Lauritzen: *Undergraduate Convexity*

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2 • Affine subspaces

General notes

REMARK 2.1. Let $f\colon \mathbb{R}^d \to \mathbb{R}^m$ be a map. Then the following properties are equivalent:

(a) There exists a matrix A and a vector b such that $f(x) = Ax + b$ for all $x \in \mathbb{R}^d$.

(b) For all $x, y \in \mathbb{R}^d$ and $t \in \mathbb{R}$, $f((1-t)x + ty) = (1-t)f(x) + tf(y)$.

(c) For all $x_1, \dots, x_k \in \mathbb{R}^d$ and $t_1, \dots, t_k \in \mathbb{R}$ with $t_1 + \cdots + t_k = 1$, $f(t_1x_1 + \cdots + t_kx_k) = t_1f(x_1) + \cdots + t_kf(x_k)$.

Such a function is called *affine*. Note that if $m = 1$, then f is affine if and only if it is both convex and concave.

The first property clearly entails the third, which then entails the second, so assume that f has the second property. First assume that $f(0) = 0$. For $\beta \in \mathbb{R}$ and $x \in \mathbb{R}^d$ be thus have

$$f(\beta x) = f(\beta x + (1-\beta)0) = \beta f(x) + (1-\beta)f(0) = \beta f(x),$$

so f is homogeneous. If also $y \in \mathbb{R}^d$, then

$$\tfrac{1}{2}f(x+y) = f\Big(\tfrac{1}{2}x + \tfrac{1}{2}y\Big) = \tfrac{1}{2}f(x) + \tfrac{1}{2}f(y),$$

so f is also additive, hence linear. For general f, simply replace f with $f - f(0)$. ⌟

REMARK 2.2. Let $M \subseteq \mathbb{R}^d$ be nonempty. Then the following properties are equivalent:

(a) For all $x, y \in M$ and $t \in \mathbb{R}$, $(1-t)x + ty \in M$.

(b) For all $x_1,\dots,x_k \in \mathbb{R}^d$ and $t_1,\dots,t_k \in \mathbb{R}$ with $t_1+\dots+t_k = 1$, $t_1x_1+\dots+t_kx_k \in M$.

(c) There is a linear subspace $W \subseteq \mathbb{R}^d$ and an element $x_0 \in M$ such that $M = x_0+W$. In this case $W = \{x-y \mid x,y \in M\}$, and the identity $M = x_0+W$ holds for all $x_0 \in M$.

(d) There is an $m\times d$ matrix A and a vector b such that $M = \{x \in \mathbb{R}^d \mid Ax = b\}$.

(e) There is an affine map $f : \mathbb{R}^d \to \mathbb{R}^m$ such that $M = \{x \in \mathbb{R}^d \mid f(x) = 0\}$.

If M has any of these properties, then M is called *affine*.

Clearly (b) implies (a), and the converse follows by noticing that when $k > 1$ at least one t_1 must be different from 1, say t_k, so

$$t_1x_1 + \dots + t_kx_k = (1-t_k)\left(\frac{t_1}{1-t_k}x_1 + \dots + \frac{t_{k-1}}{1-t_k}x_{k-1}\right) + t_kx_k.$$

Hence (b) follows by induction. Assuming (b), let $W = \{x-y \mid x,y \in M\}$, and consider $x_1,x_2,y_1,y_2 \in M$ and $\beta \in \mathbb{R}$. Then

$$\beta(x_1-y_1)+(x_2-y_2) = (\beta x_1 + x_2 + (-\beta)y_1) - y_2 \in W,$$

so W is a subspace. For any $x_0,x,y \in M$ and we have $x_0 + (x-y) \in M$, and conversely $x = x_0 + (x-x_0) \in x_0 + W$, so $M = x_0 + W$. If W' is another subspace such that $M = x_0 + W'$ for some x_0, then $W' = M - x_0 = W$.

If instead (c) holds, and $x,y \in M$ and $t \in \mathbb{R}$, then $x = x_0 + x'$ and $y = x_0 + y'$ for some $x',y' \in W$, and so

$$(1-t)x + ty = x_0 + (1-t)x' + ty' \in x_0 + W,$$

implying (a).

The equivalence between (c), (d) and (e) is obvious since W is the nullspace of some matrix, and we may choose any x_0 such that $Ax_0 = b$. (Furthermore, (e) easily implies (b).) ⌟

Remark 2.3. Let $f : \mathbb{R}^d \to \mathbb{R}^m$ be affine, and let $M \subseteq \mathbb{R}^d$ be affine. Then the image $f[M]$ is also affine. Furthermore, for any $S \subseteq \mathbb{R}^d$ we have $f[\operatorname{aff} S] = \operatorname{aff} f[S]$ (just as linear maps commute with span).

If $f(x) = Ax + b$ and $M = x_0 + W$ is affine with W a subspace, then

$$f[M] = f[x_0+W] = A[x_0+W] + b = (Ax_0+b) + A[W].$$

In particular, $\dim f[M] = \dim A[W]$. If f (and hence A) is invertible, then we thus have $\dim f[M] = \dim W = \dim M$. ⌟

REMARK 2.4. If $S \subseteq \mathbb{R}^d$ and $x_0 \in S$, then we claim that $\operatorname{aff} S = x_0 + \operatorname{span}(S - x_0)$. Consider the affine map $f(x) = x - x_0$. From the above remark we have

$$x_0 + \operatorname{aff} f[S] = x_0 + f[\operatorname{aff} S] = \operatorname{aff} S.$$

But $f[S]$ contains 0, so $\operatorname{aff} f[S]$ is a subspace and hence equal to $\operatorname{span} f[S]$. Furthermore, since f is invertible it preserves dimensions, so if $|S| = m$ and is positive and finite, then $\dim S = m - 1$ if and only if $\dim W = m - 1$, where $W = \operatorname{span}(S - x_0)$. But W is also spanned by $S \setminus \{x_0\} - x_0$ which has $m - 1$ elements, so S is affinely independent if and only if $S \setminus \{x_0\} - x_0$ is linearly independent. ⌟

REMARK 2.5. If $f : \mathbb{R}^d \to \mathbb{R}^d$ preserves the Euclidean metric, then it is affine. Replacing f with $f - f(0)$ we may assume that $f(0) = 0$, and so that f also preserves the inner product. For all $x, y, z \in \mathbb{R}^d$ and $\beta \in \mathbb{R}$ we have

$$\begin{aligned} \langle f(\beta x + y), f(z)\rangle &= \langle \beta x + y, z\rangle \\ &= \beta\langle x, z\rangle + \langle y, z\rangle \\ &= \beta\langle f(x), f(z)\rangle + \langle f(y), f(z)\rangle \\ &= \langle \beta f(x) + f(y), f(z)\rangle. \end{aligned}$$

Since f is surjective, the claim follows. ⌟

REMARK 2.6: The affine hull and dimension of the empty set.
By definition, the affine hull of the empty set is the empty set. This is the case by Definition 2.4, and this is also the standard definition. Hence the affine hull of the empty set is not an affine subspace! In particular, it cannot have a dimension in the sense of Definition 2.7. ⌟

Exercises

EXERCISE 2.10

Prove that you can have no more than $d + 1$ affinely independent vectors in $\mathbb{R}^d$.

SOLUTION. Let $\{v_1, \ldots, v_n\} \subseteq \mathbb{R}^d$ be affinely independent. By Proposition 2.9, the $n - 1$ vectors $v_2 - v_1, \ldots, v_n - v_1$ vectors are linearly independent. But then we must have $n - 1 \leq d$, i.e. $n \leq d + 1$. □

EXERCISE 2.11

Let $v_0, \ldots, v_d$ be affinely independent points in $\mathbb{R}^d$. Prove that

$$f(x) = (\lambda_0, \lambda_1, \ldots, \lambda_d)$$

is a well defined affine map $f : \mathbb{R}^d \to \mathbb{R}^{d+1}$, where

$$x = \lambda_0 v_0 + \cdots + \lambda_d v_d$$

with $\lambda_0 + \cdots + \lambda_d = 1$.

SOLUTION. If also $x = \mu_0 v_0 + \cdots + \mu_d v_d$ with $\mu_0 + \cdots + \mu_d = 1$, then

$$(\lambda_0 - \mu_0)v_0 + \cdots + (\lambda_d - \mu_d)v_d = 0,$$

so since the v_i are affinely independent we have $\lambda_i = \mu_i$ by Proposition 2.9. Hence f is well-defined.

Next we show that f is affine. Let

$$x = \lambda_0 v_0 + \cdots + \lambda_d v_d \quad \text{and} \quad y = \mu_0 v_0 + \cdots + \mu_d v_d$$

with $\lambda_0 + \cdots + \lambda_d = 1$ and $\mu_0 + \cdots + \mu_d = 1$, and let $t \in \mathbb{R}$. Then

$$\sum_{i=0}^{d} \bigl((1-t)\lambda_i + t\mu_i\bigr) = (1-t)\sum_{i=0}^{d} \lambda_i + t\sum_{i=0}^{d} \mu_i = (1-t) + t = 1.$$

Therefore, since

$$(1-t)x + ty = \sum_{i=0}^{d} \bigl((1-t)\lambda_i + t\mu_i\bigr)v_i,$$

we have

$$\begin{aligned} f((1-t)x + ty) &= \bigl((1-t)\lambda_0 + t\mu_0, \ldots, (1-t)\lambda_d + t\mu_d\bigr) \\ &= (1-t)(\lambda_0, \ldots, \lambda_d) + t(\mu_0, \ldots, \mu_d) \\ &= (1-t)f(x) + tf(y), \end{aligned}$$

so f is affine. □

EXERCISE 2.12

Prove that a non-empty open subset $U \subseteq \mathbb{R}^d$ has dimension $\dim U = d$. Show that a subset $S \subseteq \mathbb{R}^d$ with $\dim S = d$ contains a non-empty open subset.

SOLUTION. Let $x \in U$, and let $(e_1, \ldots, e_d)$ be the standard basis for $\mathbb{R}^d$. Since U is open there is some $\delta > 0$ such that $x + \delta e_i \in U$ for all i. The linear subspace $-x + \operatorname{aff} U$ has dimension d, so U has dimension d.

The latter claim is in fact false, since if S is e.g. the collection of points δe_i along with the origin, then S has dimension d (since $\operatorname{aff} S$ contains each of the coordinate axes, hence the entire $\mathbb{R}^d$) but clearly contains no non-empty open subset. □

3 • Convex subsets

Exercises

EXERCISE 3.7

We state and prove a more general result that the one given in the exercise.

For any subsets $A, B \subseteq \mathbb{R}^d$ we have

$$\operatorname{conv}(A+B) = \operatorname{conv} A + \operatorname{conv} B.$$

SOLUTION. The inclusion '$\subseteq$' is clear since the sum of convex sets is convex. For the converse direction, let $v \in \operatorname{conv} A$ and $w \in \operatorname{conv} B$, and write $v = \sum_{i=1}^n \lambda_i v_i$ and $w = \sum_{j=1}^m \mu_j w_j$ as convex combinations of elements in A and B respectively. Then

$$v + w_j = v + (\lambda_1 + \cdots + \lambda_n) w_j = \sum_{i=1}^n \lambda_i (v_i + w_j) \in \operatorname{conv}(A+B),$$

and

$$v + w = (\mu_1 + \cdots + \mu_n) v + w = \sum_{j=1}^m \mu_j (v + w_j) \in \operatorname{conv}\operatorname{conv}(A+B) = \operatorname{conv}(A+B),$$

as desired. □

EXERCISE 3.9

We state and prove a more general result that the one given in the exercise.

For $S \subseteq \mathbb{R}^d$, prove that $(\operatorname{conv} S)^\circ = S^\circ$. If $S = \{v_1, \ldots, v_n\}$ is finite, then

$$(\operatorname{conv} S)^\circ = \{\alpha \in \mathbb{R}^d \mid \alpha^\top v_1 \leq 1, \ldots, \alpha^\top v_n \leq 1\} = \{\alpha \in \mathbb{R}^d \mid A\alpha \leq 1\},$$

where A is the $n \times d$ matrix with rows $v_1^\top, \ldots, v_n^\top$. In particular, polytopes are polyhedra.

SOLUTION. Since $S \subseteq \operatorname{conv} S$, we clearly have $(\operatorname{conv} S)^\circ \subseteq S^\circ$. For the opposite inclusion, let $\alpha \in S^\circ$ and let $\lambda_1 v_1 + \cdots + \lambda_n v_n$ be a convex combination of elements in S. Then

$$\alpha^\top(\lambda_1 v_1 + \cdots + \lambda_n v_n) = \lambda_1 \alpha^\top v_1 + \cdots + \lambda_n \alpha^\top v_n \leq \lambda_1 + \cdots \lambda_n = 1,$$

so $\alpha \in (\operatorname{conv} S)^\circ$. The rest of the claims follow easily. □

Exercise 3.10

If $F \subseteq G \subseteq C$ are convex subsets of $\mathbb{R}^d$, prove that F is a face of C if F is a face of G and G is a face of C.

Solution. Let $x, y \in C$ such that the open line segment (x, y) intersects F. Then it intersects G, so since G is a face of C we have $x, y \in G$. Similarly, since F is a face of G we then have $x, y \in F$. Hence F is a face of C. $\square$

Exercise 3.12

Prove that $C \setminus F$ is a convex subset if F is a face of a convex subset C. Is it true that $F \subseteq C$ is a face if $C \setminus F$ is a convex subset?

Solution. Let $x, y \in C \setminus F$. If the open line segment (x, y) intersected F, then we would have $x, y \in F$ since F is a face of C. Thus $(x, y) \subseteq C \setminus F$, so in total $[x, y] \subseteq C \setminus F$. Hence $C \setminus F$ is convex.

The converse is false. For instance, remove (at least) two extreme points from a disk. $\square$