Piecewise linear approximations of sigmoid-type functions

Daniel Coble[a]

[a]*Department of Mechanical Engineering, University of South Carolina*

1. Introduction

An important component of neural networks is the nonlinear activation function. In edge computing, however, it is not always possible to use the computationally intensive functions which an off-line model was trained with. Therefore, there is a need to create approximations of activation functions which are computationally cheap but still yield relatively accurate results.

This work follows from a paper by Amin, Curtis and Hayes-Gill. In that paper, the authors create a piecewise linear approximation (PLAN or PWL approximation) of the sigmoid function $y = \frac{1}{1+e^{-x}}$. By choosing slopes which are powers of 2, multiplication can be replaced with shift operations (in the case of fixed-point numbers).

A natural question is what is the 'best' that we can do under this constraint? Below I create a methodology for finding a best linear approximation. I apply it to three functions: the sigmoid function above, and two other similar functions: arctan and tanh.

2. Methodology

First, we must define what we mean by the 'best' approximation. Let $f(x)$ be our sigmoidal function and $\bar{f}(x)$ be the PWL approximation. We will say that the best function is the one with the minimum *maximum* variation between $f(x)$ and $\bar{f}(x)$ (maximum across x, minimum across $f(x)$). Another way to say it is we would like to minimize the score $\max(|f(x) - \bar{f}(x)|)$.

We will use four properties which all three functions share:

1. All $f(x)$ are related by some $h(y)$ such that $f(-x) = h(f(x))$.
   arctan and tanh are odd, so $h(y) = -y$ For the sigmoid function σ we have $h(y) = 1 - y$.
2. The functions are asymptotic
   $$\lim_{x\to\infty} f(x) = c$$
3. In the positive domain the functions have a positive derivative.
4. In the positive domain the functions have a negative second derivative.

Item 1. tells us that we only have to worry about the positive domain. To perform the function on a negative value $x < 0$, we take $h(f(-x))$. Let our piecewise function take the form

$$\bar{f}(x) = \begin{cases} m_0x + b_0 & 0 \leq x < x_{c1} \\ m_1x + b_1 & x_{c1} \leq x < x_{c2} \\ \vdots & \vdots \\ m_nx + b_0 & x_{cn} \leq x < x_{cn+1} \\ c & x > x_{cn+1} \end{cases}$$

Where $m_0...m_n$ are given. For our case we have $m_{i+1} = \frac{1}{2}m_i$, and for all three functions, the derivative at 0 is a reasonable choice for m_0.

$$\frac{d}{dx}\sigma(x) = \frac{1}{4}$$

$$\frac{d}{dx}\arctan = \frac{d}{dx}\tanh = 1$$

The only thing necessary for the proof though is that m_i are given. We must choose x_{ci} and b_i to minimize $\max(|f(x) - \bar{f}(x)|)$. Still, there is a trivial way to create a 'best' $\bar{f}(x)$ by taking very small line segments. This isn't useful, so we say that $\bar{f}(x)$ must also be continuous on $x > 0$.

2.1. Lemma

Let x_{di} be the value such that

$$\frac{df}{dx}x_{di} = m_i$$

x_{di} is unique (there is only one x_{di} such that $\frac{df}{dx}x_{di} = m_i$. x_{di}). This follows from the fourth item in the list of properties.

2.2. Lemma

For a single line segment $m_i x + b_i$, x is bounded by x_{di} and x_c ($x_{di} \leq x \leq x_c$ if $x_{di} \leq x_c$, $x_{di} \geq x \geq x_c$ if $x_{di} \geq x_c$), the maximum variation can only occur at x_{di} or x_c.

It is a calculus principle that at a maximum can only occur at endpoints or where the derivative is zero.

$$\frac{d}{dx}\max(|f(x) - m_i x + b_i|) = 0$$

$$f'(x) = \pm m_i$$

The third item in the list of properties states that $f'(x) > 0$.

$$f'(x) = m_i$$

which is x_{di}, already an endpoint.

2.3. Lemma

Varying only b_i, a line segment $m_i x + b_i$, bounded by x_{di} and x_c, will be at a minimum when $f(x_{di}) - (m_i x_{di} + b_i) = (m_i x_c + b_i) - f(x_{di}) \geq 0$.

Assume that in all locations where the maximum variation occurs, $f(x) - (m_i x + b_i)$, has the same sign. Then $m_i x + b_i$ is not at a minimum because b_i can be changed to reduce the maximum variation (increased if $f(x) - (m_i x + b_i)$ is positive, decreased if $f(x) - (m_i x + b_i)$ is negative). So the maximum variation must occur at two locations (both endpoints), and $f(x) - (m_i x + b_i)$ must have different signs at those two locations.

If $x_c \leq x_{di}$, then by property 4 we have

$$\frac{d}{dx}\left(f(x_c) - (m_i x_c + b_i) = f'(x_c) - m_i\right) \geq 0$$

If $f(x_c) - (m_i x_c + b_i) \geq 0$, $f(x_{di}) - (m_i x_{di} + b_i) \geq 0$ and the line segment is not at a minimum.
If $x_c \geq x_{di}$, then we have

$$\frac{d}{dx}\left(f(x_c) - (m_i x_c + b_i) = f'(x_c) - m_i\right) \leq 0$$

If $f(x_c) - (m_i x_c + b_i) \geq 0$, $f(x_{di}) - (m_i x_{di} + b_i) \geq 0$ and the line segment is not at a minimum. Therefore, if the line segment is at a minimum, $f(x_c) - (m_i x_c + b_i) \leq 0$ (for all x_c) and $f(x_{di}) - (m_i x_{di} + b_i) \geq 0$. (It's easier to understand this through a graph)

2.4. Definition

So now with given x_c we can vary b_i to create the best line from x_{di} to x_c. We also know that the maximum variation will occur at x_c. Let's create a function $g_i(x)$ which takes as input x_c and outputs $m_i x_c + b_i$ for the best b_i. There's a few steps of math but eventually we get

$$g_i(x) = \frac{1}{2}f(x) + \frac{1}{2}f(x_{di}) - \frac{1}{2}m_i x_{di} + \frac{1}{2}m_i x$$

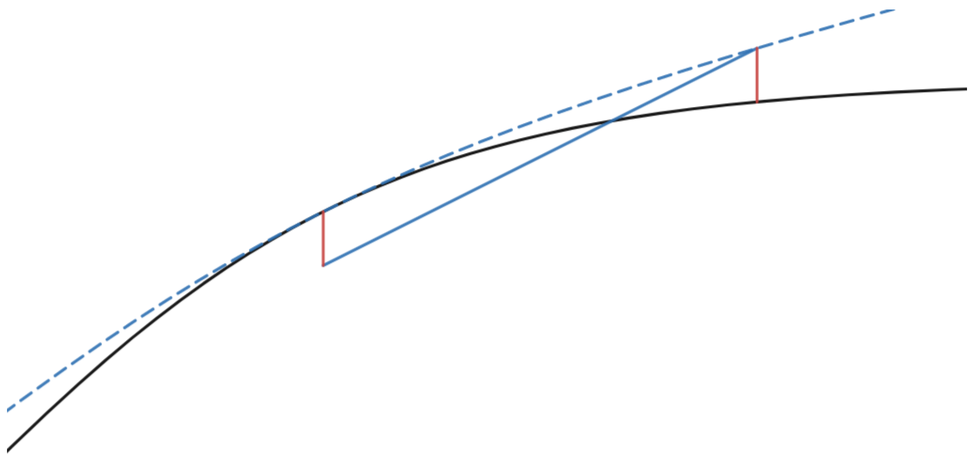

Figure 1: $f(x)$ (black) and a $g_i(x)$ (blue dashed)

And we can make another function $G(x)$ defined as

$$G(x) = \min_i(g_i(x))$$

2.5. Lemma

$\max(G(x) - f(x))$ is a lower bound to $\max(|f(x) - \bar{f}(x)|)$.

Let x_G be the value satisfying

$$G(x_G) - f(x_G) = \max(G(x) - f(x))$$

Assume we have $\bar{f}(x_G) - f(x) < G(x_G) - f(x_G)$ (or simply $\bar{f}(x_G) < G(x_G)$). Say at x_G, $\bar{f}(x) = m_i x + b_i$. If x_{di} lies within x_{ci} to x_{ci+1}, then by the definition of $G(x)$,

$$f(x_{di}) - m_i x_{di} + b_i > G(x_G) - f(x_G)$$

. If x_{di} lies outside x_{ci} to x_{ci+1}, we can see that

$$f(x) - \bar{f}(x) > f(x_{di}) - m_i x_{di} + b_i > G(x_G) - f(x_G)$$

(This is hard to explain in words, but easy to see from a graph. We rely on $\bar{f}$ being continuous.)

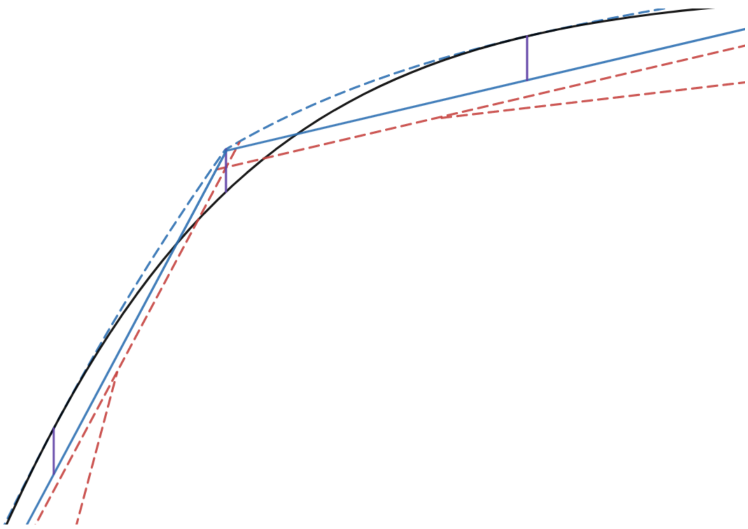

Figure 2: This cluttered diagram (not to scale) explains this argument. In blue we have what I am proposing as the best PWL function. We can see that at three different places it has a variation Δ (purple lines). If we try to propose any other PWL with a smaller variation at x_G (dashed red lines), this function must have a greater variation at one of the two x_d. If the function has another piecewise segment before reaching x_d (branches on the dashed red lines), it will have a strictly greater variation.

2.6. Final Steps

So we have that $\max(G(x) - f(x))$ is a lower bound to the maximum variation. Let's name this Δ. If we can create a PWL function which has a maximum variation of Δ then we are done. This can be done by choosing starting at x_G and expanding on either side, choosing the next x_c when $m_i x + b_i$ intersects $G(x)$. To the left, this will eventually hit the x-intercept. To the right, this can repeat until we choose to round off to the asymptote. One thing I want to include in the tables in Results is how many line segments are required so that this round off is less than Δ.

We can calculate x_G. We know x_G will occur at a point where two $g(x)$ functions intersect, say $g_i(x)$ and $g_{i+1}(x)$. Then we can show that.

$$x_G = \frac{f(x_{di}) - f(x_{di+1}) - m_i x_{d_i} + m_{i+1} x_{di+1}}{m_{i+1} - m_i}$$

$$\Delta = g_i(x_G) - f(x_G)$$

This means we can get a closed form solution if x_{di} can be explicitly solved for. Otherwise we need the use of a numerical solver.

3. Results

I want to create tables for x_c and b, also some graphs. Right now my code doesn't necessarily produce a continuous PWL function.

f(x)	Δ
σ	0.01132
arctan	0.02606
tanh	0.02265

Acknowledgments

References