# Kozen, *Automata and Computability*

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## Miscellaneous Exercises

### *Finite Automata and Regular Sets*

REMARK 1: The Pumping Lemma.
We prove the pumping lemma:

> *Let $A$ be a regular set over an alphabet $\Sigma$. Then there exists a $k \in \mathbb{N}$ such that for any $x, y, z \in \Sigma^*$ with $xyz \in A$ and $|y| \geq k$, there exist $u, v, w \in \Sigma^*$ such that $y = uvw$, $v \neq \varepsilon$, and $xuv^n wz \in A$ for all $n \in \mathbb{N}$.*

Let $M = (Q, \Sigma, \delta, s, F)$ be a DNF accepting $A$, let $k = |Q|$, and let $x, y, z$ be given as above. Write $y = y_1 \cdots y_k$ with $y_i \in \Sigma$. Let $p_0 = \hat{\delta}(s, x)$ and define $p_i = \delta(p_{i-1}, y_i)$ for $i = 1, \ldots, k$. Then since $k + 1 > |Q|$, there exist $i < j$ such that $p_i = p_j$. Let $u = y_1 \cdots y_i$, $v = y_{i+1} \cdots y_j$ and $w = y_{j+1} \cdots y_k$, and notice that $v \neq \varepsilon$ since $i < j$. Furthermore, clearly $\hat{\delta}(p_i, v) = p_j = p_i$, so by induction we have

$$\hat{\delta}(p_i, v^n) = \hat{\delta}\big(\hat{\delta}(p_i, v^{n-1}), v\big) = \hat{\delta}(p_i, v) = p_i.$$

Since $\hat{\delta}(s, xu) = p_i$, it follows that

$$\hat{\delta}(s, xuv^n wz) = \hat{\delta}\big(\hat{\delta}(p_i, v^n), wz\big) = \hat{\delta}(p_i, vwz) = \hat{\delta}(s, xyz) \in F. \qquad \square$$

EXERCISE 7

If $\Sigma$ is an alphabet, consider the triple $(2^{\Sigma^*}, \cup, \cdot)$, where $\cdot$ denotes concatenation. Since contatenation distributes over unions, this is a semiring[1] with addition $\cup$ and multiplication $\cdot$. In particular, the set of matrices with entries in $2^{\Sigma^*}$ inherits an addition and multiplication, making it into a semiring itself.

Given a DFA $M = (Q, \Sigma, \delta, s, F)$ with $n$ states, define its *transition matrix* $G$ as the $n \times n$ matrix given by

$$G_{uv} = \{a \in \Sigma \mid \delta(u, a) = v\}$$

for $u, v \in Q$.

(a) Prove that

$$(G^n)_{uv} = \{x \in \Sigma^* \mid |x| = n, \text{ and } \hat{\delta}(u,x) = v\}.$$

(b) Define the Kleene closure $A^*$ of the matrix $A$ to be the componentwise union of all the powers of $A$:

$$(A^*)_{uv} = \bigcup_{n \in \mathbb{N}} (A^n)_{uv}.$$

Prove that

$$\mathcal{L}(M) = \bigcup_{t \in F} (G^*)_{st}.$$

SOLUTION. (a) For $n = 0$ we have $x = \varepsilon$, so $\hat{\delta}(u,\varepsilon) = v$ if and only if $u = v$, and so the right-hand side above equals $I_{uv}$.

Assume that the claim holds for some $n \in \mathbb{N}$. Then

$$(G^{n+1})_{uv} = (G^n G)_{uv} = \bigcup_{w \in Q} (G^n)_{uw} G_{wv}.$$

Now notice that, by induction,

$$\begin{aligned}
(G^n)_{uw} G_{wv} &= \{x \in \Sigma^* \mid |x| = n, \text{ and } \hat{\delta}(u,x) = w\} G_{wv} \\
&= \{xa \in \Sigma^* \mid |x| = n,\ \hat{\delta}(u,x) = w, \text{ and } \delta(w,a) = v\} \\
&= \{xa \in \Sigma^* \mid |x| = n,\ \hat{\delta}(u,x) = w, \text{ and } \delta(\hat{\delta}(u,x),a) = v\} \\
&= \{xa \in \Sigma^* \mid |x| = n,\ \hat{\delta}(u,x) = w, \text{ and } \hat{\delta}(u,xa) = v\}.
\end{aligned}$$

Since the condition $\hat{\delta}(u,x) = w$ is always satisfied for some $w \in Q$, it follows that

$$\begin{aligned}
\bigcup_{w \in Q} (G^n)_{uw} G_{wv} &= \bigcup_{w \in Q} \{xa \in \Sigma^* \mid |x| = n,\ \hat{\delta}(u,x) = w, \text{ and } \hat{\delta}(u,xa) = v\} \\
&= \{xa \in \Sigma^* \mid |xa| = n+1, \text{ and } \hat{\delta}(u,xa) = v\} \\
&= (G^{n+1})_{uv}
\end{aligned}$$

as desired.

(b) A string $x \in \Sigma^*$ lies in $\mathcal{L}(M)$ if and only if there is a $t \in F$ such that $\hat{\delta}(s,x) = t$, iff $x \in (G^{|x|})_{st}$. The claim follows easily. □

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[1] Recall that, in a semiring, we require that multiplication by the additive identity annihilate any element in the semiring. In a ring this follows from additive cancellation. The set $2^{\Sigma^*}$ is in fact a Kleene algebra, but we do not need this fact below.

EXERCISE 10

Recall from Lecture 10 that an $\varepsilon$-*NFA* is a structure

$$M = (Q, \Sigma, \varepsilon, \Delta, S, F)$$

such that $\varepsilon$ is a special symbol not in $\Sigma$ and

$$M_\varepsilon = (Q, \Sigma \cup \{\varepsilon\}, \Delta, S, F)$$

is an ordinary NFA.

Define the $\varepsilon$-*closure* $C_\varepsilon(A)$ of a set $A \subseteq Q$ to be the set of all states reachable from some state in $A$ under a sequence of zero or more $\varepsilon$-transitions:

$$C_\varepsilon(A) = \bigcup_{x \in \{\varepsilon\}^*} \hat{\Delta}(A, x) = \bigcup_{k \in \mathbb{N}} \hat{\Delta}(A, \varepsilon^k).$$

(a) Using $\varepsilon$-closure, define formally acceptance for $\varepsilon$-NFAs in a way that captures the intuitive description given in Lecture 6.

(b) Prove that under your definition, $\varepsilon$-NFAs accept only regular sets.

(c) Prove that the two definitions of acceptance – the one given in part (a) involving $\varepsilon$-closure and the one given in Lecture 10 involving homomorphisms – are equivalent.

SOLUTION. (a) Notice that $\varepsilon$-closure is a map $C_\varepsilon \colon 2^Q \to 2^Q$, and that since $M_\varepsilon$ is an NFA, $\Delta$ is a map $Q \times (\Sigma \cup \{\varepsilon\}) \to 2^Q$. We define a modified transition function

$$\Delta_\varepsilon \colon Q \times (\Sigma \cup \{\varepsilon\}) \to 2^Q$$

by $\Delta_\varepsilon = C_\varepsilon \circ \Delta$. We then say that $M$ *accepts* $x \in \Sigma^*$ if the NFA

$$(Q, \Sigma \cup \{\varepsilon\}, \Delta_\varepsilon, C_\varepsilon(S), F)$$

accepts $x$. Denoting the corresponding extended transition function by $\hat{\Delta}_\varepsilon$, this means that $M$ accepts $x$ if $\hat{\Delta}_\varepsilon(C_\varepsilon(S), x) \cap F \neq \emptyset$.

(b) This is obvious from (a), since NFAs only accept regular sets by Theorem 6.4.

(c) First define

$$\begin{aligned} \mathcal{L}(M) &= h(\mathcal{L}(M_\varepsilon)) \\ \mathcal{L}'(M) &= \{x \in \Sigma^* \mid \hat{\Delta}_\varepsilon(C_\varepsilon(S), x) \cap F \neq \emptyset\}, \end{aligned}$$

where $h \colon (\Sigma \cup \{\varepsilon\})^* \to \Sigma^*$ is given by $h(\varepsilon) = \varepsilon$ and $h(a) = a$ for $a \in \Sigma$. We prove the following claim:

*For every $A \subseteq Q$ and $x \in \Sigma^*$ we have*

$$\bigcup_{y \in h^{-1}(x)} \hat{\Delta}(A, y) = \hat{\Delta}_\varepsilon(C_\varepsilon(A), x).$$

The equality $\mathcal{L}(M) = \mathcal{L}'(M)$ follows easily by letting $A = S$. We prove the claim by induction on the length of $x$. If $x = \varepsilon$, then

$$\hat{\Delta}_\varepsilon(C_\varepsilon(A), \varepsilon) = C_\varepsilon(A) = \bigcup_{k \in \mathbb{N}} \hat{\Delta}(A, \boldsymbol{\varepsilon}^k) = \bigcup_{y \in h^{-1}(\varepsilon)} \hat{\Delta}(A, y),$$

where the last equality follows since $h$ sends precisely strings on the form $\boldsymbol{\varepsilon}^k$ to $\varepsilon$. Now assume that the claim holds for some string $x \in \Sigma^*$, and let $a \in \Sigma$. Then

$$\begin{aligned}
\hat{\Delta}_\varepsilon(C_\varepsilon(A), xa) &= \hat{\Delta}_\varepsilon\big(\hat{\Delta}_\varepsilon(C_\varepsilon(A), x), a\big) \\
&= \hat{\Delta}_\varepsilon\big(C_\varepsilon \circ \hat{\Delta}_\varepsilon(C_\varepsilon(A), x), a\big) \\
&= \hat{\Delta}_\varepsilon\Bigg(C_\varepsilon\Bigg(\bigcup_{y \in h^{-1}(x)} \hat{\Delta}(A, y)\Bigg), a\Bigg) \\
&= \bigcup_{b \in h^{-1}(a)} \hat{\Delta}\Bigg(\bigcup_{y \in h^{-1}(x)} \hat{\Delta}(A, y), b\Bigg) \\
&= \bigcup_{b \in h^{-1}(a)} \bigcup_{y \in h^{-1}(x)} \hat{\Delta}\big(\hat{\Delta}(A, y), b\big) \\
&= \bigcup_{b \in h^{-1}(a)} \bigcup_{y \in h^{-1}(x)} \hat{\Delta}(A, yb) \\
&= \bigcup_{z \in h^{-1}(xa)} \hat{\Delta}(A, z),
\end{aligned}$$

proving the claim. $\square$