Electricity & Magnetism

practice session

§1 – Lorentz law

§2 – Biot–Savart law: $\vec B$

§3 – Ampere's law

§4 – Magnetization: $\vec m$, $\vec M$, $\vec A$, $\vec H$,

§5 – Linear magnetics; χ_m

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§1

$\vec F = q\,\vec v \times \vec B$

$\vec B$ uniform $\quad \vec v \perp \vec B$

$F = qvB$,

$\Rightarrow F = m a_c = m\frac{v^2}{R}$

$\frac{mv^2}{R} = qvB$

$mv = qRB$

generalizing to current $\quad \vec I = \vec v \lambda$

wire: $\quad \vec F = \int_{wire} I\, d\vec l \times \vec B$

surface current $\vec K$: $\quad \vec F = \int_{surface\ of\ current} (\vec K \times \vec B)\, da$

volume current: $\quad \vec F = \int_{volume\ of\ current} (\vec J \times \vec B)\, d\tau$

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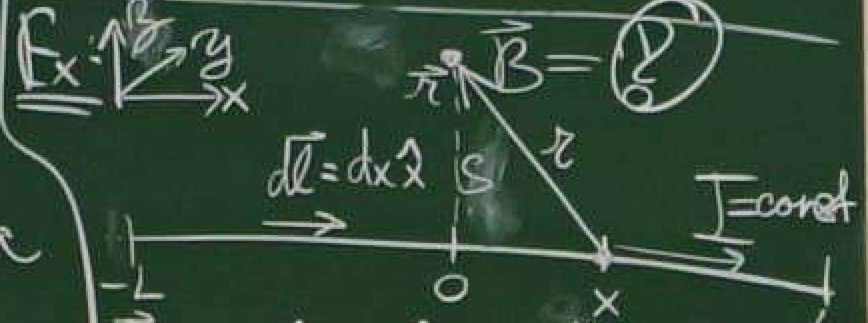
§2 magnetic field from wire of time-independent current is given by

$$\vec B(\vec r) = \frac{\mu_0}{4\pi}\int \frac{\vec I \times \hat{\mathscr r}}{\mathscr r^2}\, dl'$$

$$= \frac{\mu_0}{4\pi}\, \underbrace{I}_{const} \int \frac{d\vec l' \times \hat{\mathscr r}}{\mathscr r^2}$$

Ex: $\quad \vec B = ?$



$\vec{\mathscr r} = -x\hat x + s\hat z \qquad |\vec{\mathscr r}| = \sqrt{x^2 + s^2}$

$$d\vec l' \times \hat{\mathscr r} = \frac{1}{\mathscr r}\begin{vmatrix} \hat x & \hat y & \hat z \\ dx & 0 & 0 \\ -x & 0 & s \end{vmatrix} = -\hat y\,\frac{(s\,dx)}{\mathscr r}$$

$$\vec B = \frac{\mu_0}{4\pi} I(-\hat y)\, s \int \frac{dx}{\mathscr r^3}$$

$$= (-\hat y)\frac{\mu_0 I}{4\pi}\int_{-L}^{L}\frac{dx}{(x^2+s^2)^{3/2}}$$

$x = s\tan\theta \quad$ sub

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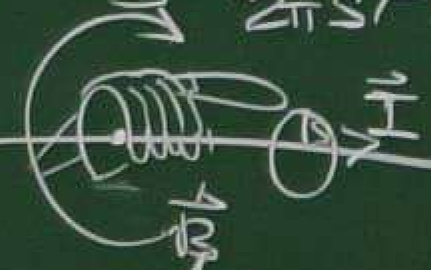
§3 Ampere's law

curve (infinite wire) $\vec I$ loop around wire

$\vec B = \frac{\mu_0 I}{2\pi s}\hat\phi$



$$\oint \vec B \cdot d\vec l = \oint \frac{\mu_0 I}{2\pi s}\, dl$$

$$= \frac{\mu_0 I}{\cancel{2\pi s}} \underbrace{\oint dl}_{\cancel{2\pi s}}$$

$$= \mu_0 I$$

By principle of superposition,

$$\boxed{\oint \vec B \cdot d\vec l = \mu_0 I_{enc}}$$

any smooth loop

$$\Longleftrightarrow \nabla \times \vec B = \mu_0 \vec J$$

Ex: wire $\odot$ s $\quad I$

$\oint \vec B \cdot d\vec l = \Leftarrow$ by symm $\begin{cases} \text{on the loop} \\ B = \text{const} \\ \text{direction along loop} \end{cases}$

$= B \cdot \underbrace{2\pi s}_{\text{length of loop}}$

By Amperes law: $\oint \vec B \cdot d\vec l = \mu_0 \underbrace{I^{enc}}_{= I}$

$\Rightarrow$ $B\, 2\pi s = \mu_0 I$

$B = \frac{\mu_0 I}{2\pi s}$

Thm (Helmholtz) any field (vector field) $\vec B$ as

$$\vec B = -\underbrace{\vec\nabla \mathcal{U}}_{\text{scalar } \mathbb{R}^3 \to \mathbb{R}} + \vec\nabla \times \underbrace{\vec A}_{\text{vector field } \mathbb{R}^3 \to \mathbb{R}^3}$$

$\mathcal{U}(\vec r) = \frac{1}{4\pi}\int \frac{\vec\nabla \cdot \vec B}{r}\, dr'$, $\quad \vec A(\vec r) = \frac{1}{4\pi}\int \frac{\vec\nabla \times \vec B}{r}\, dr'$

$\Rightarrow$ (Helmholtz Thm): $\mathcal{U}$ of magnetic field B is zero $\Rightarrow$ $\vec B = \vec\nabla \times \vec A$

$$\vec A = \frac{1}{4\pi}\int \frac{\vec\nabla \times \vec B}{r}\, dr' = \frac{1}{4\pi}\int \frac{\mu_0 \vec J(\vec r')}{r}\, dr' = \frac{\mu_0}{4\pi}\underbrace{\int \frac{\vec J(\vec r')}{r}\, dr'}$$

Typically easier to compute than B.S.

$\vec K_b = \vec M \times \hat n$

$\vec J_b = \vec\nabla \times \vec M$

given $\vec M$, find $\vec K_b$, $\vec J_b$ $\Rightarrow$ compute $\vec A \Rightarrow \vec B$

§4 Magnetization

expand $\vec A$ in $\frac{1}{r}$ with fixed O

$\Rightarrow$ $\vec A_{dip}(\vec r) = \frac{\mu_0}{4\pi}\frac{\vec m \times \hat r}{r^2}$, $\quad m = I \int da$ surface enclosing I

$\Rightarrow$ torque $\vec N = \vec m \times \vec B$

force $\vec F = \nabla(\vec m \cdot \vec B)$

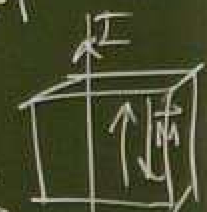
By principle of superposition, volume w/ $\vec m$ per volume we call $\vec M$ $\leftarrow$ magnetization

$\Rightarrow$ $\vec A(\vec r) = \frac{\mu_0}{4\pi}\int \frac{\vec M(\vec r') \times \hat r}{r^2}\, dr'$, $\quad \frac{\hat r}{r^2} = \vec\nabla' \frac{1}{r}$

$\uparrow$ properties of cross product

$$= \frac{\mu_0}{4\pi}\int_V \frac{1}{r}\underbrace{\vec\nabla' \times \vec M(\vec r')}_{=:\vec J_b}\, dr' + \frac{\mu_0}{4\pi}\oint_{\partial V} \frac{1}{r}\underbrace{\vec M \times \hat n}_{=:\vec K_b}\, da'$$

§5 special case of B_{free} causing $\vec M$



$\vec\nabla \times \vec B = \mu_0 \vec J_{total}$

$= \vec J_f + \vec J_b$

$\frac{1}{\mu_0}\vec\nabla \times \vec B = \vec J_f + \vec\nabla \times \vec M$

$\vec\nabla \times \underbrace{\left(\frac{1}{\mu_0}\vec B - \vec M\right)}_{=:\vec H \text{ auxiliary field}} = \vec J_f$

$\vec\nabla \times \vec H = \vec J_f$

$\Leftrightarrow$ $\oint \vec H \cdot d\vec l = I_f^{enc}$

if we know config of wires & currents we can find $\vec H$

Special case: $\vec M \propto \vec H$ $\quad \vec M = \chi_m \vec H$

$\frac{1}{\mu_0}\vec B - \vec M = \vec H$

$\vec B = \mu_0(\vec H + \vec M) = \mu_0(\vec H + \chi_m \vec H) = \underbrace{\mu_0(1+\chi_m)}_{\mu}\vec H$

$\boxed{\vec B = \mu \vec H}$ linear

Problem (linear)

1. given wire/current distrib $\Rightarrow$ compute $\vec H$
2. $\vec B = \mu \vec H$

Problem: given $\vec M$ $\Rightarrow$ compute $\vec J_b$, $\vec K_b$ $\Rightarrow \vec A \Rightarrow \vec B$