

# Approximating $\pi$ with an Infinite Series

Dave Neary

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## 1 Introduction

We can sum the infinite series:

$$f(t) = 1 - t^2 + t^4 - t^6 + \dots = \frac{1}{1 + t^2}$$

Integrating across from 0 to  $x$ :

$$\int_{t=0}^x f(t) dt = \tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

And setting  $x = 1$  (which converges in the integral, but not in the original function):  $\tan^{-1}(1) = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$

This is a nice formula for  $\pi$ , but is slow to converge. You can use the tan sum formula to find faster converging solutions:

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Let  $A = \tan^{-1} \frac{1}{2}$ ,  $B = \tan^{-1} \frac{1}{3}$ . Then apply the tangent sum formula, and take inverse tangents on both side:

$$\tan(A+B) = \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} \cdot \frac{1}{3}} = 1$$

$$\tan^{-1}(1) = \frac{\pi}{4} = \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3}$$

Then:

$$\frac{\pi}{4} = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{2i-1} \left( \frac{1}{2^{2i-1}} + \frac{1}{3^{2i-1}} \right)$$

which converges many times faster. You can do this for any other numbers you can find for  $A+B=1-AB$  which gives  $(A+1)(B+1)=2$  such as  $\frac{1}{4}$  and  $\frac{3}{5}$ . Another example, more complicated to find, is:

$$\frac{\pi}{4} = 4 \tan^{-1} \left( \frac{1}{5} \right) - \tan^{-1} \left( \frac{1}{239} \right)$$

which converges to 10 digits of  $\pi$  when you take the first 4 terms of the expansion of  $\tan^{-1} \frac{1}{5}$  and the first two terms of  $\tan^{-1} \frac{1}{239}$ .