

# Folland: *Real Analysis*

Danny Nygård Hansen

11th January 2022

## 1 • Measures

### 1.2. $\sigma$ -algebras

#### EXERCISE 1.1

Let  $\mathcal{M}$  be an infinite  $\sigma$ -algebra.

- (a)  $\mathcal{M}$  contains an infinite sequence of disjoint sets.
- (b)  $|\mathcal{M}| \geq \mathfrak{c}$ .

(a) We show by contraposition that there exists an  $A \in \mathcal{M}$  such that the restriction of  $\mathcal{M}$  to  $A^c$  is infinite. That is, assuming that no such set exists, we show that  $\mathcal{M}$  is finite. Pick any nonempty  $A \in \mathcal{M}$ . Then the restriction of  $\mathcal{M}$  to  $A$  and  $A^c$  respectively are both finite. For any  $B \in \mathcal{M}$  we can write

$$B = (B \cap A) \cup (B \cap A^c).$$

But each set in the union lies in one of the restrictions, so there are finitely many decompositions like the one above, so there are finitely many sets  $B \in \mathcal{M}$ .

Now construct the sequence: Pick  $A \in \mathcal{M}$  as above, restrict  $\mathcal{M}$  to  $A^c$ , and continue recursively.

(b) Let  $(A_n)$  be the sequence constructed above. There is an injection  $\varphi: 2^{\mathbb{N}} \rightarrow \mathcal{M}$  given by  $\varphi(I) = \bigcup_{i \in I} A_i$  (injectivity follows since the sets in the sequence are disjoint). Hence  $|\mathcal{M}| \geq |2^{\mathbb{N}}| = \mathfrak{c}$ .

### 1.3. Measures

## EXERCISE 1.14

If  $\mu$  is a semifinite measure and  $\mu(E) = \infty$ , for any  $C > 0$  there exists  $F \subseteq E$  with  $C < \mu(F) < \infty$ .

Consider

$$S = \sup\{\mu(F) \mid F \subseteq E, \mu(F) < \infty\}.$$

If  $S = \infty$ , then the result is obvious. So assume towards a contradiction that  $S < \infty$ . For  $n \in \mathbb{N}$  choose  $F_n \subseteq E$  with  $\mu(F_n) < \infty$  such that

$$S - \frac{1}{n} \leq \mu(F_n) \leq S.$$

Put  $G_k = \bigcup_{n=1}^k F_n$ . Then  $G_k \subseteq E$  and  $\mu(G_k) < \infty$ , so the same inequality holds with  $F_n$  replaced by  $G_k$ . Now putting  $G = \bigcup_{k \in \mathbb{N}} G_k$ , continuity of  $\mu$  gives

$$S - \frac{1}{n} \leq \mu(G) \leq S$$

for all  $n \in \mathbb{N}$ , so  $\mu(G) = S$ .

By assumption  $\mu(E \setminus G) = \infty$ , so  $E \setminus G$  contains a set  $G' \in \mathcal{M}$  such that  $0 < \mu(G') < \infty$ . But then

$$\mu(G \cup G') = \mu(G) + \mu(G') > S,$$

a contradiction.

## EXERCISE 1.16

Let  $(X, \mathcal{M}, \mu)$  be a measure space. A set  $E \subseteq X$  is called *locally measurable* if  $E \cap A \in \mathcal{M}$  for all  $A \in \mathcal{M}$  such that  $\mu(A) < \infty$ . Let  $\widetilde{\mathcal{M}}$  be the collection of all locally measurable sets. Clearly  $\mathcal{M} \subseteq \widetilde{\mathcal{M}}$ ; if  $\mathcal{M} = \widetilde{\mathcal{M}}$ , then  $\mu$  is called *saturated*.

- (a) If  $\mu$  is  $\sigma$ -finite, then  $\mu$  is saturated.
- (b)  $\widetilde{\mathcal{M}}$  is a  $\sigma$ -algebra.
- (c) Define  $\tilde{\mu}$  on  $\widetilde{\mathcal{M}}$  by  $\tilde{\mu}(E) = \mu(E)$  if  $E \in \mathcal{M}$  and  $\tilde{\mu}(E) = \infty$  otherwise. Then  $\tilde{\mu}$  is a saturated measure on  $\widetilde{\mathcal{M}}$ , called the *saturation* of  $\mu$ .
- (d) If  $\mu$  is complete, so is  $\tilde{\mu}$ .
- (e) Suppose that  $\mu$  is semifinite. For  $E \in \widetilde{\mathcal{M}}$  define

$$\underline{\mu}(E) = \sup\{\mu(A) \mid A \in \mathcal{M} \text{ and } A \subseteq E\}.$$

Then  $\underline{\mu}$  is a saturated measure on  $\widetilde{\mathcal{M}}$  that extends  $\mu$ .

(f) Let  $X_1, X_2$  be disjoint uncountable sets,  $X = X_1 \cup X_2$ , and  $\mathcal{M}$  the  $\sigma$ -algebra of countable or co-countable sets in  $X$ . Let  $\mu_0$  be counting measure on  $2^{X_1}$ , and define  $\mu$  on  $\mathcal{M}$  by  $\mu(E) = \mu_0(E \cap X_1)$ . Then  $\mu$  is a measure on  $\mathcal{M}$ ,  $\widetilde{\mathcal{M}} = 2^X$ , and in the notation of parts (c) and (e),  $\tilde{\mu} \neq \underline{\mu}$ .

(a) Assume that  $\mu$  is  $\sigma$ -finite, and let  $E \subseteq X$  be locally measurable. Let  $(A_n) \subseteq \mathcal{M}$  be such that  $X = \bigcup_{n \in \mathbb{N}} A_n$  and  $\mu(A_n) < \infty$ . Then  $E \cap A_n \in \mathcal{M}$ , and so  $E = \bigcup_{n \in \mathbb{N}} (E \cap A_n) \in \mathcal{M}$ .

(b) Clearly we have  $X \in \widetilde{\mathcal{M}}$ . Then let  $(E_n) \subseteq \widetilde{\mathcal{M}}$ , and let  $A \in \mathcal{M}$  with  $\mu(A) < \infty$ . Then

$$A \cap \bigcup_{n \in \mathbb{N}} E_n = \bigcup_{n \in \mathbb{N}} (A \cap E_n) \in \mathcal{M},$$

so  $\bigcup_{n \in \mathbb{N}} E_n \in \widetilde{\mathcal{M}}$ . Finally let  $E \in \widetilde{\mathcal{M}}$  and  $A \in \mathcal{M}$  with  $\mu(A) < \infty$ . Then

$$E^c \cap A = (E \cap A)^c \cap A \in \mathcal{M}$$

since  $E \cap A \in \mathcal{M}$ , so  $E^c \in \widetilde{\mathcal{M}}$ .

(c) We first show that  $\tilde{\mu}$  is a measure. Clearly  $\tilde{\mu}(\emptyset) = 0$ , so let  $(E_n)$  be a sequence of disjoint sets in  $\widetilde{\mathcal{M}}$ , and let  $E = \bigcup_{n \in \mathbb{N}} E_n$ . Say that  $E_m$  does not lie in  $\mathcal{M}$  for some  $m \in \mathbb{N}$ . Then we must have  $\tilde{\mu}(E) = \infty$ , since otherwise  $E \in \mathcal{M}$  with  $\mu(E) < \infty$ , and hence  $E_m = E_m \cap E \in \mathcal{M}$ . Thus we have

$$\sum_{n=1}^{\infty} \tilde{\mu}(E_n) \geq \tilde{\mu}(E_m) = \infty = \tilde{\mu}(E),$$

so  $\sum_{n=1}^{\infty} \tilde{\mu}(E_n) = \tilde{\mu}(E)$ . The same is obviously true if all  $E_n$  lie in  $\mathcal{M}$ .

Next we show that  $\tilde{\mu}$  is saturated, i.e. that  $\widetilde{\mathcal{M}} \subseteq \widetilde{\widetilde{\mathcal{M}}}$ , so let  $E \in \widetilde{\widetilde{\mathcal{M}}}$ . For all  $A \in \widetilde{\mathcal{M}}$  with  $\tilde{\mu}(A) < \infty$  we then have  $E \cap A \in \widetilde{\mathcal{M}}$ . By definition of  $\tilde{\mu}$  we must have  $A \in \mathcal{M}$ , so we also have

$$E \cap A = (E \cap A) \cap A \in \mathcal{M}.$$

And since this is true for all  $A \in \mathcal{M}$  with  $\mu(A) < \infty$ , it follows that  $E \in \widetilde{\mathcal{M}}$ .

(d) Assume that  $\mu$  is complete. Let  $F \subseteq X$  be such that there is a set  $E \in \widetilde{\mathcal{M}}$  with  $F \subseteq E$  and  $\tilde{\mu}(E) = 0$ . Then also  $E \in \mathcal{M}$ , and since  $\mu$  is complete we have  $F \in \mathcal{M} \subseteq \widetilde{\mathcal{M}}$  as desired.

(e) Assume that  $\mu$  is semifinite. We first show that  $\underline{\mu}$  is a measure. Clearly  $\underline{\mu}(\emptyset) = 0$ , so let  $(E_n) \subseteq \widetilde{\mathcal{M}}$  be a sequence of disjoint sets. Clearly  $\underline{\mu}$  is increasing, so sigma-additivity is obvious if any of the sets  $E_n$  have infinite measure.

Assume then that  $\mu(E_n) < \infty$  for all  $n \in \mathbb{N}$ . Let  $\varepsilon > 0$ , and choose  $A_n \in \mathcal{M}$  such that  $A_n \subseteq E_n$  and  $\underline{\mu}(E_n) \leq \mu(A_n) + \varepsilon/2^n$ . Then

$$\underline{\mu}\left(\bigcup_{n \in \mathbb{N}} E_n\right) \geq \mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) \geq \sum_{n=1}^{\infty} \mu(E_n) - \varepsilon.$$

Since this holds for all  $\varepsilon > 0$ , we obtain the first inequality. For the other inequality, let  $E = \bigcup_{n \in \mathbb{N}} E_n$ , and first assume that  $\underline{\mu}(E) = \infty$ . Pick  $A \in \mathcal{M}$  with  $A \subseteq E$ . Since  $\mu$  is semifinite, we can choose  $A$  such that  $C < \mu(A) < \infty$  for any given  $C > 0$ . Letting  $A_n = A \cap E_n \in \mathcal{M}$  we get

$$C < \mu(A) = \sum_{n=1}^{\infty} \mu(A_n) \leq \sum_{n=1}^{\infty} \underline{\mu}(E_n),$$

and since  $C$  is arbitrary, we get  $\sum_{n=1}^{\infty} \underline{\mu}(E_n) = \infty$ . If instead  $\mu(E) < \infty$ , pick  $A \subseteq E$  with  $A \in \mathcal{M}$  and  $\underline{\mu}(E) \leq \mu(A) + \varepsilon$ . Again letting  $A_n = A \cap E_n$  we get

$$\underline{\mu}(E) - \varepsilon \leq \mu(A) = \sum_{n=1}^{\infty} \mu(A_n) \leq \sum_{n=1}^{\infty} \underline{\mu}(E_n).$$

And since  $\varepsilon$  is arbitrary, we obtain the other inequality.

Next we show that  $\underline{\mu}$  is saturated. Letting  $E$  be locally  $\underline{\mu}$ -measurable, we must show that  $E$  is also locally  $\mu$ -measurable. So let  $A \in \mathcal{M}$  with  $\mu(A) < \infty$ . Then  $\underline{\mu}(A) < \infty$ , and so  $E \cap A \in \mathcal{M}$ . But then

$$E \cap A = (E \cap A) \cap A \in \mathcal{M},$$

as desired.

(f) It is pretty obvious that  $\mu$  is a measure on  $\mathcal{M}$ . Then let  $E \subseteq X$  and  $A \in \mathcal{M}$  with  $\mu(A) < \infty$ . Then  $A \cap X_1$  must be finite, and so  $A$  is not co-countable. But then it is countable, and so is  $E \cap A$ , hence  $E \cap A \in \mathcal{M}$ . Thus every subset of  $X$  is locally measurable.

Notice that  $\mu$  is semifinite. We have  $\tilde{\mu}(X_2) = \infty$  since  $X_2 \notin \mathcal{M}$ , but  $\underline{\mu}(X_2) = 0$  since every subset of  $X_2$  is disjoint from  $X_1$ , and so has measure zero.

## 1.4. Outer Measures

## Exercise 17

The inequality  $\leq$  holds by definition. For the other inequality, notice that

$$\begin{aligned}\mu^*\left(E \cap \bigcup_{j \in \mathbb{N}} A_j\right) &= \mu^*(E \cap A_1) + \mu^*\left(E \cap \bigcup_{j=2}^{\infty} A_j\right) \\ &= \sum_{j=1}^n \mu^*(E \cap A_j) + \mu^*\left(E \cap \bigcup_{j=n+1}^{\infty} A_j\right) \\ &\geq \sum_{j=1}^n \mu^*(E \cap A_j)\end{aligned}$$

for all  $n \in \mathbb{N}$ . Letting  $n \rightarrow \infty$  proves the inequality.

## EXERCISE 1.18

Let  $\mathcal{A} \subseteq 2^X$  be an algebra,  $\mathcal{A}_\sigma$  the collection of countable unions of sets in  $\mathcal{A}$ , and  $\mathcal{A}_{\sigma\delta}$  the collection of countable intersections of sets in  $\mathcal{A}_\sigma$ . Let  $\mu_0$  be a premeasure on  $\mathcal{A}$  and  $\mu^*$  the induced outer measure.

- (a) For any  $E \subseteq X$  and  $\varepsilon > 0$  there exists  $A \in \mathcal{A}_\sigma$  with  $E \subseteq A$  with  $\mu^*(A) \leq \mu^*(E) + \varepsilon$ .
- (b) If  $\mu^*(E) < \infty$ , then  $E$  is  $\mu^*$ -measurable iff there exists  $B \in \mathcal{A}_{\sigma\delta}$  with  $E \subseteq B$  and  $\mu^*(B \setminus E) = 0$ .
- (c) If  $\mu_0$  is  $\sigma$ -finite, the restriction  $\mu^*(E) < \infty$  in (b) is superfluous.

(a) Let  $E \subseteq X$  and  $\varepsilon > 0$ . The definition of  $\mu^*$  yields a sequence  $(A_n) \subseteq \mathcal{A}$  such that  $E \subseteq \bigcup_{n \in \mathbb{N}} A_n$  and  $\sum_{n=1}^{\infty} \mu_0(A_n) \leq \mu^*(E) + \varepsilon$ . It follows that

$$\mu^*(E) + \varepsilon \geq \sum_{n=1}^{\infty} \mu_0(A_n) = \sum_{n=1}^{\infty} \mu^*(A_n) \geq \mu^*\left(\bigcup_{n \in \mathbb{N}} A_n\right).$$

(b) Let  $E \subseteq X$ . For  $n \in \mathbb{N}$  there is a set  $B_n \in \mathcal{A}_\sigma$  such that  $E \subseteq B_n$  and  $\mu^*(B_n) \leq \mu^*(E) + 1/n$ . Letting  $B = \bigcap_{n \in \mathbb{N}} B_n \in \mathcal{A}_{\sigma\delta}$  we get  $\mu^*(B) \leq \mu^*(E)$ , and since  $E \subseteq B$  we also have the opposite inequality, so  $\mu^*(B) = \mu^*(E)$ .

Now assume that  $\mu^*(E) < \infty$  and that  $E$  is  $\mu^*$ -measurable. Then

$$\mu^*(B) = \mu^*(B \cap E) + \mu^*(B \cap E^c) = \mu^*(E) + \mu^*(B \setminus E),$$

from which it follows that  $\mu^*(B \setminus E) = 0$ .

Conversely, assume that there is a  $B \in \mathcal{A}_{\sigma\delta}$  with  $E \subseteq B$  and  $\mu^*(B \setminus E) = 0$ . Then  $B$  lies in the  $\sigma$ -algebra generated by  $\mathcal{A}$ , so it is  $\mu^*$ -measurable. Let  $A \subseteq X$ . Then

$$\begin{aligned}\mu^*(A \cap E^c) &\leq \mu^*(B \cap A \cap E^c) + \mu^*(B^c \cap A \cap E^c) \\ &= \mu^*(A \cap (B \cup E)^c) \\ &= \mu^*(A \cap B^c),\end{aligned}$$

and so

$$\mu^*(A \cap E) + \mu^*(A \cap E^c) \leq \mu^*(A \cap B) + \mu^*(A \cap B^c) = \mu^*(A),$$

showing that  $E$  is  $\mu^*$ -measurable. (Notice that we haven't used that  $\mu^*(E) < \infty$  for the second implication.)

(c) We only need to prove the first implication above. By  $\sigma$ -finiteness of  $\mu_0$ , let  $(E_n)$  be a sequence of subsets of  $X$  such that  $\mu^*(E_n) < \infty$  and  $E = \bigcup_{n \in \mathbb{N}} E_n$ . Let  $\varepsilon > 0$ . Then there are sets  $A_n \in \mathcal{A}_\sigma$  such that  $\mu^*(A_n) \leq \mu^*(E_n) + \varepsilon/2^n$ . Letting  $B_\varepsilon = \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}_\sigma$  we get

$$\mu^*(B_\varepsilon \setminus E) = \mu^*\left(\bigcup_{n \in \mathbb{N}} (A_n \cap E^c)\right) \leq \mu^*\left(\bigcup_{n \in \mathbb{N}} (A_n \cap E_n^c)\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n \setminus E_n) \leq \varepsilon.$$

Finally we let  $B = \bigcap_{k \in \mathbb{N}} B_{1/k} \in \mathcal{A}_{\sigma\delta}$ , and we get  $\mu^*(B \setminus E) = 0$  as desired.

**REMARK 1.1.** Notice that (b) and (c) in particular show that any Lebesgue measurable set  $E$ , and therefore any Borel set, is the union of a  $G_\delta$  set  $B$  and a Lebesgue null set  $B \setminus E$ .  $\lrcorner$

#### EXERCISE 1.20

Let  $\mu^*$  be an outer measure on  $X$ ,  $\mathcal{M}^*$  the  $\sigma$ -algebra of  $\mu^*$ -measurable sets,  $\bar{\mu} = \mu^*|_{\mathcal{M}^*}$ , and  $\mu^+$  the outer measure induced by  $\bar{\mu}$  as in (1.12) (with  $\bar{\mu}$  and  $\mathcal{M}^*$  replacing  $\mu_0$  and  $\mathcal{A}$ ).

- (a) If  $E \subseteq X$ , we have  $\mu^*(E) \leq \mu^+(E)$ , with equality iff there exists  $A \in \mathcal{M}^*$  with  $A \supseteq E$  and  $\mu^*(A) = \mu^*(E)$ .
- (b) If  $\mu^*$  is induced from a premeasure, then  $\mu^* = \mu^+$ .
- (c) If  $X = \{0, 1\}$ , there exists an outer measure  $\mu^*$  on  $X$  such that  $\mu^* \neq \mu^+$ .

(a) Recall that the definition of  $\mu^+$  means that

$$\mu^+(E) = \inf \left\{ \sum_{n=1}^{\infty} \bar{\mu}(A_n) \mid A_n \in \mathcal{M}^*, E \subseteq \bigcup_{n \in \mathbb{N}} A_n \right\},$$

and that we by definition of  $\bar{\mu}$  can replace  $\bar{\mu}$  with  $\mu^*$ . For any such sequence  $(A_n)$  we have

$$\mu^*(E) \leq \mu^*\left(\bigcup_{n \in \mathbb{N}} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n) = \sum_{n=1}^{\infty} \bar{\mu}(A_n).$$

And since  $\mu^+(E)$  is the infimum of all such sums, we have  $\mu^*(E) \leq \mu^+(E)$ .

Next assume that there is an  $A \in \mathcal{M}^*$  with  $E \subseteq A$  such that  $\mu^*(A) = \mu^*(E)$ . Using the sequence  $A_1 = A$  and  $A_n = \emptyset$  for  $n > 1$  in the definition of  $\mu^+$  yields

$$\mu^+(E) \leq \bar{\mu}(A) = \mu^*(A) = \mu^*(E).$$

Hence  $\mu^+(E) = \mu^*(E)$  as desired.

Conversely, assuming that  $\mu^*(E) = \mu^+(E)$  we have

$$\mu^*(E) = \inf \left\{ \sum_{n=1}^{\infty} \mu^*(A_n) \mid A_n \in \mathcal{M}^*, E \subseteq \bigcup_{n \in \mathbb{N}} A_n \right\}.$$

Given  $\varepsilon > 0$ , choose a sequence  $(A_n)$  such that

$$\mu^*\left(\bigcup_{n \in \mathbb{N}} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n) \leq \mu^*(E) + \varepsilon,$$

and let  $B_\varepsilon = \bigcup_{n \in \mathbb{N}} A_n$ . Letting  $A = \bigcap_{k \in \mathbb{N}} B_{1/k} \in \mathcal{M}^*$  we thus have  $\mu^*(A) \leq \mu^*(E)$ .

(b) Assume that  $\mu^*$  is induced from a premeasure on an algebra  $\mathcal{A}$ , and let  $E \subseteq X$ . Recall that  $\mathcal{A}$  consists of  $\mu^*$ -measurable sets, so  $\sigma(\mathcal{A}) \subseteq \mathcal{M}^*$ . For  $n \in \mathbb{N}$  choose, in accordance with Exercise 1.18(a), a set  $A_n \in \mathcal{A}_\sigma$  with  $E \subseteq A_n$  such that  $\mu^*(A_n) \leq \mu^*(E) + 1/n$ . Letting  $A = \bigcap_{n \in \mathbb{N}} A_n$  we have  $E \subseteq A$  and  $\mu^*(A) \leq \mu^*(E)$ . The other inequality is obvious, so  $\mu^*(A) = \mu^*(E)$ , and part (a) implies that  $\mu^*(E) = \mu^+(E)$  as desired.

#### EXERCISE 1.21

Let  $\mu^*$  be an outer measure induced from a premeasure and  $\bar{\mu}$  the restriction of  $\mu^*$  to the  $\mu^*$ -measurable sets. Then  $\bar{\mu}$  is saturated.

Let  $\mathcal{A}$  denote the algebra on which the premeasure in question is defined, and denote by  $\mathcal{M}^*$  the  $\sigma$ -algebra of  $\mu^*$ -measurable sets. Recall that  $\mathcal{A} \subseteq \mathcal{M}^*$ .

Let  $E \subseteq X$  be locally measurable. It suffices to show that

$$\mu^*(F) \geq \mu^*(F \cap E) + \mu^*(F \cap E^c)$$

for all  $F \subseteq X$  with  $\mu^*(F) < \infty$ . Given  $\varepsilon > 0$ , Exercise 1.18(a) yields a set  $A \in \mathcal{A}_\sigma$  such that  $\mu^*(A) \leq \mu^*(F) + \varepsilon$ . Then  $\mu^*(A) < \infty$ , and so  $E \cap A \in \mathcal{M}^*$ . It follows that

$$\begin{aligned} \mu^*(F) + \varepsilon &\geq \mu^*(A) = \mu^*(A \cap (E \cap A)) + \mu^*(A \cap (E \cap A)^c) \\ &= \mu^*(A \cap E) + \mu^*(A \cap E^c) \\ &\geq \mu^*(F \cap E) + \mu^*(F \cap E^c), \end{aligned}$$

and hence  $E \in \mathcal{M}^*$ . Thus  $\bar{\mu}$  is saturated.

#### EXERCISE 1.22

Let  $(X, \mathcal{M}, \mu)$  be a measure space,  $\mu^*$  the outer measure induced by  $\mu$  according to (1.12),  $\mathcal{M}^*$  the  $\sigma$ -algebra of  $\mu^*$ -measurable sets, and  $\bar{\mu} = \mu^*|_{\mathcal{M}^*}$ .

- (a) If  $\mu$  is  $\sigma$ -finite, then  $\bar{\mu}$  is the completion of  $\mu$ .
- (b) In general,  $\bar{\mu}$  is the saturation of the completion of  $\mu$ .

(a) Let  $\bar{\mathcal{M}}$  be the  $\sigma$ -algebra from Theorem 1.9 (namely, the  $\sigma$ -algebra generated by the sets in  $\mathcal{M}$  along with all  $\mu$ -null sets). This is clearly the smallest  $\sigma$ -algebra on which there can exist a complete extension of  $\mu$ , so since  $\bar{\mu}$  is also a complete extension of  $\mu$ , we must have  $\bar{\mathcal{M}} \subseteq \mathcal{M}^*$ . Theorem 1.9 yields the uniqueness of a complete extension of  $\mu$  on  $\bar{\mathcal{M}}$ , so it suffices to show that  $\mathcal{M}^* \subseteq \bar{\mathcal{M}}$ .

Now assume that  $\mu$  is  $\sigma$ -finite, and let  $E \in \mathcal{M}^*$ . Then also  $E^c \in \mathcal{M}^*$ , and Exercise 1.18(c) ensures the existence of sets  $B, D \in \mathcal{M}_{\sigma\delta} = \mathcal{M}$  with  $E \subseteq B$  and  $E^c \subseteq D$  such that

$$\mu(B \setminus E) = 0 \quad \text{and} \quad \mu^*(E \setminus D^c) = \mu^*(D \setminus E^c) = 0.$$

It follows that

$$\mu(B \setminus D^c) \leq \mu^*(B \setminus E) + \mu^*(E \setminus D^c) = 0,$$

so  $E \setminus D^c$  is a  $\mu$ -null set. Thus  $E = D^c \cup (E \setminus D^c)$  is a union of a set in  $\mathcal{M}$  and a  $\mu$ -null set, and hence  $E \in \bar{\mathcal{M}}$ .

(b) Let  $\hat{\mu}$  denote the completion of  $\mu$  on  $\bar{\mathcal{M}}$ , and let  $\widetilde{\mathcal{M}}$  denote the  $\sigma$ -algebra of locally  $\hat{\mu}$ -measurable sets. First we show that  $\widetilde{\mathcal{M}} = \mathcal{M}^*$ , so let  $E \in \widetilde{\mathcal{M}}$ . To show that  $E$  is  $\mu^*$ -measurable it suffices to show that

$$\mu^*(F) \geq \mu^*(F \cap E) + \mu^*(F \cap E^c)$$

for all  $F \subseteq X$  with  $\mu^*(F) < \infty$ . Calculations identical to the ones in the solution to Exercise 1.21 show this.

Conversely, let  $E \in \mathcal{M}^*$  and consider  $A \in \bar{\mathcal{M}}$  with  $\hat{\mu}(A) < \infty$ . Then also  $A \in \mathcal{M}^*$ , so  $E \cap A \in \mathcal{M}^*$ . The argument at the beginning of part (a) showed that



$\bar{\mu}$  is an extension of  $\hat{\mu}$ , so  $\mu^*(E \cap A) = \hat{\mu}(E \cap A) < \infty$ . The same argument as in part (a), only now using Exercise 1.18(b) instead of (c), shows that  $E \cap A \in \overline{\mathcal{M}}$ , and so  $E \in \overline{\mathcal{M}}$ .

Finally, let  $\tilde{\mu}$  denote the saturation of  $\hat{\mu}$ . We show that  $\bar{\mu} = \tilde{\mu}$ . Since the completion of  $\mu$  on  $\overline{\mathcal{M}}$  is unique, the two measures must agree here. Instead let  $E \in \overline{\mathcal{M}} \setminus \mathcal{M}$ . By definition of  $\tilde{\mu}$  we must then have  $\tilde{\mu}(E) = \infty$ . On the other hand, we just showed (for  $E \cap A$  instead of  $E$ ) that  $\mu^*(E) < \infty$  implies  $E \in \mathcal{M}$ . Since we have assumed that this is not the case, we must have  $\bar{\mu}(E) = \mu^*(E) = \infty$ . Thus  $\bar{\mu} = \tilde{\mu}$ .

## 2 • Integration

## 3 • Signed Measures and Differentiation

### 3.1. Signed Measures

#### Exercise 2

Assume that  $E$  is  $\nu$ -null, and let  $P \cup N$  be a Hahn decomposition for  $\nu$ . Then

$$\nu^+(E) = \nu(E \cap P) = 0,$$

since  $E \cap P \subseteq E$ . Similarly we get  $\nu^-(E) = 0$ , so  $|\nu|(E) = 0$ .

Conversely, assume that  $|\nu|(E) = 0$ . Then  $\nu^\pm(F) = 0$  for all measurable  $F \subseteq E$ , and so  $\nu(F) = 0$ .

The other claims follow directly from the above.

#### Exercise 3

- (a) This follows directly from the definition of  $L^1(\nu)$ .
- (b) This is a simple application of the triangle inequality.
- (c) If  $|f| \leq 1$ , then

$$\left| \int_E f \, d\nu \right| \leq \int_E |f| \, d|\nu| \leq |\nu|(E),$$

showing one inequality. For the other inequality, let  $P \cup N$  be a Hahn decomposition of  $\nu$ , and let  $f = \mathbf{1}_P - \mathbf{1}_N$ . Then  $\int_E f \, d\nu = |\nu|(E)$ .

*Exercise 4*

Let  $P \cup N$  be a Hahn decomposition for  $\nu$ . Then

$$\nu^+(E) = \nu(E \cap P) \leq \lambda(E \cap P) \leq \lambda(E),$$

and similarly for  $\nu^-$ .

*Exercise 5*

First notice that

$$\nu_1 + \nu_2 = (\nu_1^+ + \nu_2^+) - (\nu_1^- + \nu_2^-),$$

so by the previous exercise we have

$$|\nu_1 + \nu_2| = (\nu_1 + \nu_2)^+ + (\nu_1 + \nu_2)^- \leq (\nu_1^+ + \nu_2^+) + (\nu_1^- + \nu_2^-) = |\nu_1| + |\nu_2|.$$