Generating functions and counting problems

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1 Problem

How many solutions are there to the equation $n = a + b + c + d$, where $a \leq b \leq c \leq d$ and $a, b, c, d, n \in \mathbb{N}$?

2 Solution

The natural numbers $\mathbb{N} = \{1, 2, 3, \cdots\}$ — that is, the positive integers (not 0). It will be easier to work on this problem if our solutions are in the non-negative integer, including 0.

We can rewrite the equation as:

$$x_1 = a' + b' + c' + d'$$

where: $x_1 = x - 4, a' = a - 1, b' = b - 1, c' = c - 1, d' = d - 1$, and the inequality for a, b, c, d still holds for a', b', c', d'.

We can go one step further, and remove this inequality, by focusing on the differences between the variables. Since we know that $a' \leq b' \leq c' \leq d'$ we can rewrite $a_1 = a', b_1 = b' - a', c_1 = c' - b', d_1 = d' - c'$, and substituting these in to the equation, we get:

$$x_1 = 4a_1 + 3b_1 + 2c_1 + d_1$$

where each of $a_1, b_1, c_1, d_1 \geq 0$.

There's a nice method of calculating this using generating functions. Consider:

$$P(x) = (1 + x^4 + x^8 + \cdots)(1 + x^3 + x^6 + \cdots)(1 + x^2 + x^4 + \cdots)(1 + x + x^2 + \cdots)$$

This expands to an infinite series:

$$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

where each coefficient corresponds to the number of solutions for $x_1 = n, x = n + 4$.

To get the coefficient in the expansion for x^{12} for example, we need to look at all the ways that you can combine multiples of 1,2,3, and 4 to add to 12. We can quickly find that the possible solutions are:

$$(a_1, b_1, c_1, d_1) \in \{(3,0,0,0), (2,1,0,1), (2,0,2,0), (2,0,1,2), (2,0,0,4), \cdots\}$$

where the solution $(2,1,0,1)$ for example corresponds to taking an x^8 term from $(1+x^4+x^8+\cdots)$, an x^3 from $(1+x^3+x^6+\cdots)$, a 1 from $(1+x^2+x^4+\cdots)$, and an x from $(1+x+x^2+\cdots)$.

Then we can sum these geometric series to get:

$$P(x) = \frac{1}{(1-x^4)(1-x^3)(1-x^2)(1-x)}$$

We can use partial fraction decomposition on this to get a closed form for a_n.

Our denominator can be factored:

$$\begin{aligned} P(x) &= \frac{1}{(1-x^4)(1-x^3)(1-x^2)(1-x)} \\ &= \frac{1}{(1-x)^4(1+x)^2(1+x^2)(1+x+x^2)} \\ &= \frac{1}{(1-x)^4(1+x)^2(1+ix)(1-ix)(1-\omega x)(1-\omega^2 x)} \end{aligned}$$

where $\omega = \frac{1}{2}(-1+\sqrt{3}i)$, a primitive cube root of unity.

Then we can write this as:

$$\begin{aligned} P(x) = \frac{A_1}{1-x} + \frac{A_2}{(1-x)^2} + \frac{A_3}{(1-x)^3} + \frac{A_4}{(1-x)^4} \\ + \frac{A_5}{1+x} + \frac{A_6}{(1+x)^2} \\ + \frac{A_7}{1+ix} + \frac{A_8}{1-ix} + \frac{A_9}{1-\omega x} + \frac{A_{10}}{1-\omega^2 x} \end{aligned}$$

Calculating the A_i values is tedious but straightforward using the Heaviside cover-up method. Clearing denominators, we get:

$$\begin{aligned} 1 &= A_1(1-x)^3(1+x)^2(1+x^2)(1+x+x^2) + A_2(1-x)^2(1+x)^2(1+x^2)(1+x+x^2) \\ &+ A_3(1-x)(1+x)^2(1+x^2)(1+x+x^2) + A_4(1+x)^2(1+x^2)(1+x+x^2) \\ &+ A_5(1-x)^4(1+x)(1+x^2)(1+x+x^2) + A_6(1-x)^4(1+x^2)(1+x+x^2) \\ &+ A_7(1-x)^4(1+x)^2(1-ix)(1+x+x^2) + A_8(1-x)^4(1+x)^2(1+ix)(1+x+x^2) \\ &+ A_9(1-x)^4(1+x)^2(1+x^2)(1-\omega^2 x) + A_{10}(1-x)^4(1+x)^2(1+x^2)(1-\omega x) \end{aligned}$$

Now we can set x to various values to isolate and calculate the coefficients (since the equation above must hold for all values of x).

Setting $x = 1, x = -1, x = i, x = -i, x = \omega^2, x = \omega$ in order we get

$$A_4 = \frac{1}{24}, A_6 = \frac{1}{32}, A_7 = \frac{1}{16}, A_8 = \frac{1}{16}, A_9 = \frac{1-\omega}{27}, A_{10} = \frac{1-\omega^2}{27}$$

Then I set $x = 2, -2, 0, 3$ respectively to get four simultaneous equations in A_1, A_2, A_3, A_5. When all is said and done, I get:

$$\begin{aligned} P(x) = {} & \frac{17}{72(1-x)} + \frac{59}{288(1-x)^2} + \frac{1}{8(1-x)^3} \\ & + \frac{1}{24(1-x)^4} + \frac{1}{8(1+x)} + \frac{1}{32(1+x)^2} + \frac{1}{16(1-ix)} \\ & + \frac{1}{16(1+ix)} + \frac{1-\omega}{27(1-\omega x)} + \frac{1-\omega^2}{27(1-\omega^2 x)} \end{aligned}$$

And if I haven't made a mistake, after turning each of these simple fractions into its own infinite series as follows:

$$\begin{aligned} \frac{1}{1-x} &= 1 + x + x^2 + \cdots \\ \frac{1}{(1-x)^2} &= 1 + 2x + 3x^2 + 4x^3 + \cdots \\ \frac{1}{(1-x)^3} &= 1 + \binom{3}{2}x + \binom{4}{2}x^2 + \binom{5}{2}x^3 + \cdots \\ \frac{1}{(1-x)^4} &= 1 + \binom{4}{3}x + \binom{5}{3}x^2 + \binom{6}{3}x^3 + \cdots \\ \frac{1}{1+x} &= 1 - x + x^2 - \cdots \\ \frac{1}{(1+x)^2} &= 1 - 2x + 3x^2 - 4x^3 + \cdots \\ \frac{1}{1+ix} &= 1 + ix - x^2 - ix^3 + x^4 + \cdots \\ \frac{1}{1-ix} &= 1 - ix - x^2 + ix^3 + x^4 - \cdots \\ \frac{1}{1-\omega^2 x} &= 1 + \omega^2 x + \omega x^2 + x^3 + \omega^2 x^4 + \omega x^5 + x^6 + \cdots \\ \frac{1}{1-\omega x} &= 1 + \omega x + \omega^2 x^2 + x^3 + \omega x^4 + \omega^2 x^5 + x^6 + \cdots \end{aligned}$$

And when we plug everything in, we get a coefficient for a_n (reminder, this is the number of solutions for partitions in four ordered natural numbers for

$x = n + 4$) of:

$$a_n = \frac{17}{72} + \frac{59}{288}\binom{n+1}{1} + \frac{1}{8}\binom{n+2}{2} + \frac{1}{24}\binom{n+3}{3}$$
$$+ \frac{1}{8}(-1)^n + \frac{1}{32}\binom{n+1}{1}(-1)^n + \frac{1}{16}(i^n + (-i)^n)$$
$$+ \frac{1}{27}\left(\omega^n + \omega^{2n} - \omega^{n+1} - \omega^{2n+2}\right)$$

The $\frac{1}{16}(i^n + (-i)^n)$ terms equal 0 for odd terms, $\frac{1}{8}$ for terms where n is divisible by 4, and $-\frac{1}{8}$ for other even terms.

Similarly, the $\frac{1}{27}\left(\omega^n + \omega^{2n} - \omega^{n+1} - \omega^{2n+2}\right)$ terms equal zero, $-\frac{1}{9}$, or $\frac{1}{9}$, depending on whether n has a remainder of 0, 1, or 2 when divided by 3.

Let's check our closed form solution for $n = 8$ - that is, for $x = 12$. By inspection, we can see that the solutions are:

$$(a_1, b_1, c_1, d_1) \in \{(2,0,0,0), (1,1,0,1), (1,0,2,0), (1,0,1,2),$$
$$(1,0,0,4), (0,2,1,0), (0,2,0,2), (0,1,2,1), (0,1,1,3), (0,1,0,5),$$
$$(0,0,4,0), (0,0,3,2), (0,0,2,4), (0,0,1,6), (0,0,0,8)\}$$

which gives 15 solutions. Again, the solution $(0,1,2,1)$ (for example) corresponds to $a_1 = 0, b_1 = 1, c_1 = 2, d_1 = 1$, which translates to $a' = 0, b' = 1, c' = 3, d' = 4$, or $a = 1, b = 2, c = 4, d = 5$. Using our formula, we get:

$$a_8 = \frac{17}{72} + \frac{59}{288}\binom{9}{1} + \frac{1}{8}\binom{10}{2} + \frac{1}{24}\binom{11}{3}$$
$$+ \frac{1}{8}(-1)^8 + \frac{1}{32}\binom{9}{1}(-1)^8 + \frac{1}{16}(i^8 + (-i)^8)$$
$$+ \frac{1}{27}\left(\omega^8 + \omega^{16} - \omega^9 - \omega^{18}\right)$$

$$a_8 = \frac{17}{72} + \frac{59}{32} + \frac{45}{8} + \frac{55}{8} + \frac{1}{8} + \frac{9}{32} + \frac{1}{8} - \frac{1}{9} = 15$$

It's pretty amazing that this complicated fractional expression including binomial coefficients works, but it does!

3 Recurrence relation

It is possible to calculate the number of partitions also using a recurrence relation. If we define: $P_k(n)$ to be the number of ordered partitions of the number n into exactly k non-zero partitions, we can deduce the following:

- $P_0(0) = 1$ (by definition - similar to defining $0! = 1$, this ensures the recurrence relationship below terminates in all cases).

- $P_k(n) = 0$ if $k \leq 0, n \leq 0$ and k, n are not both 0.
- $P_k(n) = P_k(n-k) + P_{k-1}(n-1)$ - that is, we have a choice to increment the size of all partitions by 1 and leave $n-k$ items to distribute across exactly k buckets, or we can fix one bucket, and we have $n-1$ items to distribute across the other $k-1$ buckets.

We can come up with some quick short-cuts for $P_k(n)$ for small values of k:

- $P_1(n) = 1$ for all $n \geq 0$ - that is, with exactly 1 partition, there is only one possible representation.
- $P_n(n) = 1$ is the transposed equivalent - with n buckets and n items, there is only one way to distribute the items so that no bucket is empty.
- $P_k(n) = 0$ if $k > 0, n < k$.
- $P_2(n) = \lfloor \frac{n}{2} \rfloor$
- $P_3(n) = \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n-4}{2} \rfloor + \lfloor \frac{n-7}{2} \rfloor + \cdots$

From this, we can reproduce the result above, albeit a little awkwardly:

$$\begin{aligned}
P_4(12) &= P_4(8) + P_3(11) \\
&= P_4(4) + P_3(7) + P_3(8) + P_2(10) \\
&= 1 + \lfloor \frac{6}{2} \rfloor + \lfloor \frac{3}{2} \rfloor + \lfloor \frac{7}{2} \rfloor + \lfloor \frac{4}{2} \rfloor + \lfloor \frac{10}{2} \rfloor \\
&= 1 + 3 + 1 + 3 + 2 + 5 = 15
\end{aligned}$$

as before. I have not found any nice closed form solution to this recurrence relation, however, and the recurrence relation, while easy to calculate by computer, becomes very unwieldy when calculating by hand.