Lauritzen: *Undergraduate Convexity*

Danny Nygård Hansen

10th November 2022

2 • Affine subspaces

General notes

REMARK 2.1: Linear subspaces are null spaces.
In the proof of Proposition 2.6 we use the fact that every linear subspace W of $\mathbb{R}^d$ is the null space for some $m \times d$ matrix A with $m \leq d$. We prove a more general claim:

> *Let V be a vector space, and let U be a complemented subspace of V. Then U is the kernel of some surjective linear map $\varphi\colon V \to W$.*

Recall that U is a complemented subspace of V if V has a subspace U' such that $V = U \oplus U'$. In this case V is the coproduct of U and U', so if $0\colon U \to U'$ is the the map sending every element of U to 0, then the coproduct map $\varphi = [0, \mathrm{id}_{U'}]\colon U \oplus U' \to U'$ is a linear map with kernel U. To prove this last claim, note that we clearly have $U \subseteq \ker\varphi$, and conversely of $u + u' \in \ker\varphi$ then

$$0 = \varphi(u + u') = 0(u) + \mathrm{id}_{U'}(u') = u',$$

so $u + u' = u \in U$. Also, φ is clearly surjective.

In the case of Proposition 2.6, since $\mathbb{R}^d$ is finite-dimensional every subspace is complemented (e.g. by basis considerations), so there is a φ as above with $\ker\varphi = W$. Since φ is surjective its codomain has dimension $m \leq d$. To obtain A, simply take the standard matrix representation of φ. ⌟

Exercises

EXERCISE 2.4

Prove that $M = \{v \in \mathbb{R}^d \mid Av = b\}$ is an affine subspace of $\mathbb{R}^d$, where A is an

$m \times d$ matrix and $b \in \mathbb{R}^m$.

Solution. For $u, v \in \mathbb{R}^d$ and $t \in \mathbb{R}$ we have

$$A((1-t)u + tv) = (1-t)Au + tAv = (1-t)b + tb = b. \qquad \square$$

Exercise 2.9

Prove that $f(x) = h(x) + b$ is an affine map if $h\colon \mathbb{R}^d \to \mathbb{R}^m$ is a linear map and $b \in \mathbb{R}^m$. Prove that $h(x) = f(x) - f(0)$ is a linear map if $f : \mathbb{R}^d \to \mathbb{R}^m$ is an affine map.

Solution. First assume that h is linear, and let $u, v \in \mathbb{R}^d$ and $t \in \mathbb{R}$. Then

$$\begin{aligned} f((1-t)u + tv) &= h((1-t)u + tv) + b \\ &= (1-t)h(u) + th(v) + (1-t)b + t(b) \\ &= (1-t)(h(u) + b) + t(h(v) + b) \\ &= (1-t)f(u) + tf(v), \end{aligned}$$

so f is affine. Next assume that f is affine, and let $u, v \in \mathbb{R}^d$ and $\beta \in \mathbb{R}$. If $\beta = -1$ then

$$\begin{aligned} h((-1)u) = f((-1)u) - f(0) &= f(2 \cdot 0 + (-1)u) - f(0) \\ &= 2f(0) - f(u) - f(0) = -h(u). \end{aligned}$$

If instead $\beta \neq -1$, then

$$\begin{aligned} h(\beta u + v) &= f(\beta u + v) - f(0) \\ &= f\Big(\tfrac{\beta}{\beta+1}(\beta+1)u + \tfrac{1}{\beta+1}(\beta+1)v\Big) - f(0) \\ &= \tfrac{\beta}{\beta+1}f((\beta+1)u) + \tfrac{1}{\beta+1}f((\beta+1)v) + \Big(\tfrac{\beta^2}{\beta+1} + \tfrac{\beta}{\beta+1} - (\beta+1)\Big)f(0) \\ &= \beta\Big(\tfrac{1}{\beta+1}f((\beta+1)u) + \tfrac{\beta}{\beta+1}f(0)\Big) + \Big(\tfrac{1}{\beta+1}f((\beta+1)v) + \tfrac{\beta}{\beta+1}f(0)\Big) - (\beta+1)f(0) \\ &= \beta f(u) + f(v) - (\beta+1)f(0) \\ &= \beta(f(u) - f(0)) + (f(v) - f(0)) \\ &= \beta h(u) + h(v). \end{aligned} \qquad \square$$

Exercise 2.10

Prove that you can have no more than $d+1$ affinely independent vectors in $\mathbb{R}^d$.

Solution. Let $\{v_1, \ldots, v_n\} \subseteq \mathbb{R}^d$ be affinely independent. By Proposition 2.9, the $n-1$ vectors $v_2 - v_1, \ldots, v_n - v_1$ vectors are linearly independent. But then we must have $n - 1 \leq d$, i.e. $n \leq d+1$. $\square$

Exercise 2.11

Let $v_0, \dots, v_d$ be affinely independent points in $\mathbb{R}^d$. Prove that

$$f(x) = (\lambda_0, \lambda_1, \dots, \lambda_d)$$

is a well defined affine map $f : \mathbb{R}^d \to \mathbb{R}^{d+1}$, where

$$x = \lambda_0 v_0 + \dots + \lambda_d v_d$$

with $\lambda_0 + \dots + \lambda_d = 1$.

Solution. If also $x = \mu_0 v_0 + \dots + \mu_d v_d$ with $\mu_0 + \dots + \mu_d = 1$, then

$$(\lambda_0 - \mu_0) v_0 + \dots + (\lambda_d - \mu_d) v_d = 0,$$

so since the v_i are affinely independent we have $\lambda_i = \mu_i$ by Proposition 2.9. Hence f is well-defined.

Next we show that f is affine. Let

$$x = \lambda_0 v_0 + \dots + \lambda_d v_d \quad \text{and} \quad y = \mu_0 v_0 + \dots + \mu_d v_d$$

with $\lambda_0 + \dots + \lambda_d = 1$ and $\mu_0 + \dots + \mu_d = 1$, and let $t \in \mathbb{R}$. Then

$$\sum_{i=0}^{d} \big((1-t)\lambda_i + t\mu_i\big) = (1-t)\sum_{i=0}^{d} \lambda_i + t\sum_{i=0}^{d} \mu_i = (1-t) + t = 1.$$

Therefore, since

$$(1-t)x + ty = \sum_{i=0}^{d} \big((1-t)\lambda_i + t\mu_i\big) v_i,$$

we have

$$\begin{aligned} f((1-t)x + ty) &= \big((1-t)\lambda_0 + t\mu_0, \dots, (1-t)\lambda_d + t\mu_d\big) \\ &= (1-t)(\lambda_0, \dots, \lambda_d) + t(\mu_0, \dots, \mu_d) \\ &= (1-t) f(x) + t f(y), \end{aligned}$$

so f is affine. □

Exercise 2.12

Prove that a non-empty open subset $U \subseteq \mathbb{R}^d$ has dimension $\dim U = d$. Show that a subset $S \subseteq \mathbb{R}^d$ with $\dim S = d$ contains a non-empty open subset.

SOLUTION. Let $x \in U$, and let $(e_1, \ldots, e_d)$ be the standard basis for $\mathbb{R}^d$. Since U is open there is some $\delta > 0$ such that $x + \delta e_i \in U$ for all i. The linear subspace $-x + \operatorname{aff} U$ has dimension d, so U has dimension d.

The latter claim is in fact false, since if S is e.g. the collection of points δe_i along with the origin, then S has dimension d (since $\operatorname{aff} S$ contains each of the coordinate axes, hence the entire $\mathbb{R}^d$) but clearly contains no non-empty open subset. $\square$