Lee: *Introduction to Topological Manifolds*

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2 • Topological Spaces

Exercise 2.32

(a) Every homeomorphism is a local homeomorphism.

(b) Every local homeomorphism is continuous and open.

(c) Every bijective local homeomorphism is a homeomorphism.

Solution. (a) This is obvious since the domain is an open neighbourhood of each point, and this is mapped onto the codomain which is also open.

(b) Continuity follows from Exercise 5.3 and Problem 5.6. To prove openness, let $f\colon X \to Y$ be a local homeomorphism and $V \subseteq X$ an open set. Let $\mathcal{U}$ be a cover of V of open sets in accordance with the definition of local homeomorphisms. Then $V \cap U$ is open in U for all $U \in \mathcal{U}$, and since $f|_U$ is a homeomorphism $f(V \cap U)$ is also open in $f(U)$, hence in Y. Furthermore,

$$f(V) = \bigcup_{U \in \mathcal{U}} f(V \cap U),$$

so $f(V)$ is a union of open sets in Y.

(c) This is obvious from (b). $\square$

Problems

Problem 2.6

Suppose X and Y are topological spaces, and $f\colon X \to Y$ is any map.

(a) f is continuous if and only if $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq X$.

(b) f is closed if and only if $f(\overline{A}) \supseteq \overline{f(A)}$ for all $A \subseteq X$.

SOLUTION. (a) First assume that f is continuous, and let $A \subseteq X$. Then $A \subseteq f^{-1}(\overline{f(A)})$, which implies that $\overline{A} \subseteq f^{-1}(\overline{f(A)})$. This is equivalent to the property above.

Conversely, let $F \subseteq Y$ be closed. Then

$$f\left(\overline{f^{-1}(F)}\right) \subseteq \overline{f(f^{-1}(F))} \subseteq \overline{F} = F,$$

showing that f is continuous.

(b) The 'if' implication is obvious, so we prove the converse. Let $F \subseteq X$ be closed. Then

$$f(F) = f(\overline{F}) \supseteq \overline{f(F)},$$

which shows that $f(F) = \overline{f(F)}$, so $f(F)$ is closed. □

PROBLEM 2.21

Show that every locally Euclidean space is first countable.

SOLUTION. Let M be locally Euclidean of dimension n. If $p \in M$, then p lies in the domain of a chart (U, φ). For simplicity we may let U be homeomorphic to $\mathbb{B}^n$. For $r \in \mathbb{Q} \cap (0,1]$ let B_r be the preimage under φ of the ball $B_r(0) \subseteq \mathbb{R}^n$. We claim that the B_r constitute a neighbourhood basis at p.

Let V be an open neighbourhood of p. By intersecting it with U we may assume that $V \subseteq U$. Then $\varphi(V)$ is open in $\mathbb{B}^n$ and hence contains $B_r(0)$ for some $r \in \mathbb{Q} \cap (0,1]$. But then V contains B_r. □

PROBLEM 2.23

Show that every manifold has a basis of coordinate balls.

[TODO: Is being locally Euclidean not enough?]

SOLUTION. Let M be an n-manifold, and let $U \subseteq M$ be open. If $p \in U$, then p lies in the domain of a chart. By intersecting U with this chart domain we may assume that U is itself the domain of a chart with coordinate map φ. Since $\varphi(U) \subseteq \mathbb{R}^n$ is open it contains an open ball B with $p \in B$. Notice that φ restricts to a homeomorphism $\varphi^{-1}(B) \to B$, so this preimage is a coordinate ball lying in U and containing p. □

PROBLEM 2.25

If M is an n-dimensional manifold with boundary, then $\operatorname{Int} M$ is an open subset of M, which is itself an n-dimensional manifold without boundary.

SOLUTION. It suffices to show that $\operatorname{Int} M$ is open in M. To this end, let $p \in \operatorname{Int} M$, and let (U, φ) be an interior chart with $p \in U$. If $q \in U$, then q also lies in the domain of an interior chart, namely (U, φ). Thus $q \in \operatorname{Int} M$, so $U \subseteq \operatorname{Int} M$. □

3 • New Spaces from Old

General notes

REMARK 3.1. We rephrase Proposition 3.56 and its proof. First note that Problem 2.19 (which is used in its proof) implies that a locally Euclidean Lindelöf space is second countable. Next let X be a Lindelöf space (so for instance a second countable space), Y a locally Euclidean space, and let $f: X \to Y$ be a continuous surjection. Then Y is also Lindelöf, hence second countable.

In particular, the hypothesis that f be a quotient map is superfluous. ⌟

REMARK 3.2. We comment on the proof of Proposition 3.67, which shows that a continuous open/closed surjection is a quotient map. Let $q: X \to Y$ be such a map, and let $U \subseteq Y$. It suffices to show that if $q^{-1}(U)$ is open/closed, then so is U. But since q is open/closed, so is $q(q^{-1}(U)) = U$, where the equality follows since q is surjective.

In other words, there is no need to appeal to Proposition 3.60. ⌟

REMARK 3.3: Uniqueness of quotient spaces.
We phrase the proof of Theorem 3.75 in a slightly different way. If $q_1: X \to Y_1$ and $q_2: X \to Y_2$ are quotient maps that make the same identifications, then each map factors uniquely through the other. This yields the diagram

$$\begin{tikzcd} & X \arrow[dl, "q_1"'] \arrow[d, "q_2"] \arrow[dr, "q_1"] & \\ Y_1 \arrow[r, dashed, "\tilde{q}_2"'] & Y_2 \arrow[r, dashed, "\tilde{q}_1"'] & Y_1 \end{tikzcd}$$

Since q_1 factors uniquely through itself as the identity, we have $\tilde{q}_1 \circ \tilde{q}_2 = \mathrm{id}_{Y_1}$. By symmetry we also have $\tilde{q}_2 \circ \tilde{q}_1 = \mathrm{id}_{Y_2}$, so $Y_1 \cong Y_2$.

This in particular yields a sort of ‘first isomorphism theorem’ for topology. Let $q: X \to Y$ be a quotient map, and denote by $\ker q$ the equivalence relation on X given by $x \sim x'$ if and only if $q(x) = q(x')$. The quotient map $p: X \to$

$X/\ker q$ then trivially makes the same identifications as q, so q descends to a homeomorphism $\tilde{q}\colon X/\ker q \xrightarrow{\sim} Y$.

Notice that we do require q to be a quotient map: Say that q is only assumed to be a continuous surjection. It then induces a continuous map $\tilde{q}\colon X/\ker q \to Y$, and we have the commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X/\ker q \arrow[rr, "\tilde{q}"'] & & Y \end{tikzcd}$$

If $\tilde{q}$ is a homeomorphism then Y automatically carries the quotient topology induced by q. For a subset $U \subseteq Y$ is open if and only if $\tilde{q}^{-1}(U)$ is open, which is the case if and only if

$$p^{-1}(\tilde{q}^{-1}(U)) = (\tilde{q} \circ p)^{-1}(U) = q^{-1}(U)$$

is open. ⌟

Exercises

EXERCISE 3.32

Let $\{X_\alpha\}_{\alpha\in A}$ be a collection of topological spaces

(a) For any $\beta \in A$ and any $x_\alpha \in X_\alpha$ for $\alpha \neq \beta$, define a map $f\colon X_\beta \to \prod_{\alpha\in A} X_\alpha$ by $f(x) = (x_\alpha)_{\alpha\in A}$, where $x_\beta = x$. Then f is a topological embedding.

(b) Each canonical projection $\pi_\beta\colon \prod_{\alpha\in A} X_\alpha \to X_\beta$ is an open map.

SOLUTION. (a) It is easy to see by looking at a basis for the product space that f is continuous. By Problem 3.13 it suffices to show that f has a continuous left inverse. But the projection π_β is a left inverse for f, and this is continuous.

(b) It suffices to show that $\pi_\beta(U)$ is open for all elements U in a basis for the product topology. Hence assume that U is on the form $\prod_{\alpha\in A} U_\alpha$, where U_α is open in X_α, and $U_\alpha = X_\alpha$ except for finitely many α. But then $\pi_\beta(U) = U_\beta$, which is open. □

EXERCISE 3.55

Show that every wedge sum of Hausdorff spaces is Hausdorff.

SOLUTION. Let $\{X_\alpha\}_{\alpha\in A}$ be a collection of Hausdorff spaces, denote the base point of X_α by p_α, and let x and y be distinct points in their wedge sum.

First assume that neither point is the base point, and that $x, y \in X_\alpha$ for some $\alpha \in A$. Picking disjoint open neighbourhoods $U, V \subseteq X_\alpha$ of x and y yields disjoint open neighbourhoods $U \setminus \{p_\alpha\}$ and $V \setminus \{p_\alpha\}$ of x and y in X_α. These are clearly also disjoint and open in $\bigvee_{\alpha \in A} X_\alpha$.

Next assume that neither point is the base point, and that $x \in X_\alpha$ and $y \in X_\beta$ for some $\alpha \neq \beta$. In this case the sets $X_\alpha \setminus \{p_\alpha\}$ and $X_\beta \setminus \{p_\beta\}$ work.

Finally assume that x is the base point and that $y \in X_\beta$. Let $x \in U$ and $y \in V$ in X_β. The set

$$U \vee \bigvee_{\alpha \neq \beta} X_\alpha$$

is an open neighbourhood of x disjoint from V. □

EXERCISE 3.59

Let $f : X \to Y$ be any map. For a subset $A \subseteq X$, show that the following are equivalent:

(a) A is saturated.

(b) $A = f^{-1}(f(A))$.

(c) A is a union of fibres.

(d) If $x \in A$, then $f(x) = f(x')$ implies that $x' \in A$, for all $x' \in X$.

SOLUTION. *(a) ⇔ (b)*: First assume that A is saturated, so that $A = f^{-1}(B)$ for some $B \subseteq Y$. Then $f(A) \subseteq B$, so $f^{-1}(f(A)) \subseteq f^{-1}(B) = A$, and the opposite inclusion always holds. The opposite implication is obvious.

(a) ⇔ (c): Simply notice that

$$f^{-1}(B) = f^{-1}\left(\bigcup_{y \in B} \{y\}\right) = \bigcup_{y \in B} f^{-1}(y)$$

for any $B \subseteq Y$, so A is on the form $f^{-1}(B)$ if and only if it is a union of fibres.

(c) ⇔ (d): First assume that A is a union if fibres, and let $x \in A$. If $f(x) = f(x')$ for some $x' \in X$, then x and x' lie in the same fibre, and this fibre is either contained entirely in A or is disjoint from A. Hence $x' \in A$.

Conversely, if $x \in A$ then $f^{-1}(x) \subseteq A$, since $f(x) = f(x')$ for all x' in this preimage. □

EXERCISE 3.61

A continuous surjective map $q : X \to Y$ is a quotient map if and only if it takes saturated open subsets to open subsets, or saturated closed subsets to closed subsets.

SOLUTION. First assume that q is a quotient map, and let $U \subseteq X$ be a saturated open subset. Then since $U = q^{-1}(q(U))$, the set U is open if and only if $q(U)$ is open.

Conversely, assume that q takes saturated open subsets to open subsets, and let $V \subseteq Y$. We need to show that V is open if and only if $q^{-1}(V)$ is open. The 'only if' part follows since q is continuous. For the 'if' part, assume that $q^{-1}(V)$ is open and notice that it is saturated. But then $q(q^{-1}(V)) = V$ is also open, where the equality follows since q is surjective.

The case where q takes saturated closed subsets to closed subsets follows similarly (replace 'open' with 'closed' throughout). □

EXERCISE 3.62

Properties of quotient maps.

(a) Any composition of quotient maps is a quotient map.

(b) An injective quotient map is a homeomorphism.

(c) If $q \colon X \to Y$ is a quotient map, a subset $F \subseteq Y$ is closed if and only if $q^{-1}(F)$ is closed in X.

(d) If $q \colon X \to Y$ is a quotient map and $U \subseteq X$ is a saturated open or closed subset, then the restriction $q|_U \colon U \to q(U)$ is a quotient map.

(e) If $\{q_\alpha \colon X_\alpha \to Y_\alpha\}_{\alpha \in A}$ is an indexed family of quotient maps, then the map $q \colon \coprod_{\alpha \in A} X_\alpha \to \coprod_{\alpha \in A} Y_\alpha$ whose restriction to each X_α is equal to q_α is a quotient map.

SOLUTION. (a) This follows from the fact that final topologies compose.

(b) Let $q \colon X \to Y$ be an injective quotient map. Since q is already continuous and surjective, it suffices to show that it is open. For any subset $U \subseteq X$ we have $U = q^{-1}(q(U))$ since q is injective, so U is open if and only if $q(U)$ is. Hence q is open.

(c) This follows easily from the definition of the quotient topology by taking complements.

(d) Assume that U is open (the case where U is closed is similar). By Proposition 3.60 (or Exercise 3.61) it suffices to show that $q|_U$ takes saturated open subsets of U to open subsets, so let $V \subseteq U$ be open. Then V is also open in X, and $q(V)$ is open in Y. But then $q(V)$ is also open in $q(U)$ as desired.

(e) This also follows from the fact that final topologies compose. To be explicit, consider for each $\beta \in A$ the diagram

$$\begin{tikzcd} X_\beta \arrow[rr, "q_\beta"] \arrow[dr, "\iota_{X_\beta}"'] & & Y_\beta \arrow[dr, "\iota_{Y_\beta}"] & \\ & \coprod_{\alpha\in A} X_\alpha \arrow[rr, "q"'] & & \coprod_{\alpha\in A} Y_\alpha \end{tikzcd}$$

Each Y_β has the final topology coinduced by q_β, and $\coprod_{\alpha\in A} Y_\alpha$ has the final topology coinduced by the maps ι_{Y_β}. Since final topologies compose, this also has the final topology induced by the maps $\iota_{Y_\beta} \circ q_\beta$. But since the above diagram commutes, this is the same as the final topology coinduced by the maps $q \circ \iota_{X_\beta}$. But since $\coprod_{\alpha\in A} X_\alpha$ itself carries the final topology coinduced by the ι_{X_β}, the topology on $\coprod_{\alpha\in A} Y_\alpha$ is the same as the final topology coinduced by the map q. □

Problems

PROBLEM 3.1

Suppose M is an n-dimensional manifold with boundary. Show that ∂M is an $(n-1)$-manifold (without boundary) when endowed with the subspace topology. (We may assume invariance of the boundary.)

SOLUTION. Let $p \in \partial M$. Then p lies in the domain of a boundary chart (U, φ), and $\varphi(p) \in \partial\mathbb{H}^n$. We claim that φ maps $U \cap \partial M$ into $\partial\mathbb{H}^n$. If $q \in U \cap \partial M$, then q does not lie in $\operatorname{Int} M$ (since we are assuming invariance of the boundary). If we had $\varphi(q) \in \operatorname{Int}\mathbb{H}^n$, then we could restrict the domain of U to a smaller open set U' such that we still had $q \in U'$, and such that $\varphi(U')$ was an open subset of $\operatorname{Int}\mathbb{H}^n$. (For instance, we could simply remove $\partial\mathbb{H}^n$ from the codomain and restrict the domain accordingly.) But this is impossible since q is not an interior point, so we must have $\varphi(q) \in \partial\mathbb{H}^n$.

Hence φ restricted to $U \cap \partial M$ is a chart from ∂M to $\partial\mathbb{H}^n \cong \mathbb{R}^{n-1}$. □

PROBLEM 3.13

Suppose X and Y are topological spaces and $f : X \to Y$ is a continuous map. Prove the following:

(a) If f admits a continuous left inverse, it is a topological embedding.

(b) If f admits a continuous right inverse, it is a quotient map.

(c) TODO

SOLUTION. (a) Let $g\colon Y \to X$ be a continuous left inverse of f, and define $\tilde{f}\colon X \to f(X)$ by $\tilde{f}(x) = f(x)$. Then $g|_{f(X)}$ is continuous and a left inverse of $\tilde{f}$. But since $\tilde{f}$ is bijective it has a unique (two-sided) inverse, namely $g|_{f(X)}$, and since this is also continuous $\tilde{f}$ is a homeomorphism.

(b) Let $g\colon Y \to X$ be a continuous right inverse of f, and let $U \subseteq Y$ be such that $f^{-1}(U)$ is open. Then

$$g^{-1}(f^{-1}(U)) = (f \circ g)^{-1}(U) = U \qquad \square$$

is also open, so f is a quotient map.

(c)

REMARK 3.4. I would have liked for split mono- and epimorphisms to be open and closed, but I'm not sure this is the case. Instead, this problem and Proposition 3.69 seem to provide independent criteria under which a continuous map f is an embedding or a quotient map: In either case f must be injective/surjective. But then we further assume either that f splits, or that f is open/closed. In other words, the hypothesis that f splits can be replaced with the weaker hypothesis that f is injective/surjective, but then we must further assume that f is open or closed. ⌟

5 • Cell Complexes

General notes

REMARK 5.1: Coherent topologies.
Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{B}$ be a collection of subspaces of X whose union is X. We claim that $\mathcal{T}$ is coherent with $\mathcal{B}$ if and only if $\mathcal{T}$ is the final topology coinduced by the inclusion maps $\iota_B\colon B \to X$ for all $B \in \mathcal{B}$. For this topology has the property that a set $U \subseteq X$ is open if and only if $\iota_B^{-1}(U) = U \cap B$ is open in B for all $B \in \mathcal{B}$.

This is precisely Problem 5.5. ⌟

REMARK 5.2. We comment on Proposition 5.4, the claim that Hausdorff spaces equipped with a locally finite cell depomposition is a CW complex.

Let $\mathcal{E}$ be the cell decomposition in question, and let $\overline{\mathcal{E}} = \{\bar{e} \mid e \in \mathcal{E}\}$. Then $\overline{\mathcal{E}}$ is a locally finite closed cover of X, so Problem 5.6(b) shows that the topology of X is coherent with $\overline{\mathcal{E}}$. ⌟

Exercises

EXERCISE 5.3

Suppose X is a topological space whose topology is coherent with a family $\mathcal{B}$ of subspaces.

(a) If Y is another topological space, then a map $f\colon X \to Y$ is continuous if and only if $f|_B$ is continuous for every $B \in \mathcal{B}$.

(b) The map $q\colon \coprod_{B\in\mathcal{B}} B \to X$ induced by the inclusion of each set $B \hookrightarrow X$ is a quotient map.

SOLUTION. (a) Let $V \subseteq Y$. Then $f^{-1}(V)$ is open if and only if $(f|_B)^{-1}(V) = f^{-1}(V) \cap B$ is open in B for all $B \in \mathcal{B}$. But this precisely expresses that each $f|_B$ is continuous, so the claim follows.

Alternatively, since $f|_B = \iota_B \circ f$, this precisely expresses the universal property of the final topology induced by the inclusion maps ι_B, so this follows from Remark 5.1.

(b) Notice that $q^{-1}(U) = U \cap B$ for all $U \subseteq X$. Since the topology on X is coherent with $\mathcal{B}$, the set U is open if and only if $U \cap B$ is open for all $B \in \mathcal{B}$. But this precisely expresses that q is a quotient map.

Alternatively, this follows since X has the final topology induced by the inclusion maps, but the disjoint union $\coprod_{B\in\mathcal{B}} B$ also has a final topology, and final topologies compose. □

Problems

PROBLEM 5.5

Suppose X is a topological space and $\{X_\alpha\}$ is a family of subspaces whose union in X. Show that the topology of X is coherent with the subspaces $\{X_\alpha\}$ if and only if it is the finest topology on X for which all of the inclusion maps $i_\alpha\colon X_\alpha \hookrightarrow X$ are continuous.

SOLUTION. This follows immediately from the fact that $i_\alpha^{-1}(U) = U \cap X_\alpha$ for all $U \subseteq X$. □

PROBLEM 5.6

Suppose X is a topological space. Show that the topology of X is coherent with each of the following collections of subspaces of X:

(a) Any open over of X.

(b) Any locally finite closed cover of X.

SOLUTION. (a) Let $\mathcal{V}$ be an open cover of X. If $U \subseteq X$ is open $U \cap V$ is open for all $V \in \mathcal{V}$ (as Lee also remarks, this implication always holds). Conversely, if $U \cap V$ is open in V for all $V \in \mathcal{V}$, then since each V is open in X, $U \cap V$ is also open in X. Furthermore, because $\mathcal{V}$ is a cover of X we have

$$U = U \cap \bigcup_{V \in \mathcal{V}} V = \bigcup_{V \in \mathcal{V}} (U \cap V),$$

so U is a union of open set, hence itself open.

(b) We first prove the following lemma:

Let $\mathcal{F}$ be a locally finite collection of closed sets in a topological space X. Then the union $\mathbb{F} = \bigcup_{F \in \mathcal{F}} F$ is closed in X.

Let $x \in \mathbb{F}^c$. Then since $\mathcal{F}$ is locally finite, x has an open neighbourhood U that intersects finitely many elements from $\mathcal{F}$, say $F_1, \ldots, F_n$. Let $U' = U \setminus (F_1 \cup \cdots \cup F_n)$. Then U' is an open neighbourhood of x disjoint from $\mathbb{F}$, so $\mathbb{F}^c$ is open.

We now solve the exercise. Let $\mathcal{F}$ be a locally finite closed over of X, and let $C \subseteq X$ be such that $C \cap F$ is closed in F for all $F \in \mathcal{F}$. Then

$$C = C \cap \bigcup_{F \in \mathcal{F}} F = \bigcup_{F \in \mathcal{F}} (C \cap F).$$

The collection $\{C \cap F \mid F \in \mathcal{F}\}$ is clearly also locally finite, so since each $C \cap F$ is closed in X, the lemma shows that the above union is also closed in X. $\square$