Bertsimas, Tsitsiklis: *Introduction to Linear Optimization*

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2 • The geometry of linear programming

Exercise 2.5

(Extreme points of isomorphic polyhedra) A mapping f is called ***affine*** if it is of the form $f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$, where $\boldsymbol{A}$ is a matrix and $\boldsymbol{b}$ is a vector. Let P and Q be polyhedra in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. We say that P and Q are ***isomorphic*** if there exist affine mappings $f\colon P \to Q$ and $g\colon Q \to P$ such that $g(f(\boldsymbol{x})) = \boldsymbol{x}$ for all $\boldsymbol{x} \in P$, and $f(g(\boldsymbol{y})) = \boldsymbol{y}$ for all $\boldsymbol{y} \in Q$. (Intuitively, isomorphic polyhedra have the same shape.)

(a) If P and Q are isomorphic, show that there exists a one-to-one correspondence between their extreme points. In particular, if f and g are as above, show that $\boldsymbol{x}$ is an extreme point of P if and only if $f(\boldsymbol{x})$ is an extreme point of Q.

(b) (Introducing slack variables leads to an isomorphic polyhedron) Let $P = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$, where $\boldsymbol{A}$ is a matrix of dimensions $k \times n$. Let $Q = \{(\boldsymbol{x}, \boldsymbol{z}) \in \mathbb{R}^{n+k} \mid \boldsymbol{A}\boldsymbol{x} - \boldsymbol{z} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}, \boldsymbol{x} \geq \boldsymbol{0}\}$. Show that P and Q are isomorphic.

Solution. (a) For $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in P$ with $\boldsymbol{x} = \lambda\boldsymbol{y} + (1-\lambda)\boldsymbol{z}$, then since f is affine we have $f(\boldsymbol{x}) = \lambda f(\boldsymbol{y}) + (1-\lambda) f(\boldsymbol{z})$. Hence if $\boldsymbol{x}$ is *not* an extreme point, then neither is $f(\boldsymbol{x})$. Since f is bijective with an affine inverse, this proves the claim.

(b) Simply note that the map $f\colon P \to Q$ given by $f(\boldsymbol{x}) = (\boldsymbol{x}, \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b})$ is an isomorphism. □

EXERCISE 2.7

Suppose that $\{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{a}_i'\boldsymbol{x} \geq b_i, i = 1,\ldots,m\}$ and $\{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{g}_i'\boldsymbol{x} \geq h_i, i = 1,\ldots,k\}$ are representations of the same nonempty polyhedron P. Suppose that the vectors $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m$ span $\mathbb{R}^n$. Show that the same must be true for the vectors $\boldsymbol{g}_1,\ldots,\boldsymbol{g}_k$.

SOLUTION. If $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m$ span $\mathbb{R}^n$, then they must contain a collection of n linearly independent vectors. Then Theorem 2.6 implies that P has an extreme point. But this is a geometric property, so it does not depend on the representation. The same theorem then implies that the collection $\boldsymbol{g}_1,\ldots,\boldsymbol{g}_k$ contains n linearly independent vectors. But then they span $\mathbb{R}^n$. □

EXERCISE 2.12

Consider a nonempty polyhedron P and suppose that for each variable x_i we have either the constraint $x_i \geq 0$ or the contraint $x_i \geq 0$. Is it true that P has at least one basic feasible solution?

SOLUTION. Let I be the set of indices i such that we have the contraint $x_i \leq 0$, and let $\boldsymbol{A}$ be a diagonal matrix with a 1 as the ith entry on the diagonal if $i \notin I$, and -1 if $i \in I$. The map $\boldsymbol{x} \mapsto \boldsymbol{A}\boldsymbol{x}$ is then an isomorphism between P and the polyhedron Q which is defined by the same contraints as P, except that the contraints on the form $x_i \leq 0$ are replaced by contraints $x_i \geq 0$. In particular, P and Q have the same extreme points by [TODO ref] Exercise 2.4, hence the same basic feasible solutions by Theorem 2.3.

Next introduce slack variables to Q, yielding a polyhedron R in standard form, which is isomorphic to Q by [TODO ref] Exercise 2.5. But since R is nonempty (since it is isomorphic to P) it has a basic feasible solution by Corollary 2.2. □

TODO 2.14, 2.15

EXERCISE 2.22

Let P and Q be polyhedra in $\mathbb{R}^n$.

(a) Show that $P + Q$ is a polyhedron.

(b) Show that every extreme point of $P + Q$ is the sum of an extreme point of P and an extreme point of Q.

SOLUTION. (a) The Cartesian product $P \times Q$ is clearly a polyhedron. Its image under the linear map $(\boldsymbol{x}, \boldsymbol{y}) \mapsto \boldsymbol{x} + \boldsymbol{y}$ is exactly $P + Q$, and this is a polyhedron by Corollary 2.5.

(b) Let $\boldsymbol{z} \in P + Q$ and write $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y}$ for $\boldsymbol{x} \in P$ and $\boldsymbol{y} \in Q$. Assume that $\boldsymbol{x}$ is not an extreme point in P, so that there exist $\boldsymbol{x}_1, \boldsymbol{x}_2 \in P \setminus \{\boldsymbol{x}\}$ and a $\lambda \in [0,1]$ such that $\boldsymbol{x} = \lambda \boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2$. Letting $\boldsymbol{z}_i := \boldsymbol{x}_i + \boldsymbol{y}$ we easily find that $\boldsymbol{z} = \lambda \boldsymbol{z}_1 + (1-\lambda)\boldsymbol{z}_2$, so $\boldsymbol{z}$ is not an extreme point. $\square$