# Davey & Priesley, *Introduction to Lattices and Order*

Danny Nygård Hansen

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## 1 • Ordered sets

Remark 1.1: Duality in ordered sets.

We elaborate on the duality principle. Consider the elementary language $\mathcal{L}_{\mathbf{Pos}}$ of ordered sets. This is a (single sorted) first-order language with identity, variable symbols $x, y, z, \ldots$ and a single binary relation symbol '$\leq$'. If $\varphi$ is a wff of $\mathcal{L}_{\mathbf{Pos}}$, then its *dual* $\varphi^{\partial}$ is the wff obtained by reversing the order of all inequalities, so that '$X \leq Y$' becomes '$Y \leq X$'.

Recall that a *sentence* in a first-order language is a wff with no free variables. The duality principle then says the following:

> *Let $\varphi$ be a $\mathcal{L}_{\mathbf{Pos}}$-sentence. If the order axioms entail $\varphi$, then they also entail the dual claim $\varphi^{\partial}$.*

In other words, if $\varphi$ holds in all ordered sets, then so does $\varphi^{\partial}$. We give both a syntactic and a semantic proof of this claim. (Of course either is sufficient since the theory of ordered sets is a first-order theory.)

First notice that the axioms of ordered sets (reflexivity, antisymmetry and transitivity) are self-dual: That is, each is their own dual. For instance, the transitive axiom states that

$$\forall x \forall y \forall z (x \leq y \wedge y \leq z \Rightarrow x \leq z),$$

and this is obviously equivalent to its dual:

$$\forall x \forall y \forall z (y \leq x \wedge z \leq y \Rightarrow z \leq x).$$

Now say that there is a (first-order) proof of $\varphi$ from the order axioms. Then taking the dual of every wff in this proof yields a proof of the dual claim $\varphi^{\partial}$

from the duals of the order axioms. But these are themselves axioms, so the order axioms entail $\varphi^\partial$.

For a semantic proof, say that $\varphi$ holds in every ordered set $P$. Then $\varphi^\partial$ holds in every dual $P^\partial$ of an ordered set. But these are precisely all the ordered sets, so $\varphi^\partial$ holds in all ordered sets.

This obviously extends to set theories augmented with inequality. ⌟

Remark 1.2: The category of ordered sets.

The category **Pos** of (partially) ordered sets has as objects posets and as arrows monotone maps. We claim that **Pos** has all small products and coproducts.

Let $A$ be an index set, and let $(P_\alpha)_{\alpha\in A}$ be a collection of posets. We define an order on the Cartesian product $P = \prod_{\alpha\in A} P_\alpha$ by letting $(x_\alpha)_{\alpha\in A} \leq (y_\alpha)_{\alpha\in A}$ if and only if $x_\alpha \leq y_\alpha$ for all $\alpha \in A$. The projections $\pi_\alpha \colon P \to P_\alpha$ are then clearly monotone. Notice that the above definition means that for $x, y \in P$ we have $x \leq y$ if and only if $\pi_\alpha(x) \leq \pi_\alpha(y)$ for all $\alpha \in A$. Given monotone maps $\varphi_\alpha \colon Q \to P_\alpha$, there is a unique *set map* $\varphi \colon Q \to P$ making the diagram

$$\begin{tikzcd} & & P_\alpha \\ Q \arrow[urr, bend left, "\varphi_\alpha"] \arrow[r, dashed, "\varphi"'] & P \arrow[ur, "\pi_\alpha"'] & \end{tikzcd}$$

commute for all $\alpha \in A$. But $\varphi$ is clearly also monotone: For $x, y \in Q$ with $x \leq y$, since $\varphi_\alpha$ is monotone we have

$$\pi_\alpha(\varphi(x)) = \varphi_\alpha(x) \leq \varphi_\alpha(y) = \pi_\alpha(\varphi(y)),$$

so $\varphi(x) \leq \varphi(y)$.

Next we define an order on the disjoint union $P = \coprod_{\alpha\in A} P_\alpha$. Denoting the canonical injections by $\iota_\alpha \colon P_\alpha \to P$, each element in $P$ is on the form $\iota_\alpha(x)$ for precisely one $\alpha \in A$ and $x \in P_\alpha$. Given another element $\iota_\beta(y)$ in $P$, we thus let $\iota_\alpha(x) \leq \iota_\beta(y)$ if and only if $\alpha = \beta$ and $x \leq y$ in $P_\alpha$. Given monotone maps $\varphi_\alpha \colon P_\alpha \to Q$, there is a unique *set map* $\varphi \colon P \to Q$ making the diagram

$$\begin{tikzcd} P_\alpha \arrow[dr, "\iota_\alpha"'] \arrow[drr, bend left, "\varphi_\alpha"] & & \\ & P \arrow[r, dashed, "\varphi"'] & Q \end{tikzcd}$$

commute for all $\alpha \in A$. But $\varphi$ is clearly also monotone: If two elements in $P$ are comparable, then they are on the form $\iota_\alpha(x)$ and $\iota_\alpha(y)$ for a common $\alpha \in A$ and $x, y \in P_\alpha$. If $\iota_\alpha(x) \leq \iota_\alpha(y)$, then by definition we must have $x \leq y$ in $P_\alpha$. Since $\varphi_\alpha$ is monotone it follows that

$$\varphi(\iota_\alpha(x)) = \varphi_\alpha(x) \leq \varphi_\alpha(y) = \varphi(\iota_\alpha(y)),$$

showing that $\varphi$ is monotone.

For finite coproducts we prefer the notation $P \sqcup Q$. ⌟

Remark 1.3: Linear sums of ordered sets.
We may also define a (potentially) different order on the disjoint union $\coprod_{\alpha \in A} P_\alpha$ in the case where the index set $A$ is itself (partially) ordered. For $x, y \in P$ we let $x \leq y$ if and only if

(a) $x, y \in P_\alpha$ for a common $\alpha \in A$, and $x \leq y$ in $P_\alpha$, or

(b) $x \in P_\alpha$ and $y \in P_\beta$ for distinct $\alpha, \beta \in A$ with $\alpha \leq \beta$.

We denote the disjoint union equipped with this order by $\bigoplus_{\alpha \in A} P_\alpha$. In the case where $A = \{1, \dots n\}$ is finite and totally ordered (in the obvious way) we also write

$$P_1 \oplus \cdots \oplus P_n.$$

Despite the additive notation, this operation is clearly *not* commutative, even up to isomorphism.

Notice that if $A$ has the discrete order, then the second clause above is never satisfied, and the order reduces to the coproduct order. ⌟

Remark 1.4. An injective monotone map is not necessarily an order-embedding: For instance, the identity map $\iota \colon \overline{\mathbf{2}} \to \mathbf{2}$ is obviously injective, but it is not an embedding since $0 < 1$ in $\mathbf{2}$ but not in $\overline{\mathbf{2}}$. ⌟

Remark 1.5: The functor $\mathcal{O}$.
We claim that $\mathcal{O} \colon \mathbf{Pos}^{\mathrm{op}} \to \mathbf{Pos}$ is a (contravariant) functor, when its action on arrows is given by $\mathcal{O}(\varphi) = \varphi^{-1}$, i.e. it is the pullback of $\varphi$.

Exercise 1.24 shows that $\mathcal{O}$ is well-defined as a map between categories, and the identity

$$\mathcal{O}(\psi \circ \varphi) = (\psi \circ \varphi)^{-1} = \varphi^{-1} \circ \psi^{-1} = \mathcal{O}(\varphi) \circ \mathcal{O}(\psi),$$

where $\circ$ denotes composition in $\mathbf{Pos}$ and not in $\mathbf{Pos}^{\mathrm{op}}$, shows that it is indeed a functor (its action on identity arrows is obvious). ⌟

Exercise 1.24

Let $P$ and $Q$ be ordered sets.

(a) Show that $\varphi \colon P \to Q$ is order-preserving if and only if $\varphi^{-1}(A)$ is a down-set in $P$ whenever $A$ is a down-set in $Q$.

(b) Assume $\varphi \colon P \to Q$ is order-preserving. Then, by (i), the map $\varphi^{-1} : \mathcal{O}(Q) \to \mathcal{O}(P)$ is well defined.

(a) Show that $\varphi$ is an order-embedding if and only if $\varphi^{-1}$ maps $\mathcal{O}(Q)$ onto $\mathcal{O}(P)$.

(b) Show that $\varphi$ maps onto $Q$ if and only if the map $\varphi^{-1}\colon \mathcal{O}(Q) \to \mathcal{O}(P)$ is one-to-one.

SOLUTION. (a) First assume that $\varphi$ is order-preserving, let $A \subseteq Q$ be a down-set, and let $x \in \varphi^{-1}(A)$ and $y \in P$ with $y \leq x$. Then $\varphi(y) \leq \varphi(x)$, and since $\varphi(x)$ lies in the down-set $A$ we also have $\varphi(y) \in A$. Hence $y \in \varphi^{-1}(A)$ as claimed.

Conversely, let $x, y \in P$ with $x \leq y$. If $A$ is a down-set of $Q$ with $\varphi(y) \in A$, then $y$ lies in the down-set $\varphi^{-1}(A)$ in $P$. By Lemma 1.30, $x$ also lies in $\varphi^{-1}(A)$, so $\varphi(x) \in A$. The lemma then implies that $\varphi(x) \leq \varphi(y)$.

(b) Assume that $\varphi$ is an order-embedding, and let $B \in \mathcal{O}(P)$. We claim that $B = \varphi^{-1}(\downarrow \varphi(B))$. The inclusion '$\subseteq$' is obvious, so let $x \in P$ with $\varphi(x) \in \downarrow \varphi(B)$. Then there is a $y \in B$ such that $\varphi(x) \leq \varphi(y)$, and this implies that $x \leq y$ since $\varphi$ is an embedding. But then $x \in B$, proving the other inclusion, so $\varphi^{-1}$ is surjective.

Next assume that $\varphi^{-1}$ is surjective, and consider $x, y \in P$ such that $\varphi(x) \leq \varphi(y)$. There is a down-set $A$ in $Q$ such that $\downarrow y = \varphi^{-1}(A)$, so in particular $\varphi(y) \in A$. But then $\varphi(x) \in A$, so $x \in \downarrow y$. Hence $x \leq y$ as desired.

Now assume that $\varphi$ is surjective. For $A, B \subseteq Q$ we have

$$A = \varphi(\varphi^{-1}(A)) = \varphi(\varphi^{-1}(B)) = B,$$

so $\varphi^{-1}$ is injective even on the larger domain $2^Q$.

Finally assume that $\varphi$ is *not* surjective, and choose an element $y \in Q$ not in the image of $\varphi$. Then if $B \subseteq Q$ contains $y$, we have $\varphi^{-1}(B) = \varphi^{-1}(B \setminus \{y\})$, so $\varphi^{-1}$ is not injective on the domain $2^Q$. Letting $B = \downarrow y$ we notice that $\downarrow y \setminus \{y\}$ is also a down-set, so $\varphi^{-1}$ is also not injective on $\mathcal{O}(Q)$. ◻

## 2 • Lattices and complete lattices

REMARK 2.1: Duality in lattices.
We elaborate on the duality principle for lattices. Consider the elementary language $\mathcal{L}_{\mathbf{Lat}}$ of lattices, which is an extension of $\mathcal{L}_{\mathbf{Pos}}$: It further includes binary function symbols '$\vee$' and '$\wedge$' (not to be confused with the logical operators), which satisfy the axioms

$$\forall x \forall y \forall z \Big[ (x \leq z) \wedge (y \leq z) \Leftrightarrow x \vee y \leq z \Big]$$

and

$$\forall x \forall y \forall z \Big[ (z \leq x) \wedge (z \leq y) \Leftrightarrow z \leq x \wedge y \Big].$$

Notice that there is no uniqueness assumption in the above axioms. Of course, in any model the values of the functions corresponding to the symbols '$\vee$' and '$\wedge$' will be unique given arguments, and we may prove that the axioms above are satisfied only by these values.

If $\varphi$ is a wff of $\mathcal{L}_{\mathbf{Lat}}$, then its *dual* $\varphi^\partial$ is the wff obtained by reversing the order of all inequalities, as well as exchanging '$\vee$' and '$\wedge$'. Notice that the two axioms above are each other's duals.

The duality principle for lattices says the following:

> *Let $\varphi$ be a $\mathcal{L}_{\mathbf{Lat}}$-sentence. If the lattice axioms entail $\varphi$, then they also entail the dual claim $\varphi^\partial$.*

In other words, if $\varphi$ holds in all lattices, then so does $\varphi^\partial$. This follows just as the duality principle for ordered sets, either syntactically or semantically.

As with ordered sets, this extends to set theories as well. ⌟

Remark 2.2: The category of lattices.

The category **Lat** is the subcategory of **Pos** whose objects are lattices and whose arrows are lattice homomorphisms. We claim that this has all small products [TODO coproducts?].

Let $A$ be an index set, and let $(L_\alpha)_{\alpha\in A}$ be a collection of lattices. Then $L = \prod_{\alpha\in A} L_\alpha$ is a product in **Pos**, and it is easy to show that $L$ is also a lattice, where the lattice operations are given coordinatewise, i.e.

$$(x_\alpha) \vee (y_\alpha) = (x_\alpha \vee y_\alpha) \quad \text{and} \quad (x_\alpha) \wedge (y_\alpha) = (x_\alpha \wedge y_\alpha).$$

Furthermore, the projections $\pi_\alpha \colon L \to L_\alpha$ are also lattice homomorphisms, since if $x = (x_\alpha)$ and $y = (y_\alpha)$, then

$$\pi_\alpha(x \vee y) = \pi_\alpha\big((x_\alpha \vee y_\alpha)_{\alpha\in A}\big) = x_\alpha \vee y_\alpha = \pi_\alpha(x) \vee \pi_\alpha(y),$$

and similarly for meets. To show that $L$ is a product in **Lat**, it thus suffices to show that a collection of lattice homomorphisms $f_\alpha \colon L_\alpha \to K$ factors uniquely through $L$. Uniqueness is clear, and the product map $f \colon L \to K$ is a lattice homomorphism since

$$\begin{aligned} f(x \vee y) = f\big((x_\alpha \vee y_\alpha)_{\alpha\in A}\big) &= \big(f_\alpha(x_\alpha \vee y_\alpha)\big)_{\alpha\in A} = \big(f_\alpha(x_\alpha) \vee f_\alpha(y_\alpha)\big)_{\alpha\in A} \\ &= \big(f_\alpha(x_\alpha)\big)_{\alpha\in A} \vee \big(f_\alpha(y_\alpha)\big)_{\alpha\in A} = f(x) \vee f(y), \end{aligned}$$

and similarly for meets.

Denote by **CLat** the (full) subcategory of **Lat** whose objects are complete lattices. This category also has all small products, since if $S \subseteq L = \prod_{\alpha\in A} L_\alpha$ with all $L_\alpha$ complete, then

$$\bigvee S = \Big(\bigvee \pi_\alpha(S)\Big)_{\alpha\in A},$$

and dually for meets. ⌟

Remark 2.3: Linear sums of lattices.
Let $(L_\alpha)_{\alpha\in A}$ be a collection of lattices, where the index set $A$ is itself a lattice. We wish to find further constraints on either $A$ or the $L_\alpha$ that ensure that the linear sum $L = \bigoplus_{\alpha\in A} L_\alpha$ is also a lattice.

First assume that $A$ is totally ordered. For elements $x, y \in L$, say that $x, y \in L_\alpha$ for a common index $\alpha$. Then the join $x \vee y$ in $L$ is simply the join of $x$ and $y$ in $L_\alpha$. If instead $x \in L_\alpha$ and $y \in L_\beta$ for distinct $\alpha, \beta \in A$, then

$$x \vee y = \begin{cases} x, & \alpha > \beta, \\ y, & \alpha < \beta. \end{cases}$$

Meets are given dually. In particular, finite linear sums of lattices are themselves lattices.

Next, instead assume that each $L_\alpha$ is bounded. Then if $x \in L_\alpha$ and $y \in L_\beta$ for $\alpha \neq \beta$, then $x \vee y$ is the zero in $L_{\alpha\vee\beta}$. Dually for meets.

Finally we consider complete lattices. Assume that each $L_\alpha$ is complete, and that $A$ is also complete. Then $L$ is a lattice by the above, and we claim that it is also complete. Let $S \subseteq L$ and define

$$B = \{\alpha \in A \mid \pi_\alpha(S) \neq \emptyset\}.$$

Letting $\beta = \bigvee B$ we have $\bigvee S = \bigvee \pi_\beta(S)$. Again we in particular see that finite linear sums of complete lattices are complete. ⌟

Remark 2.4: Ideals and filters.
Ideals and filters may be defined more generally for a partially ordered set $(P, \leq)$. An *ideal* in $P$ is a nonempty subset $I$ that is

(a) upward directed: for every $x, y \in I$ there is a $z \in I$ such that $x, y \leq z$; and

(b) a down-set: for every $x \in P$ and $y \in I$, $x \leq y$ implies that $x \in I$.

Dually, a *filter* in $P$ is a nonempty, downward directed up-set. ⌟

Exercise 2.6

Let $P$ be an ordered set.

(a) Prove that if $A \subseteq P$ and $\bigwedge A$ exists in $P$, then

$$\bigcap\{\downarrow a \mid a \in A\} = \downarrow\Big(\bigwedge A\Big).$$

(b) Formulate and prove the dual result.

Solution. (a) This is obvious, since $x \leq a$ for all $a \in A$ if and only if $x \leq \bigwedge A$.

(b) The dual result is

$$\bigcap\{\uparrow a \mid a \in A\} = \uparrow\left(\bigvee A\right),$$

whenever $\bigvee A$ exists in $P$. This follows by duality or by an argument analogous to the above. $\square$

**Exercise 2.17**

Let $L$ and $K$ be lattices with 0 and 1. Show that there exist $a, b \in L \times K$ such that

(a) $\downarrow a \cong L$ and $\downarrow b \cong K$,

(b) $a \wedge b = (0,0)$ and $a \vee b = (1,1)$.

Solution. Let $a = (1,0)$ and $b = (0,1)$. Then

$$\downarrow(1,0) = \{(x,y) \in L \times K \mid x \leq 1, y \leq 0\} = L \times \{0\}.$$

The map $x \mapsto (x,0)$ is clearly an isomorphism[1] $L \to L \times \{0\}$, proving that $\downarrow(1,0) \cong L$. Similarly we find that $\downarrow(0,1) \cong K$.

Furthermore notice that

$$(1,0) \wedge (0,1) = (1 \wedge 0, 0 \wedge 1) = (0,0)$$

and

$$(1,0) \vee (0,1) = (1 \vee 0, 0 \vee 1) = (1,1)$$

as desired. $\square$

**Exercise 2.22**

Let $L$ be a lattice and let $\emptyset \neq A \subseteq L$. Show that

$$(A] := \downarrow\{a_1 \vee \cdots \vee a_n \mid n \in \mathbb{N}, a_1, \ldots, a_n \in A\}$$

is an ideal and moreover it is contained in any ideal $J$ of $L$ which contains $A$.

Solution. Clearly $(A]$ is a down-set, so it suffices to show that it is closed under binary joins. If $x, y \in (A]$, then by definition there exist $a_1, \ldots, a_n, b_1, \ldots, b_m \in A$ such that

$$x \leq a_1 \vee \cdots \vee a_n \quad \text{and} \quad y \leq b_1 \vee \cdots \vee b_m.$$

It follows that

$$x \vee y \leq a_1 \vee \cdots \vee a_n \vee b_1 \vee \cdots \vee b_m,$$

[1] Indeed, if $c \in K$ is any fixed element, then the map $L \to L \times K$ given by $x \mapsto (x,c)$ is an injective homomorphism.

so $x \vee y \in (A]$.

To show that $(A] \subseteq J$, notice that it suffices (since $J$ is a down-set) to show that all finite joins of elements in $A$ lie in $J$. But this is obvious since $A \subseteq J$ and $J$ is an ideal. □

EXERCISE 2.29

Let $P$ be a complete lattice. Prove that there is a topped $\cap$-structure $\mathfrak{L}$ on the set $P$ such that $P \cong \mathfrak{L}$.

SOLUTION. Consider the map $\varphi\colon P \to \mathcal{O}(P)$ given by $\varphi(x) = {\downarrow} x$. This is clearly an order-embedding, so restricting its codomain to its image $\mathfrak{L} := \varphi(P)$ yields an order isomorphism, hence a lattice isomorphism. It remains to be shown that $\mathfrak{L}$ is a topped $\cap$-structure.

Clearly $\mathfrak{L}$ is topped since $P = {\downarrow} 1$, so it suffices to show that it is an $\cap$-structure. Let $\{A_i\}_{i \in I}$ be a nonempty family of elements in $\mathfrak{L}$, and write $A_i = {\downarrow} x_i$ for appropriate $x_i \in P$. Now notice that Exercise 2.6 implies that

$$\bigcap_{i \in I} A_i = \bigcap_{i \in I} {\downarrow} x_i = {\downarrow}\Big(\bigwedge_{i \in I} x_i\Big) \in \mathfrak{L}$$

as desired. □

EXERCISE 2.31

Let $G_1$ and $G_2$ be finite groups such that $\gcd(|G_1|, |G_2|) = 1$. Show that

$$\operatorname{Sub}(G_1 \times G_2) \cong \operatorname{Sub} G_1 \times \operatorname{Sub} G_2,$$

where on the left we have the usual coordinatewise product of groups and on the right the coordinatewise product of ordered sets.

SOLUTION. Write $n_i = |G_i|$. If $H$ is a subgroup of $G_1 \times G_2$, then $|H| = k_1 k_2$ with $k_i \mid n_i$. Consider the subgroups $H_i = \pi_i(H)$ of $G_i$. Then $|H_i|$ divides both $n_i$ and $|H|$, so since $\gcd(n_1, n_2) = 1$ it divides $k_i$. It follows that $|H_1 \times H_2| \leq k_1 k_2 = |H|$, and the opposite inequality follows since $H \subseteq H_1 \times H_2$. Hence this inclusion is in fact an equality, so the map

$$\begin{aligned} \operatorname{Sub}(G_1 \times G_2) &\to \operatorname{Sub} G_1 \times \operatorname{Sub} G_2, \\ H &\mapsto \pi_1(H) \times \pi_2(H), \end{aligned}$$

is a bijection. It is also clearly monotone, proving the claim. □

REMARK 2.5. The assumption that $|G_1|$ and $|G_2|$ be relatively prime is necessary. This hinges on the fact that a subgroup of a product is not necessarily a product of subgroups of each factor. For instance, the subgroup $\langle(1,1)\rangle$ of $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ is clearly not a product of subgroups. It is, however, isomorphic to the subgroup $\mathbb{Z}/2\mathbb{Z}\times\{0\}$, but we can find examples of subgroups of products that are not even isomorphic to any product of subgroups:

Consider for instance the subgroup of $S_3 \times S_3$ given by the kernel $K$ of the homomorphism $\varphi\colon S_3 \times S_3 \to \{\pm 1\}$ given by $\varphi(g,h) = \operatorname{sgn}(gh)$. Since $\varphi$ is surjective we have $[S_3 \times S_3 : K] = 2$, so $|K| = 18$. Now assume towards a contradiction that $K$ is isomorphic to a product of two subgroups of $S_3$. By Lagrange's theorem, the possible orders of such a subgroup are $1,2,3,6$. Hence one factor must be $S_3$ itself, and the other must be of order 3, i.e. it must be $A_3$. Hence $K \cong S_3 \times A_3$. Now notice that $S_3 \times A_3$ contains an element of order 6, e.g. $((1\,2),(1\,2\,3))$, but $K$ does not: For an element of order 6 in $S_3 \times S_3$ must be a pair $(\tau,\sigma)$, where $\tau$ is a transposition and $\sigma$ a 3-cycle, and $\operatorname{sgn}(\tau\sigma) = -1$.⌟

EXERCISE 2.32

(a) Use the Knaster–Tarski Fixpoint Theorem to prove Banach's Decomposition Theorem:

*Let $X$ and $Y$ be sets and let $f: X \to Y$ and $g\colon Y \to X$ be maps. Then there exist disjoint subsets $X_1$ and $X_2$ of $X$ and disjoint subsets $Y_1$ and $Y_2$ of $Y$ such that $f(X_1) = Y_1$, $g(Y_2) = X_2$, $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$.*

(b) Use (i) to obtain the Schröder–Bernstein Theorem:

*Let $X$ and $Y$ be sets and suppose there exist one-to-one maps $f: X \to Y$ and $g\colon Y \to X$. Then there exists a bijective map $h$ from $X$ onto $Y$.*

SOLUTION. (a) Consider the map $F\colon 2^X \to 2^X$ given by $F(S) = X \setminus g(Y \setminus f(S))$. This is easily seen to be monotone, so the Knaster–Tarski fixpoint theorem yields the existence of a fixpoint $X_1 \subseteq X$. Letting $Y_1 = f(X_1)$, $Y_2 = Y \setminus Y_1$ and $X_2 = g(Y_2)$, the fact that $F(X_1) = X_1$ implies that

$$X_1 = X \setminus g(Y \setminus f(X_1)) = X \setminus g(Y \setminus Y_1) = X \setminus g(Y_2) = X \setminus X_2,$$

which proves the claim.

(b) Let $X_1, X_2, Y_1, Y_2$ be a Banach decomposition. Since $g$ is injective it restricts to a bijection from $Y_2$ onto $X_2$. Let $g^{-1}$ denote the inverse of this restriction and define $h$ by

$$h(x) = \begin{cases} f(x), & x \in X_1, \\ g^{-1}(x), & x \in X_2. \end{cases}$$

This is easily seen to be both injective and surjective. □

**Exercise 2.34**

Prove that $P \times Q$ satisfies (ACC) if and only if both $P$ and $Q$ do.

**Solution.** Assume that $P$ and $Q$ satisfy (ACC), and let

$$(x_1, y_1) \le (x_2, y_2) \le \cdots \le (x_n, y_n) \le \cdots$$

be a sequence in $P \times Q$. By definition of the product order, this implies that $x_1 \le x_2 \le \cdots$ and $y_1 \le y_2 \le \cdots$. But then there exist $k_1, k_2 \in \mathbb{N}$ such that $x_{k_1} = x_{k_1+1} = \ldots$ and $y_{k_2} = y_{k_2+1} = \ldots$. Letting $k = \max\{k_1, k_2\}$ we thus have

$$(x_k, y_k) = (x_{k+1}, y_{k+1}) = \ldots,$$

so $P \times Q$ also satisfies (ACC).

Conversely, assume that $P \times Q$ satisfies (ACC), and let $x_1 \le x_2 \le \cdots$ be a sequence in $P$. If $y$ is any element of $Q$, this implies that $(x_1, y) \le (x_2, y) \le \cdots$ in $P \times Q$. Hence we must have $x_k = x_{k+1} = \ldots$ for some $k \in \mathbb{N}$, so $P$ satisfies (ACC). Obviously so does $Q$. □

**Exercise 2.35**

Let $P$ and $Q$ be ordered sets of finite length. Prove that

$$\ell(P \times Q) = \ell(P) + \ell(Q).$$

**Solution.** Write $n = \ell(P)$ and $m = \ell(Q)$ and let $x_0 < x_1 < \cdots < x_n$ and $y_0 < y_1 < \cdots y_m$ be chains in $P$ and $Q$ respectively. Then

$$(x_0, y_0) < (x_1, y_0) < \cdots < (x_n, y_0) < (x_n, y_1) < \cdots < (x_n, y_m)$$

is a chain in $P \times Q$ of length $n + m$, so $\ell(P \times Q) \ge n + m$. To prove the other inequality, let

$$(x_0, y_0) < (x_1, y_1) < \cdots < (x_k, y_k)$$

be a chain in $P \times Q$ of length $k$. For each $i = 1, \ldots, k$ we must have either $x_{i-1} < x_i$ or $y_{i-1} < y_i$. Let $I$ be the set of $i$ such that the former holds, and let $J$ be the set of $i$ such that the latter holds. Then $\{x_i \mid i \in I\}$ is a chain in $P$ of length $|I|$, and $\{y_i \mid i \in J\}$ is a chain in $Q$ of length $|J|$, so we must have $|I| \le n$ and $|J| \le m$. Since each $i$ lies in either $I$ or $J$ we have

$$k \le |I| + |J| \le n + m,$$

which implies that $\ell(P \times Q) \le n + m$ as desired. □

EXERCISE 2.37

Let $L$ be a lattice.

(a) Let $J_1 \subseteq J_2 \subseteq \cdots$ be a chain of ideals of $L$. Show that their union $\bigcup_{n\in\mathbb{N}} J_n$ is an ideal of $L$.

(b) Show that every ideal of $L$ is principal if and only if $L$ satisfies (ACC).

SOLUTION. (a) If $a, b \in J = \bigcup_{n\in\mathbb{N}} J_n$, then $a \in J_n$ and $b \in J_m$ for some $n, m \in \mathbb{N}$. But then $a, b \in J_{n\vee m} \subseteq J$ as required. Furthermore, if $a \in L$ and $b \in J_m$ with $a \leq b$, then $a \in J_m \subseteq J$.

(b) First assume that every ideal in $L$ is principal, and let $x_1 \leq x_2 \leq \cdots$ be a chain in $L$. Then there is an inclusion $\downarrow x_1 \subseteq \downarrow x_2 \subseteq \cdots$ of ideals, and the above shows that $J = \bigcup_{n\in\mathbb{N}}(\downarrow x_n)$ is also an ideal. Thus there is an $a \in L$ such that $J = \downarrow a$. But then $a \in \downarrow x_n$ for some $n \in \mathbb{N}$, and since $a$ is the maximum of $J$ we must have $a = x_n$. Hence this chain is of finite length.

Now assume that $L$ satisfies (ACC) and let $J$ be an ideal in $L$. By Lemma 2.39, $J$ has a maximal element $a$. If $x \in J$, then since $J$ is an ideal we have $x \vee a \in J$. Obviously $a \leq x \vee a$, and since $a$ is maximal in $J$ we also have $x \vee a \leq a$, so $x \vee a = a$. Hence $x \leq a$, so $a$ is in fact the maximum of $J$. It follows that $J \subseteq \downarrow a$, and the opposite inequality is obvious by minimality of $\downarrow a$. □

# 3 • Formal concept analysis

# 4 • Modular, distributive and Boolean lattices

EXERCISE 4.9

Let $L$ be a distributive lattice and let $a, b, c \in L$. Prove that

$$(a \vee b = c \vee b \text{ and } a \wedge b = c \wedge b) \quad \Rightarrow \quad a = c.$$

SOLUTION. This follows from the calculation

$$\begin{aligned} a &= a \wedge (a \vee b) \\ &= a \wedge (c \vee b) \\ &= (a \wedge c) \vee (a \wedge b) \\ &= (a \wedge c) \vee (c \wedge b) \\ &= c \wedge (a \vee b) \\ &= c \wedge (c \vee b) \\ &= c. \end{aligned}$$

□

EXERCISE 4.12

(a) Prove that a lattice $L$ is distributive if and only if for each $a \in L$, the map $f_a \colon L \to {\downarrow} a \times {\uparrow} a$ defined by

$$f_a(x) = (x \wedge a, x \vee a) \quad \text{for all } x \in L$$

is a one-to-one homomorphism.

(b) Prove that, if $L$ is distributive and possesses 0 and 1, then $f_a$ is an isomorphism if and only if $a$ has a complement in $L$.

SOLUTION. (a) First assume that $L$ is distributive and fix $a \in L$. For $x, y \in L$ we then have

$$\begin{aligned} f_a(x \wedge y) &= \big((x \wedge y) \wedge a, (x \wedge y) \vee a\big) \\ &= \big((x \wedge a) \wedge (y \wedge a), (x \vee a) \wedge (y \vee a)\big) \\ &= (x \wedge a, x \vee a) \wedge (y \wedge a, y \vee a) \\ &= f_a(x) \wedge f_a(y), \end{aligned}$$

and similarly for joins. Thus $f_a$ is a homomorphism. Injectivity follows immediately from Exercise 4.9.

Conversely, given $a, b, c \in L$ and using that $f_a$ is a homomorphism, it follows that

$$f_a\big((a \wedge b) \vee (a \wedge c)\big) = \big((a \wedge b) \vee (a \wedge c), a\big),$$

and that

$$f_a\big(a \wedge (b \vee c)\big) = \big(a \wedge ((a \wedge b) \vee (a \wedge c)), a\big) = \big((a \wedge b) \vee (a \wedge c), a\big),$$

where the second equality follows since $(a \wedge b) \vee (a \wedge c) \leq a$. Distributivity then follows since $f_a$ is injective.

(b) Since $f_a$ is already shown to be an injective homomorphism, it is an isomorphism if and only if it is surjective. First, if $f_a$ is surjective then there exists a $b \in L$ such that

$$(0, 1) = f_a(b) = (b \wedge a, b \vee a),$$

which precisely says that $b$ is a complement of $a$.

Conversely, say that $a$ has a complement $a'$ in $L$, and let $c \in {\downarrow} a$ and $d \in {\uparrow} a$. Then $c \leq d$, so since $L$ is modular we have

$$x := c \vee (a' \wedge d) = (c \vee a') \wedge d.$$

We furthermore have

$$\big((c \vee a') \wedge d\big) \wedge a = \big((c \wedge a) \vee (a' \wedge a)\big) \wedge d = (c \vee 0) \wedge d = c$$

and

$$\big(c \vee (a' \wedge d)\big) \vee a = c \vee \big((a' \vee a) \wedge (d \vee a)\big) = a \vee (1 \wedge d) = d.$$

Hence $f_a(x) = (c, d)$, so $f_a$ is surjective. $\square$