

Davey & Priesley, *Introduction to Lattices and Order*

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1 • Ordered sets

REMARK 1.1: Duality in ordered sets.

We elaborate on the duality principle. Consider the elementary language $\mathcal{L}_{\mathbf{Pos}}$ of ordered sets. This is a (single sorted) first-order language with identity, variable symbols x, y, z, \dots and a single binary relation symbol ' \leq '. If φ is a wff of $\mathcal{L}_{\mathbf{Pos}}$, then its *dual* φ^∂ is the wff obtained by reversing the order of all inequalities, so that ' $X \leq Y$ ' becomes ' $Y \leq X$ '.

Recall that a *sentence* in a first-order language is a wff with no free variables. The duality principle then says the following:

Let φ be a $\mathcal{L}_{\mathbf{Pos}}$ -sentence. If the order axioms entail φ , then they also entail the dual claim φ^∂ .

In other words, if φ holds in all ordered sets, then so does φ^∂ . We give both a syntactic and a semantic proof of this claim. (Of course either is sufficient since the theory of ordered sets is a first-order theory.)

First notice that the axioms of ordered sets (reflexivity, antisymmetry and transitivity) are self-dual: That is, each is their own dual. For instance, the transitive axiom states that

$$\forall x \forall y \forall z (x \leq y \wedge y \leq z \Rightarrow x \leq z),$$

and this is obviously equivalent to its dual:

$$\forall x \forall y \forall z (y \leq x \wedge z \leq y \Rightarrow z \leq x).$$

Now say that there is a (first-order) proof of φ from the order axioms. Then taking the dual of every wff in this proof yields a proof of the dual claim φ^∂

from the duals of the order axioms. But these are themselves axioms, so the order axioms entail φ^∂ .

For a semantic proof, say that φ holds in every ordered set P . Then φ^∂ holds in every dual P^∂ of an ordered set. But these are precisely all the ordered sets, so φ^∂ holds in all ordered sets.

This obviously extends to set theories augmented with inequality. \square

REMARK 1.2: The category of ordered sets.

The category **Pos** of (partially) ordered sets has as objects posets and as arrows monotone maps. We claim that **Pos** has all small products and coproducts.

Let A be an index set, and let $(P_\alpha)_{\alpha \in A}$ be a collection of posets. We define an order on the Cartesian product $P = \prod_{\alpha \in A} P_\alpha$ by letting $(x_\alpha)_{\alpha \in A} \leq (y_\alpha)_{\alpha \in A}$ if and only if $x_\alpha \leq y_\alpha$ for all $\alpha \in A$. The projections $\pi_\alpha: P \rightarrow P_\alpha$ are then clearly monotone. Notice that the above definition means that for $x, y \in P$ we have $x \leq y$ if and only if $\pi_\alpha(x) \leq \pi_\alpha(y)$ for all $\alpha \in A$. Given monotone maps $\varphi_\alpha: Q \rightarrow P_\alpha$, there is a unique *set map* $\varphi: Q \rightarrow P$ making the diagram

$$\begin{array}{ccc} & & P_\alpha \\ & \nearrow \varphi_\alpha & \\ Q & \xrightarrow[\varphi]{} & P \end{array}$$

commute for all $\alpha \in A$. But φ is clearly also monotone: For $x, y \in Q$ with $x \leq y$, since φ_α is monotone we have

$$\pi_\alpha(\varphi(x)) = \varphi_\alpha(x) \leq \varphi_\alpha(y) = \pi_\alpha(\varphi(y)),$$

so $\varphi(x) \leq \varphi(y)$.

Next we define an order on the disjoint union $P = \coprod_{\alpha \in A} P_\alpha$. Denoting the canonical injections by $\iota_\alpha: P_\alpha \rightarrow P$, each element in P is on the form $\iota_\alpha(x)$ for precisely one $\alpha \in A$ and $x \in P_\alpha$. Given another element $\iota_\beta(y)$ in P , we thus let $\iota_\alpha(x) \leq \iota_\beta(y)$ if and only if $\alpha = \beta$ and $x \leq y$ in P_α . Given monotone maps $\varphi_\alpha: P_\alpha \rightarrow Q$, there is a unique *set map* $\varphi: P \rightarrow Q$ making the diagram

$$\begin{array}{ccc} P_\alpha & & \\ \searrow \iota_\alpha & \nearrow \varphi_\alpha & \\ & P \xrightarrow[\varphi]{} & Q \end{array}$$

commute for all $\alpha \in A$. But φ is clearly also monotone: If two elements in P are comparable, then they are on the form $\iota_\alpha(x)$ and $\iota_\alpha(y)$ for a common $\alpha \in A$ and $x, y \in P_\alpha$. If $\iota_\alpha(x) \leq \iota_\alpha(y)$, then by definition we must have $x \leq y$ in P_α . Since φ_α is monotone it follows that

$$\varphi(\iota_\alpha(x)) = \varphi_\alpha(x) \leq \varphi_\alpha(y) = \varphi(\iota_\alpha(y)),$$

showing that φ is monotone.

For finite coproducts we prefer the notation $P \sqcup Q$. \lrcorner

REMARK 1.3: Linear sums of ordered sets.

We may also define a (potentially) different order on the disjoint union $\coprod_{\alpha \in A} P_\alpha$ in the case where the index set A is itself (partially) ordered. For $x, y \in P$ we let $x \leq y$ if and only if

- (a) $x, y \in P_\alpha$ for a common $\alpha \in A$, and $x \leq y$ in P_α , or
- (b) $x \in P_\alpha$ and $y \in P_\beta$ for distinct $\alpha, \beta \in A$ with $\alpha \leq \beta$.

We denote the disjoint union equipped with this order by $\bigoplus_{\alpha \in A} P_\alpha$. In the case where $A = \{1, \dots, n\}$ is finite and totally ordered (in the obvious way) we also write

$$P_1 \oplus \dots \oplus P_n.$$

Despite the additive notation, this operation is clearly *not* commutative, even up to isomorphism.

Notice that if A has the discrete order, then the second clause above is never satisfied, and the order reduces to the coproduct order. \lrcorner

REMARK 1.4. An injective monotone map is not necessarily an order-embedding: For instance, the identity map $\iota: \bar{2} \rightarrow 2$ is obviously injective, but it is not an embedding since $0 < 1$ in 2 but not in $\bar{2}$. \lrcorner

REMARK 1.5: The functor \mathcal{O} .

We claim that $\mathcal{O}: \mathbf{Pos}^{\text{op}} \rightarrow \mathbf{Pos}$ is a (contravariant) functor, when its action on arrows is given by $\mathcal{O}(\varphi) = \varphi^{-1}$, i.e. it is the pullback of φ .

Exercise 1.24 shows that \mathcal{O} is well-defined as a map between categories, and the identity

$$\mathcal{O}(\psi \circ \varphi) = (\psi \circ \varphi)^{-1} = \varphi^{-1} \circ \psi^{-1} = \mathcal{O}(\varphi) \circ \mathcal{O}(\psi),$$

where \circ denotes composition in \mathbf{Pos} and not in \mathbf{Pos}^{op} , shows that it is indeed a functor (its action on identity arrows is obvious). \lrcorner

EXERCISE 1.24

Let P and Q be ordered sets.

- (a) Show that $\varphi: P \rightarrow Q$ is order-preserving if and only if $\varphi^{-1}(A)$ is a down-set in P whenever A is a down-set in Q .
- (b) Assume $\varphi: P \rightarrow Q$ is order-preserving. Then, by (i), the map $\varphi^{-1}: \mathcal{O}(Q) \rightarrow \mathcal{O}(P)$ is well defined.
 - (a) Show that φ is an order-embedding if and only if φ^{-1} maps $\mathcal{O}(Q)$

onto $\mathcal{O}(P)$.

- (b) Show that φ maps onto Q if and only if the map $\varphi^{-1}: \mathcal{O}(Q) \rightarrow \mathcal{O}(P)$ is one-to-one.

SOLUTION. (a) First assume that φ is order-preserving, let $A \subseteq Q$ be a down-set, and let $x \in \varphi^{-1}(A)$ and $y \in P$ with $y \leq x$. Then $\varphi(y) \leq \varphi(x)$, and since $\varphi(x)$ lies in the down-set A we also have $\varphi(y) \in A$. Hence $y \in \varphi^{-1}(A)$ as claimed.

Conversely, let $x, y \in P$ with $x \leq y$. If A is a down-set of Q with $\varphi(y) \in A$, then y lies in the down-set $\varphi^{-1}(A)$ in P . By Lemma 1.30, x also lies in $\varphi^{-1}(A)$, so $\varphi(x) \in A$. The lemma then implies that $\varphi(x) \leq \varphi(y)$.

(b) Assume that φ is an order-embedding, and let $B \in \mathcal{O}(P)$. We claim that $B = \varphi^{-1}(\downarrow \varphi(B))$. The inclusion ' \subseteq ' is obvious, so let $x \in P$ with $\varphi(x) \in \downarrow \varphi(B)$. Then there is a $y \in B$ such that $\varphi(x) \leq \varphi(y)$, and this implies that $x \leq y$ since φ is an embedding. But then $x \in B$, proving the other inclusion, so φ^{-1} is surjective.

Next assume that φ^{-1} is surjective, and consider $x, y \in P$ such that $\varphi(x) \leq \varphi(y)$. There is a down-set A in Q such that $\downarrow y = \varphi^{-1}(A)$, so in particular $\varphi(y) \in A$. But then $\varphi(x) \in A$, so $x \in \downarrow y$. Hence $x \leq y$ as desired.

Now assume that φ is surjective. For $A, B \subseteq Q$ we have

$$A = \varphi(\varphi^{-1}(A)) = \varphi(\varphi^{-1}(B)) = B,$$

so φ^{-1} is injective even on the larger domain 2^Q .

Finally assume that φ is *not* surjective, and choose an element $y \in Q$ not in the image of φ . Then if $B \subseteq Q$ contains y , we have $\varphi^{-1}(B) = \varphi^{-1}(B \setminus \{y\})$, so φ^{-1} is not injective on the domain 2^Q . Letting $B = \downarrow y$ we notice that $\downarrow y \setminus \{y\}$ is also a down-set, so φ^{-1} is also not injective on $\mathcal{O}(Q)$. \square

2 • Lattices and complete lattices

REMARK 2.1: Duality in lattices.

We elaborate on the duality principle for lattices. Consider the elementary language \mathcal{L}_{Lat} of lattices, which is an extension of \mathcal{L}_{Pos} : It further includes binary function symbols ' \vee ' and ' \wedge ' (not to be confused with the logical operators), which satisfy the axioms

$$\forall x \forall y \forall z [(x \leq z) \wedge (y \leq z) \Leftrightarrow x \vee y \leq z]$$

and

$$\forall x \forall y \forall z [(z \leq x) \wedge (z \leq y) \Leftrightarrow z \leq x \wedge y].$$

Notice that there is no uniqueness assumption in the above axioms. Of course, in any model the values of the functions corresponding to the symbols ‘ \vee ’ and ‘ \wedge ’ will be unique given arguments, and we may prove that the axioms above are satisfied only by these values.

If φ is a wff of $\mathcal{L}_{\mathbf{Lat}}$, then its *dual* φ^∂ is the wff obtained by reversing the order of all inequalities, as well as exchanging ‘ \vee ’ and ‘ \wedge ’. Notice that the two axioms above are each other’s duals.

The duality principle for lattices says the following:

Let φ be a $\mathcal{L}_{\mathbf{Lat}}$ -sentence. If the lattice axioms entail φ , then they also entail the dual claim φ^∂ .

In other words, if φ holds in all lattices, then so does φ^∂ . This follows just as the duality principle for ordered sets, either syntactically or semantically.

As with ordered sets, this extends to set theories as well. \lrcorner

REMARK 2.2: The category of lattices.

The category **Lat** is the subcategory of **Pos** whose objects are lattices and whose arrows are lattice homomorphisms. We claim that this has all small products [TODO coproducts?].

Let A be an index set, and let $(L_\alpha)_{\alpha \in A}$ be a collection of lattices. Then $L = \prod_{\alpha \in A} L_\alpha$ is a product in **Pos**, and it is easy to show that L is also a lattice, where the lattice operations are given coordinatewise, i.e.

$$(x_\alpha) \vee (y_\alpha) = (x_\alpha \vee y_\alpha) \quad \text{and} \quad (x_\alpha) \wedge (y_\alpha) = (x_\alpha \wedge y_\alpha).$$

Furthermore, the projections $\pi_\alpha: L \rightarrow L_\alpha$ are also lattice homomorphisms, since if $x = (x_\alpha)$ and $y = (y_\alpha)$, then

$$\pi_\alpha(x \vee y) = \pi_\alpha((x_\alpha \vee y_\alpha)_{\alpha \in A}) = x_\alpha \vee y_\alpha = \pi_\alpha(x) \vee \pi_\alpha(y),$$

and similarly for meets. To show that L is a product in **Lat**, it thus suffices to show that a collection of lattice homomorphisms $f_\alpha: K \rightarrow L_\alpha$ factors uniquely through L . Uniqueness is clear, and the product map $f: K \rightarrow L$ is a lattice homomorphism since

$$\begin{aligned} f(x \vee y) &= (f_\alpha(x \vee y))_{\alpha \in A} = (f_\alpha(x) \vee f_\alpha(y))_{\alpha \in A} \\ &= (f_\alpha(x))_{\alpha \in A} \vee (f_\alpha(y))_{\alpha \in A} = f(x) \vee f(y), \end{aligned}$$

and similarly for meets.

Denote by **CLat** the (full) subcategory of **Lat** whose objects are complete lattices. This category also has all small products, since if $S \subseteq L = \prod_{\alpha \in A} L_\alpha$ with all L_α complete, then it is easy to show that

$$\bigvee S = \left(\bigvee \pi_\alpha(S) \right)_{\alpha \in A},$$

and dually for meets. \lrcorner

REMARK 2.3: Linear sums of lattices.

Let $(L_\alpha)_{\alpha \in A}$ be a collection of lattices, where the index set A is itself a lattice. We wish to find further constraints on either A or the L_α that ensure that the linear sum $L = \bigoplus_{\alpha \in A} L_\alpha$ is also a lattice.

First assume that A is totally ordered. For elements $x, y \in L$, say that $x, y \in L_\alpha$ for a common index α . Then the join $x \vee y$ in L is simply the join of x and y in L_α . If instead $x \in L_\alpha$ and $y \in L_\beta$ for distinct $\alpha, \beta \in A$, then

$$x \vee y = \begin{cases} x, & \alpha > \beta, \\ y, & \alpha < \beta. \end{cases}$$

Meets are given dually. In particular, finite linear sums of lattices are themselves lattices.

Next, instead assume that each L_α is bounded. Then if $x \in L_\alpha$ and $y \in L_\beta$ for $\alpha \neq \beta$, then $x \vee y$ is the zero in $L_{\alpha \vee \beta}$. Dually for meets.

Finally we consider complete lattices. Assume that each L_α is complete, and that A is also complete. Then L is a lattice by the above, and we claim that it is also complete. Let $S \subseteq L$ and define

$$B = \{\alpha \in A \mid \pi_\alpha(S) \neq \emptyset\}.$$

Letting $\beta = \bigvee B$ we have $\bigvee S = \bigvee \pi_\beta(S)$. Again we in particular see that finite linear sums of complete lattices are complete. \lrcorner

REMARK 2.4: Ideals and filters in posets.

Ideals and filters may be defined more generally for a partially ordered set (P, \leq) . An *ideal* in P is a nonempty subset I that is

- (a) upward directed: for every $x, y \in I$ there is a $z \in I$ such that $x, y \leq z$; and
- (b) downward closed (i.e. a down-set): for every $x \in P$ and $y \in I$, $x \leq y$ implies that $x \in I$.

Dually, a *filter* in P is a nonempty, downward directed up-set.

If P is also a lattice and I is an ideal in the above sense, then it is also an ideal in the sense of Definition 2.20 (the converse is obvious). For if $x, y \in I$, then there is some $z \in I$ with $x, y \leq z$. But $x \vee y \leq z$, so since I is downward closed we also have $x \vee y \in I$. \lrcorner

REMARK 2.5: Generating ideals and filters.

If P is a partially ordered set and $A \subseteq P$, then it is easy to show that A is upward directed if and only if $\downarrow A$ is upward directed. Thus if $A \neq \emptyset$, then $\downarrow A$ is an ideal if and only if A is upward directed, and dually for filters. Clearly $\downarrow A$ is the smallest ideal containing A (compare Exercise 2.22).

In topology it is common to consider filters in powerset lattices. All filters are usually assumed to be proper in this context. If X is a set, then a *prefilter* or *filter basis* is a nonempty, downward directed collection \mathcal{B} of nonempty subsets of X . It generates the filter $\mathcal{F} = \uparrow \mathcal{B}$, and this is indeed proper since $\emptyset \notin \mathcal{F}$. By the above, a filter basis is precisely a collection $\mathcal{B} \subseteq 2^X$ such that $\uparrow \mathcal{B}$ is a (proper) filter. \square

EXERCISE 2.6

Let P be an ordered set.

- (a) Prove that if $A \subseteq P$ and $\bigwedge A$ exists in P , then

$$\bigcap \{\downarrow a \mid a \in A\} = \downarrow (\bigwedge A).$$

- (b) Formulate and prove the dual result.

SOLUTION. (a) This is obvious, since $x \leq a$ for all $a \in A$ if and only if $x \leq \bigwedge A$.

- (b) The dual result is

$$\bigcap \{\uparrow a \mid a \in A\} = \uparrow (\bigvee A),$$

whenever $\bigvee A$ exists in P . This follows by duality or by an argument analogous to the above. \square

EXERCISE 2.17

Let L and K be lattices with 0 and 1. Show that there exist $a, b \in L \times K$ such that

- (a) $\downarrow a \cong L$ and $\downarrow b \cong K$,
 (b) $a \wedge b = (0, 0)$ and $a \vee b = (1, 1)$.

SOLUTION. Let $a = (1, 0)$ and $b = (0, 1)$. Then

$$\downarrow(1, 0) = \{(x, y) \in L \times K \mid x \leq 1, y \leq 0\} = L \times \{0\}.$$

The map $x \mapsto (x, 0)$ is clearly an isomorphism¹ $L \rightarrow L \times \{0\}$, proving that $\downarrow(1, 0) \cong L$. Similarly we find that $\downarrow(0, 1) \cong K$.

Furthermore notice that

$$(1, 0) \wedge (0, 1) = (1 \wedge 0, 0 \wedge 1) = (0, 0)$$

¹ Indeed, if $c \in K$ is any fixed element, then the map $L \rightarrow L \times K$ given by $x \mapsto (x, c)$ is an injective homomorphism.

and

$$(1, 0) \vee (0, 1) = (1 \vee 0, 0 \vee 1) = (1, 1)$$

as desired. \square

EXERCISE 2.22

Let L be a lattice and let $\emptyset \neq A \subseteq L$. Show that

$$(A] := \downarrow \{a_1 \vee \cdots \vee a_n \mid n \in \mathbb{N}, a_1, \dots, a_n \in A\}$$

is an ideal and moreover it is contained in any ideal J of L which contains A .

SOLUTION. Clearly $(A]$ is a down-set, so it suffices to show that it is closed under binary joins. If $x, y \in (A]$, then by definition there exist $a_1, \dots, a_n, b_1, \dots, b_m \in A$ such that

$$x \leq a_1 \vee \cdots \vee a_n \quad \text{and} \quad y \leq b_1 \vee \cdots \vee b_m.$$

It follows that

$$x \vee y \leq a_1 \vee \cdots \vee a_n \vee b_1 \vee \cdots \vee b_m,$$

so $x \vee y \in (A]$.

To show that $(A] \subseteq J$, notice that it suffices (since J is a down-set) to show that all finite joins of elements in A lie in J . But this is obvious since $A \subseteq J$ and J is an ideal. \square

EXERCISE 2.29

Let P be a complete lattice. Prove that there is a topped \cap -structure \mathfrak{L} on the set P such that $P \cong \mathfrak{L}$.

SOLUTION. Consider the map $\varphi: P \rightarrow \mathcal{O}(P)$ given by $\varphi(x) = \downarrow x$. This is clearly an order-embedding, so restricting its codomain to its image $\mathfrak{L} := \varphi(P)$ yields an order isomorphism, hence a lattice isomorphism. It remains to be shown that \mathfrak{L} is a topped \cap -structure.

Clearly \mathfrak{L} is topped since $P = \downarrow 1$, so it suffices to show that it is an \cap -structure. Let $\{A_i\}_{i \in I}$ be a nonempty family of elements in \mathfrak{L} , and write $A_i = \downarrow x_i$ for appropriate $x_i \in P$. Now notice that [Exercise 2.6](#) implies that

$$\bigcap_{i \in I} A_i = \bigcap_{i \in I} \downarrow x_i = \downarrow \left(\bigwedge_{i \in I} x_i \right) \in \mathfrak{L}$$

as desired. \square

EXERCISE 2.31

Let G_1 and G_2 be finite groups such that $\gcd(|G_1|, |G_2|) = 1$. Show that

$$\text{Sub}(G_1 \times G_2) \cong \text{Sub } G_1 \times \text{Sub } G_2,$$

where on the left we have the usual coordinatewise product of groups and on the right the coordinatewise product of ordered sets.

SOLUTION. Write $n_i = |G_i|$. If H is a subgroup of $G_1 \times G_2$, then $|H| = k_1 k_2$ with $k_i \mid n_i$. Consider the subgroups $H_i = \pi_i(H)$ of G_i . Then $|H_i|$ divides both n_i and $|H|$, so since $\gcd(n_1, n_2) = 1$ it divides k_i . It follows that $|H_1 \times H_2| \leq k_1 k_2 = |H|$, and the opposite inequality follows since $H \subseteq H_1 \times H_2$. Hence this inclusion is in fact an equality, so the map

$$\begin{aligned} \text{Sub}(G_1 \times G_2) &\rightarrow \text{Sub } G_1 \times \text{Sub } G_2, \\ H &\mapsto \pi_1(H) \times \pi_2(H), \end{aligned}$$

is a bijection. It is also clearly monotone, proving the claim. \square

REMARK 2.6. The assumption that $|G_1|$ and $|G_2|$ be relatively prime is necessary. This hinges on the fact that a subgroup of a product is not necessarily a product of subgroups of each factor. For instance, the subgroup $\langle (1, 1) \rangle$ of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is clearly not a product of subgroups. It is, however, isomorphic to the subgroup $\mathbb{Z}/2\mathbb{Z} \times \{0\}$, but we can find examples of subgroups of products that are not even isomorphic to any product of subgroups:

Consider for instance the subgroup of $S_3 \times S_3$ given by the kernel K of the homomorphism $\varphi: S_3 \times S_3 \rightarrow \{\pm 1\}$ given by $\varphi(g, h) = \text{sgn}(gh)$. Since φ is surjective we have $[S_3 \times S_3 : K] = 2$, so $|K| = 18$. Now assume towards a contradiction that K is isomorphic to a product of two subgroups of S_3 . By Lagrange's theorem, the possible orders of such a subgroup are 1, 2, 3, 6. Hence one factor must be S_3 itself, and the other must be of order 3, i.e. it must be A_3 . Hence $K \cong S_3 \times A_3$. Now notice that $S_3 \times A_3$ contains an element of order 6, e.g. $((1\ 2), (1\ 2\ 3))$, but K does not: For an element of order 6 in $S_3 \times S_3$ must be a pair (τ, σ) , where τ is a transposition and σ a 3-cycle, and $\text{sgn}(\tau\sigma) = -1$. \perp

EXERCISE 2.32

- (a) Use the Knaster–Tarski Fixpoint Theorem to prove Banach's Decomposition Theorem:

Let X and Y be sets and let $f: X \rightarrow Y$ and $g: Y \rightarrow X$ be maps. Then there exist disjoint subsets X_1 and X_2 of X and disjoint subsets Y_1 and Y_2 of Y such that $f(X_1) = Y_1$, $g(Y_2) = X_2$, $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$.

(b) Use (i) to obtain the Schröder–Bernstein Theorem:

Let X and Y be sets and suppose there exist one-to-one maps $f: X \rightarrow Y$ and $g: Y \rightarrow X$. Then there exists a bijective map h from X onto Y .

SOLUTION. (a) Consider the map $F: 2^X \rightarrow 2^X$ given by $F(S) = X \setminus g(Y \setminus f(S))$. This is easily seen to be monotone, so the Knaster–Tarski fixpoint theorem yields the existence of a fixpoint $X_1 \subseteq X$. Letting $Y_1 = f(X_1)$, $Y_2 = Y \setminus Y_1$ and $X_2 = g(Y_2)$, the fact that $F(X_1) = X_1$ implies that

$$X_1 = X \setminus g(Y \setminus f(X_1)) = X \setminus g(Y \setminus Y_1) = X \setminus g(Y_2) = X \setminus X_2,$$

which proves the claim.

(b) Let X_1, X_2, Y_1, Y_2 be a Banach decomposition. Since g is injective it restricts to a bijection from Y_2 onto X_2 . Let g^{-1} denote the inverse of this restriction and define h by

$$h(x) = \begin{cases} f(x), & x \in X_1, \\ g^{-1}(x), & x \in X_2. \end{cases}$$

This is easily seen to be both injective and surjective. \square

EXERCISE 2.34

Prove that $P \times Q$ satisfies (ACC) if and only if both P and Q do.

SOLUTION. Assume that P and Q satisfy (ACC), and let

$$(x_1, y_1) \leq (x_2, y_2) \leq \cdots \leq (x_n, y_n) \leq \cdots$$

be a sequence in $P \times Q$. By definition of the product order, this implies that $x_1 \leq x_2 \leq \cdots$ and $y_1 \leq y_2 \leq \cdots$. But then there exist $k_1, k_2 \in \mathbb{N}$ such that $x_{k_1} = x_{k_1+1} = \cdots$ and $y_{k_2} = y_{k_2+1} = \cdots$. Letting $k = \max\{k_1, k_2\}$ we thus have

$$(x_k, y_k) = (x_{k+1}, y_{k+1}) = \cdots,$$

so $P \times Q$ also satisfies (ACC).

Conversely, assume that $P \times Q$ satisfies (ACC), and let $x_1 \leq x_2 \leq \cdots$ be a sequence in P . If y is any element of Q , this implies that $(x_1, y) \leq (x_2, y) \leq \cdots$ in $P \times Q$. Hence we must have $x_k = x_{k+1} = \cdots$ for some $k \in \mathbb{N}$, so P satisfies (ACC). Obviously so does Q . \square

EXERCISE 2.35

Let P and Q be ordered sets of finite length. Prove that

$$\ell(P \times Q) = \ell(P) + \ell(Q).$$

SOLUTION. Write $n = \ell(P)$ and $m = \ell(Q)$ and let $x_0 < x_1 < \cdots < x_n$ and $y_0 < y_1 < \cdots < y_m$ be chains in P and Q respectively. Then

$$(x_0, y_0) < (x_1, y_0) < \cdots < (x_n, y_0) < (x_n, y_1) < \cdots < (x_n, y_m)$$

is a chain in $P \times Q$ of length $n + m$, so $\ell(P \times Q) \geq n + m$. To prove the other inequality, let

$$(x_0, y_0) < (x_1, y_1) < \cdots < (x_k, y_k)$$

be a chain in $P \times Q$ of length k . For each $i = 1, \dots, k$ we must have either $x_{i-1} < x_i$ or $y_{i-1} < y_i$. Let I be the set of i such that the former holds, and let J be the set of i such that the latter holds. Then $\{x_i \mid i \in I\}$ is a chain in P of length $|I|$, and $\{y_i \mid i \in J\}$ is a chain in Q of length $|J|$, so we must have $|I| \leq n$ and $|J| \leq m$. Since each i lies in either I or J we have

$$k \leq |I| + |J| \leq n + m,$$

which implies that $\ell(P \times Q) \leq n + m$ as desired. \square

EXERCISE 2.37

Let L be a lattice.

- (a) Let $J_1 \subseteq J_2 \subseteq \cdots$ be a chain of ideals of L . Show that their union $\bigcup_{n \in \mathbb{N}} J_n$ is an ideal of L .
- (b) Show that every ideal of L is principal if and only if L satisfies (ACC).

SOLUTION. (a) If $a, b \in J = \bigcup_{n \in \mathbb{N}} J_n$, then $a \in J_n$ and $b \in J_m$ for some $n, m \in \mathbb{N}$. But then $a, b \in J_{n \vee m} \subseteq J$ as required. Furthermore, if $a \in L$ and $b \in J_m$ with $a \leq b$, then $a \in J_m \subseteq J$.

(b) First assume that every ideal in L is principal, and let $x_1 \leq x_2 \leq \cdots$ be a chain in L . Then there is an inclusion $\downarrow x_1 \subseteq \downarrow x_2 \subseteq \cdots$ of ideals, and the above shows that $J = \bigcup_{n \in \mathbb{N}} (\downarrow x_n)$ is also an ideal. Thus there is an $a \in L$ such that $J = \downarrow a$. But then $a \in \downarrow x_n$ for some $n \in \mathbb{N}$, and since a is the maximum of J we must have $a = x_n$. Hence this chain is of finite length.

Now assume that L satisfies (ACC) and let J be an ideal in L . By Lemma 2.39, J has a maximal element a . If $x \in J$, then since J is an ideal we have $x \vee a \in J$. Obviously $a \leq x \vee a$, and since a is maximal in J we also have $x \vee a \leq a$, so $x \vee a = a$. Hence $x \leq a$, so a is in fact the maximum of J . It follows that $J \subseteq \downarrow a$, and the opposite inequality is obvious by minimality of $\downarrow a$. \square

3 • Formal concept analysis

4 • Modular, distributive and Boolean lattices

EXERCISE 4.9

Let L be a distributive lattice and let $a, b, c \in L$. Prove that

$$(a \vee b = c \vee b \text{ and } a \wedge b = c \wedge b) \Rightarrow a = c.$$

SOLUTION. This follows from the calculation

$$\begin{aligned} a &= a \wedge (a \vee b) \\ &= a \wedge (c \vee b) \\ &= (a \wedge c) \vee (a \wedge b) \\ &= (a \wedge c) \vee (c \wedge b) \\ &= c \wedge (a \vee b) \\ &= c \wedge (c \vee b) \\ &= c. \end{aligned}$$

□

EXERCISE 4.12

- (a) Prove that a lattice L is distributive if and only if for each $a \in L$, the map $f_a: L \rightarrow \downarrow a \times \uparrow a$ defined by

$$f_a(x) = (x \wedge a, x \vee a) \quad \text{for all } x \in L$$

is a one-to-one homomorphism.

- (b) Prove that, if L is distributive and possesses 0 and 1, then f_a is an isomorphism if and only if a has a complement in L .

SOLUTION. (a) First assume that L is distributive and fix $a \in L$. For $x, y \in L$ we then have

$$\begin{aligned} f_a(x \wedge y) &= ((x \wedge y) \wedge a, (x \wedge y) \vee a) \\ &= ((x \wedge a) \wedge (y \wedge a), (x \vee a) \wedge (y \vee a)) \\ &= (x \wedge a, x \vee a) \wedge (y \wedge a, y \vee a) \\ &= f_a(x) \wedge f_a(y), \end{aligned}$$

and similarly for joins. Thus f_a is a homomorphism. Injectivity follows immediately from [Exercise 4.9](#).

Conversely, given $a, b, c \in L$ and using that f_a is a homomorphism, it follows that

$$f_a((a \wedge b) \vee (a \wedge c)) = ((a \wedge b) \vee (a \wedge c), a),$$

and that

$$f_a(a \wedge (b \vee c)) = (a \wedge ((a \wedge b) \vee (a \wedge c)), a) = ((a \wedge b) \vee (a \wedge c), a),$$

where the second equality follows since $(a \wedge b) \vee (a \wedge c) \leq a$. Distributivity then follows since f_a is injective.

(b) Since f_a is already shown to be an injective homomorphism, it is an isomorphism if and only if it is surjective. First, if f_a is surjective then there exists a $b \in L$ such that

$$(0, 1) = f_a(b) = (b \wedge a, b \vee a),$$

which precisely says that b is a complement of a .

Conversely, say that a has a complement a' in L , and let $c \in \downarrow a$ and $d \in \uparrow a$. Then $c \leq d$, so since L is modular we have

$$x := c \vee (a' \wedge d) = (c \vee a') \wedge d.$$

We furthermore have

$$((c \vee a') \wedge d) \wedge a = ((c \wedge a) \vee (a' \wedge a)) \wedge d = (c \vee 0) \wedge d = c$$

and

$$(c \vee (a' \wedge d)) \vee a = c \vee ((a' \vee a) \wedge (d \vee a)) = a \vee (1 \wedge d) = d.$$

Hence $f_a(x) = (c, d)$, so f_a is surjective. \square