A Trilemma for Asset Demand Estimation*

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Abstract

Portfolio choice models rely on no arbitrage to ensure the existence of smooth, low-dimensional asset demand functions that are amenable to estimation. But because no arbitrage imposes cross-asset restrictions on prices, supply shocks generically fail to generate the *ceteris paribus* price variation required to identify elasticities. This yields a trilemma for asset demand estimation: given observational data, one cannot jointly require (i) no arbitrage, (ii) preferences over state-contingent payoffs, and (iii) ceteris paribus variation from asset supply shocks. Because the ideal experiment is unobservable, model-based inference is necessarily sensitive to the assumed structure.

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1 Introduction

The availability of granular portfolio data has fueled a wave of new research estimating demand functions for financial assets. Drawing on methods from industrial organization, much of this literature uses supply shocks to measure asset-level demand elasticities, and interprets the resulting coefficients as the slopes of asset-specific demand functions. This interpretation is valid if the estimation strategy accurately identifies the demand response to *exogenous variation in a single asset price*, *holding all else fixed*. Conversely, if the demand response is contaminated by other simultaneous price changes, then the estimated elasticities do not correspond to asset-specific demand functions, and thus may not reflect structural responses to counterfactual price changes.

We argue that violations of the *ceteris paribus* assumptions are endemic to financial markets, even when researchers have access to perfectly exogenous variation in asset supply. Because investors optimize over portfolios, asset prices are linked together through a common pricing kernel. Except in highly restrictive special cases, this makes it infeasible to vary the price of a single asset without inducing other concurrent price changes.

We formalize this problem as a trilemma for asset demand estimation: given observational data, one cannot simultaneously require (i) no arbitrage, (ii) investor preferences over state-contingent payoffs, and (iii) ceteris paribus price variation from asset-level supply shocks. That is, given standard preferences and the weak requirement of no arbitrage, the ideal experiment which would identify an asset-level demand elasticity is infeasible.

Neither no arbitrage nor payoff-based preferences are easily discarded. Preferences over payoffs (as opposed to direct preferences over assets) form the basis of portfolio choice theory. Given such preferences, no arbitrage is necessary to ensure the existence of smooth demand functions that do not jump discontinuously in response to small price changes – as is required for demand estimation. Empirical applications also rely on no arbitrage to derive empirically tractable demand systems based on a small number of common characteristics and risk factors (Koijen and Yogo, 2019). This dimension reduction is necessary because asset markets may have hundreds or thousands of assets.

The problem becomes clear when we write asset prices in terms of *state prices*, which measure the cost of state-contingent consumption. Let p be the vector of asset prices, Y the payoff matrix, and q the vector of state prices. Arbitrage free pricing requires

$$p = Yq.$$

Now consider the canonical notion of an asset-level demand elasticity. This object corresponds to an *ideal experiment* in which we vary a single asset price j while holding all other prices fixed. Under no arbitrage, a shock to a single asset price implies a specific set of state price changes. For simplicity, assume for now that markets are complete and that there are no redundant assets. Then state prices are related to asset prices by $q = Y^{-1}p$, and a shock to a given asset price affects state prices according to

$$\frac{\partial q}{\partial p} = Y^{-1}.$$

The ideal experiment in which we exogenously vary a single asset price can be equivalently interpreted as an experiment in which we induce a specific set of state price changes that are *fully characterized* by the *inverse* payoff matrix Y^{-1}. Measuring the elasticity thus requires researchers to generate precisely these implied state price changes.

This presents a fundamental challenge. Under standard risk averse preference, an increase in the supply of state-contingent consumption in a given state reduces the associated state price. Asset-level supply shocks thus alter state prices in all states in which the asset pays off, and these state price changes are proportional to payoff matrix Y. Except in the knife-edge case where Y is diagonal (i.e., assets are Arrow securities), the state price variation induced by supply shocks thus differs from the state price variation in the ideal experiment, which is proportional to Y^{-1}. Indeed, they may not even have the same *sign*. Hence it is infeasible to use supply shocks to generate the ideal experiment that would allow for direct identification of the demand elasticity.

We formalize this argument in a general portfolio choice model, showing that it also generalizes to the case of incomplete markets and redundant assets. In the case of incomplete markets, supply shocks create appropriate state price variation only if *for each state, there is a unique asset with a positive payoff*. This is a much stronger condition than requiring assets to be uncorrelated conditional on risk factors, as is assumed in Koijen and Yogo (2019). We also use a two-asset example to describe how the direction and magnitude of the implied consumption shifts diverge between the ideal experiment and the actual equilibrium allocation. Across both complete and incomplete markets, the results reveal a robust obstacle to estimating asset-specific elasticities using supply shocks.

While we emphasize the disconnect between supply shocks and the ideal experiment, we stress that structural assumptions (e.g., on payoffs, preferences, or the pricing kernel) can in principle allow researchers to recover demand elasticities from equilibrium data. However, these assumptions are inherently untestable in observational data because the required ceteris paribus price variation is never observed. Thus, the credibility of model-based elasticity estimates depends on the plausibility of the assumed structure.

The remainder of the paper is organized as follows. Section 2 introduces the model and defines the key objects of interest. Section 3 clarifies the link between demand functions and no arbitrage in the context of our framework. Section 4 formalizes the inconsistency between arbitrage, state prices, and asset-level elasticities. Section 5 illustrates the logic using a fully-solved general equilibrium model. Section 6 discusses implications for estimation and provides concluding remarks. The proofs are delegated to the Appendix.

Related literature

Our paper contributes to a growing literature on asset demand estimation, particularly the empirical study of asset-level demand elasticities following Koijen and Yogo (2019). Their framework – and much subsequent work – relies on the interpretation that changes in asset prices induced by supply shocks can be used to infer structural demand elasticities. However, while Koijen and Yogo (2019) explicitly rely on a structural model to discipline demand estimation, other authors pursue more reduced-form strategies.

Our key result is that the same no arbitrage restrictions which facilitate smooth, low-dimensional demand functions also complicate the measurement of demand elasticities based on supply shocks, at least in observational data. There are a number of potential paths to circumventing the issues we raise. The first is to restrict attention to settings where one can estimate demand even without supply variation. This typically requires additional data on demand functions, as in the Canadian bond market studied by Allen, Kastl, and Wittwer (2025). The second is to abandon no arbitrage. As discussed above, this makes it more difficult to reduce the dimensionality of the portfolio choice problem, and therefore likely limits the scope of asset demand estimation to settings with a limited number of assets. The third is to assume that preferences are defined directly over assets (as opposed to state-contingent payoffs), for example because investors have non-pecuniary *tastes* over assets. Because preferences are not defined over cash flows, there is no need to worry about cross-asset spillovers through the common pricing of cash flows. However, Fuchs, Fukuda, and Neuhann (2025) shows that heterogeneous tastes may also invalidate no arbitrage pricing.

The fourth is to change the target of estimation, possibly under additional assumptions. One example is in Haddad, He, Huebner, Kondor, and Loualiche (2025), who show that a difference-in-difference estimator based on a single supply shock can identify a *relative* elasticity (the own minus cross-price elasticities) under a set of stringent symmetry assumptions. Another example is An and Huber (2024), who propose measuring demand elasticities over aggregated factor portfolios (as opposed to individual assets). Our results here suggest that these factors must be designed to generate a diagonal payoff matrix. This is a stricter requirement than just requiring factors to be mutually orthogonal.

Overall, our results suggest an important role for model-based inference in asset markets. When the ideal experiment is infeasible, restrictions imposed by structural models allow researchers to learn about investor behavior from equilibrium data. This interpretation supports the general approach laid out in Koijen and Yogo (2019). However, because these restrictions can never be fully validated in the data, the assumed models must be plausible ex-ante. Fuchs, Fukuda, and Neuhann (2025) argues that logit demand systems do not sufficiently account for cross-asset interactions in portfolio choice.

2 Setup

We consider a canonical model with a set I of potentially heterogeneous investors. Each investor $i \in I$ must choose how much to consume at date 0 and across Z states of the world at date 1. To acquire a desired state-contingent consumption profile, the investor can invest in J assets. Investor i's *portfolio* is a vector $a^i \equiv (a^i_j)_{j=1}^J \in \mathbb{R}^J$ of asset positions, where each element a^i_j denotes the investor's holdings of asset j. Asset j has payoff $y_j(z)$ in state z. We denote by $Y \equiv (y_j(z))_{j,z}$ the $J \times Z$ matrix of cash flows. In line with the literature, we assume that the payoff matrix is known to the investor but unobserved by the econometrician. This is because the payoff matrix reflects expected returns, which is latent. Prices are observed by both the investor and the econometrician.

We treat time-zero consumption as the numeraire good (or, equivalently, as the *outside asset*) whose price is normalized to 1. Investor i is endowed with e^i_j units of asset j and e^i_0 units of the numeraire. The budget constraints at date 0 and in state z are given by

$$c^i_0 = e^i_0 - \sum_{j=1}^{J} p_j(a^i_j - e^i_j) \quad \text{and}$$
$$c^i_z = \sum_{j=1}^{J} y_j(z)a^i_j \quad \text{for all } z.$$

Each investor i has standard preferences over consumption given by

$$U^i(a^i) \equiv (1-\delta^i)u^i(c^i_0) + \delta^i \sum_{z=1}^{Z} \pi_z u^i(c^i_z),$$

where $\delta^i \in (0,1)$ is the discount factor, u^i is a strictly increasing and strictly concave von-Neumann Morgenstern utility function, and $\pi_z \in (0,1)$ is the probability of state z.

Investors may face constraints on portfolio formation beyond the budget constraint. Let $\mathcal{A}^i$ denote the set of feasible portfolios of investor i, and assume that $\mathcal{A}^i$ is a closed convex subset of $\mathbb{R}^J$. The investor's decision problem is:

$$\sup_{a^i \in \mathcal{A}^i} U^i(a^i). \tag{1}$$

When necessary, we close the model using the standard notion of competitive equilibrium, whereby investors form optimal portfolios given prices and asset markets clear.

3 The Importance of No Arbitrage for Demand Analysis

Demand analysis in financial markets faces two basic challenges. The first is the large number of assets under consideration. For example, in US equities markets alone, investors can choose among many *thousands* of assets, which creates a curse of dimensionality in demand estimation. The second is that demand functions must be sufficiently well-behaved. For example, demand elasticities are partial derivatives of demand with respect to an asset price. Hence the demand elasticity can be used to describe demand only if demand functions are smooth functions of asset prices.

Both challenges can be addressed by relying on the principle of no arbitrage. With respect to the first challenge, Koijen and Yogo (2019) implicitly rely on Ross's arbitrage pricing theory to argue that asset demand can be summarized by a relatively small number of asset characteristics and common risk factors, leading to a low-dimensional representation. (We leave aside here the concern that several common characteristics, such as book-to-market ratios, are themselves endogenous to demand.) With respect to the second challenge, it is well-established that arbitrage opportunities can lead to discontinuous changes in demand functions with respect to arbitrarily small price changes. Imposing no arbitrage rules out such discontinuities, thereby facilitating an analysis of demand elasticities. For demand analysis, no arbitrage is thus not only a constraint on *equilibrium* prices, but an important restriction on investors' decision problems themselves.

In the following, we briefly recapitulate the link between demand functions and no arbitrage in the context of our model. Since much of the empirical literature emphasizes institutional investors' constraints on portfolio choice when designing instruments, we explicitly incorporate these into our analysis as well. We also establish the standard result that no arbitrage allows for an analysis of asset prices (and thus demand) using *state prices*.

To this end, we begin by defining *unbounded* arbitrage opportunities as those that can be exploited using arbitrarily large asset positions. Standard definitions of arbitrage always consider unbounded arbitrage opportunities (Duffie, 2001). Hence this definition differs only in that we permit *bounded* arbitrages. (We discuss this case below.)

Definition 1 (No Unbounded Arbitrage for Investor i) *Investor i has an unbounded arbitrage opportunity if, for any $m > 0$, there exists a portfolio $a^i \in \mathcal{A}^i$ such that either (i) $p \cdot a^i \leq 0$, $Y^{\mathrm{T}} a^i \geq 0$, and $(Y^{\mathrm{T}} a^i)_z \geq m$ for some z or (ii) $p \cdot a^i \leq -m$ and $Y^{\mathrm{T}} a^i \geq 0$. Otherwise, investor i has no unbounded arbitrage opportunity.*

Proposition 1 shows that well-defined decision problem requires the absence of unbounded arbitrage opportunities. The simple reason is that unbounded arbitrage precludes the existence of a solution to the investor's problem. Hence the absence of unbounded arbitrages is thus a minimal requirement for any analysis of investor demand functions. This is a well-known result based on textbook treatments (e.g, Duffie, 2001).

Proposition 1 (Duffie (2001): No arbitrage and the investor's problem) *If there is a solution to (1), then there is no unbounded arbitrage opportunity for investor i. If U^i is continuous and there is no unbounded arbitrage opportunity for investor i, then there is a solution to (1).*

Proof. See Appendix. ■

Under weak additional assumptions, asset prices and demand can then be analyzed using *state prices*, which measure the marginal cost of a unit of state-contingent consumption. In particular, if the union of investors' feasible sets covers the entire space of feasible portfolios $\mathbb{R}^J$, the absence of unbounded arbitrage implies the existence of state prices such asset prices are payoff-weighted sums of state prices.

Lemma 1 (Existence of state prices) *If there exists a subset I_0 of investors such that every investor $i \in I_0$ does not have an unbounded arbitrage opportunity and $\mathbb{R}^J = \bigcup_{i \in I_0} \mathcal{A}^i$, then there exist state prices $q \in \mathbb{R}^Z_{++}$ such that asset prices are payoff-weighted sums of state prices:*

$$p = Yq. \tag{2}$$

Taken together, these results show that the absence of unbounded arbitrage is required for well-defined demand functions, and that, along with a weak condition on portfolio constraints, it implies the existence of strictly positive state prices. Hence we can recast both the interpretation and measurement of demand elasticities in terms of the cost of state-contingent consumption. This is useful because, given preferences over consumption, asset prices affect consumption plans to the extent that they affect state prices.

Bounded arbitrage. The same basic consideration apply to bounded arbitrages, whereby investors can only exploit mispricing up to an exogenous constraint on asset positions. In particular, it remains optimal to exploit the arbitrage to the extent possible, and, as the example below illustrates, this can lead to discontinuous changes in demand in response to arbitrarily small price changes.

Example 1 (Discontinuous demand functions) *Suppose there are two states of the world at date 1, and three assets. Given some $\epsilon \in (0, \frac{1}{2})$, let a cash flow matrix Y be given by*

$$\begin{bmatrix} \frac{1}{2}+\epsilon & \frac{1}{2}-\epsilon \\ \frac{1}{2}-\epsilon & \frac{1}{2}+\epsilon \\ 1 & 1 \end{bmatrix}.$$

Now consider the demand functions for some investor i with continuous utility function U^i.

(i) Suppose that $\mathcal{A}^i = \mathbb{R}^3$. The absence of unbounded arbitrage requires that $p_3 = p_1 + p_2$. Given this restriction on prices, there exist well-defined demand functions for all three assets, with the investor taking weakly positive quantities in all three assets. Now suppose that, starting from an initial benchmark where no arbitrage pricing holds, p_3 increases slightly. Then, investor i's problem (1) is no longer well-defined, and well-defined demand functions no longer exist.

(ii) Suppose instead that investor i faces the short-sale constraint $a^i_j \geq -\chi$ for some $\chi > 0$. Given $p_3 = p_1 + p_2$, there still exist well-defined demand functions for all three assets, with the investor taking weakly positive quantities in all three assets. Now suppose that p_3 increases slightly. Then it is optimal for the investor to jump to a portfolio allocation where $a_3 = -\chi$. This can trigger discontinuities in optimal demand.

It is clear that such discontinuities prevent an analysis of demand elasticities. Since any infinitesimal price change triggers an arbitrage for redundant assets, for the remainder we focus on the more interesting case without redundant assets.

Assumption 1 (No redundant assets) *$Z \geq J$ and $\text{rank}(Y) = J$.*

4 The Trilemma

We now turn to our main result, which pertains to the difficulty in measuring demand elasticities for individual assets in settings governed by no arbitrage restrictions. In line with the literature, we focus primarily on the case where the demand elasticity for a given asset is estimated by instrumenting for its price using a shock to the (residual) supply curve for that asset. Section 4.1 considers instruments constructed from multiple shocks.

Ideal experiment. The notion of an asset-level demand elasticity pertains to an *ideal experiment* in which one traces out an investor's demand response to *ceteris paribus* variation in a single asset price. As such, the demand elasticity is widely interpreted as the slope of an asset-specific demand function. We show that no arbitrage restrictions imply sharp conditions on the exact nature of state price changes associated with the ideal experiment, and that asset-level supply shocks do not satisfy these restrictions.

It is useful to describe the ideal experiment in terms of state prices, as these ultimately determine the investor's optimal consumption plans through the cost of consumption. Say that the investor observes asset prices p and payoff matrix Y. Then equation (2) allows the investor to infer the vector of state prices implied by prevailing asset prices:

$$q = Y^{+} p, \tag{3}$$

where Y^{+} is the Moore-Penrose pseudo-inverse of Y. If Y is square, as when markets are complete, then $Y^{+} = Y^{-1}$ and there is a unique vector of state prices. If markets are incomplete ($J < Z$), then there are many feasible state price vectors. We focus on the minimum norm solution, whereby the Moore-Penrose pseudo-inverse is $Y^{+} = Y^{\mathrm{T}}(YY^{\mathrm{T}})^{-1}$.

Implied state prices. Equation (3) determines the vector of state price changes that occur when we vary asset price p_j. That is, if we impose no (unbounded) arbitrage on the investor's problem to obtain well-behaved demand functions, then the thought experiment in which there is ceteris paribus variation in a given asset price is formally equivalent to one in which we vary the set of *state prices* that determine the cost of consumption.

Lemma 2 *Let v_j denote the unit vector in $\mathbb{R}^J$ with 1 in the j-th position and zeros elsewhere. Then the changes in state prices given the exogenous variation in a single price p_j are*

$$\Delta \mathbf{q}_j^{\text{ideal}} \equiv \frac{\partial q}{\partial p_j} = Y^{+} v_j.$$

Proof. The assertion follows immediately from equation (3). ■

These state price changes then induce a change in the optimal consumption plan, which can then be mapped into a change in desired portfolio holdings. Estimating the demand elasticity therefore requires an empirical setting in which one can generate the state price variation $\Delta \mathbf{q}_j^{\text{ideal}}$ associated with the ideal experiment. The challenge is that these restrictions can be tight: in particular, when markets are complete, the ideal experiment requires a *unique* set of state price changes.

Measurement using supply variation. In practice, one does not generally observe exogenous price shocks. Instead, one observes shocks to an economic environment that might trigger equilibrium price changes. As such, empirical approaches to estimating asset demand elasticities typically rely on suitably exogenous variation in asset supply (e.g., through flows or institutional holdings) and interpret the resulting change in an asset price as identifying a local demand response. However, in demand systems where the demand for one asset depends on the prices of others (as in financial markets, either through preferences or arbitrage relations), obtaining exogenous variation in a single price is insufficient. Instead, one must ensure that other asset prices remain unchanged.

We now show that these conditions are generically not satisfied even when researchers have access to quasi-experimental variation in asset supply. The key problem is that asset prices are linked to state prices through no arbitrage restrictions, and that supply shocks alter state prices in all states of the world in which the asset pays off. Hence, the prices of other assets that pay off in the same states will also change. Unless the asset has a unique state-contingent payoff (i.e., Y contains a diagonal matrix), there is no reason to expect only p_j to change.

To establish this result, we must describe how supply shocks affect state prices. To cover a broad range of models, we impose only the weak requirement that a positive supply shock to asset j reduces state prices in all states where asset j has a strictly positive payoff. In standard models, this condition holds if the marginal investor has a strictly increasing and strictly concave utility function.

Definition 2 (Downward-sloping consumption demand) *Let $E = (E_j)_{j=1}^{J} \in \mathbb{R}_{++}^{J}$ denote the vector of aggregate asset endowments. An economy has downward-sloping consumption demand if there exists a diagonal $Z \times Z$ matrix V with strictly positive elements such that*

$$\Delta \mathbf{q}_j^{\text{supply}} \equiv \frac{\partial q}{\partial E_j} = -V y_j^{\mathrm{T}} \qquad \textit{for all assets } j,$$

where y_j^{T} is the transpose of the j-th row $y_j \equiv (y_j(z))_{z=1}^{Z}$ of Y.

Under this definition, price changes are proportional to the induced change in consumption, as determined by the payoff matrix Y, multiplied by the marginal change in valuations induced by this shift, as measured by V. Matrix V thus captures the slope of the marginal investor's demand function, and V is diagonal because marginal utility depends on consumption. The next example computes V in a simple benchmark.

Example 2 (Representative Agent Model) *In a standard representative-agent model, state prices* *relate to marginal utility over aggregate consumption,*

$$\frac{\partial q_z}{\partial E_j} = \frac{\delta}{1-\delta} \pi_z \frac{u''(C_z)}{u'(C_0)} y_j(z) < 0,$$

where C_0 and C_z are aggregate consumption at date 0 *and in state z. The matrix V is*

$$-\frac{\delta}{1-\delta} \text{diag} \left(\pi_1 \frac{u''(C_1)}{u'(C_0)}, \ldots, \pi_z \frac{u''(C_z)}{u'(C_0)}, \ldots, \pi_Z \frac{u''(C_Z)}{u'(C_0)} \right).$$

We can now state two formal conditions which ensure that supply shocks create the type of state price variation required for the ideal experiment underlying a demand elasticity. The first is that the supply shock generates precisely required price variation, up to a scalar transformation that allows for a change in the size of the shock.

Condition 1 (Identical variation) *A supply shock to asset j generates the ideal state price variation for asset j if there exists some scalar k_j such that*

$$\Delta \mathbf{q}_j^{\text{ideal}} = k_j \Delta \mathbf{q}_j^{\text{supply}}.$$

This condition holds for all assets if

$$Y^+ = -VY^{\mathrm{T}}K, \quad \textit{where } K \equiv \text{diag}(k_1, \ldots, k_J).$$

While natural, one might argue that Condition 1 is too strict. The supply shock may still provide useful variation if it does not depart too much from the ideal experiment. Hence we also consider a much weaker condition, namely the state price variation generated by a supply shock has the same *sign* as the state price changes in the ideal experiment.

Condition 2 (Variation of the same sign) *The supply shock generates state price variation of the same sign if $\Delta \mathbf{q}_j^{\text{ideal}}$ has the same sign as $\Delta \mathbf{q}_j^{\text{supply}}$ element by element. Given that Y has only weakly positive entries, this condition holds for every j if Y^+ has only weakly positive entries.*

We can state our main result, which is that Conditions 1 and 2 require stringent conditions on the payoff matrix. In particular, if supply shocks generate useful variation, then for every state of the world there must exist a *unique* asset which offers a positive payoff in the world. Strikingly, *both conditions require the same stringent restrictions.*

Definition 3 (Overlapping payoffs) *Assets j and j' have overlapping payoffs if there exists at least one state of the world z such that $y_j(z) > 0$ and $y_{j'}(z) > 0$.*

Theorem 1 (Trilemma) *If Condition 1 or Condition 2 is satisfied, then YY^{T} is diagonal, and:*

(i) If YY^{T} is diagonal, then there are no assets with overlapping payoffs.

(ii) If markets are complete, then YY^{T} is diagonal if and only if Y is diagonal up to permutations.

This result shows that supply shocks fail to generate the price variation required by the ideal experiment. Under the maintained assumption of no arbitrage, it is thus infeasible to estimate asset-level demand elasticities from asset-level supply shocks. The only exception is when the payoff matrix is diagonal, as when the underlying assets are Arrow securities. In this case, there is no distinction between demand for assets and demand for state-contingent consumption. While this circumvents the problem of cross-asset spillovers, it is unlikely in practice that a given payoff matrix is indeed diagonal.

4.1 Constructing instruments from multiple shocks

One potential solution to the problems discussed above is to construct price instruments from multiple supply shocks. For example, if a researcher has access to supply shocks for every asset, then a *combination* of these shocks may generate the right state price variation. To fix ideas, suppose that, for a given asset j, there exists a vector of coefficients $\psi \in \mathbb{R}^J$ such that the ideal variation $\Delta\mathbf{q}_j^{\text{ideal}}$ is a linear combination of the vectors of supply shocks:

$$\Delta\mathbf{q}_j^{\text{ideal}} = \Delta\mathbf{q}^{\text{supply}}\psi.$$

Then an instrument constructed from ψ-weighted combination of asset-level supply shocks would be suitable for estimating the demand elasticity for asset j. However, while this is theoretically possible, the main practical challenge is that the weights ψ necessarily depend on the payoff matrix Y, and this matrix is *unobserved* to the econometrician. Hence, constructing instruments in this manner does not present a practical solution unless one is willing to rely on strong assumptions about the payoff matrix. This also explains why much of the existing literature relies only on asset-level supply shocks: *if* there were no cross-asset restrictions to worry about, then individual supply shocks would be valid instruments even without knowledge of the payoff matrix.

5 Illustration in a general equilibrium model

We now illustrate our results in an example economy with a log-utility representative investor based on Fuchs, Fukuda, and Neuhann (2025). Derivations are in Appendix A.3.

There are two assets and two states of the world, both denoted by g (green) and r (red). The probability of state $z \in \{g, r\}$ is $\pi_z \in (0, 1)$. The payoff profile of asset $j \in \{g, r\}$ is $y_j = (y_j(g), y_j(r))$. The aggregate endowment satisfy $E_r = 1$ and $E_g = 1 + s_g$, where s_g is a supply shock to the green asset.

Table 1 depicts the payoff matrix. Markets are complete, and parameter $\epsilon \in (0, 1)$ determines the degree of complementarity between green and red assets. In the limit $\epsilon \to 0$, green and red assets are perfect substitutes with respect to their cash flows. The assets become more complementary as ϵ increases. In the limit $\epsilon \to 1$, the red and green assets are Arrow securities paying exactly one unit in one state of the world.

	State g (π_g)	State r (π_r)
Asset g	$\frac{1}{2}(1+\epsilon)$	$\frac{1}{2}(1-\epsilon)$
Asset r	$\frac{1}{2}(1-\epsilon)$	$\frac{1}{2}(1+\epsilon)$

Table 1: Payoff matrix.

Prices. Denote by q_g and q_r the state prices measuring the cost of a unit of consumption in states g and r, respectively. Under no (unbounded) arbitrage, asset prices satisfy

$$\begin{bmatrix} p_g \\ p_r \end{bmatrix} = \begin{bmatrix} y_g(g) & y_g(r) \\ y_r(g) & y_r(r) \end{bmatrix} \begin{bmatrix} q_g \\ q_r \end{bmatrix}. \tag{4}$$

We can invert this expression to solve for state prices as a function of the asset prices:

$$\begin{bmatrix} q_g \\ q_r \end{bmatrix} = \frac{1}{2\epsilon} \begin{bmatrix} (1+\epsilon) & -(1-\epsilon) \\ -(1-\epsilon) & (1+\epsilon) \end{bmatrix} \begin{bmatrix} p_g \\ p_r \end{bmatrix}.$$

Demand. Since markets are complete, we can solve the decision problem in terms of state-contingent consumption. Let c_z denote quantities of Arrow securities chosen by the investor, and let q_z the associated state prices. The decision problem is:

$$\begin{aligned}
\max_{(c_0,c_g,c_r)} \quad & (1-\delta)u(c_0) + \delta\pi_g u(c_g) + \delta\pi_r u(c_r) \\
\text{s.t.} \quad & c_0 + \sum_{z\in\{g,r\}} q_z c_z = e_0 + \sum_{z\in\{g,r\}} q_z \Big(y_g(z)e_g + y_r(z)e_r\Big).
\end{aligned}$$

The necessary and sufficient optimality condition for Arrow security $z \in \{g,r\}$ is

$$q_z = \pi_z \frac{\delta}{1-\delta}\frac{u'(c_z)}{u'(c_0)}. \tag{5}$$

Given the budget constraint, this condition determines optimal consumption as a function of Arrow prices. Consumption can then be mapped back into asset positions.

State prices in the ideal experiment. Consider the ideal experiment where a given investor faces an exogenous increase in the price of the green asset p_g while p_r remains fixed. Consistently with Lemma 2, the induced change in state prices is

$$\frac{\partial}{\partial p_g}\begin{bmatrix} q_g \\ q_r \end{bmatrix} = \frac{1}{y_g(g)y_r(r) - y_g(r)y_r(g)}\begin{bmatrix} y_r(r) \\ -y_r(g) \end{bmatrix} = \frac{1}{2\epsilon}\begin{bmatrix} 1+\epsilon \\ -(1-\epsilon) \end{bmatrix}.$$

A pure shock to p_g thus raises the cost of consumption in state g, but *lowers* it in state r. The reason is that replicating an Arrow security on the green asset requires going long the green asset and shorting the red asset, while replicating a red Arrow security requires going long the red asset and shorting the green asset. Holding p_r fixed, a change in p_g thus has the opposite effect on state prices in the two states of the world. Estimating the demand elasticity associated with this experiment thus requires a shock that triggers precisely this price variation.

State prices after a supply shock. We now show that supply shocks do not create the ideal state price variation. Market clearing requires consumption to equal available resources in every state:

$$c_z = y_g(z)(1+s_g) + y_r(z).$$

Hence equilibrium state prices as a function of supply s_g are:

$$q_g = \pi_g \frac{\delta}{1-\delta} \cdot \frac{1}{1 + \frac{1+\epsilon}{2} s_g} \qquad \text{and} \qquad q_r = \pi_r \frac{\delta}{1-\delta} \cdot \frac{1}{1 + \frac{1-\epsilon}{2} s_g}. \tag{6}$$

It contrast to the ideal experiment, it is apparent that a negative supply shock to the green asset increases *both* state prices. In particular, differentiating q_z with respect to s_g and evaluating in the limit $s_g \to 0$ yields

$$\left.\frac{\partial q_g}{\partial s_g}\right|_{s_g \to 0} = -\pi_g \frac{\delta}{1-\delta} \frac{1+\epsilon}{2} < 0 \qquad \text{and} \qquad \left.\frac{\partial q_r}{\partial s_g}\right|_{s_g \to 0} = -\pi_r \frac{\delta}{1-\delta} \frac{1-\epsilon}{2} < 0.$$

The reason is that the green asset pays off in both states of the world. Unfortunately, this means that the supply shock generates a state price change that is of the *wrong sign* compared to the ideal experiment.

The only exception is when $\epsilon = 1$, so that the payoff matrix is the identity matrix. In line with our theoretical results, this is because the underlying assets *are* Arrow assets, and these do not generate cross-asset spillovers to other assets. However, for all other ϵ even a purely exogenous supply shock does not generate the right variation.

Implications for demand. The fact that the supply shock generates the wrong type of state price variation dramatically affects the observed demand response. We illustrate this effect by computing the response to the consumption ratio c_g/c_r to both the ideal experiment and the supply shock. Given log utility, it follows from the first-order conditions (5) that the relative consumption process satisfies:

$$\frac{c_g}{c_r} = \frac{\pi_g}{\pi_r} \frac{q_r}{q_g}. \tag{7}$$

Relative consumption in turn determines the desired holdings of red and green assets.

Consider first the ideal experiment with a pure hypothetical price shock. Differentiating the optimality with respect to p_g and evaluating in the limit $s_g \to 0$ yields:

$$-\left.\frac{\partial}{\partial p_g}\left(\frac{c_g}{c_r}\right)\right|_{s_g \to 0} = \frac{1-\delta}{\delta} \frac{(1-\epsilon)\pi_g + (1+\epsilon)\pi_r}{2\pi_g \pi_r \epsilon}. \tag{8}$$

This derivative diverges to infinity as $\epsilon \to 0$. As the two assets are perfect substitutes in

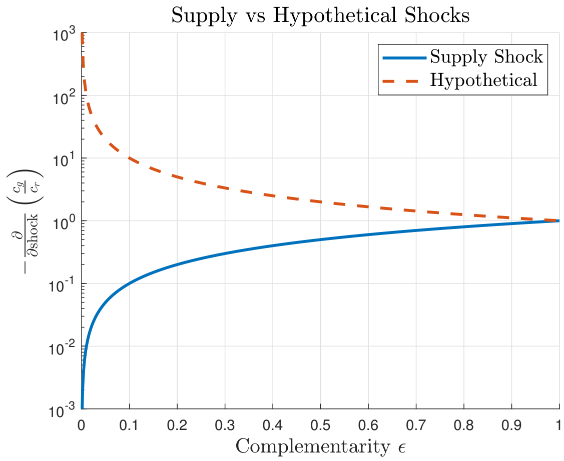

Figure 1: Optimal change in consumption ratio c_g/c_r on log scale. Parameters: $\pi_g = \pi_r = \frac{1}{2}$ and $\delta = \frac{1}{3}$.

this limit, a small *price* shock triggers a rapid reallocation from green to red assets.

Next, consider the response to the supply shock. In the limit as $s_g \to 0$,

$$\frac{\partial}{\partial s_g}\left(\frac{c_g}{c_r}\right)\bigg|_{s_g \to 0} = \epsilon. \tag{9}$$

which converges to *zero* in the limiting case of perfect substitutes as $\epsilon \to$. When the two assets are perfect substitutes, a supply shock has identical effects in both states. As such, it results in *zero* difference in the optimal consumption ratio across the two states.

Figure 1 depicts the optimal investor-level response to the hypothetical price shock (8) and the response to the supply shock (9) on log scale (the Appendix provides the derivations of these expressions). The difference in responses diverges to infinity as $\epsilon \to 0$. The only point of overlap occurs when the two assets are both Arrow securities. In line with our theory, this is the case where there can be no spillovers across assets.

6 Conclusion

We show that asset-level supply shocks generally fail to generate the ceteris paribus price variation required to identify demand elasticities in asset markets. This reflects a fundamental trilemma: no arbitrage, payoff-based preferences, and identification from supply shocks cannot all hold simultaneously in observational data. As a result, elasticity estimates must rely on structural assumptions that cannot be directly validated. Our findings highlight the limitations of reduced-form approaches and underscore the importance of carefully specified models in empirical asset demand estimation.

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A Appendix

A.1 Section 2

Proof of Proposition 1. For the first statement, let $a^{*i} \in \mathcal{A}^i$ be a solution to (1). For ease of exposition, we allow 0 to be in the domain of u^i (this is not essential). Suppose to the contrary that there is an unbounded arbitrage opportunity. Since u^i is strictly increasing, there exists $m > 0$ such that

$$U^i(a^{*i}) < (1-\delta^i)u^i(e_0^i + p \cdot e^i) + \delta^i \pi_z u^i(m) + \delta^i(1-\pi_z)u^i(0) \text{ for some } z$$

and

$$U^i(a^{*i}) < (1-\delta^i)u^i(e_0^i + p \cdot e^i + m) + \delta^i u^i(0),$$

where $e^i \equiv (e_j^i)_{j=1}^J$. Since there is an unbounded arbitrage opportunity, for this $m > 0$, there exists $a^i \in \mathcal{A}^i$ such that either (i) $p \cdot a^i \leq 0$, $Y^{\mathrm{T}} a^i \geq 0$, and $(Y^{\mathrm{T}} a^i)_z \geq m$, in which case

$$U^i(a^{*i}) < (1-\delta^i)u^i(e_0^i + p \cdot e^i) + \delta^i \pi_z u^i(m) + \delta^i(1-\pi_z)u^i(0) \leq U^i(a^i)$$

or (ii) $p \cdot a^i \leq -m$ and $Y^{\mathrm{T}} a^i \geq 0$, in which case

$$U^i(a^{*i}) < (1-\delta^i)u^i(e_0^i + p \cdot e^i + m) + \delta^i u^i(0) \leq U^i(a^i).$$

In either way, $a^{*i} \in \mathcal{A}^i$ does not solve (1), a contradiction.

For the second statement, since there is no unbounded arbitrage opportunity, there exists $m > 0$ such that, for any $a^i \in \mathcal{A}^i$,

$$U^i(a^i) < (1-\delta^i)u^i(e_0^i + p \cdot e^i + m) + \delta^i u^i(m).$$

Thus, we obtain:

$$\sup_{a^i \in \mathcal{A}^i} U^i(a^i) \leq (1-\delta^i)u^i(e_0^i + p \cdot e^i + m) + \delta^i u^i(m) < \infty.$$

Then, there exists a sequence $(a^{n,i})_{n \in \mathbb{N}}$ from $\mathcal{A}^i$ such that

$$\sup_{a^i \in \mathcal{A}^i} U^i(a^i) - \frac{1}{n} < U^i(a^{n,i}) \leq \sup_{a^i \in \mathcal{A}^i} U^i(a^i) < \infty \text{ for all } n \in \mathbb{N}.$$

Since $\sup_{a^i \in \mathcal{A}^i} U^i(a^i) < \infty$, it follows that

$$\sup_{n \in \mathbb{N}} |a_j^{n,i}| < \infty \text{ for all } j \in \{1, \dots, J\}.$$

Since $\mathcal{A}^i$ is closed, it follows that there exists a convergent subsequence $(a^{n_k,i})_{k \in \mathbb{N}}$ of $(a^{n,i})_{n \in \mathbb{N}}$ such that $a^{n_k,i} \to a^{*i} \in \mathcal{A}^i$. Since U^i is continuous, it follows that

$$U^i(a^{*i}) = \sup_{a^i \in \mathcal{A}^i} U^i(a^i),$$

as desired. ■

Proof of Lemma 1. Suppose the conditions in the statement of the lemma. The proof consists of seven steps. First, for each $i \in I_0$, we define a subset M^i of $\mathbb{R}^{Z+1}$:

$$M^i \equiv \{(-p \cdot a^i, Y^{\mathrm{T}} a^i) \in \mathbb{R}^{Z+1} \mid a^i \in \mathcal{A}^i\}.$$

Then, for each $i \in I_0$, since investor i does not have an unbounded arbitrage opportunity, it follows that

$$M^i \cap \mathbb{R}_+^{Z+1} = \{0\}.$$

Note that $\mathbb{R}_+^{Z+1}$ is a closed convex cone in $\mathbb{R}^{Z+1}$ and does not contain any linear subspace other than $\{0\}$.

Second, let

$$M \equiv \bigcup_{i \in I_0} M^i.$$

It follows from the assumption

$$\mathbb{R}^J = \bigcup_{i \in I_0} \mathcal{A}^i$$

that

$$M = \{(-p \cdot a^i, Y^{\mathrm{T}} a^i) \in \mathbb{R}^{Z+1} \mid a^i \in \mathbb{R}^J\}$$

is a linear subspace.

Third, since

$$M \cap \mathbb{R}_+^{Z+1} = \{0\},$$

it follows from the separating hyperplane theorem (which is referred to as "Linear separation of Cones" in Duffie (2001)), there exists $\overline{q} \in \mathbb{R}^{Z+1} \setminus \{0\}$ such that

$$\overline{q} \cdot t < \overline{q} \cdot x \text{ for all } t \in M \text{ and } x \in \mathbb{R}^{Z+1}_{+}.$$

Fourth, we show that $\overline{q} \in \mathbb{R}^{Z+1}_{++}$. Since $0 \in M$, it follows that

$$0 = \overline{q} \cdot 0 < \overline{q} \cdot x \text{ for all } x \in \mathbb{R}^{Z+1}_{+}.$$

Taking x as standard unit vectors in $\mathbb{R}^{Z+1}_{++}$ yields $\overline{q}_z > 0$ for all z.

Fifth, we show that

$$0 = \overline{q} \cdot t \text{ for all } t \in M.$$

Suppose to the contrary that $0 \neq \overline{q} \cdot t$ for some $t \in M$. Since M is a linear subspace, we can assume, without loss, that

$$\overline{q} \cdot t > 0.$$

However, this leads to a contradiction because, for any given $x \in \mathbb{R}^{Z+1}_{++}$, there exists $\lambda \in \mathbb{R}$ such that $\lambda t \in M$ and

$$\overline{q} \cdot x \leq \lambda(\overline{q} \cdot t) = \overline{q} \cdot (\lambda t).$$

Sixth, we show that

$$\overline{q}^{\mathrm{T}} \begin{bmatrix} -p^{\mathrm{T}} \\ Y^{\mathrm{T}} \end{bmatrix} = 0.$$

It follows from the fifth step that

$$\overline{q}^{\mathrm{T}} \begin{bmatrix} -p^{\mathrm{T}} \\ Y^{\mathrm{T}} \end{bmatrix} a = 0 \text{ for all } a \in \mathbb{R}^{J} = \bigcup_{i \in I_0} \mathcal{A}^{i}.$$

If

$$\overline{q}^{\mathrm{T}} \begin{bmatrix} -p^{\mathrm{T}} \\ Y^{\mathrm{T}} \end{bmatrix} \neq 0,$$

then letting

$$a = \left(\overline{q}^{\mathrm{T}} \begin{bmatrix} -p^{\mathrm{T}} \\ Y^{\mathrm{T}} \end{bmatrix}\right)^{\mathrm{T}} \in \mathbb{R}^{J} = \bigcup_{i} \mathcal{A}^{i}$$

yields

$$\overline{q}^{T} \begin{bmatrix} -p^{\mathrm{T}} \\ Y^{\mathrm{T}} \end{bmatrix} a > 0,$$

a contradiction.

Seventh, then, denoting by

$$\overline{q} = (q_0, q_{-0}),$$

we have

$$\overline{q}_0 p^{\mathrm{T}} = q_{-0}^{\mathrm{T}} Y^{\mathrm{T}}, \text{ that is, } p = Y \frac{q_{-0}}{q_0}.$$

Letting $q = \frac{q_{-0}}{q_0} \in \mathbb{R}^Z_{++}$, we finally obtain

$$p = Yq,$$

as desired. ∎

A.2 Section 4

Proof of Theorem 1. First, we show that Condition 1 implies that YY^{T} is diagonal. Suppose $Y^{+} = -VY^{\mathrm{T}}K$ for some diagonal matrix $K \equiv \mathrm{diag}(k_1, \ldots, k_J)$.

Operating Y on both sides from the left,

$$I = -YVY^{\mathrm{T}}K,$$

where both sides are a $J \times J$ matrix. The (j, j') element of the right-hand side is

$$\begin{cases} -\sum_{z=1}^{Z} y_j(z) v_z y_{j'}(z) k_{j'} = 1 & \text{if } j = j' \\ -\sum_{z=1}^{Z} y_j(z) v_z y_{j'}(z) k_{j'} = 0 & \text{if } j \neq j' \end{cases}.$$

This implies that $k_j \neq 0$ for all j. Then,

$$\begin{cases} \sum_{z=1}^{Z} y_j(z) v_z y_{j'}(z) \neq 0 & \text{if } j = j' \\ \sum_{z=1}^{Z} y_j(z) v_z y_{j'}(z) = 0 & \text{if } j \neq j' \end{cases}.$$

Since $y_j(z), y_{j'}(z) \geq 0$, and $v_z > 0$, it follows that

$$\begin{cases} \sum_{z=1}^{Z} y_j(z) y_{j'}(z) \neq 0 & \text{if } j = j' \\ \sum_{z=1}^{Z} y_j(z) y_{j'}(z) = 0 & \text{if } j \neq j' \end{cases}.$$

Hence, YY^{T} is diagonal.

Second, we show that Condition 2 implies that YY^{T} is diagonal. Since Y is a $J \times Z$ matrix with $J \leq Z$ and $\mathrm{rank}(Y) = J$, the Moore-Penrose pseudo-inverse is given by $Y^{+} = Y^{\mathrm{T}}(YY^{\mathrm{T}})^{-1}$. By Plemmons and Cline (1972, Theorem 1), the pseudo-inverse Y^{+} is non-negative if and only if there exists a diagonal matrix with positive elements $D \equiv \mathrm{diag}(d_1, \ldots, d_Z)$ such that

$$Y^{+} = DY^{\mathrm{T}}.$$

Then, operating Y from the left,

$$I = YY^{\mathrm{T}}(YY^{\mathrm{T}})^{-1} = YDY^{\mathrm{T}}.$$

Then, extracting the (j, k) element (with $j \neq k$) from each of both sides,

$$0 = \sum_{z=1}^{Z} y_j(z) d_z y_k(z).$$

Since $y_j(z) \geq 0$, $d_z > 0$, and $y_k(z) \geq 0$ for all $z \in \{1, \ldots, Z\}$, it follows that

$$y_j(z) y_k(z) = 0 \text{ for all } z \in \{1, \ldots, Z\}.$$

This implies that the (j, k) element (with $j \neq k$) of YY^{T} is 0:

$$0 = \sum_{z=1}^{Z} y_j(z) y_k(z). \tag{10}$$

Thus, YY^{T} is a diagonal matrix.

We remark that the converse also holds. Suppose that YY^{T} is a diagonal matrix. Since YY^{T} is invertible under Assumption 1, $(YY^{\mathrm{T}})^{-1}$ is a diagonal matrix with positive entries. Since Y is non-negative, so is Y^{T}. Then, $Y^{+} = Y^{\mathrm{T}}(YY^{\mathrm{T}})^{-1}$ is non-negative.

Third, we show that, given that YY^{T} is diagonal, there are no assets with overlapping payoffs. Since YY^{T} is invertible, it is a diagonal matrix with positive elements. Equation (10) implies that, for any $z \in \{1, \ldots, Z\}$, there exists at most one $j \in \{1, \ldots, J\}$ such that $y_j(z) > 0$.

Fourth, we show that if markets are complete then YY^{T} is diagonal if and only if Y has exactly one non-zero element in each row and in each column (so that Y is a diagonal matrix up a re-ordering of rows or columns). If YY^{T} is diagonal, then its (j, k) element is:

$$\begin{cases} \sum_{z=1}^{Z} y_j(z) y_j(z) > 0 & \text{if } j = k \\ \sum_{z=1}^{Z} y_j(z) y_k(z) = 0 & \text{if } j \neq k \end{cases}.$$

Hence, for each row j, there exists exactly one element z such that $y_j(z) > 0$. Thus, Y has J non-zero elements. Since Y is square and invertible, for each column z, there exists exactly one element j such that $y_j(z) > 0$.

Conversely, if Y has exactly one non-zero element in each row and in each column, then

$$\begin{cases} \sum_{z=1}^{Z} y_j(z) y_j(z) > 0 & \text{if } j = k \\ \sum_{z=1}^{Z} y_j(z) y_k(z) = 0 & \text{if } j \neq k \end{cases}.$$

Thus, YY^{T} is diagonal. ■

A.3 Section 5

First, we derive equation (8). Since the Arrow prices q can be expressed as a function of the asset prices p through equation (4), the consumption ratio (7) can be written as:

$$\frac{c_g}{c_r} = \frac{\pi_g}{\pi_r} \frac{(1+\epsilon) p_r - (1-\epsilon) p_g}{(1+\epsilon) p_g - (1-\epsilon) p_r}.$$

Thus, differentiating it with respect to the price p_g, we have:

$$\frac{\partial}{\partial p_g} \left(\frac{c_g}{c_r} \right) = -\frac{\pi_g}{\pi_r} \frac{4 \epsilon p_r}{((1+\epsilon) p_g - (1-\epsilon) p_r)^2}. \tag{11}$$

In contrast, substituting the Arrow prices (6) into equation (4), we obtain:

$$\begin{aligned} p_g &= \frac{1+\epsilon}{2} \pi_g \frac{\delta}{1-\delta} \frac{1}{1 + \frac{1+\epsilon}{2} s_g} + \frac{1-\epsilon}{2} \pi_r \frac{\delta}{1-\delta} \frac{1}{1 + \frac{1-\epsilon}{2} s_g}; \\ p_r &= \frac{1-\epsilon}{2} \pi_g \frac{\delta}{1-\delta} \frac{1}{1 + \frac{1+\epsilon}{2} s_g} + \frac{1+\epsilon}{2} \pi_r \frac{\delta}{1-\delta} \frac{1}{1 + \frac{1-\epsilon}{2} s_g}. \end{aligned}$$

Substituting the asset prices p at $s_g = 0$ into equation (11), we obtain equation (8). When

$\delta = \frac{1}{3}$ and $\pi_g = \pi_r = \frac{1}{2}$, equation (8) reduces to:

$$-\frac{\partial}{\partial p_g}\left(\frac{c_g}{c_r}\right)\bigg|_p = \frac{1}{\epsilon}.$$

Second, we derive equation (9). Substituting the Arrow prices (6) into the consumption ratio (7) yields

$$\frac{c_g}{c_r} = \frac{1 + \frac{1+\epsilon}{2} s_g}{1 + \frac{1-\epsilon}{2} s_g}.$$

Thus, differentiating it with respect to the supply shock s_g, we obtain

$$\frac{\partial}{\partial s_g}\left(\frac{c_g}{c_r}\right) = \frac{\epsilon}{\left(1 + \frac{1-\epsilon}{2} s_g\right)^2}.$$

In the limit as $s_g \to 0$, we get equation (9).