Bertsimas, Tsitsiklis: *Introduction to Linear Optimization*

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2 The geometry of linear programming

REMARK 2.1. Let $f \colon \mathbb{R}^n \to \mathbb{R}^m$ be a map. Then the following properties are equivalent:

(a) There exists a matrix A and a vector b such that $f(x) = Ax + b$ for all $x \in \mathbb{R}^n$.

(b) For all $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$, $f((1-t)x + ty) = (1-t)f(x) + tf(y)$.

(c) For all $x_1, \ldots, x_k \in \mathbb{R}^n$ and $t_1, \ldots, t_k \in \mathbb{R}$ with $t_1 + \cdots + t_k = 1$, $f(t_1 x_1 + \cdots + t_k x_k) = t_1 f(x_1) + \cdots + t_k f(x_k)$.

Such a function is called *affine*. Note that if $m = 1$, then f is affine if and only if it is both convex and concave.

The first property clearly entails the third, which then entails the second, so assume that f has the second property. First assume that $f(0) = 0$. For $\beta \in \mathbb{R}$ and $x \in \mathbb{R}^n$ be thus have

$$f(\beta x) = f(\beta x + (1-\beta)0) = \beta f(x) + (1-\beta)f(0) = \beta f(x),$$

so f is homogeneous. If also $y \in \mathbb{R}^n$, then

$$\tfrac{1}{2}f(x+y) = f\left(\tfrac{1}{2}x + \tfrac{1}{2}y\right) = \tfrac{1}{2}f(x) + \tfrac{1}{2}f(y),$$

so f is also additive, hence linear. For general f, simply replace f with $f - f(0)$. ⌟

REMARK 2.2. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be affine, and let $M \subseteq \mathbb{R}^n$ be affine. Then the image $f(M)$ is also affine.   ⌐

REMARK 2.3. If $f : \mathbb{R}^n \to \mathbb{R}^n$ preserves the Euclidean metric, then it is affine. Replacing f with $f - f(\mathbf{0})$ we may assume that $f(\mathbf{0}) = \mathbf{0}$, and so that f also preserves the inner product. For all $x, y, z \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$ we have

$$\langle f(\beta x + y), f(z) \rangle = \langle \beta x + y, z \rangle$$
$$= \beta \langle x, z \rangle + \langle y, z \rangle$$
$$= \beta \langle f(x), f(z) \rangle + \langle f(y), f(z) \rangle$$
$$= \langle \beta f(x) + f(y), f(z) \rangle.$$

Since f is surjective, the claim follows.   ⌐

REMARK 2.4: *Adjacent basic solutions and adjacent bases.*
Let $P = \{ x \in \mathbb{R}^n \mid Ax = b, x \geq 0 \}$ be a polyhedron in standard form.

> *If x and y are adjacent basic solutions, then they can be obtained from two adjacent bases.*

Assume that $x_i = 0$ for $i \neq B(1), \ldots, B(m)$. Each of the m equality constraints are active at both x and y, so there must be exactly one k such that $y_{B(k)} \neq 0$. Then x can be obtained from the basis with basic indices $\{B(1), \ldots, B(m)\}$ and y from a basis with basic indices $\{B(1), \ldots, B(k-1), j, B(k+1), \ldots, B(m)\}$ where $j \neq B(k)$. Hence these two bases are adjacent as claimed.

> *If two adjacent bases lead to distinct basic solutions x and y, then these are adjacent.*

Similarly, there are (at least) $n-1$ constraints that are active at both x and y (the equality constraints along with all but one of the constraints given by the $B(k)$). Since $x \neq y$ there must be some k such that $x_{B(k)} = 0$ and $y_{B(k)} \neq 0$, so there are exactly $n-1$ constraints that are active at both x and y.   ⌐

EXERCISE 2.5

(Extreme points of isomorphic polyhedra) A mapping f is called *affine* if it is of the form $f(x) = Ax + b$, where A is a matrix and b is a vector. Let P and Q be polyhedra in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. We say that P and Q are *isomorphic* if there exist affine mappings $f : P \to Q$ and $g : Q \to P$ such that $g(f(x)) = x$ for all $x \in P$, and $f(g(y)) = y$ for all $y \in Q$. (Intuitively, isomorphic polyhedra have the same shape.)

(a) If P and Q are isomorphic, show that there exists a one-to-one correspondence between their extreme points. In particular, if f and g are

as above, show that x is an extreme point of P if and only if $f(x)$ is an extreme point of Q.

(b) (Introducing slack variables leads to an isomorphic polyhedron) Let $P = \{x \in \mathbb{R}^n \mid Ax \geq b, x \geq 0\}$, where A is a matrix of dimensions $k \times n$. Let $Q = \{(x, z) \in \mathbb{R}^{n+k} \mid Ax - z = b, x \geq 0, x \geq 0\}$. Show that P and Q are isomorphic.

SOLUTION. (a) For $x, y, z \in P$ with $x = \lambda y + (1 - \lambda)z$, then since f is affine we have $f(x) = \lambda f(y) + (1 - \lambda)f(z)$. Hence if x is *not* an extreme point, then neither is $f(x)$. Since f is bijective with an affine inverse, this proves the claim.

(b) Simply note that the map $f : P \to Q$ given by $f(x) = (x, Ax - b)$ is an isomorphism. $\qquad\square$

EXERCISE 2.7

Suppose that $\{x \in \mathbb{R}^n \mid a_i'x \geq b_i, i = 1, \ldots, m\}$ and $\{x \in \mathbb{R}^n \mid g_i'x \geq h_i, i = 1, \ldots, k\}$ are representations of the same nonempty polyhedron P. Suppose that the vectors $a_1, \ldots, a_m$ span $\mathbb{R}^n$. Show that the same must be true for the vectors $g_1, \ldots, g_k$.

SOLUTION. If $a_1, \ldots, a_m$ span $\mathbb{R}^n$, then they must contain a collection of n linearly independent vectors. Then Theorem 2.6 implies that P has an extreme point. But this is a geometric property, so it does not depend on the representation. The same theorem then implies that the collection $g_1, \ldots, g_k$ contains n linearly independent vectors. But then they span $\mathbb{R}^n$. $\qquad\square$

EXERCISE 2.12

Consider a nonempty polyhedron P and suppose that for each variable x_i we have either the constraint $x_i \geq 0$ or the contraint $x_i \geq 0$. Is it true that P has at least one basic feasible solution?

SOLUTION. Let I be the set of indices i such that we have the contraint $x_i \leq 0$, and let A be a diagonal matrix with a 1 as the ith entry on the diagonal if $i \notin I$, and -1 if $i \in I$. The map $x \mapsto Ax$ is then an isomorphism between P and the polyhedron Q which is defined by the same contraints as P, except that the contraints on the form $x_i \leq 0$ are replaced by contraints $x_i \geq 0$. In particular, P and Q have the same extreme points by [TODO ref] Exercise 2.4, hence the same basic feasible solutions by Theorem 2.3.

Next introduce slack variables to Q, yielding a polyhedron R in standard form, which is isomorphic to Q by [TODO ref] Exercise 2.5. But since R is

nonempty (since it is isomorphic to P) it has a basic feasible solution by Corollary 2.2. $\square$

TODO 2.14, 2.15

EXERCISE 2.21

Suppose that Fourier–Motzkin elimination is used in the manner described at the end of Section 2.8 to find the optimal cost in a linear programming problem. Show how this approach can be augmented to obtain an optimal solution as well.

SOLUTION. Let $P \subseteq \mathbb{R}^n$ be a polyhedron, and assume that we wish to minimise the quantity $c'x$ subject to $x \in P$. Define

$$P_n = \{(x_0, x) \in \mathbb{R}^{n+1} \mid x_0 = c'x \text{ and } x \in P\},$$

which is clearly a polyhedron in $\mathbb{R}^{n+1}$. To minimise $c'x$ with $x \in P$ we thus find the minimal value of x_0 among elements $(x_0, x) \in P_n$. Letting $P_{n-k} = \Pi_k(P_n)$ for $k = 1, \dots, n$ we have $Q := P_0 = \Pi_n(P_n) = \{x_0 \in \mathbb{R} \mid \exists x \in P : (x_0, x) \in P_n\}$, so this is equivalent to finding the minimum of Q. Given the minimum of Q we wish to find a (not necessarily unique) $x \in P$ such that $(\min Q, x) \in P_n$.

We do this by recursively finding a suitable value for the variable we eliminate in each step of the algorithm. In the notation of the Elimination algorithm on page 72, let $P \subseteq \mathbb{R}^n$ be a polyhedron, and let $Q = \Pi_{n-1}(P)$ be the polyhedron obtained when eliminating the variable x_n. Let $\bar{x}$ be an element of Q, i.e. a vector satisfying the inequalities

$$d_j + f_j' \bar{x} \geq d_i + f_i' \bar{x}, \qquad \text{if } a_{in} > 0 \text{ and } a_{jn} < 0,$$
$$0 \geq d_k + f_k' \bar{x}, \qquad \text{if } a_{kn} = 0,$$

for $i, j, k = 1, \dots, n$. Now choose any $x_n \in \mathbb{R}$ that lies in all the intervals $[d_i + f_i' \bar{x}, d_j + f_j' \bar{x}]$. Then the vector $(\bar{x}, x_n)$ satisfies the inequalities

$$x_n \geq d_i + f_i' \bar{x}, \qquad \text{if } a_{in} > 0,$$
$$d_j + f_j' \bar{x} \geq x_n, \qquad \text{if } a_{jn} < 0,$$
$$0 \geq d_k + f_k' \bar{x}, \qquad \text{if } a_{kn} = 0.$$

Such an x_n exists since these inequalities are a representation of P, and the former inequalities are a representation of Q, which is the projection of P by Theorem 2.10. Thus $(\bar{x}, x_n) \in P$ as desired. $\square$

EXERCISE 2.22

Let P and Q be polyhedra in $\mathbb{R}^n$.

(a) Show that $P + Q$ is a polyhedron.

(b) Show that every extreme point of $P + Q$ is the sum of an extreme point of P and an extreme point of Q.

SOLUTION. (a) The Cartesian product $P \times Q$ is clearly a polyhedron. Its image under the linear map $(x, y) \mapsto x + y$ is exactly $P + Q$, and this is a polyhedron by Corollary 2.5.

(b) Let $z \in P + Q$ and write $z = x + y$ for $x \in P$ and $y \in Q$. Assume that x is not an extreme point in P, so that there exist $x_1, x_2 \in P \setminus \{x\}$ and a $\lambda \in [0, 1]$ such that $x = \lambda x_1 + (1 - \lambda)x_2$. Letting $z_i := x_i + y$ we easily find that $z = \lambda z_1 + (1 - \lambda)z_2$, so z is not an extreme point. $\square$