Descartes' Circle Theorem

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December 11, 2021

1 Statement of theorem

Descartes' Circle Theorem relates to the relationship between packed "kissing" circles - that is, groups of circles that are mutually tangent to each other. The modern form of the theorem (slightly different from the statement of Descartes) states that, given 3 pairwise tangent circles, there are two circles that are tangent to all of the circles (see figure 1).

Defining the curvature c of a circle as the inverse of its radius, and setting the curvature of a circle which is surrounding the 3 other circles to be negative, the curvatures of four circles which are all mutually tangent to each other satisfy the relation:

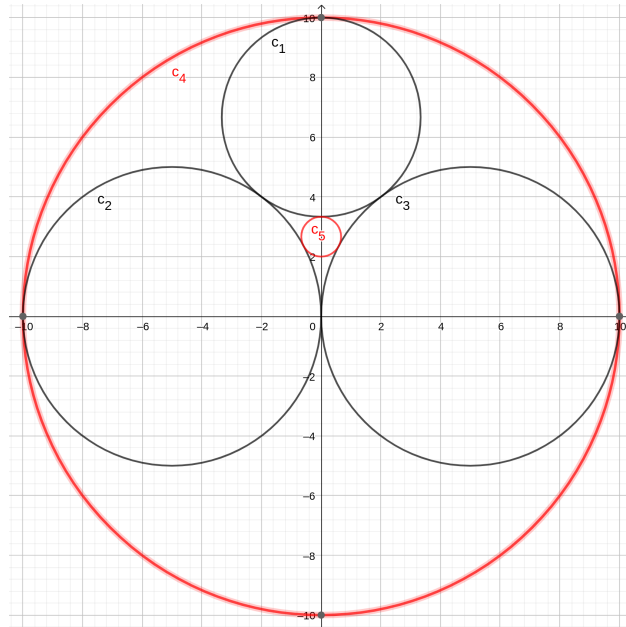$$2(c_1^2 + c_2^2 + c_3^2 + c_4^2) = (c_1 + c_2 + c_3 + c_4)^2$$

Figure 1: Two circles c_4 and c_5 are mutually tangent to c_1, c_2, c_3

This paper is a presentation of a proof of this theorem which uses circle inversion.

2 Circle inversion

Circle inversion is an involution (that is, a function which is also its own inverse) of the plane. A circle C with center O and radius r maps a point P to a point P' such that P, and P' are both on a ray with endpoint O and:

$$|OP| \cdot |OP'| = r^2$$

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This maps points on the circle C to themselves, and maps points inside the circle to points outside the circle, and vice versa.

P'

$|OP'| = 2$

P

$|OP| = \frac{1}{2}$

O
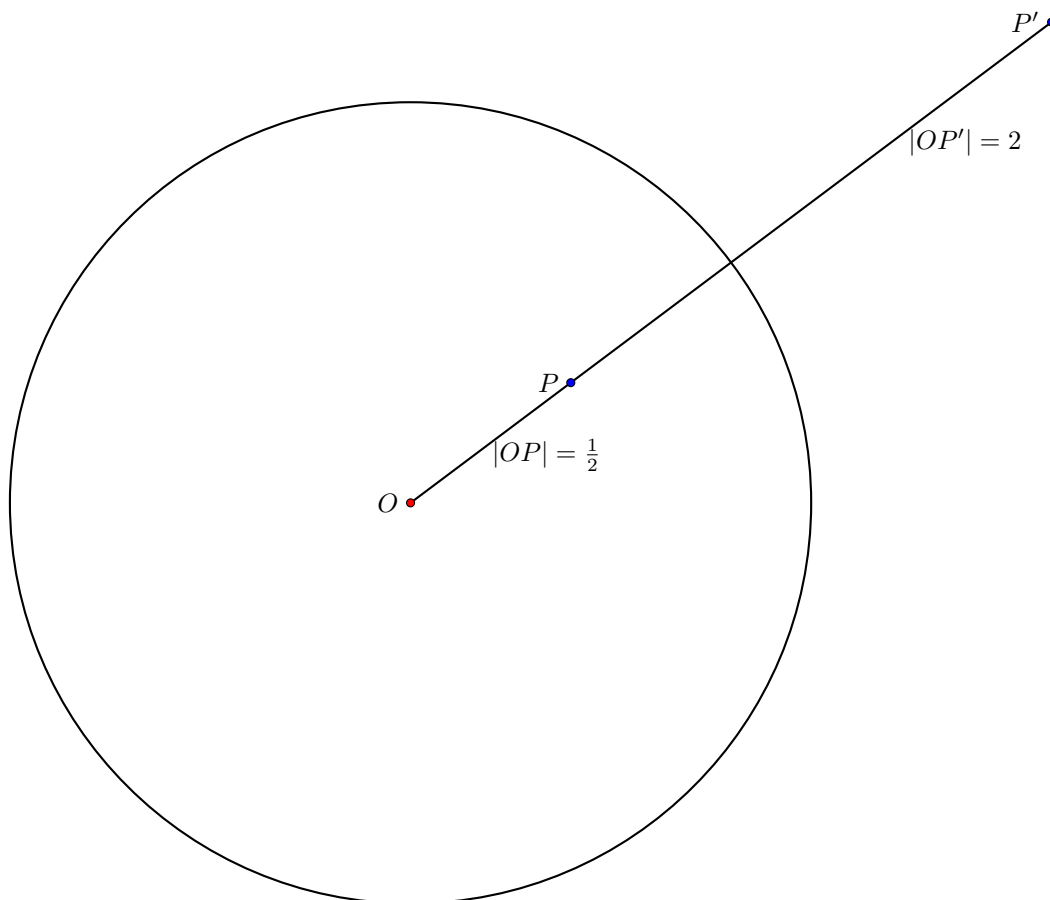
Where things start to get very interesting is when we start to look at inversions of circles and lines.

- A circle inside the circle of inversion maps to a circle outside, and vice versa.

- A circle straddling the circle of inversion will map to another circle sharing the intersection points with the circle of inversion.

- The diameter of a circle coincident with the ray through the center of the circle of inversion inverts to the diameter of the circle it maps to. (note: circle centers do **not** invert to the centers of the circles they invert to).

- Lines which do not go through the center of the circle of inversion invert to circles which go through the center of the circle of inversion, and vice versa.

- Lines through the center of the circle of inversion invert to themselves.