

Bertsimas, Tsitsiklis: *Introduction to Linear Optimization*

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2 The geometry of linear programming

REMARK 2.1. Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a map. Then the following properties are equivalent:

- (a) There exists a matrix A and a vector b such that $f(x) = Ax + b$ for all $x \in \mathbb{R}^n$.
- (b) For all $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$, $f((1-t)x + ty) = (1-t)f(x) + tf(y)$.
- (c) For all $x_1, x_k \in \mathbb{R}^n$ and $t_1, \dots, t_k \in \mathbb{R}$ with $t_1 + \dots + t_k = 1$, $f(t_1 x_1 + \dots + t_k x_k) = t_1 f(x_1) + \dots + t_k f(x_k)$.

Such a function is called *affine*. Note that if $m = 1$, then f is affine if and only if it is both convex and concave.

The first property clearly entails the third, which then entails the second, so assume that f has the second property. First assume that $f(0) = 0$. For $\beta \in \mathbb{R}$ and $x \in \mathbb{R}^n$ we thus have

$$f(\beta x) = f(\beta x + (1-\beta)0) = \beta f(x) + (1-\beta)f(0) = \beta f(x),$$

so f is homogeneous. If also $y \in \mathbb{R}^n$, then

$$\frac{1}{2}f(x+y) = f\left(\frac{1}{2}x + \frac{1}{2}y\right) = \frac{1}{2}f(x) + \frac{1}{2}f(y),$$

so f is also additive, hence linear. For general f , simply replace f with $f - f(0)$. \lrcorner

REMARK 2.2. Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be affine, and let $M \subseteq \mathbb{R}^n$ be affine. Then the image $f(M)$ is also affine. \lrcorner

REMARK 2.3. If $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ preserves the Euclidean metric, then it is affine. Replacing f with $f - f(\mathbf{0})$ we may assume that $f(\mathbf{0}) = \mathbf{0}$, and so that f also preserves the inner product. For all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$ we have

$$\begin{aligned} \langle f(\beta\mathbf{x} + \mathbf{y}), f(\mathbf{z}) \rangle &= \langle \beta\mathbf{x} + \mathbf{y}, \mathbf{z} \rangle \\ &= \beta\langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle \\ &= \beta\langle f(\mathbf{x}), f(\mathbf{z}) \rangle + \langle f(\mathbf{y}), f(\mathbf{z}) \rangle \\ &= \langle \beta f(\mathbf{x}) + f(\mathbf{y}), f(\mathbf{z}) \rangle. \end{aligned}$$

Since f is surjective, the claim follows. \lrcorner

REMARK 2.4: Adjacent basic solutions and adjacent bases.

Let $P = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ be a polyhedron in standard form.

If \mathbf{x} and \mathbf{y} are adjacent basic solutions, then they can be obtained from two adjacent bases.

Assume that $x_i = 0$ for $i \neq B(1), \dots, B(m)$. Each of the m equality constraints are active at both \mathbf{x} and \mathbf{y} , so there must be exactly one k such that $y_{B(k)} \neq 0$. Then \mathbf{x} can be obtained from the basis with basic indices $\{B(1), \dots, B(m)\}$ and \mathbf{y} from a basis with basic indices $\{B(1), \dots, B(k-1), j, B(k+1), \dots, B(m)\}$ where $j \neq B(k)$. Hence these two bases are adjacent as claimed.

If two adjacent bases lead to distinct basic solutions \mathbf{x} and \mathbf{y} , then these are adjacent.

Similarly, there are (at least) $n - 1$ constraints that are active at both \mathbf{x} and \mathbf{y} (the equality constraints along with all but one of the constraints given by the $B(k)$). Since $\mathbf{x} \neq \mathbf{y}$ there must be some k such that $x_{B(k)} = 0$ and $y_{B(k)} \neq 0$, so there are exactly $n - 1$ constraints that are active at both \mathbf{x} and \mathbf{y} . \lrcorner

EXERCISE 2.5

(Extreme points of isomorphic polyhedra) A mapping f is called **affine** if it is of the form $f(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$, where A is a matrix and \mathbf{b} is a vector. Let P and Q be polyhedra in \mathbb{R}^n and \mathbb{R}^m , respectively. We say that P and Q are **isomorphic** if there exist affine mappings $f: P \rightarrow Q$ and $g: Q \rightarrow P$ such that $g(f(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x} \in P$, and $f(g(\mathbf{y})) = \mathbf{y}$ for all $\mathbf{y} \in Q$. (Intuitively, isomorphic polyhedra have the same shape.)

- (a) If P and Q are isomorphic, show that there exists a one-to-one correspondence between their extreme points. In particular, if f and g are

as above, show that \mathbf{x} is an extreme point of P if and only if $f(\mathbf{x})$ is an extreme point of Q .

- (b) (Introducing slack variables leads to an isomorphic polyhedron) Let $P = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{Ax} \geq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$, where \mathbf{A} is a matrix of dimensions $k \times n$. Let $Q = \{(\mathbf{x}, \mathbf{z}) \in \mathbb{R}^{n+k} \mid \mathbf{Ax} - \mathbf{z} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}, \mathbf{z} \geq \mathbf{0}\}$. Show that P and Q are isomorphic.

SOLUTION. (a) For $\mathbf{x}, \mathbf{y}, \mathbf{z} \in P$ with $\mathbf{x} = \lambda \mathbf{y} + (1 - \lambda) \mathbf{z}$, then since f is affine we have $f(\mathbf{x}) = \lambda f(\mathbf{y}) + (1 - \lambda) f(\mathbf{z})$. Hence if \mathbf{x} is *not* an extreme point, then neither is $f(\mathbf{x})$. Since f is bijective with an affine inverse, this proves the claim.

(b) Simply note that the map $f: P \rightarrow Q$ given by $f(\mathbf{x}) = (\mathbf{x}, \mathbf{Ax} - \mathbf{b})$ is an isomorphism. \square

EXERCISE 2.7

Suppose that $\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}'_i \mathbf{x} \geq b_i, i = 1, \dots, m\}$ and $\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{g}'_i \mathbf{x} \geq h_i, i = 1, \dots, k\}$ are representations of the same nonempty polyhedron P . Suppose that the vectors $\mathbf{a}_1, \dots, \mathbf{a}_m$ span \mathbb{R}^n . Show that the same must be true for the vectors $\mathbf{g}_1, \dots, \mathbf{g}_k$.

SOLUTION. If $\mathbf{a}_1, \dots, \mathbf{a}_m$ span \mathbb{R}^n , then they must contain a collection of n linearly independent vectors. Then Theorem 2.6 implies that P has an extreme point. But this is a geometric property, so it does not depend on the representation. The same theorem then implies that the collection $\mathbf{g}_1, \dots, \mathbf{g}_k$ contains n linearly independent vectors. But then they span \mathbb{R}^n . \square

EXERCISE 2.12

Consider a nonempty polyhedron P and suppose that for each variable x_i we have either the constraint $x_i \geq 0$ or the constraint $x_i \leq 0$. Is it true that P has at least one basic feasible solution?

SOLUTION. Let I be the set of indices i such that we have the constraint $x_i \leq 0$, and let \mathbf{A} be a diagonal matrix with a 1 as the i th entry on the diagonal if $i \notin I$, and -1 if $i \in I$. The map $\mathbf{x} \mapsto \mathbf{Ax}$ is then an isomorphism between P and the polyhedron Q which is defined by the same constraints as P , except that the constraints on the form $x_i \leq 0$ are replaced by constraints $x_i \geq 0$. In particular, P and Q have the same extreme points by [TODO ref] Exercise 2.4, hence the same basic feasible solutions by Theorem 2.3.

Next introduce slack variables to Q , yielding a polyhedron R in standard form, which is isomorphic to Q by [TODO ref] Exercise 2.5. But since R is

nonempty (since it is isomorphic to P) it has a basic feasible solution by Corollary 2.2. \square

TODO 2.14, 2.15

EXERCISE 2.21

Suppose that Fourier–Motzkin elimination is used in the manner described at the end of Section 2.8 to find the optimal cost in a linear programming problem. Show how this approach can be augmented to obtain an optimal solution as well.

SOLUTION. Let $P \subseteq \mathbb{R}^n$ be a polyhedron, and assume that we wish to minimise the quantity $c'x$ subject to $x \in P$. Define

$$P_n = \{(x_0, x) \in \mathbb{R}^{n+1} \mid x_0 = c'x \text{ and } x \in P\},$$

which is clearly a polyhedron in \mathbb{R}^{n+1} . To minimise $c'x$ with $x \in P$ we thus find the minimal value of x_0 among elements $(x_0, x) \in P_n$. Letting $P_{n-k} = \Pi_k(P_n)$ for $k = 1, \dots, n$ we have $Q := P_0 = \Pi_n(P_n) = \{x_0 \in \mathbb{R} \mid \exists x \in P: (x_0, x) \in P_n\}$, so this is equivalent to finding the minimum of Q . Given the minimum of Q we wish to find a (not necessarily unique) $x \in P$ such that $(\min Q, x) \in P_n$.

We do this by recursively finding a suitable value for the variable we eliminate in each step of the algorithm. In the notation of the Elimination algorithm on page 72, let $P \subseteq \mathbb{R}^n$ be a polyhedron, and let $Q = \Pi_{n-1}(P)$ be the polyhedron obtained when eliminating the variable x_n . Let \bar{x} be an element of Q , i.e. a vector satisfying the inequalities

$$\begin{aligned} d_j + f_j' \bar{x} &\geq d_i + f_i' \bar{x}, & \text{if } a_{in} > 0 \text{ and } a_{jn} < 0, \\ 0 &\geq d_k + f_k' \bar{x}, & \text{if } a_{kn} = 0, \end{aligned}$$

for $i, j, k = 1, \dots, n$. Now choose any $x_n \in \mathbb{R}$ that lies in all the intervals $[d_i + f_i' \bar{x}, d_j + f_j' \bar{x}]$. Then the vector (\bar{x}, x_n) satisfies the inequalities

$$\begin{aligned} x_n &\geq d_i + f_i' \bar{x}, & \text{if } a_{in} > 0, \\ d_j + f_j' \bar{x} &\geq x_n, & \text{if } a_{jn} < 0, \\ 0 &\geq d_k + f_k' \bar{x}, & \text{if } a_{kn} = 0. \end{aligned}$$

Such an x_n exists since these inequalities are a representation of P , and the former inequalities are a representation of Q , which is the projection of P by Theorem 2.10. Thus $(\bar{x}, x_n) \in P$ as desired. \square

EXERCISE 2.22

Let P and Q be polyhedra in \mathbb{R}^n .

- (a) Show that $P + Q$ is a polyhedron.
- (b) Show that every extreme point of $P + Q$ is the sum of an extreme point of P and an extreme point of Q .

SOLUTION. (a) The Cartesian product $P \times Q$ is clearly a polyhedron. Its image under the linear map $(x, y) \mapsto x + y$ is exactly $P + Q$, and this is a polyhedron by Corollary 2.5.

(b) Let $z \in P + Q$ and write $z = x + y$ for $x \in P$ and $y \in Q$. Assume that x is not an extreme point in P , so that there exist $x_1, x_2 \in P \setminus \{x\}$ and a $\lambda \in [0, 1]$ such that $x = \lambda x_1 + (1 - \lambda)x_2$. Letting $z_i := x_i + y$ we easily find that $z = \lambda z_1 + (1 - \lambda)z_2$, so z is not an extreme point. \square