

Folland: *Real Analysis*

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0 • Introduction

- We use the letter d to denote vector spaces dimensions, freeing up n to be used as an index, e.g. in sequences. In particular we write \mathbb{R}^d and \mathbb{C}^d .
- The Lebesgue measure on \mathbb{R}^d is denoted λ_d , and $\lambda = \lambda_1$.
- The symbol \mathbb{K} denotes either the real or complex numbers.
- The unit sphere in \mathbb{R}^{n+1} is denoted \mathbb{S}^n .
- We denote the power set of a set X by 2^X .
- The restriction of a function $f: X \rightarrow Y$ to a subset $A \subseteq X$ is denoted $f|_A$.
- Whenever we need to make the distinction, $\mathcal{L}^p(\mu)$ refers to the space of μ - p -integrable functions, while $L^p(\mu)$ denotes the quotient of $\mathcal{L}^p(\mu)$ with the subspace of functions that are zero μ -a.e.
- The space of bounded operators between normed spaces X and Y is denoted $\mathcal{B}(X, Y)$.
- The bounded and continuous complex-valued functions on a topological space X is denoted $C_b(X)$.
- A vector space equipped with an inner product is called an *inner product space*.

1 • Measures

1.2. σ -algebras

EXERCISE 1.1

Let \mathcal{M} be an infinite σ -algebra.

- (a) \mathcal{M} contains an infinite sequence of disjoint sets.

(b) $|\mathcal{M}| \geq \mathfrak{c}$.

Of course part (a) is trivial unless we require the sets to be nonempty.

SOLUTION. (a) We show by contraposition that there exists a nonempty set $A \in \mathcal{M}$ such that the restriction of \mathcal{M} to A^c is infinite. That is, assuming that no such set exists, we show that \mathcal{M} is finite. Pick any nonempty $A \in \mathcal{M}$. Then the restriction of \mathcal{M} to A and A^c respectively are both finite. For any $B \in \mathcal{M}$ we can write

$$B = (B \cap A) \cup (B \cap A^c).$$

But each set in the union lies in one of the restrictions, so there are finitely many decompositions like the one above, so there are finitely many sets $B \in \mathcal{M}$.

Now construct the sequence: Pick $A \in \mathcal{M}$ as above, restrict \mathcal{M} to A^c , and continue recursively.

(b) Let (A_n) be the sequence constructed above. There is an injection $\varphi: 2^{\mathbb{N}} \rightarrow \mathcal{M}$ given by $\varphi(I) = \bigcup_{i \in I} A_i$ (injectivity follows since the sets in the sequence are disjoint). Hence $|\mathcal{M}| \geq |2^{\mathbb{N}}| = \mathfrak{c}$. \square

1.3. Measures

EXERCISE 1.14

If μ is a semifinite measure and $\mu(E) = \infty$, for any $C > 0$ there exists $F \subseteq E$ with $C < \mu(F) < \infty$.

SOLUTION. Consider

$$S = \sup\{\mu(F) \mid F \subseteq E, \mu(F) < \infty\}.$$

If $S = \infty$, then the result is obvious. So assume towards a contradiction that $S < \infty$. For $n \in \mathbb{N}$ choose $F_n \subseteq E$ with $\mu(F_n) < \infty$ such that

$$S - \frac{1}{n} \leq \mu(F_n) \leq S.$$

Put $G_k = \bigcup_{n=1}^k F_n$. Then $G_k \subseteq E$ and $\mu(G_k) < \infty$, so the same inequality holds with F_n replaced by G_k . Now putting $G = \bigcup_{k \in \mathbb{N}} G_k$, continuity of μ gives

$$S - \frac{1}{n} \leq \mu(G) \leq S$$

for all $n \in \mathbb{N}$, so $\mu(G) = S$.

By assumption $\mu(E \setminus G) = \infty$, so $E \setminus G$ contains a set $G' \in \mathcal{M}$ such that $0 < \mu(G') < \infty$. But then

$$\mu(G \cup G') = \mu(G) + \mu(G') > S,$$

a contradiction. \square

EXERCISE 1.16

Let (X, \mathcal{M}, μ) be a measure space. A set $E \subseteq X$ is called *locally measurable* if $E \cap A \in \mathcal{M}$ for all $A \in \mathcal{M}$ such that $\mu(A) < \infty$. Let $\widetilde{\mathcal{M}}$ be the collection of all locally measurable sets. Clearly $\mathcal{M} \subseteq \widetilde{\mathcal{M}}$; if $\mathcal{M} = \widetilde{\mathcal{M}}$, then μ is called *saturated*.

- (a) If μ is σ -finite, then μ is saturated.
- (b) $\widetilde{\mathcal{M}}$ is a σ -algebra.
- (c) Define $\tilde{\mu}$ on $\widetilde{\mathcal{M}}$ by $\tilde{\mu}(E) = \mu(E)$ if $E \in \mathcal{M}$ and $\tilde{\mu}(E) = \infty$ otherwise. Then $\tilde{\mu}$ is a saturated measure on $\widetilde{\mathcal{M}}$, called the *saturation* of μ .
- (d) If μ is complete, so is $\tilde{\mu}$.
- (e) Suppose that μ is semifinite. For $E \in \widetilde{\mathcal{M}}$ define

$$\underline{\mu}(E) = \sup\{\mu(A) \mid A \in \mathcal{M} \text{ and } A \subseteq E\}.$$

Then $\underline{\mu}$ is a saturated measure on $\widetilde{\mathcal{M}}$ that extends μ .

- (f) Let X_1, X_2 be disjoint uncountable sets, $X = X_1 \cup X_2$, and \mathcal{M} the σ -algebra of countable or co-countable sets in X . Let μ_0 be counting measure on 2^{X_1} , and define μ on \mathcal{M} by $\mu(E) = \mu_0(E \cap X_1)$. Then μ is a measure on \mathcal{M} , $\widetilde{\mathcal{M}} = 2^X$, and in the notation of parts (c) and (e), $\tilde{\mu} \neq \underline{\mu}$.

SOLUTION. (a) Assume that μ is σ -finite, and let $E \subseteq X$ be locally measurable. Let $(A_n) \subseteq \mathcal{M}$ be such that $X = \bigcup_{n \in \mathbb{N}} A_n$ and $\mu(A_n) < \infty$. Then $E \cap A_n \in \mathcal{M}$, and so $E = \bigcup_{n \in \mathbb{N}} (E \cap A_n) \in \mathcal{M}$.

(b) Clearly we have $X \in \widetilde{\mathcal{M}}$. Then let $(E_n) \subseteq \widetilde{\mathcal{M}}$, and let $A \in \mathcal{M}$ with $\mu(A) < \infty$. Then

$$A \cap \bigcup_{n \in \mathbb{N}} E_n = \bigcup_{n \in \mathbb{N}} (A \cap E_n) \in \mathcal{M},$$

so $\bigcup_{n \in \mathbb{N}} E_n \in \widetilde{\mathcal{M}}$. Finally let $E \in \widetilde{\mathcal{M}}$ and $A \in \mathcal{M}$ with $\mu(A) < \infty$. Then

$$E^c \cap A = A \setminus E = A \setminus (E \cap A) = (E \cap A)^c \cap A \in \mathcal{M}$$

since $E \cap A \in \mathcal{M}$, so $E^c \in \widetilde{\mathcal{M}}$.

(c) We first show that $\tilde{\mu}$ is a measure. Clearly $\tilde{\mu}(\emptyset) = 0$, so let (E_n) be a sequence of disjoint sets in $\widetilde{\mathcal{M}}$, and let $E = \bigcup_{n \in \mathbb{N}} E_n$. Say that E_m does not lie in \mathcal{M} for some $m \in \mathbb{N}$. Then we must have $\tilde{\mu}(E) = \infty$, since otherwise $E \in \mathcal{M}$ with $\mu(E) < \infty$, and hence $E_m = E_m \cap E \in \mathcal{M}$. Thus we have

$$\sum_{n=1}^{\infty} \tilde{\mu}(E_n) \geq \tilde{\mu}(E_m) = \infty = \tilde{\mu}(E),$$

so $\sum_{n=1}^{\infty} \tilde{\mu}(E_n) = \tilde{\mu}(E)$. The same is obviously true if all E_n lie in \mathcal{M} .

Next we show that $\tilde{\mu}$ is saturated, i.e. that $\widetilde{\widetilde{\mathcal{M}}} \subseteq \widetilde{\mathcal{M}}$, so let $E \in \widetilde{\widetilde{\mathcal{M}}}$. For all $A \in \widetilde{\mathcal{M}}$ with $\tilde{\mu}(A) < \infty$ we then have $E \cap A \in \widetilde{\mathcal{M}}$. By definition of $\tilde{\mu}$ we must have $A \in \mathcal{M}$, so we also have

$$E \cap A = (E \cap A) \cap A \in \mathcal{M}.$$

And since this is true for all $A \in \mathcal{M}$ with $\mu(A) < \infty$, it follows that $E \in \widetilde{\mathcal{M}}$.

In some sense, the fact that $\tilde{\mu}$ is saturated is obvious: The more sets of finite measure, the harder it is to be saturated, and vice-versa. On the other hand, the sets of infinite measure are irrelevant, so since the only new sets in $\widetilde{\mathcal{M}}$ have infinite measure, they cannot affect whether the measure is saturated or not.

(d) Assume that μ is complete. Let $F \subseteq X$ be such that there is a set $E \in \widetilde{\mathcal{M}}$ with $F \subseteq E$ and $\tilde{\mu}(E) = 0$. Then also $E \in \mathcal{M}$, and since μ is complete we have $F \in \mathcal{M} \subseteq \widetilde{\mathcal{M}}$ as desired. Or more succinctly: Saturating a measure only introduces sets of infinite measure, so it does not introduce any null-sets.

(e) Assume that μ is semifinite. We first show that $\underline{\mu}$ is a measure. Clearly $\underline{\mu}(\emptyset) = 0$, so let $(E_n) \subseteq \widetilde{\mathcal{M}}$ be a sequence of disjoint sets. Clearly $\underline{\mu}$ is increasing, so sigma-additivity is obvious if any of the sets E_n have infinite measure. Assume then that $\mu(E_n) < \infty$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$, and choose $A_n \in \mathcal{M}$ such that $A_n \subseteq E_n$ and $\underline{\mu}(E_n) \leq \mu(A_n) + \varepsilon/2^n$. Then

$$\underline{\mu}\left(\bigcup_{n \in \mathbb{N}} E_n\right) \geq \underline{\mu}\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) \geq \sum_{n=1}^{\infty} \underline{\mu}(E_n) - \varepsilon.$$

Since this holds for all $\varepsilon > 0$, we obtain the first inequality. For the other inequality, let $E = \bigcup_{n \in \mathbb{N}} E_n$, and first assume that $\underline{\mu}(E) = \infty$. Pick $A \in \mathcal{M}$ with $A \subseteq E$. Since μ is semifinite, we can choose A such that $C < \mu(A) < \infty$ for any given $C > 0$. Letting $A_n = A \cap E_n \in \mathcal{M}$ we get

$$C < \mu(A) = \sum_{n=1}^{\infty} \mu(A_n) \leq \sum_{n=1}^{\infty} \underline{\mu}(E_n),$$

and since C is arbitrary, we get $\sum_{n=1}^{\infty} \underline{\mu}(E_n) = \infty$. If instead $\underline{\mu}(E) < \infty$, pick $A \subseteq E$ with $A \in \mathcal{M}$ and $\underline{\mu}(E) \leq \underline{\mu}(A) + \varepsilon$. Again letting $A_n = A \cap \bar{E}_n$ we get

$$\underline{\mu}(E) - \varepsilon \leq \underline{\mu}(A) = \sum_{n=1}^{\infty} \underline{\mu}(A_n) \leq \sum_{n=1}^{\infty} \underline{\mu}(E_n).$$

And since ε is arbitrary, we obtain the other inequality.

Next we show that $\underline{\mu}$ is saturated. Letting E be locally $\underline{\mu}$ -measurable, we must show that E is also locally $\underline{\mu}$ -measurable. So let $A \in \bar{\mathcal{M}}$ with $\underline{\mu}(A) < \infty$. Then $\underline{\mu}(A) < \infty$, and so $E \cap A \in \bar{\mathcal{M}}$. But then

$$E \cap A = (E \cap A) \cap A \in \mathcal{M},$$

as desired.

(f) It is pretty obvious that μ is a measure on \mathcal{M} . Then let $E \subseteq X$ and $A \in \mathcal{M}$ with $\mu(A) < \infty$. Then $A \cap X_1$ must be finite, and so A is not co-countable. But then it is countable, and so is $E \cap A$, hence $E \cap A \in \mathcal{M}$. Thus every subset of X is locally measurable.

Notice that μ is semifinite. We have $\tilde{\mu}(X_2) = \infty$ since $X_2 \notin \mathcal{M}$, but $\underline{\mu}(X_2) = 0$ since every subset of X_2 is disjoint from X_1 , and so it has measure zero. \square

1.4. Outer Measures

EXERCISE 1.18

Let $\mathcal{A} \subseteq 2^X$ be an algebra, \mathcal{A}_σ the collection of countable unions of sets in \mathcal{A} , and $\mathcal{A}_{\sigma\delta}$ the collection of countable intersections of sets in \mathcal{A}_σ . Let μ_0 be a premeasure on \mathcal{A} and μ^* the induced outer measure.

- (a) For any $E \subseteq X$ and $\varepsilon > 0$ there exists $A \in \mathcal{A}_\sigma$ with $E \subseteq A$ with $\mu^*(A) \leq \mu^*(E) + \varepsilon$.
- (b) If $\mu^*(E) < \infty$, then E is μ^* -measurable iff there exists $B \in \mathcal{A}_{\sigma\delta}$ with $E \subseteq B$ and $\mu^*(B \setminus E) = 0$.
- (c) If μ_0 is σ -finite, the restriction $\mu^*(E) < \infty$ in (b) is superfluous.

SOLUTION. (a) Let $E \subseteq X$ and $\varepsilon > 0$. The definition of μ^* yields a sequence $(A_n) \subseteq \mathcal{A}$ such that $E \subseteq \bigcup_{n \in \mathbb{N}} A_n$ and $\sum_{n=1}^{\infty} \mu_0(A_n) \leq \mu^*(E) + \varepsilon$. It follows that

$$\mu^*(E) + \varepsilon \geq \sum_{n=1}^{\infty} \mu_0(A_n) = \sum_{n=1}^{\infty} \mu^*(A_n) \geq \mu^*\left(\bigcup_{n \in \mathbb{N}} A_n\right).$$

(b) Let $E \subseteq X$. For $n \in \mathbb{N}$ there is a set $B_n \in \mathcal{A}_\sigma$ such that $E \subseteq B_n$ and $\mu^*(B_n) \leq \mu^*(E) + 1/n$. Letting $B = \bigcap_{n \in \mathbb{N}} B_n \in \mathcal{A}_{\sigma\delta}$ we get $\mu^*(B) \leq \mu^*(E)$, and since $E \subseteq B$ we also have the opposite inequality, so $\mu^*(B) = \mu^*(E)$.

Now assume that $\mu^*(E) < \infty$ and that E is μ^* -measurable. Then

$$\mu^*(B) = \mu^*(B \cap E) + \mu^*(B \cap E^c) = \mu^*(E) + \mu^*(B \setminus E),$$

from which it follows that $\mu^*(B \setminus E) = 0$.

Conversely, assume that there is a $B \in \mathcal{A}_{\sigma\delta}$ with $E \subseteq B$ and $\mu^*(B \setminus E) = 0$. Then B lies in the σ -algebra generated by \mathcal{A} , so it is μ^* -measurable. Let $A \subseteq X$. Then

$$\begin{aligned} \mu^*(A \cap E^c) &\leq \mu^*(A \cap E^c \cap B) + \mu^*(A \cap E^c \cap B^c) \\ &= \mu^*(A \cap (B \cup E)^c) \\ &= \mu^*(A \cap B^c), \end{aligned}$$

and so

$$\mu^*(A \cap E) + \mu^*(A \cap E^c) \leq \mu^*(A \cap B) + \mu^*(A \cap B^c) = \mu^*(A),$$

showing that E is μ^* -measurable. (Notice that we haven't used that $\mu^*(E) < \infty$ for the second implication.)

(c) We only need to prove the first implication above. By σ -finiteness of μ_0 , let (E_n) be a sequence of subsets of X such that $\mu^*(E_n) < \infty$ and $E = \bigcup_{n \in \mathbb{N}} E_n$. Let $\varepsilon > 0$. Then there are sets $A_n \in \mathcal{A}_\sigma$ such that $\mu^*(A_n) \leq \mu^*(E_n) + \varepsilon/2^n$. Letting $B_\varepsilon = \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}_\sigma$ we get

$$\mu^*(B_\varepsilon \setminus E) = \mu^*\left(\bigcup_{n \in \mathbb{N}} (A_n \cap E^c)\right) \leq \mu^*\left(\bigcup_{n \in \mathbb{N}} (A_n \cap E_n^c)\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n \setminus E_n) \leq \varepsilon.$$

Finally we let $B = \bigcap_{k \in \mathbb{N}} B_{1/k} \in \mathcal{A}_{\sigma\delta}$, and we get $\mu^*(B \setminus E) = 0$ as desired. \square

REMARK 1.1. Notice that (b) and (c) in particular show that any Lebesgue measurable set E , and therefore any Borel set, is the intersection of a G_δ set B and a Lebesgue null set $B \setminus E$. \lrcorner

EXERCISE 1.20

Let μ^* be an outer measure on X , \mathcal{M}^* the σ -algebra of μ^* -measurable sets, $\bar{\mu} = \mu^*|_{\mathcal{M}^*}$, and μ^+ the outer measure induced by $\bar{\mu}$ as in (1.12) (with $\bar{\mu}$ and \mathcal{M}^* replacing μ_0 and \mathcal{A}).

- (a) If $E \subseteq X$, we have $\mu^*(E) \leq \mu^+(E)$, with equality iff there exists $A \in \mathcal{M}^*$ with $A \supseteq E$ and $\mu^*(A) = \mu^*(E)$.
- (b) If μ^* is induced from a premeasure, then $\mu^* = \mu^+$.

(c) If $X = \{0, 1\}$, there exists an outer measure μ^* on X such that $\mu^* \neq \mu^+$.

SOLUTION. (a) Recall that the definition of μ^+ means that

$$\mu^+(E) = \inf \left\{ \sum_{n=1}^{\infty} \bar{\mu}(A_n) \mid A_n \in \mathcal{M}^*, E \subseteq \bigcup_{n \in \mathbb{N}} A_n \right\},$$

and that we by definition of $\bar{\mu}$ can replace $\bar{\mu}$ with μ^* . For any such sequence (A_n) we have

$$\mu^*(E) \leq \mu^* \left(\bigcup_{n \in \mathbb{N}} A_n \right) \leq \sum_{n=1}^{\infty} \mu^*(A_n) = \sum_{n=1}^{\infty} \bar{\mu}(A_n).$$

And since $\mu^+(E)$ is the infimum of all such sums, we have $\mu^*(E) \leq \mu^+(E)$.

Next assume that there is an $A \in \mathcal{M}^*$ with $E \subseteq A$ such that $\mu^*(A) = \mu^*(E)$. Using the sequence $A_1 = A$ and $A_n = \emptyset$ for $n > 1$ in the definition of μ^+ yields

$$\mu^+(E) \leq \bar{\mu}(A) = \mu^*(A) = \mu^*(E).$$

Hence $\mu^+(E) = \mu^*(E)$ as desired.

Conversely, assuming that $\mu^*(E) = \mu^+(E)$ we have

$$\mu^*(E) = \inf \left\{ \sum_{n=1}^{\infty} \mu^*(A_n) \mid A_n \in \mathcal{M}^*, E \subseteq \bigcup_{n \in \mathbb{N}} A_n \right\}.$$

Given $\varepsilon > 0$, choose a sequence (A_n) such that

$$\mu^* \left(\bigcup_{n \in \mathbb{N}} A_n \right) \leq \sum_{n=1}^{\infty} \mu^*(A_n) \leq \mu^*(E) + \varepsilon,$$

and let $B_\varepsilon = \bigcup_{n \in \mathbb{N}} A_n$. Letting $A = \bigcap_{k \in \mathbb{N}} B_{1/k} \in \mathcal{M}^*$ we thus have $\mu^*(A) \leq \mu^*(E)$.

(b) Assume that μ^* is induced from a premeasure on an algebra \mathcal{A} , and let $E \subseteq X$. Recall that \mathcal{A} consists of μ^* -measurable sets, so $\sigma(\mathcal{A}) \subseteq \mathcal{M}^*$. For $n \in \mathbb{N}$ choose, in accordance with Exercise 1.18(a), a set $A_n \in \mathcal{A}_\sigma$ with $E \subseteq A_n$ such that $\mu^*(A_n) \leq \mu^*(E) + 1/n$. Letting $A = \bigcap_{n \in \mathbb{N}} A_n$ we have $E \subseteq A$ and $\mu^*(A) \leq \mu^*(E)$. The other inequality is obvious, so $\mu^*(A) = \mu^*(E)$, and part (a) implies that $\mu^*(E) = \mu^+(E)$ as desired. \square

EXERCISE 1.21

Let μ^* be an outer measure induced from a premeasure and $\bar{\mu}$ the restriction of μ^* to the μ^* -measurable sets. Then $\bar{\mu}$ is saturated.

SOLUTION. Let \mathcal{A} denote the algebra on which the premeasure in question is defined, and denote by \mathcal{M}^* the σ -algebra of μ^* -measurable sets. Recall that $\mathcal{A} \subseteq \mathcal{M}^*$.

Let $E \subseteq X$ be locally measurable. It suffices to show that

$$\mu^*(F) \geq \mu^*(F \cap E) + \mu^*(F \cap E^c)$$

for all $F \subseteq X$ with $\mu^*(F) < \infty$. Given $\varepsilon > 0$, Exercise 1.18(a) yields a set $A \in \mathcal{A}_\sigma$ such that $\mu^*(A) \leq \mu^*(F) + \varepsilon$. Then $\mu^*(A) < \infty$, and so $E \cap A \in \mathcal{M}^*$. It follows that

$$\begin{aligned} \mu^*(F) + \varepsilon &\geq \mu^*(A) = \mu^*(A \cap (E \cap A)) + \mu^*(A \cap (E \cap A)^c) \\ &= \mu^*(A \cap E) + \mu^*(A \cap E^c) \\ &\geq \mu^*(F \cap E) + \mu^*(F \cap E^c), \end{aligned}$$

and hence $E \in \mathcal{M}^*$. Thus $\bar{\mu}$ is saturated. \square

EXERCISE 1.22

Let (X, \mathcal{M}, μ) be a measure space, μ^* the outer measure induced by μ according to (1.12), \mathcal{M}^* the σ -algebra of μ^* -measurable sets, and $\bar{\mu} = \mu^*|_{\mathcal{M}^*}$.

- (a) If μ is σ -finite, then $\bar{\mu}$ is the completion of μ .
- (b) In general, $\bar{\mu}$ is the saturation of the completion of μ .

SOLUTION. (a) Let $\bar{\mathcal{M}}$ be the σ -algebra from Theorem 1.9 (namely, the σ -algebra generated by the sets in \mathcal{M} along with all μ -null sets). This is clearly the smallest σ -algebra on which there can exist a complete extension of μ , so since $\bar{\mu}$ is also a complete extension of μ , we must have $\bar{\mathcal{M}} \subseteq \mathcal{M}^*$. Theorem 1.9 yields the uniqueness of a complete extension of μ on $\bar{\mathcal{M}}$, so it suffices to show that $\mathcal{M}^* \subseteq \bar{\mathcal{M}}$.

Now assume that μ is σ -finite, and let $E \in \mathcal{M}^*$. Then also $E^c \in \mathcal{M}^*$, and Exercise 1.18(c) ensures the existence of sets $B, D \in \mathcal{M}_{\sigma\delta} = \mathcal{M}$ with $E \subseteq B$ and $E^c \subseteq D$ such that

$$\mu^*(B \setminus E) = 0 \quad \text{and} \quad \mu^*(E \setminus D^c) = \mu^*(D \setminus E^c) = 0.$$

It follows that

$$\mu(B \setminus D^c) \leq \mu^*(B \setminus E) + \mu^*(E \setminus D^c) = 0,$$

so $E \setminus D^c$ is a μ -null set. Thus $E = D^c \cup (E \setminus D^c)$ is a union of a set in \mathcal{M} and a μ -null set, and hence $E \in \bar{\mathcal{M}}$.

(b) Let $\hat{\mu}$ denote the completion of μ on $\overline{\mathcal{M}}$, and let $\widetilde{\mathcal{M}}$ denote the σ -algebra of locally $\hat{\mu}$ -measurable sets. First we show that $\widetilde{\mathcal{M}} = \mathcal{M}^*$, so let $E \in \widetilde{\mathcal{M}}$. To show that E is μ^* -measurable it suffices to show that

$$\mu^*(F) \geq \mu^*(F \cap E) + \mu^*(F \cap E^c)$$

for all $F \subseteq X$ with $\mu^*(F) < \infty$. Calculations identical to the ones in the solution to Exercise 1.21 show this.

Conversely, let $E \in \mathcal{M}^*$ and consider $A \in \overline{\mathcal{M}}$ with $\hat{\mu}(A) < \infty$. Then also $A \in \mathcal{M}^*$, so $E \cap A \in \mathcal{M}^*$. The argument at the beginning of part (a) showed that $\bar{\mu}$ is an extension of $\hat{\mu}$, so $\mu^*(E \cap A) = \hat{\mu}(E \cap A) < \infty$. The same argument as in part (a), only now using Exercise 1.18(b) instead of (c), shows that $E \cap A \in \overline{\mathcal{M}}$, and so $E \in \widetilde{\mathcal{M}}$.

Finally, let $\tilde{\mu}$ denote the saturation of $\hat{\mu}$. We show that $\bar{\mu} = \tilde{\mu}$. Since the completion of μ on $\overline{\mathcal{M}}$ is unique, the two measures must agree here. Instead let $E \in \widetilde{\mathcal{M}} \setminus \overline{\mathcal{M}}$. By definition of $\tilde{\mu}$ we must then have $\tilde{\mu}(E) = \infty$. On the other hand, we just showed (for $E \cap A$ instead of E) that $\mu^*(E) < \infty$ implies $E \in \overline{\mathcal{M}}$. Since we have assumed that this is not the case, we must have $\bar{\mu}(E) = \mu^*(E) = \infty$. Thus $\bar{\mu} = \tilde{\mu}$. \square

1.5. Borel Measures on the Real Line

EXERCISE 1.25

If $E \subseteq \mathbb{R}$, the following are equivalent.

- (a) $E \in \mathcal{M}_\mu$.
- (b) $E = V \setminus N_1$ where V is a G_δ set and $\mu(N_1) = 0$.
- (c) $E = H \cup N_2$ where H is an F_σ set and $\mu(N_2) = 0$.

SOLUTION. Folland proves this claim when $\mu(E) < \infty$, so assume that $\mu(E) = \infty$. Since μ is σ -finite, there is a sequence $(E_n)_{n \in \mathbb{N}}$ in \mathcal{M}_μ with $\mu(E_n) < \infty$ for all $n \in \mathbb{N}$ and $E = \bigcup_{n \in \mathbb{N}} E_n$. Then there are sequences (H_n) of F_σ sets and (N_n) of null sets such that $E_n = H_n \cup N_n$. Then $H = \bigcup_{n \in \mathbb{N}} H_n$ is also an F_σ set and $N = \bigcup_{n \in \mathbb{N}} N_n$ a null set, and $E = H \cup N$.

Applying this to E^c yields a similar decomposition $E^c = H \cup N$. But then $E = H^c \setminus N$, and H^c is a G_δ set. \square

2 • Integration

2.1. Measurable Functions

EXERCISE 2.10

The following implications are valid iff the measure μ is complete:

- (a) If f is measurable and $f = g$ μ -a.e., then g is measurable.
- (b) If f_n is measurable for $n \in \mathbb{N}$ and $f_n \rightarrow f$ μ -a.e., then f is measurable.

SOLUTION. (a) Assume that μ is complete, and let $f, g: (X, \mathcal{E}, \mu) \rightarrow (Y, \mathcal{F})$ be functions from a measure space to a measurable space where f is $(\mathcal{E}, \mathcal{F})$ -measurable. Let $N = \{f \neq g\}$ and assume that $\mu(N) = 0$. Given $B \in \mathcal{F}$ we must show that $g^{-1}(B) \in \mathcal{E}$. But notice that

$$g^{-1}(B) = f^{-1}(B) \cup \{f \notin B, g \in B\} \setminus \{f \in B, g \notin B\},$$

and that the latter two sets are subsets of N , hence measurable. Thus $g^{-1}(B)$ is also measurable.

Conversely, let $N \subseteq X$ be a μ -null set. Then $\mathbf{1}_N = 0$ μ -a.e., so $\mathbf{1}_N$ and therefore N is measurable. Hence μ is complete.

(b) Assume that μ is complete, and consider the set A of points $x \in X$ such that $f_n(x)$ does not converge to $f(x)$. Then $f_n \mathbf{1}_{A^c} \rightarrow f \mathbf{1}_{A^c}$ pointwise everywhere, so Proposition 2.7 (or Corollary 2.9) implies that $f \mathbf{1}_{A^c}$ is measurable. By assumption we have $\mu(A) = 0$, so $f \mathbf{1}_{A^c} = f$ μ -a.e. and part (a) implies that f is measurable.

Conversely, let $N \subseteq X$ be a μ -null set and consider the sequence $(f_n)_{n \in \mathbb{N}}$ with $f_n = 0$ for all $n \in \mathbb{N}$. This converges μ -a.e. to $\mathbf{1}_N$ so N is measurable. Hence μ is complete. \square

2.7. Integration in Polar Coordinates

REMARK 2.1. We give a heuristic derivation of the radial measure ρ_d . Let dA be an area element in \mathbb{R}^2 . In polar coordinates (r, θ) this has a radial size of dr and an angular size of $r d\theta$. Notice that since θ is an angle, we multiply it by the distance r from the origin. Hence

$$dA = r d\theta dr = (r dr) d\theta.$$

Going up one dimension we introduce another angular coordinate φ , which contributes a factor $f(\theta, \varphi) r d\varphi$ to the volume element, where f is some function of the angular coordinates. Similarly when going up yet another dimension: this again introduces a factor r , and now f is a function of yet another angular coordinate. In d dimensions we have $d-1$ angular coordinates $\theta_1, \dots, \theta_{d-1}$, so the volume element is on the form

$$dV = f(\theta_1, \dots, \theta_{d-1}) r^{d-1} dr d\theta_1 \cdots d\theta_{d-1}.$$

The radial part is thus $r^{d-1} dr$, so it makes sense to define the radial measure ρ_d on $(0, \infty)$ by

$$\rho_d(E) = \int_E r^{d-1} dr. \quad \lrcorner$$

3 • Signed Measures and Differentiation

3.1. Signed Measures

EXERCISE 3.2

If ν is a signed measure, E is ν -null iff $|\nu|(E) = 0$. Also, if ν and μ are signed measures, $\nu \perp \mu$ iff $|\nu| \perp \mu$ iff $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.

SOLUTION. Assume that E is ν -null, and let $P \cup N$ be a Hahn decomposition for ν . Then

$$\nu^+(E) = \nu(E \cap P) = 0,$$

since $E \cap P \subseteq E$. Similarly we get $\nu^-(E) = 0$, so $|\nu|(E) = 0$. Conversely, assume that $|\nu|(E) = 0$. Then $\nu^\pm(F) = 0$ for all measurable $F \subseteq E$, and so $\nu(F) = 0$.

The other claims follow directly from the above. \square

EXERCISE 3.3

Let ν be a signed measure on (X, \mathcal{M}) .

(a) $L^1(\nu) = L^1(|\nu|)$.

(b) If $f \in L^1(\nu)$,

$$\left| \int f d\nu \right| \leq \int |f| d|\nu|.$$

(c) If $E \in \mathcal{M}$,

$$|\nu|(E) = \sup \left\{ \left| \int_E f d\nu \right| : |f| \leq 1 \right\}.$$

SOLUTION. (a) This follows directly from the definition of $L^1(\nu)$.

(b) For $f \in L^1(\nu)$ we have

$$\left| \int f d\nu \right| = \left| \int f d\nu^+ - \int f d\nu^- \right| \leq \int |f| d\nu^+ + \int |f| d\nu^- = \int |f| d|\nu|,$$

since $|\nu| = \nu^+ + \nu^-$.

(c) If $|f| \leq 1$, then

$$\left| \int_E f \, d\nu \right| \leq \int_E |f| \, d|\nu| \leq |\nu|(E),$$

showing one inequality. For the other inequality, let $P \cup N$ be a Hahn decomposition for ν , and let $f = \mathbf{1}_P - \mathbf{1}_N$. Then

$$\begin{aligned} \int_E f \, d\nu &= \int_E (\mathbf{1}_P - \mathbf{1}_N) \, d\nu^+ - \int_E (\mathbf{1}_P - \mathbf{1}_N) \, d\nu^- \\ &= \nu^+(E \cap P) - \nu^+(E \cap N) - \nu^-(E \cap P) + \nu^-(E \cap N) \\ &= \nu^+(E) + \nu^-(E) = |\nu|(E). \end{aligned} \quad \square$$

EXERCISE 3.4

If ν is a signed measure and λ, μ are positive measures such that $\nu = \lambda - \mu$, then $\lambda \geq \nu^+$ and $\mu \geq \nu^-$.

SOLUTION. Let $P \cup N$ be a Hahn decomposition for ν . Then

$$\nu^+(E) = \nu(E \cap P) = \lambda(E \cap P) - \mu(E \cap P) \leq \lambda(E \cap P) \leq \lambda(E),$$

and similarly for ν^- . \square

EXERCISE 3.5

If ν_1, ν_2 are signed measures that both omit the value ∞ or $-\infty$, then $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$.

SOLUTION. First notice that

$$\nu_1 + \nu_2 = (\nu_1^+ + \nu_2^+) - (\nu_1^- + \nu_2^-),$$

so by the previous exercise we have

$$|\nu_1 + \nu_2| = (\nu_1 + \nu_2)^+ + (\nu_1 + \nu_2)^- \leq (\nu_1^+ + \nu_2^+) + (\nu_1^- + \nu_2^-) = |\nu_1| + |\nu_2|. \quad \square$$

EXERCISE 3.7

Suppose that ν is a signed measure on (X, \mathcal{M}) and $E \in \mathcal{M}$.

(a) $\nu^+(E) = \sup\{\nu(F) \mid F \in \mathcal{M}, F \subseteq E\}$ and $\nu^-(E) = -\inf\{\nu(F) \mid F \in \mathcal{M}, F \subseteq E\}$.

(b) We have

$$|\nu|(E) = \sup \left\{ \sum_{i=1}^n |\nu(E_i)| \mid n \in \mathbb{N}, E_1, \dots, E_n \text{ disjoint}, \bigcup_{i=1}^n E_i = E \right\}.$$

SOLUTION. (a) We prove the first identity, the second is proved similarly. Denote the supremum on the right-hand side by $\mu(E)$, and let $P \cup N$ be a Hahn decomposition for ν . Since $E \cap P \subseteq E$ we have

$$\nu^+(E) = \nu(E \cap P) \leq \mu(E).$$

Furthermore, for $F \in \mathcal{M}$ with $F \subseteq E$ notice that

$$\nu(F) = \nu^+(F) - \nu^-(F) \leq \nu^+(F) \leq \nu^+(E),$$

showing that $\mu(E) \leq \nu^+(E)$.

(b) Denote the quantity on the right-hand side by $\rho(E)$, and let $P \cup N$ be a Hahn decomposition for ν . The disjoint union $E = (E \cap P) \cup (E \cap N)$ yields

$$\rho(E) \geq |\nu(E \cap P)| + |\nu(E \cap N)| = \nu^+(E) + \nu^-(E) = |\nu|(E).$$

Conversely, let E_1, \dots, E_n be disjoint sets in \mathcal{M} such that $\bigcup_{i=1}^n E_i = E$. For $i = 1, \dots, n$ we have

$$|\nu(E_i)| = |\nu^+(E_i) - \nu^-(E_i)| \leq \nu^+(E_i) + \nu^-(E_i) = |\nu|(E_i),$$

implying that

$$\sum_{i=1}^n |\nu(E_i)| \leq \sum_{i=1}^n |\nu|(E_i) = |\nu|(E).$$

It follows that $\rho(E) \leq |\nu|(E)$. □

4 • Point Set Topology

4.4. Compact Spaces

EXERCISE 4.38

Suppose that (X, \mathcal{T}) is a compact Hausdorff space and \mathcal{T}' is another topology on X . If \mathcal{T}' is strictly stronger than \mathcal{T} , then (X, \mathcal{T}') is Hausdorff but not compact. If \mathcal{T}' is strictly weaker than \mathcal{T} , then (X, \mathcal{T}') is compact but not Hausdorff.

SOLUTION. First assume that $\mathcal{T} \subseteq \mathcal{T}'$, and further assume that (X, \mathcal{T}') is compact. If $U \in \mathcal{T}'$ we then must show that $U \in \mathcal{T}$. Notice that $X \setminus U$ is closed in \mathcal{T}' and hence compact, so it is also compact in the weaker topology \mathcal{T} . Since \mathcal{T} is Hausdorff $X \setminus U$ is closed, but then $U \in \mathcal{T}$.

Next assume that $\mathcal{T}' \subseteq \mathcal{T}$ and that (X, \mathcal{T}') is Hausdorff. Let $U \in \mathcal{T}$ and fix $x \in U$. For each $y \notin U$ there is a pair of disjoint neighbourhoods V_y of x and W_y of y in \mathcal{T}' . The collection $\{W_y \mid y \notin U\}$ is an open cover of $X \setminus U$, and since this is closed in \mathcal{T} it is also compact, so there is a finite subcover W_{y_1}, \dots, W_{y_n} . Letting $V_x = V_{y_1} \cap \dots \cap V_{y_n}$, the set V_x is completely contained in U . But then U is the union of the sets V_x as x ranges over U , so U is a union of elements from \mathcal{T}' . Hence it is itself open in \mathcal{T}' . \square

4.5. Locally Compact Hausdorff Spaces

EXERCISE 4.49

Let X be a compact Hausdorff space and $E \subseteq X$.

- (a) If E is open, then E is locally compact in the relative topology.
- (b) b
- (c) c

SOLUTION. (a) Since every point in X has a compact neighbourhood (namely X itself), X is locally compact. So if $x \in E$ and E is open, then Proposition 4.30 yields a compact neighbourhood $K \subseteq E$ of x . But then K is also a compact neighbourhood of x in E , showing that E is locally compact.

(b) b

(c) c \square

REMARK 4.1. Let X be a compact Hausdorff space, and let $x_0 \in X$ be any point in X . Exercise 4.49(a) then shows that $X \setminus \{x_0\}$ is locally compact, so we may consider the one-point compactification $(X \setminus \{x_0\})^*$. We claim that $(X \setminus \{x_0\})^* \cong X$.

Denote the adjoined point by ∞ and consider the inclusion map $i: X \setminus \{x_0\} \rightarrow X$ extended to $(X \setminus \{x_0\})^*$ by letting $i(\infty) = x_0$. This is a bijection, and restricted to $X \setminus \{x_0\}$ it is a homeomorphism onto its image. We claim that i is itself a homeomorphism, and since both its domain and codomain are compact Hausdorff it suffices to show that it is continuous. So let $U \subseteq X$ be open. If $x_0 \notin U$ then $i^{-1}(U) = (i|_{X \setminus \{x_0\}})^{-1}(U)$, which is open in $X \setminus \{x_0\}$, hence open in $(X \setminus \{x_0\})^*$. Otherwise $x_0 \in U$ then $\infty \in i^{-1}(U)$, and we need to show that $i^{-1}(U)^c = i^{-1}(U^c)$ is compact. But U^c is a closed, hence compact, subset of $X \setminus \{0\}$, so its preimage under the homeomorphism $i|_{X \setminus \{x_0\}}$ is also compact. \square

EXERCISE 4.52

The one-point compactification of \mathbb{R}^n is homeomorphic to the sphere \mathbb{S}^n .

SOLUTION. Let $x_0 \in \mathbb{S}^n$ be any point on the sphere. By stereographic projection, $\mathbb{S}^n \setminus \{x_0\}$ and \mathbb{R}^n are homeomorphic. But then [Remark 4.1](#) shows that

$$(\mathbb{R}^n)^* \cong (\mathbb{S}^n \setminus \{x_0\})^* \cong \mathbb{S}$$

as desired. □

EXERCISE 4.55

Every open set in a second countable LCH space is σ -compact.

SOLUTION. Let X be a second countable LCH space, and let $U \subseteq X$ be open. By Proposition 4.31, for all $x \in U$ there is a precompact open set $V \subseteq X$ such that $x \in V \subseteq \overline{V} \subseteq U$. The collection of such V is an open cover of U , so there is a countable subcover \mathcal{V} by second countability (inherited to U , so U is Lindelöf). But then

$$U = \bigcup_{V \in \mathcal{V}} \overline{V},$$

so U is a countable union of compact sets. □

4.6. Two Compactness Theorems

REMARK 4.2: Completeness of uniform convergence on compact sets.

Let X be a σ -compact LCH space, and consider the metric from Exercise 56d on \mathbb{C}^X of uniform convergence on compact sets. We claim that $C(X)$ is complete in this metric.

Let $(f_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $C(X)$. For any compact $K \subseteq X$ the sequence $(f_n|_K)$ is then Cauchy in $C(K)$ (equipped with the uniform metric), hence converges to a unique function f_K by Proposition 4.13. Notice that $f_K = f_{K'}$ on $K \cap K'$ for any other compact set K' . The function $f: X \rightarrow \mathbb{C}$ given by $f(x) = f_K(x)$ if $x \in K$ is thus well-defined.

Since X is a k -space¹ by Lemma 4.37, it follows that f is continuous. Alternatively let $(K_j)_{j \in \mathbb{N}}$ be an exhaustion of X by compact sets, and note that f is continuous on K_j° . Since the K_j° form an open cover of X , f is then continuous on X . And f_n converges uniformly to f on compact sets since any compact set is covered by finitely many K_j° .

This result can be generalised to the space $C(X, Y)$, where X is a k -space and Y is a complete topological space, in that we note that the assumptions

¹ Recall that a topological space X is called a k -space if its topology is coherent with the set of compact subsets of X .

of σ -compactness (and Hausdorff) and $Y = \mathbb{R}$ are only required for uniform convergence on compact sets to be given by a metric. See e.g. Theorem 43.11 in Willard. \lrcorner

EXERCISE 4.62

Rephrase Theorem 4.44 in a form similar to Theorem 4.43 by using the metric in Exercise 56d.

SOLUTION. Theorem 4.44 may be phrased as follows:

Let X be a σ -compact LCH space. If \mathcal{F} is an equicontinuous, pointwise bounded subset of $C(X)$, then \mathcal{F} is totally bounded in the metric of uniform convergence on compact sets, and the closure of \mathcal{F} in $C(X)$ is compact.

Theorem 4.44 says that every sequence in \mathcal{F} has a subsequence converging to an element in $C(X)$, so $\overline{\mathcal{F}}$ is sequentially compact, hence compact. Then it is in particular totally bounded, so \mathcal{F} is also totally bounded. \square

4.7. The Stone–Weierstrass Theorem

REMARK 4.3. Notice that we never use the Hausdorff assumption in the proof of the Stone–Weierstrass theorem. However, if X is a topological space and there exists a family \mathcal{F} of functions in $C(X)$ or $C(X, \mathbb{R})$ that separates points in X , then X is automatically Hausdorff: For let $x \neq y$ be points in X , and let $f \in \mathcal{F}$ be such that $f(x) \neq f(y)$. Choosing disjoint neighbourhoods U_x and U_y of x and y respectively, $f^{-1}(U_x)$ and $f^{-1}(U_y)$ are disjoint neighbourhoods of x and y in X . Hence X is Hausdorff.

In other words, Hausdorff is not a necessary condition in the statement of the theorem, but rather follows from the other hypotheses.

In contrast, the compactness hypothesis is used very explicitly in the proof of Lemma 4.49. \lrcorner

EXERCISE 4.66

Let $1 - \sum_{n=1}^{\infty} c_n t^n$ be the Maclaurin series for $(1 - t)^{1/2}$.

- The series converges absolutely and uniformly on compact subsets of $(-1, 1)$, as does the termwise differentiated series $-\sum_{n=1}^{\infty} n c_n t^{n-1}$. Thus, if $f(t) = 1 - \sum_{n=1}^{\infty} c_n t^n$, then $f'(t) = -\sum_{n=1}^{\infty} n c_n t^{n-1}$.
- By explicit calculation, $f(t) = -2(1 - t)f'(t)$, from which it follows that $(1 - t)^{-1/2}f(t)$ is constant. Since $f(0) = 1$, $f(t) = (1 - t)^{1/2}$.

SOLUTION. (a) We first compute the coefficients c_n . If $g(t) = (1-t)^{1/2}$, then we claim that

$$g^{(n)}(t) = -\frac{(2n-3)(2n-5)\cdots(3)(1)}{2^n}(1-t)^{-(2n-1)/2}$$

for $n \in \mathbb{N}$ and $t \in (-1, 1)$. Indeed, this follows easily by induction. Hence

$$c_n = \frac{1}{n!}g^{(n)}(0) = -\frac{1}{n!}\frac{(2n-3)(2n-5)\cdots(3)(1)}{2^n}.$$

Now let $\rho \in (0, 1)$. Then

$$\left| \frac{c_{n+1}\rho^{n+1}}{c_n\rho^n} \right| = \frac{n!}{(n+1)!} \frac{2n-1}{2} \rho = \frac{2n-1}{2n} \rho \xrightarrow{n \rightarrow \infty} \rho < 1.$$

The ratio test then implies that the series $\sum_{n=1}^{\infty} c_n \rho^n$ converges, so it follows from the Weierstrass M-test that the series $1 - \sum_{n=1}^{\infty} c_n t^n$ converges absolutely and uniformly on the interval $[-\rho, \rho]$, and hence on all compact subsets of $(-1, 1)$. We similarly find that

$$\left| \frac{(n+1)c_{n+1}\rho^n}{nc_n\rho^{n-1}} \right| = \frac{n!}{(n+1)!} \frac{n+1}{n} \frac{2n-1}{2} \rho = \frac{n+1}{n} \frac{2n-1}{2n} \rho \xrightarrow{n \rightarrow \infty} \rho < 1,$$

so the series $-\sum_{n=1}^{\infty} nc_n t^{n-1}$ also converges as claimed.

(b) Notice that

$$\begin{aligned} -2(1-t)f'(t) &= 2(1-t) \sum_{n=1}^{\infty} nc_n t^{n-1} = 2 \sum_{n=1}^{\infty} nc_n t^{n-1} - 2 \sum_{n=1}^{\infty} nc_n t^n \\ &= 2 \sum_{n=0}^{\infty} (n+1)c_{n+1} t^n - 2 \sum_{n=1}^{\infty} nc_n t^n \\ &= 2 \sum_{n=0}^{\infty} ((n+1)c_{n+1} - nc_n) t^n. \end{aligned}$$

A short calculation shows that $(n+1)c_{n+1} - nc_n = c_n/2$, so the above equals $f(t)$ as claimed. Thus we have

$$\frac{d}{dt}(1-t)^{-1/2}f(t) = (1-t)^{-1/2}f'(t) + \frac{1}{2}(1-t)^{-3/2}f(t) = 0,$$

showing that $(1-t)^{-1/2}f(t)$ is constant. But $f(0) = 1$, so it follows that $f(t) = (1-t)^{1/2} = g(t)$. \square

5.1. Normed Vector Spaces

REMARK 5.1. We give a slightly different proof of Proposition 5.2.

Clearly if $T: X \rightarrow Y$ is continuous, then it is continuous at 0. And if this is so, then there is a $\delta > 0$ such that $\|h\| \leq \delta$ implies $\|Th\| \leq 1$, for $h \in X$. For all $x \in X$ we thus have

$$\|Tx\| = \frac{\|x\|}{\delta} \left\| T \left(\delta \frac{x}{\|x\|} \right) \right\| \leq \delta^{-1} \|x\|,$$

so T is bounded.

We let

$$\|T\| = \sup \{ \|Tx\| \mid x \in X, \|x\| \leq 1 \}$$

If T is bounded, then clearly $\|T\| < \infty$. If conversely $\|T\| < \infty$, then

$$\|Tx\| = \left\| T \frac{x}{\|x\|} \right\| \|x\| \leq \|T\| \|x\|$$

for all $x \neq 0$, so T is bounded. Furthermore, if $K > 0$ is such that $\|Tx\| \leq K\|x\|$ for all $x \in X$, then $\|Tx\| \leq K$ whenever $\|x\| \leq 1$. But then $\|T\| \leq K$. \square

REMARK 5.2: Riesz' lemma.

The statement of the lemma is as follows:

Let X be a normed vector space and M a proper closed subspace of X .

For $\alpha \in (0, 1)$ there exists an $x \in X$ with $\|x\| = 1$ such that

$$\inf_{m \in M} \|x - m\| \geq \alpha.$$

Since the quotient norm on X/M is given by $\|x + M\| = \inf_{m \in M} \|x - m\|$, this is precisely the statement of Exercise 5.12(b) [TODO: reference].

In Exercise 5.19(b) [TODO: reference] we use this to show that an infinite-dimensional normed vector space is not locally compact. It is easy to show that this is equivalent to the closed unit ball $\bar{B}_1(0)$ being compact.

Conversely, every normed space $(X, \|\cdot\|)$ of dimension $d < \infty$ is locally compact: Choose a linear isomorphism $T: \mathbb{C}^d \rightarrow X$ and let it induce a norm $\|\cdot\|'$ on X . With this norm T is an isometry, hence a homeomorphism, so the local compactness of \mathbb{C}^d is transferred to $(X, \|\cdot\|')$. But all norms on finite-dimensional vector spaces are equivalent, so $(X, \|\cdot\|)$ is also locally compact.

This equivalence of local compactness and finite-dimensionality generalises to Hausdorff topological vector spaces. This is known as *F. Riesz' theorem*. \square

EXERCISE 5.3

If Y is complete, so is $\mathcal{B}(X, Y)$.

SOLUTION. We prove the following lemma:

Let X and Y be normed spaces, and let $(T_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{B}(X, Y)$. If (T_n) is Cauchy in the operator norm and converges to some $T: X \rightarrow Y$ in the strong operator topology, then $T \in \mathcal{B}(X, Y)$ and $T_n \rightarrow T$ in the operator norm.

The map T is clearly linear. Choose $N \in \mathbb{N}$ such that $m, n \geq N$ implies that $\|T_n - T_m\| \leq \varepsilon$. For $x \in X$ and $n \geq N$ we then have

$$\|(T_n - T)x\| = \lim_{m \rightarrow \infty} \|(T_n - T_m)x\| \leq \limsup_{m \rightarrow \infty} \|T_n - T_m\| \|x\| \leq \varepsilon \|x\|.$$

Hence $T_n - T$ is bounded, and then so is T . Furthermore, $\|T_n - T\| \leq \varepsilon$, so $T_n \rightarrow T$ in the operator norm.

To prove the initial claim it thus suffices to produce, given a Cauchy sequence $(T_n)_{n \in \mathbb{N}}$, a map $T: X \rightarrow Y$ such that $\text{s-lim}_{n \rightarrow \infty} T_n = T$. But notice that we for $x \in X$ we have

$$\|T_n x - T_m x\| \leq \|T_n - T_m\| \|x\|$$

for $m, n \in \mathbb{N}$, so $(T_n x)_{n \in \mathbb{N}}$ is a Cauchy sequence in Y . Defining $T: X \rightarrow Y$ by $Tx = \lim_{n \rightarrow \infty} T_n x$, T is the strong limit of T_n by construction. \square

EXERCISE 5.4

If X and Y are normed spaces, the map $(T, x) \mapsto Tx$ is continuous from $\mathcal{B}(X, Y) \times X$ to Y .

SOLUTION. If $T, S \in \mathcal{B}(X, Y)$ and $x, y \in X$, then

$$\|Tx - Sy\| \leq \|Tx - Ty\| + \|Ty - Sy\| \leq \|T\| \|x - y\| + \|T - S\| \|y\|.$$

The claim follows.

Notice that this proof is identical to the proof that multiplication in a Banach algebra is continuous, but the Banach inequality is replaced with the inequality $\|Tx\| \leq \|T\| \|x\|$. The proof is also almost identical to the proof that multiplication on \mathbb{R} or \mathbb{C} is continuous, except here we have the *equality* $|xy| = |x||y|$. \square

EXERCISE 5.6

Suppose that X is a finite-dimensional vector space. Let (e_1, \dots, e_d) be a basis for X , and define $\|\sum_{i=1}^d a_i e_i\|_1 = \sum_{i=1}^d |a_i|$.

- (a) $\|\cdot\|_1$ is a norm on X .
- (b) The map $T: (a_1, \dots, a_d) \mapsto \sum_{i=1}^d a_i e_i$ is continuous from \mathbb{K}^d with the usual Euclidean topology to X with the topology defined by $\|\cdot\|_1$.
- (c) The set $S = \{x \in X \mid \|x\|_1 = 1\}$ is compact in the topology defined by $\|\cdot\|_1$.
- (d) All norms on X are equivalent.

SOLUTION. (a) This is obvious.

(b) If we equip \mathbb{K}^d with the 1-norm, then T is an isometry and thus continuous (in fact a homeomorphism since it is surjective).

(c) Since the unit sphere in \mathbb{K}^d (with respect to the 1-norm) is compact and T is continuous, S is also compact.

(d) If $\|\cdot\|$ is any norm on X , we need to find $C_1, C_2 > 0$ such that

$$C_1 \|x\|_1 \leq \|x\| \leq C_2 \|x\|_1 \quad (5.1)$$

for all $x \in X$. This is obvious for $x = 0$, and if $x \neq 0$ we may divide through by $\|x\|_1$. The claim is then that

$$C_1 \leq \|x\| \leq C_2$$

for all $x \in X$ with $\|x\|_1 = 1$, i.e. all $x \in S$. We first show that $\|\cdot\|$ is continuous with respect to $\|\cdot\|_1$. For $x = \sum_{i=1}^d a_i e_i$ and $y = \sum_{i=1}^d b_i e_i$ in X we have

$$\|x - y\| = \left\| \sum_{i=1}^d (a_i - b_i) e_i \right\| \leq \sum_{i=1}^d |a_i - b_i| \|e_i\| \leq \|x - y\|_1 \max_{1 \leq i \leq d} \|e_i\|.$$

Continuity of $\|\cdot\|$ now follows from the reverse triangle inequality. (In fact, this calculation also proves the second inequality of (5.1), but we give a second argument below.)

Since $\|\cdot\|$ is continuous and S is compact with respect to $\|\cdot\|_1$, $\|\cdot\|$ has a minimum and maximum on S . That is, there exist $x_0, x_1 \in S$ such that

$$\|x_0\| \leq \|x\| \leq \|x_1\|$$

for all $x \in S$. And since both of x_0 and x_1 are nonzero then so are their norms, proving the claim. \square

EXERCISE 5.9

Let $C^k([0, 1])$ be space of functions on $[0, 1]$ possessing continuous derivatives up to order k on $[0, 1]$, including onesided derivatives at the endpoints.

- (a) If $f \in C([0, 1])$, then $f \in C^k([0, 1])$ iff f is k times continuously differentiable on $(0, 1)$ and $f^{(j)}(0+) = \lim_{x \downarrow 0} f^{(j)}(x)$ and $f^{(j)}(1-) = \lim_{x \uparrow 1} f^{(j)}(x)$ exist for $j \leq k$.
- (b) $\|f\| = \sum_{j=0}^k \|f^{(j)}\|_\infty$ is a norm on $C^k([0, 1])$ that makes $C^k([0, 1])$ into a Banach space.

SOLUTION. (a) The ‘only if’ part is obvious. Conversely, we show by induction in j that $f \in C^j([0, 1])$ for $j = 0, \dots, k$. This is true for $j = 0$ by assumption, so assume that it is true for some j . For $x \in (0, 1)$ there is a $\xi \in (0, x)$ such that $f^{(j)}(x) - f^{(j)}(0) = f^{(j+1)}(\xi)(x - 0)$. It follows that

$$\frac{f^{(j)}(x) - f^{(j)}(0)}{x - 0} = f^{(j+1)}(\xi) \xrightarrow{x \downarrow 0} f^{(j+1)}(0+).$$

Thus $f^{(j)}$ has a one-sided derivative at 0, and since the derivative is precisely the limit $f^{(j+1)}(0+)$, this also shows that $f^{(j+1)}$ is continuous at 0. Similarly at 1, so $f \in C^{j+1}([0, 1])$ as desired.

(b) Let $(f_n)_{n \in \mathbb{N}}$ be a sequence in $C^1([0, 1])$ converging to a function f , such that the sequence (f'_n) converges uniformly in $C([0, 1])$ to a function g . Let $\varepsilon > 0$, and choose $N \in \mathbb{N}$ such that $n \geq N$ implies that $\|f'_n - g\|_\infty < \varepsilon$. For $n \geq N$ and fixed $x \in [0, 1]$ we then have

$$\left| \int_0^x f'_n(t) dt - \int_0^x g(t) dt \right| \leq \int_0^x |f'_n(t) - g(t)| dt \leq \varepsilon x.$$

It follows that

$$f(x) - f(0) = \lim_{n \rightarrow \infty} (f_n(x) - f_n(0)) = \lim_{n \rightarrow \infty} \int_0^x f'_n(t) dt = \int_0^x g(t) dt.$$

Thus we see that $f \in C^1([0, 1])$ with $f' = g$.

Now let $(f_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $C^k([0, 1])$. Then the sequences $(f_n^{(j)})$ are Cauchy sequences in $C([0, 1])$ for $j = 0, \dots, k$, and so the sequences have uniform limits. But then we are in the situation above, so it follows by induction that $f_n^{(j)} \rightarrow f^{(j)}$ uniformly for all j . Hence $f_n \rightarrow f$ in $C^k([0, 1])$, so this is a Banach space. \square

REMARK 5.3. As an application of the above we consider the following: Let $D: C^k([0, 1]) \rightarrow C^{k-1}([0, 1])$ be the differential operator $f \mapsto f'$. We claim that this is bounded with respect to the above norm. For $f \in C^k([0, 1])$ we have

$$\|Df\| = \sum_{j=0}^{k-1} \|(Df)^{(j)}\|_{\infty} = \sum_{j=0}^{k-1} \|f^{(j+1)}\|_{\infty} = \sum_{j=1}^k \|f^{(j)}\|_{\infty} \leq \|f\|.$$

The usual counterexamples to the boundedness of D on e.g. $(C^1([0, 1]), \|\cdot\|_{\infty})$ do not work here. The norm $\|\cdot\|$ in effect takes into account the fact that functions that take on similar values may have derivatives that vary wildly. \lrcorner

EXERCISE 5.12

Let X be a normed vector space and M a proper closed subspace of X .

- (a) $\|x + M\| = \inf_{m \in M} \|x + m\|$ is a norm on X/M .
- (b) For any $\varepsilon > 0$ there exists $x \in X$ such that $\|x\| = 1$ and $\|x + M\| \geq 1 - \varepsilon$.
- (c) The projection map $\pi: X \rightarrow X/M$ has norm 1.
- (d) If X is complete, so is X/M .
- (e) The topology defined by the quotient norm is the quotient topology.

SOLUTION. (a) Assume first that M is not necessarily closed. For $x \in X$ and $\alpha \in \mathbb{K} \setminus \{0\}$ we have

$$\|\alpha(x + M)\| = \|\alpha x + M\| = \inf_{m \in M} \|\alpha x + m\| = |\alpha| \inf_{m \in M} \|x + \alpha^{-1}m\| = |\alpha| \|x + M\|,$$

where the last equality follows since every element in M is on the form $\alpha^{-1}m$ for some $m \in M$. Hence the map $\|\cdot\|$ on X/M is absolutely homogeneous.

For the triangle inequality, let $x, y \in X$ and $m, m' \in M$. Then

$$\|(x + M) + (y + M)\| = \|(x + y) + M\| \leq \|x + y + m + m'\| \leq \|x + m\| + \|y + m'\|,$$

which implies that

$$\|(x + M) + (y + M)\| \leq \|x + M\| + \|y + M\|$$

as desired. Hence $\|\cdot\|$ is a seminorm on X/M for any M .

Finally assume that M is closed, and let $x \in X \setminus M$. Then there exists an $r > 0$ such that $B_r(x) \cap M = \emptyset$, so $\|x - m\| \geq r$ for all $m \in M$. Hence $\|x + M\| \geq r > 0$ as desired.

(b) Let $\varepsilon > 0$, and pick some $y \in X \setminus M$. By definition of the quotient norm there exists an $m \in M$ such that

$$\frac{\|y + M\|}{\|y - m\|} \geq 1 - \varepsilon.$$

Letting $x = (y - m)/\|y - m\|$ we have $\|x\| = 1$ and

$$\|x + M\| = \left\| \frac{y - m}{\|y - m\|} + M \right\| = \frac{\|y + M\|}{\|y - m\|} \geq 1 - \varepsilon$$

as desired.

(c) For any $x \in X$ we have $\|x + M\| \leq \|x + 0\|$, so $\|\pi\| \leq 1$. But given $\varepsilon > 0$, (b) shows that $\|x + M\| \geq 1 - \varepsilon$ for some $x \in X$ with $\|x\| = 1$, so $\|\pi\| \geq 1 - \varepsilon$. Since ε was arbitrary, $\|\pi\| \geq 1$.

(d) We use Theorem 5.1. Let $\sum_{n=1}^{\infty} \xi_n$ be an absolutely convergent series with terms in X/M . For each $n \in \mathbb{N}$ there exists an $x_n \in X$ such that $\xi_n = x_n + M$ and such that $\|x_n\| \leq \|\xi_n\| + 2^{-n}$. It follows that

$$\sum_{n=1}^{\infty} \|x_n\| \leq \sum_{n=1}^{\infty} (\|\xi_n\| + 2^{-n}) = \sum_{n=1}^{\infty} \|\xi_n\| + 1 < \infty,$$

so by completeness of X , Theorem 5.1 implies that the series $\sum_{n=1}^{\infty} x_n$ converges to some $x \in X$. Since π is continuous, it follows that $\sum_{n=1}^{\infty} x_n + M$ converges to $x + M$, so X/M is complete by Theorem 5.1.

(e) The projection map $\pi: X \rightarrow X/M$ is continuous in the norm topology by (c), so the quotient topology is coarser than the norm topology. To prove the opposite inclusion we show that π is a quotient map. It suffices to show that π is open. To this end we prove the following lemma:

Let X be a (semi)normed vector space and M a subspace of X . For $r > 0$ and $x \in X$ we have

$$B_r(\pi(x)) = \pi(B_r(x)).$$

By homogeneity it suffices to consider the case $x = 0$. By (b) we have $\|\pi\| = 1$, so for $y \in B_r(0)$ we have

$$\|y + M\| \leq \|y\| < r,$$

and so $y + M \in B_r(0 + M)$, proving the inclusion ' \supseteq '. For the opposite inclusion, for $y + M \in B_r(0 + M)$ we have

$$\inf_{m \in M} \|y + m\| = \|y + M\| < r,$$

so there is an $m \in M$ such that $\|y + m\| < r$. Hence $y + m \in B_r(0)$, and so

$$y + M = \pi(y + m) \in \pi(B_r(0)),$$

proving the inclusion ' \subseteq '.

In particular, the image of an open ball under π is an open ball. It now follows that π is open, since every open set is a union of open balls. \square

EXERCISE 5.15

Suppose that X and Y are normed vector spaces and $T \in \mathcal{B}(X, Y)$. Let $\mathcal{N}(T) = \{x \in X \mid Tx = 0\}$.

- (a) $\mathcal{N}(T)$ is a closed subspace of X
- (b) Let M be a closed subspace of X contained in $\mathcal{N}(T)$. There is a unique bounded $\tilde{T}: X/M \rightarrow Y$ such that $T = \tilde{T} \circ \pi$, where $\pi: X \rightarrow X/M$ is the projection. Moreover, $\|\tilde{T}\| = \|T\|$.

Our version of [TODO: link] (b) is more general than Folland's.

SOLUTION. (a) This is obvious since T is continuous.

(b) Basic linear algebra yields a unique (not necessarily bounded) linear map $\tilde{T}: X/M \rightarrow Y$ such that $T = \tilde{T} \circ \pi$. To compute its norm we use the lemma from the solution to [TODO: Exercise 5.12(e)]. Let $B = B_1(0) \subseteq X$ and $\tilde{B} = \pi(B) = B_1(0 + M) \subseteq X/M$. Then

$$\begin{aligned} \|\tilde{T}\| &= \sup\{\|\tilde{T}\xi\| \mid \xi \in \tilde{B}\} \\ &= \sup\{\|\tilde{T}\xi\| \mid \xi \in \pi(B)\} \\ &= \sup\{\|\tilde{T}(\pi(x))\| \mid x \in B\} \\ &= \sup\{\|Tx\| \mid x \in B\} \\ &= \|T\|. \end{aligned}$$

Here we use the fact that for an operator $T: X \rightarrow Y$ it suffices to consider $x \in X$ with $\|x\| < 1$ in computing its norm: For assume that $\|x\| = 1$, and let $\varepsilon_n = 1 - 1/n$. Then $\|\varepsilon_n x\| < 1$, and

$$\|Tx\| = \frac{1}{\varepsilon_n} \|T(\varepsilon_n x)\| \leq \frac{1}{\varepsilon_n} \sup\{\|Ty\| \mid y \in B\} \xrightarrow{n \rightarrow \infty} \sup\{\|Ty\| \mid y \in B\}.$$

Hence $\|T\| \leq \sup\{\|Ty\| \mid y \in B\}$, and the opposite equality is obvious. \square

5.2. Linear Functionals

EXERCISE 5.18

Let X be a normed vector space.

- (a) If M is a closed subspace and $x \in X \setminus M$, then $M + \mathbb{C}x$ is closed.
- (b) Every finite-dimensional subspace of X is closed.

SOLUTION. (a) Let $(y_n)_{n \in \mathbb{N}}$ and $(\lambda_n)_{n \in \mathbb{N}}$ be sequences in M and \mathbb{C} respectively such that $y_n + \lambda_n x$ converges to some $z \in X$. By Theorem 5.8(b) there is a $\varphi \in X^*$ such that $\varphi(x) \neq 0$ and $\varphi|_M = 0$. Applying φ to the above sequence yields

$$\varphi(z) = \lim_{n \rightarrow \infty} (\varphi(y_n) + \lambda_n \varphi(x)) = \left(\lim_{n \rightarrow \infty} \lambda_n \right) \varphi(x),$$

which implies that λ_n converges to $\varphi(z)/\varphi(x)$. The sequence (y_n) is then also convergent with limit in M , and so

$$\lim_{n \rightarrow \infty} (y_n + \lambda_n x) = \lim_{n \rightarrow \infty} \left(y_n + \frac{\varphi(z)}{\varphi(x)} x \right) = \lim_{n \rightarrow \infty} y_n + \frac{\varphi(z)}{\varphi(x)} x,$$

which lies in $M + \mathbb{C}x$ as desired.

(b) We give two different arguments. If U is a finite-dimensional subspace of X and (e_1, \dots, e_d) is a basis for U , then $U = \sum_{i=1}^d \mathbb{C}e_i$. Since $\{0\}$ is a closed subspace of X , the desired result follows from the above by induction.

We may also argue as follows: It suffices to show that U is complete. To this end, let $(x_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in U and write $x_n = \lambda_{n1}e_1 + \dots + \lambda_{nd}e_d$. We claim that the sequence $(\lambda_{ni})_{n \in \mathbb{N}}$ is a Cauchy sequence for all i . For the norm $\|\cdot\|$ on U inherited from X is equivalent to the 1-norm $\|\cdot\|_1$, so

$$\|x_m - x_n\| \geq C\|x_m - x_n\|_1 \geq C|\lambda_{mi} - \lambda_{ni}|$$

for some $C > 0$. Since \mathbb{C} is complete, the sequence $(\lambda_{ni})_{n \in \mathbb{N}}$ converges to some $\lambda_i \in \mathbb{C}$. Letting $x = \lambda_1 e_1 + \dots + \lambda_d e_d$, we claim that $x_n \rightarrow x$ as $n \rightarrow \infty$. This follows since (choosing the e_i to be unit vectors)

$$\begin{aligned} \|x_n - x\| &= \|(\lambda_{n1} - \lambda_1)e_1 + \dots + (\lambda_{nd} - \lambda_d)e_d\| \\ &\leq |\lambda_{n1} - \lambda_1| + \dots + |\lambda_{nd} - \lambda_d|, \end{aligned}$$

and the right-hand side converges to zero. □

EXERCISE 5.19

Let X be an infinite-dimensional normed vector space.

- (a) There is a sequence $(x_n)_{n \in \mathbb{N}}$ in X such that $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and $\|x_n - x_m\| \geq 1/2$ for $m \neq n$.

(b) X is not locally compact.

SOLUTION. (a) First pick any unit vector $x_1 \in X$. By Exercise 5.18 the subspace $M_1 = \mathbb{C}x_1$ is closed, so Exercise 5.12(b) yields a unit vector $x_2 \notin M_1$ such that $\|x_2 + M_1\| \geq 1/2$. Since $x_1 \in M_1$ we in particular have $\|x_2 - x_1\| \geq 1/2$. Similarly, letting $M_2 = M_1 + \mathbb{C}x_2$ we get a unit vector $x_3 \notin M_2$ with $\|x_3 + M_2\| \geq 1/2$. Since both x_1 and x_2 lie in M_2 we have $\|x_3 - x_1\| \geq 1/2$ and $\|x_3 - x_2\| \geq 1/2$. Continuing this process yields the desired sequence. [TODO: Exercise references]

(b) Assume towards a contradiction that X is locally compact. Then $0 \in X$ has a compact neighbourhood K , and by multiplying with an appropriate scalar we may assume that K contains the closed unit ball $\bar{B}_1(0)$. Thus K contains the sequence (x_n) constructed in part (a). Now Theorem 0.25 implies that K is sequentially compact, so (x_n) has a convergent subsequence. But this is impossible since $\|x_n - x_m\| \geq 1/2$ for $m \neq n$, so X is not locally compact. \square

REMARK 5.4. Let X and Y be normed spaces, and let $T \in \mathcal{B}(X, Y)$. If T is an isometry, then clearly $\|T\| = 1$. It is easy to think that the converse is also true, perhaps if T is also assumed to be boundedly invertible, but this is not the case: For instance, equip \mathbb{R}^2 with the supremum norm² and consider the operator $S: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by $S(x, y) = (x, y/2)$. Then $\|S\| = 1$, and $S^{-1}(x, y) = (x, 2x)$ is also bounded with $\|S^{-1}\| = 2$. But S is clearly not an isometry, since e.g.

$$\|S(0, 2)\|_\infty = \|(0, 1)\|_\infty = 1 \neq 2 = \|(0, 2)\|_\infty.$$

The problem is already apparent, in that the norm of S^{-1} is *not* 1, so S^{-1} cannot be an isometry. This motivates the following result:

Let $T \in \mathcal{B}(X, Y)$ be a boundedly invertible map between normed spaces such that $\|T\| = \|T^{-1}\| = 1$. Then T is an isometry.

For if $x \in X$ and $y = Tx$, then

$$\|Tx\| \leq \|T\|\|x\| = \|x\| = \|T^{-1}y\| \leq \|T^{-1}\|\|y\| = \|y\| = \|Tx\|,$$

so $\|Tx\| = \|x\|$. \lrcorner

REMARK 5.5: The categories \mathbf{Nor} and \mathbf{Nor}_1 of normed spaces.

A map $f: (S, \rho) \rightarrow (T, \delta)$ between metric spaces having the property that

$$\delta(f(x), f(y)) \leq \rho(x, y)$$

² In Remark 5.5 we will see that this makes \mathbb{R}^2 into the categorical product of \mathbb{R} and \mathbb{R} . This has no relevance to the present discussion, as far as I know.

for all $x, y \in S$ is variously called a *short map*, a *metric map*, *nonexpansive* or *-expanding*, a *weak contraction*, or just a Lipschitz function with Lipschitz constant 1. We consider the category \mathbf{Nor}_1 whose objects are normed spaces and whose arrows are linear maps that are also short maps. Notice that a linear map $T: X \rightarrow Y$ between normed spaces is short just when $\|T\| \leq 1$. Hence \mathbf{Nor}_1 is a subcategory of the category \mathbf{Nor} of normed spaces and bounded linear maps.

A bounded linear map $T: X \rightarrow Y$ is an isomorphism in \mathbf{Nor} just when it is boundedly invertible. In \mathbf{Nor}_1 the situation is slightly more complicated: The map S in [Remark 5.4](#) is a short map but its inverse is not short. Hence the isomorphisms in \mathbf{Nor}_1 are the boundedly invertible maps with short inverses, and this latter assumption cannot be removed. Furthermore, we claim that in this case T is in fact an isometry. If T has a bounded inverse T^{-1} , then

$$1 = \|\mathrm{id}_X\| = \|T^{-1}T\| \leq \|T^{-1}\| \|T\|.$$

Hence if both T and T^{-1} are short maps, then $\|T\| = \|T^{-1}\| = 1$. But then [Remark 5.4](#) implies that T is an isometry. Conversely, surjective isometries³ are clearly short maps whose inverses are also short, so any surjective isometry is an isomorphism in \mathbf{Nor}_1 .

If X and Y are normed spaces we may equip the Cartesian product $X \times Y$ with different norms, two of which are of particular importance here, namely the supremum norm⁴ $\|(x, y)\|_\infty = \max\{\|x\|, \|y\|\}$ and the 1-norm $\|(x, y)\|_1 = \|x\| + \|y\|$. We reserve the notation $X \times Y$ for the Cartesian product equipped with the supremum norm, and we use the notation $X \oplus Y$ when we equip the Cartesian product with the 1-norm.

We claim that $X \times Y$ is a categorical product of X and Y . First notice that the projections $\pi_X: X \times Y \rightarrow X$ and $\pi_Y: X \times Y \rightarrow Y$ are indeed short maps. For instance,

$$\|\pi_X(x, y)\| = \|x\| \leq \max\{\|x\|, \|y\|\} = \|(x, y)\|_\infty.$$

Given short linear maps $T_X: Z \rightarrow X$ and $T_Y: Z \rightarrow Y$, the map $T: Z \rightarrow X \times Y$ given by $Tz = (T_X z, T_Y z)$ is certainly linear. It is also short, for

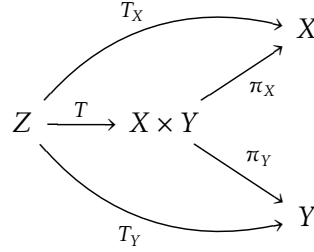
$$\|Tz\|_\infty = \|(T_X z, T_Y z)\|_\infty = \max\{\|T_X z\|, \|T_Y z\|\} \leq \|z\|.$$

Notice that the 1-norm would not in general make T into a short map, but that the supremum norm is in some sense natural: Bounding a pair (x, y) just

³ An isometry is in particular injective, so surjective isometries are bijective. The inverse is also clearly bounded.

⁴ We denote any norm on a vector space other than $X \times Y$ by $\|\cdot\|$, relying on context to distinguish.

means bounding *both* x and y separately. Furthermore, it clearly makes the diagram



commute, and it is (even in **Set**) unique with this property, so $X \times Y$ is indeed a product of X and Y .

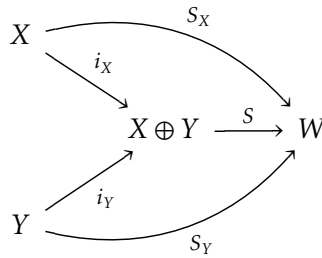
Next we claim that $X \oplus Y$ is a coproduct of X and Y . The inclusion maps $i_X: X \rightarrow X \oplus Y$ and $i_Y: Y \rightarrow X \oplus Y$ are given by $i_X(x) = (x, 0)$ and $i_Y(0, y)$. Notice that e.g.

$$\|i_X(x)\|_1 = \|(x, 0)\|_1 = \|x\| + \|0\| = \|x\|,$$

so the inclusion maps are isometries, in particular short maps. Furthermore, if $S_X: X \rightarrow W$ and $S_Y: Y \rightarrow W$ are short linear maps, we define a map $S: X \oplus Y \rightarrow W$ by $S(x, y) = S_X x + S_Y y$. This is then clearly linear, and it is also short since

$$\|S(x, y)\| = \|S_X x + S_Y y\| \leq \|S_X x\| + \|S_Y y\| \leq \|x\| + \|y\| = \|(x, y)\|_1.$$

Again notice that the supremum norm would not make S into a short map. But the 1-norm is natural in the sense that elements of $X \oplus Y$ are to be thought of, in some sense, *sums* of elements in X and Y . Hence the norm of such a sum is (naturally) the sum of the norms. Finally, it clearly makes the diagram



commute, and so $X \oplus Y$ is a coproduct of X and Y as claimed.

For completeness we note that the categories **Ban** and **Ban**₁ of Banach spaces and, respectively, bounded and short linear maps are full subcategories of **Nor** and **Nor**₁. If X and Y are Banach spaces, then so are $X \times Y$ and $X \oplus Y$: If $((x_n, y_n))_{n \in \mathbb{N}}$ is a Cauchy sequence in either, then (x_n) and (y_n) are Cauchy in X and Y respectively, converging to $x \in X$ and $y \in X$. We then have

$$\|(x_n, y_n) - (x, y)\|_\infty = \|(x_n - x, y_n - y)\|_\infty = \max\{\|x_n - x\|, \|y_n - y\|\},$$

which goes to zero as $n \rightarrow \infty$. We similarly have

$$\|(x_n, y_n) - (x, y)\|_1 = \|x_n - x\| + \|y_n - y\|,$$

which similarly goes to zero. In either case (x_n, y_n) converges to (x, y) . Thus $X \times Y$ and $X \oplus Y$ are also a product and coproduct in **Ban** and **Ban**₁.

Furthermore, if X and Y are Banach spaces and $T \in \mathcal{B}(X, Y)$ is bijective, then the Open Mapping Theorem implies that T^{-1} is bounded. The isomorphisms in **Ban** are thus simply the bijections. However, the example in [Remark 5.4](#) shows that an isomorphism in **Ban** with norm 1 might have an inverse with norm greater than 1. Thus there does not seem to be a simpler characterisation of the isomorphisms of **Ban**₁ than the bijections T such that both T and T^{-1} have norm 1. \lrcorner

EXERCISE 5.21

If X and Y are normed vector spaces, define $\alpha: X^* \oplus Y^* \rightarrow (X \times Y)^*$ by

$$\alpha(\varphi, \psi)(x, y) = \varphi(x) + \psi(y).$$

Then α is an isometric isomorphism.

This says that the dual functor $(-)^*: \mathbf{Nor} \rightarrow \mathbf{Nor}$ sends products to coproducts. [TODO: Is this more properly a functor on **Nor**₁? And what about the dual space, can it contain functionals with norm > 1 ?]

SOLUTION. We first show that α is surjective, so let $\chi \in (X \times Y)^*$ and define $\varphi(x) = \chi(x, 0)$ and $\psi(y) = \chi(0, y)$. These are then bounded linear functionals: e.g.,

$$|\varphi(x)| = |\chi(x, 0)| \leq \|\chi\| \|(x, 0)\| = \|\chi\| \|x\|,$$

and $\alpha(\varphi, \psi)(x, y) = \varphi(x) + \psi(y) = \chi(x, y)$, so α is surjective.

Next we show that α is an isometry. We have

$$\begin{aligned} |\alpha(\varphi, \psi)(x, y)| &= |\varphi(x) + \psi(y)| \\ &\leq |\varphi(x)| + |\psi(y)| \\ &\leq \|\varphi\| \|x\| + \|\psi\| \|y\| \\ &\leq (\|\varphi\| + \|\psi\|) \max\{\|x\|, \|y\|\} \\ &= \|(\varphi, \psi)\| \|(x, y)\|, \end{aligned}$$

so $\|\alpha(\varphi, \psi)\| \leq \|(\varphi, \psi)\|$. Next, let $x \in X$ and $y \in Y$ be unit vectors. Theorem 5.8(b) then furnishes $\varphi \in X^*$ and $\psi \in Y^*$ with $\|\varphi\| = \|\psi\| = 1$, $\varphi(x) = \|x\| = 1$ and

$\psi(y) = \|y\| = 1$. We thus have

$$\begin{aligned}
 |\alpha(\varphi, \psi)(x, y)| &= |\varphi(x) + \psi(y)| \\
 &= \|x\| + \|y\| \\
 &= 2 \cdot 1 \\
 &= (\|\varphi\| + \|\psi\|) \max\{\|x\|, \|y\|\} \\
 &= \|(\varphi, \psi)\| \|(x, y)\|,
 \end{aligned}$$

showing that $\|\alpha(\varphi, \psi)\| \geq \|(\varphi, \psi)\|$. In total, α is an isometry. Hence it is also injective and thus an isomorphism. \square

REMARK 5.6. Let X be a vector space over a field k , and let X^* be the algebraic dual of X . If U is a subspace of X , then the *annihilator* of U is the subspace U^0 of X^* consisting of those functionals φ such that $\varphi(u) = 0$ for all $u \in U$. We use U^0 to describe the algebraic dual U^* of U .

Let $i_U: U \rightarrow X$ be the inclusion map, and consider its pullback

$$\beta = i_U^*: X^* \rightarrow U^*$$

given by precomposition with i_U . This is surjective, since if $\psi \in U^*$ then we may extend this to a linear functional on X by letting $\psi(v) = 0$ for all $v \in V$, where V is any complement of U in X . Furthermore, a functional $\varphi \in X^*$ lies in the kernel of β just if φ vanishes on U , i.e. if $\varphi \in U^0$. The first isomorphism theorem then yields a linear isomorphism

$$\tilde{\beta}: X^*/U^0 \rightarrow U^*.$$

\lrcorner

EXERCISE 5.23

Suppose that X is a Banach space. If M is a closed subspace of X and N is a closed subspace of X^* , let $M^0 = \{\varphi \in X^* \mid \varphi|_M = 0\}$ and $N^\perp = \{x \in X \mid \varphi(x) = 0 \text{ for all } \varphi \in N\}$.

- (a) M^0 and N^\perp are closed subspaces of X^* and X , respectively.
- (b) $(M^0)^\perp = M$ and $(N^\perp)^0 \supseteq N$. If X is reflexive, $(N^\perp)^0 = N$.
- (c) c
- (d) Define $\beta: X^* \rightarrow M^*$ by $\beta(\varphi) = \varphi|_M$; then β induces a map $\tilde{\beta}: X^*/M^0 \rightarrow M^*$, and $\tilde{\beta}$ is an isometric isomorphism.

SOLUTION. (a) First assume that X is a normed vector space over \mathbb{K} , and assume that M and N are merely (not necessarily closed) *subsets* of X and X^* . Then M^0 and N^\perp are clearly subspaces. Consider the inclusion map

$i_M: M \rightarrow X$ and its pullback $\beta = i_M^*: X^* \rightarrow M^*$. The former clearly has norm 1, so for $\varphi \in X^*$ the composition $\varphi \circ i_M$ is bounded. It follows that

$$\|\beta(\varphi)\| = \|\varphi \circ i_M\| \leq \|\varphi\| \|i_M\| = \|\varphi\|$$

so β is bounded. But notice that $M^0 = \ker \beta$, so M^0 is closed. Furthermore, notice that

$$N^\perp = \bigcap_{\varphi \in N} \ker \varphi,$$

so N^\perp is an intersection of closed sets, hence is closed.

(b) Let $x \in M$. Then $\varphi(x) = 0$ for all $\varphi \in M^0$, and so $x \in (M^0)^\perp$. Conversely, assume that M is now a closed subspace of M , and assume that $x \notin M$. Theorem 5.8(b) then yields a functional $\varphi \in X^*$ such that $\varphi_M = 0$ and $\varphi(x) \neq 0$. But this means that $\varphi \in M^0$ and that $x \notin (M^0)^\perp$.

Furthermore, if $\varphi \in N$ then clearly $\varphi(x) = 0$ for all $x \in N^\perp$, so $\varphi \in (N^\perp)^0$ (even if N is neither closed or a subspace).

TODO: X reflexive.

(c) c

(d) Since $\ker \beta = M^0$ and β is surjective, $\tilde{\beta}$ is a linear isomorphism, and it is bounded since β is. It remains to be shown that it is an isometry. First let $\varphi \in X^*$ and notice that

$$\|\tilde{\beta}(\varphi + M^0)\| = \|\beta(\varphi)\| = \|\varphi \circ i_M\| \leq \|\varphi\|,$$

since $\|i_M\| = 1$ (unless $M = 0$, but in this case the claim is trivial). Since $\|\varphi + M^0\|$ is the infimum of $\|\psi\|$ over all $\psi \in X^*$ such that $\psi + M^0 = \varphi + M^0$, it follows that $\|\tilde{\beta}(\varphi + M^0)\| \leq \|\varphi + M^0\|$. For the opposite inequality, consider the seminorm

$$p(x) = \|\tilde{\beta}(\varphi + M^0)\| \|x\|$$

on X . For $x \in M$ we have

$$|\varphi(x)| = |\beta(\varphi)(x)| = |\tilde{\beta}(\varphi + M^0)(x)| \leq p(x),$$

so the Hahn–Banach theorem furnishes a $\psi \in X^*$ that extends $\varphi|_M$ and satisfies⁵ $|\psi| \leq p$, i.e. $\|\psi\| \leq \|\tilde{\beta}(\varphi + M^0)\|$. In other words, $\psi|_M = \varphi|_M$ or equivalently $\psi + M^0 = \varphi + M^0$. It follows that

$$\|\varphi + M^0\| = \|\psi + M^0\| \leq \|\psi\| \leq \|\tilde{\beta}(\varphi + M^0)\|.$$

In total, $\tilde{\beta}$ is an isometry. □

⁵ In the real case this follows since p is a seminorm.

5.4. Topological vector spaces

REMARK 5.7: Induced vector space topologies.

Let X be a vector space and Y a normed vector space over \mathbb{K} , and let $\mathcal{L}(X, Y)$ be the vector space of all linear maps $X \rightarrow Y$. Any collection $\mathcal{F} \subseteq \mathcal{L}(X, Y)$ of course induces an initial topology on X . On the other hand, each map $T \in \mathcal{F}$ defines a seminorm p_T on X given by $p_T(x) = \|Tx\|$. We claim that the initial topology on X induced by \mathcal{F} is the same as the seminorm topology induced by the family $\{p_T\}_{T \in \mathcal{F}}$ of seminorms as in Theorem 5.14.

To see this, notice that, for $x_0 \in X$ and $\varepsilon > 0$,

$$\begin{aligned} U_{x_0 T \varepsilon} &= \{x \in X \mid p_T(x - x_0) < \varepsilon\} \\ &= \{x \in X \mid \|Tx - Tx_0\| < \varepsilon\} \\ &= T^{-1}(B_\varepsilon(Tx_0)). \end{aligned}$$

The initial topology on X induced by \mathcal{F} is generated by the sets on the right-hand side.⁶ On the other hand, the seminorm topology induced by $\{p_T\}_{T \in \mathcal{F}}$ is generated by the sets on the left-hand side. Hence the two topologies agree.

The most common application of the above is when U is a subspace of $\mathcal{L}(X, Y)$ and \mathcal{F} is the set of evaluation maps $\text{ev}_x: U \rightarrow Y$ given by $\text{ev}_x(T) = Tx$ for $x \in X$. It is easy to show that the evaluation maps are in fact linear. Since the evaluation maps obviously separate points in $\mathcal{L}(X, Y)$, the resulting topology is Hausdorff (hence T_3 since topological groups are automatically regular; in fact they are completely regular, though this is not trivial to prove). Notice also that the product topology on Y^X is precisely induced by the evaluation maps, so $U \subseteq Y^X$ in fact carries the subspace topology and is thus a topology of pointwise convergence. We give some examples of this:

- (a) Let X be a topological vector space with topological dual X^* . Then the *weak*-topology* on X^* is the initial topology induced by the collection of evaluation maps $\text{ev}_x: X^* \rightarrow \mathbb{K}$. Since \mathbb{K} is itself a normed space, the above shows that the weak*-topology is a seminorm topology.
- (b) Let X and Y be normed spaces, and consider the space $\mathcal{B}(X, Y)$ of bounded linear maps $X \rightarrow Y$. We equip this space with the *strong operator topology*, defined as the initial topology induced by the evaluation maps $\text{ev}_x: \mathcal{B}(X, Y) \rightarrow Y$. More concretely, the topology is induced by seminorms $T \mapsto \|Tx\|$, so a net (T_i) in $\mathcal{B}(X, Y)$ converges to T iff $\|T_i x - Tx\| \rightarrow 0$ for all $x \in X$.

⁶ This is clear if each T is surjective. But $T: X \rightarrow Y$ is continuous iff the corresponding map with codomain $T(X)$ is continuous, so it suffices to consider balls in Y with centres in $T(X)$.

Notice that the SOT is coarser than the norm topology, since if $T_i \rightarrow T$ in the norm topology, then

$$\|T_i x - T x\| \leq \|T_i - T\| \|x\| \rightarrow 0,$$

so $T_i \rightarrow T$ in the SOT.

- (c) In the same setup, the *weak operator topology* on $\mathcal{B}(X, Y)$ is the initial topology induced by maps $\Phi_{x, \varphi} = \varphi \circ \text{ev}_x: \mathcal{B}(X, Y) \rightarrow \mathbb{K}$ given by $\Phi_{x, \varphi}(T) = \varphi(Tx)$ for $x \in X$ and $\varphi \in Y^*$, where Y^* is the topological dual of Y . That is, contrary to the strong operator topology we do not require the evaluation maps ev_x themselves to be continuous, only the compositions $\varphi \circ \text{ev}_x$. Hence the WOT is coarser than the SOT, since if ev_x is continuous then so is $\varphi \circ \text{ev}_x$.

We claim that the WOT is also Hausdorff. This is not immediate from the above since the generating functions are not evaluation maps. But notice that Y^* separates points in Y by Theorem 5.8c. For distinct $T, S \in \mathcal{L}(X, Y)$ there is a $x \in X$ with $Tx \neq Sx$, and then a $\varphi \in Y^*$ with $\varphi(Tx) \neq \varphi(Sx)$. Hence the functions $\Phi_{x, \varphi}$ separate points in $\mathcal{B}(X, Y)$.

If \mathcal{H} is a Hilbert space, then the weak operator topology on $\mathcal{B}(X, \mathcal{H})$ is also induced by maps $T \mapsto \langle Tx, y \rangle$ by the Riesz–Fréchet theorem (Theorem 5.25). In this case a net (T_i) converges to T iff $\langle T_i x, y \rangle \rightarrow \langle Tx, y \rangle$ for all $x \in X$ and $y \in \mathcal{H}$. \lrcorner

5.5. Hilbert Spaces

REMARK 5.8. We give a different proof of the Cauchy–Schwarz inequality using (very) basic properties of orthogonal projections:

If X is a inner product space over \mathbb{K} , then

$$|\langle x, y \rangle| \leq \|x\| \|y\| \tag{5.2}$$

for all $x, y \in X$, with equality if and only if x and y are linearly dependent.

This is obvious if $y = 0$, so assume not. The *projection* of x on y is the unique vector $p \in \text{span}(y)$ such that $y \perp x - p$. This exists and is unique, for notice that for $\alpha \in \mathbb{K}$ we have

$$0 = \langle x - \alpha y, y \rangle = \langle x, y \rangle - \alpha \langle y, y \rangle$$

if and only if

$$\alpha = \frac{\langle x, y \rangle}{\langle y, y \rangle},$$

so $p = \alpha y$. Notice that p has the property that $x = p$ if and only if x and y are linearly dependent. The ‘only if’ part is obvious, and the converse follows since if $x = \beta y$ for some $\beta \in \mathbb{K}$ then, plugging in above, we find that $\alpha = \beta$.

Also notice that $p \perp x - p$. Writing $x = p + (x - p)$, Pythagoras’ theorem thus implies that

$$\|x\|^2 = \|p\|^2 + \|x - p\|^2 \geq \|p\|^2, \quad (5.3)$$

with equality just when $x = p$, i.e. when x and y are linearly dependent. Inserting the formula above for p , the inequality (5.3) is equivalent to

$$\|x\| \geq \|p\| = \frac{|\langle x, y \rangle|}{\|y\|} = \frac{|\langle x, y \rangle|}{\|y\|^2} \|y\|$$

which in turn is equivalent to (5.2). \lrcorner

6 • L^p -spaces

6.1. Basic Theory of L^p -spaces

REMARK 6.1: The space $L^\infty(\mu)$.

Let (X, \mathcal{E}, μ) be a measure space, and let $f \in \mathcal{M}(\mathcal{E})$. We prefer to define the essential supremum of f with respect to μ as

$$\|f\|_\infty = \inf\{R > 0 \mid |f| \leq R \text{ } \mu\text{-a.e.}\},$$

which is clearly equivalent to Folland’s definition. If $R > \|f\|_\infty$ then $\mu(\{|f| > R\}) = 0$, so the set

$$E := \{|f| > \|f\|_\infty\} = \bigcup_{n \in \mathbb{N}} \left\{ |f| > \|f\|_\infty + \frac{1}{n} \right\}$$

is also a null set.

The function $\tilde{f} = f \mathbf{1}_{K^c}$ then equals f a.e., and we clearly have $\|\tilde{f}\|_\infty = \|f\|_\infty$. Furthermore, since $|\tilde{f}| \leq \|f\|_\infty$ everywhere, we also have $\|\tilde{f}\|_{\sup} \leq \|\tilde{f}\|_\infty$. The opposite inequality follows since $\{|\tilde{f}| > R\}$ has positive measure for all $R < \|\tilde{f}\|_\infty$. Hence $\|\tilde{f}\|_{\sup} = \|\tilde{f}\|_\infty$, so if we only consider functions up to null sets we may replace any $f \in \mathcal{M}(\mathcal{E})$ by a function $\tilde{f} \in \mathcal{M}(\mathcal{E})$ whose supremum agrees with its essential supremum. Furthermore, if $\|f\|_\infty < \infty$, i.e. if $f \in L^\infty(\mu)$, then \tilde{f} is bounded.

This yields another interpretation of $L^\infty(\mu)$ -functions, namely as those functions that arise from bounded measurable functions by altering them on a (measurable) μ -null set.

It might seem possible to alter a function f on a null set and allow it to *attain* its essential supremum. This is only possible if the measure space

(X, \mathcal{E}, μ) has nonempty null sets. However, consider for instance the space $(\mathbb{N}, 2^{\mathbb{N}}, \tau)$, where τ is the counting measure. On this space the function $f(n) = 1 - 1/n$ has (essential) supremum 1, but it does not attain it. It also cannot be altered on a nonempty null set, since there are no such sets. \lrcorner

6.3. Some Useful Inequalities

REMARK 6.2: Minkowski's inequality for integrals.

We give a different proof of Theorem 6.19 that does not require duality. Assume that $f \geq 0$ and let $p \in (1, \infty)$ and $H(x) = \int_Y f(x, y) d\nu(y)$. Notice that the left-hand side of the inequality is $\|H\|_p$. Then Tonelli's theorem and Hölder's inequality imply that

$$\begin{aligned} \|H\|_p^p &= \int_X \int_Y f(x, y) d\nu(y) H(x)^{p-1} d\mu(x) \\ &= \int_Y \int_X f(x, y) H(x)^{p-1} d\mu(x) d\nu(y) \\ &\leq \int_Y \left(\int_X f(x, y)^p d\mu(x) \right)^{1/p} \left(\int_X H(x)^{q(p-1)} d\mu(x) \right)^{1/q} d\nu(y) \\ &= \int_Y \left(\int_X f(x, y)^p d\mu(x) \right)^{1/p} \|H\|_p^{p-1} d\nu(y). \end{aligned}$$

If $\|H\|_p < \infty$ then the claim follows, so assume that $\|H\|_p = \infty$. Choose sequences $(A_n)_{n \in \mathbb{N}}$ and $(B_m)_{m \in \mathbb{N}}$ of measurable subsets of X and Y , respectively, such that $A_n \uparrow X$ and $B_m \uparrow X$, and such that $\mu(A_n) < \infty$ and $\nu(B_m) < \infty$. For $k \in \mathbb{N}$ let $f_k = f \wedge k$ and notice that replacing f in the definition of H with $\mathbf{1}_{A_n} \mathbf{1}_{B_m} f_k$ yields $\|H\|_p < \infty$, so we may apply Minkowski's inequality to this function:

$$\left(\int_{A_n} \left(\int_{B_m} f_k(x, y) d\nu(y) \right)^p d\mu(x) \right)^{1/p} \leq \int_{B_m} \left(\int_{A_n} f_k(x, y)^p d\mu(x) \right)^{1/p} d\nu(y).$$

Letting $n, m, k \rightarrow \infty$, monotone convergence yields the theorem.

The second part for $p \in [1, \infty)$ follows by applying the first part to $|f|$ and using the triangle inequality for integrals. \lrcorner

8 • Elements of Fourier Analysis

8.1. Preliminaries

REMARK 8.1: Completeness of the Schwartz space.

We elaborate on the proof of Proposition 8.2. Let $(f_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in \mathcal{S} . Then the sequence $(\partial^\alpha f_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in the uniform norm

for all multi-indices α , since $\|\partial^\alpha f_n\|_{\text{sup}} = \|f_n\|_{(0,\alpha)}$. Hence $\partial^\alpha f_n$ converges uniformly to a function g_α by completeness in the uniform norm. We then have

$$f_n(x + te_i) - f_n(x) = \int_0^t \partial_i f_n(x + se_i) ds.$$

Letting $n \rightarrow \infty$, $\partial_i f_n$ converges to g_{e_i} uniformly, so it follows that

$$g_0(x + te_i) - g_0(x) = \int_0^t g_{e_i}(x + se_i) ds.$$

The fundamental theorem of calculus then implies that $\partial_j g_0$ exists and equals g_{e_j} , so it follows by induction that $g_\alpha = \partial^\alpha g_0$. It remains to be shown that $\|f_n - g_0\|_{(N,\alpha)} \rightarrow 0$ for all N and α .

To this end, let $\varepsilon > 0$ and choose $M \in \mathbb{N}$ such that $m, n \geq M$ implies that $\|f_n - f_m\|_{(N,\alpha)} < \varepsilon$. For every $x \in \mathbb{R}^d$ we thus have

$$(1 + \|x\|)^N |\partial^\alpha f_n(x) - \partial^\alpha f_m(x)| < \varepsilon.$$

Letting $m \rightarrow \infty$ we get

$$(1 + \|x\|)^N |\partial^\alpha f_n(x) - \partial^\alpha g_0(x)| \leq \varepsilon.$$

Taking the supremum we find that $n \geq M$ implies that $\|f_n - g_0\|_{(N,\alpha)} \leq \varepsilon$, showing that $f_n \rightarrow g_0$ in \mathcal{S} . \lrcorner

8.2. Convolutions

REMARK 8.2: Associativity of convolution.

If $f, g, h \in \mathcal{M}(\mathcal{B}(\mathbb{R}^d))$, then we define the function $k: \mathbb{R}^{3d} \rightarrow \mathbb{C}$ by

$$k(x, y, z) = f(y)g(x - y - z)h(z).$$

This is clearly measurable, so we may consider the function $K: \mathbb{R}^d \rightarrow [0, \infty]$ given by

$$K(x) = \int_{\mathbb{R}^{2d}} |k(x, \cdot, \cdot)| d\lambda_{2d}.$$

By Tonelli's theorem K is also measurable, so the set

$$\Delta(f, g, h) = \{x \in \mathbb{R}^d \mid k(x, \cdot, \cdot) \in \mathcal{L}^1(\lambda_{2d})\} = \{x \in \mathbb{R}^d \mid K(x) < \infty\}$$

is measurable. For $x \in \Delta(f, g, h)$, Fubini's theorem thus implies that

$$\begin{aligned}
 (f * g) * h(x) &= (g * f) * h(x) \\
 &= \int_{\mathbb{R}^d} g * f(x - z) h(z) \, dz \\
 &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} f(y) g(x - z - y) \, dy \right) h(z) \, dz \\
 &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} f(y) g(x - z - y) h(z) \, dz \right) dy \\
 &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} f(y) g(x - y - z) h(z) \, dz \right) dy \\
 &= \int_{\mathbb{R}^d} f(y) \left(\int_{\mathbb{R}^d} g(x - y - z) h(z) \, dz \right) dy \\
 &= \int_{\mathbb{R}^d} f(y) h * g(x - y) \, dy \\
 &= f * (h * g)(x) \\
 &= f * (g * h)(x).
 \end{aligned}$$

Thus convolution is associative on $\Delta(f, g, h)$. If $f, g, h \in \mathcal{L}^1(\lambda_d)$, then it is easy to show that $\Delta(f, g, h)^c$ is a Lebesgue null-set. However, it is not clear whether (and I don't see why it should be true that) $\Delta(f, g, h)$ is the same as $\Delta(f * g, h)$ or $\Delta(f, g * h)$. \perp

REMARK 8.3: Approximate identities and pointwise convergence.

We prove the following version of Theorem 8.15:

*Let φ be an approximate identity. If $f \in L^\infty$ and f is continuous at some $x \in \mathbb{R}^d$, then $f * \varphi_t(x) \rightarrow f(x)$ as $t \rightarrow 0$.*

Since φ is an approximate identity, we have

$$\begin{aligned}
 f * \varphi_t(x) - f(x) &= \int_{\mathbb{R}^d} (f(x - y) - f(x)) \varphi_t(y) \, dy \\
 &= \int_{\mathbb{R}^d} (f(x - y) - f(x)) t^{-1} \varphi_1(y/t) \, dy \\
 &= \int_{\mathbb{R}^d} (f(x - tz) - f(x)) \varphi_1(z) \, dz.
 \end{aligned}$$

The last integrand is dominated by $\|f\|_\infty \varphi_1(z)$, so the claim follows from the dominated convergence theorem. \perp

8.3. The Fourier Transform

REMARK 8.4: Uniform continuity of Fourier transforms.

Let $f \in L^1(\mathbb{R}^d)$. For $\xi, \eta \in \mathbb{R}^d$ we then have

$$\begin{aligned} |\hat{f}(\xi) - \hat{f}(\eta)| &\leq \int_{\mathbb{R}^d} |f(x)| |e^{-2\pi i \langle \xi, x \rangle} - e^{-2\pi i \langle \eta, x \rangle}| dx \\ &= \int_{\mathbb{R}^d} |f(x)| |e^{-2\pi i \langle \xi - \eta, x \rangle} - 1| dx. \end{aligned}$$

Since $2|f|$ is integrable and dominates the integrand, the dominated convergence theorem implies that the above goes to zero as $\xi - \eta \rightarrow 0$. \lrcorner

REMARK 8.5: The Plancherel theorem.

We give a different proof of the Plancherel theorem, based on Rudin, that does not make use of the Schwartz space.

If $f \in L^1 \cap L^2$, then $\hat{f} \in L^2$ and $\|\hat{f}\|_2 = \|f\|_2$.

Let $\tilde{f}(x) = \overline{f(-x)}$ and define a function $g: \mathbb{R}^d \rightarrow \mathbb{C}$ by $g = f * \tilde{f}$. Then $g \in L^1$ by Young's inequality, and we also have $g(x) = \langle \tau_{-x}f, f \rangle$, so g is continuous by Proposition 8.5. Furthermore,

$$|g(x)| \leq \|\tau_{-x}f\|_2 \|f\|_2 = \|f\|_2^2$$

by the Cauchy–Schwarz inequality so g is bounded. Finally notice that $\hat{g} = |\hat{f}|^2$ by Theorem 8.22(c). Letting $\varphi(x) = e^{-\pi \|x\|^2}$, Remark 8.3 implies that

$$\lim_{t \rightarrow 0} g * \varphi_t(0) = g(0) = \|f\|_2^2.$$

On the other hand, as in the proof of the inversion theorem we have, by the monotone convergence theorem,

$$\lim_{t \rightarrow 0} g * \varphi_t(0) = \lim_{t \rightarrow 0} \int_{\mathbb{R}^d} e^{-\pi t^2 \|\xi\|^2} \hat{g}(\xi) d\xi = \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 d\xi = \|\hat{f}\|_2^2,$$

proving the claim.

$L^1 \cap L^2$ is dense in L^2 .

Let $f \in L^2$, and let $f_n = f \mathbf{1}_{[-n, n]}$. Then $f_n \in L^1 \cap L^2$, and $f_n \rightarrow f$ pointwise. By the dominated convergence theorem f_n also converges to f in L^2 .

Let $Y = \{\hat{f} \mid f \in L^1 \cap L^2\}$. Then Y is dense in L^2 .

Notice that $Y \subseteq L^2$ by the above. Let $w \in Y^\perp$, and let $\psi_{t,x}(y) = \varphi_t(x-y)$. Then $\psi_{t,x}$ is the Fourier transform of the function $\xi \mapsto \exp(2\pi i \langle \xi, x \rangle - \pi t^2 \|\xi\|^2)$ by the proof of the inversion theorem, and this is a function from $L^1 \cap L^2$ so $\psi_{t,x} \in Y$. Furthermore,

$$\varphi_t * \bar{w}(x) = \int_{\mathbb{R}^d} \varphi_t(x-y) \overline{w(y)} dy = \langle \psi_{t,x}, w \rangle = 0.$$

On the other hand, $\varphi_t * \bar{w} \rightarrow \bar{w}$ in L^1 as $t \rightarrow 0$, so $w = 0$. This proves the claim.

The Fourier transform $\mathcal{F}|_{L^1 \cap L^2}$ extends uniquely to a bounded operator $\tilde{\mathcal{F}}$ on L^2 , and $\tilde{\mathcal{F}}$ is a unitary isomorphism on L^2 .

The BLT theorem yields a unique bounded operator $\tilde{\mathcal{F}}$ on L^2 that extends $\mathcal{F}|_{L^1 \cap L^2}$. This is an isometry, since if $f \in L^2$ and (f_n) is a sequence in $L^1 \cap L^2$ converging to f , then

$$\|\tilde{\mathcal{F}}f\| = \lim_{n \rightarrow \infty} \|\tilde{\mathcal{F}}f_n\| = \lim_{n \rightarrow \infty} \|f_n\| = \|f\|,$$

by continuity of the norm. Furthermore, if $g \in L^2$ then there is a sequence (g_n) in Y converging to g . But then $g_n = \mathcal{F}f_n$ for some $f_n \in L^1 \cap L^2$, and (f_n) is a Cauchy sequence since \mathcal{F} is an isometry, and hence it converges to some $f \in L^2$. But then

$$g = \lim_{n \rightarrow \infty} g_n = \lim_{n \rightarrow \infty} \mathcal{F}f_n = \mathcal{F}f,$$

since \mathcal{F} is continuous. Thus $\tilde{\mathcal{F}}$ is surjective. \lrcorner

9 • Elements of Distribution Theory

9.1. Distributions

REMARK 9.1: The fundamental lemma of the calculus of variations.

We prove the following result:

Let $U \subseteq \mathbb{R}^d$ be open, and let $f \in L^1_{\text{loc}}(U)$. If

$$\int_U f(x) \varphi(x) dx = 0$$

for all $\varphi \in C_c^\infty(U)$, then $f = 0$ a.e.

First assume that $f \in L^1(U)$. Let $\varphi \in C_c^\infty(U)$ have unit integral, and notice that the hypothesis implies that

$$f * \varphi_t(x) = \int_U f(y) \varphi_t(x-y) dy = 0$$

for all $x \in U$. But Theorem 8.14(a) implies that $f * \varphi_t \rightarrow f$ in L^1 as $t \rightarrow 0$, so $f = 0$ a.e.

Next assume that $f \in L^1_{\text{loc}}(U)$. Since U is σ -compact there is an increasing sequence $(K_n)_{n \in \mathbb{N}}$ of compact subsets of U such that $U = \bigcup_{n \in \mathbb{N}} K_n$. By Theorem 8.18, let ψ_n be a smooth Urysohn function supported in U that is 1 on K_n . Then $f\psi_n \in L^1(U)$, and notice that if $\varphi \in C_c^\infty(U)$, then also $\psi_n\varphi \in C_c^\infty(U)$. It follows that

$$\int_U f(x)\psi_n(x)\varphi(x) \, dx = 0$$

for all such φ , so the above implies that $f\psi_n = 0$ a.e. But since $\psi_n \uparrow \mathbf{1}_U$, this shows that $f = 0$ a.e. ┘