1 Approximation of square roots

We can estimate square roots of numbers that are close to perfect squares. For example, $18^2 = 324$ and

$$\begin{aligned} (18-x)^2 &= 321 \\ 324 - 36x + x^2 &= 321 \\ 3 &= 36x - x^2 \end{aligned}$$

Now, since x will be small, x^2 will be negligible for a reasonable estimate. We can thus discard the x^2 term, giving:

$$x \approx \frac{3}{36} = \frac{1}{12}$$

for small values of x. So $\sqrt{321} \approx 18 - \frac{1}{12}$.

In general, if a number is close to a perfect square, we can write it as $n^2 + m$ for $m \in \mathbb{Z}$, and you can approximate the square root with:

$$\sqrt{n^2 + m} \approx n + \frac{m}{2n}$$

This also works for higher powers! If a number is close to n^k, we write it as $n^k + m$, and:

$$\sqrt[k]{n^k + m} \approx n + \frac{m}{kn^{k-1}}$$

This is the method that Richard Feynmann used to approximate the cube root of 1729 when competing against an abacus salesman. $12^3 = 1728$ (which he knew from cubic inches in a cubic foot), so

$$\sqrt[3]{1729} \approx 12 + \frac{1}{3 \times 12^2} = 12 + \frac{1}{432}$$

We can also use the Binomial theorem extended to real exponents to get even closer approximations.

$$\begin{aligned} \sqrt{321} &= \sqrt{324 - 3} \\ &= 18\left(1 - \frac{1}{108}\right)^{\frac{1}{2}} \\ &= 18\left(1 + \binom{\frac{1}{2}}{1}\left(\frac{-1}{108}\right) + \binom{\frac{1}{2}}{2}\left(\frac{1}{108^2}\right) + \cdots)\right) \\ &\approx 18\left(1 - \frac{1}{216} - \frac{1}{93312} - \cdots\right) \\ &\approx 18 - \frac{1}{12} - \frac{1}{5184} - \cdots \end{aligned}$$

where $\binom{x}{r} = \frac{(x)(x-1)(\cdots)(x-r+1)}{r!}$ for $x \in \mathbb{R}$.