Lines in the projective plane $\mathbb{P}^2$

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1 Introduction

The projective plane $\mathbb{P}^2$ is the set of points on the affine plane, plus the points at infinity representing different slopes. The addition of the points at infinity guarantees that any two distinct lines intersect at one point exactly. It can also be describes by the set of triples $[X;Y;Z]$ not all zero, which are equivalent up to multiplication by a scalar - $[1;2;3]$ and $[2;4;6]$ refer to the same point.

For a point $[X;Y;Z] \in \mathbb{P}^2$, if $Z \neq 0$ then this maps to the point in the Cartesian (or affine) plane $(\frac{X}{Z}, \frac{Y}{Z})$. If $Z = 0$, the point corresponds to a point at infinity (the intersection point of all lines with the slope $-\frac{X}{Y}$).

2 Lines in the projective plane

Common results in Euclidean geometry hold also in the projective plane, with some differences.

- Given two distinct points in the projective plane $P_1 = [a_1;a_2;a_3], P_2 = [b_1;b_2;b_3]$ there is a unique line $L : \alpha X + \beta Y + \gamma Z = 0$ which goes through both points.

- Given two distinct lines in the projective plane, they intersect at exactly one point.

Recall that if $P_1 = \lambda P_2$ for some scalar factor then $P_1 = P_2$ (if it helps, think of these as equivalence classes like fractions: $\frac{4}{2} = \frac{6}{3}$). So the triples $(a_1, a_2, a_3), (b_1, b_2, b_3)$ are linearly independent if they are distinct points in $\mathbb{P}^2$.

Then:

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

has a solution, unique up to multiplication by a scalar, with $[\alpha;\beta;\gamma] \in \mathbb{P}^2$. As we have seen, a point in $\mathbb{P}^2$ is essentially the same as a line through the origin in $\mathbb{A}^3$. Given $P = [X; Y; Z]$ we can write that line in parametric form with:

$$L : \vec{p} = \lambda \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

Given this, you can see that the two linearly independent vectors P_1, P_2 define a plane through the origin in $\mathbb{A}^3$ defined by:

$$C : \vec{p} = \lambda_1 \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda_2 \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

This maps to the line in $\mathbb{P}^2$:

$$L : \alpha X + \beta Y + \gamma Z = 0$$

where the vector

$$P_3 = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

satisfies the relationships $P_1 \cdot P_3 = 0$ and $P_2 \cdot P_3 = 0$ - that is, P_3 is orthogonal to both P_1 and P_2. In other words, it is a scalar multiple of $P_1 \times P_2$.

We can calculate α, β, γ with:

$$\begin{vmatrix} \alpha & \beta & \gamma \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0$$

or:

$$L : \alpha \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \beta \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \gamma \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = 0$$

Similarly, given two lines $L_1 : \alpha_1 X + \beta_1 Y + \gamma_1 Z = 0, L_2 : \alpha_2 X + \beta_2 Y + \gamma_2 Z = 0$, L_1, L_2 are linearly independent, and there is a unique solution $[X_1; Y_1; Z_1]$ (up to multiplication by a scalar) for the system:

$$\begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{pmatrix} \begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which means that any two homogeneous lines intersect at exactly one point in $\mathbb{P}^2$.