Pythagorean triples

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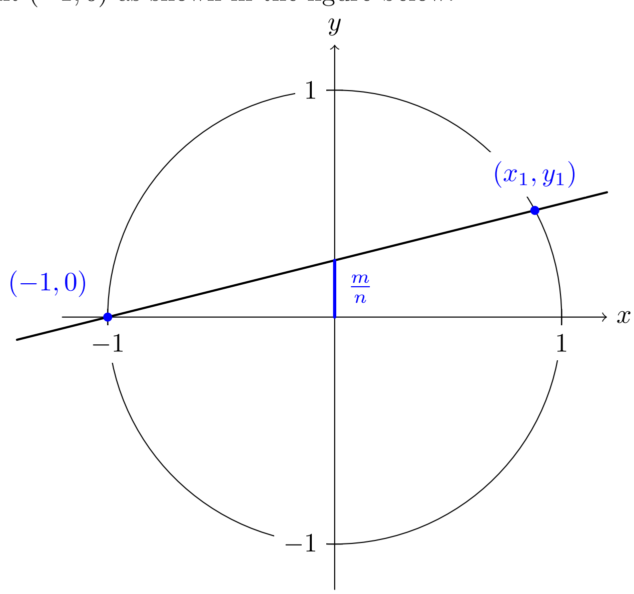
1 Finding all Pythagorean triples

A Pythagorean triple is a set of three positive integers which are the side lengths of a right-angled triangle - that is, by Pythagoras's Theorem, the triple (a, b, c) satisfies $a^2 + b^2 = c^2$. There are many well-known examples of these, including $(3, 4, 5), (5, 12, 13), (7, 40, 41)$ - and so on. Our challenge is to find a way by which we can identify all possible Pythagorean triples.

To do this, we are going to consider lines with a rational slope, through the point $(-1, 0)$ as shown in the figure below.



A line with a rational slope $\frac{m}{n}$ through $(-1, 0)$ will intersect the y-axis at the point $(0, \frac{m}{n})$. It will also intersect the unit circle $x^2 + y^2 = 1$ in two points: $(-1, 0)$ and (x_1, y_1). Since the unit circle is a quadratic curve, and the slope of the line is rational, then if one intersection point is a rational point, the second intersection must also be rational.

We will find its coordinates in terms of the slope $\frac{m}{n}$. In slope-intercept form, the equation of the line is:

$$y = \frac{m}{n}(x+1)$$

Substituting this expression for y into the equation for the circle, we get the intersection points:

$$\begin{aligned}
x^2 + \left(\frac{m}{n}(x+1)\right)^2 &= 1 \\
x^2 + \frac{m^2}{n^2}x^2 + \frac{2m^2}{n^2}x + \frac{m^2}{n^2} - 1 &= 0 \\
\left(\frac{m^2+n^2}{n^2}\right)x^2 + \frac{2m^2}{n^2}x + \frac{m^2-n^2}{n^2} &= 0 \\
(x+1)\left(\frac{m^2+n^2}{n^2}x + \frac{m^2-n^2}{n^2}\right) &= 0
\end{aligned}$$

So we have intersection points when $x = -1$ and $x = \frac{n^2-m^2}{n^2+m^2}$. Substituting this value for x back into the line equation gives:

$$\begin{aligned}
y_1 &= \frac{m}{n}\left(\frac{n^2-m^2}{n^2+m^2} + 1\right) \\
&= \frac{m}{n}\left(\frac{2n^2}{n^2+m^2}\right) \\
&= \frac{2mn}{n^2+m^2}
\end{aligned}$$

So for any rational slope $\frac{m}{n}$ we can find a rational point on the unit circle $\left(\frac{n^2-m^2}{n^2+m^2}, \frac{2nm}{n^2+m^2}\right)$ which, in turn, gives us a Pythagorean triple, by setting $(a, b, c) = (n^2 - m^2, 2nm, n^2 + m^2)$.

Since $a^2 + b^2 = c^2$ is equivalent to $\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$, we can always turn a Pythagorean triple into a rational point on the unit circle. And since all lines between $(-1, 0)$ and a rational point on the unit circle have a rational slope, and our construction above works for all rational numbers, every Pythagorean triple will be of the form above, or an integer multiple of a point of the form above.