MCEN90028 Robotics Systems Assignment 1 Report

Group 3

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1 Introduction

Following on the first part of the project of creating a robot arm that is capable of picking and moving the chess piece (report 1), this next part of the project will investigate the relation of the end-effector velocity and the joint velocity in the workspace based on the previously determined physical properties of the structure. The evaluation of this study can indicate the mechanical feasibility of the chosen electronics components as well as of the linkage design for the robot arm. These following objectives will be explicitly address throughout the report.

- To demonstrate the proficiency in deriving a Jacobian matrix $J(Q)$ for the robot that maps the joint velocities into the end-effector velocity expressed in the inertial frame.
- To evaluate if joint torques required to lift up the inertia of the robot in addition to the load (the weight of the heaviest chess piece) within all required workspace, evaluated in a quasi static manner, are within the torque range that can be produced by the motors.

2 Jacobian Matrix Derivation for Joint Velocity

Following from the report 1 studying about the forward and inverse kinematics of the robot arm, Figure 1 shows the schematic of the robot arm and the final expression of forward kinematic is presented by Equation 1. These will be used later when deriving the expression of the end-effector related to joints 1, 2, 3, 4 in the inertial frame.

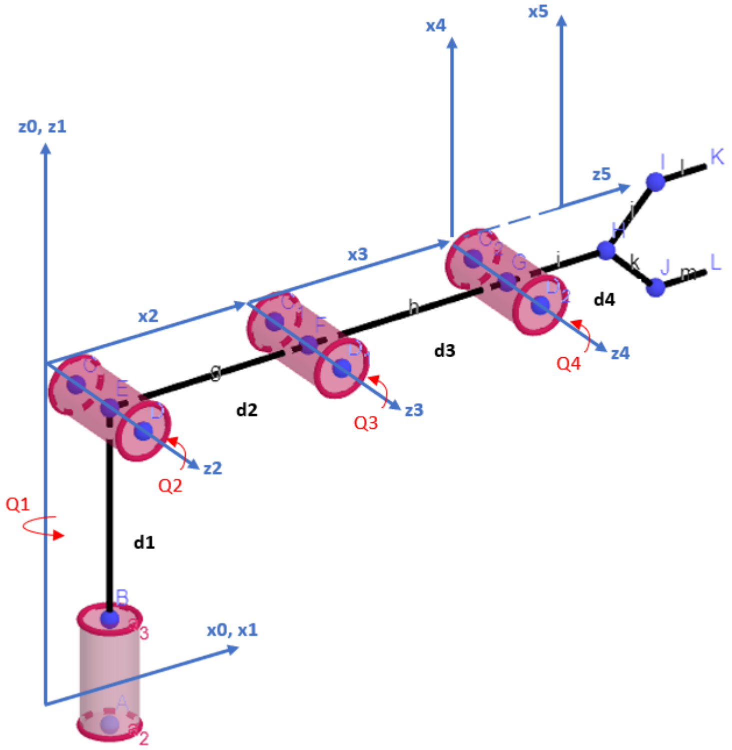

Figure 1: The schematic of the robot arm

The transformation matrix of the end-effector in frame $\{0\}$ is presented as

$$
\begin{aligned}
{}^0_E T &= {}^0_5 T = {}^0_1 T \cdot {}^1_2 T \cdot {}^2_3 T \cdot {}^3_4 T \cdot {}^4_5 T \\
&\simeq \begin{bmatrix} \sigma_{11} & \sin(Q_1) & \sigma_{13} & \sigma_{14} \\ \sigma_{21} & -\cos(Q_2) & \sigma_{23} & \sigma_{24} \\ \cos(Q_2+Q_3+Q_4) & 0 & \sin(Q_2+Q_3+Q_4) & \sigma_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned} \tag{1}
$$

where:

$$
\begin{aligned}
\sigma_{11} &= -0.5\sin(Q_2 - Q_1 + Q_3 + Q_4) - 0.5\sin(Q_1 + Q_2 + Q_3 + Q_4) \\
\sigma_{13} &= 0.5\cos(Q_2 - Q_1 + Q_3 + Q_4) + 0.5\cos(Q_1 + Q_2 + Q_3 + Q_4) \\
\sigma_{14} &= \cos(Q_1) * \big(d_3\cos(Q_2 + Q_3) + d_2\cos(Q_2) + d_4\cos(Q_2 + Q_2 + Q_4)\big) \\
\sigma_{21} &= 0.5\cos(Q_1 + Q_2 + Q_3 + Q_4) - 0.5\cos(Q_2 - Q_1 + Q_3 + Q_4) \\
\sigma_{23} &= 0.5\sin(Q_1 + Q_2 + Q_3 + Q_4) - 0.5\sin(Q_2 - Q_1 + Q_3 + Q_4) \\
\sigma_{24} &= \sin(Q_1) * \big(d_3\cos(Q_2 + Q_3) + d_2\cos(Q_2) + d_4\cos(Q_2 + Q_3 + Q_4)\big) \\
\sigma_{34} &= d_1 + d_3\sin(Q_2 + Q_3) + d_2\sin(Q_2) + d_4\sin(Q_2 + Q_3 + Q_4)
\end{aligned}
$$

Whereas, ${}^0p_E = \begin{bmatrix}\sigma_{14} & \sigma_{24} & \sigma_{34}\end{bmatrix}^T$ presents the pose of end-effector relative to point O in frame $\{0\}$.

2.1 Jacobian matrix derivation

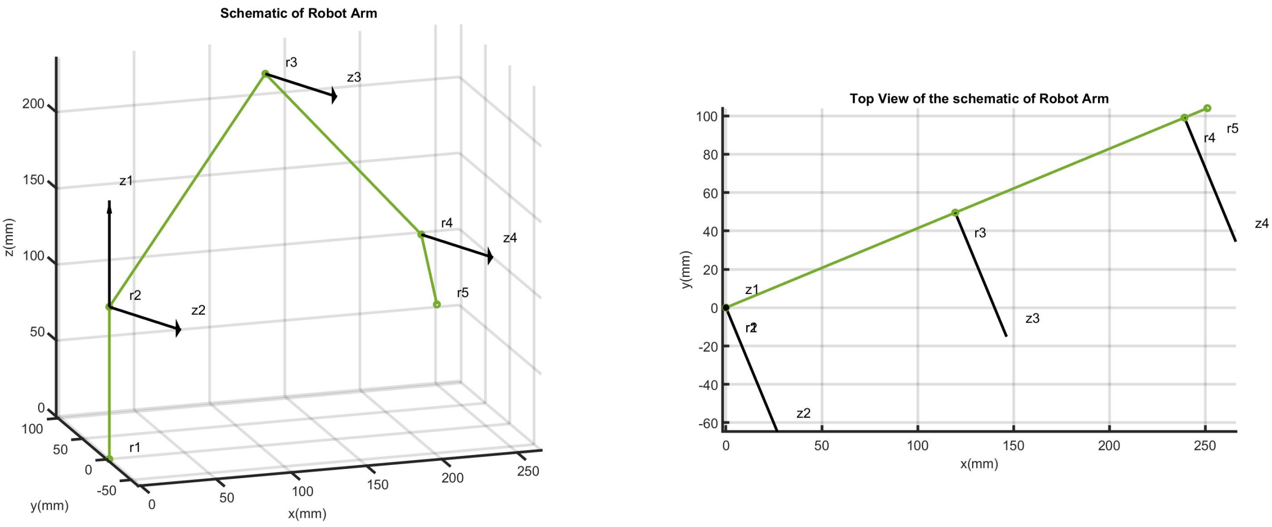

Figure 2: The schematic of the robot arm

The translational ($\vec{v}_E$) and angular velocity ($\vec{\omega}_E$) of the end-effector can be obtained using superposition as shown in Equation 2 such that velocities generated due to movement of joint i ($i = 1, 2, 3, 4$) will be added up in vector form, expressed in the inertial frame.

$$^0\vec{v}_E =^0 \vec{v}_1 +^0 \vec{v}_2 +^0 \vec{v}_3 +^0 \vec{v}_4 \qquad ^0\vec{\omega}_E =^0 \vec{\omega}_1 +^0 \vec{\omega}_2 +^0 \vec{\omega}_3 +^0 \vec{\omega}_4 \tag{2}$$

As the robot arm only uses revolute joints, the translational and angular velocities of the end-effector are generated due to rotational motion of links d_1, d_2, d_3, d_4 about joints 1, 2, 3, 4 respectively, which are calculated as

$$\begin{aligned} ^0\vec{\omega}_i &= \dot{Q}_i\, {}^0\hat{\vec{z}}_i \\ ^0\vec{v}_i &=^0 \vec{\omega}_i \times^0 \vec{p}_{iE} = \dot{Q}_i\, {}^0\hat{\vec{z}}_i \times^0 \vec{p}_{iE} \qquad for\ i = 1, 2, 3, 4 \end{aligned} \tag{3}$$

where

$^0\vec{\omega}_i$: *angular velocity of the end-effector due to rotational motion of joint i expressed in frame {0}*
$^0\vec{v}_i$: *translational velocity of the end-effector due to rotational motion of joint i expressed in frame {0}*
$\dot{Q}_i$: *angular speed of the rotational motion about joint i in frame {0}*
$\hat{\vec{z}}_i$: *unit vector of z_i-axis in frame {i}*
$^0\vec{p}_{iE}$: *displacement vector from the origin of frame {i} to the end-effector expressed in frame {0}*

From Equation 2 and 3, we can come up with detail expressions of $\vec{\omega}_E$ and $\vec{v}_E$ in matrix forms such that

$$\begin{aligned} ^0\vec{\omega}_E &= \dot{Q}_1\, {}^0\hat{\vec{z}}_1 + \dot{Q}_2\, {}^0\hat{\vec{z}}_2 + \dot{Q}_3\, {}^0\hat{\vec{z}}_3 + \dot{Q}_4\, {}^0\hat{\vec{z}}_4 \\ &= \begin{bmatrix} ^0\hat{\vec{z}}_1 & ^0\hat{\vec{z}}_2 & ^0\hat{\vec{z}}_3 & ^0\hat{\vec{z}}_4 \end{bmatrix} \cdot \begin{bmatrix} \dot{Q}_1 & \dot{Q}_2 & \dot{Q}_3 & \dot{Q}_4 \end{bmatrix}^T \end{aligned} \tag{4}$$

$$\begin{aligned} ^0\vec{v}_E &= \dot{Q}_1\, {}^0\hat{\vec{z}}_1 \times^0 \vec{p}_{1E} + \dot{Q}_2\, {}^0\hat{\vec{z}}_2 \times^0 \vec{p}_{2E} + \dot{Q}_3\, {}^0\hat{\vec{z}}_3 \times^0 \vec{p}_{3E} + \dot{Q}_4\, {}^0\hat{\vec{z}}_4 \times^0 \vec{p}_{4E} \\ &= \begin{bmatrix} ^0\hat{\vec{z}}_1 \times^0 \vec{p}_{1E} & ^0\hat{\vec{z}}_2 \times^0 \vec{p}_{3E} & ^0\hat{\vec{z}}_3 \times^0 \vec{p}_{3E} & ^0\hat{\vec{z}}_4 \times^0 \vec{p}_{4E} \end{bmatrix} \cdot \begin{bmatrix} \dot{Q}_1 & \dot{Q}_2 & \dot{Q}_3 & \dot{Q}_4 \end{bmatrix}^T \end{aligned} \tag{5}$$

And therefore, combining Equation 4 and 5, we can get the expression of the Jacobian matrix for joint velocity $J(Q)$.

$$\begin{bmatrix} {}^0\vec{v}_E \\ {}^0\vec{\omega}_E \end{bmatrix} = \begin{bmatrix} {}^0\hat{\vec{z}}_1 \times {}^0\vec{p}_{1E} & {}^0\hat{\vec{z}}_2 \times {}^0\vec{p}_{3E} & {}^0\hat{\vec{z}}_3 \times {}^0\vec{p}_{3E} & {}^0\hat{\vec{z}}_4 \times {}^0\vec{p}_{4E} \\ {}^0\hat{\vec{z}}_1 & {}^0\hat{\vec{z}}_2 & {}^0\hat{\vec{z}}_3 & {}^0\hat{\vec{z}}_4 \end{bmatrix} \cdot \begin{bmatrix} \dot{Q}_1 & \dot{Q}_2 & \dot{Q}_3 & \dot{Q}_4 \end{bmatrix}^T$$
$$= \begin{bmatrix} J_v \\ J_\omega \end{bmatrix} \cdot \begin{bmatrix} \dot{Q}_1 & \dot{Q}_2 & \dot{Q}_3 & \dot{Q}_4 \end{bmatrix}^T \quad (6)$$

Observing Equation 6, there are expressions of ${}^0\hat{\vec{z}}_i$ and ${}^0\vec{p}_{iE}$ that need to be defined. During the process of deriving the forward kinematic for the robot arm, we have already constructed the transformation matrix from frame {i-1} to {i} as well as from frame {i} to {0}. Refer to the expression of those transformation and rotation matrices, we can obtain the expression of ${}^0\hat{\vec{z}}_i$ and ${}^0\vec{p}_i$ using Equation 7. In addition to that, using Equation 8 presenting spatial description of the end-effector, we can calculate the displacement of the origin of frame {i} and the end-effector, ${}^0\vec{p}_{iE}$.

$${}^0_iT = \begin{bmatrix} {}^0_iR & {}^0\vec{p}_i \\ \vec{0}_{(1\times3)} & 1 \end{bmatrix} \qquad {}^0_iR = \begin{bmatrix} {}^0\hat{x}_i & {}^0\hat{y}_i & {}^0\hat{z}_i \end{bmatrix} \quad (7)$$

$${}^0\vec{p}_{iE} = {}^0\vec{p}_E - {}^0\vec{p}_i = {}^0_ER \cdot {}^E\vec{p}_E - {}^0\vec{p}_i \qquad for\ i = 1,2,3,4 \quad (8)$$

where

$${}^0\vec{p}_i : \textit{pose of the origin of frame } i \textit{ expressed in frame 0}$$

Using MATLAB, the symbolic expression of the Jacobian matrix for joint velocity $J(Q)$ is obtained

$$J(Q) = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix} = \begin{bmatrix} \sigma_8 & -\cos(Q_1)\,\sigma_2 & -\cos(Q_1)\,\sigma_7 & -d_4\cos(Q_1)\,\sigma_4 \\ \sigma_9 & -\sin(Q_1)\,\sigma_2 & -\sin(Q_1)\,\sigma_7 & -d_4\sin(Q_1)\,\sigma_4 \\ 0 & \sigma_3 + d_4\,\sigma_1 & \sigma_6 + d_4\,\sigma_1 & d_4\,\sigma_1 \\ 0 & \sin(Q_1) & \sin(Q_1) & \sin(Q_1) \\ 0 & -\cos(Q_1) & -\cos(Q_1) & -\cos(Q_1) \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad (9)$$

where:

$$\sigma_1 = \cos(Q_2 + Q_3 + Q_4) \qquad \sigma_6 = d_3\cos(Q_2 + Q_3)$$
$$\sigma_2 = \sigma_5 + d_2\sin(Q_2) + d_4\,\sigma_4 \qquad \sigma_7 = \sigma_5 + d_4\,\sigma_4$$
$$\sigma_3 = \sigma_6 + d_2\cos(Q_2) \qquad \sigma_8 = -\sin(Q_1)\,\sigma_3 - d_4\,\sigma_1\sin(Q_1)$$
$$\sigma_4 = \sin(Q_2 + Q_3 + Q_4) \qquad \sigma_9 = \cos(Q_1)\,\sigma_3 + d_4\cos(Q_1)\,\sigma_1$$
$$\sigma_5 = d_3\sin(Q_2 + Q_3)$$

2.2 Justification

In the section 2.1, the task space velocity was derived by using superposition, from which the Jacobian matrix has been derived based on the relation of the joint velocity and task space velocity. Since we had the expression of the end-effector pose in frame {0} (Equation 10), the translational velocity can be obtained simply by taking derivative of Equation 10, whose result is shown in Equation 11.

$${}^0\vec{p}_E = f(Q)$$
$$= \begin{bmatrix} \cos(Q_1)\,(d_3\cos(Q_2+Q_3) + d_2\cos(Q_2)) + d_4\cos(Q_1)\cos(Q_2+Q_3+Q_4) \\ \sin(Q_1)\,(d_3\cos(Q_2+Q_3) + d_2\cos(Q_2)) + d_4\cos(Q_2+Q_3+Q_4)\sin(Q_1) \\ d_1 + d_3\sin(Q_2+Q_3) + d_2\sin(Q_2) + d_4\sin(Q_2+Q_3+Q_4) \end{bmatrix} \quad (10)$$

$$
\begin{aligned}
{}^{0}\dot{\vec{p}}_E &= \frac{\partial f}{\partial Q}\frac{\mathrm{d}Q}{\mathrm{d}t} \\
&= \begin{bmatrix} -\cos(Q_1)\ (\tau_2) - \dot{Q}_1\ \sin(Q_1)\ (\tau_3) - d_4\ \cos(Q_1)\ \sin(\tau_1)\ (\tau_5) - d_4\,\dot{Q}_1\ \sin(Q_1)\ \cos(\tau_1) \\ \dot{Q}_1\ \cos(Q_1)\ (\tau_3) - \sin(Q_1)\ (\tau_2) - d_4\ \sin(Q_1)\ \sin(\tau_1)\ (\tau_5) + d_4\,\dot{Q}_1\ \cos(Q_1)\ \cos(\tau_1) \\ d_2\,\dot{Q}_2\ \cos(Q_2) + d_3\ \cos(\tau_4)\ \left(\dot{Q}_2 + \dot{Q}_3\right) + d_4\ \cos(\tau_1)\ (\tau_5) \end{bmatrix} \qquad (11) \\
&= J_v(Q)[\dot{Q}]
\end{aligned}
$$

where

$$
\begin{aligned}
\tau_1 &= Q_2 + Q_3 + Q_4 \\
\tau_2 &= d_2\,\dot{Q}_2\ \sin(Q_2) + d_3\ \sin(\tau_4)\ \left(\dot{Q}_2 + \dot{Q}_3\right) \\
\tau_3 &= d_2\ \cos(Q_2) + d_3\ \cos(\tau_4) \\
\tau_4 &= Q_2 + Q_3 \\
\tau_5 &= \dot{Q}_2 + \dot{Q}_3 + \dot{Q}_4
\end{aligned}
$$

The resulting matrix $\frac{\partial f}{\partial Q}$ is termed the Jacobian matrix. To obtain the expression for the Jacobian matrix, we use function *equationsToMatrix* from MATLAB on the result in Equation 11 to convert the equation into matrix form, ${}^{0}\dot{\vec{p}}_E = J_v(Q)\begin{bmatrix}\dot{Q}_1 & \dot{Q}_2 & \dot{Q}_3 & \dot{Q}_4\end{bmatrix}^T$. Compare with the expression of J_v from Equation 6, we can confirm that our solution of the Jacobian matrix that relates the translation velocity and joint velocity is correct.

In addition, the Jacobian matrix for angular velocity depends on the expression of the unit vector $\hat{\vec{z}}_1, \hat{\vec{z}}_2, \hat{\vec{z}}_3, \hat{\vec{z}}_4$. Observing from Figure 2, $\hat{\vec{z}}_1$ and $\hat{\vec{z}}_0$ are always parallel to each other regardless of joint displacements. Similarly for $\hat{\vec{z}}_2, \hat{\vec{z}}_3$ and $\hat{\vec{z}}_4$ which are parallel to each other and perpendicular to $x_1 - z_1$ plane. Therefore, the expression of $\hat{\vec{z}}_2, \hat{\vec{z}}_3$ and $\hat{\vec{z}}_4$ in frame $\{0\}$ will only depend on the joint displacement Q_1, which can be proved by using MATLAB such that

$$
{}^{0}\hat{\vec{z}}_2 = {}^{0}\ \hat{\vec{z}}_3 = {}^{0}\ \hat{\vec{z}}_4 = \begin{bmatrix}\sin(Q_1) & -\cos(Q_1) & 0\end{bmatrix}^T
$$

Hence, the Jacobian matrix for angular velocity J_ω is correctly derived. Combining with the Jacobian matrix for translational velocity, we have

$$
J(Q) = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix} = \begin{bmatrix} \sigma_8 & -\cos(Q_1)\ \sigma_2 & -\cos(Q_1)\ \sigma_7 & -d_4\ \cos(Q_1)\ \sigma_4 \\ \sigma_9 & -\sin(Q_1)\ \sigma_2 & -\sin(Q_1)\ \sigma_7 & -d_4\ \sin(Q_1)\ \sigma_4 \\ 0 & \sigma_3 + d_4\,\sigma_1 & \sigma_6 + d_4\,\sigma_1 & d_4\,\sigma_1 \\ 0 & \sin(Q_1) & \sin(Q_1) & \sin(Q_1) \\ 0 & -\cos(Q_1) & -\cos(Q_1) & -\cos(Q_1) \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad (12)
$$

where:

$$
\begin{aligned}
\sigma_1 &= \cos(Q_2 + Q_3 + Q_4) & \sigma_6 &= d_3\ \cos(Q_2 + Q_3) \\
\sigma_2 &= \sigma_5 + d_2\ \sin(Q_2) + d_4\,\sigma_4 & \sigma_7 &= \sigma_5 + d_4\,\sigma_4 \\
\sigma_3 &= \sigma_6 + d_2\ \cos(Q_2) & \sigma_8 &= -\sin(Q_1)\ \sigma_3 - d_4\,\sigma_1\ \sin(Q_1) \\
\sigma_4 &= \sin(Q_2 + Q_3 + Q_4) & \sigma_9 &= \cos(Q_1)\ \sigma_3 + d_4\ \cos(Q_1)\ \sigma_1 \\
\sigma_5 &= d_3\ \sin(Q_2 + Q_3)
\end{aligned}
$$

3 Joint Torque Evaluation

In this section, we will focus on calculating the torque generated on each joint of the robot arm. The purpose of doing so is to determine the maximum torque generated due to the structure's mass when the robot is operating. From there, we will iterate through our workspace and evaluate the torque at each pose to get the maximum torque. Finally, we will compare with the nominal torque of the motor, specified by the manufacture and make adjustment to the design if necessary.

3.1 Jacobian matrix derivation

Having verified our solution of the Jacobian matrix for the end-effector, we can now generalise Equation 6 to obtain the Jacobian matrices J_i that relate the joint velocities to the velocities of the centres of mass M_i of links $i = 1, 2...4$.

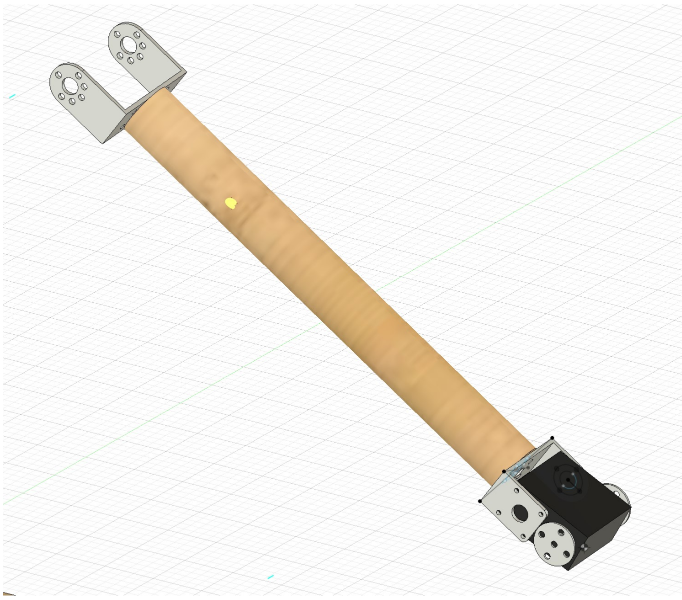

Figure 3: 3D model of a link consisting of a servo motor and 2 brackets connected by a pine wood shaft

By first defining the distance of M_i of each link from its body frame as a vector ${}^{i}\vec{C}_i$, we get the displacement vector of M_i from the origin of frame $\{j\}$ expressed in the inertial frame $\{0\}$ to be:

$$ {}^{0}\vec{p}_{jMi} = ({}^{0}\vec{p}_i +^{0}_{i} R \cdot^{i} \vec{C}_i) -^{0} \vec{p}_j, \quad \text{for } \; i, j \; = \; 1, 2...4 \tag{13} $$

where ${}^{0}\vec{p}_i$ & ${}^{0}\vec{p}_j$ are the displacement vectors of frame $\{i\}$ & frame $\{j\}$ from the origin of the inertial frame $\{0\}$. This can then be used to obtain the following generalised expression for the translational and angular velocities of M_i:

$$ \begin{aligned} \begin{bmatrix} {}^{0}\vec{v}_i \\ {}^{0}\vec{\omega}_i \end{bmatrix} &= \begin{bmatrix} {}^{0}\hat{\vec{z}}_1 \times^{0} \vec{p}_{1Mi} & \sigma_1 *^{0} \hat{\vec{z}}_2 \times^{0} \vec{p}_{2Mi} & \sigma_2 *^{0} \hat{\vec{z}}_3 \times^{0} \vec{p}_{3Mi} & \sigma_3 *^{0} \hat{\vec{z}}_4 \times^{0} \vec{p}_{4Mi} \\ {}^{0}\hat{\vec{z}}_1 & \sigma_1 *^{0} \hat{\vec{z}}_2 & \sigma_2 *^{0} \hat{\vec{z}}_3 & \sigma_3 *^{0} \hat{\vec{z}}_4 \end{bmatrix} \cdot \begin{bmatrix} \dot{Q}_1 & \dot{Q}_2 & \dot{Q}_3 & \dot{Q}_4 \end{bmatrix}^T \\ &= \begin{bmatrix} J_{vi} \\ J_{\omega i} \end{bmatrix} \cdot \begin{bmatrix} \dot{Q}_1 & \dot{Q}_2 & \dot{Q}_3 & \dot{Q}_4 \end{bmatrix}^T \end{aligned} \tag{14} $$

where:

$$ \sigma_1 = \max\left\{\frac{i-1}{|i-1|}, 0\right\},\ \sigma_2 = \max\left\{\frac{i-2}{|i-2|}, 0\right\},\ \sigma_3 = \max\left\{\frac{i-3}{|i-3|}, 0\right\} $$

From Equation 14, we see that the cross product and ${}^0\hat{\vec{z}}_i$ terms will simply be 3×3 zero vectors for cases where the centre of mass is nearer than the origin of frame {j} to the origin of frame {0} (i.e. when $j > i$). This is because both the translational and angular velocities for a centre of mass M_i will only affected by the velocities of the joints that precede it. The symbolic representations for the Jacobian matrices can then be derived using MATLAB:

$$J_1(Q) = \begin{bmatrix} J_{v1} \\ J_{\omega 1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{15}$$

$$J_2(Q) = \begin{bmatrix} J_{v2} \\ J_{\omega 2} \end{bmatrix} = \begin{bmatrix} c_2\,(\cos(Q_2)\,\sin(Q_1)) & -c_2\,(\cos(Q_1)\,\sin(Q_2)) & 0 & 0 \\ c_2\,(\cos(Q_1)\,\cos(Q_2)) & c_2\,(\sin(Q_1)\,\sin(Q_2)) & 0 & 0 \\ 0 & c_2\,\cos(Q_2) & 0 & 0 \\ 0 & \sin(Q_1) & 0 & 0 \\ 0 & -\cos(Q_1) & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{16}$$

$$J_3(Q) = \begin{bmatrix} J_{v3} \\ J_{\omega 3} \end{bmatrix} = \begin{bmatrix} \sigma_1 & -\cos(Q_1)\,\sigma_3 & -c_3\sin(Q_2+Q_3)\,\cos(Q_1) & 0 \\ \sigma_2 & -\sin(Q_1)\,\sigma_3 & -c_3\sin(Q_2+Q_3)\,\sin(Q_1) & 0 \\ 0 & \sigma_4 + d_2\cos(Q_2) & \sigma_4 & 0 \\ 0 & \sin(Q_1) & \sin(Q_1) & 0 \\ 0 & -\cos(Q_1) & -\cos(Q_1) & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

where:

$$\begin{aligned} \sigma_1 &= -c_3\,(0.5\sin(Q_1+Q_2+Q_3) - 0.5\sin(\sigma_5)) - d_2\cos(Q_2)\,\sin(Q_1) \\ \sigma_2 &= c_3\,(0.5\cos(Q_1+Q_2+Q_3) + 0.5\cos(\sigma_5)) + d_2\cos(Q_1)\,\cos(Q_2) \\ \sigma_3 &= c_3\sin(Q_2+Q_3) + d_2\sin(Q_2) \\ \sigma_4 &= c_3\cos(Q_2+Q_3) \\ \sigma_5 &= Q_2 - Q_1 + Q_3 \end{aligned} \tag{17}$$

$$J_4(Q) = \begin{bmatrix} J_{v4} \\ J_{\omega 4} \end{bmatrix} = \begin{bmatrix} \sigma_1 & -\cos(Q_1)\ \sigma_6 & -\cos(Q_1)\ (\sigma_8 + c_4\,\sigma_7) & -c_4\,\sigma_7 \cos(Q_1) \\ \sigma_2 & -\sin(Q_1)\ \sigma_6 & -\sin(Q_1)\ (\sigma_8 + c_4\,\sigma_7) & -c_4\,\sigma_7 \sin(Q_1) \\ 0 & \sigma_4 + d_2\cos(Q_2) + \sigma_9 & \sigma_4 + \sigma_9 & \sigma_9 \\ 0 & \sin(Q_1) & \sin(Q_1) & \sin(Q_1) \\ 0 & -\cos(Q_1) & -\cos(Q_1) & -\cos(Q_1) \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

where:

$$\begin{aligned} \sigma_1 &= c_4\,(0.5\sin(\sigma_3) - 0.5\sin(Q_1+Q_2+Q_3+Q_4)) - \ d_3\,(0.5\sin(Q_1+Q_2+Q_3) - 0.5\sin(\sigma_5)) \\ &\quad - d_2\cos(Q_2)\ \sin(Q_1) \\ \sigma_2 &= c_4\,(0.5\cos(\sigma_3) + 0.5\cos(Q_1+Q_2+Q_3+Q_4)) + d_3\,(0.5\cos(Q_1+Q_2+Q_3) + 0.5\cos(\sigma_5)) \\ &\quad + d_2\cos(Q_1)\ \cos(Q_2) \\ \sigma_3 &= Q_2 - \ Q_1 + Q_3 + Q_4 \\ \sigma_4 &= d_3\cos(Q_2+Q_3) \\ \sigma_5 &= Q_2 - \ Q_1 + Q_3 \\ \sigma_6 &= \sigma_8 + d_2\sin(Q_2) + c_4\,\sigma_7 \\ \sigma_7 &= \sin(Q_2+Q_3+Q_4) \\ \sigma_8 &= d_3\sin(Q_2+Q_3) \\ \sigma_9 &= c_4\cos(Q_2+Q_3+Q_4) \end{aligned} \tag{18}$$

with c_i being the scalar distance of M_i from the origin of its body frame.

3.2 Center of mass (CoM) determination

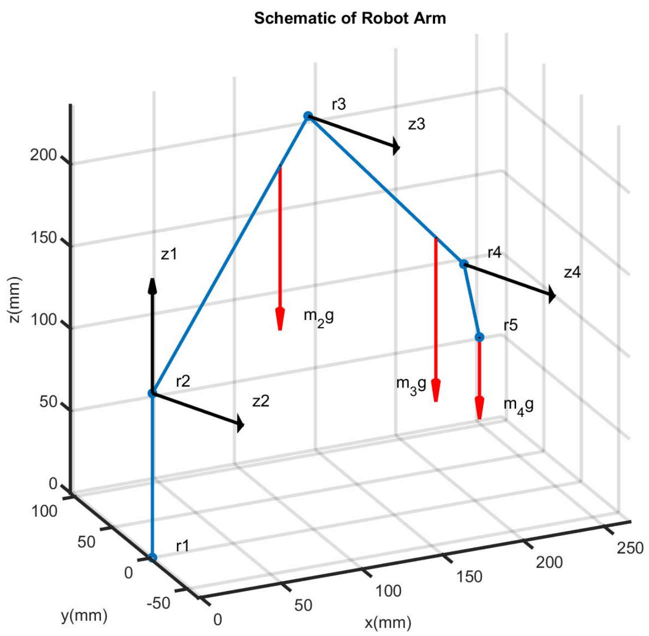

Figure 4: Schematic of the robot arm

The coordinates ${}^{i}\vec{C}_i$ of the center of mass M_i for each link from its body frame can be determined using the AutoDesk Fusion 360 software. ${}^{i}\vec{C}_i$ can be computed using the `Center of Mass` option from the `Inspect` panel in Fusion 360, and the resulting coordinates can then be substituted into Equation 13 to obtain the displacement vector ${}^{0}\vec{p}_{jMi}$ of M_i from a frame {j}. Note that for the case of link 4, the coordinates of its center of mass ${}^{4}\vec{C}_4$ will be determined by both the link itself and the end effector with a chess piece attached to it through the following relationship:

$$ {}^{4}\vec{C}_4 = \frac{m_{link}\,{}^{4}\vec{C}_{link} + m_E\,{}^{4}\vec{C}_E}{m_{link} + m_E} \tag{19} $$

where m_{link} & m_E are the masses of the link and the end effector respectively. From Equation 19, we can infer that ${}^{4}\vec{C}_4$ will vary depending on the mass of the chess piece being held by the end-effector. For the purposes of the following section, we will assume the worst-case scenario (i.e. end effector is holding the heaviest chess piece) when performing our torque calculations.

The following assumptions have been made on Fusion 360 regarding the materials of the links (Figure 3), brackets and motors, to obtain the closest approximation of the actual center of mass:

Part	**Material**
Links	Pine wood
Motors	ABS
Brackets	Aluminium
Chess Piece	Wood

Table 1: Connections for the GPS module

According to [1], the motor mass is 56g. In order to minimize error in the calculation, ABS material is chosen for the motor such that the motor model's mass is 52g.

A link has been defined as an assembly of bracket → wooden link → bracket → motor. Given below are the screenshots of the COM for the links d_1, d_2, d_3 and d_4.

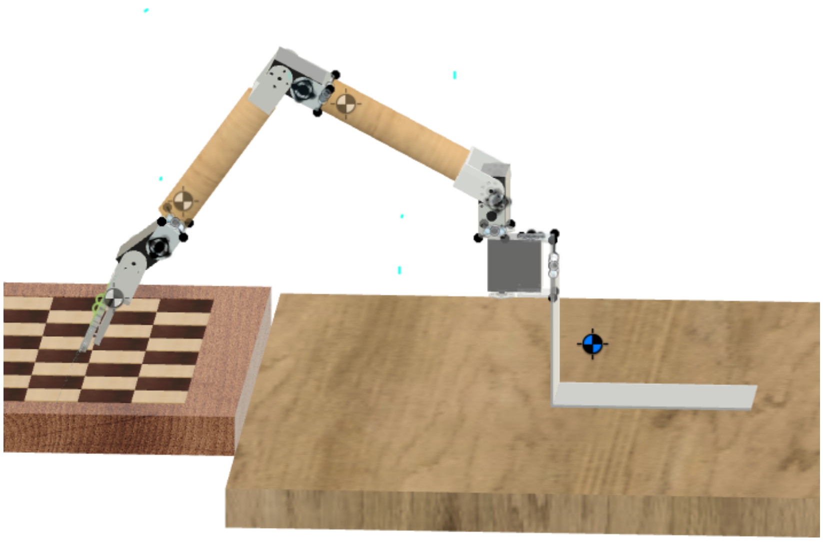

Figure 5: Center of mass for link d1

For the above figure, the COM is 50mm and includes 2 motors, 2 brackets and one L-bracket. In order to calculate the COM of link 2 (as per the figure):

$$ r_{c2} = 100 + 25 = 125mm \tag{20} $$

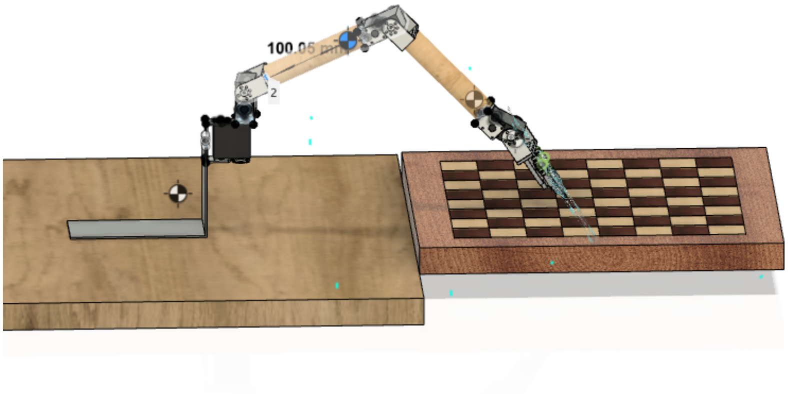

Figure 6: Center of mass for link d2

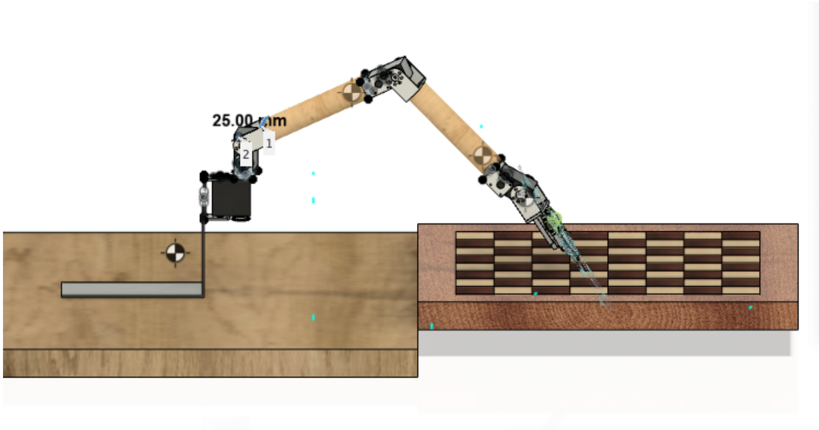

Figure 7: Center of mass for link d2

Since the length of the links d_2 and d_3 are equal, the location of the COM would also be at an equal length, i.e., 125mm.

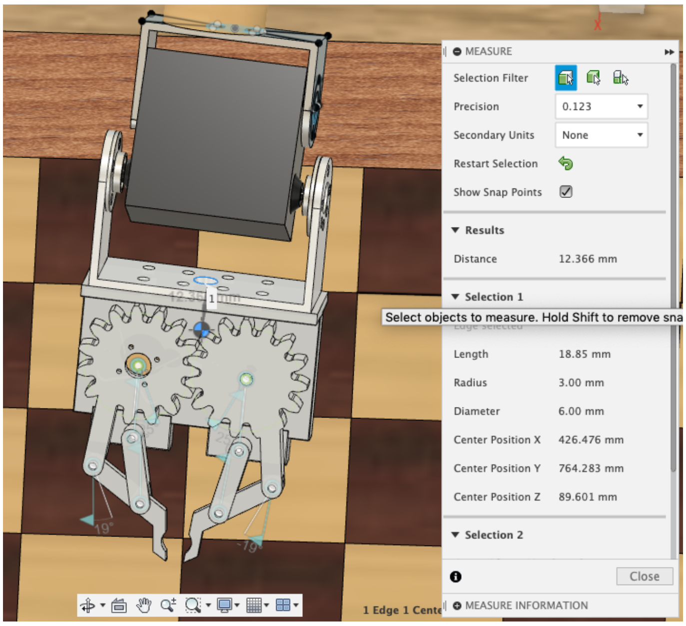

Figure 8: Center of mass for link d2

The masses of the materials can be found from the properties section under bodies on Fusion 360. The masses of links, brackets and the motor have been added to the appendix section.

3.3 Torque calculations

Having obtained our center of mass coordinates, we can now evaluate the joint torques required to overcome the gravity component for the task of lifting a chess piece. Using the Jacobian matrices obtained in section 3.1, we can derive an expression for the torques $\tau_i = [\tau_{i1}, \tau_{i2}...\tau_{i4}]^T$ needed to result in a task space force ${}^0\vec{f}_{Mi}$ at the center of mass M_i for all 4 joints of our robot:

$$\tau_i = J_{vi}^T \cdot {}^0\vec{f}_{Mi}, \ \ i = 1, 2...4 \tag{21}$$

such that J_{vi} is the translational component of the Jacobian matrix J_i for M_i. With Equation 21, we can now calculate the joint torques needed to overcome the gravity component of the task, to account for the weight of both the chess piece and the robot itself across the necessary workspace (i.e. the whole chessboard). Replacing f_{Mi} with the gravity component of each center of mass and summing the resultant torques needed at each joint we get:

$$\begin{aligned} G &= \sum J_{vi}^T \cdot -m_i g \, {}^0\hat{\vec{z}}_0 \\ &= \sum J_{vi}^T \cdot \begin{bmatrix} 0 \\ 0 \\ -m_i g \end{bmatrix}, \quad i = 1, 2...4 \end{aligned} \tag{22}$$

where $G = [G_1 \ G_2 \ G_3 \ G_4]^T$ is the static load of the robot due to the gravity forces acting on the centers of mass of the robot. To ensure that our robot can generate sufficient torque for lifting the heaviest chess piece at any point in the workspace, we will iterate through different values of the joint angles Q_i (within their operating boundaries) to determine the maximum torque needed to overcome the gravity component at each respective M_i. Note that for any Q_1 within range, the weight acting on the centers of masses for each link will be the same for a fixed $[Q_2, Q_3, Q_4]$ and hence will not have any affect on the values of G. Therefore, we will only need to vary Q_2, Q_3 & Q_4 when implementing the search algorithm on MATLAB. For the set of parameters below:

Link lengths	**Value (mm)**	**Center of mass displacement**	**Value (mm)**	**Link masses (rad)**	**Value (kg)**
d_1	50	c_1	25	m_1	0.067
d_2	183	c_2	125	m_2	0.111
d_3	183	c_3	125	m_3	0.111
d_4	50	c_4	50	m_4	0.035

We obtain the maximum torque needed for each individual joint across the entire workspace to be:

$$G = \begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.5802 \\ -0.1978 \\ -0.0132 \end{bmatrix} Nm \tag{23}$$

$$\tau_{max} = -G$$

Comparing this with the nominal torque $\tau_n = 1.618Nm$ [1] of our given motors, we can conclude that our robot will be able to generate sufficient torque to lift the chess piece at any point on the chessboard.

4 Conclusion

This report covered the derivation of the Jacobian matrix for the robot that maps the joint velocities into the end-effector velocity. From there, the torques acting on the joints due to gravity are calculated following the similar approach to calculate the Jacobian matrix. Hence, we can determine the maximum generated torque on the motor shaft and justify if the chosen motor can withstand that much amount of torque. The evaluation shows that the maximum torque generated on the motor shaft is 0.58Nm, which is less than the nominal torque of the motor, 1.618Nm.

References

[1] Rapidonline, 2021. [Online]. Available: https://www.rapidonline.com/feetech-scs15-smart-control-digital-servo-with-metal-gears-and-mounting-brackets-37-1332