Dynkin systems and independence

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1 • Introduction

These notes are basically a proof of Theorem 6.3 in Bauer.

2 • Dynkin systems

Definition 2.1: *Dynkin systems*

A collection $\mathcal{D}$ of subsets of a set X is called a *Dynkin system* in X if

(i) $X \in \mathcal{D}$,

(ii) $B \setminus A \in \mathcal{D}$ for $A, B \in \mathcal{D}$ with $A \subseteq B$, and

(iii) $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{D}$ for every increasing sequence $(A_n)_{n\in\mathbb{N}}$ of sets in $\mathcal{D}$.

A Dynkin system is also variously called a Dynkin class, δ-system, d-system, or λ-system. Clearly every σ-algebra is a Dynkin system.

The motivation for considering Dynkin systems is twofold. First of all they are significantly simpler than σ-algebras, and working with Dynkin systems instead of σ-algebras can often we done with no loss of generality, as the following fundamental result shows:

Theorem 2.2: *Dynkin's Lemma*

Let $\mathcal{S}$ be a collection of subsets of a set X that is closed under finite intersections. Then

$$\delta(\mathcal{S}) = \sigma(\mathcal{S}).$$

Also known as 'Dynkin's π-λ theorem' since a non-empty collection of sets that is closed under finite intersections is also called a π-system.

PROOF. Bauer Theorem 2.3 (different definition of Dynkin systems), Cohn Theorem 1.6.2. □

Another source of motivation comes from the following result about finite measures which is important when proving uniqueness of properties of finite or σ-finite measures:

LEMMA 2.3

Let μ and ν be finite measures on a measurable space $(X, \mathcal{E})$ such that $\mu(X) = \nu(X)$. The family $\mathcal{D} \subseteq \mathcal{E}$ of sets on which μ and ν agree is a Dynkin system.

PROOF. By assumption $X \in \mathcal{D}$. Let $A_1, A_2 \in \mathcal{D}$ with $A_1 \subseteq A_2$. Then

$$\mu(A_2 \setminus A_1) = \mu(A_2) - \mu(A_1) = \nu(A_2) - \nu(A_1) = \nu(A_2 \setminus A_1),$$

since μ and ν are finite. Finally assume that $(A_n)_{n \in \mathbb{N}}$ is an increasing sequence of elements in $\mathcal{D}$. Then by continuity we have

$$\mu\Big(\bigcup_{n \in \mathbb{N}} A_n\Big) = \lim_{n \to \infty} \mu(A_n) = \lim_{n \to \infty} \nu(A_n) = \nu\Big(\bigcup_{n \in \mathbb{N}} A_n\Big).$$

Thus $\mathcal{D}$ is a Dynkin system as claimed. □

3 • Independence

Below we let $(\Omega, \mathcal{F}, \mathbb{P})$ be a fixed probability space.

DEFINITION 3.1: *Independence I*

Let I be a non-empty index set. A family $(A_i)_{i \in I}$ of events from $\mathcal{F}$ is called *independent* relative to $\mathbb{P}$ if

$$\mathbb{P}\Big(\bigcap_{\nu=1}^{n} A_{i_\nu}\Big) = \prod_{\nu=1}^{n} \mathbb{P}(A_{i_\nu}),$$

for any finite subset $\{i_1, \dots, i_n\}$ of distinct elements of I.

DEFINITION 3.2: *Independence II*

Let $(\mathcal{A}_i)_{i \in I}$ be a family of sets $\mathcal{A}_i \subseteq \mathcal{F}$ of events. This family is called *independent* if

$$\mathbb{P}\Big(\bigcap_{\nu=1}^{n} A_{i_\nu}\Big) = \prod_{\nu=1}^{n} \mathbb{P}(A_{i_\nu}),$$

for any choice of events $A_{i_\nu} \in \mathcal{A}_{i_\nu}$ and any finite subset $\{i_1, \dots, i_n\}$ of distinct elements of I.

Notice that this definition reduces to the first one if all $\mathcal{A}_i$ are singletons.

REMARK 3.3. Finite subfamilies independent is sufficient. ⌟

PROPOSITION 3.4

Let $(\mathcal{A}_i)_{i\in I}$ be an independent family of sets of events from $\mathcal{F}$. Then the family $(\delta(\mathcal{A}_i))_{i\in I}$ is also independent. In particular, if the $\mathcal{A}_i$ are closed under intersection, the family $(\sigma(\mathcal{A}_i))_{i\in I}$ is independent.

PROOF. By Remark 3.3 we may assume that I is finite.

Fix an index $i_0 \in I$, and choose sets $A_{i_\nu} \in \mathcal{A}_{i_\nu}$ for distinct indices $i_1, \dots, i_n \in I \setminus \{i_0\}$. Then define measures $\mathbb{P}_1$ and $\mathbb{P}_2$ on $\mathcal{F}$ by

$$\mathbb{P}_1(A) = \mathbb{P}\Big(A \cap \bigcap_{\nu=1}^{n} A_{i_\nu}\Big) \quad \text{and} \quad \mathbb{P}_2(A) = \mathbb{P}(A)\prod_{\nu=1}^{n}\mathbb{P}(A_{i_\nu}),$$

for $A \in \mathcal{F}$. Notice that $\mathbb{P}_1$ and $\mathbb{P}_2$ agree on $\mathcal{A}_{i_0}$ by independence, so since $\mathbb{P}_1(\Omega) = \mathbb{P}_2(\Omega)$, Lemma 2.3 implies that they also agree on $\delta(\mathcal{A}_{i_0})$. It follows that

$$\mathbb{P}\Big(A \cap \bigcap_{\nu=1}^{n} A_{i_\nu}\Big) = \mathbb{P}(A)\prod_{\nu=1}^{n}\mathbb{P}(A_{i_\nu})$$

for all choices of sets $A_{i_\nu} \in \mathcal{A}_{i_\nu}$. But this precisely expresses the independence of the family $(\mathcal{A}_i)_{i\in I}$ with $\mathcal{A}_{i_0}$ replaced by $\delta(\mathcal{A}_{i_0})$.

Performing a finite number of such replacements, once for each index in I, proves the first claim. The second claim follows by Dynkin's lemma. □