Rational Functions and Limits

Rational Functions

A function, $f(x)$, that can be expressed as a ratio of two other functions. $p(x)$ and $q(x)$:

$$f(x) = \frac{p(x)}{q(x)}$$

Two important conditions:

1. $p(x)$ and $q(x)$ have no common factors, and
2. $q(x) \neq 0$.

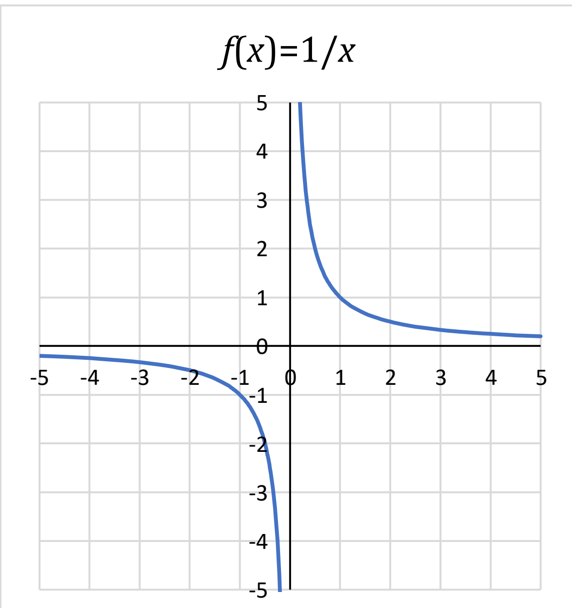
Vertical Asymptotes

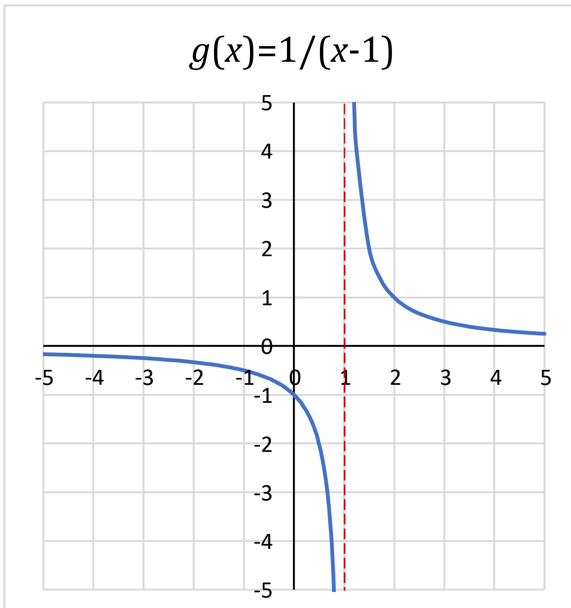
The vertical line, $x = a$, where the value of the function increases or decreases without bound, that is, the value goes to $+\infty$ or $-\infty$ as $x \to a$.

Classic example: $f(x) = {}^{1}/_{x}$

Vertical asymptote at $x = 0$.

This is where the denominator, or $q(x)$ when referring to rational functions, is 0.





Example: $g(x) = \frac{1}{x-1}$ .Find the location of the vertical asymptote.

What value of x sets the denominator to zero? What does this look like?

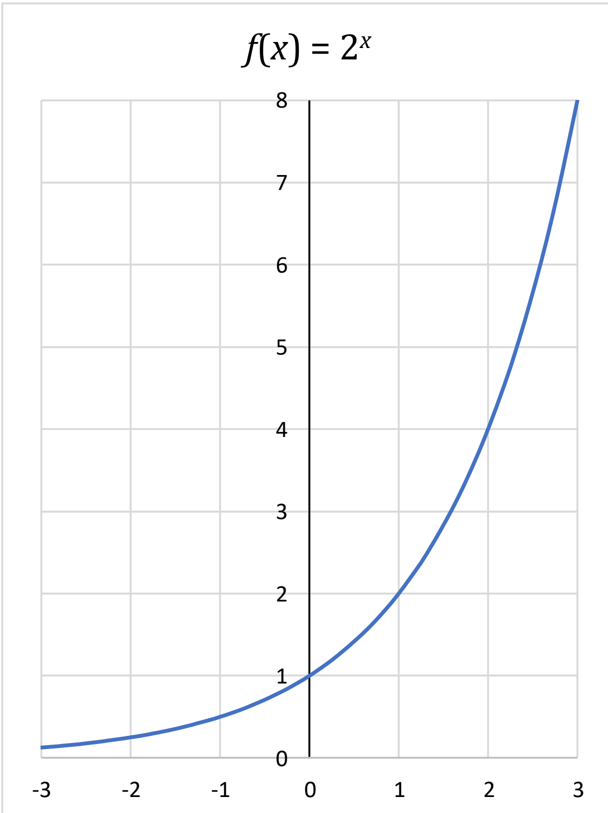
Horizontal Asymptotes

The horizontal line, $y = b$, where the value of the function approaches b as x increases or decreases without bound, that is, x goes to $+\infty$ or $-\infty$.

Example: $f(x) = 2^x$

On the positive x side, the value of the function goes to $+\infty$ as $x \to +\infty$.

But on the negative x side, the value of the function goes to zero as $x \to -\infty$. That is, the function is asymptotic to $y = 0$.



Limits

There are three basic kinds:

1. The function is undefined at one or more discrete points,
2. The value of the function goes to $+\infty$ or $-\infty$ as x approaches a particular value (vertical asymptotes) or itself goes to $+\infty$ or $-\infty$.
3. The value of the function approaches a particular value as x goes to $+\infty$ or $-\infty$ (horizontal asymptotes).

The Obvious Ones

1. Lines: $y \propto x$
2. Parabolas: $y \propto x^2$
3. Cubics: $y \propto x^3$
4. Hyperbolas: $y \propto 1/x$

Determining Limits

Basic Rule

The highest order term dominates.

Example: Determine $\lim_{x\to\infty} \frac{3x^2}{2x}$

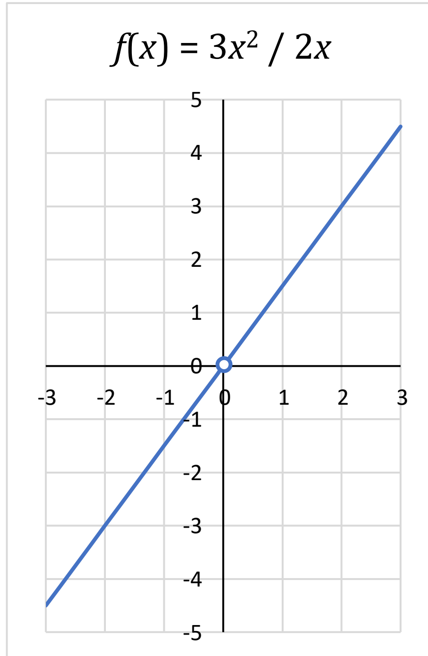
The numerator, $3x^2$, is the highest order and dominates (seen with basic algebra)

Now in the form which is a line with positive slope, therefore as $x \to \infty$ then the function approaches ∞.

EXCEPT

In this case, because of the denominator, the function is undefined at $x = 0$.

This must be indicated on the graph with an open circle for this ***discontinuity***.



Limits with Rational Expressions

Consider:

$$\lim_{x\to\infty} \frac{8x^2 + 5x}{4x^2 + 7}$$

There is a temptation to say the value approaches infinity, but this is not quite as simple.

As with above, the rule that the highest order terms dominate still applies, but we need to work on it a little.

Rule: Divide both the numerator and denominator by the highest order variable, in this case x^2.

$$\lim_{x\to\infty} \frac{8x^2 + 5x}{4x^2 + 7} \cdot \frac{\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)} \qquad \lim_{x\to\infty} \frac{8 + \frac{5}{x}}{4 + \frac{7}{x^2}} \qquad \lim_{x\to\infty} \frac{8}{4} \qquad = \quad 2$$

These go to zero because x is in the denominator