The Wonderful World of Pell's Equations

Dave Neary

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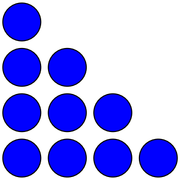
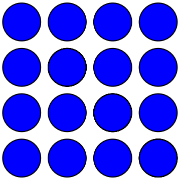
1. A number puzzle
2. Approximating square roots
3. Finding smallest solutions for $m^2 - dn^2 = 1$



Triangular numbers



Square numbers

Can you find triangular numbers (of the form $1 + 2 + 3 + \cdots + a$) that are also perfect squares (b^2 for an integer b)?

Can we find a formula for the ath triangular number:

$$T_a = 1 + 2 + 3 + \cdots + a$$

We are looking for solutions to:

$$\begin{aligned}\frac{1}{2}(a)(a+1) &= b^2\\ a^2+a &= 2b^2\\ 4a^2+4a &= 8b^2\\ (2a+1)^2-8b^2 &= 1\end{aligned}$$

Setting $m = 2a+1, n = 2b$, we get the Pell's equation:

$$m^2 - 2n^2 = 1$$

Pell's Equations: $m^2 - dn^2 = 1$

$$m^2 - dn^2 = 1$$

Equations of this form are called Pell's equations after John Pell, who translated a text on the equation into English, and Euler thought it was original work, and named the equation after him.

Equations of this form have been studied for centuries (especially by Indian mathematicians Brahmagupta and Bhāskara II).

Geometrically, the curve is a hyperbola, but tonight, we will explore its connections to number theory.

By inspection, we can find $m = 3, n = 2$ works:

$$3^2 - 2(2^2) = 9 - 8 = 1$$

But can we find other solutions?
It's a bit trickier!

What does $m^2 - 2n^2$ remind you of?

$$m^2 - 2n^2 = (m - \sqrt{2}n)(m + \sqrt{2}n)$$

But now if we have two numbers of the form $m^2 - 2n^2$ what happens when we multiply these factors?

$$\begin{aligned}(a - \sqrt{2}b)(c - \sqrt{2}d) &= ac - \sqrt{2}bc - \sqrt{2}ad + 2bd \\ &= (ac + 2bd) - \sqrt{2}(bc + ad)\end{aligned}$$

$$\begin{aligned}(a + \sqrt{2}b)(c + \sqrt{2}d) &= ac + \sqrt{2}bc + \sqrt{2}ad + 2bd \\ &= (ac + 2bd) - \sqrt{2}(bc + ad)\end{aligned}$$

Numbers $a + \sqrt{2}b$ are closed under multiplication

This means is that when we multiply numbers of the form $a + \sqrt{2}b$ together, they give a result of the same form!

These numbers are an extension to the integers - we can add, subtract, and multiply numbers $a + \sqrt{2}b$ together and the result is in the same form.

What's more:

$$(a - \sqrt{2}b)(c - \sqrt{2}d) = A - \sqrt{2}B$$

$$(a + \sqrt{2}b)(c + \sqrt{2}d) = A + \sqrt{2}B$$

What this means is that when we multiply numbers of the form $a^2 - 2b^2$, we can make the product into the form $A^2 - 2B^2$ for some integers A, B.
But we can now generate larger solutions to our problem!

$$(3 - 2\sqrt{2})^2(3 + 2\sqrt{2})^2 = (17 - 12\sqrt{2})(17 + 12\sqrt{2}) = 1^2$$

$$(3 - 2\sqrt{2})^3(3 + 2\sqrt{2})^3 = (99 - 70\sqrt{2})(99 + 70\sqrt{2}) = 1^3$$

There are an infinite number of solutions!

Notice anything?

$$3^2 - 2(2^2) = 1, \frac{3}{2} = 1.5$$

$$17^2 - 2(12^2) = 1, \frac{17}{12} = 1.41666\cdots$$

$$99^2 - 2(70^2) = 1, \frac{99}{70} = 1.414285\cdots$$

These numbers are getting close to a well known constant... why?

If we take large solutions to:

$$a^2 - 2b^2 = 1$$

then:

$$\left(\frac{a}{b}\right)^2 = 2 + \frac{1}{b^2}$$

and as b gets large, $\frac{a}{b}$ gets very close to $\sqrt{2}$.

Bonus trick: if one estimate is $\frac{a}{b}$, can you prove that the next one is $\frac{3a+4b}{2a+3b}$ and explain why?

Choosing a different value for d

This works just as well for other values of d. For example, with $d = 5$:

$$m^2 - 5n^2 = 1$$

has a solution at $a = 9, b = 4$: $9^2 - 5(4^2) = 1$.
We can get very good estimates for $\sqrt{5}$ by raising this solution to different powers:

$$(9 - 4\sqrt{5})^2 = 161 - 72\sqrt{5}$$

$$\frac{161}{72} = 2.2361\cdots, \sqrt{5} = 2.23606\cdots$$

Finding the smallest solution

Finding $a = 9, b = 4$ for $d = 5$ was a little trickier than for $d = 2$ - in general, how can we find the smallest solution without checking every number?

For example: what is the smallest solution for $d = 10$? Or $d = 13$?

Continuous fractions!

Enter continued fractions! Any positive real number can be expressed as a continuous fraction of the form:

$$x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}$$

where every one of the $\{a_i\}$ are whole numbers.

When written like this, we can find good rational approximations to the number x by stopping the calculation after a few numbers. These approximations are called "convergents".

Calculating a continued fraction for $\sqrt{10}$

Since $3 < \sqrt{10} < 4$:

$$\sqrt{10} = 3 + (\sqrt{10} - 3)$$

We found on the last slide that:

$$\sqrt{10} = 3 + \cfrac{1}{6 + \frac{1}{6 + \frac{1}{6 + \cdots}}}$$

We can now calculate the first few convergents:

$$3 + \frac{1}{6} = \frac{19}{6}$$

$$19^2 - 10(6^2) = 361 - 10(36) = 1$$

And we have our smallest solution!

Complicated solutions

There are some famously difficult first solutions, for example:
$d = 13$

$$\sqrt{13} = 3 + \cfrac{1}{1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{6+\cdots}}}}}$$

The convergents for this continued fraction are:

$$\frac{3}{1}, \frac{4}{1}, \frac{7}{2}, \frac{11}{3}, \frac{18}{5}, \frac{119}{33}, \frac{137}{38}, \frac{256}{71}, \frac{393}{109}, \frac{649}{180}, \ldots$$

The smallest solution to $m^2 - 13n^2 = 1$ is

$$649^2 - 13(180^2) = 1$$

Challenge: Can you find the smallest solution to $m^2 - 41n^2 = 1$?

AoPS

Conclusion

The Pell's equations pull together diverse fields of mathematics:

- Number theory - extensions to the integers
- Rational approximations for irrational numbers
- Continued fractions
- Algebraic geometry

These equations continue to be relevant today!