Concrete topological spaces

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1 • Introduction

In this note we elaborate on some of the examples of topological spaces discussed in John M. Lee's *Introduction to Topological Manifolds*, henceforth denoted [TM]. References are placed in brackets at the end of each text unit. All references are to [TM].

We follow the notation used in [TM], except that we write $X \cong Y$ to denote that topological spaces X and Y are homeomorphic. Furthermore, norms are denoted $\|\cdot\|$.

Furthermore, we make frequent use of a few central results without explicit reference: First of all the closed map lemma [TM, Lemma 4.50], which says that continuous maps from compact spaces into Hausdorff spaces are closed. Secondly the uniqueness of quotient spaces [TM, Theorem 3.75], which says that if two quotient maps make the same identifications, then the associated quotient spaces are homeomorphic.

2 • Basic spaces

EXAMPLE 2.1: $\mathbb{B}^n \cong \mathbb{R}^n$. The unit ball $\mathbb{B}^n$ is homemorphic to $\mathbb{R}^n$ through the map $F\colon \mathbb{B}^n \to \mathbb{R}^n$ given by

$$F(x) = \frac{x}{1 - \|x\|}.$$

Direct computation shows that F has inverse $G\colon \mathbb{R}^n \to \mathbb{B}^n$ given by

$$G(y) = \frac{y}{1 + \|y\|}.$$

[2.25] ⌟

EXAMPLE 2.2: Homeomorphic spheres. Let $n \in \mathbb{N}$, and let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on $\mathbb{R}^{n+1}$. For $p \in \{0,1\}$, define the 'sphere' $\mathbb{S}^n_p = \{x \in \mathbb{R}^{n+1} \mid \|x\|_p = 1\}$. If also $q \in \{0,1\}$, then we claim that $\mathbb{S}^n_p \cong \mathbb{S}^n_q$. For define maps $\varphi_{q,p}\colon \mathbb{S}^n_p \to \mathbb{S}^n_q$ by

$$\varphi_{q,p}(x) = \frac{x}{\|x\|_q}.$$

This is clearly continuous, and notice that

$$\varphi_{p,q} \circ \varphi_{q,p}(x) = \varphi_{p,q}\left(\frac{x}{\|x\|_q}\right) = \frac{x/\|x\|_q}{\big\|x/\|x\|_q\big\|_p} = \frac{x}{\|x\|_p} = x,$$

since $x \in \mathbb{S}^n_p$. [2.26] ⌟

EXAMPLE 2.3: Graphs of continuous functions. Let $U \subseteq \mathbb{R}^n$ be open, and let $f\colon U \to \mathbb{R}^k$ be continuous. Then U is homeomorphic to the graph $\Gamma(f)$ of f through the homeomorphism $\Phi_f\colon U \to \mathbb{R}^{n+k}$ given by $\Phi_f(x) = (x, f(x))$. Thus $\Gamma(f)$ is a manifold. [3.20] ⌟

EXAMPLE 2.4: Spheres as manifolds. The sphere $\mathbb{S}^n$ is a manifold. We produce two sets of charts:

For $i = 1, \dots, n+1$, let

$$U_i^\pm = \{x \in \mathbb{R}^{n+1} \mid \pm x_i > 0\}.$$

These sets cover $\mathbb{S}^n$. On $U_i^\pm$ we can solve the equation $\|x\| = 1$ and find that x_i is a continuous function of the other coordinates. Thus $\mathbb{S}^n \cap U_i^\pm$ is the graph of a continuous function.

Stereographic projection: Let $N = (0, \dots, 0, 1)$ be the 'north pole' in $\mathbb{S}^n$, and define the stereographic projection $\sigma\colon \mathbb{S}^n \setminus \{N\} \to \mathbb{R}^n$ as the map that sends $x \in \mathbb{S}^n \setminus \{N\}$ to a point $u \in \mathbb{R}^n$ such that $U = (u, 0)$ is the intersection of the line through N and x with the subspace where $x_{n+1} = 0$. Note that $u = \sigma(x)$ can be written $U - N = \lambda(x - N)$ for some $\lambda \in \mathbb{R}$. Solving this for λ gives an explicit form for σ, whose inverse can also be found explicitly. In particular, this provides a Euclidean neighbourhood of every point of $\mathbb{S}^n$ except for N. [3.21] ⌟

3 • Quotient spaces

EXAMPLE 3.1: Spheres as quotients. Define a map $q\colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{S}^n$ by

$$q(x) = \frac{x}{\|x\|}.$$

This is continuous and surjective, and its fibres are open rays in $\mathbb{R}^{n+1} \setminus \{0\}$, so the saturated sets are unions of open rays. Let $U \subseteq \mathbb{R}^{n+1} \setminus \{0\}$ be an open saturated set. Then clearly $q(U) = U \cap \mathbb{S}^n$, and this is open in the subspace topology. Thus q sends saturated open sets to open subsets of $\mathbb{S}^n$, and so q is a quotient map. Thus $\mathbb{S}^n$ is obtained from $\mathbb{R}^{n+1} \setminus \{0\}$ by collapsing open rays to a point. [3.64] ⌟

Example 3.2: $\mathbb{T}^n$ as a quotient of $\mathbb{R}^n$. The subgroup $\mathbb{Z}^n$ of acts freely on $\mathbb{R}^n$ by translation, and gives rise to the coset space $\mathbb{R}^n/\mathbb{Z}^n$. We claim that this is homeomorphic to $\mathbb{T}^n$.

Consider the exponential map $\varepsilon\colon \mathbb{R} \to \mathbb{T}$ given by $\varepsilon(r) = \mathrm{e}^{2\pi \mathrm{i} r}$ and its n-fold product $\varepsilon^n\colon \mathbb{R}^n \to \mathbb{T}^n$. Since ε is a covering map, then so is ε^n by Proposition 11.1, and the same proposition then says that ε^n is a quotient map. But it makes the same identifications as the quotient map $\mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$, so the claim follows by the uniqueness of quotient spaces. [3.92] ⌟

Example 3.3: $\mathbb{T}^n$ as a quotient of I^n. Define an equivalence relation $\sim$ on I^n by letting $(x_1,\dots,x_n) \sim (y_1,\dots,y_n)$ if there is an $i \in \{1,\dots n\}$ such that $x_i = 0$ and $y_i = 1$, and such that $x_j = y_j$ when $j \neq i$. (That is, we identify points on the sides of the cube that are immediately opposite each other.) Then define a map $q\colon I^n \to \mathbb{T}^n$ by

$$q(r_1,\dots,r_n) = (\mathrm{e}^{2\pi \mathrm{i} r_1},\dots,\mathrm{e}^{2\pi \mathrm{i} r_n}).$$

This is open as in Example 3.2, hence a quotient map. Furthermore, it makes the same identifications as $\sim$, so $\mathbb{T}^n$ is homeomorphic to $I^n/\sim$. ⌟

Example 3.4: The doughnut surface. Consider the doughnut surface $D \subseteq \mathbb{R}^3$ obtained by revolving the circle $(x-2)^2 + z^2 = 1$ around the z-axis. It is characterised by the equation $(r-2)^2 + z^2 = 1$, where $r = \sqrt{x^2+y^2}$. Thus there is an angle φ such that $z = \sin\varphi$ and $r - 2 = \cos\varphi$. It follows that

$$x = r\cos\theta = (2+\cos\varphi)\cos\theta \quad \text{and} \quad y = r\sin\theta = (2+\cos\varphi)\sin\theta,$$

for some angle θ. Making the substitutions $\varphi = 2\pi u$ and $\theta = 2\pi v$ yields a surjective map $F\colon \mathbb{R}^2 \to D$ given by

$$F(u,v) = \big((2+\cos 2\pi u)\cos 2\pi v, (2+\cos 2\pi u)\sin 2\pi v, \sin 2\pi u\big).$$

This map is clearly not injective, but restricting it to the square $I \times I$ yields a surjective map that makes the same identifications as the quotient map $I \times I \to \mathbb{T}^2$. By the closed map lemma it is a quotient map, so D and $\mathbb{T}^2$ are homeomorphic. [3.22, 3.49, 4.52, 4.53] ⌟

EXAMPLE 3.5: Real projective space. Let $\mathbb{P}^n$ be the set of one-dimensional (linear) subspaces of $\mathbb{R}^{n+1}$, and let $q\colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ send points to their span. Give $\mathbb{P}^n$ the quotient topology with respect to q. Alternatively define an equivalence relation on $\mathbb{R}^{n+1} \setminus \{0\}$ by declaring that x and y be equivalent if $x = \lambda y$ for some $y \in \mathbb{R} \setminus \{0\}$. Except for the point 0, the equivalence classes are the fibres of q. Notice that $\mathbb{P}^n$ is precisely the orbit space of the action of $\mathbb{R}^*$ on $\mathbb{R}^{n+1} \setminus \{0\}$.

We claim that $\mathbb{P}^n$ is Hausdorff: Given two distinct points $u, v \in \mathbb{P}^n$, notice that there exist $x, y \in \mathbb{R}^{n+1}$ such that

$$\{x, -x\} = q^{-1}(u) \cap \mathbb{S}^n \quad \text{and} \quad \{y, -y\} = q^{-1}(v) \cap \mathbb{S}^n.$$

Let $\varepsilon = \frac{1}{2}\min\{\|x-y\|, \|x+y\|\}$, and let $U = B(x, \varepsilon)$ and $V = B(y, \varepsilon)$. Then no one-dimensional subspace intersects both U and V, so $q(U)$ and $q(V)$ are disjoint. But q is open by Problem 3.22, so these are disjoint open neighbourhoods of $q(x)$ and $q(y)$ respectively.

We can also represent $\mathbb{P}^n$ as a quotient of $\mathbb{S}^n$ by identifying antipodal points. Let $\sim$ denote this equivalence relation, and let $p\colon \mathbb{S}^n \to \mathbb{S}^n/{\sim}$ be the associated quotient map. Consider the composite map

$$\mathbb{S}^n \xhookrightarrow{\iota} \mathbb{R}^{n+1} \setminus \{0\} \xrightarrow{q} \mathbb{P}^n,$$

where ι is the inclusion. The composition $q \circ \iota$ is a quotient map by the closed map lemma, and it makes the same identifications as p, so $\mathbb{P}^n$ is homeomorphic to $\mathbb{S}^n/{\sim}$. This also shows that $\mathbb{P}^n$ is compact. [3.51, 3.91, 4.54] ⌟

EXAMPLE 3.6: Collapsing $\partial\overline{\mathbb{B}}^n$ to a point. Let $\overline{\mathbb{B}}^n/\mathbb{S}^{n-1}$ be the quotient space obtained by collapsing the boundary of $\overline{\mathbb{B}}^n$ to a point. We show that this is homeomorphic to $\mathbb{S}^n$. To this end, let $q\colon \overline{\mathbb{B}}^n \to \mathbb{S}^n$ be the map given by

$$q(x) = \Big(2\sqrt{1-\|x\|^2}\,x, 2\|x\|^2 - 1\Big).$$

Computing the norm of $q(x)$ shows that this is well-defined. The map is surjective and continuous, and a quotient map by the closed map lemma. Notice that it is injective on $\mathbb{B}^n$ and constant on $\partial\mathbb{B}^n$, so it makes the same identifications as the quotient map $\overline{\mathbb{B}}^n \to \overline{\mathbb{B}}^n/\mathbb{S}^{n-1}$. Hence this quotient space is homeomorphic to $\mathbb{S}^n$. [3.52, 4.55] ⌟

EXAMPLE 3.7: $C\mathbb{S}^n \cong \overline{\mathbb{B}}^{n+1}$. The map $F\colon \mathbb{S}^n \times I \to \overline{\mathbb{B}}^{n+1}$ defined by $F(x,s) = sx$ is continuous and surjective, and by the closed map lemma it is a quotient map. It maps $\mathbb{S}^n \times \{0\}$ to $0 \in \overline{\mathbb{B}}^{n+1}$ and is injective elsewhere, so it makes the same identifications as the quotient map $\mathbb{S}^n \times I \to C\mathbb{S}^n$. Hence $C\mathbb{S}^n$ is homeomorphic to $\overline{\mathbb{B}}^{n+1}$. [3.65, 4.56] ⌟

4 • Adjunction spaces

EXAMPLE 4.1. Let (X, x) and (Y, y) be pointed topological spaces. Let $A = \{y\}$ and define $f: A \to X$ by $f(y) = x$. Then the adjunction space $X \cup_f Y$ and the wedge sum $X \vee Y$ are identical. [3.78(a)] ⌟

EXAMPLE 4.2: Gluing two balls together. Let $A = \mathbb{S}^1 \subseteq \overline{\mathbb{B}}^2$, and let $f: A \hookrightarrow \overline{\mathbb{B}}^2$ be the inclusion map. Then the adjunction space $\overline{\mathbb{B}}^2 \cup_f \overline{\mathbb{B}}^2$ is homeomorphic to $\mathbb{S}^2$: Let $q: \overline{\mathbb{B}}^2 \sqcup \overline{\mathbb{B}}^2 \to \mathbb{S}^2$ be the map that sends the first copy of $\overline{\mathbb{B}}^2$ to the upper hemisphere of $\mathbb{S}^2$ and the second copy to the lower hemisphere. This makes the same identifications as f, so the adjunction space is homeomorphic to $\mathbb{S}^2$. [3.78(b)] ⌟