VII.3 Recombination

It's worth now ensuring that we're all on the same page when it comes to understanding the basics of hydrogen recombination. Let's first go through a few basic points:

- Recombination can occur to any level n
- If this electron has kinetic energy E_k, then the recombination releases a photon of energy $h\nu = E_k + E_b$, where E_b is the binding energy.
- If the photon leaves the nebula, then this recombination has removed energy from the system, thereby *cooling* it.
- If, on the other hand, the system is optically thick to the transport of these photons, we can invoke the 'on-the-spot' assumption that the photon is immediately re-absorbed (thus, re-ionizing another atom). In this case, the recombination has effectively no impact on the total ionization state of the system.

This leads us to two limits:

- **Case A Recombination:** Here, the nebula is optically thin. Every ionizing photon that is emitted due to a recombination escapes. In this case, we say that the radiative capture coefficient, α is summed over all levels: $\sum_n \alpha_n$. α has units of cm^3 s^{-1}. Every recombination decreases the overall ionization state of the gas, therefore, and cools the gas radiatively. This situation is common in the diffuse interstellar medium (or, sometimes, the circumgalactic medium), but not as relevant in more dense, star-forming regions.
- **Case B Recombination:** Here, $\tau >> 1$ for all radiation above $h\nu > 13.6$ eV, and the on the spot approximation applies. In this case, recombinations down to $n = 1$ do not reduce the ionization state of the gas. Only recombinations to $n \geq 2$ act to reduce the ionization (because, to first order, all electrons are in the ground state, so the liklihood of that photon seeing another atom is very small).)

The recombination rates for these are as follows:

$$\alpha_A(T) = \sum_{n=1}^{\infty}\sum_{l=0}^{n-1}\alpha_{nl}(T) \tag{VII.22}$$

$$\alpha_B(T) = \sum_{n=2}^{\infty}\sum_{l=0}^{n-1}\alpha_{nl}(T) = \alpha_A(T) - \alpha_{1s}(T) \tag{VII.23}$$

where, here l is the spin angular momentum of the electron. Draine gives good approximations for the hydrogen recombination rates between temperatures $\sim 30 - 3\times10^4$ K:

$$\alpha_A(T) \approx 4.13\times10^{-13} Z^2 \left(T_4/Z^2\right)^{-0.7131-0.0115\,\ln\left(T_4/Z^2\right)}\ \mathrm{cm^3 s^{-1}} \tag{VII.24}$$

$$\alpha_B(T) \approx 2.54\times10^{-13} Z^2 \left(T_4/Z^2\right)^{-0.8163-0.0208\,\ln\left(T_4/Z^2\right)}\ \mathrm{cm^3 s^{-1}} \tag{VII.25}$$

$$\tag{VII.26}$$

VII.4 Building a Photoionization Model

We'll frame our discussion of photoionization in terms of what it might be like to build a photoionization model. Some popular versions of such codes are CLOUDY and MAPPINGS.

VII.4.1 Assumptions

Assumptions: We first have to make a number of assumptions that are worth spelling out so that you have the right physical picture.

1. Steady State: We assume that the number of ionizations at any point in time are balanced by recombinations.
2. Radiative Equilibrium: in other words, the *radiative* energy in = *radiative* energy out.

Input Parameters: We additionally need to know a few things about our region being ionized:

1. Geometry (i.e. are we in a plane-parallel geometry? spherical? other?)
2. Incident flux/SED – we need to know what the ionizing spectrum looks like!
3. Distance of region to star
4. n_H and N_H (or, equivalently, n_H and the size of the region, R)

And then, in order to understand the emergent spectrum, a number of things must be calculated:

1. Statistical Equilibrium (to understand the distribution of bound level populations)
2. Ionization Equilibrium (to know how many photoionizations there are; i.e. the Saha equation)
3. Temperature Equilibrium: Here, the heating from photoionization (i.e. the KE given to an ejected e^-) has to balance the cooling from line radiation
4. Radiative transfer through the HII regions

VII.4.2 The Ionization Parameter

This is a problem you would have gotten to do in class work in groups. Please play the mental excercise of trying thi sout to see what you think would happen: Imagine we want to balance the number of photoionizations out of a single level (say, $n = 1$) with the number of recombinations, and the units on both sides should be number/s/cm^3, what does that equation look like? Some hints: one side will need an ionization cross section (cm^{-2}), and one side will need a recombination rate coefficient (1/s)

Let's assume that the region being blasted by ionizing photons is very optically thick: $\tau_\nu >> 1$ for Lyman continuum photons. Then, we can assume that every incident photon produces a photoionization.

Ionization balance everywhere in the nebula implies:

$$n_H \int_{\nu_1}^{\infty} \frac{F_\nu}{h\nu} \sigma_\nu d\nu = n_e n_p \alpha \left(T\right) \tag{VII.27}$$

Where:

- ν_1 corresponds to a threshold energy for ionization (i.e. $h\nu_1 = 13.6\text{eV}$)
- F_ν = incident flux (erg/s/cm^2)
- σ_ν = HI bound-free photoionization cross section for $n = 1$ (cm^{-2})
- $\alpha\left(T\right)$ = total HI recombionation rate coefficient, to all levels n (cm^3/s)

Now, the flux of the H-ionizing photons hitting this cloud is given by:

$$\Phi\left(H\right) = \int_{\nu_1}^{\infty} \frac{F_\nu}{h\nu} d\nu = \int_{\nu_1}^{\infty} \frac{L_\nu}{h\nu 4\pi r^2} d\nu \tag{VII.28}$$

Then, we can re-write the ionization balance simply as:

$$n_H \sigma_\nu \Phi\left(H\right) = n_e n_p \alpha\left(T\right) \tag{VII.29}$$

How can we parameterize the degree of ionization then? We can either say it as n_e/n_H or n_p/n_H, since we only create an electron-proton pair upon an ionization of a neutral H. If we parameterize it as the latter, we have:

$$\frac{n_p}{n_H} = \frac{\sigma\Phi\left(H\right)}{n_e \alpha\left(T\right)} = U \frac{\sigma c}{\alpha\left(T\right)} \tag{VII.30}$$

Where we have defined the **Ionization Parameter**:

$$U \equiv \frac{\Phi\left(H\right)}{n_H c} = \frac{n_{\rm ph}}{n_H} \tag{VII.31}$$

where the Ionization Parameter is the dimensionless ratio of hydrogen-ionizing photon density, $n_{\rm ph}$ to H particle (HI and HII) density. Note, the units seem wonky, but remember Φ has units of $1/s/cm^3$.

Some properties of U:

- The definition of U assumes that $n_H \approx n_e$ when the hydrogen is fully ionized (i.e. there are no extra free electrons or protons hanging around)
- The degree of ionization increases directly with $\Phi\left(H\right)$ = flux, but inversely with n_H
- The ionization parameter directly indicates the level of ionization

VII.4.3 Stromgren Radius and General Ionization Structure

Let's now ask *how big* the expected ionization region is? Let's assume a uniform (constant density) nebula surrounds the ionizing source.

Let's first do this the simple way. We'll define Q_0 as the total *rate* of hydrogen-ionizing photons:

$$Q_0 = \int_{\nu 1}^{\infty} \frac{L_\nu}{h\nu} d\nu \tag{VII.32}$$

We define the edge of the ionization region as the radius at which the total number of recombinations/s equals the total number of ionizations/s. i.e.:

$$\int_{\nu 1}^{\infty} \frac{L_\nu}{h\nu} d\nu = Q_0 = \left(\frac{4}{3}\pi r_s^3\right) n_e n_p \alpha\left(T\right) \tag{VII.33}$$

Then, the **Stromgren Radius** is the total radius of the HII region:

$$R_s = \left(\frac{3Q_0}{4\pi n_H^2 \alpha}\right)^{1/3} \tag{VII.34}$$

Where we assume $n_H = n_e = n_p$ Note, the units work out for this again becuase $\alpha =$cm^3 s^{-1} .

That was fun, but not as much fun as we *could* be having! Let's do the same thing, but a little bit more mathy (following Osterbrock). The cool thing about this method is that it allows us to introduce a few different concepts regarding recombination. We start with our normal equation for ionization equilibrium (in terms of column densities instead of volume densities):

$$N_H \int_{\nu_0}^{\infty} \frac{4\pi J_\nu}{h\nu} \sigma_\nu d\nu = N_p N_e \alpha_A\left(T\right) \tag{VII.35}$$

Importantly, α_A is the *total* recombination rate into all levels, and σ_ν is the *1 dimensional* absorption coefficient (i.e. the same as α_ν in normal radiative transfer – we can do this because we've turned the ionization balance equation into one with column densities instead of volume densities.)

Now our intensities (be they J_ν or the non-angle averaged I_ν can be thought of as a sum of two terms: the stellar component and the diffuse component:

$$I_\nu = I_{\nu,s} + I_{\nu,d} \tag{VII.36}$$

where:

- The stellar radiation can be thought of as a point source that decreases both with an inverse square law with the radius, and with optical depth.
- The diffuse term owes to ionization radiation released when an electron is recaptured by a proton. Because we assume that we're in equilibrium, this radiation must be absorbed somewhere else in the nebula.

Let's concentrate on the stellar component first. This can be written as

$$4\pi J_{\nu,s} = \pi F_{\nu,s} R^2 \frac{e^{-\tau_\nu}}{r^2} \tag{VII.37}$$

Where $\pi F_{\nu,s} R^2$ is the surface luminosity.

For the diffuse radiation, we can say that the total number of photons generated by recombinations to the ground level is given by:

$$4\pi \int_{\nu_0}^{\infty} \frac{j_{\nu,d}}{h\nu} d\nu = N_p N_e \alpha_1 \tag{VII.38}$$

where α_1 is the recombination coefficient just to the ground state (all we're concerned about here). (Note, here, we're using σ_ν instead of α_ν, but the units work out otherwise).

Now, if in the diffuse nebula, we can assume the 'on-the-spot' assumption...i.e. that any ionizing photon created from a recombination is immediately absorbed, then we can write:

$$4\pi \int \frac{j_{\nu,d}}{h\nu} dV = 4\pi \int N_H \frac{\sigma_\nu J_{\nu,d}}{h\nu} dV \tag{VII.39}$$

where this integration is over the entire volume of the nebula[2]. This means, then that:

$$J_{\nu,d} = \frac{j_{\nu,d}}{N_H \sigma_\nu} \tag{VII.40}$$

Taking, then, our stellar intensity (Equation VII.37), our diffuse intensity (Equation VII.38), and the relationship between $j_{\nu,d}$ and $J_{\nu,d}$ (Equation VII.40), we have (with some algebra that we'll skip):

$$\frac{N_H R^2}{r^2} \int_{\nu_0}^{\infty} \frac{\pi F_\nu(R)}{h\nu} \sigma_\nu e^{-\tau_\nu} d\nu = N_p N_e \alpha_B \tag{VII.41}$$

where we've defined

$$\alpha_B = \alpha_A - \alpha_1 = \sum_2^\infty \alpha_n$$

(i.e. α_B is the "Case-B recombination coefficient", and is the coefficient governing the rates of recombination down to the $n = 2$ level.

Now, if we remind ourselves of the definition of the optical depth:

$$\frac{d\tau_\nu}{dr} = N_H \sigma_\nu \tag{VII.42}$$

substituting this in, and integrating over r:

$$R^2 \int_{\nu_0}^{\infty} \frac{\pi F_\nu(R)}{h\nu} d\nu \int_0^\infty d\left(-e^{-\tau_\nu}\right) = \int_0^\infty N_p N_e \alpha_B r^2 dr \tag{VII.43}$$

where the integral $\int_0^\infty d(-e^{-\tau_\nu}) = 1$

So, what is r? r is a radius from the ionizing source. Let's define r_s as the radius where gas is fully ionized: that is, $N_p = N_e = N_H$. Then, we can simplify to say:

$$R^2 \int_{\nu_0}^{\infty} \frac{\pi F_\nu(R)}{h\nu} d\nu = \int_{\nu_0}^{\infty} \frac{L_\nu}{h\nu} d\nu = \frac{4\pi}{3} r_s^3 N_H^2 \alpha_B \tag{VII.44}$$

[2] The derivation of this last equation is left as an excercise to you, my awesome student. Since I know this problem will nag at you, and keep you from having fun, and you're probably up late working on this, let me give you a hint: you need to remember that $J_\nu = \frac{1}{4\pi} \int I_\nu d\Omega$, and the solutoin to the equation of radiative transfer: $\frac{dI}{ds} = -N_H \sigma_\nu I_\nu + j_\nu$

and we can solve as:

$$Q_0 = \frac{4\pi}{3} r_s^3 N_H^2 \alpha_B \tag{VII.45}$$

What's a good number for a size of a Stromgren sphere? Following Draine (pg 163), if we assume the recombination is dominated by case-B recombination (what this means is that we're in the optically thick scenario, and that we can employ the 'on-the-spot' approximation, meaning as soon as an electron is ejected, it is recombined), then we can say (without derivation):

$$r_s = 9.77 \times 10^{18} Q_{0,49}^{1/3} n_2^{-2/3} T_4^{-0.28} \mathrm{cm} \tag{VII.46}$$

Where $Q_{0,49} = Q_0/10^{49}$ s^{-1}, $n_2 = n_H/10^2$ cm^{-3}, and $\alpha = 2.56 \times 10^{-13} T_4^{-0.83}$ cm^3 s^{-1}. In short, an HII region is about 3 pc under normal conditions.