Lauritzen: *Undergraduate Convexity*

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2 • Affine subspaces

General notes

REMARK 2.1: Linear subspaces are null spaces.
In the proof of Proposition 2.6 we use the fact that every linear subspace W of $\mathbb{R}^d$ is the null space for some $m \times d$ matrix A with $m \leq d$. We prove a more general claim:

> *Let V be a vector space, and let U be a complemented subspace of V. Then U is the kernel of some surjective linear map $\varphi \colon V \to W$.*

Recall that U is a complemented subspace of V if V has a subspace U' such that $V = U \oplus U'$. In this case V is the coproduct of U and U', so if $0 \colon U \to U'$ is the the map sending every element of U to 0, then the coproduct map $\varphi = [0, \mathrm{id}_{U'}] \colon U \oplus U' \to U'$ is a linear map with kernel U. To prove this last claim, note that we clearly have $U \subseteq \ker \varphi$, and conversely of $u + u' \in \ker \varphi$ then

$$0 = \varphi(u + u') = 0(u) + \mathrm{id}_{U'}(u') = u',$$

so $u + u' = u \in U$. Also, φ is clearly surjective.

In the case of Proposition 2.6, since $\mathbb{R}^d$ is finite-dimensional every subspace is complemented (e.g. by basis considerations), so there is a φ as above with $\ker \varphi = W$. Since φ is surjective its codomain has dimension $m \leq d$. To obtain A, simply take the standard matrix representation of φ. ⌟

REMARK 2.2: Solutions to systems of linear equations.
Let A be an $m \times d$ matrix. Recall that the solution set of the equation $Ax = b$ is $x_0 + N(A)$, where x_0 is a particular solution to the equation. Conversely, for any $x_0 \in \mathbb{R}^d$ there is an equation with solution set $x_0 + N(A)$, namely $Ax = Ax_0$. This is clear, since x_0 is trivially a solution, hence the solution set is $x_0 + N(A)$.

In Proposition 2.6 we first show that any affine subspace M is on the form $x_0 + W$ for some linear subspace W. By Remark 2.1 $W = N(A)$ for some matrix A, so $x_0 + W$ is the solution set of the equation $Ax = Ax_0$. ⌟

Remark 2.3: The affine hull and dimension of the empty set.
By definition, the affine hull of the empty set is the empty set. This is the case by Definition 2.4, and this is also the standard definition. Hence the affine hull of the empty set is not an affine subspace! In particular, it cannot have a dimension in the sense of Definition 2.7. ⌟

Exercises

Exercise 2.4

Prove that $M = \{v \in \mathbb{R}^d \mid Av = b\}$ is an affine subspace of $\mathbb{R}^d$, where A is an $m \times d$ matrix and $b \in \mathbb{R}^m$.

Solution. For $u, v \in \mathbb{R}^d$ and $t \in \mathbb{R}$ we have

$$A((1-t)u + tv) = (1-t)Au + tAv = (1-t)b + tb = b. \qquad \square$$

Exercise 2.9

Prove that $f(x) = h(x) + b$ is an affine map if $h\colon \mathbb{R}^d \to \mathbb{R}^m$ is a linear map and $b \in \mathbb{R}^m$. Prove that $h(x) = f(x) - f(0)$ is a linear map if $f : \mathbb{R}^d \to \mathbb{R}^m$ is an affine map.

Solution. First assume that h is linear, and let $u, v \in \mathbb{R}^d$ and $t \in \mathbb{R}$. Then

$$\begin{aligned} f((1-t)u + tv) &= h((1-t)u + tv) + b \\ &= (1-t)h(u) + th(v) + (1-t)b + t(b) \\ &= (1-t)(h(u) + b) + t(h(v) + b) \\ &= (1-t)f(u) + tf(v), \end{aligned}$$

so f is affine. Next assume that f is affine, and let $u, v \in \mathbb{R}^d$ and $\beta \in \mathbb{R}$. If $\beta = -1$ then

$$\begin{aligned} h((-1)u) = f((-1)u) - f(0) &= f(2 \cdot 0 + (-1)u) - f(0) \\ &= 2f(0) - f(u) - f(0) = -h(u). \end{aligned}$$

If instead $\beta \neq -1$, then

$$\begin{aligned}
h(\beta u + v) &= f(\beta u + v) - f(0) \\
&= f\Big(\tfrac{\beta}{\beta+1}(\beta+1)u + \tfrac{1}{\beta+1}(\beta+1)v\Big) - f(0) \\
&= \tfrac{\beta}{\beta+1} f((\beta+1)u) + \tfrac{1}{\beta+1} f((\beta+1)v) + \Big(\tfrac{\beta^2}{\beta+1} + \tfrac{\beta}{\beta+1} - (\beta+1)\Big) f(0) \\
&= \beta\Big(\tfrac{1}{\beta+1} f((\beta+1)u) + \tfrac{\beta}{\beta+1} f(0)\Big) + \Big(\tfrac{1}{\beta+1} f((\beta+1)v) + \tfrac{\beta}{\beta+1} f(0)\Big) - (\beta+1) f(0) \\
&= \beta f(u) + f(v) - (\beta+1) f(0) \\
&= \beta(f(u) - f(0)) + (f(v) - f(0)) \\
&= \beta h(u) + h(v).
\end{aligned}$$

□

Exercise 2.10

Prove that you can have no more than $d+1$ affinely independent vectors in $\mathbb{R}^d$.

Solution. Let $\{v_1, \dots, v_n\} \subseteq \mathbb{R}^d$ be affinely independent. By Proposition 2.9, the $n-1$ vectors $v_2 - v_1, \dots, v_n - v_1$ vectors are linearly independent. But then we must have $n - 1 \leq d$, i.e. $n \leq d+1$. □

Exercise 2.11

Let $v_0, \dots, v_d$ be affinely independent points in $\mathbb{R}^d$. Prove that

$$f(x) = (\lambda_0, \lambda_1, \dots, \lambda_d)$$

is a well defined affine map $f\colon \mathbb{R}^d \to \mathbb{R}^{d+1}$, where

$$x = \lambda_0 v_0 + \dots + \lambda_d v_d$$

with $\lambda_0 + \dots + \lambda_d = 1$.

Solution. If also $x = \mu_0 v_0 + \dots + \mu_d v_d$ with $\mu_0 + \dots + \mu_d = 1$, then

$$(\lambda_0 - \mu_0)v_0 + \dots + (\lambda_d - \mu_d)v_d = 0,$$

so since the v_i are affinely independent we have $\lambda_i = \mu_i$ by Proposition 2.9. Hence f is well-defined.

Next we show that f is affine. Let

$$x = \lambda_0 v_0 + \dots + \lambda_d v_d \quad \text{and} \quad y = \mu_0 v_0 + \dots + \mu_d v_d$$

with $\lambda_0 + \dots + \lambda_d = 1$ and $\mu_0 + \dots + \mu_d = 1$, and let $t \in \mathbb{R}$. Then

$$\sum_{i=0}^{d} \Big((1-t)\lambda_i + t\mu_i\Big) = (1-t)\sum_{i=0}^{d} \lambda_i + t\sum_{i=0}^{d} \mu_i = (1-t) + t = 1.$$

Therefore, since

$$(1-t)x + ty = \sum_{i=0}^{d} \big((1-t)\lambda_i + t\mu_i\big) v_i,$$

we have

$$\begin{aligned} f((1-t)x+ty) &= \big((1-t)\lambda_0 + t\mu_0, \dots, (1-t)\lambda_d + t\mu_d\big) \\ &= (1-t)(\lambda_0,\dots,\lambda_d) + t(\mu_0,\dots,\mu_d) \\ &= (1-t)f(x) + tf(y), \end{aligned}$$

so f is affine. □

EXERCISE 2.12

Prove that a non-empty open subset $U \subseteq \mathbb{R}^d$ has dimension $\dim U = d$. Show that a subset $S \subseteq \mathbb{R}^d$ with $\dim S = d$ contains a non-empty open subset.

SOLUTION. Let $x \in U$, and let $(e_1,\dots,e_d)$ be the standard basis for $\mathbb{R}^d$. Since U is open there is some $\delta > 0$ such that $x + \delta e_i \in U$ for all i. The linear subspace $-x + \operatorname{aff} U$ has dimension d, so U has dimension d.

The latter claim is in fact false, since if S is e.g. the collection of points δe_i along with the origin, then S has dimension d (since $\operatorname{aff} S$ contains each of the coordinate axes, hence the entire $\mathbb{R}^d$) but clearly contains no non-empty open subset. □

3 • Convex subsets

Exercises

EXERCISE 3.7

We state and prove a more general result that the one given in the exercise.

For any subsets $A, B \subseteq \mathbb{R}^d$ we have

$$\operatorname{conv}(A+B) = \operatorname{conv} A + \operatorname{conv} B.$$

SOLUTION. The inclusion '⊆' is clear since the sum of convex sets is convex. For the converse direction, let $v \in \operatorname{conv} A$ and $w \in \operatorname{conv} B$, and write $v = \sum_{i=1}^n \lambda_i v_i$ and $w = \sum_{j=1}^m \mu_j w_j$ as convex combinations of elements in A and B respectively. Then

$$v + w_j = v + (\lambda_1 + \cdots + \lambda_n) w_j = \sum_{i=1}^{n} \lambda_i (v_i + w_j) \in \operatorname{conv}(A+B),$$

and

$$v + w = (\mu_1 + \cdots + \mu_n)v + w = \sum_{j=1}^{m} \mu_j(v + w_j) \in \operatorname{conv}\operatorname{conv}(A + B) = \operatorname{conv}(A + B),$$

as desired. □

EXERCISE 3.9

We state and prove a more general result that the one given in the exercise.

For $S \subseteq \mathbb{R}^d$, prove that $(\operatorname{conv} S)^\circ = S^\circ$. If $S = \{v_1, \ldots, v_n\}$ is finite, then

$$(\operatorname{conv} S)^\circ = \{\alpha \in \mathbb{R}^d \mid \alpha^\top v_1 \le 1, \ldots, \alpha^\top v_n \le 1\} = \{\alpha \in \mathbb{R}^d \mid A\alpha \le 1\},$$

where A is the $n \times d$ matrix with rows $v_1^\top, \ldots, v_n^\top$. In particular, polytopes are polyhedra.

SOLUTION. Since $S \subseteq \operatorname{conv} S$, we clearly have $(\operatorname{conv} S)^\circ \subseteq S^\circ$. For the opposite inclusion, let $\alpha \in S^\circ$ and let $\lambda_1 v_1 + \cdots + \lambda_n v_n$ be a convex combination of elements in S. Then

$$\alpha^\top(\lambda_1 v_1 + \cdots + \lambda_n v_n) = \lambda_1 \alpha^\top v_1 + \cdots + \lambda_n \alpha^\top v_n \le \lambda_1 + \cdots \lambda_n = 1,$$

so $\alpha \in (\operatorname{conv} S)^\circ$. The rest of the claims follow easily. □

EXERCISE 3.10

If $F \subseteq G \subseteq C$ are convex subsets of $\mathbb{R}^d$, prove that F is a face of C if F is a face of G and G is a face of C.

SOLUTION. Let $x, y \in C$ such that the open line segment (x, y) intersects F. Then it intersects G, so since G is a face of C we have $x, y \in G$. Similarly, since F is a face of G we then have $x, y \in F$. Hence F is a face of C. □

EXERCISE 3.12

Prove that $C \setminus F$ is a convex subset if F is a face of a convex subset C. Is it true that $F \subseteq C$ is a face if $C \setminus F$ is a convex subset?

SOLUTION. Let $x, y \in C \setminus F$. If the open line segment (x, y) intersected F, then we would have $x, y \in F$ since F is a face of C. Thus $(x, y) \subseteq C \setminus F$, so in total $[x, y] \subseteq C \setminus F$. Hence $C \setminus F$ is convex.

The converse is false. For instance, remove (at least) two extreme points from a disk. □