Folland: *Real Analysis*

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1 • Measures

1.2. σ-algebras

Exercise 1.1

Let $\mathcal{M}$ be an infinite σ-algebra.

(a) $\mathcal{M}$ contains an infinite sequence of disjoint sets.

(b) $|\mathcal{M}| \geq \mathfrak{c}$.

Of course part (a) is trivial unless we require the sets to be nonempty.

Solution. (a) We show by contraposition that there exists a nonempty set $A \in \mathcal{M}$ such that the restriction of $\mathcal{M}$ to A^c is infinite. That is, assuming that no such set exists, we show that $\mathcal{M}$ is finite. Pick any nonempty $A \in \mathcal{M}$. Then the restriction of $\mathcal{M}$ to A and A^c respectively are both finite. For any $B \in \mathcal{M}$ we can write

$$B = (B \cap A) \cup (B \cap A^c).$$

But each set in the union lies in one of the restrictions, so there are finitely many decompositions like the one above, so there are finitely many sets $B \in \mathcal{M}$.

Now construct the sequence: Pick $A \in \mathcal{M}$ as above, restrict $\mathcal{M}$ to A^c, and continue recursively.

(b) Let (A_n) be the sequence constructed above. There is an injection $\varphi\colon 2^{\mathbb{N}} \to \mathcal{M}$ given by $\varphi(I) = \bigcup_{i \in I} A_i$ (injectivity follows since the sets in the sequence are disjoint). Hence $|\mathcal{M}| \geq |2^{\mathbb{N}}| = \mathfrak{c}$. □

1.3. Measures

EXERCISE 1.14

If μ is a semifinite measure and $\mu(E) = \infty$, for any $C > 0$ there exists $F \subseteq E$ with $C < \mu(F) < \infty$.

SOLUTION. Consider

$$S = \sup\{\mu(F) \mid F \subseteq E, \mu(F) < \infty\}.$$

If $S = \infty$, then the result is obvious. So assume towards a contradiction that $S < \infty$. For $n \in \mathbb{N}$ choose $F_n \subseteq E$ with $\mu(F_n) < \infty$ such that

$$S - \frac{1}{n} \leq \mu(F_n) \leq S.$$

Put $G_k = \bigcup_{n=1}^k F_n$. Then $G_k \subseteq E$ and $\mu(G_k) < \infty$, so the same inequality holds with F_n replaced by G_k. Now putting $G = \bigcup_{k\in\mathbb{N}} G_k$, continuity of μ gives

$$S - \frac{1}{n} \leq \mu(G) \leq S$$

for all $n \in \mathbb{N}$, so $\mu(G) = S$.

By assumption $\mu(E \setminus G) = \infty$, so $E \setminus G$ contains a set $G' \in \mathcal{M}$ such that $0 < \mu(G') < \infty$. But then

$$\mu(G \cup G') = \mu(G) + \mu(G') > S,$$

a contradiction. □

EXERCISE 1.16

Let $(X, \mathcal{M}, \mu)$ be a measure space. A set $E \subseteq X$ is called *locally measurable* if $E \cap A \in \mathcal{M}$ for all $A \in \mathcal{M}$ such that $\mu(A) < \infty$. Let $\widetilde{\mathcal{M}}$ be the collection of all locally measurable sets. Clearly $\mathcal{M} \subseteq \widetilde{\mathcal{M}}$; if $\mathcal{M} = \widetilde{\mathcal{M}}$, then μ is called *saturated*.

(a) If μ is σ-finite, then μ is saturated.

(b) $\widetilde{\mathcal{M}}$ is a σ-algebra.

(c) Define $\tilde{\mu}$ on $\widetilde{\mathcal{M}}$ by $\tilde{\mu}(E) = \mu(E)$ if $E \in \mathcal{M}$ and $\tilde{\mu}(E) = \infty$ otherwise. Then $\tilde{\mu}$ is a saturated measure on $\widetilde{\mathcal{M}}$, called the *saturation* of μ.

(d) If μ is complete, so is $\tilde{\mu}$.

(e) Suppose that μ is semifinite. For $E \in \widetilde{\mathcal{M}}$ define

$$\underline{\mu}(E) = \sup\{\mu(A) \mid A \in \mathcal{M} \text{ and } A \subseteq E\}.$$

Then $\underline{\mu}$ is a saturated measure on $\widetilde{\mathcal{M}}$ that extends μ.

(f) Let X_1, X_2 be disjoint uncountable sets, $X = X_1 \cup X_2$, and $\mathcal{M}$ the σ-algebra of countable or co-countable sets in X. Let μ_0 be counting measure on 2^{X_1}, and define μ on $\mathcal{M}$ by $\mu(E) = \mu_0(E \cap X_1)$. Then μ is a measure on $\mathcal{M}$, $\widetilde{\mathcal{M}} = 2^X$, and in the notation of parts (c) and (e), $\tilde{\mu} \neq \underline{\mu}$.

Solution. (a) Assume that μ is σ-finite, and let $E \subseteq X$ be locally measurable. Let $(A_n) \subseteq \mathcal{M}$ be such that $X = \bigcup_{n\in\mathbb{N}} A_n$ and $\mu(A_n) < \infty$. Then $E \cap A_n \in \mathcal{M}$, and so $E = \bigcup_{n\in\mathbb{N}}(E \cap A_n) \in \mathcal{M}$.

(b) Clearly we have $X \in \widetilde{\mathcal{M}}$. Then let $(E_n) \subseteq \widetilde{\mathcal{M}}$, and let $A \in \mathcal{M}$ with $\mu(A) < \infty$. Then

$$A \cap \bigcup_{n\in\mathbb{N}} E_n = \bigcup_{n\in\mathbb{N}} (A \cap E_n) \in \mathcal{M},$$

so $\bigcup_{n\in\mathbb{N}} E_n \in \widetilde{\mathcal{M}}$. Finally let $E \in \widetilde{\mathcal{M}}$ and $A \in \mathcal{M}$ with $\mu(A) < \infty$. Then

$$E^c \cap A = (E \cap A)^c \cap A \in \mathcal{M}$$

since $E \cap A \in \mathcal{M}$, so $E^c \in \widetilde{\mathcal{M}}$.

(c) We first show that $\tilde{\mu}$ is a measure. Clearly $\tilde{\mu}(\emptyset) = 0$, so let (E_n) be a sequence of disjoint sets in $\widetilde{\mathcal{M}}$, and let $E = \bigcup_{n\in\mathbb{N}} E_n$. Say that E_m does not lie in $\mathcal{M}$ for some $m \in \mathbb{N}$. Then we must have $\tilde{\mu}(E) = \infty$, since otherwise $E \in \mathcal{M}$ with $\mu(E) < \infty$, and hence $E_m = E_m \cap E \in \mathcal{M}$. Thus we have

$$\sum_{n=1}^{\infty} \tilde{\mu}(E_n) \geq \tilde{\mu}(E_m) = \infty = \tilde{\mu}(E),$$

so $\sum_{n=1}^{\infty} \tilde{\mu}(E_n) = \tilde{\mu}(E)$. The same is obviously true if all E_n lie in $\mathcal{M}$.

Next we show that $\tilde{\mu}$ is saturated, i.e. that $\widetilde{\widetilde{\mathcal{M}}} \subseteq \widetilde{\mathcal{M}}$, so let $E \in \widetilde{\widetilde{\mathcal{M}}}$. For all $A \in \widetilde{\mathcal{M}}$ with $\tilde{\mu}(A) < \infty$ we then have $E \cap A \in \widetilde{\mathcal{M}}$. By definition of $\tilde{\mu}$ we must have $A \in \mathcal{M}$, so we also have

$$E \cap A = (E \cap A) \cap A \in \mathcal{M}.$$

And since this is true for all $A \in \mathcal{M}$ with $\mu(A) < \infty$, it follows that $E \in \widetilde{\mathcal{M}}$.

(d) Assume that μ is complete. Let $F \subseteq X$ be such that there is a set $E \in \widetilde{\mathcal{M}}$ with $F \subseteq E$ and $\tilde{\mu}(E) = 0$. Then also $E \in \mathcal{M}$, and since μ is complete we have $F \in \mathcal{M} \subseteq \widetilde{\mathcal{M}}$ as desired.

(e) Assume that μ is semifinite. We first show that $\underline{\mu}$ is a measure. Clearly $\underline{\mu}(\emptyset) = 0$, so let $(E_n) \subseteq \widetilde{\mathcal{M}}$ be a sequence of disjoint sets. Clearly $\underline{\mu}$ is increasing, so sigma-additivity is obvious if any of the sets E_n have infinite measure.

Assume then that $\underline{\mu}(E_n) < \infty$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$, and choose $A_n \in \mathcal{M}$ such that $A_n \subseteq E_n$ and $\underline{\mu}(E_n) \leq \mu(A_n) + \varepsilon/2^n$. Then

$$\underline{\mu}\Big(\bigcup_{n\in\mathbb{N}} E_n\Big) \geq \mu\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) = \sum_{n=1}^{\infty} \mu(A_n) \geq \sum_{n=1}^{\infty} \mu(E_n) - \varepsilon.$$

Since this holds for all $\varepsilon > 0$, we obtain the first inequality. For the other inequality, let $E = \bigcup_{n\in\mathbb{N}} E_n$, and first assume that $\underline{\mu}(E) = \infty$. Pick $A \in \mathcal{M}$ with $A \subseteq E$. Since μ is semifinite, we can choose A such that $C < \mu(A) < \infty$ for any given $C > 0$. Letting $A_n = A \cap E_n \in \mathcal{M}$ we get

$$C < \mu(A) = \sum_{n=1}^{\infty} \mu(A_n) \leq \sum_{n=1}^{\infty} \underline{\mu}(E_n),$$

and since C is arbitrary, we get $\sum_{n=1}^{\infty} \underline{\mu}(E_n) = \infty$. If instead $\underline{\mu}(E) < \infty$, pick $A \subseteq E$ with $A \in \mathcal{M}$ and $\underline{\mu}(E) \leq \mu(A) + \varepsilon$. Again letting $A_n = A \cap E_n$ we get

$$\underline{\mu}(E) - \varepsilon \leq \mu(A) = \sum_{n=1}^{\infty} \mu(A_n) \leq \sum_{n=1}^{\infty} \underline{\mu}(E_n).$$

And since ε is arbitrary, we obtain the other inequality.

Next we show that $\underline{\mu}$ is saturated. Letting E be locally $\underline{\mu}$-measurable, we must show that E is also locally μ-measurable. So let $A \in \mathcal{M}$ with $\mu(A) < \infty$. Then $\underline{\mu}(A) < \infty$, and so $E \cap A \in \widetilde{\mathcal{M}}$. But then

$$E \cap A = (E \cap A) \cap A \in \mathcal{M},$$

as desired.

(f) It is pretty obvious that μ is a measure on $\mathcal{M}$. Then let $E \subseteq X$ and $A \in \mathcal{M}$ with $\mu(A) < \infty$. Then $A \cap X_1$ must be finite, and so A is not co-countable. But then it is countable, and so is $E \cap A$, hence $E \cap A \in \mathcal{M}$. Thus every subset of X is locally measurable.

Notice that μ is semifinite. We have $\tilde{\mu}(X_2) = \infty$ since $X_2 \notin \mathcal{M}$, but $\underline{\mu}(X_2) = 0$ since every subset of X_2 is disjoint from X_1, and so is has measure zero. $\square$

1.4. Outer Measures

Exercise 1.18

Let $\mathcal{A} \subseteq 2^X$ be an algebra, $\mathcal{A}_\sigma$ the collection of countable unions of sets in $\mathcal{A}$, and $\mathcal{A}_{\sigma\delta}$ the collection of countable intersections of sets in $\mathcal{A}_\sigma$. Let μ_0 be a premeasure on $\mathcal{A}$ and μ^* the induced outer measure.

(a) For any $E \subseteq X$ and $\varepsilon > 0$ there exists $A \in \mathcal{A}_\sigma$ with $E \subseteq A$ with $\mu^*(A) \leq$

$\mu^*(E) + \varepsilon$.

(b) If $\mu^*(E) < \infty$, then E is μ^*-measurable iff there exists $B \in \mathcal{A}_{\sigma\delta}$ with $E \subseteq B$ and $\mu^*(B \setminus E) = 0$.

(c) If μ_0 is σ-finite, the restriction $\mu^*(E) < \infty$ in (b) is superfluous.

SOLUTION. (a) Let $E \subseteq X$ and $\varepsilon > 0$. The definition of μ^* yields a sequence $(A_n) \subseteq \mathcal{A}$ such that $E \subseteq \bigcup_{n\in\mathbb{N}} A_n$ and $\sum_{n=1}^{\infty} \mu_0(A_n) \leq \mu^*(E) + \varepsilon$. It follows that

$$\mu^*(E) + \varepsilon \geq \sum_{n=1}^{\infty} \mu_0(A_n) = \sum_{n=1}^{\infty} \mu^*(A_n) \geq \mu^*\Big(\bigcup_{n\in\mathbb{N}} A_n\Big).$$

(b) Let $E \subseteq X$. For $n \in \mathbb{N}$ there is a set $B_n \in \mathcal{A}_\sigma$ such that $E \subseteq B_n$ and $\mu^*(B_n) \leq \mu^*(E) + 1/n$. Letting $B = \bigcap_{n\in\mathbb{N}} B_n \in \mathcal{A}_{\sigma\delta}$ we get $\mu^*(B) \leq \mu^*(E)$, and since $E \subseteq B$ we also have the opposite inequality, so $\mu^*(B) = \mu^*(E)$.

Now assume that $\mu^*(E) < \infty$ and that E is μ^*-measurable. Then

$$\mu^*(B) = \mu^*(B \cap E) + \mu^*(B \cap E^c) = \mu^*(E) + \mu^*(B \setminus E),$$

from which it follows that $\mu^*(B \setminus E) = 0$.

Conversely, assume that there is a $B \in \mathcal{A}_{\sigma\delta}$ with $E \subseteq B$ and $\mu^*(B \setminus E) = 0$. Then B lies in the σ-algebra generated by $\mathcal{A}$, so it is μ^*-measurable. Let $A \subseteq X$. Then

$$\begin{aligned}\mu^*(A \cap E^c) &\leq \mu^*(A \cap E^c \cap B) + \mu^*(A \cap E^c \cap B^c)\\ &= \mu^*(A \cap (B \cup E)^c)\\ &= \mu^*(A \cap B^c),\end{aligned}$$

and so

$$\mu^*(A \cap E) + \mu^*(A \cap E^c) \leq \mu^*(A \cap B) + \mu^*(A \cap B^c) = \mu^*(A),$$

showing that E is μ^*-measurable. (Notice that we haven't used that $\mu^*(E) < \infty$ for the second implication.)

(c) We only need to prove the first implication above. By σ-finiteness of μ_0, let (E_n) be a sequence of subsets of X such that $\mu^*(E_n) < \infty$ and $E = \bigcup_{n\in\mathbb{N}} E_n$. Let $\varepsilon > 0$. Then there are sets $A_n \in \mathcal{A}_\sigma$ such that $\mu^*(A_n) \leq \mu^*(E_n) + \varepsilon/2^n$. Letting $B_\varepsilon = \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}_\sigma$ we get

$$\mu^*(B_\varepsilon \setminus E) = \mu^*\Big(\bigcup_{n\in\mathbb{N}} (A_n \cap E^c)\Big) \leq \mu^*\Big(\bigcup_{n\in\mathbb{N}} (A_n \cap E_n^c)\Big) \leq \sum_{n=1}^{\infty} \mu^*(A_n \setminus E_n) \leq \varepsilon.$$

Finally we let $B = \bigcap_{k\in\mathbb{N}} B_{1/k} \in \mathcal{A}_{\sigma\delta}$, and we get $\mu^*(B \setminus E) = 0$ as desired. $\square$

REMARK 1.1. Notice that (b) and (c) in particular show that any Lebesgue measurable set E, and therefore any Borel set, is the union of a G_δ set B and a Lebesgue null set $B \setminus E$. ⌟

EXERCISE 1.20

Let μ^* be an outer measure on X, $\mathcal{M}^*$ the σ-algebra of μ^*-measurable sets, $\overline{\mu} = \mu^*|_{\mathcal{M}^*}$, and μ^+ the outer measure induced by $\overline{\mu}$ as in (1.12) (with $\overline{\mu}$ and $\mathcal{M}^*$ replacing μ_0 and $\mathcal{A}$).

(a) If $E \subseteq X$, we have $\mu^*(E) \le \mu^+(E)$, with equality iff there exists $A \in \mathcal{M}^*$ with $A \supseteq E$ and $\mu^*(A) = \mu^*(E)$.

(b) If μ^* is induced from a premeasure, then $\mu^* = \mu^+$.

(c) If $X = \{0,1\}$, there exists an outer measure μ^* on X such that $\mu^* \neq \mu^+$.

SOLUTION. (a) Recall that the definition of μ^+ means that

$$\mu^+(E) = \inf\left\{\sum_{n=1}^{\infty} \overline{\mu}(A_n) \,\middle|\, A_n \in \mathcal{M}^*, E \subseteq \bigcup_{n\in\mathbb{N}} A_n\right\},$$

and that we by definition of $\overline{\mu}$ can replace $\overline{\mu}$ with μ^*. For any such sequence (A_n) we have

$$\mu^*(E) \le \mu^*\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) \le \sum_{n=1}^{\infty} \mu^*(A_n) = \sum_{n=1}^{\infty} \overline{\mu}(A_n).$$

And since $\mu^+(E)$ is the infimum of all such sums, we have $\mu^*(E) \le \mu^+(E)$.

Next assume that there is an $A \in \mathcal{M}^*$ with $E \subseteq A$ such that $\mu^*(A) = \mu^*(E)$. Using the sequence $A_1 = A$ and $A_n = \emptyset$ for $n > 1$ in the definition of μ^+ yields

$$\mu^+(E) \le \overline{\mu}(A) = \mu^*(A) = \mu^*(E).$$

Hence $\mu^+(E) = \mu^*(E)$ as desired.

Conversely, assuming that $\mu^*(E) = \mu^+(E)$ we have

$$\mu^*(E) = \inf\left\{\sum_{n=1}^{\infty} \mu^*(A_n) \,\middle|\, A_n \in \mathcal{M}^*, E \subseteq \bigcup_{n\in\mathbb{N}} A_n\right\}.$$

Given $\varepsilon > 0$, choose a sequence (A_n) such that

$$\mu^*\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) \le \sum_{n=1}^{\infty} \mu^*(A_n) \le \mu^*(E) + \varepsilon,$$

and let $B_\varepsilon = \bigcup_{n\in\mathbb{N}} A_n$. Letting $A = \bigcap_{k\in\mathbb{N}} B_{1/k} \in \mathcal{M}^*$ we thus have $\mu^*(A) \le \mu^*(E)$.

(b) Assume that μ^* is induced from a premeasure on an algebra $\mathcal{A}$, and let $E \subseteq X$. Recall that $\mathcal{A}$ consists of μ^*-measurable sets, so $\sigma(\mathcal{A}) \subseteq \mathcal{M}^*$. For $n \in \mathbb{N}$ choose, in accordance with Exercise 1.18(a), a set $A_n \in \mathcal{A}_\sigma$ with $E \subseteq A_n$ such that $\mu^*(A_n) \leq \mu^*(E) + 1/n$. Letting $A = \bigcap_{n \in \mathbb{N}} A_n$ we have $E \subseteq A$ and $\mu^*(A) \leq \mu^*(E)$. The other inequality is obvious, so $\mu^*(A) = \mu^*(E)$, and part (a) implies that $\mu^*(E) = \mu^+(E)$ as desired. □

Exercise 1.21

Let μ^* be an outer measure induced from a premeasure and $\overline{\mu}$ the restriction of μ^* to the μ^*-measurable sets. Then $\overline{\mu}$ is saturated.

Solution. Let $\mathcal{A}$ denote the algebra on which the premeasure in question is defined, and denote by $\mathcal{M}^*$ the σ-algebra of μ^*-measurable sets. Recall that $\mathcal{A} \subseteq \mathcal{M}^*$.

Let $E \subseteq X$ be locally measurable. It suffices to show that

$$\mu^*(F) \geq \mu^*(F \cap E) + \mu^*(F \cap E^c)$$

for all $F \subseteq X$ with $\mu^*(F) < \infty$. Given $\varepsilon > 0$, Exercise 1.18(a) yields a set $A \in \mathcal{A}_\sigma$ such that $\mu^*(A) \leq \mu^*(F) + \varepsilon$. Then $\mu^*(A) < \infty$, and so $E \cap A \in \mathcal{M}^*$. It follows that

$$\begin{aligned} \mu^*(F) + \varepsilon \geq \mu^*(A) &= \mu^*(A \cap (E \cap A)) + \mu^*(A \cap (E \cap A)^c) \\ &= \mu^*(A \cap E) + \mu^*(A \cap E^c) \\ &\geq \mu^*(F \cap E) + \mu^*(F \cap E^c), \end{aligned}$$

and hence $E \in \mathcal{M}^*$. Thus $\overline{\mu}$ is saturated. □

Exercise 1.22

Let $(X, \mathcal{M}, \mu)$ be a measure space, μ^* the outer measure induced by μ according to (1.12), $\mathcal{M}^*$ the σ-algebra of μ^*-measurable sets, and $\overline{\mu} = \mu^*|_{\mathcal{M}^*}$.

(a) If μ is σ-finite, then $\overline{\mu}$ is the completion of μ.

(b) In general, $\overline{\mu}$ is the saturation of the completion of μ.

Solution. (a) Let $\overline{\mathcal{M}}$ be the σ-algebra from Theorem 1.9 (namely, the σ-algebra generated by the sets in $\mathcal{M}$ along with all μ-null sets). This is clearly the smallest σ-algebra on which there can exist a complete extension of μ, so since $\overline{\mu}$ is also a complete extension of μ, we must have $\overline{\mathcal{M}} \subseteq \mathcal{M}^*$. Theorem 1.9 yields the uniqueness of a complete extension of μ on $\overline{\mathcal{M}}$, so it suffices to show that $\mathcal{M}^* \subseteq \overline{\mathcal{M}}$.

Now assume that μ is σ-finite, and let $E \in \mathcal{M}^*$. Then also $E^c \in \mathcal{M}^*$, and Exercise 1.18(c) ensures the existence of sets $B, D \in \mathcal{M}_{\sigma\delta} = \mathcal{M}$ with $E \subseteq B$ and $E^c \subseteq D$ such that

$$\mu^*(B \setminus E) = 0 \quad \text{and} \quad \mu^*(E \setminus D^c) = \mu^*(D \setminus E^c) = 0.$$

It follows that

$$\mu(B \setminus D^c) \leq \mu^*(B \setminus E) + \mu^*(E \setminus D^c) = 0,$$

so $E \setminus D^c$ is a μ-null set. Thus $E = D^c \cup (E \setminus D^c)$ is a union of a set in $\mathcal{M}$ and a μ-null set, and hence $E \in \overline{\mathcal{M}}$.

(b) Let $\hat{\mu}$ denote the completion of μ on $\overline{\mathcal{M}}$, and let $\widetilde{\mathcal{M}}$ denote the σ-algebra of locally $\hat{\mu}$-measurable sets. First we show that $\widetilde{\mathcal{M}} = \mathcal{M}^*$, so let $E \in \widetilde{\mathcal{M}}$. To show that E is μ^*-measurable it suffices to show that

$$\mu^*(F) \geq \mu^*(F \cap E) + \mu^*(F \cap E^c)$$

for all $F \subseteq X$ with $\mu^*(F) < \infty$. Calculations identical to the ones in the solution to Exercise 1.21 show this.

Conversely, let $E \in \mathcal{M}^*$ and consider $A \in \overline{\mathcal{M}}$ with $\hat{\mu}(A) < \infty$. Then also $A \in \mathcal{M}^*$, so $E \cap A \in \mathcal{M}^*$. The argument at the beginning of part (a) showed that $\overline{\mu}$ is an extension of $\hat{\mu}$, so $\mu^*(E \cap A) = \hat{\mu}(E \cap A) < \infty$. The same argument as in part (a), only now using Exercise 1.18(b) instead of (c), shows that $E \cap A \in \overline{\mathcal{M}}$, and so $E \in \widetilde{\mathcal{M}}$.

Finally, let $\tilde{\mu}$ denote the saturation of $\hat{\mu}$. We show that $\overline{\mu} = \tilde{\mu}$. Since the completion of μ on $\overline{\mathcal{M}}$ is unique, the two measures must agree here. Instead let $E \in \widetilde{\mathcal{M}} \setminus \overline{\mathcal{M}}$. By definition of $\tilde{\mu}$ we must then have $\tilde{\mu}(E) = \infty$. On the other hand, we just showed (for $E \cap A$ instead of E) that $\mu^*(E) < \infty$ implies $E \in \overline{\mathcal{M}}$. Since we have assumed that this is not the case, we must have $\overline{\mu}(E) = \mu^*(E) = \infty$. Thus $\overline{\mu} = \tilde{\mu}$. ☐

EXERCISE 1.25

If $E \subseteq \mathbb{R}$, the following are equivalent.

(a) $E \in \mathcal{M}_\mu$.

(b) $E = V \setminus N_1$ where V is a G_δ set and $\mu(N_1) = 0$.

(c) $E = H \cup N_2$ where H is an F_σ set and $\mu(N_2) = 0$.

SOLUTION. Folland proves this claim when $\mu(E) < \infty$, so assume that $\mu(E) = \infty$. Since μ is σ-finite, there is a sequence $(E_n)_{n\in\mathbb{N}}$ in $\mathcal{M}_\mu$ with $\mu(E_n) < \infty$ for all $n \in \mathbb{N}$ and $E = \bigcup_{n\in\mathbb{N}} E_n$. Then there are sequences (H_n) of F_σ sets and (N_n)

of null sets such that $E_n = H_n \cup N_n$. Then $H = \bigcup_{n\in\mathbb{N}} H_n$ is also an F_σ set and $N = \bigcup_{n\in\mathbb{N}} N_n$ a null set, and $E = H \cup N$.

Applying this to E^c yields a similar decomposition $E^c = H \cup N$. But then $E = H^c \setminus N$, and H^c is a G_δ set. □

2 • Integration

2.1. Measurable Functions

EXERCISE 2.10

The following implications are valid iff the measure μ is complete:

(a) If f is measurable and $f = g$ μ-a.e., then g is measurable.

(b) If f_n is measurable for $n \in \mathbb{N}$ and $f_n \to f$ μ-a.e., then f is measurable.

SOLUTION. (a) Let $f, g\colon (X, \mathcal{E}, \mu) \to (Y, \mathcal{F})$ be functions from a measure space to a measurable space where f is $(\mathcal{E}, \mathcal{F})$-measurable. Let $N = \{f \neq g\}$ and assume that $\mu(N) = 0$. Given $B \in \mathcal{F}$ we must show that $g^{-1}(B) \in \mathcal{E}$. But notice that

$$g^{-1}(B) = f^{-1}(B) \cup \{f \notin B, g \in B\} \setminus \{f \in B, g \notin B\},$$

and that the latter two sets are subsets of N, hence measurable. Thus $g^{-1}(B)$ is also measurable.

Conversely, let μ be a measure on a measurable space $(X, \mathcal{E})$ that is not complete, and let $N \subseteq X$ be a non-measurable μ-null set. Then $\mathbf{1}_N = 0$ μ-a.e., but $\mathbf{1}_N$ is not measurable.

(b) Consider the set A of points $x \in X$ such that $f_n(x)$ does not converge to $f(x)$. Then $f_n \mathbf{1}_{A^c} \to f \mathbf{1}_{A^c}$ pointwise everywhere, so Proposition 2.7 (or Corollary 2.9) implies that $f \mathbf{1}_{A^c}$ is measurable. By assumption $\mu(A) = 0$, so $f \mathbf{1}_{A^c} = f$ μ-a.e. and part (a) implies that f is measurable.

Conversely □

3 • Signed Measures and Differentiation

3.1. Signed Measures

EXERCISE 3.2

If ν is a signed measure, E is ν-null iff $|\nu|(E) = 0$. Also, if ν and μ are signed measures, $\nu \perp \mu$ iff $|\nu| \perp \mu$ iff $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.

SOLUTION. Assume that E is ν-null, and let $P \cup N$ be a Hahn decomposition for ν. Then

$$\nu^+(E) = \nu(E \cap P) = 0,$$

since $E \cap P \subseteq E$. Similarly we get $\nu^-(E) = 0$, so $|\nu|(E) = 0$. Conversely, assume that $|\nu|(E) = 0$. Then $\nu^{\pm}(F) = 0$ for all measurable $F \subseteq E$, and so $\nu(F) = 0$.

The other claims follow directly from the above. □

EXERCISE 3.3

Let ν be a signed measure on $(X, \mathcal{M})$.

(a) $L^1(\nu) = L^1(|\nu|)$.

(b) If $f \in L^1(\nu)$,

$$\left|\int f \,\mathrm{d}\nu\right| \leq \int |f| \,\mathrm{d}|\nu|.$$

(c) If $E \in \mathcal{M}$,

$$|\nu|(E) = \sup\left\{\left|\int_E f \,\mathrm{d}\nu\right| \,\middle|\, |f| \leq 1\right\}$$

SOLUTION. (a) This follows directly from the definition of $L^1(\nu)$.

(b) For $f \in L^1(\nu)$ we have

$$\left|\int f \,\mathrm{d}\nu\right| = \left|\int f \,\mathrm{d}\nu^+ - \int f \,\mathrm{d}\nu^-\right| \leq \int |f| \,\mathrm{d}\nu^+ + \int |f| \,\mathrm{d}\nu^- = \int |f| \,\mathrm{d}|\nu|,$$

since $|\nu| = \nu^+ + \nu^-$.

(c) If $|f| \leq 1$, then

$$\left|\int_E f \,\mathrm{d}\nu\right| \leq \int_E |f| \,\mathrm{d}|\nu| \leq |\nu|(E),$$

showing one inequality. For the other inequality, let $P \cup N$ be a Hahn decomposition for ν, and let $f = \mathbf{1}_P - \mathbf{1}_N$. Then

$$\begin{aligned}
\int_E f \,\mathrm{d}\nu &= \int_E (\mathbf{1}_P - \mathbf{1}_N) \,\mathrm{d}\nu^+ - \int_E (\mathbf{1}_P - \mathbf{1}_N) \,\mathrm{d}\nu^- \\
&= \nu^+(E \cap P) - \nu^+(E \cap N) - \nu^-(E \cap P) + \nu^-(E \cap N) \\
&= \nu^+(E) + \nu^-(E) = |\nu|(E).
\end{aligned}$$

□

Exercise 3.4

If ν is a signed measure and λ, μ are positive measures such that $\nu = \lambda - \mu$, then $\lambda \geq \nu^+$ and $\mu \geq \nu^-$.

Solution. Let $P \cup N$ be a Hahn decomposition for ν. Then

$$\nu^+(E) = \nu(E \cap P) = \lambda(E \cap P) - \mu(E \cap P) \leq \lambda(E \cap P) \leq \lambda(E),$$

and similarly for ν^-. □

Exercise 3.5

If ν_1, ν_2 are signed measures that both omit the value ∞ or $-\infty$, then $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$.

Solution. First notice that

$$\nu_1 + \nu_2 = (\nu_1^+ + \nu_2^+) - (\nu_1^- + \nu_2^-),$$

so by the previous exercise we have

$$|\nu_1 + \nu_2| = (\nu_1 + \nu_2)^+ + (\nu_1 + \nu_2)^+ \leq (\nu_1^+ + \nu_2^+) + (\nu_1^- + \nu_2^-) = |\nu_1| + |\nu_2|.$$ □

Exercise 3.7

Suppose that ν is a signed measure on $(X, \mathcal{M})$ and $E \in \mathcal{M}$.

(a) $\nu^+(E) = \sup\{\nu(F) \mid F \in \mathcal{M}, F \subseteq E\}$ and $\nu^-(E) = -\inf\{\nu(F) \mid F \in \mathcal{M}, F \subseteq E\}$.

(b) We have

$$|\nu|(E) = \sup\left\{ \sum_{i=1}^{n} |\nu(E_i)| \;\middle|\; n \in \mathbb{N}, E_1, \ldots, E_n \text{ disjoint}, \bigcup_{i=1}^{n} E_i = E \right\}.$$

Solution. (a) We prove the first identity, the second is proved similarly. Denote the supremum on the right-hand side by $\mu(E)$, and let $P \cup N$ be a Hahn decomposition for ν. Since $E \cap P \subseteq E$ we have

$$\nu^+(E) = \nu(E \cap P) \leq \mu(E).$$

Furthermore, for $F \in \mathcal{M}$ with $F \subseteq E$ notice that

$$\nu(F) = \nu^+(F) - \nu^-(F) \leq \nu^+(F) \leq \nu^+(E),$$

showing that $\mu(E) \leq \nu^+(E)$.

(b) Denote the quantity on the right-hand side by $\rho(E)$, and let $P \cup N$ be a Hahn decomposition for ν. The disjoint union $E = (E \cap P) \cup (E \cap N)$ yields

$$\rho(E) \geq |\nu(E \cap P)| + |\nu(E \cap N)| = \nu^+(E) + \nu^-(E) = |\nu|(E).$$

Conversely, let $E_1, \ldots, E_n$ be disjoint sets in $\mathcal{M}$ such that $\bigcup_{i=1}^n E_i = E$. For $i = 1, \ldots, n$ we have

$$|\nu(E_i)| = |\nu^+(E_i) - \nu^-(E_i)| \leq \nu^+(E_i) + \nu^-(E_i) = |\nu|(E_i),$$

implying that

$$\sum_{i=1}^n |\nu(E_i)| \leq \sum_{i=1}^n |\nu|(E_i) = |\nu|(E).$$

It follows that $\rho(E) \leq |\nu|(E)$. $\square$