A Trilemma for Asset Demand Estimation

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Introduction

- There is a wealth of new data on portfolio choices by important investors.

  How and what can we learn from this data?

- Paradigmatic approach today follows IO: **estimate demand functions for financial assets**.

  Use "supply shocks" (or instruments) to estimate the asset-level price elasticity of demand

$$\mathcal{E}_{jk} = -\frac{\partial a_j(\vec{a}_{-j}, \vec{p})}{\partial p_k} \times \frac{p_k}{a_j(\vec{a}_{-j}, \vec{p})}.$$

- Promise: inform our understanding of price determination, policy transmission, and counterfactuals.

Two questions

1. Is asset-level demand estimation the right paradigm for empirical analysis of portfolio data?

   Which specific thought experiments and/or behavioral parameters does this capture?

2. When can asset-level elasticities be identified from observational data?

   – Can ideal asset-level supply shocks reveal substitution patterns beyond those implied by theory?

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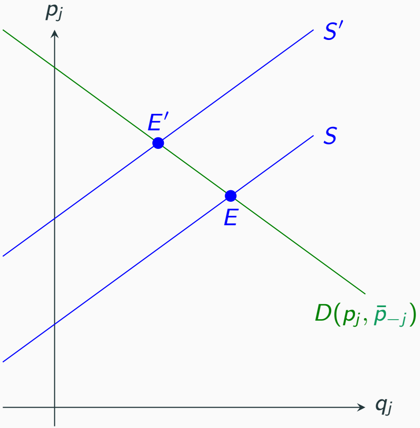
Finance is different: asset demand is *instrumental*, and there is flexible bundling and unbundling.

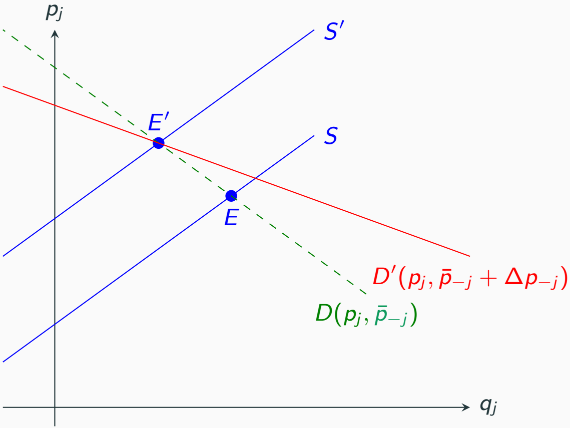
Exceptions: the asset menu consists of Arrow securities, or one has *many* independent experiments.

Beyond these edge cases, demand elasticities reflect a-priori theoretical restrictions.

Summary graph



(a) Identification given **stable** demand for asset j.



(b) Identification failure with endogenous demand shifts.

Environment

- Two dates, $t = 0, 1$. At date 1, one of Z states of the world is realized.

- Investors choose among J assets. Y is the $J \times Z$ payoff matrix, $y_j(z) \geq 0$ is j's payoff in state z.
- Markets can be complete or incomplete.

- p is the vector of asset prices, q is the vector of state prices (need not be unique).

- Potentially heterogeneous investors indexed by i.
- Investor i has endowment e_j^i of asset j. Aggregate endowment of asset j is $E_j = \sum_i e_j^i$.

Standard decision problem

Investors maximize utility over consumption at dates zero and one:

$$V^i = \sup_{a^i \in \mathcal{A}^i} \quad (1-\delta^i)u^i(c_0^i) + \delta^i \sum_{z=1}^{Z} \pi_z u^i(c_z^i) \qquad \text{(Utility over payoffs)}$$

$$\text{s.t.} \quad c_0^i = e_0^i - \sum_{j=1}^{J} p_j(a_j^i - e_j^i) \qquad \text{(Date 0 Budget)}$$

$$c_z^i = \sum_{j=1}^{J} y_j(z) a_j^i \quad \text{for all } z \qquad \text{(State } z \text{ Budget)}$$

NB: We treat c_0 as the numeraire good. Could also allow additional “taste” parameters.

Why care about no arbitrage in demand estimation?

1. With flexible portfolio formation, no arbitrage is required to ensure well-defined demand functions.
2. It is *practically important* to reduce dimensionality of choice sets (e.g., using characteristics).
3. It is a weak requirement on equilibrium play that ensures consistency in counterfactuals.

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Assumption 1. There is no arbitrage,

$$p = Yq.$$

We use a strict definition, but similar issues are at play with "approximate arbitrage."

What do demand elasticities capture?

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"How does asset demand change if asset price p_j changes but all other asset prices remain fixed?"

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$\Rightarrow$ *"How do investors respond to the state price changes induced by an shock to asset prices?"*

State price changes in an ideal experiment

Let Y^+ denote the Moore-Penrose inverse. Given no arbitrage, $q = Y^+ p$.

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Lemma. Let v_j denote the unit vector in $\mathbb{R}^J$ with 1 in the j-th position and zeros elsewhere. Then the changes in state prices given the exogenous variation in a single price p_j satisfy

$$\boxed{\Delta\mathbf{q}^{\text{ideal}} \equiv \frac{\partial q}{\partial p_j} = Y^+ v_j.}$$

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Elasticity measurement requires state price variation proportional to the **inverse** payoff matrix.

Identification from supply shocks

What do supply shocks buy us?

Assume that we have an "ideal" laboratory with exogenous shocks to the supply of some asset j.

- For example, an outside investor helicopter drops assets for purely exogenous reasons.

Does this variation identify a demand elasticity with respect to price p_j?

Minimal assumption: downward-sloping aggregate demand for consumption

To understand equilibrium effects, we impose a minimal condition satisfied by "standard models."

Definition (Downward-sloping consumption demand)

Consumption demand is downward sloping if there exists a strictly positive $Z \times Z$ diagonal matrix V s.t.

$$\Delta \mathbf{q}_j^{\text{supply}} \equiv \frac{\partial q}{\partial E_j} = -V y_j^{\mathrm{T}} \qquad \text{for all assets } j,$$

In standard models, V captures the marginal utility of the marginal investor.

State price changes induced by supply shocks are proportional to the payoff matrix itself.

Do supply shocks generate the "ideal" state price variation? (Strict version)

Condition 1 (Identical Variation)

A supply shock to asset j **generates the ideal state price variation** if there exists a scalar k_j such that

$$\Delta \mathbf{q}_j^{\mathrm{ideal}} = k_j \Delta \mathbf{q}_j^{\mathrm{supply}}.$$

Do supply shocks generate the "ideal" state price variation? (Weaker version)

Condition 2 (Variation of the same sign)

The supply shock generates **state price variation of the same sign** if, element by element,

$$\mathrm{sign}(\Delta \mathbf{q}_j^{\text{ideal}}) = \mathrm{sign}(\Delta \mathbf{q}_j^{\text{supply}})$$

Since Y has weakly positive entries, this condition holds for all j if Y^+ has only weakly positive entries.

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Theorem (Trilemma). If Conditions 1 or 2 are satisfied, then YY^{T} is diagonal, and:

(i) If YY^{T} is diagonal, then **there are no assets with overlapping payoffs.**

(ii) If markets are complete, then YY^{T} is diagonal if and only if Y **is diagonal up to permutations.**

"Proof." Plemmons and Cline (PAMS, 1972), $Y \neq Y^{+}$.

Interpretation

1. Asset supply shocks affect behavior because change the cost of consumption (i.e., state prices).
2. Under no arbitrage, supply shocks affects the price of other assets with overlapping payoffs.
3. Only exception: Arrow securities (or a suitable generalization to incomplete markets).
4. **No overlapping payoffs is much stronger than orthogonal payoff distributions.**

Back to IO: Arrow securities eliminate the distinction between asset demand and consumption demand.

Illustration

Simple equilibrium economy

- Representative investor with log utility. Date-zero aggregate endowment $= 1$.
- Two equally likely states and two assets: $j, z \in \{g, r\}$. Aggregate supply $E_g = 1 + s_g$ and $E_r = 1$.

	State g	State r
Asset g	$\frac{1}{2}(1+\epsilon)$	$\frac{1}{2}(1-\epsilon)$
Asset r	$\frac{1}{2}(1-\epsilon)$	$\frac{1}{2}(1+\epsilon)$

- Standard optimality condition: optimal consumption depends on **relative state prices**,

$$q_z = \frac{1}{2}\frac{c_0}{c_z} \qquad \Rightarrow \qquad \frac{c_r}{c_g} = \frac{q_g}{q_r}.$$

Δq^{ideal}: State price changes given a hypothetical asset price change

We can back out implied state prices from asset prices:

$$\begin{pmatrix} q_g \\ q_r \end{pmatrix} = \frac{1}{8\epsilon} \begin{pmatrix} (1+\epsilon)p_g - (1-\epsilon)p_r \\ -(1-\epsilon)p_g + (1+\epsilon)p_r \end{pmatrix}.$$

In the **ideal experiment** where we vary p_g exogenously, induced state price changes are

$$\frac{\partial}{\partial p_g} \begin{pmatrix} q_g \\ q_r \end{pmatrix} = \frac{1}{4\epsilon} \begin{pmatrix} 1+\epsilon \\ -(1-\epsilon) \end{pmatrix}.$$

For any $\epsilon < 1$, **the green state becomes expensive and the red state becomes cheap.**

Δq^{supply}: Equilibrium state prices and supply shocks

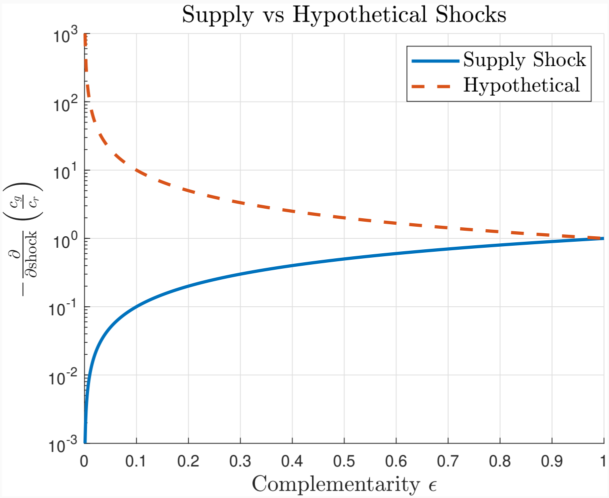
Impose market clearing. Then **equilibrium state prices** satisfy

$$q_g^*(s_g) = \frac{1}{2} \cdot \frac{1}{1 + \frac{s_g}{2}(1+\epsilon)} \qquad \text{and} \qquad q_r^*(s_g) = \frac{1}{2} \cdot \frac{1}{1 + \frac{s_g}{2}(1-\epsilon)}.$$

For any $\epsilon < 1$, **both state prices are decreasing in green supply s_g.** Hence **one is of the wrong sign.**

(Only exception is $\epsilon = 1$, in which case we recover Arrow securities.)

Optimal investor-level change in consumption ratio c_g/c_r (log scale)



“Calibration”

In the symmetric benchmark, the tracking error between red and green satisfies

$$TE = \frac{\epsilon}{p}$$

where $p = p_g = p_r \approx 1/2$.

Rolling window tracking errors of, say, Pepsi and Coca-Cola are on the order of 5% $\Rightarrow \epsilon \approx 2.5\%$.

Overcoming the trilemma?

Suppose you were willing to work with linear demand. Write investor i's asset demand function as

$$a_i \;=\; \bar{a}_i \;+\; S_i\, p \;+\; \varepsilon_i, \tag{1}$$

Suppose we have N **distinct experiments** generating data on prices and quantities for investor i,

$$G \equiv \left[\, \Delta p^{(1)}, \ldots, \Delta p^{(N)} \,\right] \in \mathbb{R}^{J \times N}, \tag{2}$$

$$\Delta A_i \equiv \left[\, \Delta a_i^{(1)}, \ldots, \Delta a_i^{(N)} \,\right] \in \mathbb{R}^{J \times N}. \tag{3}$$

Then we can write this as the linear system:

$$\Delta A_i \;=\; S_i\, G \;+\; U_i, \tag{4}$$

Complete Identifcation given $N = J$

Let $N = J$. The unique ordinary least-squares estimator of S_i is

$$\hat{S}_i \; = \; \Delta A_i \, G^{+}, \tag{5}$$

where $\hat{S}_i$ is an **unbiased and consistent estimator** of S_i. When $U_i = 0$, $\hat{S}_i = S_i$.

Incomplete Identifcation with $N < J$

Let $P_G \equiv GG^+$ be the orthogonal projector onto col(G), the column space of the matrix of observed price changes G. Then the general solution to the least-squares problem is

$$S_i = \Delta A_i G^+ + B_i(I - P_G), \quad B_i \in \mathbb{R}^{J \times J} \tag{6}$$

where B_i is an **arbitrary matrix that is entirely unrestricted by the data.**

Only the projection onto observed shocks is identified; elasticities in the null space are unbounded.

2. Theoretical restrictions

The other approach is to impose structural restrictions (*aka* a model).

These must be evaluated on first-principles: the data is silent.

In particular, structural models must capture the cross-asset spillovers which are endemic to asset pricing.

(Preliminary) conclusions

Three central properties of asset pricing and canonical demand estimation cannot jointly hold:
(i) no arbitrage, (ii) preferences over cash flows, and (iii) identification from supply shocks.

What can be done?

1. Certain assets may not be vulnerable to these issues. **Can check this using the payoff matrix**.
2. We can change the estimand (e.g., HHHKL's "relative elasticities").
3. Controls or factor approaches can make the payoff matrix diagonal (but also change the estimand).
4. If you observe a demand curve, you don't need supply shocks (e.g., Allen, Kastl and Wittwer).
5. ...