Approximating π with an Infinite Series

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1 Introduction

We can sum the infinite series:

$$f(t) = 1 - t^2 + t^4 - t^6 + \cdots = \frac{1}{1+t^2}$$

Integrating across from 0 to x:

$$\int_{t=0}^{x} f(t)dt = \tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

And setting $x = 1$ (which converges in the integral, but not in the original function): $\tan^{-1}(1) = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} - \cdots$

This is a nice formula for π, but is slow to converge. You can use the tan sum formula to find faster converging solutions:

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Let $A = \tan^{-1}\frac{1}{2}, B = \tan^{-1}\frac{1}{3}$ Then apply the tangent sum forula, and take inverse tangents on both side:

$$\tan(A+B) = \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}} = 1$$

$$\tan^{-1}(1) = \frac{\pi}{4} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$$

Then:

$$\frac{\pi}{4} = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{2i-1}\left(\frac{1}{2^{2i-1}} + \frac{1}{3^{2i-1}}\right)$$

which converges many times faster. You can do this for any other numbers you can find for $A+B = 1-AB$ which gives $(A+1)(B+1) = 2$ such as $\frac{1}{4}$ and $\frac{3}{5}$. Another example, more complicated to find, is:

$$\frac{\pi}{4} = 4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right)$$

which converges to 10 digits of π when you take the first 4 terms of the expansion of $\tan^{-1}\frac{1}{5}$ and the first two terms of $\tan^{-1}\frac{1}{239}$.