x'' + δ x' + x + c x^3 + q x^2. The value of q is maintained as 0.0 and c is 0.1. This is a candidate for Poincare-Lindstedt method. For now, the qualitative results and trends obnserved here will be extrapolated to analyse the fourth order film evolution equation with and without a porous substrate.

```
In[109]:= Needs["HierarchicalClustering`"]
SetOptions[{Plot, MatrixPlot}, ImageSize → 250 {1, 1}];
{tmax = 200, δ = 0.05, c = 0.1, q = 1.}

Out[111]= {200, 0.05, 0.1, 1.}
```

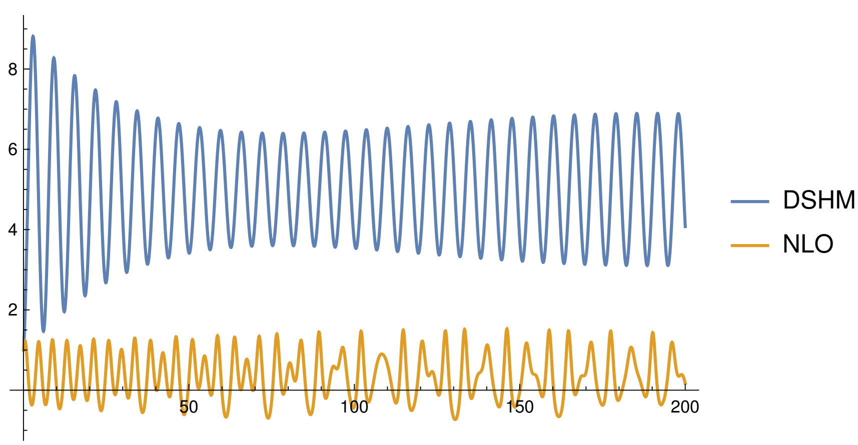
An experiment on simple harmonic motion with a multimode forcing function vs a non-linear oscillator with the same multimode forcing function.

```
In[2]:= shm = NDSolveValue[{x''[t] + x[t] + 0.05 x'[t] == 5 + 0.1 Sin[t / 100] Sin[t],
    x[0] == 1, x'[0] == 1}, x, {t, 0, tmax}, Method → "LSODA"]
nlo = NDSolveValue[{x''[t] + x[t] + 0.01 x'[t] + 0.5 (x[t]) ^3 + (x[t]) ^2 ==
     1 + Sin[t / 100] Sin[t], x[0] == 1, x'[0] == 1}, x, {t, 0, tmax}, Method → "LSODA"]
Plot[{(*Cos[t]+Sin[t],*)shm[t], nlo[t]}, {t, 0, tmax},
 PlotLegends → {"DSHM", "NLO"}, ImageSize → Medium]

Out[2]= InterpolatingFunction[ Domain: {{0., 200.}} Output: scalar ]

Out[3]= InterpolatingFunction[ Domain: {{0., 200.}} Output: scalar ]
```

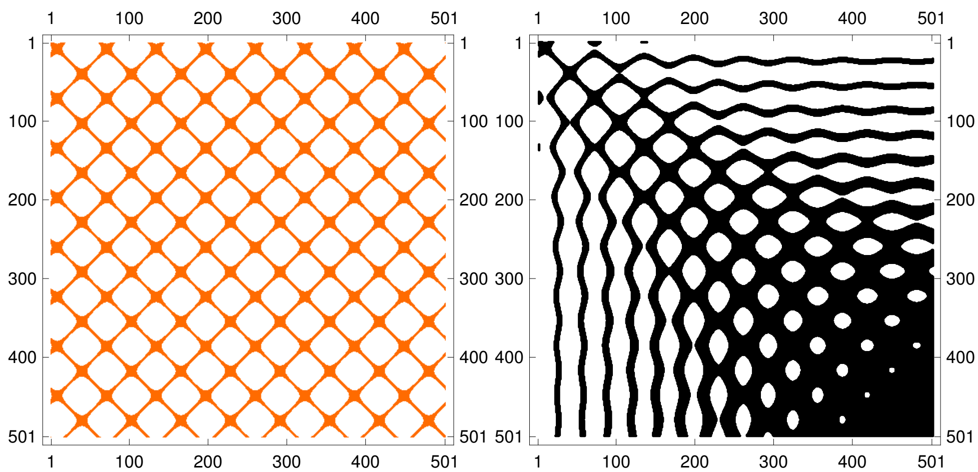
Out[4]=

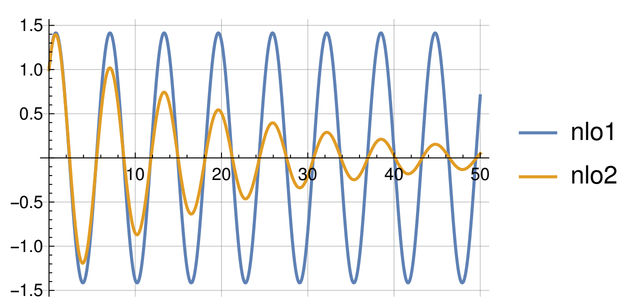


An undamped and a damped oscillator's recurrence plots are compared. No forcing function is used.

In[112]:=
```
Module[{tmax = 50, δ, c = 0., q = 0, x, t, nlo1, nlo2},
 δ = 0.0;
 nlo1 = NDSolveValue[{x''[t] + x[t] + δ x'[t] + c (x[t]) ^3 + q (x[t]) ^2 == 0,
    x[0] == 1, x'[0] == 1}, x, {t, 0, tmax}, Method → "LSODA"];
 δ =
  0.1;
 nlo2 = NDSolveValue[{x''[t] + x[t] + δ x'[t] + c (x[t]) ^3 + q (x[t]) ^2 == 0,
    x[0] == 1, x'[0] == 1}, x, {t, 0, tmax}, Method → "LSODA"];
 Plot[{nlo1[t], nlo2[t]}, {t, 0, tmax}, GridLines -> Automatic,
   PlotLegends → {"nlo1", "nlo2"}]
  With[{stepSize = 0.1, end = tmax, nn = 0.1},
   MatrixPlot[UnitStep[nn - DistanceMatrix[nlo1[Range[0, end, stepSize]]]]]
    MatrixPlot[UnitStep[nn - DistanceMatrix[nlo2[Range[0, end, stepSize]]]],
     ColorFunction → "Monochrome"]
  ]]
```

Out[112]=

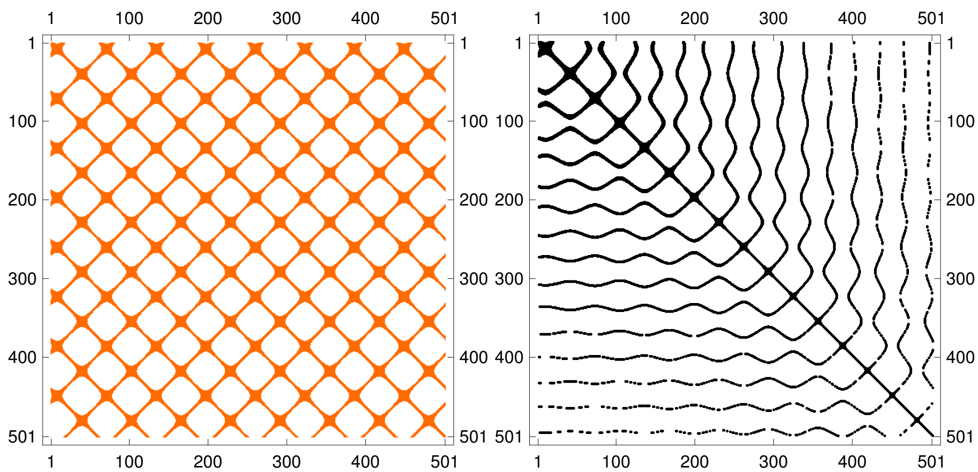


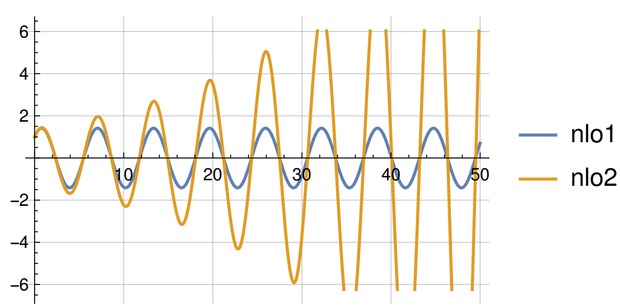


An undamped oscillator’s recurrence plot is compared with that of an unstable oscillator

In[115]:=
```
Module[{tmax = 50, δ, c = 0., q = 0, x, t, nlo1, nlo2},
 δ = 0.;
 nlo1 = NDSolveValue[{x''[t] + x[t] + δ x'[t] + c (x[t]) ^ 3 + q (x[t]) ^ 2 == 0,
    x[0] == 1, x'[0] == 1}, x, {t, 0, tmax}, Method → "LSODA"];
 δ =
  -0.1;
 nlo2 = NDSolveValue[{x''[t] + x[t] + δ x'[t] + c (x[t]) ^ 3 + q (x[t]) ^ 2 == 0,
    x[0] == 1, x'[0] == 1}, x, {t, 0, tmax}, Method → "LSODA"];
 Plot[{nlo1[t], nlo2[t]}, {t, 0, tmax}, GridLines -> Automatic,
   PlotLegends → {"nlo1", "nlo2"}]
  With[{stepSize = .1, end = tmax, nn = 0.1},
   MatrixPlot[UnitStep[nn - DistanceMatrix[nlo1[Range[0, end, stepSize]]]]]
    MatrixPlot[UnitStep[nn - DistanceMatrix[nlo2[Range[0, end, stepSize]]]],
     ColorFunction → "Monochrome"]
  ]]
```

Out[115]=

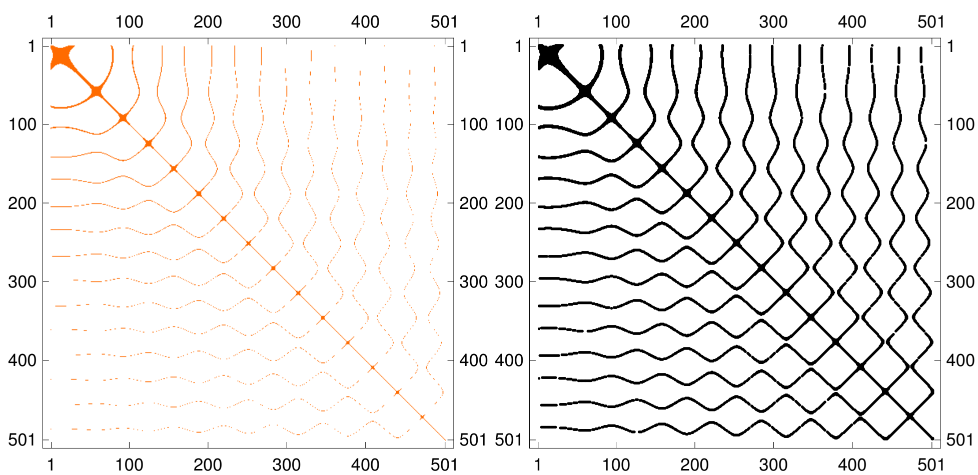


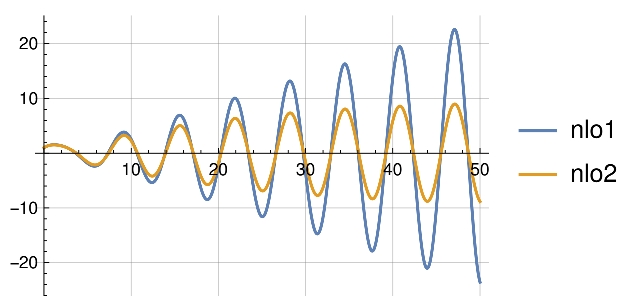


An undamped and a damped oscillator are compared. A single mode forcing function is used. The recurrence plot shows nearly similar characteristic. The undamped oscillator grows as a result of the forcing function.

```
In[120]:= Module[{tmax = 50, δ, c = 0., q = 0, x, t, nlo1, nlo2},
  δ = 0.;
  nlo1 = NDSolveValue[{x''[t] + x[t] + δ x'[t] + c (x[t])^3 + q (x[t])^2 == Sin[t],
     x[0] == 1, x'[0] == 1}, x, {t, 0, tmax}, Method → "LSODA"];
  δ =
   0.1;
  nlo2 = NDSolveValue[{x''[t] + x[t] + δ x'[t] + c (x[t])^3 + q (x[t])^2 == Sin[t],
     x[0] == 1, x'[0] == 1}, x, {t, 0, tmax}, Method → "LSODA"];
  Plot[{nlo1[t], nlo2[t]}, {t, 0, tmax}, GridLines -> Automatic,
    PlotLegends → {"nlo1", "nlo2"}]
   With[{stepSize = .1, end = tmax, nn = 0.1},
    MatrixPlot[UnitStep[nn - DistanceMatrix[nlo1[Range[0, end, stepSize]]]]]
     MatrixPlot[UnitStep[nn - DistanceMatrix[nlo2[Range[0, end, stepSize]]]],
      ColorFunction → "Monochrome"]
   ]]
```

Out[120]=

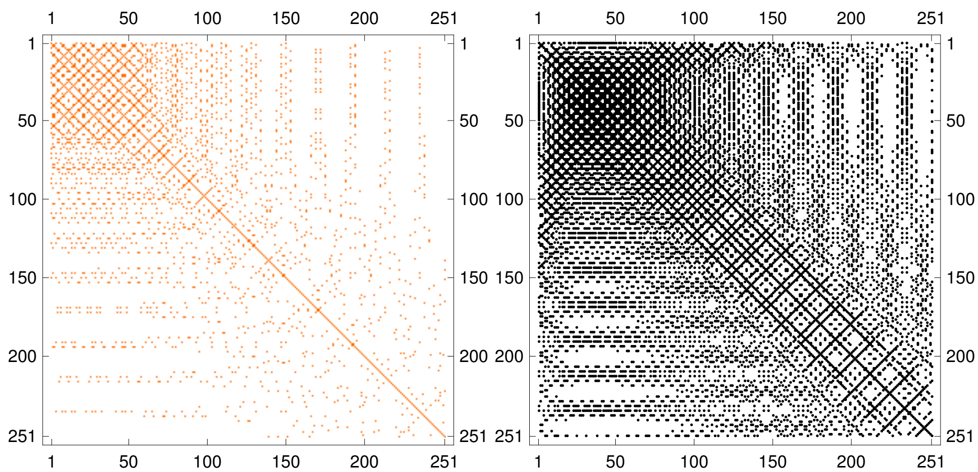


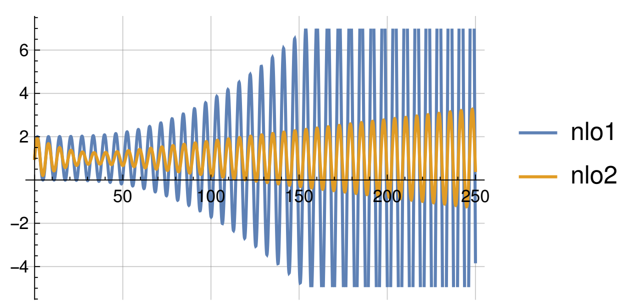


Again, an undamped oscillator and damped oscillators with a multimode forcing function are compared through recurrence plots.

In[127]:=
```
Module[{tmax = 250, δ, c = 0., q = 0, x, t, nlo1, nlo2},
 δ = 0.;
 nlo1 = NDSolveValue[{x''[t] + x[t] + δ x'[t] + c (x[t]) ^3 + q (x[t]) ^2 ==
     1 + Sin[t / 1000] Sin[t], x[0] == 1, x'[0] == 1}, x, {t, 0, tmax}, Method → "LSODA"];
 δ =
  0.1;
 nlo2 = NDSolveValue[{x''[t] + x[t] + δ x'[t] + c (x[t]) ^3 + q (x[t]) ^2 ==
     1 + Sin[t / 1000] Sin[t], x[0] == 1, x'[0] == 1}, x, {t, 0, tmax}, Method → "LSODA"];
 Plot[{nlo1[t], nlo2[t]}, {t, 0, tmax}, GridLines -> Automatic,
   PlotLegends → {"nlo1", "nlo2"}]
  With[{stepSize = 1, end = tmax, nn = 0.1},
   MatrixPlot[UnitStep[nn - DistanceMatrix[nlo1[Range[0, end, stepSize]]]]]
    MatrixPlot[UnitStep[nn - DistanceMatrix[nlo2[Range[0, end, stepSize]]]],
     ColorFunction → "Monochrome"]
  ]]
```

Out[127]=

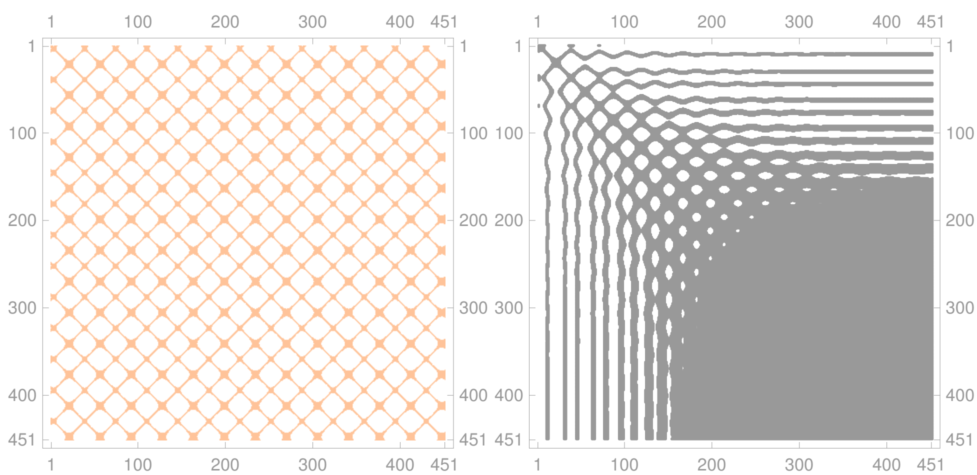


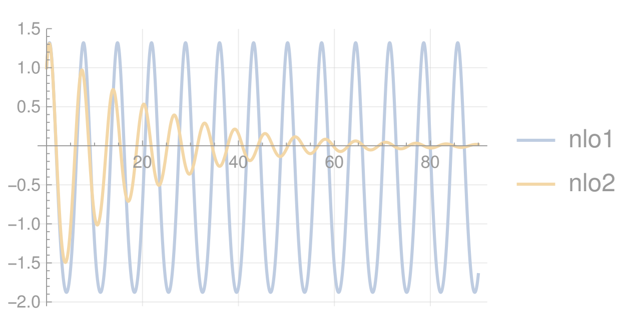


Non linear versions of the undamped and damped oscillators without a forcing function are compared. x'' + x + δ x' + c x^3 + q x^2=0 with small q. The non-linear damped oscillator has its amplituded completely damped. At q=0.5, the equation becomes stiff with c=0.

```
Module[{tmax = 90, δ, c = 0., q = 0.1, x, t, nlo1, nlo2},
 δ = 0.;
 nlo1 = NDSolveValue[{x''[t] + x[t] + δ x'[t] + c (x[t]) ^3 + q (x[t]) ^2 == 0,
    x[0] == 1, x'[0] == 1}, x, {t, 0, tmax}, Method → "LSODA"];
 δ =
  0.1;
 nlo2 = NDSolveValue[{x''[t] + x[t] + δ x'[t] + c (x[t]) ^3 + q (x[t]) ^2 == 0,
    x[0] == 1, x'[0] == 1}, x, {t, 0, tmax}, Method → "LSODA"];
 Plot[{nlo1[t], nlo2[t]}, {t, 0, tmax}, GridLines -> Automatic,
   PlotLegends → {"nlo1", "nlo2"}]
  With[{stepSize = 0.2, end = tmax, nn = 0.1},
   MatrixPlot[UnitStep[nn - DistanceMatrix[nlo1[Range[0, end, stepSize]]]]]
    MatrixPlot[UnitStep[nn - DistanceMatrix[nlo2[Range[0, end, stepSize]]]],
     ColorFunction → "Monochrome"]
  ]]
```

Out[135]=

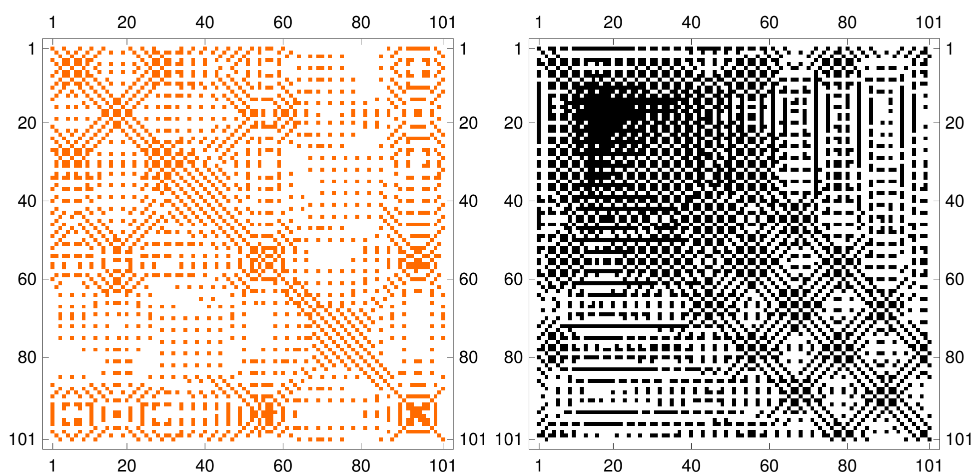


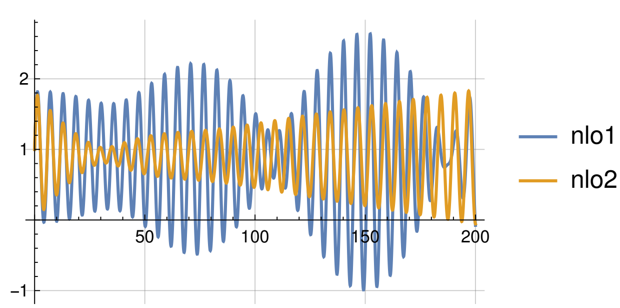


The non-linear undamped and damped oscillators with q=0.1 are now compared with a multimode forcing function. Multimode characteristics are seen for long times (t->100 or more). With tmax ≈500, stiffness sets in. The following code, with c=0, q=0.1, should be run for tmax=200.

In[146]:=
```
Module[{tmax = 200, δ, c = 0., q = 0.1, x, t, nlo1, nlo2},
 δ = 0.;
 nlo1 = NDSolveValue[{x''[t] + x[t] + δ x'[t] + c (x[t]) ^3 + q (x[t]) ^2 == 1 +
      Sin[t / 1000] Sin[t / 1], x[0] == 1, x'[0] == 1}, x, {t, 0, tmax}, Method → "LSODA"];
 δ =
  0.1;
 nlo2 = NDSolveValue[{x''[t] + x[t] + δ x'[t] + c (x[t]) ^3 + q (x[t]) ^2 == 1 +
      Sin[t / 1000] Sin[t / 1], x[0] == 1, x'[0] == 1}, x, {t, 0, tmax}, Method → "LSODA"];
 Plot[{nlo1[t], nlo2[t]}, {t, 0, tmax}, GridLines -> Automatic,
   PlotLegends → {"nlo1", "nlo2"}]
  With[{stepSize = 2, end = tmax, nn = 0.1},
   MatrixPlot[UnitStep[nn - DistanceMatrix[nlo1[Range[0, end, stepSize]]]]]
    MatrixPlot[UnitStep[nn - DistanceMatrix[nlo2[Range[0, end, stepSize]]]],
     ColorFunction → "Monochrome"]
  ]]
```

Out[146]=





A larger value of δ damps the mixed modes at an earlier time period.

In[147]:=
```
Module[{tmax = 200, δ, c = 0., q = .1, x, t, nlo1, nlo2},
 δ = 0.0;
 nlo1 = NDSolveValue[{x''[t] + x[t] + δ x'[t] + c (x[t]) ^3 + q (x[t]) ^2 == 1 +
      Sin[t / 1000] Sin[t / 1], x[0] == 1, x'[0] == 1}, x, {t, 0, tmax}, Method → "LSODA"];
 δ =
  0.2;
 nlo2 = NDSolveValue[{x''[t] + x[t] + δ x'[t] + c (x[t]) ^3 + q (x[t]) ^2 == 1 +
      Sin[t / 1000] Sin[t / 1], x[0] == 1, x'[0] == 1}, x, {t, 0, tmax}, Method → "LSODA"];
 Plot[{nlo1[t], nlo2[t]}, {t, 0, tmax}, GridLines -> Automatic,
   PlotLegends → {"nlo1", "nlo2"}]
  With[{stepSize = 2, end = tmax, nn = 0.1},
   MatrixPlot[UnitStep[nn - DistanceMatrix[nlo1[Range[0, end, stepSize]]]]]
    MatrixPlot[UnitStep[nn - DistanceMatrix[nlo2[Range[0, end, stepSize]]]],
     ColorFunction → "Monochrome"]
  ]]
```

Out[147]=