Category theory notes

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1 • Basic definitions

DEFINITION 1.1: *Categories*

A *category* $\mathcal{C}$ consists of a collection of *objects* and a collection of *arrow* satisfying the following axioms:

(i) To each arrow f are associated unique objects called the *source* and *target* of f. We write $f: A \to B$ to notate that f is an arrow with source A and target B.

(ii) For two arrows $f: A \to B$ and $g: B \to C$ there exists a *composite* arrow $g \circ f: A \to C$.

(iii) For each object there is an arrow $1_A: A \to A$ called the *identity arrow* on A.

(iv) Composition is associative, and identity arrows behave as identities with respect to composition.

We note that identity arrows on a given object are unique, and the identity arrows on different objects are distinct.

For two objects A, B in some category $\mathcal{C}$, we write $\mathcal{C}(A, B)$ for the collection of arrows $A \to B$ in $\mathcal{C}$.

EXAMPLE 1.2.

(i) There is a category **Mon** whose objects are monoids and arrows are monoid homomorphisms.

(ii) Similarly there is a category **Ord** whose objects are preordered collections (not necessarily sets, I guess) and whose arrows are monotone maps.

(iii) There are also categories of groups, abelian groups, rings, commutative rings, posets, totally ordered sets, topological spaces, etc. ⌟

Example 1.3.

(i) Every monoid can be considered a category: This category has a single (arbitrary) object, and its arrows are the elements of the monoid.

(ii) A preordered collected $(M, \leq)$ can also be considered a category. The objects of the category are the elements of M, and there is a single arrow $A \to B$ if and only $A \leq B$. Composition is well-defined by transitivity. Notice that there is at most one arrow between two objects, so any two such arrows must be equal. ⌟

Definition 1.4: *Opposite category*

Given a category $\mathcal{C}$, the *opposite* or *dual* category $\mathcal{C}^{\mathrm{op}}$ is the category such that:

(i) The objects of $\mathcal{C}^{\mathrm{op}}$ are just the objects of $\mathcal{C}$.

(ii) The arrows are given by $\mathcal{C}^{\mathrm{op}}(A,B) = \mathcal{C}(B,A)$.

(iii) The identity arrows are the same.

(iv) Composition is defined by $f \circ^{\mathrm{op}} g = g \circ f$.

It is clear that $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}$, so every category is the opposite of some category.

2 • Construction of other categories

Definition 2.1: *Subcategories*

Given a category $\mathcal{C}$, a *subcategory* $\mathcal{S}$ of $\mathcal{C}$ consists of some of the objects a...
Full subcategory

Definition 2.2: *Product categories*

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Definition 2.3: *Quotient categories*

Let $\mathcal{C}$ be a category, and let $\sim$ be a *congruence* on the arrows of $\mathcal{C}$, i.e., let $\sim$ be an equivalence relation on arrows such that:

(i) If $f \sim g$ then f and g have the same source and target.

(ii) If $f \sim g$ then $f \circ h \sim f \circ g$ and $k \circ f \sim k \circ g$.

Then $\mathcal{C}/{\sim}$ is the category whose objects are the same as the objects in $\mathcal{C}$, and whose arrows are $\sim$-equivalence classes.

EXAMPLE 2.4. Let $\sim$ be the congruence on the arrows of **Top** that holds between two arrows if they are homotopic. Then **Top**$/{\sim}$ is the homotopy category **hTop**. ⌟

DEFINITION 2.5: *Arrow category*

Given a category $\mathcal{C}$, the *arrow category* $\mathcal{C}^{\rightarrow}$ is the category with the following data:

(i) The objects of $\mathcal{C}^{\rightarrow}$ are the arrows of $\mathcal{C}$.

(ii) Given objects $f_1\colon X_1 \to Y_1$ and $f_2\colon X_2 \to Y_2$ in $\mathcal{C}^{\rightarrow}$, an arrow is a pair (j,k) of $\mathcal{C}$-arrows such that the following diagram commutes:

$$\begin{tikzcd} X_1 \arrow[r, "j"] \arrow[d, "f_1"'] & X_2 \arrow[d, "f_2"] \\ Y_1 \arrow[r, "k"] & Y_2 \end{tikzcd}$$

The identity arrows and composition are defined in the obvious way.

DEFINITION 2.6: *Slice category*

Given a category $\mathcal{C}$ and an object I of $\mathcal{C}$, the *slice category* over I is the category $\mathcal{C}/I$ whose objects are $\mathcal{C}$-arrows $f: A \to I$ and whose arrows are commuting triangles in $\mathcal{C}$.

The *co-slice category* $I/\mathcal{C}$ (or the slice category *under* I) is defined dually.

EXAMPLE 2.7.

(i) Pick a one-element set 1 and consider the co-slice category 1/**Set**. This is (in a way we make precise later) essentially the same as the category of pointed sets $\mathbf{Set}_*$. (Reference?)

(ii) Given a (nonempty) set I we can think of the slice category **Set**$/I$ as a category of indexed sets, where the indexes are given by I. ⌟

3 • Types of arrows

Definition 3.1: *Monomorphisms and epimorphisms*

An arrow f in a category $\mathcal{C}$ is a *monomorphism* if

$$f \circ g = f \circ h \quad \Rightarrow \quad g = h$$

for all appropriate arrows g, h. Similarly, f is an *epimorphism* if

$$g \circ f = h \circ f \quad \Rightarrow \quad g = h.$$

Example 3.2.

(i) In **Set**, the monomorphisms are precisely the injective functions, and the epimorphisms are the surjective functions.

(ii) In **Grp**, the monomorphisms are similarly the injective homomorphisms (equivalently the homomorphisms with trivial kernel). Surjective homomorphisms are obviously epimorphisms, but the converse is also true.

In Example 6.2(iv) we show that every subgroup of a group is an equaliser subgroup, and the above is an easy corollary: Let $f: G \to H$ be an epimorphism in **Grp** (or in **FinGrp**, the argument is the same), and put $A = f(G)$. Since A is a subgroup of H, there are group homomorphisms $g_1, g_2 : H \to K$ (where K is finite if we are working in **FinGrp**) such that A is the equaliser subgroup of g_1 and g_2. But then $g_1 \circ f = g_2 \circ f$, and since f is epic we have $g_1 = g_2$, so the two homomorphisms in fact agree on H. Hence $A = H$ and f is surjective.

(iii) An arrow that is both a monomorphism and an epimorphism need not be an isomorphism. In **Mon**, the inclusion arrow $\mathbb{N} \to \mathbb{Z}$ is both monic and epic but is clearly not an isomorphism. ⌟

4 • Initial and terminal objects

Definition 4.1: *Initial and terminal objects*

An object I of a category $\mathcal{C}$ is *initial* if for every object X there is a unique arrow $I \to X$. If an initial object exist, we often denote it 0.

Dually, an object T is *terminal* if for every object X there is a unique arrow $X \to T$. If a terminal object exist, we often denote it 1.

An object that is both initial and terminal is called a *null object*.

Example 4.2.

(i) A poset $(S, \leq)$ treated as a category has an initial (terminal) object iff it has a minimum (maximum).

(ii) In **Set** the only initial object is the empty set, and every singleton is terminal. Similarly in **Poset** and **Top**.

(iii) In the category $\mathbf{Set}_*$ of pointed sets every singleton is both initial and terminal. Similarly in $\mathbf{Top}_*$ and in **Grp**.

(iv) In the category **Ring** of rings, $\mathbb{Z}$ is initial and the zero ring (where $0 = 1$) is final.

(v) In a slice category $\mathcal{C}/X$ the identity arrow 1_X is terminal. (What about initials?) ⌟

DEFINITION 4.3: *Elements of an object*

In a category $\mathcal{C}$ with a terminal object 1, an *element*, *global element* or *point* of an object X is an arrow $f : 1 \to X$.

In any category, a *generalised element* (of shape S) of X is an arrow $e : S \to X$.

Note that parallel arrows are equal iff they act identically on all generalised elements. In some categories we don't need to look at all generalised elements to check if two arrows are the same; it is enough to only consider point elements:

DEFINITION 4.4: *Well-pointed category*

Let $\mathcal{C}$ be a category with a terminal object 1. Suppose for any objects X, Y and parallel arrows $f, g : X \to Y$ we have that $f = g$ if $f \circ \boldsymbol{x} = g \circ \boldsymbol{x}$ for all $\boldsymbol{x} : 1 \to X$. Then $\mathcal{C}$ is called *well-pointed*.

Note that the choice of terminal object does not matter.

EXAMPLE 4.5. The category **Set** of sets is well-pointed, but the category **Grp** of groups is not, since any homomorphism must send the single element of 1 to the identity, even if $f \neq g$. ⌟

5 • Products and coproducts

DEFINITION 5.1: *Products*

In any category $\mathcal{C}$ a *product* of a class of objects X_j indexed by a collection J of indices is an object P along with projection arrows $\pi_j : P \to X_j$, such that for any object Z and arrows $f_j : Z \to X_j$ there is a unique arrow $u : Z \to P$ such that the diagram

$$\begin{tikzcd}
& Z \arrow[dl, "f_j"'] \arrow[d, dashed, "u"] \\
X_j & P \arrow[l, "\pi_j"]
\end{tikzcd}$$

commutes for all $j \in J$. The product object P is denoted $\times_{j \in J} X_j$.

If the collection of objects consists of two objects X and Y, we denote their product $X \times Y$ and the mediating arrow u is denoted $\langle f_1, f_2 \rangle$.

Products may also be defined as follows. We consider only binary products for simplicity. Let X and Y be objects in a category $\mathcal{C}$, and define the *wedge category* $\mathcal{C}_{W(XY)}$ to be the category whose objects are wedges

$$\begin{tikzcd}
& X \\
Z \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & \\
& Y
\end{tikzcd}$$

and composition of wedges is given by a commutative diagram:

$$\begin{tikzcd}
& & X \\
Z \arrow[r, "h"] \arrow[urr, bend left, "f_1"] \arrow[drr, bend right, "f_2"'] & W \arrow[ur, "g_1"] \arrow[dr, "g_2"'] & \\
& & Y
\end{tikzcd}$$

A binary product of X and Y in $\mathcal{C}$ is then a terminal object in $\mathcal{C}_{W(XY)}$. It follows that binary products are unique up to unique isomorphism in the wedge category, so P is unique up to unique isomorphism if we consider those isomorphisms that commute with the projections.

Given two arrows $f \colon X \to X'$ and $g \colon Y \to Y'$, we define the arrow $f \times g \colon X \times Y \to X' \times Y'$ (assuming the products exist) to be the unique arrow such that the diagram

$$\begin{tikzcd}
X \arrow[d, "f"'] & X \times Y \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "f \times g"] & Y \arrow[d, "g"] \\
X' & X' \times Y' \arrow[l, "\pi_1'"'] \arrow[r, "\pi_2'"] & Y'
\end{tikzcd}$$

commutes, i.e. $f \times g = \langle f \circ \pi_1, g \circ \pi_2 \rangle$.

DEFINITION 5.2: *Small products*

A category $\mathcal{C}$ *has all small products* if for any $\mathcal{C}$-objects X_j for $j \in J$, where J is some index set, these objects have a product.

REMARK 5.3. To see why this definition of small products is significant, consider the following example: Let $\mathcal{C}$ be a category with all products, and let J be the class (which might be a set) of arrows in $\mathcal{C}$. If $\mathcal{C}$ has two distinct parallel arrows $f, g \colon X \to Y$, then there are at least $2^{|J|}$ arrows $X \to \times_{j\in J} Y$, which is a contradiction.

In a category like **Set** whose objects form a proper class, this definition is thus useful. ⌟

EXAMPLE 5.4.

(i) In **Set**, or in categories of structured sets like **Grp** and **Top**, products are the usual product constructions: Cartesian products, direct products and product spaces.

(It seems like the existence of all small products in **Set**, and categories of structured sets, relies on the axiom of choice?)

(ii) Given a first-order language $\mathcal{L}$, consider the category $\mathbf{Prop}_{\mathcal{L}}$ whose objects are wffs of $\mathcal{L}$, and there is a unique arrow from X to Y if X (semantically) entails Y. Then the product of X and Y is the logical product, i.e. the conjunction $X \wedge Y$, and the projections are obvious by uniqueness of arrows.

(iii) In a poset $(P, \leq)$ considered as a category, the product of two elements $p, q \in P$ is their meet or infimum $p \wedge q$. Hence a category need not in general have products, since two elements in a poset need not have a meet.

(iv) The category **Meas** of measurable spaces has all small products.

Let $\{(X_\alpha, \mathcal{E}_\alpha) \mid \alpha \in A\}$ be a class of measurable spaces indexed by a set A, and equip $X = \times_{\alpha\in A} X_\alpha$ with the product σ-algebra $\mathcal{E} = \bigotimes_{\alpha\in A} \mathcal{E}_\alpha$. We denote the coordinate projections $\pi_\alpha \colon X \to X_\alpha$.

Let $(Y, \mathcal{F})$ be a measurable space, and consider for each $\alpha \in A$ a measurable function $f_\alpha \colon Y \to X_\alpha$. Since X is a product in **Set** there is a unique mediating arrow $f \colon Y \to X$, and we only need to show that f is measurable. Considering a generator $\pi_\alpha^{-1}(B)$ of $\mathcal{E}$ for some $B \in \mathcal{E}_\alpha$ we then have

$$f^{-1}(\pi_\alpha^{-1}(B)) = (\pi_\alpha \circ f)^{-1}(B) = f_\alpha^{-1}(B) \in \mathcal{F},$$

since f_α is measurable. Thus f is measurable. ⌟

EXAMPLE 5.5. Coproducts are defined dually to products. The coproduct of two objects X and Y is denoted $X \sqcup Y$.

(i) In **Set**, coproducts are disjoint unions $X \sqcup Y$.

(ii) In **Grp**, coproducts are free products $G * H$. But in the category **Ab** of abelian groups, coproducts are direct products, so products and coproducts coincide (or at least the objects do, they are of course equipped with projections or injections as well). When considered as a coproduct, the direct product of abelian groups is often called the *direct sum* and is denoted $G \oplus H$.

(iii) In $\mathbf{Prop}_{\mathcal{L}}$, coproducts are disjunctions $X \vee Y$.

(iv) In a poset $(P, \leq)$, coproducts are joins or suprema, $p \vee q$. This again shows that not all categories have all coproducts. ⌟

REMARK 5.6. By the following exercise, in a category with a terminal object 1 we have $1 \times X \cong X$ for any object X. If a category has an initial object 0, we do not necessarily have $0 \times X \cong 0$. For instance, in a category with a null object $0 = 1$ we would then have for all objects X,

$$X \cong 1 \times X \cong 0 \times X \cong 0.$$

In **Grp** we do have a null object, but not all groups are trivial. ⌟

6 • Equalisers and co-equalisers

In a category $\mathcal{C}$, a *fork* is a commuting diagram

$$S \xrightarrow{k} X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

i.e. $f \circ k = g \circ k$.

DEFINITION 6.1: *Equalisers*

Let $\mathcal{C}$ be a category and let $f, g \colon X \to Y$ be parallel arrows. The object E and the arrow $e \colon E \to X$ is an *equaliser* for f and g if and only if $f \circ e = g \circ e$ (i.e. they form a fork), and for any fork

$$S \xrightarrow{k} X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

there is a unique mediating arrow $u \colon S \to E$ such that the following diagram commutes:

$$\begin{tikzcd} S \arrow[d, dashed, "u"'] \arrow[dr, "k"] & & \\ E \arrow[r, "e"'] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

As with products, we can define a category $\mathcal{C}_{F(fg)}$ of forks such that equalisers are terminal objects.

Example 6.2.

(i) In **Set**, let $f, g\colon X \to Y$ be a pair of parallel arrows. Let $E = \{x \in X \mid f(x) = g(x)\}$ be the subset of X on which f and g agree, and let $e\colon E \to X$ be inclusion. Then $[E, e]$ is an equaliser of f and g.

Conversely, if $S \subseteq X$ is an arbitrary subset, this can also be characterised as an equaliser of two arrows in **Set**: Simply take the indicator function of S and a function that is 1 everywhere.

(ii) In categories of structured sets, like **Mon** and **Top**, there are also equalisers, and they are given similarly as above. In the first case the above subset E of a monoid is also a monoid, and in the second we give E the subspace topology.

(iii) Consider the category **Grp**. Let $f\colon X \to Y$ be a group homomorphism, and let $o\colon X \to Y$ send everything to the identity of Y (i.e. o is the unique composite $X \to 1 \to Y$). Now let $K = \ker f$ and let $i\colon K \to X$ be inclusion. Then $f \circ i = o \circ i$, so we have a fork. Given any other fork

$$\begin{tikzcd} S \arrow[r, "k"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "o"'] & Y \end{tikzcd}$$

we have $f \circ k = o \circ k$, so $k(S) \subseteq K$. Then let $k'\colon S \to K$ agree with k on S. Then k' makes the diagram

$$\begin{tikzcd} S \arrow[d, "k'"'] \arrow[dr, "k"] & & \\ K \arrow[r, "i"'] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "o"'] & Y \end{tikzcd}$$

commute, and k' is unique with this property. Thus K with inclusion is the equaliser of f and o, and the kernel of a homomorphism can be described categorically in terms of equalisers.

This shows that all normal subgroups are equaliser subgroups; hence *all* subgroups in **Ab** are also equaliser subgroups.

More generally, let $\mathcal{C}$ be a category with a null object 1, and let $o\colon X \to Y$ be the unique composite $X \to 1 \to Y$ as above. The kernel of an arrow $f\colon X \to Y$ is then (by definition) the equaliser of f and o, if it exists. (The existence of a null object is not strictly necessary.)

(iv) In fact, all subgroups in **Grp** are equaliser subgroups. To show this, let H be a subgroup of G, and let K be the permutation group of the set $G/H \cup \{\hat{H}\}$, where $\hat{H}$ is some element not in G/H. Let $\rho \in K$ be the permutation that exchanges eH and $\hat{H}$ and leaves the rest of the set fixed. Then define maps $f_1, f_2\colon G \to H$ by

$$f_1(g)(S) = \begin{cases} gg'H, & S = g'H, \\ \hat{H}, & S = \hat{H}, \end{cases}$$

for $g \in G$ and $S \in G/H \cup \{\hat{H}\}$, and $f_2(g) = \rho \circ f_1(g) \circ \rho^{-1}$. Easy (but a bit tedious) calculations show that f_1 and f_2 are in fact homomorphisms, and that H is the equaliser subgroup of f_1 and f_2.

Notice that if G is finite then K is also finite, so this argument shows that the category **FinGrp** of finite groups also has this property.

(v) The category $\mathbf{Top}_*$ has null objects (one-point spaces), so it also has kernels. The kernel of a map $f\colon (X, x_0) \to (Y, y_0)$ is $(f^{-1}(y_0), x_0)$. As far as I know this has no applications. ⌟

Smith doesn't talk much about co-equalisers, so I'll skip them for now.

7 • Limits and colimits

DEFINITION 7.1: *Diagrams*

Given categories $\mathbf{J}$ and $\mathcal{C}$, we say that a functor $D\colon \mathbf{J} \to \mathcal{C}$ is a *diagram (of shape* $\mathbf{J}$*)* in $\mathcal{C}$.

REMARK 7.2. We use the convention ⌟

DEFINITION 7.3: *Cones*

Let $D\colon \mathbf{J} \to \mathcal{C}$ be a diagram. A *cone over* D is an object $C \in \mathcal{C}$ (the *vertex* or *apex* of the cone) together with an arrow $c_J\colon C \to D(J)$ for each $\mathbf{J}$-object J, such that for any $\mathbf{J}$-arrow $d\colon K \to L$, $c_L = D(d) \circ c_K$, i.e. the following diagram commutes:

$$\begin{tikzcd} & C \arrow[dl, "c_K"'] \arrow[dr, "c_L"] & \\ D(K) \arrow[rr, "D(d)"'] & & D(L) \end{tikzcd}$$

We write $[C, c_J]$ for such a cone.

The *category of cones over D*, written $\mathcal{C}_{C(D)}$, has the following data:

(i) Its objects are the cones $[C, c_J]$ over D.

(ii) An arrow from $[C, c_J]$ to $[C', c'_J]$ is a $\mathcal{C}$-arrow $f\colon C \to C'$ such that $c'_J \circ f = c_J$ for all $J \in \mathbf{J}$. That is, the triangles

$$\begin{tikzcd} & C \arrow[dl, "c_J"'] \arrow[d, "f"] \\ D(J) & C' \arrow[l, "c'_J"] \end{tikzcd}$$

all commute.

Composition in $\mathcal{C}_{C(D)}$ is given by composition in $\mathcal{C}$.

DEFINITION 7.4: *Limit cones and limits*

Given a diagram $D\colon \mathbf{J} \to \mathcal{C}$, a *limit cone for D* in $\mathcal{C}$ is a terminal object in $\mathcal{C}_{C(D)}$. We denote the limit object at the vertex of the limit cone by

$$\varprojlim_{\mathbf{J}} D.$$

We note that limit cones in $\mathcal{C}_{C(D)}$ are unique up to unique isomorphism, and since isomorphisms in $\mathcal{C}_{C(D)}$ are just isomorphism in $\mathcal{C}$ that commute with the cone's arrows, two limit objects are also isomorphic. This isomorphism is not unique in $\mathcal{C}$ without the requirement that it commutes with the cone's arrows: For instance, it is easy to find non-trivial automorphisms on $X \times Y$ for sets X and Y, for instance non-trivial permutations of elements in either set, but these do not commute with the canonical projections.

EXAMPLE 7.5.

(i) Take the empty diagram in a category $\mathcal{C}$: A cone over this is a single object without any other conditions, so an arrow between cones is just an arrow $C \to C'$ in $\mathcal{C}$. Thus the limit object is a terminal object if it exists.

(ii) Consider a diagram with two objects D_1 and D_2 but no mediating arrows. A cone over this diagram is a wedge, so limit objects are products $D_1 \times D_2$.

(iii) Now consider a diagram with two objects and two parallel arrows between them:

$$\begin{tikzcd} D_1 \arrow[r, shift left, "d"] \arrow[r, shift right, "d'"'] & D_2 \end{tikzcd}$$

A cone over this diagram is a commutative diagram:

$$\begin{tikzcd} & C \arrow[dl, "c_1"'] \arrow[dr, "c_2"] & \\ D_1 \arrow[rr, shift left, "d"] \arrow[rr, shift right, "d'"'] & & D_2 \end{tikzcd}$$

The commutativity of such a diagram is equivalent to the condition $d \circ c_1 = d' \circ c_1$ since this lets us define c_2 as above. So this diagram is equivalent to the fork

$$\begin{tikzcd} C \arrow[r, "c_1"] & D_1 \arrow[r, shift left, "d"] \arrow[r, shift right, "d'"'] & D_2 \end{tikzcd}$$

so the limit is an equaliser of d and d'. ⌟

8 • Pullbacks and pushouts

DEFINITION 8.1: *Pullbacks*

A limit for a corner diagram

$$\begin{tikzcd} & D_2 \arrow[d, "e"] \\ D_1 \arrow[r, "d"] & D_3 \end{tikzcd}$$

is a *pullback*. A pullback is also called a *fibered product*.

EXAMPLE 8.2.

(i) In **Set**, consider the corner:

$$\begin{tikzcd} & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

The limit object must be reminiscent of a product, and to get the rest of the diagram to commute, we must have something like $L = \{(x, y) \in X \times Y \mid f(x) = g(y)\}$.

In particular, if $X, Y \subseteq Z$ and f and g are inclusions, we get the pullback square

$$\begin{tikzcd} L \arrow[r] \arrow[d] & Y \arrow[d, "i_2"] \\ X \arrow[r, "i_1"] & Z \end{tikzcd}$$

and $L = X \cap Y$ (up to isomorphism).

A pullback object looks like an equaliser of a product!

(ii) In **Set** we also have the pullback square:

$$\begin{tikzcd} f^{-1}(Z) \arrow[r] \arrow[d] & Z \arrow[d, "1_Z"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

So preimages are also pullback objects, and we get the preimage $f^{-1}(Z)$ by pulling Z back along f. Or more precisely, we get the arrow $f^{-1}(Z) \to X$ by pulling the identity 1_Z back along f. ⌟

9 • Subobjects

There are (at least) two different notions of subobject. The first one is:

DEFINITION 9.1: *Subobjects, first definition*

A *subobject-1* of an object X in a category $\mathcal{C}$ is a monomorphism $S \rightarrowtail X$.

If $f : A \rightarrowtail X$ and $g : B \rightarrowtail X$ are subobjects-1 of X, then f *is included in* g, written $f \subseteq g$, if f factors through g, i.e. if there is an arrow $h : A \to B$ such that $f = g \circ h$.

REMARK 9.2. It might seem more natural to assume that h is also a monomorphism, but **??** shows that h is automatically monic since f is monic. ⌟

EXAMPLE 9.3.

(i) In **Set**, a monomorphism is just an injective function, so a monomorphism $f : S \rightarrowtail X$ sets up an isomorphism $S \cong f(S) \subseteq X$. So subobjects-1 in **Set** in some sense correspond to subsets.

However, take for instance a singleton $\{1\}$. This has infinitely many subobjects since there are infinitely many singletons in **Set** (indeed there are too many to form a set).

(ii) For two subobjects-1 $f: A \rightarrowtail X$ and $g: B \rightarrowtail X$, if $f \subseteq g$ and $g \subseteq f$ then it follows from **??** that $A \cong B$. But it is easy to find examples, for instance in **Set**, of this occuring even when $f \neq g$. In other words, the subobjects-1 of X ordered by inclusion need not form a poset. ⌟

DEFINITION 9.4: *Subobjects, second definition*

A *subobject-2* of an object X in a category $\mathcal{C}$ is a class of subobjects-1 that factor through each other.

If $[\![f]\!]$ and $[\![g]\!]$ are subobjects-2 of X, then $[\![f]\!]$ *is included in* $[\![g]\!]$, written $[\![f]\!] \Subset [\![g]\!]$, if $f \subseteq g$.

This definition of $\Subset$ is clearly independent of the choice of representatives and makes the class of subobjects-2 of X into a poset.

10 • Exponentials

DEFINITION 10.1: *Exponentials*

Let $\mathcal{C}$ be a category with binary products. An *exponential* of an object C by an object B is given by the pair[1] $[C^B, ev]$, where C^B is an object and $ev: C^B \times B \to C$ an arrow, that satisfies the following property: For every object A and arrow $g: A \times B \to C$ in $\mathcal{C}$ there is a unique arrow $\overline{g}: A \to C^B$ such that the diagram

$$\begin{tikzcd}
A \times B \arrow[d, dashed, "\overline{g}\times 1_B"'] \arrow[dr, "g"] & \\
C^B \times B \arrow[r, "ev"'] & C
\end{tikzcd}$$

commutes. The arrow $\overline{g}$ is called the *(exponential) transpose* of g.

As with limits we can define a category in which the exponential is a terminal object. Given objects B and C from a category $\mathcal{C}$, define a category $\mathcal{C}_{E(B,C)}$ with the following data:

(1) Objects are pairs $[A, g]$, where A is an object $g: A \times B \to C$ an arrow in $\mathcal{C}$.

[1] The square brackets indicate that we are not necessarily thinking of the pair as more than a virtual pair; it is just convenient notation.

(2) An arrow from $[A, g]$ to $[A', g']$ is a $\mathcal{C}$-arrow $h\colon A \to A'$ such that the diagram

$$\begin{tikzcd} A \times B \arrow[d, dashed, "h\times 1_B"'] \arrow[dr, "g"] & \\ A' \times B \arrow[r, "g'"'] & C \end{tikzcd}$$

commutes.

Identity arrows and composition are inherited from $\mathcal{C}$. The exponential $[C^B, ev]$ is then terminal in this category, and so it is unique up to isomorphism compatible with the evaluation arrows.

Note that an exponential $[C^B, ev]$ in a category $\mathcal{C}$ is a limit in the derived category $\mathcal{C}_{E(B,C)}$ but *not* in $\mathcal{C}$, since $[C^B, ev]$ is not even an object in $\mathcal{C}$.

Example 10.2.

(i) In **Set**, the exponential of C by B is the set C^B of functions $B \to C$. The category **FinSet** of finite sets has the same exponentials, since C^B is finite if B and C are.

(ii) However, the category **Count** of *countable* sets does not have all exponentials. The intuition is of course that the exponential C^B in **Set** might not be countable even if B and C are countable.

To prove this we use the fact that $\mathbf{Count}(A \times B, C)$ by **??** is in one-to-one correspondence with $\mathbf{Count}(A, C^B)$ if C^B exists. Assume that C has at least two elements, let B be countably infinite, and assume (for convenience) that A is a singleton. Then $\mathbf{Count}(A \times B, C)$ is in bijection with the (uncountable) set of maps $B \to C$. But $\mathbf{Count}(A, C^B)$ is in bijection with C^B, so C^B must also be uncountable.

(iii) In $\mathbf{Prop}_{\mathcal{L}}$, the implication $B \to C$ provides an exponential object C^B, and the evaluation map $ev\colon C^B \times B \to C$ reflecting the modus ponens rule, $B \to C, B \vDash C$.

Since $A \wedge B \vDash C$ implies $A \vDash B \to C$, there is a unique arrow from A to $B \to C$. And since trivially $B \vDash B$, we have $A \wedge B \vDash (B \to C) \wedge B$. We therefore get the commutative diagram

$$\begin{tikzcd} A \wedge B \arrow[d, dashed] \arrow[dr] & \\ (B \to C) \wedge B \arrow[r] & C \end{tikzcd}$$

where the dashed arrow is the product of the arrows corresponding to the two implications above. ⌟

Definition 10.3: *Cartesian closed categories*

A category $\mathcal{C}$ is *Cartesian closed* if it has all finite products and all exponentials.

Example 10.4.

(i) The categories **Set** and $\mathbf{Prop}_{\mathcal{L}}$ are Cartesian closed.

(ii) But **Grp** is not Cartesian closed: Recall that **Grp** has a null object, so if it were Cartesian closed, **??** would imply that all groups were isomorphic, which they are not. ⌟

Theorem 10.5

If $\mathcal{C}$ is a Cartesian closed category, then for $A, B, C \in \mathcal{C}$ we have:

(i) *If $B \cong C$, then $A^B \cong A^C$.*

(ii) $(A^B)^C \cong A^{B \times C}$.

(iii) $(A \times B)^C \cong A^C \times B^C$.

Proof. All three claims have elementary proofs, but we give neater proofs once we have developed enough machinery. (Reference later.) ☐

11 • Functors

Definition 11.1: *Functors*

Given categories $\mathcal{C}$ and $\mathcal{D}$, a *(covariant) functor* $F \colon \mathcal{C} \to \mathcal{D}$ consists of the following data:

(i) A map of object whose value at the $\mathcal{C}$-object A is a $\mathcal{D}$-object $F(A)$.

(ii) A map of arrows whose value at the $\mathcal{C}$-arrow $f \colon A \to B$ is a $\mathcal{D}$-arrow $F(f) \colon F(A) \to F(B)$.

Furthermore, F must preserve identities and respect composition of arrows.

A functor $F \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ is called a *contravariant functor* from $\mathcal{C}$ to $\mathcal{D}$.

Remark 11.2. Note that functors compose in the obvious way, and we thus get the notion of a category of categories. While a category of all categories may be problematic, we can without issue consider the category **Cat** of all *small* categories and the category $\mathbf{Cat}^*$ of all *locally small* categories.

But even if the categories we are considering are not locally small, we might still be able to form a category containing them if there are not too many of them. Let us denote such a category by **Cat**.

Thus, two categories are isomorphic if they are isomorphic as elements in some category **Cat**. ⌟

Example 11.3: Covariant functors.

(i) There are many examples of so-called *forgetful* functors that 'forget' (some of) the structure of a category. For instance, there is a forgetful functor **Mon** $\to$ **Set** that sends a monoid to its underlying set.

Some functors only forget some of the structure, like the functor **Ring** $\to$ **Grp** that sends a ring to its underlying additive group, or the functor **Ab** $\to$ **Grp** that 'forgets' that a group is abelian.

(ii) The powerset functor $P\colon \mathbf{Set} \to \mathbf{Set}$ maps a set to its powerset, and maps set functions $f\colon X \to Y$ to the map $U \mapsto f(U)$ for $U \subseteq X$.

(iii) Any homomorphism between monoids or groups is exactly a functor between the corresponding monoids or groups considered as one-object categories.

Similarly, a functor between posets considered as categories is just a monotone map. ⌟

Example 11.4: Contravariant functors.

(i) There is a contravariant version $\overline{P}\colon \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ of the powerset functor that sends a function $f\colon X \to Y$ to the map $V \mapsto f^{-1}(V)$ for $V \subseteq Y$.

(ii) Given a field k, we denote the category of vector spaces over k by $\mathbf{Vect}_k$. Let $\mathbf{Hom}(V,W)$ be the vector space of linear maps $V \to W$. Fix a vector space W. We define a functor

$$\mathbf{Hom}(-,W)\colon \mathbf{Vect}_k^{\mathrm{op}} \to \mathbf{Vect}_k$$

that sends a vector space V to $\mathbf{Hom}(V,W)$, and that sends a linear map $f\colon V \to V'$ to the map $f^*\colon \mathbf{Hom}(V',W) \to \mathbf{Hom}(V,W)$ given by $f^*(g) = g \circ f$.

If $W = k$, then this is the dualising functor $(-)^* = \mathbf{Hom}(-,k)$ that sends a vector space to its dual. ⌟

Definition 11.5

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor, and let P be some property of arrows. Then:

(i) F *preserves* P if, for every $\mathcal{C}$-arrow f, $F(f)$ has the property P if f does.

(ii) F *reflects* P if, for every $\mathcal{C}$-arrow f, if $F(f)$ has the property P then so does f.

A functor is *conservative* if it reflects all isomorphisms.

DEFINITION 11.6: *Faithfulness and fulness*

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is *faithful* if for any pair of $\mathcal{C}$-objects C, C' and arrows $f, g\colon C \to C'$, if $F(f) = F(g)$, then $f = g$.

F is *full* if for any pair of $\mathcal{C}$-objects C, C' and arrow $g\colon F(C) \to F(C')$, there is an arrow $f\colon C \to C'$ such that $F(f) = g$.

If F is both faithful and full, we say that it is *fully faithful*.

Notions of injectivity and surjectivity on objects are less interesting, since we are usually not interested in objects except up to isomorphism. More useful is the following:

DEFINITION 11.7: *Essentially surjective*

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is *essentially surjective on objects* if for any $\mathcal{D}$-object D there is a $\mathcal{C}$-object C such that $F(C) \cong D$.

EXAMPLE 11.8.

(i) The forgetful functor $F\colon \mathbf{Mon} \to \mathbf{Set}$ is faithful but not full, since there are many set functions that do not correspond to monoid homomorphisms. (Smith claims it is essentially surjective on objects, but what about the empty set?)

(ii) The forgetful functor $F\colon \mathbf{Ab} \to \mathbf{Set}$ is fully faithful but not essentially surjective on objects.

(iii) Let $\mathcal{M}$ and $\mathcal{N}$ be the categories corresponding to monoids M and N. Let $f\colon M \to N$ be a monoid homomorphism, and let $F\colon \mathcal{M} \to \mathcal{N}$ be the corresponding functor. Then f is injective (surjective) iff F is faithful (full).

(iv) Consider the 'total collapse' functor $\Delta_0\colon \mathbf{Set} \to \mathbf{1}$ that sends every object to the sole object of $\mathbf{1}$ and all arrows to the identity. Then this is *not* full: Take $C \neq \emptyset$ and $C' = \emptyset$, so there is no arrow $C \to C'$.

(v) An inclusion functor $F\colon \mathcal{S} \to \mathcal{C}$ is faithful. If $\mathcal{S}$ is a full subcategory of $\mathcal{C}$, then it is also full. ⌟

12 • Functors and limits