Some results and open problems in relational verification

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∀∀ relational property for imperative programs

$\boxed{c :\ p \rightsquigarrow q}$ partial correctness, commonly written $\{p\}\, c\, \{q\}$

$\boxed{c \mid d : \mathcal{R} \approx\!> \mathcal{S}}$ ∀∀-correctness, sometimes written $\{\mathcal{R}\}\, c \sim d\, \{\mathcal{S}\}$ and called 2-safety

$$\forall \quad \begin{tikzcd} \sigma \arrow[r, mapsto, "c"] \arrow[d, "\mathcal{R}"'] & \tau \\ \sigma' \arrow[r, mapsto, "d"] & \tau' \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} \tau \arrow[d, "\mathcal{S}"] \\ \tau' \end{tikzcd}$$

$$\forall \sigma\ \sigma'\ \tau\ \tau'.\ \ \sigma \mathcal{R} \sigma' \land \sigma [\![\, c \,]\!] \tau \land \sigma' [\![\, d \,]\!] \tau' \Rightarrow \tau \mathcal{S} \tau'$$

∀∀ relational property for imperative programs

$\boxed{c \mid d : \mathcal{R} \approx\!\!> \mathcal{S}}$

$$\forall \quad \begin{tikzcd} \sigma \arrow[r, mapsto, "c"] \arrow[d, "\mathcal{R}"'] & \tau \\ \sigma' \arrow[r, mapsto, "d"] & \tau' \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} \tau \arrow[d, "\mathcal{S}"] \\ \tau' \end{tikzcd}$$

$c \mid c : \overleftarrow{low} = \overrightarrow{low} \approx\!\!> \overleftarrow{low} = \overrightarrow{low}$ (c noninterferent)

$c \mid c : \overleftarrow{low} = \overrightarrow{low} \wedge (Fri \Rightarrow \overleftarrow{exp} = \overrightarrow{exp}) \approx\!\!> \overleftarrow{low} = \overrightarrow{low}$ (declassify)

$c \mid c : \overleftarrow{in} \leq \overrightarrow{in} \approx\!\!> \overleftarrow{out} \leq \overrightarrow{out}$ (monotonic)

$c; c \mid c : \overleftarrow{x} = \overrightarrow{x} \approx\!\!> \overleftarrow{x} = \overrightarrow{x}$ (idempotent)

$c \mid d : \overleftarrow{x} = \overrightarrow{x} \approx\!\!> \overleftarrow{x} = \overrightarrow{x}$ (d extensionally equivalent to c)

$c \mid d : \overleftarrow{x} = \overrightarrow{x} \approx\!\!> \overleftarrow{x} \leq \overrightarrow{x}$ (d majorizes c)

Verification example

Example: $c_0 \mid c_0 : \; 0 \leq \overleftarrow{x} \wedge \overleftarrow{x} = \overrightarrow{x} \;\approx\!\!> \; \overleftarrow{z} = \overrightarrow{z}$

$$c_0 \quad \widehat{=} \quad y := x; z := 1; \mathsf{while}\ y \neq 0\ \mathsf{do}\ z := z * y; y := y - 1\ \mathsf{od}$$

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Use renamed copy to reduce to partial correctness:
$c_0; c_0' : x = x' \wedge x \geq 0 \leadsto z = z'$ (invariant includes $x! = z * y!$).

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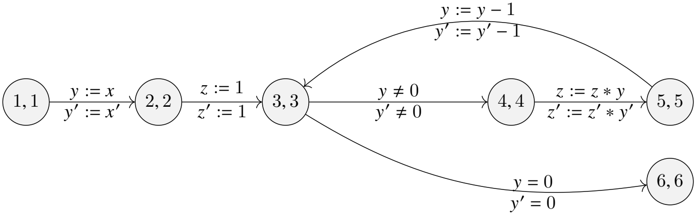
Better: lockstep aligned product (invariant: $\overleftarrow{y} = \overrightarrow{y} \wedge \overleftarrow{z} = \overrightarrow{z}$).



Outline

Goal of this work: theory for foundational & usable auto-active verification tools. For general programs; maximizing automation.

Outline of this talk:

- deductive & automata-based ∀∀ reasoning (sequential programs, standard interpretation)
- alignment completeness
- ∀∃ automata, deduction, completeness
- logics & completeness beyond inductive assertion method
- (if time) reducing ∀∀ to ∀∃

Lockstep alignment of different loops

```
procedure fact (x: int) returns (z: int)
{ var y: int; y := 0; z := 1;
  while (y < x) { y := y+1; z := z*y; } }

procedure fact' (x: int) returns (z: int)
{ var y: int; y := 1; z := 1;
  while (y ≤ x) { z := z*y; y := y+1; } }

procedure fact_eq (x, x': int) returns (z, z': int)
  requires x == x' ∧ x ≥ 0;    ensures z == z';
{ var y, y': int;
  y := 0; y' := 1; z := 1; z' := 1;
  while (y < x)
    invariant y' == y + 1 ∧ 0 ≤ y ∧ y ≤ x;
    invariant z == z' ∧ z > 0;
    invariant y < x ⟺ y' ≤ x; // adequacy
    { y := y+1; z := z * y; z' := z' * y'; y' := y'+1; }
}
```

Relational Hoare logic [Francez'83, Benton'04, Yang'07, Barthe...]

Lockstep alignment:

$$\frac{c \mid c' : \mathcal{P} \approx\!> \mathcal{R} \qquad d \mid d' : \mathcal{R} \approx\!> \mathcal{Q}}{c; d \mid c'; d' : \mathcal{P} \approx\!> \mathcal{Q}} \qquad x := e \mid x' := e' : \mathcal{R}^{x|x'}_{e|e'} \approx\!> \mathcal{R}$$

$$\frac{\mathcal{P} \Rightarrow \overleftarrow{e} = \overrightarrow{e'} \qquad c \mid c' : \mathcal{P} \wedge \overleftarrow{e} \wedge \overrightarrow{e'} \approx\!> \mathcal{P}}{\textsf{while } e \textsf{ do } c \textsf{ od} \mid \textsf{while } e' \textsf{ do } c' \textsf{ od} : \mathcal{P} \approx\!> \mathcal{P} \wedge \neg\overleftarrow{e} \wedge \neg\overrightarrow{e'}}$$

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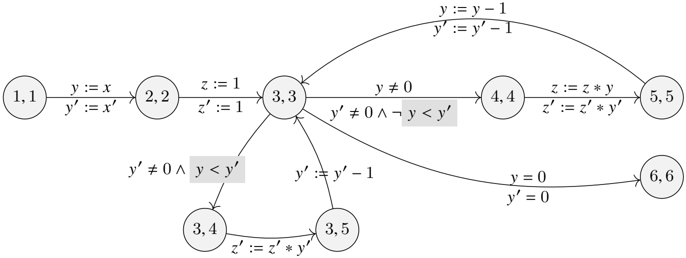
Sequential alignment:

$$\frac{c \mid \mathsf{skip} : \mathcal{P} \approx\!> \mathcal{R} \qquad \mathsf{skip} \mid c' : \mathcal{R} \approx\!> \mathcal{Q}}{c \mid c' : \mathcal{P} \approx\!> \mathcal{Q}}$$

Data dependent alignment

Example $c_0 \mid c_0 : 0 \leq \overleftarrow{x} \leq \overrightarrow{x} \approx\!> \overleftarrow{z} \leq \overrightarrow{z}$

$c_0 \quad \widehat{=} \quad y := x; z := 1;$ while $y \neq 0$ do $z := z * y; y := y - 1$ od



Proof uses $an(3,3)\ \widehat{=}\ 0 \leq \overleftarrow{y} \leq \overrightarrow{y} \wedge 0 \leq \overleftarrow{z} \leq \overrightarrow{z}$ invariant at $(3, 3)$

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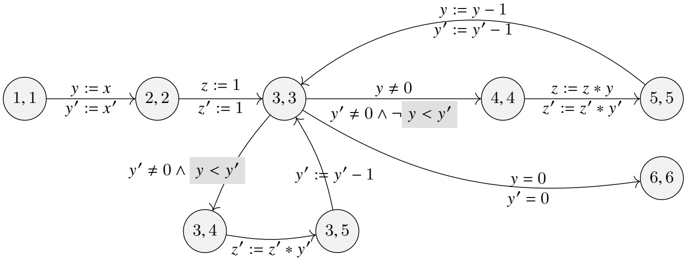

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The automaton is *adequate*: covers all pairs of runs.

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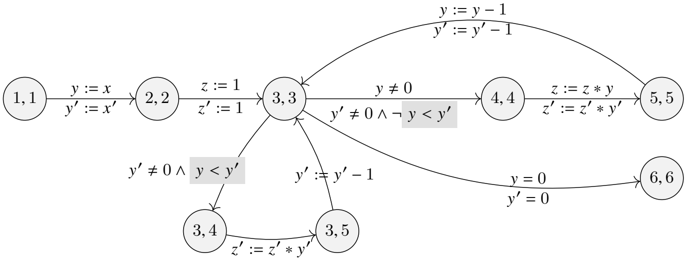

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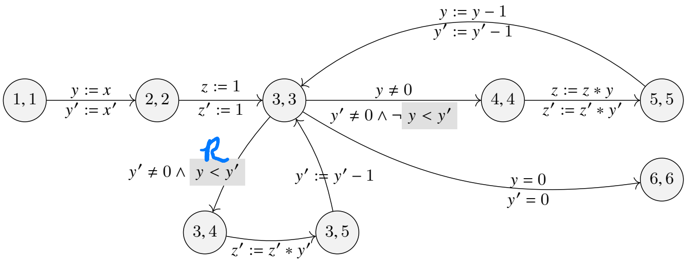
Alignment given by three state relations: when to do *L*eft-only, *R*ight-only, or *J*oint transitions.

Conditionally aligned loop rule



$$
\dfrac{
\begin{array}{c}
c \mid \mathsf{skip} : \mathcal{Q} \wedge \mathcal{L} \wedge \overleftarrow{e} \mathrel{\approx\!>} \mathcal{Q} \qquad \mathsf{skip} \mid c' : \mathcal{Q} \wedge \mathcal{R} \wedge \overrightarrow{e'} \mathrel{\approx\!>} \mathcal{Q} \\
c \mid c' : \mathcal{Q} \wedge \overleftarrow{e} \wedge \overrightarrow{e'} \wedge \neg\mathcal{L} \wedge \neg\mathcal{R} \mathrel{\approx\!>} \mathcal{Q} \\
\mathcal{Q} \Rightarrow (\overleftarrow{e} = \overrightarrow{e'}) \vee (\mathcal{L} \wedge \overleftarrow{e}) \vee (\mathcal{R} \wedge \overrightarrow{e'}) \quad \text{adequacy}
\end{array}
}{
\mathsf{while}\ e\ \mathsf{do}\ c\ \mathsf{od} \mid \mathsf{while}\ e'\ \mathsf{do}\ c'\ \mathsf{od} : \mathcal{Q} \mathrel{\approx\!>} \mathcal{Q} \wedge \neg\overleftarrow{e} \wedge \neg\overrightarrow{e'}
}
$$

For example: $\mathcal{L} := \mathit{false}$, $\mathcal{R} := \overleftarrow{y} < \overrightarrow{y}$.

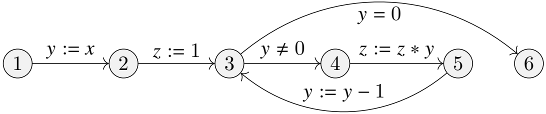
Cook Completeness

Theorem: the sequential alignment rule plus "unary Hoare logic" is complete (relative to assertion logic).

That is, $\models c \mid c' : \mathcal{P} \approx\!> \mathcal{Q}$ implies $\vdash c \mid c' : \mathcal{P} \approx\!> \mathcal{Q}$.

Problem: poor criterion, doesn't motivate the alignment rules.

Floyd completeness of Hoare logic

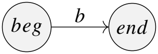


n	$an(n)$
1	$0 \leq x$
3	$x! = z * y!$
6	$x! = z$

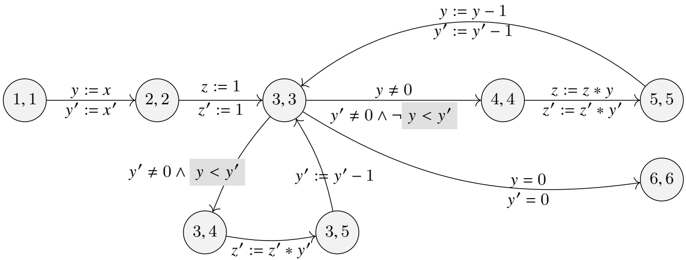
annotation for $P \rightsquigarrow Q$ assigns formulas to points in control-flow graph, cutting loops, with P initial and Q final

verification conditions: validity along paths

Thm[NN21]: given a valid annotation, an, for $c : P \rightsquigarrow Q$, there is a HL proof using only judgments

$b : an(beg) \rightsquigarrow an(end)$ for subprograms  of c.

Alignment completeness for certain automata



Theorem: for alignment products like the above, conditional loop + lockstep + one-side rules are alignment complete: Automata proof gives rise to deductive proof using “the same” assertions.

The associated judgments look like
$z := 1 \mid z := 1 : an(2,2) \approx\!> an(3,3)$.

Alignment completeness (∀∀)

Theorem: Given an adequate L, R, J-alignment automaton for $aut(c)$ and $aut(c')$, with a valid annotation an proving $\mathcal{P} \approx\!> \mathcal{Q}$, then there is a proof of $c \mid c' : \mathcal{P} \approx\!> \mathcal{Q}$ using "the same" assertions, by the RHL rules plus…

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Proof idea: Rewrite c and c' to an automaton-like normal form using explicit pc variable. The verification conditions imply this is invariant: $\land_{i,j}\,(\overleftarrow{pc} = i \land \overrightarrow{pc} = j \Rightarrow an(i,j))$.

control point label

$y :=^{1} x; z :=^{2} 1; \mathsf{do}^{3}\ y \neq 0 \rightarrow z :=^{4} z * y; y :=^{5} y - 1\ \mathsf{od}$

$$
\begin{array}{ll}
pc := 1; & \mathsf{do}\ pc = 1 \rightarrow y := x; pc := 2 \\
& [\!]\ pc = 2 \rightarrow x := 1; pc := 3 \\
& [\!]\ pc = 3 \land y \neq 0 \rightarrow pc := 4 \\
& [\!]\ pc = 3 \land y = 0 \rightarrow pc := 6 \qquad (*\ term.\ *) \\
& [\!]\ pc = 4 \rightarrow z := z * y; pc := 5 \\
& [\!]\ pc = 5 \rightarrow y := y - 1; pc := 3 \quad \mathsf{od}
\end{array}
$$

Additional rules for alignment completeness

Need disjunction, ghost elimination, and rewriting.

$$\frac{c \mid c' : \mathcal{R} \approx\!> \mathcal{S} \qquad [\![c]\!] = [\![d]\!] \qquad [\![c']\!] = [\![d']\!]}{d \mid d' : \mathcal{R} \approx\!> \mathcal{S}}\ \textsc{Lemma}$$

$$\frac{c \mid c' : \mathcal{R} \approx\!> \mathcal{S} \qquad \text{KAT,AX} \vdash c = d \qquad \text{KAT,AX} \vdash c' = d'}{d \mid d' : \mathcal{R} \approx\!> \mathcal{S}}\ \textsc{rRewrite}$$

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For any c there is $\hat{c}$ in automaton normal form such that KAT,AX $\vdash addPC(c) = \hat{c}$.

∀∃ properties for nondeterminism

$$\boxed{c \mid d : \mathcal{R} \overset{\exists}{\leadsto} \mathcal{S}} \qquad \forall \quad \begin{tikzcd} \sigma \arrow[r, mapsto, "c"] \arrow[d, "\mathcal{R}"'] & \tau \\ \sigma' & \end{tikzcd} \qquad \exists \quad \begin{tikzcd} & \tau \arrow[d, "\mathcal{S}"] \\ \sigma' \arrow[r, mapsto, "d"] & \tau' \end{tikzcd}$$

$$\forall \sigma\ \sigma'\ \tau.\ \ \sigma \mathcal{R} \sigma' \wedge \sigma [\![\, c \,]\!] \tau \Rightarrow \exists \tau'.\ \ \sigma' [\![\, d \,]\!] \tau' \wedge \tau \mathcal{S} \tau'$$

∀∃ properties for nondeterminism

$\boxed{c \mid d : \mathcal{R} \overset{\exists}{\rightsquigarrow} \mathcal{S}}$

$$\forall \quad \begin{array}{ccc} \sigma & \overset{c}{\longmapsto} & \tau \\ \mathcal{R}\big\downarrow & & \\ \sigma' & & \end{array} \qquad \exists \quad \begin{array}{ccc} & & \tau \\ & & \big\downarrow\mathcal{S} \\ \sigma' & \overset{d}{\longmapsto} & \tau' \end{array}$$

$c \mid c : \mathbb{A}low \overset{\exists}{\rightsquigarrow} \mathbb{A}low$ (possibilistic noninterference)

$\mathbb{A}x$ means $\overleftarrow{x} = \overrightarrow{x}$

$c \mid d : \mathbb{A}vars \overset{\exists}{\rightsquigarrow} \mathbb{A}vars$ (c refines d)

$\mathsf{skip} \mid c : \overrightarrow{p} \overset{\exists}{\rightsquigarrow} \overrightarrow{q}$ (forward underapprox, may term.)

$$\frac{b \mid c : \mathcal{P} \overset{\exists}{\rightsquigarrow} \mathcal{Q} \qquad c \mid d : \mathcal{R} \overset{\exists}{\rightsquigarrow} \mathcal{S}}{b \mid d : (\mathcal{P};\mathcal{R}) \overset{\exists}{\rightsquigarrow} (\mathcal{Q};\mathcal{S})}$$ (transitivity, vertical composition)

Note: $(\mathbb{A}x;\mathbb{A}x) \Leftrightarrow \mathbb{A}x$

Filtered alignment automata

Recall: automaton on state pairs, with state relations for *L*eft-only, *R*ight-only, or *J*oint transitions. Add: state relation K to block certain transitions.

Example: $\mathsf{hav}\ x \mid \mathsf{hav}\ y : \mathit{true} \overset{\exists}{\approx>} \overleftarrow{x} = \overrightarrow{y}$

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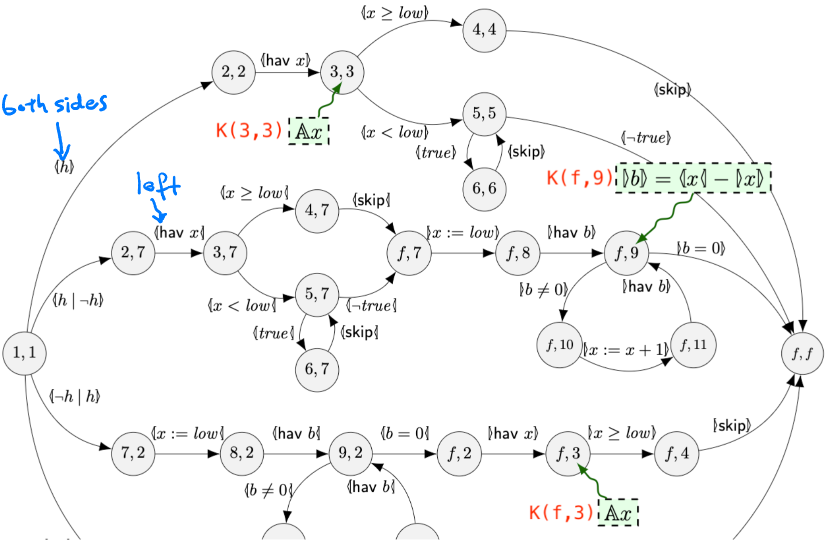
Example [Unno et al]: possibilistic NI $\quad \mathbb{A}low \overset{\exists}{\approx\!>} \mathbb{A}x$

$\mathsf{if}\ h \neq 0 \qquad$ (∗ branch on high∗)
$\mathsf{then}\ \{\mathsf{hav}\ x;\ \mathsf{if}\ x < low\ \mathsf{then\ while}\ \mathit{true}\ \mathsf{do\ skip};\ \}$
$\mathsf{else}\ \{x := low;\ \mathsf{hav}\ b;\ \mathsf{while}\ b \neq 0\ \mathsf{do}\ x := x + 1;\ \mathsf{hav}\ b;\}$

A filtered alignment automaton for [Unno et al]



Completeness of filtered automata

Theorem (soundness): Given (L, R, J, K)-automaton

- with annotation *an* that is inductive, for pre $\mathcal{P}$ post $\mathcal{Q}$
- and K satisfies local adequacy conditions w.r.t. *an*

then $\mathcal{P} \overset{\exists}{\approx\!\!>} \mathcal{Q}$ holds for the underlying programs.

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Inductive assertion method plus "relative liveness" checks.

A logic of forward simulation

$$\mathsf{skip} \mid \mathsf{hav}\ x : (\exists |x.\, \mathcal{P}) \overset{\exists}{\rightsquigarrow} \mathcal{P} \qquad \mathsf{hav}\ x \mid \mathsf{skip} : (\forall x|.\, \mathcal{P}) \overset{\exists}{\rightsquigarrow} \mathcal{P}$$

on right

on left

$$\frac{\begin{array}{c} c \mid \mathsf{skip} : \mathcal{Q} \wedge \overleftarrow{e} \wedge \mathcal{L} \overset{\exists}{\rightsquigarrow} \mathcal{Q} \\ \mathsf{skip} \mid c' : \mathcal{Q} \wedge \overrightarrow{e'} \wedge \mathcal{R} \wedge \boxed{n = E} \overset{\exists}{\rightsquigarrow} \mathcal{Q} \wedge \boxed{0 \leq E < n} \text{ for all } n \in \mathbb{Z} \\ c \mid c' : \mathcal{Q} \wedge \overleftarrow{e} \wedge \overrightarrow{e'} \wedge \neg\mathcal{L} \wedge \neg\mathcal{R} \overset{\exists}{\rightsquigarrow} \mathcal{Q} \\ \mathcal{Q} \Rightarrow (\overleftarrow{e} = \overrightarrow{e'}) \vee (\mathcal{L} \wedge \overleftarrow{e}) \vee (\mathcal{R} \wedge \overrightarrow{e'}) \end{array}}{\mathsf{while}\ e\ \mathsf{do}\ c\ \mathsf{od} \mid \mathsf{while}\ e'\ \mathsf{do}\ c'\ \mathsf{od} : \mathcal{Q} \overset{\exists}{\rightsquigarrow} \mathcal{Q} \wedge \neg\overleftarrow{e} \wedge \neg\overrightarrow{e'}}$$

Consequence, assign, lockstep loop, etc – same as for ∀∀.

Alignment completeness (∀∃)

Theorem: Given an adequate L, R, J, K-alignment automaton with a valid annotation proving $\mathcal{P} \overset{\exists}{\approx\!>} \mathcal{Q}$, then there is a proof using “the same” assertions, by the rules above plus disjunction, rewrite, and ghost elimination.

Proof idea: as for ∀∀, relying on ∀∃ adequacy.

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Problem: our proofs of alignment completeness don't imply a practical technique.

Beyond IAM and alignment (and back to ∀∀)

Far beyond: arbitrary trace relations

Slightly beyond: rule of conjunction; frame rule

Hypotheses:

- relational procedure specs $\quad f(x) : \mathcal{P}(x) \approx\!> \mathcal{Q}(x) \vdash c : \mathcal{R} \approx\!> \mathcal{S}$
- commutativity $\quad c; d \mid d; c : \mathbb{A}x \approx\!> \mathbb{A}x$,

$$c \mid c : \mathbb{A}x \approx\!> \mathbb{A}x, \quad d \mid d : \mathbb{A}x \approx\!> \mathbb{A}x$$

$$\models \quad c; d; d \mid d; d; c : \mathbb{A}x \approx\!> \mathbb{A}x$$

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- relational procedure specs $f(x) : \mathcal{P}(x) \approx\!> \mathcal{Q}(x) \vdash c : \mathcal{R} \approx\!> \mathcal{S}$
- commutativity
$$\begin{array}{rl} & \boxed{c;d \mid d;c : \mathbb{A}x \approx\!> \mathbb{A}x}, \\ & c \mid c : \mathbb{A}x \approx\!> \mathbb{A}x, \quad d \mid d : \mathbb{A}x \approx\!> \mathbb{A}x \\ \models & c;d;d \mid d;d;c : \mathbb{A}x \approx\!> \mathbb{A}x \end{array}$$
- idempotence
$$\begin{array}{rl} & \boxed{c \mid c : \overleftarrow{x} = v \approx\!> \overrightarrow{x} = v \Rightarrow \overleftarrow{x} = v}, \\ & c \mid c : \mathbb{A}x \approx\!> \mathbb{A}x \\ \models & c \mid c;c : \mathbb{A}x \approx\!> \mathbb{A}x \end{array}$$
- hoisting an idempotent out of a loop

[D'Osualdo et al'22]

Rules for D'Osualdo examples

D'Osualdo et al propose an k-safety logic with varying k (and a semantic rule for rewriting); use 3 for the examples.

For commutativity:

$$\frac{b \mid c : \mathcal{P} \approx\!> \mathcal{Q} \qquad c \mid d : \mathcal{R} \approx\!> \mathcal{S} \qquad \boxed{\mathsf{skip} \mid c : \mathcal{P} \overset{\exists}{\approx\!>} \mathit{true}}}{b \mid d : (\mathcal{P};\mathcal{R}) \approx\!> (\mathcal{Q};\mathcal{S})}$$

For idempotence:

$$\frac{c \mid c' : \mathcal{P} \approx\!> \mathsf{wp}(d \mid \mathsf{skip})(\mathcal{Q})}{c;d \mid c' : \mathcal{P} \approx\!> \mathcal{Q}} \qquad \frac{c \mid c' : \mathcal{R} \approx\!> \mathcal{S} \qquad \mathsf{skip} \mid c' : \overrightarrow{p} \approx\!> \overrightarrow{q}}{c \mid c' : \mathcal{R} \wedge \overrightarrow{p} \approx\!> \mathcal{S} \wedge \overrightarrow{q}}$$

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D'Osualdo et al propose an k-safety logic with varying k (and a semantic rule for rewriting); use 3 for the examples.

For commutativity:

$$\frac{b \mid c : \mathcal{P} \approx\!> \mathcal{Q} \qquad c \mid d : \mathcal{R} \approx\!> \mathcal{S} \qquad \boxed{\mathsf{skip} \mid c : \mathcal{P} \overset{\exists}{\approx\!>} \mathit{true}}}{b \mid d : (\mathcal{P}; \mathcal{R}) \approx\!> (\mathcal{Q}; \mathcal{S})}$$

For idempotence:

$$\frac{c \mid c' : \mathcal{P} \approx\!> \mathsf{wp}(d \mid \mathsf{skip})(\mathcal{Q})}{c; d \mid c' : \mathcal{P} \approx\!> \mathcal{Q}} \qquad \frac{c \mid c' : \mathcal{R} \approx\!> \mathcal{S} \qquad \mathsf{skip} \mid c' : \overrightarrow{p} \approx\!> \overrightarrow{q}}{c \mid c' : \mathcal{R} \land \overrightarrow{p} \approx\!> \mathcal{S} \land \overrightarrow{q}}$$

$$\frac{\mathsf{skip} \mid c' : \overrightarrow{p} \approx\!> \mathcal{Q}}{\mathsf{skip} \mid c' : \overrightarrow{p} \approx\!> \overrightarrow{\mathsf{diag}(\mathcal{Q})}}$$

So what?

Entailment completeness

$$\boxed{\text{If } hyps \models c \mid c' : \mathcal{P} \approx\!> \mathcal{Q} \text{ then } hyps \vdash c \mid c' : \mathcal{P} \approx\!> \mathcal{Q}.}$$

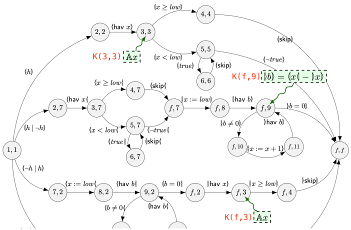
Q: What rules (and judgments) suffice for entailment complete ∀∀ logic? ∀∀ together with ∀∃?

Q: How prove that, say RHL+, isn't entailment complete?

Precursors:

- trivial with implication operator (Reynolds' Spec Logic; shallow emb.)
- KAT, Propositional Dynamic Logic
- Propositional Hoare Logic [Kozen,Tiuryn'01]

Filter automata in program form



Specialize filter to right-side havoc.

```
if (high ≠ 0 ∧ high' ≠ 0) {
   havoc x;
   assert (∃ v:int • x == v); // added by chk
   havoc x'; assume x == x';
   if (x ≥ low ∨ x' ≥ low') { /*skip*/ }
   else { while (true) { /*skip*/ } }
}
```

Filter adequacy transformation

Theorem: if $chk(prod) : \mathcal{P} \approx\!> \mathcal{Q}$ then $c \,|\, c' : \mathcal{P} \overset{\exists}{\approx\!>} \mathcal{Q}$ for underlying c, c' of $prod$. (WhyRel implementation, Rocq formalization)

Auto-active solver hacking…

```
else if (high ≠ 0 ∧ high' == 0) {
   havoc x;
   if (x ≥ low) { /*skip*/ }
   else { while (true) { /*skip*/ } }
    x' ≔ low;
    assert inst(x - x');
    assert (∃ v:int • {inst(v)} v == x - x'); // added by chk
    havoc b'; assume b' == x - x';
    while (b' ≠ 0)
       invariant x ≥ x' ∧ b' == x - x'; /* variant b' */
    {  bsnap' ≔ b'; // added by chk
       x' ≔ x' + 1;
       assert inst(x - x');
       assert (∃ v:int • {inst(v)} v == x - x'); // added by chk
       havoc b'; assume b' == x - x';
       assert b' < bsnap'; // added by chk
    }
 }
```

Summary

- Syntax-oriented logic can express alignment-based relational verification (inductive assertion method), and guide design of auto-active tools.
- Open Q: minimizing rewriting for proof of alignment completeness. (Compare the impact of wlp, versus Gödel encoding, for Cook completeness proofs.)
- Related Q: deriving a syntactic alignment product from alignment conditions inferred for unstructured transition system.
- Entailment of correctness judgments motivates additional rules like vertical composition, wlp assertions…
- Qpen Q: Is there an entailment complete system? for ∀∀ together with ∀∃? Must it go beyond 2-properties? Robust proof for interpreted logic?

Credits

Cook completeness of RHLs [Francez'83, Barthe et al'04, ...]

Relational logic with framing, procedures, abstraction [Banerjee, Nagasamudram, Nikouei, Naumann TOPLAS'22, TACAS'23]

Alignment complete relational Hoare logics for some and all. [Nagasamudram, Banerjee, Naumann; to appear in LMCS; see v5 for D'Osualdo examples `https://arxiv.org/abs/2307.10045v5`.]

Proving Hypersafety Compositionally. [D'Osualdo, Farzan, Dreyer OOPSLA'22; cf. Bao et al POPL'25.]

Forall-exists relational verification by filtering to forall-forall. [Nagasamudram, Banerjee, Naumann; `https://arxiv.org/abs/2509.04777`.]

Example conditional alignment

```
procedure hiccupSum (n: int) returns (r: int)
{ var h: bool; var i: int;
  i, r := 0, 0;
  havoc h;
  while (i < n) {
    if (!h) { r := r+i; i := i+1; }
    havoc h;
  }
}
procedure hiccupSum_eq (n, n': int) returns (r, r': int)
  requires n == n';
  ensures r == r';
{
  var h, h': bool; var i, i': int;
  i, r := 0, 0;   i', r' := 0, 0;
  havoc h; havoc h';

  while (i < n ∨ i' < n') // left/right alignment conditions h/h'
    invariant i == i';
    invariant r == r';
    {
      if (h ∧ i < n) { // left body only
        if (!h) { r := r+i; i := i+1; } havoc h;
      } else if (h' ∧ i' < n') { // right body only
        if (!h') { r' := r'+i'; i' := i'+1; } havoc h';
      } else if (i < n ∧ i' < n' ∧ !h ∧ !h') { // both
        if (!h) { r := r+i; i := i+1; } havoc h;
        if (!h') { r' := r'+i'; i' := i'+1; } havoc h';
      } else {
        assert false; // adequacy condition
      }
```