1 Deriving the Christoffel function for the metric g of $\mathcal{Q}_{\mathbb{R}_*^{n\times k}}$

Differentiate the equation for $\mathsf{g}\mathring{\omega}_2$

$$\mathsf{g}\mathring{\omega}_2 := (-\frac{1}{2K^2}\operatorname{Tr}(\bar{\omega}_2^{\mathsf{T}}xU)yU^{\mathsf{T}} + \frac{1}{2K}\bar{\omega}_2U^{\mathsf{T}}, -\frac{1}{2K^2}(\operatorname{Tr}(\omega_2^{\mathsf{T}}yU^{\mathsf{T}})xU + \frac{1}{2K}\omega_2U) \tag{1.1}$$

in direction $\mathring{\omega}$, and use the relation $\mathrm{D}_\omega U^T = -U^{\mathsf{T}}\mathrm{D}_\omega UU^{\mathsf{T}}$, we get $\mathring{\mathrm{D}}_{\mathring{\omega}}\mathsf{g}\mathring{\omega}_2 = (d_1, d_2)$ where

$$\begin{aligned}
d_1 = &\frac{1}{K^3}\operatorname{Tr}(\mathrm{D}_\omega\Sigma + \bar{\mathrm{D}}_{\bar{\omega}}\,\Sigma)\operatorname{Tr}(\bar{\mathrm{D}}_{\bar{\omega}_2}\,\Sigma)yU^{\mathsf{T}} - \frac{1}{2K^2}\operatorname{Tr}(\omega^{\mathsf{T}}\bar{\omega}_2U^{\mathsf{T}})yU^{\mathsf{T}}\\
&+\frac{1}{2K^2}\operatorname{Tr}(x^{\mathsf{T}}\bar{\omega}_2U^{\mathsf{T}}(\mathrm{D}_\omega U + \bar{\mathrm{D}}_{\bar{\omega}}\,U)U^{\mathsf{T}})yU^{\mathsf{T}} - \frac{1}{2K^2}\operatorname{Tr}(\bar{\mathrm{D}}_{\bar{\omega}_2}\,\Sigma)\bar{\omega}U^{\mathsf{T}}\\
&+\frac{1}{2K^2}\operatorname{Tr}(\bar{\mathrm{D}}_{\bar{\omega}_2}\,\Sigma)yU^{\mathsf{T}}(\mathrm{D}_\omega U + \bar{\mathrm{D}}_{\bar{\omega}}\,U)U^{\mathsf{T}} - \frac{1}{2K^2}\operatorname{Tr}(\mathrm{D}_\omega\Sigma + \bar{\mathrm{D}}_{\bar{\omega}}\,\Sigma)\bar{\omega}_2U^{\mathsf{T}}\\
&-\frac{1}{2K}\bar{\omega}_2U^{\mathsf{T}}(\mathrm{D}_\omega U + \bar{\mathrm{D}}_{\bar{\omega}}\,U)U^{\mathsf{T}},\\
d_2 = &\frac{1}{K^3}\operatorname{Tr}(\mathrm{D}_\omega\Sigma + \bar{\mathrm{D}}_{\bar{\omega}}\,\Sigma)\operatorname{Tr}(\mathrm{D}_{\omega_2}\Sigma)xU - \frac{1}{2K^2}\operatorname{Tr}(\bar{\omega}^{\mathsf{T}}\omega_2U)xU\\
&-\frac{1}{2K^2}\operatorname{Tr}(y^{\mathsf{T}}\omega_2(\mathrm{D}_\omega U + \bar{\mathrm{D}}_{\bar{\omega}}\,U))xU - \frac{1}{2K^2}\operatorname{Tr}(\mathrm{D}_{\omega_2}\Sigma)\omega U\\
&-\frac{1}{2K^2}\operatorname{Tr}(\mathrm{D}_{\omega_2}\Sigma)x(\mathrm{D}_\omega U + \bar{\mathrm{D}}_{\bar{\omega}}\,U) - \frac{1}{2K^2}\operatorname{Tr}(\mathrm{D}_\omega\Sigma + \bar{\mathrm{D}}_{\bar{\omega}}\,\Sigma)\omega_2U\\
&+\frac{1}{2K}\omega_2(\mathrm{D}_\omega U + \bar{\mathrm{D}}_{\bar{\omega}}\,U).
\end{aligned}$$

Thus, for $\mathring{\omega} = (\omega, \bar{\omega}), \mathring{\omega}_1 = (\omega_1, \bar{\omega}_1), \mathring{\omega}_2 = (\omega_2, \bar{\omega}_2)$

$$\chi_{\mathsf{g}}(\mathring{\omega}_2, \mathring{\omega}_1).\mathring{\omega} = \mathring{\mathrm{D}}\mathsf{g}_{\mathring{\omega}}\mathring{\omega}_2.\mathring{\omega}_1 = d_1.\omega_1 + d_2.\bar{\omega}_1$$

Collecting the terms with ω and D_ω to $\operatorname{Tr}\omega^{\mathsf{T}}\chi_{\mathsf{g}}(\mathring{\omega}_2, \mathring{\omega}_1)_x$ and $\bar{\mathrm{D}}_{\bar{\omega}}$ to $\operatorname{Tr}\bar{\omega}^{\mathsf{T}}\chi_{\mathsf{g}}(\mathring{\omega}_2, \mathring{\omega}_1)_y$

$$\begin{aligned}
\operatorname{Tr}\omega^{\mathsf{T}}\chi_{\mathsf{g}}(\mathring{\omega}_2, \mathring{\omega}_1)_x = &\frac{1}{K^3}\operatorname{Tr}(\mathrm{D}_\omega\Sigma)\operatorname{Tr}(\bar{\mathrm{D}}_{\bar{\omega}_2}\,\Sigma)\operatorname{Tr}yU^{\mathsf{T}}\omega_1^{\mathsf{T}} - \frac{1}{2K^2}\operatorname{Tr}(\omega^{\mathsf{T}}\bar{\omega}_2U^{\mathsf{T}})\operatorname{Tr}yU^{\mathsf{T}}\omega_1^{\mathsf{T}}\\
&+\frac{1}{2K^2}\operatorname{Tr}(x^{\mathsf{T}}\bar{\omega}_2U^{\mathsf{T}}\mathrm{D}_\omega UU^{\mathsf{T}}\operatorname{Tr}yU^{\mathsf{T}}\omega_1^{\mathsf{T}} + \frac{1}{2K^2}\operatorname{Tr}(\bar{\mathrm{D}}_{\bar{\omega}_2}\,\Sigma)\operatorname{Tr}yU^{\mathsf{T}}(\mathrm{D}_\omega U)U^{\mathsf{T}}\omega_1^{\mathsf{T}}\\
-\frac{1}{2K^2}\operatorname{Tr}(\mathrm{D}_\omega\Sigma)\operatorname{Tr}\bar{\omega}_2U^{\mathsf{T}}\omega_1^{\mathsf{T}} - &\frac{1}{2K}\operatorname{Tr}\bar{\omega}_2U^{\mathsf{T}}(\mathrm{D}_\omega U)U^{\mathsf{T}}\omega_1^{\mathsf{T}} + \frac{1}{K^3}\operatorname{Tr}(\mathrm{D}_\omega\Sigma)\operatorname{Tr}(\mathrm{D}_{\omega_2}\Sigma)\operatorname{Tr}xU\bar{\omega}_1^{\mathsf{T}}\\
&-\frac{1}{2K^2}\operatorname{Tr}(y^{\mathsf{T}}\omega_2(\mathrm{D}_\omega U))\operatorname{Tr}xU\bar{\omega}_1^{\mathsf{T}} - \frac{1}{2K^2}\operatorname{Tr}(\mathrm{D}_{\omega_2}\Sigma)\operatorname{Tr}\omega U\bar{\omega}_1^{\mathsf{T}}\\
&-\frac{1}{2K^2}\operatorname{Tr}(\mathrm{D}_{\omega_2}\Sigma)\operatorname{Tr}x(\mathrm{D}_\omega U)\bar{\omega}_1^{\mathsf{T}} - \frac{1}{2K^2}\operatorname{Tr}(\mathrm{D}_\omega\Sigma)\operatorname{Tr}\omega_2U\bar{\omega}_1^{\mathsf{T}} + \frac{1}{2K}\operatorname{Tr}\omega_2(\mathrm{D}_\omega U)\bar{\omega}_1^{\mathsf{T}}.
\end{aligned}$$

For $A \in \mathbb{R}^{k\times k}$ and $\mathring{\omega} = (\omega, \bar{\omega}) \in (\mathbb{R}^{n\times k})^2$, using

$$\begin{aligned}
\operatorname{Tr}(\mathrm{D}_\omega UA) &= 2\operatorname{Tr}(\omega^{\mathsf{T}}yU^{\mathsf{T}}L_\Sigma^{-1}(UA)_{\mathrm{skew}}),\\
\operatorname{Tr}\bar{\mathrm{D}}_{\bar{\omega}}\,UA) &= -2\operatorname{Tr}(\bar{\omega}^{\mathsf{T}}xL_\Sigma^{-1}(UA)_{\mathrm{skew}}U),
\end{aligned}$$

$$
\begin{aligned}
\operatorname{Tr}\omega^\intercal\chi_{\mathsf{g}}(\mathring{\omega}_2,\mathring{\omega}_1)_x &= \frac{1}{K^3}\operatorname{Tr}(\omega^\intercal yU^\intercal)\operatorname{Tr}(\bar{\mathrm{D}}_{\bar{\omega}_2}\Sigma)\operatorname{Tr}\mathrm{D}_{\omega_1}\Sigma - \frac{1}{2K^2}\operatorname{Tr}(\omega^\intercal\bar{\omega}_2U^\intercal)\operatorname{Tr}\mathrm{D}_{\omega_1}\Sigma \\
&+\frac{2}{2K^2}\operatorname{Tr}\omega^\intercal yU^\intercal L_\Sigma^{-1}(x^\intercal\bar{\omega}_2U^\intercal)_{\text{skew}}\operatorname{Tr}yU^\intercal\omega_1^\intercal + \frac{2}{2K^2}\operatorname{Tr}(\bar{\mathrm{D}}_{\bar{\omega}_2}\Sigma)\operatorname{Tr}\omega^\intercal yU^\intercal L_\Sigma^{-1}(\omega_1^\intercal yU^\intercal)_{\text{skew}} \\
&-\frac{1}{2K^2}\operatorname{Tr}(\omega^\intercal yU^\intercal)\operatorname{Tr}\bar{\omega}_2U^\intercal\omega_1^\intercal - \frac{2}{2K}\operatorname{Tr}\omega^\intercal yU^\intercal L_\Sigma^{-1}(\omega_1^\intercal\bar{\omega}_2U^\intercal)_{\text{skew}} \\
&+\frac{1}{K^3}\operatorname{Tr}(\omega^\intercal yU^\intercal)\operatorname{Tr}(\mathrm{D}_{\omega_2}\Sigma)\operatorname{Tr}\bar{\mathrm{D}}_{\bar{\omega}_1}\Sigma - \frac{2}{2K^2}\operatorname{Tr}((\omega^\intercal yU^\intercal L_\Sigma^{-1}(Uy^\intercal\omega_2)_{\text{skew}}\operatorname{Tr}\bar{\mathrm{D}}_{\bar{\omega}_1}\Sigma \\
&-\frac{1}{2K^2}\operatorname{Tr}(\mathrm{D}_{\omega_2}\Sigma)\operatorname{Tr}\omega^\intercal\bar{\omega}_1U^\intercal - \frac{2}{2K^2}\operatorname{Tr}(\mathrm{D}_{\omega_2}\Sigma)\operatorname{Tr}\omega^\intercal yU^\intercal L_\Sigma^{-1}(U\bar{\omega}_1^\intercal x)_{\text{skew}} \\
&-\frac{1}{2K^2}\operatorname{Tr}(\omega^\intercal yU^\intercal)\operatorname{Tr}\omega_2U\bar{\omega}_1^\intercal + \frac{2}{2K}\operatorname{Tr}\omega^\intercal yU^\intercal L_\Sigma^{-1}(U\bar{\omega}_1^\intercal\omega_2)_{\text{skew}}.
\end{aligned}
$$

Thus, if $\mathring{\omega}_1 = \mathring{\omega}_2$

$$
\begin{aligned}
\operatorname{Tr}\omega^\intercal\chi_{\mathsf{g}}(\mathring{\omega}_1,\mathring{\omega}_1)_x &= \frac{2}{K^3}\operatorname{Tr}(\omega^\intercal yU^\intercal)\operatorname{Tr}(\bar{\mathrm{D}}_{\bar{\omega}_1}\Sigma)\operatorname{Tr}\mathrm{D}_{\omega_1}\Sigma - \frac{1}{K^2}\operatorname{Tr}(\omega^\intercal\bar{\omega}_1U^\intercal)\operatorname{Tr}\mathrm{D}_{\omega_1}\Sigma \\
&+\frac{1}{K^2}\operatorname{Tr}(\omega^\intercal yU^\intercal\bar{\mathrm{D}}_{\bar{\omega}_1}UU^\intercal)\operatorname{Tr}yU^\intercal\omega_1^\intercal + \frac{1}{K^2}\operatorname{Tr}(\bar{\mathrm{D}}_{\bar{\omega}_1}\Sigma)\operatorname{Tr}\omega^\intercal yU^\intercal\mathrm{D}_{\omega_1}UU^\intercal \\
&-\frac{1}{K^2}\operatorname{Tr}(\omega^\intercal yU^\intercal)\operatorname{Tr}\bar{\omega}_1U^\intercal\omega_1^\intercal - \frac{2}{K}\operatorname{Tr}\omega^\intercal yU^\intercal L_\Sigma^{-1}(\omega_1^\intercal\bar{\omega}_1U^\intercal)_{\text{skew}},
\end{aligned}
$$

$$
\begin{aligned}
\chi_{\mathsf{g}}(\mathring{\omega}_1,\mathring{\omega}_1)_x &= \frac{2}{K^3}\operatorname{Tr}(\bar{\mathrm{D}}_{\bar{\omega}_1}\Sigma)\operatorname{Tr}(\mathrm{D}_{\omega_1}\Sigma)yU^\intercal - \frac{1}{K^2}\operatorname{Tr}(\mathrm{D}_{\omega_1}\Sigma)\bar{\omega}_1U^\intercal \\
&+\frac{1}{K^2}\operatorname{Tr}(\mathrm{D}_{\omega_1}\Sigma)yU^\intercal\bar{\mathrm{D}}_{\bar{\omega}_1}UU^\intercal + \frac{1}{K^2}\operatorname{Tr}(\bar{\mathrm{D}}_{\bar{\omega}_1}\Sigma)yU^\intercal\mathrm{D}_{\omega_1}UU^\intercal \\
&-\frac{1}{K^2}\operatorname{Tr}(\omega_1^\intercal\bar{\omega}_1U^\intercal)yU^\intercal - \frac{2}{K}yU^\intercal L_\Sigma^{-1}(\omega_1^\intercal\bar{\omega}_1U^\intercal)_{\text{skew}}.
\end{aligned}
$$

As $\Gamma^{\mathcal{Q}}$ is torsion free, it suffies to first compute $(2\mathring{\mathrm{D}}_{\mathring{\omega}}\mathsf{g}\mathring{\omega} - \chi_{\mathsf{g}}(\omega,\omega))$. The x-component is

$$
\begin{aligned}
(2\mathring{\mathrm{D}}_{\mathring{\omega}}\mathsf{g}\mathring{\omega} - \chi_{\mathsf{g}}(\omega,\omega))_x &= \frac{2}{K^3}\operatorname{Tr}(\mathrm{D}_\omega\Sigma + \bar{\mathrm{D}}_{\bar{\omega}}\Sigma)\operatorname{Tr}(\bar{\mathrm{D}}_{\bar{\omega}}\Sigma)yU^\intercal - \frac{1}{K^2}\operatorname{Tr}(\omega^\intercal\bar{\omega}U^\intercal)yU^\intercal \\
&+\frac{1}{K^2}\operatorname{Tr}(x^\intercal\bar{\omega}U^\intercal(\mathrm{D}_\omega U + \bar{\mathrm{D}}_{\bar{\omega}}U)U^\intercal)yU^\intercal - \frac{1}{K^2}\operatorname{Tr}(\bar{\mathrm{D}}_{\bar{\omega}}\Sigma)\bar{\omega}U^\intercal \\
&+\frac{1}{K^2}\operatorname{Tr}(\bar{\mathrm{D}}_{\bar{\omega}}\Sigma)yU^\intercal(\mathrm{D}_\omega U + \bar{\mathrm{D}}_{\bar{\omega}}U)U^\intercal - \frac{1}{K^2}\operatorname{Tr}(\mathrm{D}_\omega\Sigma + \bar{\mathrm{D}}_{\bar{\omega}}\Sigma)\bar{\omega}U^\intercal \\
&-\frac{1}{K}\bar{\omega}U^\intercal(\mathrm{D}_\omega U + \bar{\mathrm{D}}_{\bar{\omega}}U)U^\intercal \\
&-\frac{2}{K^3}\operatorname{Tr}(\bar{\mathrm{D}}_{\bar{\omega}}\Sigma)\operatorname{Tr}(\mathrm{D}_\omega\Sigma)yU^\intercal + \frac{1}{K^2}\operatorname{Tr}(\mathrm{D}_\omega\Sigma)\bar{\omega}U^\intercal \\
&-\frac{1}{K^2}\operatorname{Tr}(\mathrm{D}_\omega\Sigma)yU^\intercal\bar{\mathrm{D}}_{\bar{\omega}}UU^\intercal - \frac{1}{K^2}\operatorname{Tr}(\bar{\mathrm{D}}_{\bar{\omega}}\Sigma)yU^\intercal\mathrm{D}_\omega UU^\intercal \\
&+\frac{1}{K^2}\operatorname{Tr}(\omega^\intercal\bar{\omega}U^\intercal)yU^\intercal + \frac{2}{K}yU^\intercal L_\Sigma^{-1}(\omega^\intercal\bar{\omega}U^\intercal)_{\text{skew}}
\end{aligned}
$$

$$
\begin{aligned}
(2\mathring{\mathrm{D}}_{\mathring{\omega}}\mathsf{g}\mathring{\omega} - \chi_{\mathsf{g}}(\omega,\omega))_x &= \frac{2}{K^3}\operatorname{Tr}(\bar{\mathrm{D}}_{\bar{\omega}}\,\Sigma)\operatorname{Tr}(\bar{\mathrm{D}}_{\bar{\omega}}\,\Sigma) y U^{\mathsf{T}} \\
&+\frac{1}{K^2}\operatorname{Tr}(x^{\mathsf{T}}\bar{\omega}U^{\mathsf{T}}(\mathrm{D}_\omega U + \bar{\mathrm{D}}_{\bar{\omega}}\,U)U^{\mathsf{T}}) y U^{\mathsf{T}} - \frac{2}{K^2}\operatorname{Tr}(\bar{\mathrm{D}}_{\bar{\omega}}\,\Sigma)\bar{\omega}U^{\mathsf{T}} \\
&+\frac{1}{K^2}\operatorname{Tr}(\bar{\mathrm{D}}_{\bar{\omega}}\,\Sigma - \mathrm{D}_\omega\Sigma) y U^{\mathsf{T}}(\bar{\mathrm{D}}_{\bar{\omega}}\,U)U^{\mathsf{T}} - \frac{1}{K}\bar{\omega}U^{\mathsf{T}}(\mathrm{D}_\omega U + \bar{\mathrm{D}}_{\bar{\omega}}\,U)U^{\mathsf{T}} + \frac{2}{K} y U^{\mathsf{T}} L_{\Sigma}^{-1}(\omega^{\mathsf{T}}\bar{\omega}U^{\mathsf{T}})_{\mathrm{skew}}).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\operatorname{Tr} x^{\mathsf{T}}(2\mathring{\mathrm{D}}_{\mathring{\omega}}\mathsf{g}\mathring{\omega} - \chi_{\mathsf{g}}(\omega,\omega))_x &= \frac{2}{K^3}\operatorname{Tr}(\bar{\mathrm{D}}_{\bar{\omega}}\,\Sigma)\operatorname{Tr}(\bar{\mathrm{D}}_{\bar{\omega}}\,\Sigma)\operatorname{Tr}\Sigma \\
&+\frac{1}{K^2}\operatorname{Tr}(x^{\mathsf{T}}\bar{\omega}U^{\mathsf{T}}(\mathrm{D}_\omega U + \bar{\mathrm{D}}_{\bar{\omega}}\,U)U^{\mathsf{T}})\operatorname{Tr}\Sigma - \frac{2}{K^2}\operatorname{Tr}(\bar{\mathrm{D}}_{\bar{\omega}}\,\Sigma)\operatorname{Tr} x^{\mathsf{T}}\bar{\omega}U^{\mathsf{T}} \\
&+\frac{1}{K^2}\operatorname{Tr}(\bar{\mathrm{D}}_{\bar{\omega}}\,\Sigma - \mathrm{D}_\omega\Sigma)\operatorname{Tr}\Sigma(\bar{\mathrm{D}}_{\bar{\omega}}\,U)U^{\mathsf{T}} - \frac{1}{K}\operatorname{Tr} x^{\mathsf{T}}\bar{\omega}U^{\mathsf{T}}(\mathrm{D}_\omega U + \bar{\mathrm{D}}_{\bar{\omega}}\,U)U^{\mathsf{T}} + \frac{2}{K}\operatorname{Tr}\Sigma L_{\Sigma}^{-1}(\omega^{\mathsf{T}}\bar{\omega}U^{\mathsf{T}})_{\mathrm{skew}}) \\
&= -\frac{2}{K^3}\operatorname{Tr}(\bar{\mathrm{D}}_{\bar{\omega}}\,\Sigma)\operatorname{Tr}(\bar{\mathrm{D}}_{\bar{\omega}}\,\Sigma)\alpha - \frac{1}{K^2}\operatorname{Tr}(x^{\mathsf{T}}\bar{\omega}U^{\mathsf{T}}(\mathrm{D}_\omega U + \bar{\mathrm{D}}_{\bar{\omega}}\,U)U^{\mathsf{T}})\alpha
\end{aligned}
$$

using $K = \alpha + \operatorname{Tr}\Sigma$ and trace inner products between symmetric and antisymmetric matrices are zero. Hence, $\frac{1}{2}(\mathsf{g}^{-1}(2\mathring{\mathrm{D}}_{\mathring{\omega}}\mathsf{g}\mathring{\omega} - \chi_{\mathsf{g}}(\omega,\omega)))_y$ is given by

$$
\begin{aligned}
&K\{\frac{2}{K^3}\operatorname{Tr}(\bar{\mathrm{D}}_{\bar{\omega}}\,\Sigma)\operatorname{Tr}(\bar{\mathrm{D}}_{\bar{\omega}}\,\Sigma) y + \frac{1}{K^2}\operatorname{Tr}(x^{\mathsf{T}}\bar{\omega}U^{\mathsf{T}}(\mathrm{D}_\omega U + \bar{\mathrm{D}}_{\bar{\omega}}\,U)U^{\mathsf{T}}) y - \frac{2}{K^2}\operatorname{Tr}(\bar{\mathrm{D}}_{\bar{\omega}}\,\Sigma)\bar{\omega} \\
&+\frac{1}{K^2}\operatorname{Tr}(\bar{\mathrm{D}}_{\bar{\omega}}\,\Sigma - \mathrm{D}_\omega\Sigma) y U^{\mathsf{T}}(\bar{\mathrm{D}}_{\bar{\omega}}\,U) - \frac{1}{K}\bar{\omega}U^{\mathsf{T}}(\mathrm{D}_\omega U + \bar{\mathrm{D}}_{\bar{\omega}}\,U) + \frac{2}{K} y U^{\mathsf{T}} L_{\Sigma}^{-1}(\omega^{\mathsf{T}}\bar{\omega}U^{\mathsf{T}})_{\mathrm{skew}})U\} \\
&\quad +K(-\frac{2}{K^3}\operatorname{Tr}(\bar{\mathrm{D}}_{\bar{\omega}}\,\Sigma)\operatorname{Tr}(\bar{\mathrm{D}}_{\bar{\omega}}\,\Sigma) - \frac{1}{K^2}\operatorname{Tr}(x^{\mathsf{T}}\bar{\omega}U^{\mathsf{T}}(\mathrm{D}_\omega U + \bar{\mathrm{D}}_{\bar{\omega}}\,U)U^{\mathsf{T}})) y \\
&\quad = -\frac{2}{K}\operatorname{Tr}(\bar{\mathrm{D}}_{\bar{\omega}}\,\Sigma)\bar{\omega} + \frac{1}{K}\operatorname{Tr}(\bar{\mathrm{D}}_{\bar{\omega}}\,\Sigma - \mathrm{D}_\omega\Sigma) y U^{\mathsf{T}}(\bar{\mathrm{D}}_{\bar{\omega}}\,U) \\
&\quad -\bar{\omega}U^{\mathsf{T}}(\mathrm{D}_\omega U + \bar{\mathrm{D}}_{\bar{\omega}}\,U) + 2 y U^{\mathsf{T}} L_{\Sigma}^{-1}(\omega^{\mathsf{T}}\bar{\omega}U^{\mathsf{T}})_{\mathrm{skew}})U,
\end{aligned}
$$

which is the $\Gamma^{\mathcal{Q}}_y$ component of the Christoffel function in equation (7.18) in the paper. The $\Gamma^{\mathcal{Q}}_x$ component is derived similarly.