Lee: *Introduction to Topological Manifolds*

Danny Nygård Hansen

1st April 2022

5 • Cell Complexes

Exercise 5.3

Suppose X is a topological space whose topology is coherent with a family $\mathcal{B}$ of subspaces.

(a) If Y is another topological space, then a map $f: X \to Y$ is continuous if and only if $f|_B$ is continuous for every $B \in \mathcal{B}$.

(b) The map $q\colon \coprod_{B \in \mathcal{B}} B \to X$ induced by the inclusion of each set $B \hookrightarrow X$ is a quotient map.

Solution. (a) Let $V \subseteq Y$. Then $f^{-1}(V)$ is open if and only if $(f|_B)^{-1}(V) = f^{-1}(V) \cap B$ is open in B for all $B \in \mathcal{B}$. But this precisely expresses that each $f|_B$ is continuous, so the claim follows.

(b) Notice that $q^{-1}(U) = U \cap B$ for all $U \subseteq X$. Since the topology on X is coherent with $\mathcal{B}$, the set U is open if and only if $U \cap B$ is open for all $B \in \mathcal{B}$. But this precisely expresses that q is a quotient map. □

Problems

Problem 5.5

Suppose X is a topological space and $\{X_\alpha\}$ is a family of subspaces whose union in X. Show that the topology of X is coherent with the subspaces $\{X_\alpha\}$ if and only if it is the finest topology on X for which all of the inclusion maps $i_\alpha : X_\alpha \hookrightarrow X$ are continuous.

Solution. This follows immediately from the fact that $i_\alpha^{-1}(U) = U \cap X_\alpha$ for all $U \subseteq X$. □

PROBLEM 5.6

Suppose X is a topological space. Show that the topology of X is coherent with each of the following collections of subspaces of X:

(a) Any open over of X.

(b) Any locally finite closed cover of X.

SOLUTION. (a) Let $\mathcal{V}$ be an open cover of X. If $U \subseteq X$ is open $U \cap V$ is open for all $V \in \mathcal{V}$ (as Lee also remarks, this implication always holds). Conversely, if $U \cap V$ is open in V for all $V \in \mathcal{V}$, then since each V is open in X, $U \cap V$ is also open in X. Furthermore, because $\mathcal{V}$ is a cover of X we have

$$U = U \cap \bigcup_{V \in \mathcal{V}} V = \bigcup_{V \in \mathcal{V}} (U \cap V),$$

so U is a union of open set, hence itself open.

(b) We first prove the following lemma:

> *Let $\mathcal{F}$ be a locally finite collection of closed sets in a topological space X. Then the union $\mathbb{F} = \bigcup_{F \in \mathcal{F}} F$ is closed in X.*

Let $x \in \mathbb{F}^c$. Then since $\mathcal{F}$ is locally finite, x has an open neighbourhood U that intersects finitely many elements from $\mathcal{F}$, say $F_1, \ldots, F_n$. Let $U' = U \setminus (F_1 \cup \cdots \cup F_n)$. Then U' is an open neighbourhood of x disjoint from $\mathbb{F}$, so $\mathbb{F}^c$ is open.

We now solve the exercise. Let $\mathcal{F}$ be a locally finite closed over of X, and let $C \subseteq X$ be such that $C \cap F$ is closed in F for all $F \in \mathcal{F}$. Then

$$C = C \cap \bigcup_{F \in \mathcal{F}} F = \bigcup_{F \in \mathcal{F}} (C \cap F).$$

The collection $\{C \cap F \mid F \in \mathcal{F}\}$ is clearly also locally finite, so since each $C \cap F$ is closed in X, the lemma shows that the above union is also closed in X. $\square$