Logarithms

The Base

Let's review using exponential functions.

The exponential function $f(x)$ with base b in simplest form:

$$f(x) = b^x \quad or \quad y = b^x$$

The base b must be a **positive constant** and x is any real number. Note that while the base can technically be 1, that would be meaningless because 1 raised to any power is always 1. This implies the following are ***not* exponential functions**:

$f(x) = x^2$	$g(x) = 1^x$	$h(x) = x^x$
x is the base	base of 1	x is both base and exponent

Common bases are 10 (what we are used to dealing with), 2 (called binary and used in computer science), and the natural base, e, or Euler's number or sometimes Napier's constant.

> Note that "Euler" is a German name, Leonhard Euler (1707–1783), a Swiss mathematician and scientist, and is pronounced OY-ler, as "eu" in German is pronounced "OY."

$$e = 2.718281828459045235360287471352662497757247093699995\ldots$$

Logarithm as Inverse of Exponent

A logarithm is an inverse function of an exponent. For some base b and exponent x:

$$y = b^x \quad \text{then} \quad x = \log_b y$$

Note the notation of the logarithm function: "log" followed by the base as a subscript. Note that many times when using a logarithmic base of 10 (referred to as common log), the 10 subscript may be omitted. When the base is Euler's number, e, the logarithm function is referred to as the natural logarithm, or just "natural log," and uses "ln" without a subscript instead of "$\log_e$."

Examples

Base 10: $10^3 = 1{,}000 \quad \text{therefore} \quad \log_{10} 1000 = 3$

Base 2: $2^4 = 16 \quad \text{therefore} \quad \log_2 16 = 4$

Note this is the basis for using hexadecimal (base 16) numbers in computers as a shorthand for binary, where each value represents a set of four bits.

Natural Logarithm, base e: $e^2 = 7.389 \quad \text{therefore} \quad \ln 7.389 = 2$

Bases and Place Values

The base is how many times we count before we then move to the next higher place value. We do this everyday but don't really think about it. Let's review number bases and place value tables.

Base 10

In base 10, we count 9 items and track them (the "ones" place), then the 10^{th} item is tracked in the "tens" place with the "ones" place reset to zero. We call this the "carry." Think back to addition of numbers larger than ten:

$$18 + 5 = 23$$

First add the “ones” column, 8 +5 = 13. We write the 3 then “carry” the 1 (really, it’s a 10) to the next column to the left, the “tens” column” and add that to the one before the 8 in 18.

In reality, we’re breaking up the 18 into 10 + 8 then adding 8 + 5 first. We then take the carried 1 from 13, which is really 10, because that 1 is in the “tens” column, and add it to the 10 part of the original 18, which gives us 2 in the “tens” column, or 20. Add the 20 to the 3 to get 23.

	Tens	Ones	
	1	8	
+		5	
	1	3	Add the ones first, note the carry in the tens column
	2	0	Now add the tens, note the place holder zero in the ones column
	2	3	Add the tens and ones together for the final answer

In base 10, we have 10 symbols used to count, 0 through 9, because again, a count of “ten” is a carry to the next greater column with zero in the current column. We say we’re “counting by tens.”

With exponents of base 10, because this is the system we use every day, increasing the exponent effectively adds a zero to the right because each column in the place value chart is a power of ten.

Ten Thousands	Thousands	Hundreds	Tens	Ones	
10^4	10^3	10^2	10^1	10^0	
				1	$1 = 10^0$
				2	$2 = 2 \times 10^0$ or “2 ones”
			1	0	$10 = 10^1$ or “1 ten and 0 ones”
			4	9	49 is “4 tens and 9 ones.” What happens when you add 1?
	5	0	0	0	$5{,}000 = 5 \times 10^3$ or “5 thousands” or just “five thousand”
3	2	7	6	8	$32{,}768 = 3 \times 10^4 + 2 \times 10^3 + 7 \times 10^2 + 6 \times 10^1 + 8 \times 10^0$

Notice the many uses of 10 raised to some integer power. Logarithms give you the powers. Common logarithms give you the powers for base 10, like above.

Base 2

Base 2, commonly known as binary, is used in electronics and computers because the two symbols, 0 and 1, represent “off” and “on,” or any two-state system. In binary, the carry happens when the value of the current place value increases to 2. This means our 2 is represented by “10.”. Each successive column in the place value table is an increasing power of two, which is doubling the value from the preceding column.

Sixteens	Eights	Fours	Twos	Ones	
2^4	2^3	2^2	2^1	2^0	
				1	$1 = 2^0$
			1	0	$2 = 2^1$
			1	1	$3 = 1 \times 2^1 + 1 \times 2^0$ or “1 two and 1 ones”
	1	0	1	0	Ten is “1 eight, 0 fours, 1 two, and 0 ones.”
	1	1	1	1	Fifteen. What happens when you add one?
1	0	0	0	0	Sixteen. Notice this is the “carry” from adding 1 to fifteen.

You will not normally encounter logarithms for base 2, but I show it here to demonstrate how place value tables can be a powerful tool.

Base 16

16 symbols for the numbers 0 through 15: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A (ten), B (eleven), C (twelve), D (thirteen), E (fourteen), F (fifteen). Sixteen triggers a carry so is written as 10 ($1 \times 16^1 + 0 \times 16^0$).

Notations

Logarithms of base 10 and base *e* are special cases and have their own names.

Common Log

Common logs are logarithms with a base of 10. Base 10 is our everyday, *common* base, so logarithms of base 10 are referred to as *common logs*. In the notation, the subscript 10 may be left out, so these are equivalent:

$$\log_{10} 100 = 2 \quad \text{and} \quad \log 100 = 2$$

The common logarithm key on your calculator is called LOG.

Natural Log

A logarithm with a base of *e*, Euler's number (approximately 2.718) is called a *natural logarithm* due to Euler's number describing many phenomena in nature. A shorthand notation for $\log_e$ is ln, for "logarithm, natural." These are equivalent:

$$\log_e 7.389 = 2 \quad \text{and} \quad \ln 7.389 = 2$$

Note that $e^2 \approx 7.389$, which you can determine with your calculator. Make sure you know where your *e* key and ln keys are.

Logarithm Properties

Basic Properties

Because raising any base to the power of one to obtain itself ($x^1 = x$) and raising any base to the power of zero yields 1 ($x^0 = 1$). By extension, the same holds true for common logs and natural logs.

$$\log_b b = 1 \quad \log 10 = 1 \quad \ln e = 1$$

$$\log_b 1 = 0 \quad \log 1 = 0 \quad \ln 1 = 0$$

Inverse Properties

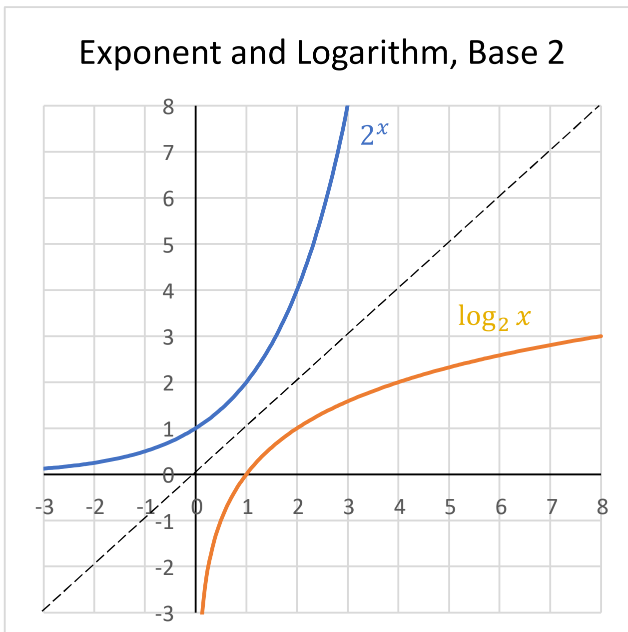
For any base b greater than zero and not equal to one:

$$\log_b b^x = x \quad \log 10^x = x \quad \ln e^x = x$$

$$b^{\log_b x} = x \quad 10^{\log x} = x \quad e^{\ln x} = x$$

Graph

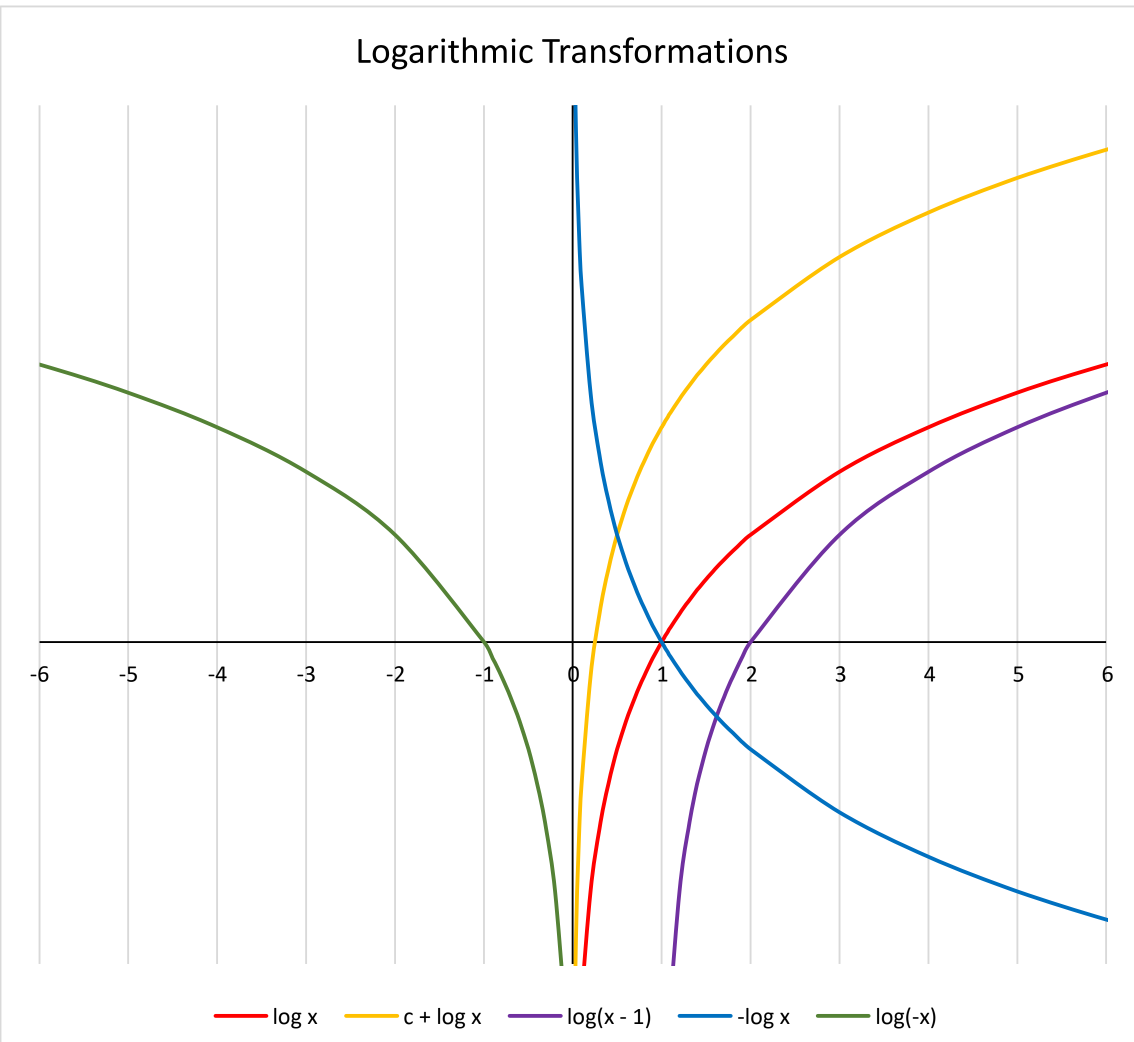
Note how the graphs of the logarithm and exponent for any base are symmetrical about the 45° line. The graph uses a base of 2 as an example.



Transformations

Given the equation f(x) = $\log_b x$, $g(x)$ represents the transformed functions.

Transformation	Equations	Description
Vertical	$g(x) = \log_b x + c$ $g(x) = \log_b x - c$	Shift graph up by *c* units Shift graph down by *c* units
Horizontal	$g(x) = \log_b (x + c)$ $g(x) = \log_b (x - c)$	Shift graph left by *c* units Shift graph right by *c* units
Reflection	$g(x) = -\log_b x$ $g(x) = \log_b (-x)$	Reflect about the *x*-axis Reflect about the *y*-axis
Vertical stretch or shrink	$g(x) = c \log_b x$	$c > 1$: stretch $0 < c < 1$: shrink



Rules

The following rules are predicated on all expressions working with the same base, not equal to 1, except for the *change of base* rule. These rules are analogous to multiplying and dividing powers in exponential expressions.

Product Rule

Let b, M, N be positive real numbers with $b \neq 1$.

$$\log_b(MN) = \log_b M + \log_b N$$

Quotient Rule

Let b, M, N be positive real numbers with $b \neq 1$.

$$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$$

Power Rule

Let b, M be positive real numbers with $b \neq 1$ and p any real number.

$$\log_b M^p = p \log_b M$$

Note that this rule may change the domain of the expression from $(0, +\infty)$ to $(-\infty, +\infty)$.

Change of Base Rule

For any logarithmic bases a and b, $M > 1$. Change from base a to base b:

$$\log_b M = \frac{\log_a M}{\log_a b}$$

This same rule can be used to convert to the common log (base 10) and natural log (base e).

Changing Base to Common Logarithm

$$\log_b M = \frac{\log M}{\log b} \qquad \log M = \frac{\log_b M}{\log_b 10}$$

Changing Base to Natural Logarithm

$$\log_b M = \frac{\ln M}{\ln b} \qquad \ln M = \frac{\log_b M}{\log_b e}$$