1 When is $1+p+p^2+p^3+p^4$ prime for prime p?

Problem. *Given a prime number p, when is the sum of all of the positive factors of p^4 a perfect square?*

Proof. Since p is a prime number, the only factors of p^4 are of the form $p^i, i \in \{0,1,2,3,4\}$. The sum of the factors is thus:

$$1+p+p^2+p^3+p^4 = \frac{p^5-1}{p-1}$$

Let's call this sum of powers of p $f(p)$ for convenience. We begin our search with some exploration. When $n = 2,3,5,7,11$ we obtain the following values:

p	$f(p)$	round$(\sqrt{f(p)})^2$
2	31	$6^2 = 36$
3	121	$11^2 = 121$
5	781	$28^2 = 784$
7	2801	$53^2 = 2809$
11	16105	$127^2 = 16129$

3 is the only prime that produces a perfect square so far, and it seems like there is always a square close above the other odd primes. It's also interesting to note that the square close to $f(p)$ is a little more than the square of the square of our prime (not surprising given the p^4 term which will dominate as p grows larger): $28 = 5^2+3, 53 = 7^2+4, 127 = 11^2+6$.

Let's see if we can get close to $f(p)$ with the square of a quadratic expression. Clearly such a quadratic will be of the form:

$$(p^2+ap+b)^2 = p^4 + 2ap^3 + (a^2+2b)p^2 + 2abp + b^2$$

It makes sense to let $a = \frac{1}{2}$, then:

$$(p^2+\frac{p}{2}+b)^2 = p^4+p^3+(\frac{1}{4}+2b)p^2+bp+b^2$$

Given that p is an odd prime, we can turn this into an integer expression by setting $b = \frac{1}{2}$:

$$\begin{aligned}(p^2+\frac{p+1}{2})^2 &= p^4+p^3+\frac{5p^2}{4}p^2+\frac{p}{2}+\frac{1}{4} \\ &= \sum_{i=0}^{4} p^i + \frac{1}{4}p^2 - \frac{p}{2} - \frac{3}{4}\end{aligned}$$

So we can write:

$$(p^2+\frac{p+1}{2})^2 = \sum_{i=0}^{4} p^i + \frac{1}{4}(p+1)(p-3)$$

or

$$(p^2 + \frac{p+1}{2})^2 = \sum_{i=0}^{4} p^i + \frac{1}{4}((p-1)^2 - 4)$$

So this is bigger than $\sum_{i=0}^{4} p^i$ for all $p > 3$ (and, as we saw earlier, is equal for $p = 3$). If we reduce the square by 1, can we prove that this is always smaller than $\sum_{i=0}^{4} p^i$?

$$\begin{aligned}
(p^2 + \frac{p-1}{2})^2 &= p^4 + p^3 - \frac{3p^2}{4} - \frac{p}{2} + \frac{1}{4} \\
&= \sum_{i=0}^{4} p^i - \frac{7}{4}p^2 - \frac{3}{2}p - \frac{3}{4} \\
&< \sum_{i=0}^{4} p^i
\end{aligned}$$

since $p > 0$.

So we have shown that:

$$(p^2 + \frac{p-1}{2})^2 < \sum_{i=0}^{4} p^i \le (p^2 + \frac{p+1}{2})^2$$

with equality if and only if $\frac{1}{4}(p+1)(p-3) = 0$, or when $p = 3$

$\square$

2 When is $p^2 + q^2 + 2017$ a perfect square?

Problem. *How many prime numbers p, q make $p^2 + q^2 + 2017$ a perfect square?*

Proof. Consider the numbers $p^2, q^2, 2017 \pmod 4$.

$$2017 \equiv 1 \pmod 4$$

If p is odd, $p^2 \equiv 1 \pmod 4$, and if p is even, $p^2 \equiv 0 \pmod 4$.

Then:

$$2017 + p^2 + q^2 \pmod 4 \in \{1, 2, 3\}$$

with the value 1 when both p, q are event, 2 if one of them is odd, and 3 if both are odd.

For any integer a, $a^2 \equiv 0$ or $1 \pmod 4$ - so both p, q must be even. The only even prime number is 2, so the only number we need to check is

$$2017 + 2^2 + 2^2 = 2025 = 45^2$$

So the only answer is $(p, q) = (2, 2)$. $\square$